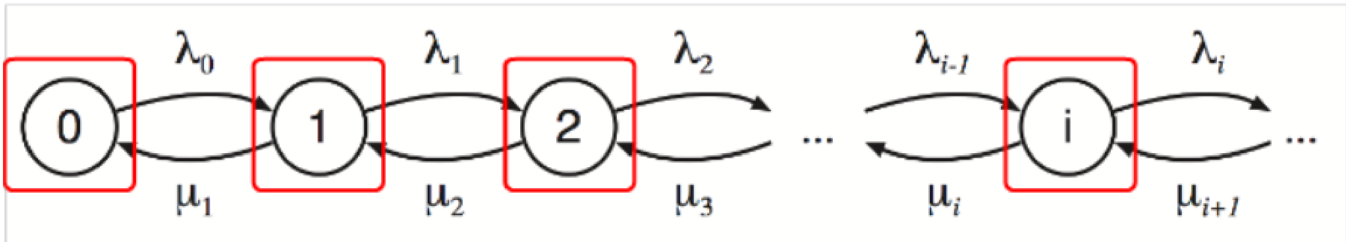
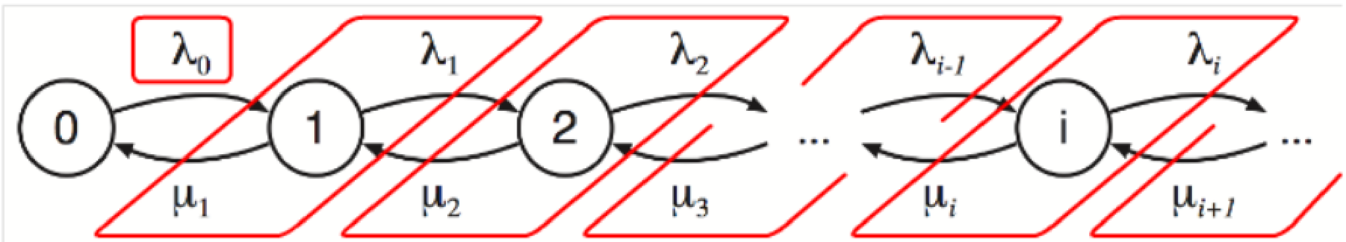
14.States classification and Birth Death Processes

Birth-death processes are special types of CTMC that studies the evolution of the number of the members of a population. Their state corresponds toe the count of the population in a given time instant.

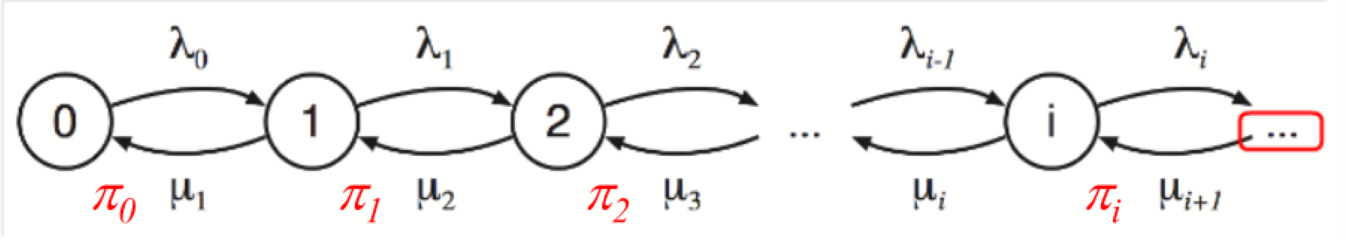


In each state i, the population can increase to state i+1 after a state depended exponentially distributed time, characterized by a rate λ_i (measured for example in s^{-1}). Such transitions are called births.
The population in each non-zero state i can also decrease to the state i-1 after a state dependent exponentially distributed time of rate μ_i (also measured in s^{-1}). Such transitions are called deaths.
Note that all rates can be different in each state. Excluding state 0, both birth and death can occur: the system will move to the next state according to which of the two events occurs first (race policy).



The number of states can be infinite: however if the system is stable, the probability of being in a state tends to zero as i tends to infinity, and only the first few states will have a non-negligible steady state probability.
The infinitesimal generator of the CTMC is a simple tri-diagonal matrix:

$$Q = \begin{vmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & & \\ & \mu_2 & -\lambda_2-\mu_2 & \lambda_2 & \\ & \ddots & \ddots & \ddots & \\ & \mu_i & -\lambda_i-\mu_i & \lambda_i & \\ & & \ddots & \ddots & \ddots \end{vmatrix}$$



The steady state probabilities of the CTMC can be explicitlycomputed solving the so called balance equations:

$$\pi Q = 0 \Rightarrow \pi_i(\lambda_i + \mu_i) = \pi_{i-1}\lambda_{i-1} + \pi_{i+1}\mu_{i+1}$$

It can be proven that the probability π_n of being in a state n is proportional to the following value:

$$\pi_n \sim \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

That is, it exists a constant a such that:

$$\pi_0 = \alpha$$
$$\pi_1 = \alpha \frac{\lambda_0}{\mu_1}$$
$$\pi_2 = \alpha \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}$$
$$\dots$$
$$\pi_n = \alpha \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

Required algebraic results

A few standard algebraic relations involving finite and infinite sums allows to analyze the special types of birth-death models we are going to consider next. They will allow to compute the previous formula in a simple closed form way for many interesting systems.
We will now summarize them:

$$\pi_n = \frac{\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}}{\sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}}$$

The finite sum of n+1 terms raised at an increasing power is:

$$1 - x^{n+1} = (1-x)\overbrace{(1 + x + x^2 + \cdots + x^n)}^{\sum_{i=0}^{n} x^i}$$

$$\boxed{\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}}$$

The infinite sum of a term raised at an increasing index is:

$$if\ 0 \leq x < 1 \qquad \lim_{n\to\infty} \sum_{i=0}^{n} x^i = \frac{1 - \lim_{n\to\infty} x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}}$$

This expression can be computed only if $x < 1$, so that $\lim_{n\to\infty} x^{n+1} = 0$.

The infinite sum of a term raised at an increasing index and multiplied by that index is:

i=0 can be included in the sum since 0 · 1 = 0

$$\frac{d}{dx}\sum_{i=0}^{\infty} x^i = \sum_{i=1}^{\infty} i \cdot x^{i-1} = \frac{1}{x}\sum_{i=1}^{\infty} i \cdot x^i = \frac{1}{x}\sum_{i=0}^{\infty} i \cdot x^i \qquad \sum_{i=0}^{\infty} i \cdot x^i = x\frac{d}{dx}\sum_{i=0}^{\infty} x^i$$

$$\frac{d}{dx}\sum_{i=0}^{\infty} x^i = \frac{d}{dx}\left(\frac{1}{1-x}\right) = -\frac{-1}{(1-x)^2}$$

$$\boxed{\sum_{i=0}^{\infty} i \cdot x^i = \frac{x}{(1-x)^2}}$$

Finite sum of a term raised at an increasing index and multiplied by the index is:

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$$\frac{d}{dx}\sum_{i=0}^{n} x^i = \sum_{i=1}^{n} i \cdot x^{i-1} = \frac{1}{x}\sum_{i=1}^{n} i \cdot x^i = \frac{1}{x}\sum_{i=0}^{n} i \cdot x^i \qquad \sum_{i=0}^{n} i \cdot x^i = x\frac{d}{dx}\sum_{i=0}^{n} x^i$$

$$\frac{d}{dx}\sum_{i=0}^{n} x^i = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right) = \frac{(-(n+1)x^n)\cdot(1-x)-(1-x^{n+1})\cdot(-1)}{(1-x)^2}$$
$$= \frac{(n+1)x^{n+1}-(n+1)x^n+1-x^{n+1}}{(1-x)^2} = \frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$$

$$\boxed{\sum_{i=0}^{n} i \cdot x^i = x\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}}$$

The infinite sum of a term raised at an increasing index is:

$$if\ 0 \leq x < 1 \qquad \lim_{n\to\infty} \sum_{i=0}^{n} x^i = \frac{1 - \lim_{n\to\infty} x^{n+1}}{1-x} = \frac{1}{1-x}$$

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}}$$

This expression can be computed only if $x < 1$, so that $\lim_{n\to\infty} x^{n+1} = 0$.

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