12.Stochastic Processes and Markov Chains

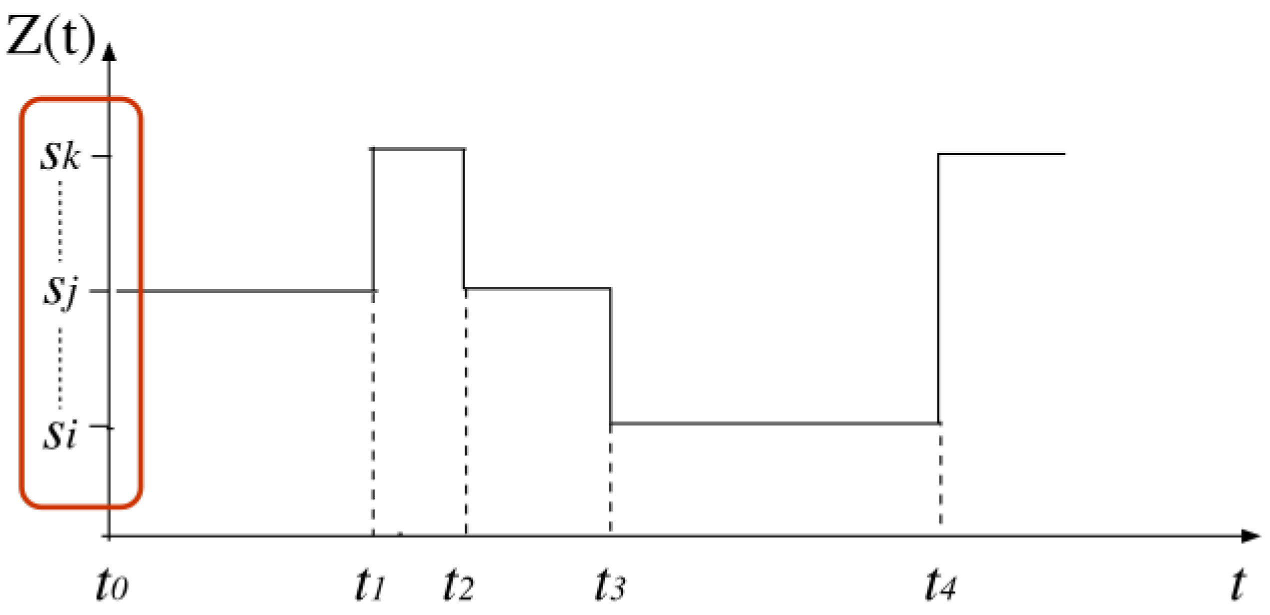
A stochastic process is a set of random variables $X_1, X_2, \ldots, X_n$ that operates over the same set.
The variables have an indexes(discrete or continuous number). From the second elements the value of the second random variable is conditioned by the previous one.

$$P(X_i = a_i | X_1 = a_1, \ldots, X_{i-1} = a_{i-1})$$

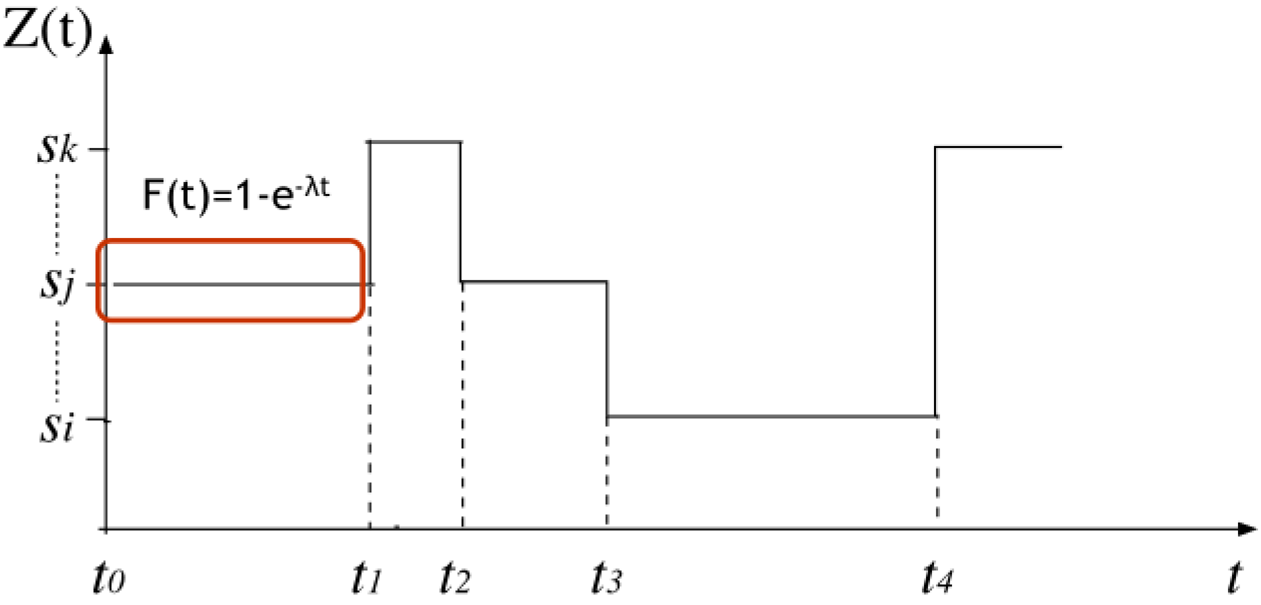
Things however can be simplified a lot by considering smaller levels of correlations among the random variables. If the index is continuous, it usually corresponds to the time. The set of outcomes of the random variables can also be discrete or continuous. In performance studies, the index is usually continuous, denoting the time, and the set of outcomes is discrete, corresponding to the states of a state machine.

Continuous Time Markov Chains(CMTC)

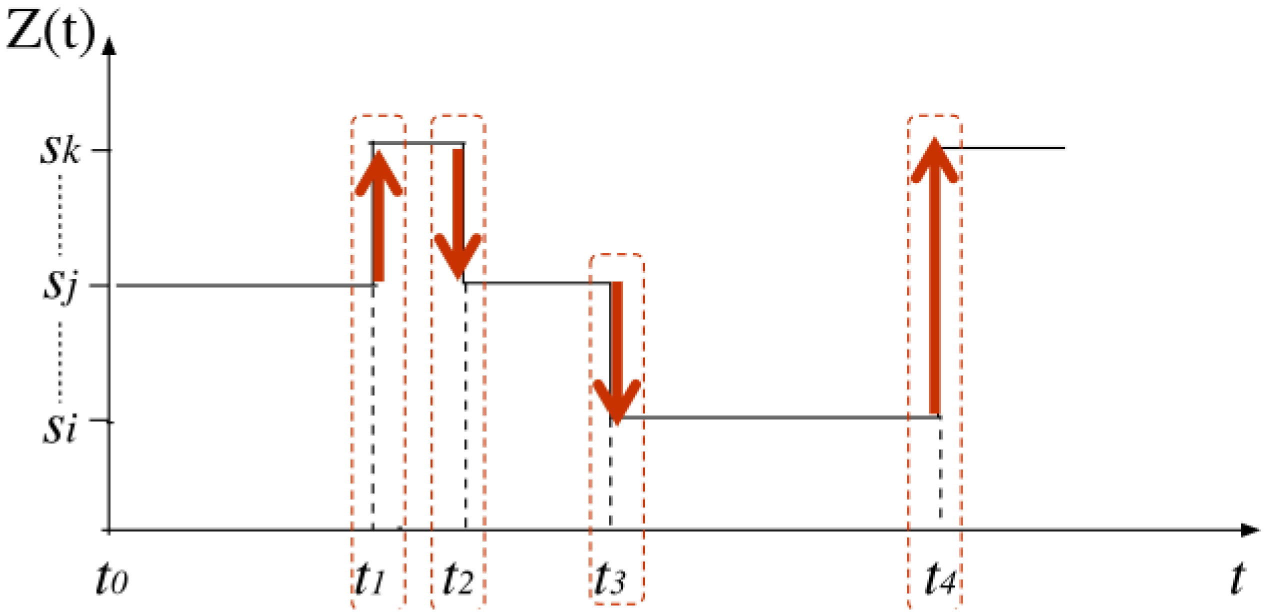
The special type of stochastic processes with discrete state and continuous time used in performance modelling are called Continuous Time Markov Chains. The (finite or infinite) set of outcomes for the random variables, are called states $\{s_1, \ldots, s_N\}$.



The system remains in a state s_j for a random exponentially distributed amount of time.



After that, it jumps to another state, which is also randomly chosen.



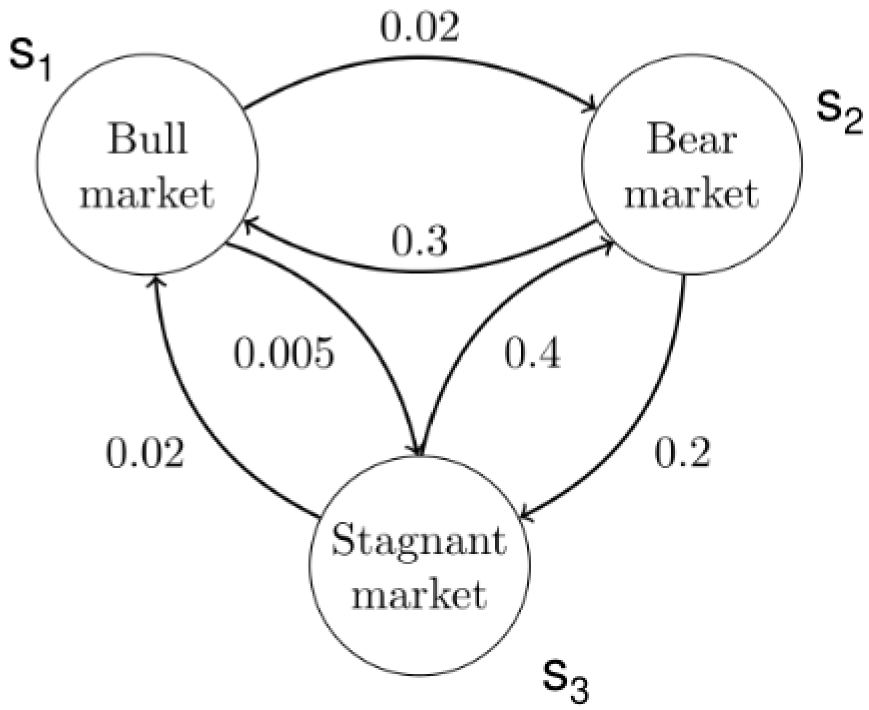
Thanks to the memory-less property of the exponential distribution, Markov Chains are stochastic processes where the probability of the state at time t_m depends only on the state at which the system was in a previous time t_{m-1} (and the total time passed $t_m - t_{m-1}$) :

$$(0 < t_1 < t_2 < \ldots < t_{m-1} < t_m)$$

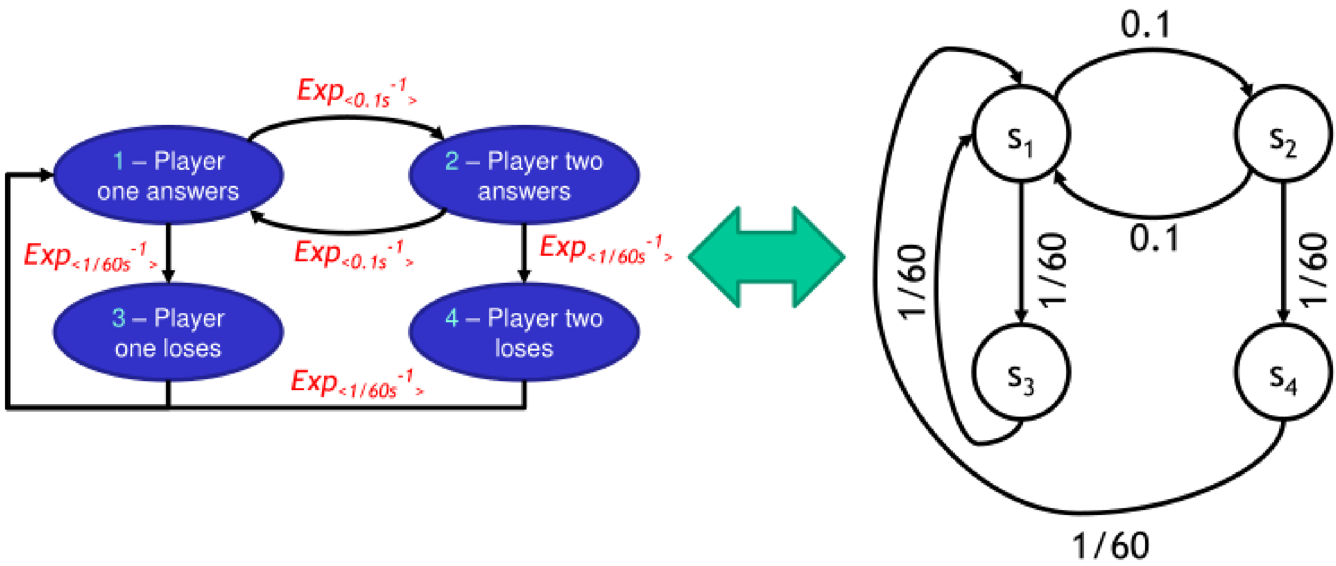
To simplify the definition, we will introduce the following notation:

$$\pi_i = Pr\{Z(t) = s_i\}$$

Usually CTMC are drawn as graphs, where nodes represent the states, and edges the possible transitions among the states. Arcs are labelled with the rate of an exponential distribution.



There is a one-to-one correspondence between a State Machine with exponential transition times, and a Continuous Time Markov Chain. The idea is that each state has associated one rate.



Each transition from state s_i to state s_j has associated a rate q_{ij} which corresponds to the rate of an exponential random variable. The system in state s_i jumps to state s_j after an exponentially distributed random amount of time with rate q_{ij} . If there is no arc connecting state s_i to state s_j, then $q_{ij} = 0$. If there are more than one arc exiting from state s_i , the system follows the evolution along the path of the event that happens first (race policy).

State Space

The set $\Omega = \{s_i, ...\}$ of all the possible states the system can assume, is called the state space of the model.

Transition Rate

The transition rate q_{ij} can be seen as the limit of the probability that the system performs a jump in a small time Δt , (divided by Δt):

q_{ij}=\lim_{\Delta t \rightarrow 0}\frac{\text{prob("System jumps from s_i to s_j in \Delta t")}}{\Delta t

If Δt is small enough, the previous definition can be inverted:

\text{*prob* “System jumps from *s_i* to *s_j* in }Δ*t*” = *q_{ij}* · Δ*t*

The Chapman-Kolmogorov equation

CTMC analysis allows to compute the probability $\pi_i(t)$ that the system is in each state s_i of Ω at time t. More formally, from $\pi_i(t)$ we can compute the probability $\pi_i(t+\Delta t)$ at time $t+\Delta t$ as the sum of two probabilities:

1. Being in state s_i at time t and not leaving state s_i in Δt
2. Being in another state s_j at time t (with $j \neq i$) and jumping from state s_j to state s_i in Δt (for every possible state s_j)

$$\pi_i(t+\Delta t) = \overbrace{\pi_i(t)}^{\text{Being instate } s_i \text{ at } t} \cdot \underbrace{\left(1-\sum_{j\neq i}^{\dots} q_{ij}\cdot\Delta t\right)}_{\text{Not leaving state } s_i \text{ in } \Delta t \text{ (= 1 - leaving state } s_i \text{ to another state } s_j\text{)}} + \sum_{j\neq i}^{\dots} \underbrace{\pi_j(t)}_{\text{Being instate } s_j \text{ at } t} \cdot \overbrace{q_{ji}}^{\text{Jumping from } s_j \text{ to } s_i \text{ in } \Delta t} \cdot \Delta t$$

To simplify the equations we define q_{ii} as: $q_{ii} = -\sum_{j\neq i} q_{ij}$

Now, q_{ii} can be included in the summation. The equation becomes:

$$\pi_i(t+\Delta t) = \pi_i(t) + \pi_i(t)\cdot q_{ii}\cdot \Delta t + \sum_{j\neq i} q_{ji}\cdot \pi_j(t)\cdot \Delta t$$

$$\pi_i(t+\Delta t) = \pi_i(t) + \sum_j q_{ji}\cdot \pi_j(t)\cdot \Delta t$$

With some computation we can find:

$$\frac{\pi_i(t+\Delta t) - \pi_i(t)}{\Delta t} = \sum_j q_{ji}\cdot \pi_j(t)$$

Taking the limit of Δt to 0, we have:

$$\frac{d\pi_i(t)}{dt} = \sum_j q_{ji}\cdot \pi_j(t)$$

This equation is known as the *Chapman-Kolmogorov master equation*

We need to remember what the equation means.
This equation need as many terms as states.

The terms q_{ij} can be collected in a matrix Q :

$$Q = \begin{vmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{vmatrix}$$

Q is called the Infinitesimal generator.
Due to the definition of q_{ii}, the elements of all the rows of matrix Q must sum up to 0.
If we collect all the probabilities in a row vector p(t), the Chapman-Kolmogorov equation in matrix form becomes:

$$\frac{d\pi(t)}{dt} = \pi(t)Q$$

If we know the initial state distribution $\pi(0)$, we can compute the probability distribution at time t of the system being in each of its states, by solving the differential equation using $\pi(0)$ as the initial condition. This is usually addressed as the transient solution of the model.
In general, we can create the Q matrix from its graphical representation, by first enumerating the states. Each state corresponds to a row and the a column of matrix Q.