SSY281 Model Predictive Control 2023
PSS 5 - MPT: Explicit MPC and Minimum-time Control

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Lecture refresh

Explicit MPC: procedure to find explicit control laws for constrained RHC problems.

General idea: start from the batch approach. Remember that:

$$\boldsymbol{x} = \Omega x + \Gamma \boldsymbol{u}. \tag{1}$$

We can "get rid" of the state vector $\boldsymbol{x}$: condensed formulation. For a fixed initial state x, the LQ-type MPC can be expressed as:

$$V_N(x, \boldsymbol{u}) = \min_{\boldsymbol{u}} \; \frac{1}{2}(\boldsymbol{u}^\top \tilde{R} \boldsymbol{u} + x^\top \tilde{Q} x) + \boldsymbol{u}^\top S x \tag{2a}$$

$$\text{s.t.} \quad F\boldsymbol{u} \leq Gx + h. \tag{2b}$$

This is a parametric optimization problem: different solutions $\boldsymbol{u}^*(x)$ for different starting points x. For a given $\boldsymbol{u}^*(x)$, some constraints are active and the others are not. (**Example**: assume we have only box constraints of the sort: $u_{\min} \leq u(k) \leq u_{\max},\ k = 0, ..., N-1$, and think of active constraints in $\boldsymbol{u}^*(x)$ as "where $u(k)$ hits the bounds $u_{\min}$ or $u_{\max}$")

In the feasible set $\mathcal{X}_N$, one can define a region R_x^* with $x \in R_x^*$ such that every other starting point w in this region results in a command $u^*(w)$ with the same active set as $u^*(x)$ (cf Lecture Notes for the complete derivation).

Bottom line: optimization problem (2) can be written with only equality constraints. Remember the unconstrained LQ problem? We can get an explicit control law inside R_x^* now !

An EMPC controller works in two steps:

1. **Off-line phase**: Partition $\mathcal{X}_N$ in many R_x^* regions and compute an explicit control law for each.
2. **On-line phase**: Find the region in which the current x is and apply the corresponding control feedback.

Tasks

Question 1: Design an MPC with MPT, cf file *MPC_controller.m*. To do that, you should design the stage cost matrices Q, R and the horizon length N. Plot and investigate the closed-loop trajectories.

Question 2: Uncomment and run the parts related to explicit MPC.

Can you make sense of what the EMPC regions represent? Do the MPC and EMPC closed-loop trajectories differ? What observations can you make on the computation times needed for the online and offline phases of each (using *tic* and *toc*)?

Minimum-time Control

Algorithm from the book: F Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems* (Chapter 11, section 5).

Minimum time control problem for some starting point x:

$$\begin{aligned} J^*(x) = \min_{\boldsymbol{u},N} \quad & N && && \text{(3a)} \\ \text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k), && k = 0, ..., N-1, && \text{(3b)} \\ & x(k) \in \mathbb{X}, \quad u(k) \in \mathbb{U}, && k = 0, ..., N-1, && \text{(3c)} \\ & x(0) = x, \quad x(N) \in \mathbb{X}_f, && && \text{(3d)} \end{aligned}$$

Assume we want an explicit control law to solve (3). Better to go for 1 N-horizon EMPC, or N 1-horizon EMPC?

Making use of the j-step controllable sets $\{\mathcal{K}_j(\mathbb{X}_f)\}$, we formulate N 1-horizon

EMPC of the form:

$$
\begin{aligned}
\min_{u_0} \quad & c(x(0), u_0) && \text{(4a)}\\
\text{s.t.} \quad & x(1) = Ax(0) + Bu_0, && \text{(4b)}\\
& x(0) \in \mathbb{X}, \quad u_0 \in \mathbb{U}, && \text{(4c)}\\
& x(1) \in \mathcal{K}_{j-1}(\mathbb{X}_f), && \text{(4d)}
\end{aligned}
$$

By definition, feasible set of each subproblem is $\mathcal{K}_j(\mathbb{X}_f)$ (cf Figure 1).

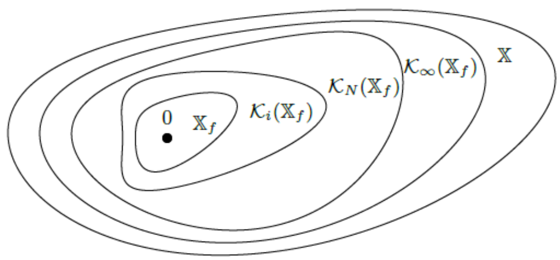

Figure 1: Illustration of the j-step controllable sets.

Algorithm 1: Minimum-time control algorithm.

```
Input: x_0
Off-line: solve each subproblem j, starting from X_f.
x ← x_0.
On-line:
while x ∉ X_f do
    Find smallest controller j s.t. x ∈ K_j(X_f).
    Find controller region R_{j,i} s.t. x ∈ R_{j,i}.
    Apply the corresponding control feedback, i.e. x ← (A + BK_{j,i})x.
end
```

MPT function tips:

- *envelope*: Translate a collection of sets (e.g. a EMPC partition..) into a set object.
- *contains*: Check if a point in a given set.
- *evaluate*: Get command from an EMPC controller.