CSC336 Assignment 3

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1.(a)

$$A=\begin{pmatrix} 1 & -4 & 5 & 4 \\ 1 & 0 & 2 & 0 \\ 2 & -2 & 4 & 2 \\ -1 & 0 & 1 & -2 \end{pmatrix}$$

Interchange row 1&3 since 2 *is the largest element in col* 1

$$P_1=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & -4 & 5 & 4 \\ -1 & 0 & 1 & -2 \end{pmatrix},\ M_1=I-m_1e_1{}^T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

$$M_1P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 3 & 3 \\ 0 & -1 & 3 & -1 \end{pmatrix}$$

No need of interchanging

$$P_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},P_2M_1P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 3 & 3 \\ 0 & -1 & 3 & -1 \end{pmatrix},M_2=I-m_2e_2{}^T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ -3 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$M_2P_2M_1P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & -2 \end{pmatrix}$$

No need of interchanging

$$P_3=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},P_3M_2P_2M_1P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & -2 \end{pmatrix},M_3=I-m_3e_3{}^T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$M_3P_3M_2P_2M_1P_1A=\begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}=U$$

$1.\ Let\ P_3 M_2 = \widehat{M_2} P_3 \quad For\ some\ \widehat{M_2} \in \mathbb{R}^4$

$Since\ P_3 = I,\ \widehat{M_2} = M_2$

$Then\ M_3 \widehat{M_2} P_3 P_2 M_1 P_1 A = U$

$2.\ Let\ P_3 P_2 M_1 = \widehat{M_1} P_3 P_2\ For\ some\ \widehat{M_1} \in \mathbb{R}^4$

$Since\ P_3 = P_2 = I,\ \widehat{M_1} = M_1$

$Then\ M_3 \widehat{M_2} \widehat{M_1} P_3 P_2 P_1 A = M_3 M_2 M_1 P_3 P_2 P_1 A = U$

$$M_3^{-1} M_2^{-1} M_1^{-1} = I + m_3 e_3^T + m_2 e_2^T + m_1 e_1^T$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -3 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -3 & 1 & 0 \\ -\frac{1}{2} & -1 & 1 & 1 \end{pmatrix} = L$$

$$P_3 P_2 P_1 = IIP_1 = P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P$$

$Since\ M_3 M_2 M_1 P_3 P_2 P_1 A = U$

$Then\ \left(P_3 P_2 P_1 \right) A = PA = LU = \left(M_3^{-1} M_2^{-1} M_1^{-1} \right) U$

(b)

$Ax = b \Rightarrow PAx = Pb \Rightarrow LUx = Pb$

$Let\ Ux = y,\ Pb = \widehat{b},\ Then\ Ly = \widehat{b}$

$$Then\ Pb = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 5 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -3 & 1 & 0 \\ -\frac{1}{2} & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdot \widehat{b}_1 = 0 \Rightarrow \widehat{b}_1 = 0 \\ 1 \cdot \widehat{b}_2 = 0 \Rightarrow \widehat{b}_2 = 0 \\ 1 \cdot \widehat{b}_3 = 3 \Rightarrow \widehat{b}_3 = 3 \\ 1 \cdot 3 + 1 \cdot \widehat{b}_4 = 5 \Rightarrow \widehat{b}_4 = 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 5 \end{pmatrix}$$

$$Then\ Ux = \begin{pmatrix} 2 & -2 & 4 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 \cdot x_1 + (-2) \cdot x_2 + 4 \cdot x_3 + 2 \cdot x_4 = 0 \Rightarrow x_1 = -2 \\ 1 \cdot x_2 + (-1) \cdot x_4 = 0 \Rightarrow x_2 = -1 \\ 3 \cdot x_3 = 3 \Rightarrow x_3 = 1 \\ -2 \cdot x_4 = 2 \Rightarrow x_4 = -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \end{pmatrix},\ Thus\ x = \begin{pmatrix} -2 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

2.

(a) and (b)

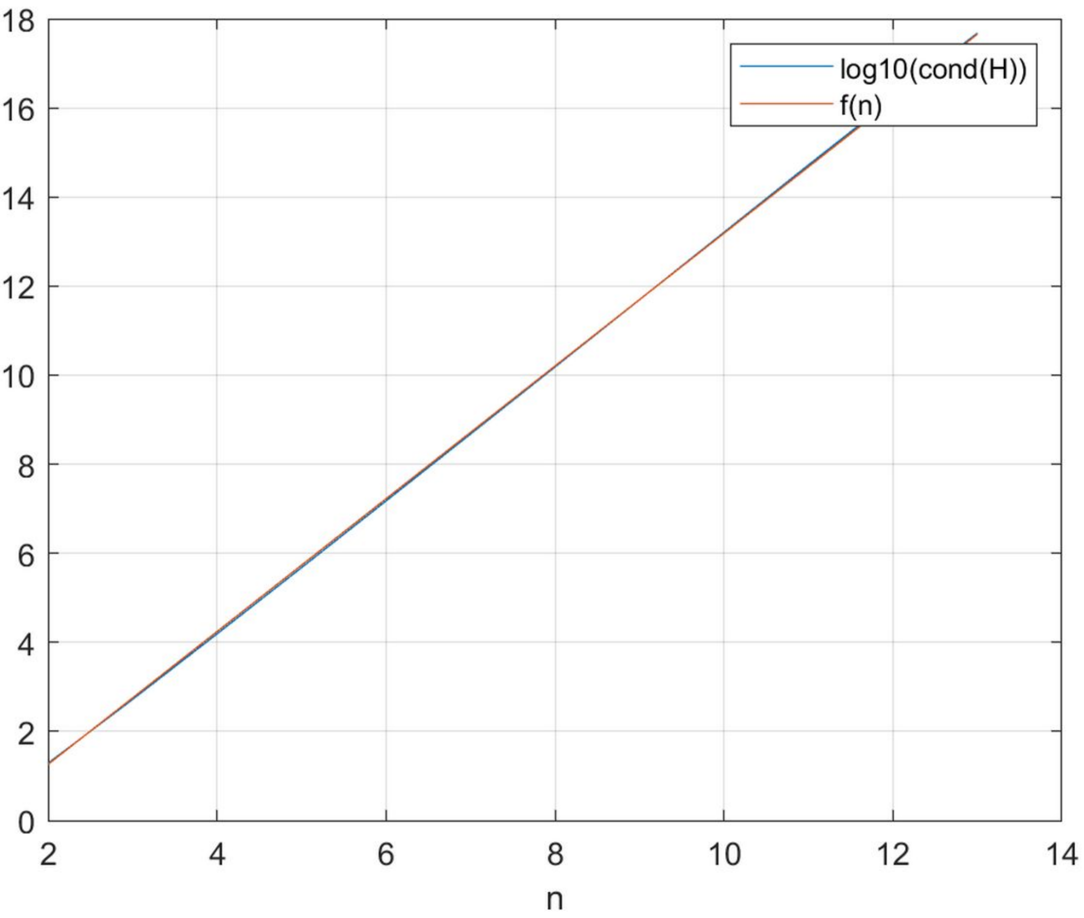
```
a = sqrt(2)/2;

b=[0;10;0;0;0;0;0;15;0;20;0;0;0];

A = [0 1 0 0 0 -1 0 0 0 0 0 0 0;
     0 0 1 0 0 0 0 0 0 0 0 0 0;
     a 0 0 -1 -a 0 0 0 0 0 0 0 0;
     a 0 1 0 a 0 0 0 0 0 0 0 0;
     0 0 0 1 0 0 0 -1 0 0 0 0 0;
     0 0 0 0 0 0 1 0 0 0 0 0 0;
     0 0 0 0 a 1 0 0 -a -1 0 0 0;
     0 0 0 0 a 0 1 0 a 0 0 0 0;
     0 0 0 0 0 0 0 0 0 1 0 0 -1;
     0 0 0 0 0 0 0 0 0 0 1 0 0;
     0 0 0 0 0 0 0 1 a 0 0 -a 0;
     0 0 0 0 0 0 0 0 a 0 1 a 0;
     0 0 0 0 0 0 0 0 0 0 0 a 1;];
 f = A \ b
 condition_number_A = cond(A, inf);
 infinity_norm_b = norm(b, inf);
 b_hat = A * f;
 r = b_hat - b;
 infinity_norm_r = norm(r,inf);
 upper_bound = condition_number_A * (infinity_norm_r/infinity_norm_b)
 lower_bound = (1/condition_number_A) * (infinity_norm_r/infinity_norm_b)
```

```
f =

  -28.2843
   20.0000
   10.0000
  -30.0000
   14.1421
   20.0000
         0
  -30.0000
    7.0711
   25.0000
   20.0000
  -35.3553
   25.0000


upper_bound =

   5.6100e-15


lower_bound =

   5.6247e-18
```

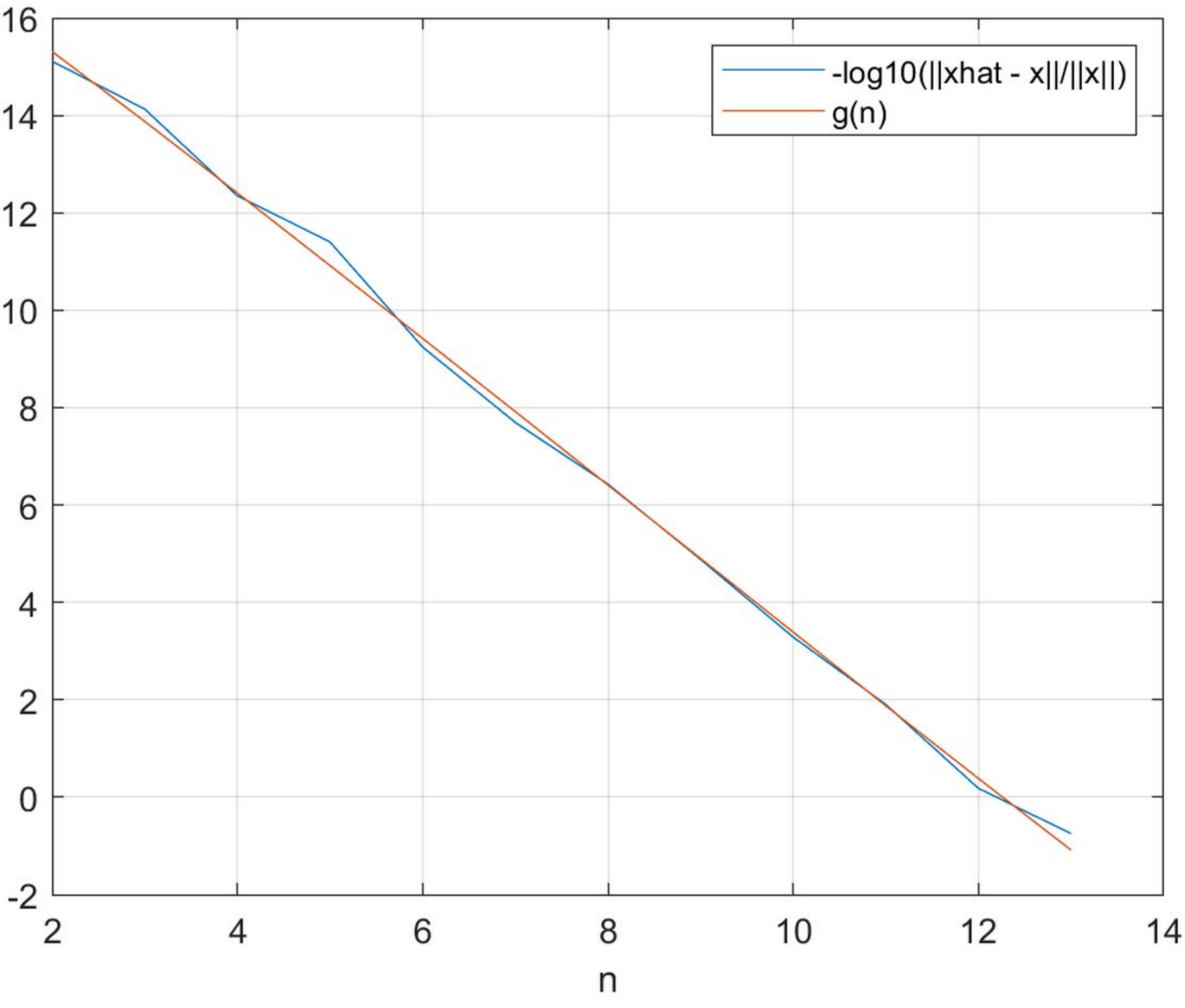
3.

```
T = table;
for n = 2:13
    x = ones(n, 1);
    H = hilb(n);
    b = H * x;
    x_hat = H \ b;

    f_n = log10(10^(1.49044545*(n)-1.72));
    condition_number = log10(cond(H));

    relative_error = norm(x_hat - x, inf)/norm(x, inf);
    g_n = -log10(cond(H)) + 16.5941;
    logrr = -log10(relative_error);

    T = [T; table(n, f_n, condition_number, relative_error, g_n, logrr)];
end
disp(T)
plot(T.n, T.condition_number, T.n, T.f_n), xlabel('n'), legend('log10(cond(H))', 'f(n)'), grid on
plot(T.n, T.logrr, T.n, T.g_n), xlabel('n'), legend('-log10(||x_hat-x||/||x||)', 'g(n)'), grid on
```

n	f_n	condition_number	relative_error	g_n	logrr
2	1.2609	1.2851	7.7716e-16	15.309	15.109
3	2.7513	2.7194	7.4385e-15	13.875	14.129
4	4.2418	4.1907	4.5053e-13	12.403	12.346
5	5.7322	5.6782	3.9924e-12	10.916	11.399
6	7.2227	7.1747	5.724e-10	9.4194	9.2423
7	8.7131	8.677	2.019e-08	7.9171	7.6949
8	10.204	10.183	3.7286e-07	6.4106	6.4285
9	11.694	11.693	1.3105e-05	4.9011	4.8825
10	13.184	13.205	0.00051646	3.3893	3.287
11	14.675	14.718	0.012488	1.8764	1.9035
12	16.165	16.21	0.65878	0.38427	0.18126
13	17.656	17.68	5.5991	-1.0859	-0.74812



log10(cond(H))
f(n)
n



-log10(||xhat - x||/||x||)
g(n)
n