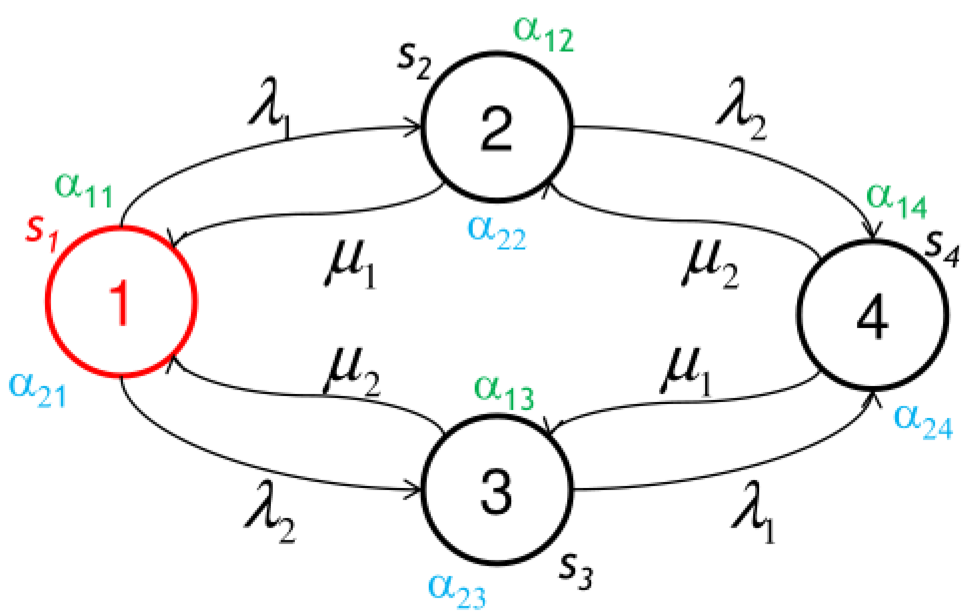
13.Steady State and Transient Analysis of CTMC

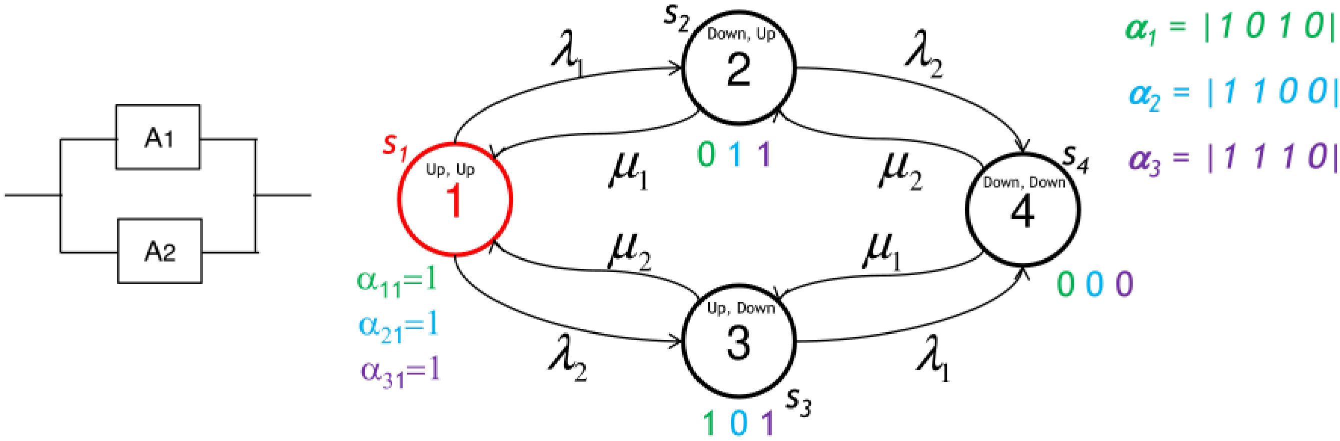
Steady state will tell us about some characteristic of the system at each state. We can compute the steady state solution as the solution is stationary(doesn't change in time) so the derivative will be null.
On Markov chains, performance metrics can be computed associating rewards to the states and the transitions. States rewards represent measures that are proportional to the time the system spends in each state. Transitions rewards account instead for performance indices that are proportional to the number of events that occur per time unit.
Examples of measures computed using state rewards are:

- Utilization
- Average number of jobs in the system / in the queue
- Average energy consumption
- Availability
- Time dependent costs / revenues
  Examples of transition rewards are instead:
- Throughput
- Loss rate / blocking rate (presented in a future lesson)
- Failure and repair rates
- Event dependent costs / revenues
  A model can have several state rewards a_k , each one defined by a vector that has a component per state of the CTMC $a_k = |a_{k1} \dots a_{kN}|$. a_k represents the reward that is obtained when the system is in state k. It can be positive, negative or zero, depending that it represents a gain, a loss, or the fact that the no reward is obtained in the state. A positive value in a state means that we are gaining something, a negative value means we are loosing something and a zero value means that we are stable.



For example, if our model represents a system with two repairable components in parallel, breaking at rate λ_1, and repaired at rate μ_i, the reward rates a_{ki} could be Boolean values that represent whether a component or the entire system are working in state i.

- a_1 : Component 1 is working
- a_2 : Component 2 is working
- a_3 : The parallel system is working



Starting from the steady state solution π_i :

- for state rewards, we can compute their average value:

$$E[\alpha_k] = \sum_{i=1} \pi_i \alpha_{ki}$$

- for transition reward, we can compute their average per time unit:

$$E[\xi_k] = \sum_{i=1} \pi_i \sum_{i \neq 1} q_{ij} \xi_{k,ij}$$

  Both measures can be computed in steady state to asses long-time performance metrics that give a general view of the system or in transient state to evaluate the corresponding property in a specific point in time.

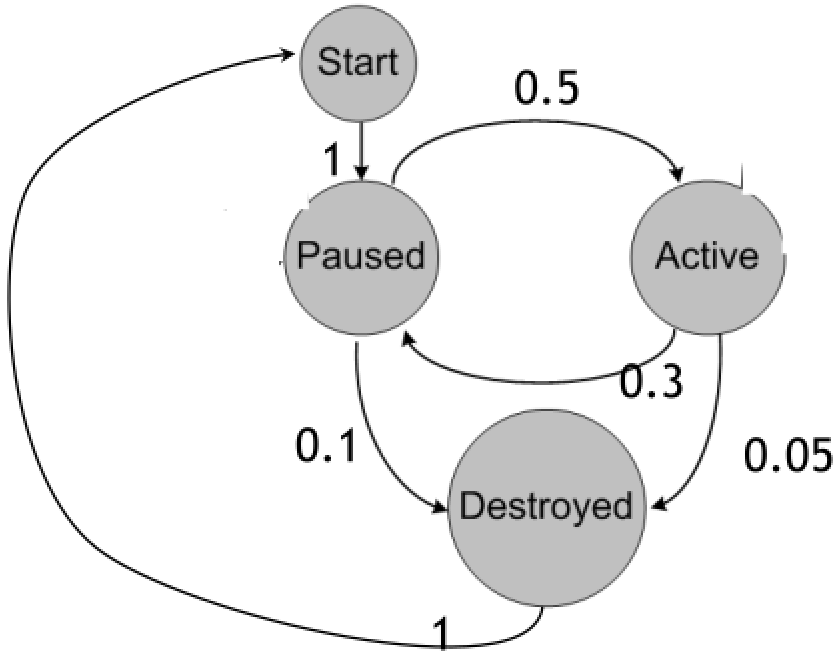
State Classification

In a CTMC, a state can either be:

- Ergodic
- Transient
- Absorbing
  Knowing these type of states can help to identify possible modelling errors, and understand issues that can arise when studying the model.
  This is useful to know as the steady state computation of a distribution cannot be computed in some cases and state classification helps to discover them.

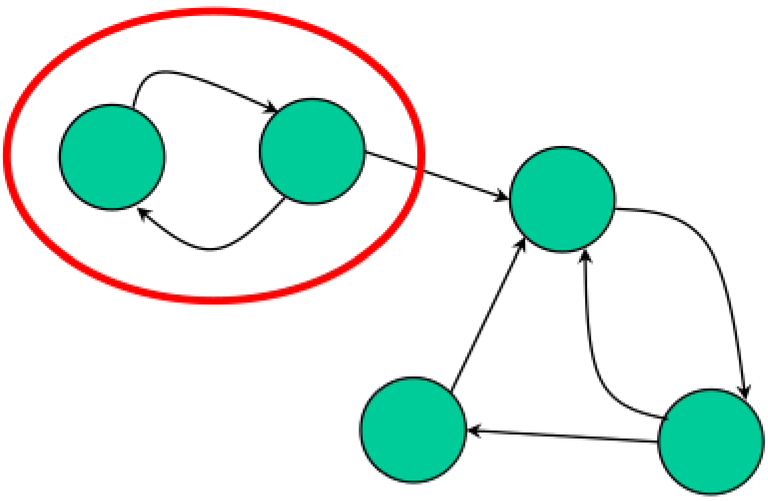
Ergodic States

An Erogodic (or recurrent) state, is a state into which the system will always return in the future. Ergodic are the “normal” states of a system. For ergodic states it is meaningful to compute the steady-state distribution.

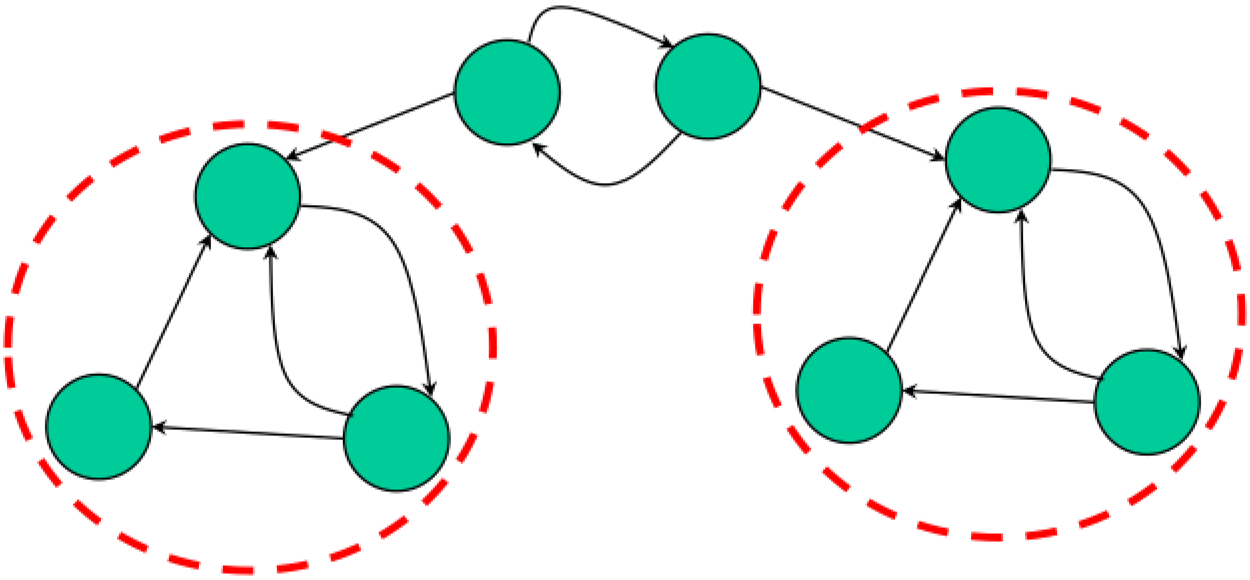


Transient States

Transient states are the ones for which their probability tends to zero as the time goes to infinity. They are the states that sooner or later will be abandoned, and in which the system will never be able to return again.

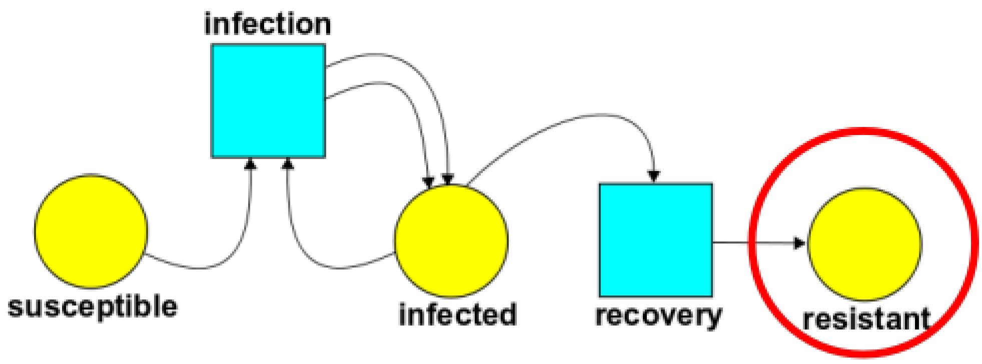


After the system leave a transient state it will be trapped in a Strong Component(subset of the states where the system might be “trapped” inside). If there are more than one strong component, the steady-state solution is not unique, and depends on the initial state of the model. Moreover it has a different meaning since, once one of the components has been chosen, the others will have zero probability.

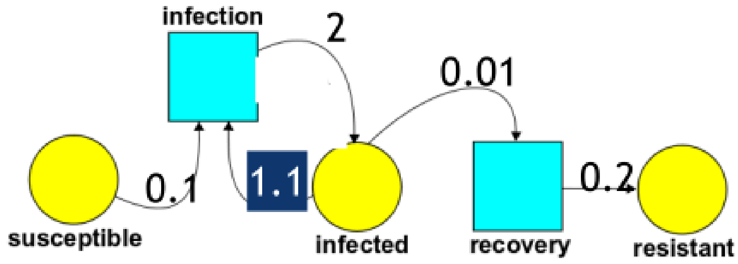


Absorbing States

A state in which there are no output transitions is called an Absorbing state. An absorbing state is a strong component, composed by a single state. When the system reaches an absorbing state, it will be trapped inside it forever. It is a strong component composed of a single state.



In the infinitesimal generator of a CTMC, the presence of absorbing states corresponds to lines entirely composed of zeros.



$$Q = \begin{vmatrix} -0.1 & 0.1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1.1 & -1.11 & 0.01 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

If there is a single absorbing state, its steady-state probability is (usually) 1, and the steady-state probability of all the other states is 0.

If there are more than one absorbing state, since they are strong components, their steady state distribution depends on the initial state of the model.

Even if their steady-state distribution is trivial, the evolution of the states probabilities over time can be used to study interesting properties of a system: for example, it can be used to determine the reliability in a dependability study.

For time constraints, we will not investigate this topic further.