Systems Biology from Genes to Ecosystems

Pattern Formation

(Lecture by Associate Prof. Christian Fleck[1], Period 2; 2014/15)

Exercise sheet

1) Solving simple partial differential equations

A (1 points): Find the general solution of $\partial_x u(x, y) = 0$

B (1 points): Find the general solution of $\partial_x \partial_y u(x, y) = 0$

2) Diffusion with advection Diffusion plus advection (transport by a flux) is described by the equation:

$$D\partial_x^2 C = v\partial_x C$$

Here $C(x)$ is the concentration of molecules and v being the velocity of the flux and D the diffusion constant. The boundary conditions on a 1-d domain read:

$$\begin{aligned} D\,\partial_x C|_{x=0} &= -\alpha + vC \\ D\,\partial_x C|_{x=1} &= -\beta C + vC \end{aligned}$$

A (2 points): Find the general solution. *Hint: A constant solves the PDE as well as* $e^{v/Dx}$

B (2 points): Specify the general solution by determining the unknown constants using the boundary conditions.

C (2 points): What is the result for $C(x = 0)$ and $C(x = 1)$?

3) Eigenvalues and eigenvectors Solve the following system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x - 2y \end{aligned}$$

Rewrite this in vector notation. Make the Ansatz $\vec{x}(t) = (x, y)^T = \vec{v}e^{\lambda t}$.

A (2 points): Determine the Eigenvalues.

B (2 points): Determine the Eigenvectors.

C (2 points): Determine the full solution using the initial condition $\vec{x}(t = 0) = (x_0, y_0)^T = (2, -3)^T$.

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4) Turing instability of a reaction-diffusion system Consider the following system:

$$\begin{aligned}\frac{\partial u}{\partial t} &= au^2v - bu + D_u\Delta u\\ \frac{\partial v}{\partial t} &= c - au^2v + D_v\Delta v\end{aligned}$$

A (4 points): Rescale length by the systems size L and time by L^2/D_u. *Hint:* $\partial_t \to D_u/L^2\partial_t$ *and* $\Delta \to 1/L^2\Delta$. Rewrite the equations further by rescaling u and v by c/b. Bring the system into the form:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \gamma f(u,v) + \Delta u\\ \frac{\partial v}{\partial t} &= \gamma g(u,v) + d\Delta v\end{aligned}$$

with $\gamma = L^2b/D$ and

$$\begin{aligned}f(u,v) &= \alpha u^2v - u\\ g(u,v) &= 1 - \alpha u^2 v\end{aligned}$$

How do the parameters α and d read in terms of the original parameters a, b and c?

B (2 points): Determine the homogeneous steady states (u_0, v_0).

C (2 points): Determine the Jacobian J. What is the sign structure of it?

D (2 points): What are the constraints on the parameters to ensure a stable homogeneous steady state?

E (2 points): What are the necessary and sufficient conditions on the parameters to ensure a diffusive (Turing) instability?

F (2 points): The solution of the linearized systems in terms of the eigenfunctions of the Laplace operator (Fourier modes, $\Delta\vec{W}_k = -k^2\vec{W}_k$) lead to the equation $[\gamma J - k^2D]\vec{W}_k = \lambda\vec{W}_k$. Determine the eigenvalues $\lambda(k)$.

G (2 points): Find the zeros of the dispersion relation, i.e, $\lambda(k) = 0$. *Hint: This is equivalent to find the zeros of* $h(k) = dk^4 - \gamma(df_u + g_v)k^2 + \gamma^2(f_ug_v - f_vg_u)$. *Why?*