1. A) $\partial_x u(x,y)=0 \Rightarrow u(x,y)$ is independent of $x \Rightarrow u(x,y)=g(y)$

any function of y will solve the PDE

B) $\partial_x\partial_y u(x,y)=0 \Rightarrow u(x,y)=f(x)+g(y)$

$$\partial_x\left[f(x)+g(y)\right]=\partial_x f\ ;\quad \partial_y\left[f(x)+g(y)\right]=\partial_y g$$

$$\partial_x\partial_y\left[f(x)+g(y)\right]=\partial_x\left[\partial_y g(y)\right]=0$$

$$\partial_y\partial_x\left[f(x)+g(y)\right]=\partial_y\left[\partial_x f(x)\right]=0$$

2.

$$D\partial_x^2 C=v\partial_x C$$

$$D\partial_x C\big|_{x=0}=-\alpha+vC$$

$$D\partial_x C\big|_{x=1}=-\beta C+vC$$

A)

$$C=a+be^{v/D\,x}$$

Test: $\partial_x C=\frac{v}{D}be^{v/D\,x}$

$$\partial_x^2 C=\left(\frac{v}{D}\right)^2 be^{v/D\,x}$$

$$\Rightarrow D\partial_x^2 C-v\partial_x C=\frac{v^2}{D}be^{v/D\,x}-\frac{v^2}{D}be^{v/D\,x}=0$$

✓

B) Determine the constants a, b

$$\left.\begin{aligned} D\frac{v}{D}b&=-\alpha+v(a+b)\\ D\frac{v}{D}be^{v/D}&=-(\beta-v)\left(a+be^{v/D}\right)\end{aligned}\right\}\Rightarrow \begin{aligned} a&=\frac{\alpha}{v}\\ b&=-\frac{\alpha}{v}\left(1-\frac{v}{\beta}\right)e^{-v/D}\end{aligned}$$

$$\boxed{c(x) = \frac{\alpha}{v} + \frac{\alpha}{v}\left(\frac{v}{\beta} - 1\right) e^{\frac{v}{D}(x-1)}}$$

c)

$$\boxed{c(0) = \frac{\alpha}{v} + \frac{\alpha}{v}\left(\frac{v}{\beta} - 1\right) e^{-v/D}}$$

$v \to \infty \quad c(0) = \frac{\alpha}{v}$

$v \to 0 \quad c(0) = \frac{\alpha}{\beta} + \frac{\alpha}{D}$

$$\boxed{c(1) = \frac{\alpha}{\beta}}$$

$$c(1) - c(0) = \left(\frac{\alpha}{\beta} - \frac{\alpha}{v}\right)\left(1 - e^{-v/D}\right)$$

$$c(1) - c(0) = \alpha \underbrace{\left(1 - e^{-v/D}\right)}_{\geq 0} \left(\frac{1}{\beta} - \frac{1}{v}\right)$$

$\Rightarrow$

$$c(1) > c(0) \quad \text{for} \quad v > \beta$$

$$c(1) < c(0) \quad \text{for} \quad \beta > v$$

$$c(0) = \frac{\alpha}{v}\left(1 - e^{-v/D}\right) + \frac{\alpha}{\beta} e^{-v/D}$$

$$c(0) = \frac{\alpha}{v}\left(1 - 1 + \frac{v}{D} - \frac{1}{2}\frac{v^2}{D^2} + \dots\right) + \frac{\alpha}{\beta}\left(1 - \frac{v}{D} + \frac{1}{2}\frac{v^2}{D^2} - \dots\right)$$

$$c(0) = \frac{\alpha}{D} - \frac{\alpha v}{2D^2} + \dots + \frac{\alpha}{\beta} - \frac{\alpha v}{\beta D} \dots$$

$$\lim_{v \to 0} c(0) = \frac{\alpha}{\beta} + \frac{\alpha}{D}$$

3

$$\dot{\vec{x}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} \quad ; \quad \vec{x}(t) = \vec{v} e^{\lambda t}$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{v} = \lambda \vec{v} \quad ; \quad \underbrace{\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}}_{A} \vec{v} = 0$$

A) $\det A = 0$

$$\Rightarrow -(1-\lambda)(2+\lambda) - 4 = 0$$

$$\lambda^2 + 2\lambda - \lambda - 2 - 4 = 0 \Leftrightarrow \lambda^2 + \lambda = 6 \Rightarrow \boxed{\lambda_+ = 2 \; ; \; \lambda_- = -3}$$

B) $\lambda_+ = 2$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \Rightarrow \boxed{\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\lambda_- = -3$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \Rightarrow \boxed{\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}}$$

c) $\Rightarrow \quad \vec{x}(t) = c_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\lambda_+ t} + c_- \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{\lambda_- t}$

$$x(0) = c_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_- \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \boxed{c_+ = c_- = 1}$$

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4 A) $\partial_t u = a u^2 v - b u + D_u \nabla^2 u$

$\partial_t v = c - a u^2 v + D_v \nabla^2 u$

(omit the tilde)

$\Rightarrow \frac{D_u}{L^2} \partial_t u = a u^2 v - b u + \frac{D_u}{L^2} \nabla^2 u$

$\frac{D_u}{L^2} \partial_t v = c - a u^2 v + \frac{D_v}{L^2} \nabla^2 v$

Rescale time and length

$\frac{\partial}{\partial t} = \frac{D_u}{L^2} \frac{\partial}{\partial \tilde{t}}$

$\nabla^2 = \frac{1}{L^2} \tilde{\nabla}^2$

$\Leftrightarrow \quad \partial_t u = \frac{L^2 b}{D} \left[\frac{a}{b} u^2 v - u \right] + \nabla^2 u$

$\partial_t v = \frac{L^2 b}{D} \left[\frac{c}{b} - \frac{a}{b} u^2 v \right] + \frac{D_v}{D_u} \nabla^2 v$

$\gamma := \frac{L^2 b}{D}\ ; \quad d = \frac{D_v}{D_u}$

Divide u and v by $\frac{c}{b}$: $\tilde{u} = \frac{ub}{c}$; $\tilde{v} = \frac{vb}{c}$

$$\partial_t \tilde{u} = \gamma \left[\frac{a}{b} \left(\frac{c}{b} \right)^2 \tilde{u}^2 \tilde{v} - \tilde{u} \right] + \nabla^2 u$$

$$\partial_t \tilde{v} = \gamma \left[1 - \frac{a}{b} \left(\frac{c}{b} \right)^2 \tilde{u}^2 v \right] + d \nabla^2 \tilde{v}$$

$$d := \frac{D_v}{D_u}$$

$$\alpha := \frac{ac^2}{b^3}$$

Test units of α: $[a] = \frac{V^2}{S}$; $[b] = \frac{1}{S}$; $[c] = \frac{1}{VS}$

$$[\alpha] = \frac{V^2}{S} \left(\frac{1}{VS} \right)^2 S^3 = \frac{V^2 S^3}{S^3 V^2} = 1 \checkmark$$

omit the tilde

$$\Rightarrow \boxed{\begin{aligned} \partial_t u &= \gamma \left[\alpha u^2 v - u \right] + \nabla^2 u \\ \partial_t v &= \gamma \left[1 - \alpha u^2 v \right] + d \nabla^2 v \end{aligned}}$$

$$\gamma = \frac{L^2 b}{D}$$

$$\alpha = \frac{ac^2}{b^3}$$

$$d = \frac{D_v}{D_u}$$

3)
$$\left. \begin{aligned} \alpha u^2 v &= u \\ \alpha u^2 v &= 1 \end{aligned} \right\} \Rightarrow \boxed{\begin{aligned} u &= 1 \\ v &= \frac{1}{\alpha} \end{aligned}}$$

c)

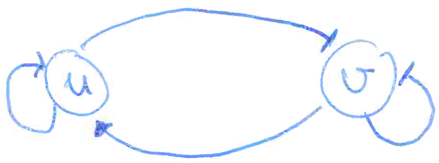
$$f(u,v) = \alpha u^2 v - u \Rightarrow f_u = 2\alpha u v - 1; \quad f_v = \alpha u^2$$

$$g(u,v) = 1 - \alpha u^2 v \qquad g_u = -2\alpha u v; \quad g_v = -\alpha u^2$$

Jacobian at steady state:

$$f_u\Big|_{\substack{u=u_0\\ v=v_0}} = 1; \quad f_v = \alpha; \quad g_u = -2; \quad g_v = -\alpha$$

$$J = \begin{pmatrix} 1 & \alpha \\ -2 & -\alpha \end{pmatrix} \Rightarrow \boxed{J = \begin{pmatrix} + & + \\ - & - \end{pmatrix} \text{ Substrate Depletion}}$$



D)

$$f_u + g_v < 0 \qquad \qquad 1 - \alpha < 0 \Rightarrow \boxed{\alpha > 1}$$

$$f_u g_v - f_v g_u > 0 \Rightarrow \cancel{-\alpha + 2\alpha}$$

$$-\alpha + 2\alpha > 0 \quad \text{always true for } \alpha > 0$$

E)

$$d f_u + g_v > 0 \Rightarrow d - \alpha > 0$$

$$(d f_u + g_v)^2 > 4d\,(f_u g_v - f_v g_u) \quad (d-\alpha)^2 > 4d\alpha \Rightarrow \boxed{\begin{array}{l} d > \alpha \\ d > (3+\sqrt{8})\,\alpha \end{array}}$$

$$\underbrace{\gamma \mathcal{J} - \kappa^2 \mathcal{D} - \lambda \mathbb{1}}_{M} = \begin{pmatrix} \gamma - \kappa^2 - \lambda & \alpha\gamma \\ -2\gamma & -\alpha\gamma - \kappa^2 d - \lambda \end{pmatrix}$$

$$\det M = 0 \Rightarrow -(\gamma - \kappa^2 - \lambda)(\alpha\gamma + \kappa^2 d + \lambda) + 2\alpha\gamma^2 = 0$$

$$\Leftrightarrow \lambda^2 - \underbrace{\left[\gamma - \alpha\gamma - (1+d)\kappa^2\right]}_{a(\kappa)}\lambda + \underbrace{d\kappa^4 - \gamma\,(d-\alpha)\,\kappa^2 + \alpha\gamma^2}_{h(\kappa)} = 0$$

$$\Rightarrow \lambda_\pm = \pm\sqrt{\frac{a^2}{4} - h} + \frac{a}{2}$$

$$\boxed{\lambda_\pm = \pm\sqrt{\frac{\left(\gamma(1-\alpha) - (1+d)\kappa^2\right)^2}{4} - d\kappa^4 + \gamma(d-\alpha)\kappa^2 - \alpha\gamma^2} + \frac{\gamma(1-\alpha) - (1+d)\kappa^2}{2}}$$

$$\lambda_\pm(\kappa=0) = \pm\sqrt{\frac{\gamma^2(1-\alpha)^2}{4} - \alpha\gamma^2} + \frac{\gamma(1-\alpha)}{2}$$

$$\lambda_\pm(\kappa=0) = \frac{\gamma}{2}\left[\pm\sqrt{(1-\alpha)^2 - 4\alpha} + 1 - \alpha\right]$$

6) $\lambda_\pm = \pm\sqrt{\frac{a^2}{4} - h} + \frac{a}{2}$; $a < 0$

$\Rightarrow \lambda_+ = \sqrt{\frac{a^2}{4} - h} - \frac{|a|}{2} \Rightarrow \lambda_+ = 0$ for $h = 0$

$h = d\kappa^4 - \gamma(d-\alpha)\kappa^2 + \alpha\gamma^2 = 0$ Solve for $\kappa^2 = \rho$

$\rho^2 - \frac{\gamma}{d}(d-\alpha)\rho + \frac{\alpha}{d}\gamma^2 = 0$

$\Rightarrow \rho_\pm = \pm\sqrt{\left(\frac{\gamma}{d}\right)^2 \frac{(d-\alpha)^2}{4} - \frac{\alpha}{d}\gamma^2} + \frac{\gamma}{2d}(d-\alpha)$

$$\boxed{\rho_\pm = \frac{\gamma}{d2}\left[\pm\sqrt{(d-\alpha)^2 - 4d\alpha} + d-\alpha\right]}$$

$\rho_+ = \rho_-$ for $d = (3+\sqrt{8})\alpha$