1. The Fibonacci sequence is the list of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

where each number is the sum of the previous two numbers.

Write a piece of computer code to **print out** the first 100 numbers in the Fibonacci sequence.
SOLN: There are 5 key components that the code should have:

- a data structure to store the numbers in the list
- initialization to get $x_0 = 0, x_1 = 1$
- a loop to compute the rest of the numbers
- the formula relating x_n to the sum of the previous two numbers
- a print statement

Here are some sample answers:

Using a list:

```
list = [0,1]
for n in range(100):
   list.append(list[-1]+list[-2])
print(list)
```

Using a numpy array:

```
list = zeros(100)
list[0] = 0; list[1] = 1;
for n in range(2,100):
    list[n] = list[n-1]+list[n-2]
print(list)
```

Using only 3 data elements:

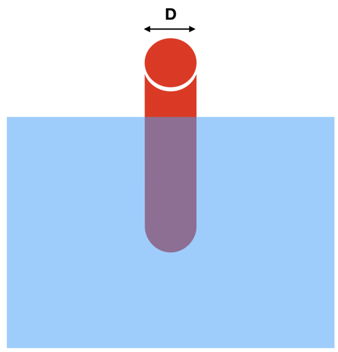
```
x0 = 0;x1 = 1;
print(x0,x1)
for n in range(2,100):
     x2 = x0 + x1
     x0 = x1
     x1 = x2
     print(x2)
```

Using recursion (WARNING: really slow!):

```
def fib(n):
   if n<2:
      return n
   return fib(n-1) + fib(n-2)

for n in range(0,101):
   print(fib(n))
```

2. Take a weighted cylinder, floating upright in water.



It you displace it slightly, it will oscillate back and forth with frequency ω. Suppose ω depends on the diameter D of the cylinder, mass m of the cylinder, and specific weight σ of the water.

Use dimensional analysis to find ω as a function of the other three parameters D, m, σ. State what happens if mass m increases – does ω increase or decrease?

SOLN: There are 4 parameters ω, D, m, σ and three physical units (time, length, and mass). By Buckingham Pi, we expect 4-3 = 1 dimensionless quantity.

A surprising number of people did not know that frequency has dimensions of $1/time$. Think of "cycles per second" or "heartbeats per minute" or "number of showers per week." These are all frequencies, all in units of number per time interval.

The specific weight $\sigma = \rho g$ is in units of Newtons per meters cubed, or using the density time acceleration form, in units of $kg/m^3 \cdot m/sec^2 = kg/(m^2 sec^2)$.

Personally, I would solve this by inspection. Start with σ. To cancel out the kg units, divide by mass m. Then σ/m has units $1/(m^2 sec^2)$ so multiply by D^2 to cancel the $meter^2$ and divide by ω^2 to get rid of the time units.

So the dimensionless quantity is

$$\frac{\sigma D^2}{\omega^2 m} = constant.$$

Solving for ω we have

$$\omega = kD\sqrt{\frac{\sigma}{m}}.$$

Note as mass increases, ω decreases. i.e. a heavier cylinder will oscillate more slowly.

If you like the matrix method, here is the matrix. Use four columns for the parameters, 3 rows for the units:

$$\begin{array}{cccc|l} \omega & D & m & \sigma & \\ \hline -1 & & & -2 & sec \\ & 1 & & -2 & meter \\ & & 1 & 1 & kg \end{array}$$

The kernel vector is $[-2, 2, -1, 1]$ which corresponds to $\omega^{-2}D^2m^{-1}\sigma^1$ just as above.

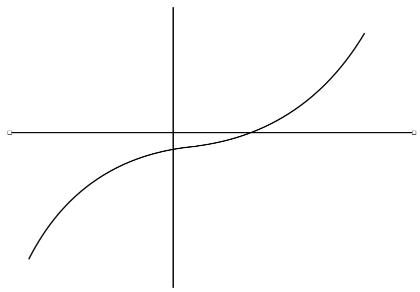
Note: specific weight $\sigma = g\rho$, where g is acceleration due to gravity, ρ is the density of water.

3. Find a series expansion in powers of $\epsilon << 1$, for the single root to the equation

$$x^3 + \epsilon x - 1 = 0.$$

Find the **first three** non-zero terms in the expansion.

SOLN: It helps to sketch the cubic, although you don't need to.



At $\epsilon = 0$ the cubic becomes $x^3 - 1 = 0$ which has one root at $x = 1$. For ϵ small, it is clear that the function $x^3 + \epsilon x - 1$ has the middle term very small, while the x^3 term will have to balance the 1 term.

We expect a regular expansion for the root, so we define a general form

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots .$$

(we can guess the first term is 1, but if you like, you can replace 1 with unknown x_0).

Plug this into the cubic

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots)^3 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) - 1 = 0.$$

Expand, up to order ϵ^3:

$$(1 + \epsilon^3 x_1^3 + 3\epsilon x_1 + 3\epsilon^2 x_1^2 + 3\epsilon^2 x_2 + 3\epsilon^3 x_3 + \cdots) + \epsilon + \epsilon^2 x_1 + \epsilon^3 x_2 + \cdots - 1 = 0.$$

Group the terms by order of ϵ and set to zero:

$$\begin{aligned}
\epsilon^0 &: \; 1 - 1 = 0 && (1)\\
\epsilon^1 &: \; 3\epsilon x_1 + \epsilon = \epsilon(3x_1 + 1) = 0 && (2)\\
\epsilon^2 &: \; 3\epsilon^2 x_1^2 + 3\epsilon^2 x_2 + \epsilon^2 x_1 = \epsilon^2(3x_1^2 + 3x_2 + x_1) = 0 && (3)\\
\epsilon^3 &: \; \epsilon^3 x_1^3 + 3\epsilon^3 x + 3 + \epsilon^3 x_2) = \epsilon^3(x_1^3 + 3x_3 + x_2) = 0 && (4)
\end{aligned}$$

Eqn(1) just tells us we guessed the first term $x_0 = 1$ correctly. Eqn(2) is solved to find $x_1 = -1/3$. Eqn(3) is solved to find $x_2 = 0$. We need one more non-zero term, so we solve Eqn(4) to find $x_3 = -x_1^3/3 = 1/81$. So we have the expansion:

$$x = 1 - \frac{1}{3}\epsilon + \frac{1}{81}\epsilon^3 + \cdots .$$

4. A weight on a spring will satisfy a differential equation

$$\ddot{x} + x = 0,$$

which has a (dimensionless) solution $x_0(t) = \sin(t)$. By perturbing the above equation, we can obtain the non-linear spring differential equation

$$\ddot{x} + x + \epsilon x^3 = 0,$$

which will have a perturbed solution of the form

$$x(t) = \sin(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots .$$

Find the differential equation that is satisfied by the first term $x_1(t)$ in the regular expansion of the perturbed solution.

SOLN: Plug this $x(t) = \sin(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$ into the perturbed differential equation to get

$$(-\sin(t)+\epsilon\ddot{x}_1(t)+\epsilon^2\ddot{x}_2(t)+\cdots)+(\sin(t)+\epsilon x_1(t)+\epsilon^2 x_2(t)+\cdots)+\epsilon(\sin(t)+\epsilon x_1(t)+\epsilon^2 x_2(t)+\cdots)^3 = 0.$$

The sines cancel, so now look at the order ϵ terms, set to zero:

$$\epsilon\ddot{x}_1(t) + \epsilon x_1(t) + \epsilon\sin^3(t) = 0.$$

Canceling the epsilon, we get the ODE

$$\ddot{x}_1(t) + x_1(t) + \sin^3(t) = 0.$$

Note: although you are not asked to solve this, you might like to try. It's really not hard!

Note: Don't try to **solve** the differential equation. Just find the differential equation..