Theory of Probability

Basic Terminologies

- Random Experiment
- Sample Space
- Trail and Event
- Equally Likely Events
- Mutually Exclusive Events
- Favourable Events or Cases
- Exhaustive Events or Cases
- Dependent Events
- Independent Events

- Sample Space (S or Ω)

Ex: tossing a fair (unbiased) coin

Possible outcomes; head and tail

S = { Head, Tail}

Ex: tossing a fair coin twice

Possible outcomes: HH, HT, TH, TT   S = {HH, HT, TH, TT}

- Rolling a die

Let E = getting 1 (simple event)

Favourable case = 1.

Let E = getting an odd number (compound event)

Favourable cases: 1, 3, 5

Total possible cases or Exhaustive cases: 1, 2, 3, 4, 5, 6

Ex: tossing a coin, S = {head, tail}

Let E = getting head

Favourable case = head (1)

Exhaustive case = head and tail (2)

Ex: tossing a coin twice

S = {HH, HT, TH, TT}

Let E = getting a head

Favourable cases = HT, TH (2), Exhaustive cases = HH, HT, TH, TT (4)

Let E = even number in first die

Ex: Rolling two dice simultaneously

Sample space:

(1,1), (1,2), ………, (1,6)

(2,1), (2,2) , ………,(2,6)

…………

………..

……….

(6,1), (6,2) , ……., (6,6)   exhaustive cases = 36

Favourable cases = 18

Laws of Set Theory

Let A, B and C be the subsets of the universal set S, then

- Commutative laws

$$A \cup B = B \cup A \quad and \; A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C \quad and \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \; and$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continue…

- Difference laws

$$A - B = A \cap \bar{B} = A - (A \cap B)$$
$$B - A = B \cap \bar{A} = B - (A \cap B)$$

- Complementary laws,

$$A \cup \bar{A} = S, \qquad A \cap \bar{A} = \emptyset$$
$$A \cup S = S, \qquad A \cap S = A$$
$$A \cup \emptyset = A, \qquad A \cap \emptyset = \emptyset$$

Continue…

- De – Morgan’s law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cup B \cup C} = \bar{A} \cap \bar{B} \cap \bar{C}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$$

Various Approaches of Probability

- Mathematical or Priori or Classical Definition of Probability
- Statistical or Empirical Definition of Probability
- Subjective Definition of Probability
- Axiomatic or Modern Definition of Probability

Classical Definition

If there are 'n' number of exhaustive, mutually exclusive and equally likely outcomes of a random experiment and 'm' number of them are favourable to an event A. Then the probability of happening or occurring an event A, denoted by P(A), is defined as

$$P(A) = \frac{Favourable\ number\ of\ cases}{Exhaustive\ number\ of\ cases} = \frac{m}{n}$$

- Ex : rolling a die
- Sample space : {1, 2, 3, 4, 5, 6}
- Let E = getting an even number
- Favourable cases : 2, 4, 6
- Favourable number of cases for getting an even number (m) = 3
- Exhaustive cases : 1, 2, 3, 4, 5, 6
- Exhaustive number of cases (n) = 6
- $P(E) = m / n = 3/ 6 = 1/2$

Ex : rolling two dice simultaneously

Sample space:

Favourable no. of cases (m) = 18

Exhaustive no. of cases (n) = 36

Let E = sum of faces is greater than 9 ,

Let A = sum of faces is 7 , P(A) = 6 / 36 =1 / 6

Let B = sum of faces is less than 5 , P(B) = 1 /6

Let C = even number in first die

Continue….

- From this definition, we have

$$P(\bar{A}) = 1 - P(A)$$

- Remarks:

1. $P(A) \geq 0$ and $P(\bar{A}) \geq 0$ such that $P(A) + P(\bar{A}) = 1$ $i.e. p + q = 1$
2. $0 \leq P(A) \leq 1$ since $m \leq n$
3. $P(S) = 1$ and $P(\emptyset) = 0$

Continue…

4. If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

For '*n*' number of mutually exclusive events;

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

If n = 3 $$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Ex : head and tail are mutually exclusive events,

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$

$P(H \cup T) = P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$

Empirical or Statistical Definition

If an event A occurs ‘m’ times in ‘n’ repetitions of a random experiment. In the limiting case when ‘n’ become sufficiently large, then the probability of happening an event A is finite and unique. That is;

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Modern or Axiomatic Definition

Let S be a sample space of a random experiment and A be any event defined on sample space S, then P(A) is the probability function satisfying the following axioms;

1. P(A) ≥ 0 i.e. Non – negativity Axiom
2. P(S) = 1 i.e. Axiom of Certainty
3. Let $A_1, A_2, \dots\dots\dots, A_n$ be any finite or infinite sequence of mutually exclusive events. Then Axiom of additive

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \;\; or \;\; P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Rules of Counting (Combinatorial Analysis)

When an experiment is performed in two or many stages, there may be large number of possible outcomes at each stage and in this case, the counting or listing of all possible outcomes becomes very difficult. Thus, it can be made simple by rules called fundamental rules of counting.

The fundamental rules of counting consist **Addition** and **Multiplication** rules:

Addition Rule:

When event A occurs in n_1 ways and event B occurs in n_2 ways are **mutually exclusive**, then the event A or B (at least one event) can occur in $n_1 + n_2$ ways.

For Example:

A bag contains 5 red balls and 4 black balls, then the total number of cases for either red or black ball is 5 + 4 = 9 .

Multiplication Rule:

When event A occurs in n_1 ways and event B occurs in n_2 ways then A and B can occur **simultaneously** in $n_1 . n_2$ ways.

For example:

On rolling two fair dice simultaneously, first die has 6 outcomes and the second die has 6 outcomes so that the total number of possible cases is $6 \times 6 = 36$.

Two Basic Methods of Counting Rules

1. Permutation

2. combination

Permutation

- The arrangement of objects
- The order matters in the arrangement of objects i.e. if the position of the objects within the group is taken in to account, then we use permutation for the selection.

*A permutation of n **different** objects taken r at a time is an ordered arrangement of only r objects out of n objects, denoted by $P(n, r) = n_{P_r}$ and is given by*

$$^{n}P_{r} = \frac{n!}{(n-r)!}\ ; r \leq n$$

- $n! = n * (n-1) *(n-2) * …….. 3*2*1$
- For example $10! = 10 *9*8*7*6*5*4*3*2*1$
- $0! = 1$
- $1! = 1$

Continue…

For repeated cases, the permutation of n units in which $p, q, and\ r$ times units are repeated.

$$\therefore Permutation = \frac{n!}{p!\,q!\,r!}$$

- Example: in how many ways the word **statistics** can be arranged?
- P = 3, q = 3 and r = 2 , n = 10
- Permutation =
- $\therefore Permutation = \frac{10!}{3!3!2!} =$

Combination

- The selection of objects
- Combination is concern with the selection only, not the order of the selection.

A combination of n different objects taken r at a time is the selection only r objects out of the n objects (w/o any regard to the order of arrangement), is denoted by $C(n,r) = \binom{n}{r} = {}^{n}C_{r}$ *and is given by*

$$\boldsymbol{{}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}\ ;r \leq n}$$

Examples

- If you flip a **fair** coin twice, what is the probability of getting at least one head?
- From a group of 15 chess players, 8 are selected by lot to represent a group at a convention. What is the probability that the selected include 3 of the 4 best players in the group?
- A lot of integrated circuit chips consist of 10 good, 4 with minor defects and 2 with major defects. Two chips are selected randomly from the lot. What is the probability that at least one chip is good?

Random Experiment : Flip a fair coin twice

Sample space = {HH, HT, TH, TT}

Let E = getting at least one head

$P(E) = m / n$

Favourable number of cases, $m = 3$

Exhaustive number of cases, $n = 4$

$P(E) = 3/4$

Solution 1

Sample space, S = {HH, HT, TH, TT}

Favourable no. of cases, m = 3

Exhaustive no. of cases, n = 4

Now, Probability of getting at least one head is given by

P(at least one head) = m / n = 3 / 4

3.

Solution:

Here, no. of good circuit chips = 10

No. of minor defect circuit chips = 4

No. of major defect circuit chips = 2

Total no. of circuit chips = 16

Prob. that at least one chip is good = ?

Exhaustive no. of cases, n = $\binom{16}{2} = 120$

P(one good and one other or 2 good)

= P(one good and one other) + P(2good)

Continue…

- For 1 good and 1 other

Out of 10 good chips, 1 chip can be drawn in $\binom{10}{1}$ = 10 ways

Similarly,

Out of 6 chips, 1 chip can be drawn in $\binom{6}{1}$ = 6 ways

Favourable cases for 1 good and 1 other = 6 * 10 = 60
P(1 good and 1 other) = m/n = 60 /120

Also, for 2 good chips,

Out of 10 good chips, 2 chips can be drawn is $\binom{10}{2}$ = 45 ways.

Continue…

Favourable cases for 2 good chips = 45

Therefore, P(2 good chips) = m / n = 45 / 120

Now,

P(at least one good chip) = P(1 good and 1 other) or P(2 good chips)

= 60/120 + 45 / 120

= 105/120

1

Solution:

Possible outcomes : HH, HT, TH, TT

Sample space = {HH, HT, TH, TT}

No. of exhaustive cases (n) = 4

Favourable outcomes for at least one head: HH, HT, TH

No. of favourable outcomes (m) = 3

Prob. of getting at least one head , P(at least one head) or

P(A) = m/n = 3/4

2.

Solution:

Here, total no. of players = 15

Out of 15 players, 8 players are selected in $\binom{15}{8}$ ways i.e. 6435

Exhaustive no. of cases (n) = 6435

No. of best player = 4

Then, 3 out of 4 best players is selected in $\binom{4}{3}$ ways = 4 ways,

Also, no. of ordinary players (remaining players) = 11

5 players are selected from 11 ordinary players in $\binom{11}{5}$ ways = 462 ways

Continue…

No. of favourable cases for 3 best players and 5 ordinary players

(m) = 4 * 462 = 1848

Now, the probability that selected group include 3 best players and 5 ordinary players is given by

P(3 best player in a group of 8) = m / n =1848 / 6435 =

Examples

1. A bag contains 3 red and 2 green balls. Two balls are drawn at random from the bag. Find the probability that both balls have the same colour. P(both have same) = P(both red or both green)
2. If a box contains 75 good IC chips and 25 defective chips, and 12 chips are selected at random, what is the probability that at least one chip is defective? P(at least one is defective)

- Total no. balls = 5
- No. of red balls = 3
- No. of green balls = 2
- Exhaustive no. of cases = C(5,2) = 10
- Both balls are red = C(3,2) = 3
- P(both are red) = 3/ 10
- Both are green = C(2,2) = 1
- P(Both are green) = 1 / 10
- P(both are of same colour) = P(both red) + P(both green)
- = 3/ 10 + 1/10 = 4 /10 = 2/ 5

1.

Solution:

Here, no. of red balls = 3

And no. of green balls = 2

Therefore, total no. of balls = 5

Out of 5 balls, 2 balls can be selected from the bag in $\binom{5}{2}$ ways =10ways

i.e. the exhaustive no. of cases (n) = 10

Both balls are of same colour, i.e. Both red or both green

Now, 2 red balls are selected from 3 red balls in $\binom{3}{2}$ ways = 3 ways

Continue…

$\therefore$ P(both red) = 3/10

And 2 green balls are selected from 2 green balls in $\binom{2}{2}$ ways = 1 way

$\therefore$ P(both green) = 1/10

Now, the probability of getting both balls of same colour is given by

P(both balls have same colour) = P(both red or both green)

= P(both red) + P(both green)

= 3/10 + 1/10

= 4/10

2.

Solution

Here, no. of good IC chips = 75

And no. of defective IC chips = 25

Total no. of IC chips = 100

Out of 100 IC chips, 12 can be selected in $\binom{100}{12}$ ways

$\therefore$ Exhaustive cases, (n) = $\binom{100}{12}$

Let E be the event that at least one chip is defective, then its complementary event i.e. $\bar{E}$ is the event that there is no defective chips.

Continue…

$\therefore$ out of 75 good chips, 12 good chips can be drawn in $\binom{75}{12}$ ways

i.e. favourable no. of cases for no defective chips (m) = $\binom{75}{12}$

Thus, the probability that there is no defective chips is given by

$P(\bar{E})$ = m / n = $\binom{75}{12} / \binom{100}{12}$ =

Now, the required probability that there is at least one defective chip is given by

$P(E) = 1 - P(\bar{E})$

$= 1 - \binom{75}{12} / \binom{100}{12}$ complete it.

Practice

- A box with fifteen integrated circuit chips contains five defectives. If a random sample of three chips is drawn, what is the probability that all three are defectives?
- An urn contains 4 white and 6 red balls. Two balls are drawn out together. What is the probability that both are red balls?

- Total chips = 15
- Good chips = 10
- Defective chips = 5
- P(All three chips are defective) = m / n = 10 / 455
- n = C(15, 3)
- m = C(5,3) = 10

Laws of Probability

- Additive law of probability
- Multiplicative law of probability

Additive law

1. Let A and B are two mutually exclusive events. Then the probability of occurrence of either A or B (at least one event) is given by

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B)$$

Similarly, for three mutually exclusive events A, B and C

$$P(A\ or\ B\ or\ C) = P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Continue…

2. If A and B are not mutually exclusive events, then the probability of occurrence of at least one of them is given by

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Similarly, for three not mutually exclusive events A, B and C

$$P(A\ or\ B\ or\ C) = P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Examples

- The probability that a new airport will get an award for its design, and award for its efficient use of materials and both the awards is, respectively, 0.16, 0.24 and 0.11. Find the probability that new airport get at least one award. Also What is the probability that it will get only one of the two awards?

- The probability that a contactor will get a plumbing contract is $\frac{2}{3}$ and the probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting at least one contract is $\frac{4}{5}$. What is the probability that he will get both the contracts?

1.

Solution:

Let A and B denotes the events that the new airport gets award for its design and its efficient use of materials respectively.

Given that,

P(A) = 0.16

P(B) = 0.24

$P(A \cap B)$ = 0.11

The probability of getting at least one award is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Continue…

$$P(A \cup B) = 0.16 + 0.24 - 0.11 = 0.29$$

Also, the probability that it will get only one award is given by

P(A only or B only) = P(A only) + P(B only)

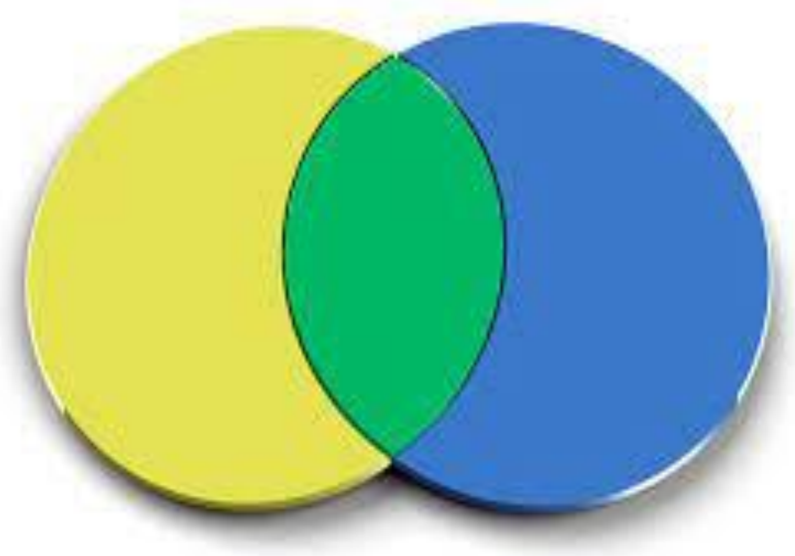
For this, P(A only) = P(A) – P(A∩B) = 0.16 – 0.11 = 0.05

And P(B only) = P(B) – P(A∩B) = 0.24 – 0.11 = 0.13

Now,

P(A only or B only) = P(A only) + P(B only)

= 0.05 + 0.13 = 0.18



2.

Solution:

Let A and B denotes the events that a contractor will get the plumbing contract and electric contract respectively. Then from the given data,

We have

P(A) = 2/3

P($\bar{B}$) = 5/9

P(A∪B) = 4/5

∴ P(B) = 1 – P($\bar{B}$) = 1 – 5/9 = 4/9

Continue….

P(A∩B) =?

Using additive law, we have

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Or, 4/5 = 2/3 + 4/9 - $P(A \cap B)$

Or, $P(A \cap B)$ = 14 / 45.

Practice

- In a city , three English dailies A, B and C are published, and a recent survey of readers indicates the following: 30% read A, 20% read B, 15% read C, 10% read both A and B, 8% read both B and C, 4% read both A and C, and 2% read all A, B and C. Compute the probability that at least one paper among A, B and C is read by randomly selected person in the city.

Let A, B and C

P(A) =0.30

P(B) = 0.20

P(C) = 0.15

P(A and B) =P(A∩B)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Multiplicative Law

1. Let A and B are two independent events, then the probability of occurrence of both the events is given by

$$P(A\ and\ B) = P(A \cap B) = P(A).P(B)$$

Similarly, for three independent events A, B and C

$$P(A\ and\ B\ and\ C) = P(A \cap B \cap C) = P(A).P(B).P(C)$$

Continue…

2. Suppose A and B are two dependent events, then

$$P(A \cap B) = \begin{cases} P(A).P\left({}^{B}/_{A}\right) \\ P(B).P\left({}^{A}/_{B}\right) \end{cases}$$

Where, $P\left({}^{B}/_{A}\right)$ is the probability of the occurrence of event B given that event A has already occurred.

$P\left({}^{A}/_{B}\right)$ is the probability of occurrence of event A given that event B has already occurred.

Conditional Probability

1. The conditional probability of event A when event B has already known (occurred) is given by

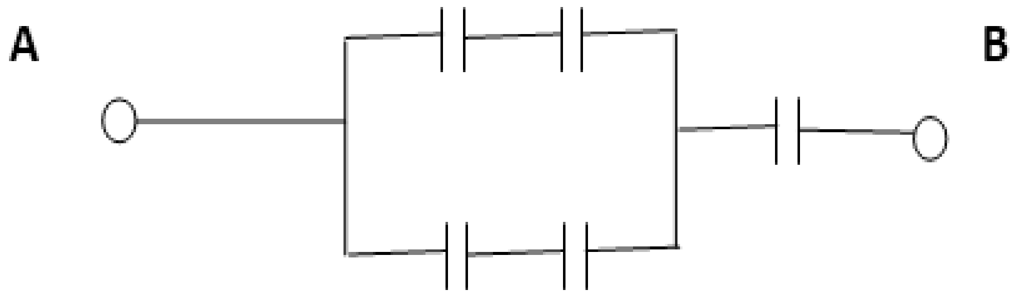
$$P\left({}^{A}/_{B}\right) = \frac{P(A \cap B)}{P(B)}$$

2. The conditional probability of event B when event A has already known (occurred) is given by

$$P\left({}^{B}/_{A}\right) = \frac{P(A \cap B)}{P(A)}$$

Examples

- For the circuit given below, the probability of closing each relay of the circuit is known to be 0.6. Assume that the relays act **independently**. What is the probability that a current will exist between the terminals A and B.

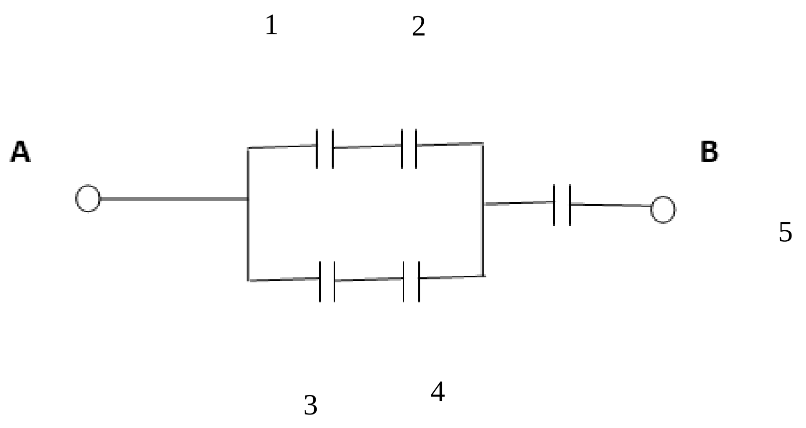


Solution:

Given that the probability of closing each relay of the circuit is 0.60

That is, $P_1 = P_2 = P_3 = P_4 = P_5 = 0.60$

The given circuit is



Continue….

Let E_1 be the event that relay 1, 2, and 5 closes and E_2 be the event that relay 3, 4, and 5 closes. Since relays act independently, then we have

P(E_1) = P(1 and 2 and 5) = $P_1 . P_2 . P_5$ = 0.6 * 0.6 * 0.6 = 0.216

Similarly, P(E_2) = P(3,4,5) = $P_3 . P_4 . P_5$ = 0.216

Continue…

And P($E_1 \cap E_2$) = P(1, 2, 3, 4, 5) = $P_1 . P_2 . P_3 . P_4 . P_5$ = 0.07776

Now, the probability that current flow between A and B is given by

P($E_1 \cup E_2$) = P(E_1) + P(E_2) – P($E_1 \cap E_2$)

= 0.216 + 0.216 – 0.07776

= 0.35424

Examples

There are three switches in a college network namely A, B and C working independently. These switches are configured in series so that all switches should be 'on' to have successful transmission of data. The individual probability of being switch 'on' for these are 1/2, 3/4 and 1/4 respectively. Find the probability that

- There will be a successful data transfer.
- There will not be a successful data transfer.

Examples

1. A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

2. A manufacturer of airplane parts knows that the probability is 0.8 that an order will be ready for shipment on time, and it is 0.7 that an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that it was also ready for shipment on time? P(deliver on time / ready for shipment) = ?

1.

Solution:

Here, no. of good tubes = 6

And no. of bad tubes = 4

Therefore, total no. of tubes = 10

Let A and B denotes the events that the first tube is good and the second tube is good respectively.

Continue…

From the given data,

P(A) = 6 / 10

Also, out of 10 tubes, 2 tubes can be drawn in $\binom{10}{2}$ ways = 45 ways

If both are good tubes then out of 6 good tubes, 2 good tubes can be drawn in $\binom{6}{2}$ ways = 15 ways.

Continue…

Therefore, $P(A \cap B) = \frac{15}{45} = \frac{1}{3}$

Hence, the required probability is given by

$$P\left({}^{B}/_{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{5}{9}$$

2.

Solution:

Let A = an order will be ready for shipment on time

B = an order will be delivered on time.

From given data,

$P(A) = 0.80$

$P(A \cap B) = 0.70$

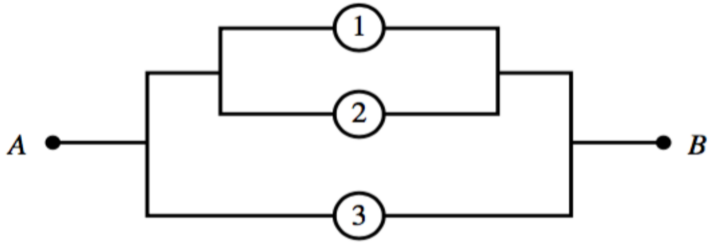
Continue..

Now, the required probability is given by

$$P\left({}^{B}/_{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{7}{8}$$

Example

Consider the following portion of an electric circuit with three relays. Current will flow from point A to point B if there is **at least one closed path** when the relays are activated. The relays may malfunction and not close when activated. Suppose that the relays act independently of one another and close properly when activated, with a probability of .90.



a. What is the probability that current will flow when the relays are activated?
b. Given that current flowed when the relays were activated, what is the probability that relay 1 functioned?

Solution:

Given that the probability that relay close properly when activated is 0.90.

That is, $p_1 = p_2 = p_3 = 0.90$

Where $p_1,\ p_2\ , and\ p_3$ are the probabilities of relay 1, 2, and 3 being closed properly respectively.

Then we have,

$q_1 = q_2 = q_3 = 0.10$

Continue…

Where $q_1,\ q_2,\ and\ q_3$ are the probabilities of relay 1, 2, and 3 being open respectively.

1. Since all relays act independently

P(all relays are open) = $q_1 . q_2 . q_3$ = 0.1*0.1*0.1 = 0.001

Hence the probability of current flow from A to B is

P(current flows) = P(at least one relay is closed)

= 1 – P(all relays are open) = 1 – 0.001 = 0.999

Continue…

2. Let A be the event that current flows and B be the event that relay 1 closed properly.

Then, $P({}^{B}/{}_{A}) = \frac{P(A\cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{0.9}{0.999} = 0.9009$ since B $\subset$ A

Examples

- The odds in favour of A solving a mathematical problem are 3 to 4 and the odds against B solving the problem are 5 to 7. Find the probability that the problem will be solved by at least one of them. P(A or B) = P(A) + P(B)
- A problem in statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{3}{4}\ and\ \frac{1}{4}$ respectively. What is the probability that the problem will solved if all of them try independently?

Hint:

- Odds in favour A is 3 to 4
- P(A) = 3/3+4
- Odds against A is 3 to 4
- P(A) = 4/3+4

Examples

1. A, B and C, in order, toss a coin. The first one to throw a head wins. If A starts, find their respective chances of winning.

2. A and B throw alternately with a pair of dice. A will win if he throws 6 before B throws 7 and B will win if he throws 7 before A throws 6. If A begins, show that his chance of winning is $\frac{30}{61}$.

Solution 1 :

Remarks:

1. If A be an event, then its complementary event is denoted by $\bar{A}$ and the probability of complementary event $\bar{A}$ is given by
$$P(\bar{A}) = 1 - P(A)$$
2. For any two events A and B, then
$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
Similarly, $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
3. Also, $P(A - B) = P(A) - P(B)$
Similarly, $P(B - A) = P(B) - P(A)$

Continue…

4. If A and B are independent, then $\overline{A}$ and $\overline{B}$ are also independent

i.e. $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = P(\bar{A}) \, . \, P(\bar{B})$

Similarly, we have

- $P(A \cap \bar{B}) = P(A) \, . \, P(\bar{B})$
- $P(\bar{A} \cap B) = P(\bar{A}) \, . \, P(B)$

Examples

- In a certain group of computer personnel, 65% have insufficient knowledge of hardware, 45% have inadequate idea of software and 70% are in either one or both of the two categories. What is the probability of people who know software among those who have a sufficient knowledge of hardware?

Solution:

Let A and B denotes the events that the computer personnel have sufficient knowledge of hardware and software respectively.

Given that,

P($\bar{A}$) = 0.65 , P($\bar{B}$) = 0.45 and $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 0.70$

Now, we have

P(A) = 1 – P($\bar{A}$) = 0.35 , P(B) = 1 – P($\bar{B}$) = 0.60

Continue…

And $P(A \cap B) = 1 - P(\overline{A \cap B}) = 0.30$

Hence the required probability is given by

$$P({}^{B}/_{A}) = \frac{P(A \cap B)}{P(A)} = \frac{6}{7}$$

Bayes' Theorem

Statement: (important)

Let $E_1, E_2, \dots \dots \dots, E_n$ be 'n' mutually exclusive events defined in the sample space (S) with $P(E_i) \neq 0;\ \ i = 1, 2, \dots \dots \dots, n$. For any arbitrary event A which is the subset of S such that P(A) > 0, then the probability of happening of E_i when event A has already occurred is given by

$$P\left({}^{E_i}\!/_{A}\right) = \frac{P(E_i).P\left({}^{A}\!/_{E_i}\right)}{\sum_{i=1}^{n} P(E_i).P\left({}^{A}\!/_{E_i}\right)}\ ;1 \leq i \leq n$$

Proof:

From the definition of conditional probability, we have

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(E_i).P\left({}^{A}/_{E_i}\right)}{P(A)} \quad \ldots\ldots\ldots\ldots (1)$$

Since, A is the subset of S and sample space S is given by

$$S = E_1 \cup E_2 \cup E_3 \cup \ \ldots\ldots\ldots \cup E_n$$

Then, we can write

$$A = A \cap S = A \cap (E_1 \cup E_2 \cup E_3 \cup \ \ldots\ldots\ldots \cup E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup \ \ldots\ldots\ldots \cup (A \cap E_n)$$

[using Distributive law]

Continue…

Since $E_1, E_2, \dots \dots \dots, E_n$ be mutually exclusive events,

Hence,

$(A \cap E_1),\ (A \cap E_2), \dots \dots \dots,\ (A \cap E_n)$ are also mutually exclusive events.

Now, taking probability on both sides, we get

$$
\begin{aligned}
\mathrm{P}(A) &= P\{(A \cap E_1) \cup (A \cap E_2) \cup \dots \dots \dots \cup (A \cap E_n)\} \\
&= P(A \cap E_1) + P(A \cap E_2) + \dots \dots \dots + P(A \cap E_n) \\
&= \sum_{i=1}^{n} P(A \cap E_i)
\end{aligned}
$$

Continue…

$$\therefore P(A) = \sum_{i=1}^{n} P(E_i).P\left({}^{A}/_{E_i}\right) \quad \dots \dots \dots \dots \dots \dots \dots (2)$$

Substituting the value of P(A) from (2) in (1), we get

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i).P\left({}^{A}/_{E_i}\right)}{\sum_{i=1}^{n} P(E_i).P\left({}^{A}/_{E_i}\right)}$$

Hence Proved…..

Notes:

- Here $P(E_i)$ are called prior probabilities, since these probabilities exist before performing the experiment.
- $P\left({}^{E_i}/_{A}\right)$ are called posterior probabilities since they are determined after the result of the experiment are known.
- $P\left({}^{A}/_{E_i}\right)$ are called conditional probabilities

Example 21

In a bolt factory machine A, B, and C manufacture respectively 25%, 35% and 40% of the total, out of their output 5%, 4%, 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A?

Solution:

Let A, B and C denotes the events that the selected bolt come from machine A, machine B and Machine C respectively.

Then we have,

$P(A) = 0.25$

$P(B) = 0.35$

$P(C) = 0.40$

Let D be the defective bolt, then

$P(D/A) = 0.05$

$P(D/B) = 0.04$

$P(D/C) = 0.02$

The probability that bolt is from machine A given that it is a defective bolt, is given by

$$P\left({}^{A}/_{D}\right) = \frac{P(A).P\left({}^{D}/_{A}\right)}{P(A).P\left({}^{D}/_{A}\right) + P(B).P\left({}^{D}/_{B}\right) + P(C).P\left({}^{D}/_{C}\right)}$$

Example 22

A manufacturing firm produces steel pipes in three plants with daily production volume of 500, 1000, and 2000 units respectively. According to the past experience, it is known that the fraction of defective outputs produced by the three plants are respectively 0.005, 0.008 and 0.010. If pipe is selected from a day's total production and found to be defective. From which plant the defective pipe is expected to have been produced?

P(A/D) = ? , P(B/D) = ? And P(C/D) = ?

Solution:

Let A, B, and C denote the event that the pipe selected is produced by the plant I, II and III respectively. Then we have

P(A) = 500 / 3500 = 0.143

P(B) = 1000 / 3500 = 0.286

P(C) = 2000 / 3500 = 0.571

Let D be the defective pipe, so we have

P(D/A) = 0.005

P(D/B) = 0.008

P(D/C) = 0.010

Continue…

Therefore, the probability of selected pipe is from Plant I given that pipe is defective, is given by

$$P(A/D) = \frac{P(A).P(D/A)}{P(A).P(D/A)+P(B).P(D/B)+P(C).P(D/C)} = 0.2$$

Similarly, the probability of selected pipe is from plant II given that pipe is defective, is given by

$$P(B/D) = \frac{P(B).P(D/B)}{P(A).P(D/A)+P(B).P(D/B)+P(C).P(D/C)} = 0.26$$

Continue…

And the probability of selected pipe is from plant III given that pipe is defective, given by

$$P\left({}^{C}/_{D}\right) = \frac{P(C).P\left({}^{D}/_{C}\right)}{P(A).P\left({}^{D}/_{A}\right)+P(B).P\left({}^{D}/_{B}\right)+P(C).P\left({}^{D}/_{C}\right)} = 0.65$$

Hence, the selected pipe is expected to have been produced by Plant III.

Example 23

State Bayes' theorem. In a class of 75 students, 15 students were considered to be very intelligent, 45 as medium and rest were below the average. The probability that a very intelligent student **failing** examination is 0.005, the medium student failing has probability 0.05 and corresponding probability for a below average is 0.15. If a student is known to have **passed** the examination. What is the probability that he is below the average? P(below Average / Pass the examination) = ?

Solution:

Let us define the following events:

A = student of class is very intelligent

B = student is medium

C = student is below average

Then we have.

$P(A) = 15 / 75 =$

$P(B) = 45 / 75 =$

$P(C) = 15 / 75 =$

Continue…

Also, let E be the event that the student passes the examination.

Then we have

P(E/A) = 1 – 0.005 = 0.995

P(E/B) = 1 – 0.05 = 0.95

P(E/C) = 1 – 0.15 = 0.85

P(C/E) = ?

Now, the probability that the student is below average given that he passed the examination is given by

$$P\left({}^{C}/_{E}\right) = \frac{P(C).P\left({}^{D}/_{E}\right)}{P(A).P\left({}^{E}/_{A}\right)+P(B).P\left({}^{E}/_{B}\right)+P(C).P\left({}^{E}/_{C}\right)}$$

Ans: 0.181

Example 24

The contents of urns I, II and III are as follows:

1 white, 2 black and 3 red balls; 2 white, 1 black and 1 red balls and 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls are drawn. They are found to be white and red. What is the probability that they come from urns I?, **II? and III**?

Solution:

Let A, B and C denote the events of selecting I, II, and III urn respectively.

P(A) = 1/3

P(B) = 1/3

P(C) = 1/3

Let E be the event that two balls selected from urn are white and red. Then we have

P(E/A) = $\frac{{}^{1}C_{1} \times {}^{3}C_{1}}{{}^{6}C_{2}} =$

Continue…

Similarly,

P(E/B) = $\frac{{}^2C_1 \times {}^1C_1}{{}^4C_2}$ =

And, P(E/C) = $\frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2}$ =

Therefore, the probability that the selected white and red balls are come from urn I is

P(A/E) = ?

Also, the probability that the selected white and red balls are from urn II is

P(B/E) = ?

And the probability that the selected white and red balls are from urn III is

And P(C/E) = ?

Example 25

Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

i. What is the probability that a product attains a good review?

ii. If a new design attains a good review, what is the probability that it will be a highly successful product?

Solution:

Let us define the following events:

A = highly successful products

B = moderately successful products

C = poor products

Then we have

P(A) = 0.40

P(B) = 0.35

P(C) = 0. 25

Continue..

Again, let G be the event of getting good reviews about the products

P(G/A) = 0.95

P(G/B) = 0.60

P(G/C) = 0.10

i. Probability that the product attains the good reviews

P(G) = $P(G \cap A) + P(G \cap B) + P(G \cap C)$

$= P(A).P({}^{G}/_{A}) + P(B).P({}^{G}/_{B}) + P(C).P({}^{G}/_{C})$

$=$

Practice 1

- State Bayes' theorem. An aircraft emergency locator transmitter (ELT) is a device designed to transmit a signal in the case of a crash. The Altigauge manufacturing company makes 80% of the ELTs, the Bryant Company makes 15% of them, and the Chartair Company makes the other 5%. The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and Chartair ELTs have a 9% rate of defects.

1. If a randomly selected ELT is then tested and is found to be defective, find the probability that it was made by the Altigauge manufacturing company.
2. If a randomly selected ELT is then tested and is found to be no defective, find the probability that it was made by the Bryant manufacturing company.

Practice 2

- Four technicians regularly make repairs when breakdown occurs on an automated production line. Janet, who services 20% of the breakdowns, make an incomplete repair 1 time in 20; Tom, who services 60% of the breakdowns, make an incomplete repair of 1 time in 10; Georgia, who services 15% of the breakdowns, make an incomplete repair of 1 time in 10; and Peter, who services 5% of the breakdown, make an incomplete repair of 1 time in 20. For the next problem with the production line diagnosed as being due to an initial repair that was incomplete, what is the probability that this initial repair was made by Janet?

Practice 3

- A given lot of IC chips contains 2% defective chips. Each chip is tested before delivery. The tester itself is not totally reliable. Probability of tester says the chip is good when it is really good is 0.95 and the chip is defective when it is actually defective is 0.94. If a tested device is indicated to be defective, what is the probability that it is actually defective?