CSC336 Assignment 4

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1.(a)

Let $\hat{x}_i = e_i$, *where* $i = 1, 2, \ldots, n$

$$Thus,\ \hat{x}_i \in \mathbb{R}^n\ and\ \hat{x}_i \neq \vec{0}\ \text{since}\ \hat{x}_i = \begin{pmatrix} 0 \\ \vdots \\ 1(\,i^{th}\ element) \\ \vdots \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Since A is real symmetric positive − definite,

$\hat{x}_i^{\,T} A \hat{x}_i = e_i^{\,T} A e_i = 1 \cdot A_{i,i} \cdot 1 = A_{i,i}$ *will be greater than* 0.

(b)

$$M_1 = I - m_1 e_1^{\,T} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{A_{2,1}}{A_{1,1}} \\ \vdots \\ \frac{A_{n,1}}{A_{1,1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{A_{2,1}}{A_{1,1}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{A_{n,1}}{A_{1,1}} & 0 & \cdots & 1 \end{pmatrix}$$

$$M_1^{\,T} = \begin{pmatrix} 1 & -\frac{A_{2,1}}{A_{1,1}} & \cdots & -\frac{A_{n,1}}{A_{1,1}} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$M_1 A M_1^{\,T} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{A_{2,1}}{A_{1,1}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{A_{n,1}}{A_{1,1}} & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} 1 & -\frac{A_{2,1}}{A_{1,1}} & \cdots & -\frac{A_{n,1}}{A_{1,1}} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ 0 & \frac{-A_{2,1} \cdot A_{1,2}}{A_{1,1}} + A_{2,2} & \cdots & \frac{-A_{2,1} \cdot A_{1,n}}{A_{1,1}} + A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{-A_{n,1} \cdot A_{1,2}}{A_{1,1}} + A_{n,2} & \cdots & \frac{-A_{n,1} \cdot A_{1,n}}{A_{1,1}} + A_{n,n} \end{pmatrix} \begin{pmatrix} 1 & -\frac{A_{2,1}}{A_{1,1}} & \cdots & -\frac{A_{n,1}}{A_{1,1}} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$=\begin{pmatrix} A_{1,1} & -A_{2,1}+A_{1,2} & \cdots & -A_{n,1}+A_{1,n} \\ 0 & \widehat{A}_{2,2} & \cdots & \widehat{A}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widehat{A}_{n,2} & \cdots & \widehat{A}_{n,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} & 0 & \cdots & 0 \\ 0 & \widehat{A}_{2,2} & \cdots & \widehat{A}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widehat{A}_{n,2} & \cdots & \widehat{A}_{n,n} \end{pmatrix}$$

Since $A = A^T$, so $A_{i,j} = A_{j,i}$, and then $-A_{i,j} + A_{j,i} = 0$ for all i = 1, 2, ... n and j = 1, 2, ... , n

(c)

Facts:

1. M_1 is a lower triangular matrix (form shown in (b))
2. M_1 is an upper triangular matrix since 1 (form shown in (b))
3. $A = \widehat{L}\widehat{L}^T$ since A is real symmetrix positive-definite
4. $\widehat{L}$ is lower triangular matrix and $\widehat{L}^T$ is an upper triangular matrix
5. **The product of two upper/lower triangular matrices is still a upper/lower triangular matrix**

Since $A_1 = M_1 A M_1^T$

Then $A_1 = M_1 \widehat{L}\widehat{L}^T M_1^T$

Let $\widetilde{L} = M_1 \widehat{L}$ *and* $\widetilde{L}^T = \widehat{L}^T M_1^T$, *since* $(AB)^T = B^T A^T$ *for all matrices*

Then $A_1 = \widetilde{L}\widetilde{L}^T$, *by* 5, $\widetilde{L}$ *is a lower triangular matrix, and* $\widetilde{L}^T$ *is an upper triangular matrix*

and they are also $n \times n$ since M_1 *and* $\widehat{L}$ *are both* $n \times n$

Therefore A_1 *is real symmetric positive − definite*

(d)

For $i = 2, 3, \ldots, n$ & $j = 2, 3, \ldots, n$

$$\widehat{A}_{i,j} = A_{1_{i,j}} = \left(\frac{-A_{i,1} \cdot A_{1,j}}{A_{1,1}} + A_{i,j} \right) = \left(\frac{-A_{j,1} \cdot A_{1,i}}{A_{1,1}} + A_{j,i} \right) = A_{1_{j,i}} = \widehat{A}_{j,i}$$

Since their values don't vary after calculating $M_1 A$, *so we will only count operations in* $M_1 A$

For every $\widehat{A}_{i,j}$, 1 *multiplication required for* $\frac{-A_{i,1}}{A_{1,1}} \cdot A_{1,j}$

1 *add required for* $\frac{-A_{i,1}}{A_{1,1}} \cdot A_{i,j} + A_{i,j}$

So that's 1 add and multiplication for one $\widehat{A}_{i,j}$

Since there are $(n-1)$ *rows in* M_1 *that have* $\frac{-A_{i,1}}{A_{1,1}}$ *in it,* $(\mathbf{n-1})$ ***divisions*** *are required for them.*

multiplying M_1 *to* A *(which has* (n) *columns) would require* $n(n-1)\times 1$ *adds and multiplications.*

However, since A is symmetric, the upper/lower part can be set at the same time after each add and multiplication, thus, a total number of $\left(\frac{n(n-1)}{2}\right)$ ***adds and multiplications*** *are required.*

(e)

$M_{n-1}M_{n-2}...M_2M_1AM_1^TM_2^T...M_{n-2}^TM_{n-1}^T = D$

$\mathbf{(M_{n-1}^{-1})}M_{n-1}M_{n-2}...M_2M_1AM_1^TM_2^T...M_{n-2}^TM_{n-1}^T = \mathbf{(M_{n-1}^{-1})}D$

$IM_{n-2}...M_2M_1AM_1^TM_2^T...M_{n-2}^TM_{n-1}^T = M_{n-1}^{-1}D$

...

$AM_1^TM_2^T...M_{n-2}^TM_{n-1}^T = M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1}D$

$AM_1^TM_2^T...M_{n-2}^TM_{n-1}^T\mathbf{(M_{n-1}^T)^{-1}} = M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1}D\mathbf{(M_{n-1}^T)^{-1}}$

$AM_1^TM_2^T...M_{n-2}^TI = M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1}D(M_{n-1}^T)^{-1}$

...

$A = M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1}D(M_{n-1}^T)^{-1}(M_{n-2}^T)^{-1}...(M_1^T)^{-1}$

Let $L = M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1}$ and

$L^T = (M_{n-1}^T)^{-1}(M_{n-2}^T)^{-1}...(M_1^T)^{-1}$ ①

Then $A = LDL^T$

Justification for ①:

Like question (c), all the M_i/M_i^T for i = 1, 2, ..., ,n-1 are lower/upper triangular matrices and the inverse matrices of them are still lower/upper triangular matrices. Also, the product of lower/upper triangular matrices is still a lower/upper triangular matrix.

Thus, $M_1^{-1}, ..., M_{n-2}^{-1}, M_{n-1}^{-1}$ are lower triangular matrices

⇒ L is still a lower triangular matrix and

$(M_{n-1}^T)^{-1}, (M_{n-2}^T)^{-1}, ..., (M_1^T)^{-1}$ are upper triangular matrices

⇒ L^T is still an upper triangular matrix and

L^T is the transpose of L since $(A_1A_2...A_n)^T = A_n^TA_{n-1}^T...A_1^T$ and $(A^T)^{-1} = (A^{-1})^T$ for any matrix $A, A_1, A_2, ..., A_n$.

So, L need in (3) is just $\mathbf{(M_1^{-1}...M_{n-2}^{-1}M_{n-1}^{-1})}$, where it will just copy items in each column of M_i^{-1} to a new matrix, but copying items will NOT be considered as additional arithmetic work. It is considered as additional "storage" work though, but "storage" work is not arithmetic.

2.

Code part:

```
p = [5, 4, 9, 10, 6, 8, 10, 9, 10];
x = [1:10]';
y1 = perm_a(p, x);
q = perm_b(p);
y2 = perm_c(q, x);

fprintf('y1 = (\n');
fprintf('%i\n', y1);
fprintf(')\n\n');

fprintf('q = ( ');
fprintf('%i ', q);
fprintf(')\n\n');

fprintf('y2 = (\n');
fprintf('%i\n', y2);
fprintf(')\n');

function f1 = perm_a(p, x)
    for i = 1:length(p)
        one = x(i);
        x(i) = x(p(i));
        x(p(i)) = one;
    end
    f1 = x;
end

function f2 = perm_b(p)
    q_init = [1:length(p)+1];
    for i = 1:length(p)
        two = q_init(i);
        q_init(i) = q_init(p(i));
        q_init(p(i)) = two;
    end
    f2 = q_init;
end

function f3 = perm_c(q, x)
    three = x;
    for i = 1:length(q)
        three(i) = x(q(i));
    end
    f3 = three;
end
```

(a)

```
y1 = (
5
4
9
10
6
8
2
3
7
1
)
```

(b)

```
q = ( 5 4 9 10 6 8 2 3 7 1 )
```

(c)

```
y2 = (
5
4
9
10
6
8
2
3
7
1
)
```

3.(a)

$g_1(x) = \left(x^2+2\right)/3,\ g_1'(x) = 2x/3,\ \left|g_1'(2)\right| = 4/3 > 1 \Rightarrow diverges;$

$g_2(x) = \sqrt{3x-2},\ g_2'(x) = 3/2\sqrt{3x-2},\ \left|g_2'(2)\right| = 3/4 < 1 \ and > 0 \Rightarrow \approx linearly\ converges;$

$g_3(x) = 3 - 2/x,\ g_3'(x) = 2/x^2,\ \left|g_3'(2)\right| = 1/2 < 1 \ and > 0 \Rightarrow \approx linearly\ converges;$

$g_4(x) = \frac{\left(x^2-2\right)}{(2x-3)},\ g_4'(x) = \frac{2\left(x^2-3x+2\right)}{(2x-3)^2},\ \left|g_4'(2)\right| = 0 \Rightarrow quaduatically\ converges;$

(b)

Code part:

```
x1 = 2.1;
x2 = 1.5;
x3 = 1.5;
x4 = 100;
T1 = table;
T2 = table;
T3 = table;
T4 = table;
for n = 0:10
    err1 = abs(x1-2);
    err2 = abs(x2-2);
    err3 = abs(x3-2);
    err4 = abs(x4-2);
    g1 = ((x1^2) + x1)/3;
    g2 = sqrt(3*x2 - 2);
    g3 = 3 - (2/x3);
    g4 = ((x4^2) - 2)/(2*x4 - 3);
    x1 = g1;
    x2 = g2;
    x3 = g3;
    x4 = g4;
    T1 = [T1; table(n, g1, err1)];
    T2 = [T2; table(n, g2, err2)];
    T3 = [T3; table(n, g3, err3)];
    T4 = [T4; table(n, g4, err4)];
end
```

For g_1

iteration	g1	error
0	2.17	0.1
1	2.293	0.17
2	2.5169	0.29297
3	2.9505	0.51689
4	3.8854	0.95054
5	6.3272	1.8854
6	15.454	4.3272
7	84.758	13.454
8	2422.9	82.758
9	1.9576e+06	2420.9
10	1.2774e+12	1.9576e+06

It diverges quickly even when the guess is pretty close to root.

For g_2

iteration	g2	error
0	1.5811	0.5
1	1.6563	0.41886
2	1.7231	0.34367
3	1.7802	0.27693
4	1.8278	0.21977
5	1.8664	0.17224
6	1.8971	0.13365
7	1.9213	0.10288
8	1.9401	0.078711
9	1.9545	0.059931
10	1.9656	0.045465

It converges, and the error shrinks about 25% each time when the guess is close to root.

For g_3

iteration	g3	error
0	1.6667	0.5
1	1.8	0.33333
2	1.8889	0.2
3	1.9412	0.11111
4	1.9697	0.058824
5	1.9846	0.030303
6	1.9922	0.015385
7	1.9961	0.0077519
8	1.9981	0.0038911
9	1.999	0.0019493
10	1.9995	0.00097561

It converges, and the error shrinks about 50% each time when the guess is close to root.

For g_4

iteration	g4	error
0	50.751	98
1	26.128	48.751
2	13.819	24.128
3	7.6697	11.819
4	4.6051	5.6697
5	3.0928	2.6051
6	2.3749	1.0928
7	2.0803	0.37489
8	2.0056	0.080319
9	2	0.0055583
10	2	3.0555e-05

It converges, and the error shrinks about 50% each time even though the guess is not even close to root.

4.(a)

```
syms x;
f = (x^2)-2;
df = diff(f);
x_ = 1;
fprintf('n              x(n)                     x(n)-sqrt(2)\n');
fprintf('-------------------------------------------\n')
for n = 0:5
    fprintf('%i %20.15f %20.15f\n', n, x_, x_-sqrt(2));
    x_ = x_ - subs(f, x, x_) / subs(df, x, x_);
end
```

```
>> A4Q4a
n           x(n)                x(n)-sqrt(2)
-------------------------------------------
0     1.000000000000000    -0.414213562373095
1     1.500000000000000     0.085786437626905
2     1.416666666666667     0.002453104293572
3     1.414215686274510     0.000002123901415
4     1.414213562374690     0.000000000001595
5     1.414213562373095     0.000000000000000
```

(b)

```
syms x;
f = (x^2)-2;
xn_2 = 1;
xn_1 = 2;
fprintf('n           x(n)               x(n)-sqrt(2)\n');
fprintf('-------------------------------------------\n')
fprintf('0 %20.15f %20.15f\n', double(xn_2), double(xn_2 - sqrt(2)));
fprintf('1 %20.15f %20.15f\n', double(xn_1), double(xn_1 - sqrt(2)));
xn = (xn_2*subs(f, x, xn_1) - xn_1*subs(f, x, xn_2))/(subs(f, x, xn_1) - subs(f, x, xn_2));
for n = 2:7
    fprintf('%i %20.15f %20.15f\n', n, double(xn), double(xn-sqrt(2)));
    xn_2 = xn_1;
    xn_1 = xn;
    xn = (xn_2*subs(f, x, xn_1) - xn_1*subs(f, x, xn_2))/(subs(f, x, xn_1) - subs(f, x, xn_2));
end
```

```
n           x(n)               x(n)-sqrt(2)
-------------------------------------------
0   1.000000000000000   -0.414213562373095
1   2.000000000000000    0.585786437626905
2   1.333333333333333   -0.080880229039762
3   1.400000000000000   -0.014213562373095
4   1.414634146341463    0.000420583968368
5   1.414211438474870   -0.000002123898225
6   1.414213562057320   -0.000000000315775
7   1.414213562373095    0.000000000000000
```

5.

```
a = 3.592;
b = 0.04267;
R = 0.082054;
T = 300;
Tb = table;
for p = [1, 10, 100]
    v = (R*T) / p;
    left = v - 1;
    right = v + 1;
    zero_interval = [left right];
    f = @(v)(p + a/(v^2))*(v-b) - R*T;
    v_sol = fzero(f, zero_interval);
    Tb = [Tb; table(p, v, v_sol)];
end

Tb.Properties.VariableNames = {'pressure', 'solution_from_initial_guess', 'solution_from_fzero'};

disp(Tb);
```

```
>> A4Q5
```

pressure	solution_from_initial_guess	solution_from_fzero
1	24.616	24.513
10	2.4616	2.3545
100	0.24616	0.079511

6.(a)

Since (i) *and* (ii)

Then $f(x)$ *is always concaving up (by definition of concavity on* 2*nd derivative)*

Then $\forall x, y \in R,\ x > y \Rightarrow f'(x) > f'(y)$ (1)

Since (iii)

Then $\forall x \in R,\ x > \hat{x} \Rightarrow f'(x) > f(\hat{x}) = 0$ (2)

since $\infty > \hat{x}$ *there is no limit to* $f'(x)$ *(the growth of* x*) after* $x = \hat{x}$

Then as $x \to \infty$, $f'(x) \to \infty$ *and* $f(x) \to \infty$

Since $f(\hat{x}) < 0$ *and* $\lim_{x \to \infty} f(x) \to \infty$ *and* $\infty > 0$

Then in order to get to ∞, *value of* f *will have to surpass* 0 *at some point*

Then $\exists x^* \in R,\ x^* > \hat{x} \wedge f(x^*) = 0$

Also Since (1),

The tagent line of $(x^*, f(x^*))$ *will be unique (this slope happens once only) for all x*

Thus, $\forall x \in R,\ (f(x) = 0 \wedge x > \hat{x}) \Rightarrow x = x^*$, *so* $(x^*, f(x^*) = 0)$ *is unique.*

(b)

Show $x^* \leqslant x_n$, *for all n = 1, 2, ... by induction on n*

Base case: $n = 1$

We know that: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} \Rightarrow (x_1 - x_0) f'(x_0) + f(x_0) = 0$

Since f *is convex function, we have* $f(x_1) \geq (x_1 - x_0) f'(x_0) + f(x_0) = 0$

We will have 2 cases in the Base case in order to show $x_1 > \hat{x}$

SInce only when x_1 *is on the increaseing side of* $f\left(x_1 > \hat{x}\right)$, *we can conclude the root is valid*

Case 1: $\hat{x} < x_0 \leq x^*$ *(NOTE:* $f(x^*) = 0$*)*

Then we have $f(x_0) \leq 0$, *but* $f'(x_0) > 0$

$$\textit{Thus, } x_1 = x_0 - \underbrace{\frac{f(x_0)}{f'(x_0)}}_{\uparrow \leq 0} \geq x_0 > \hat{x}$$

$Case\ 2: \hat{x} < x^* < x_0\ (NOTE: f(x^*) = 0)$

$Then\ we\ have\ f'(x_0) > 0,\ f(x_0) > 0$

$Prove\ x_1 > \hat{x}\ by\ contradiction: we\ assume\ x_1 \le \hat{x}$

$Then\ x_0 f'(x_0) - f(x_0) \le \hat{x} f'(x_0) \Rightarrow f(x_0) + (\hat{x} - x_0) f'(x_0) \ge 0$

$But\ f(\hat{x}) \ge f(x_0) + f'(x_0)(\hat{x} - x_0) \ge 0\ while\ f(\hat{x}) < 0\ is\ not\ possible!$

$Thus,\ x_1 > \hat{x}$

Therefore, we can truly conclude that $x_1 \ge x^*$, base case holds.

$Inductive\ Step\ (n > 1): Assume\ x_n \ge x^*,\ we\ need\ to\ show\ that\ x_{n+1} \ge x^*$

$$Then\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0$$

$By\ convexity,\ we\ know\ f(x_{n+1}) \ge f'(x_n)(x_{n+1} - x_n) + f(x_n) = f(x^*) = 0$

$Also,\ with\ similar\ proof\ of\ base\ cases,\ we\ can\ conclude\ that\ x^* > \hat{x}$

Thus, $x_{n+1} \ge x^*$

Therefore, $x^* \le x_n$ *for all* $n = 1, 2, \ldots$

Show $x_{n+1} \le x_n$, *for* $n = 1, 2, \ldots$

$$Then\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Then $f(x_n) \ge 0$ *and* $f'(x_n)$ *(by conclusion from previous proof)*

Then $x_n \ge x_n -$ *(something* ≥ 0*)*

Therefore $x_n \ge x_{n+1}$

(c)

Since x_n *is a decreasing sequence on n and f is increasing after* $x = \hat{x}$

And $x_n \geqslant x^* > \hat{x}$ *for all* $n = 1, 2, \ldots$ *(by 6(b))*

So $f(x_n)$ *is also a decreasing sequence as n increases.*

Also, we know that x_n *is bounded by* x^* *(given) , so* $f(x_n)$ *will never decrease below* $f(x^*)$

Thus, $f(x_n)$ *is bounded by* $f(x^*)$

Therefore, $\lim_{n \to \infty} f(x_n) = f(x^*) = 0 \Leftrightarrow \lim_{n \to \infty} x_n = y^* = x^*$ *is true for such decreasing sequence.*