Random Variable

Definition

- Random variable is a real valued function associated with each outcome of a random experiment.
- In other words, it is a real valued function defined on a sample space of the random experiment.
- It is denoted by capital letters X, Y, Z, etc. and the value taken by a random variable is denoted by small letters x, y, z, etc.
- For example:

Consider a random experiment of tossing a coin, assign 1 if head appears and 0 if tail appears. Hence the number of head is a random variable taking value 1 and 0 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ respectively.

- In tossing a fair coin twice,

$S = \{ HH, HT, TH, TT\}$

X = Number of heads = 0, 1, 2

$P(X=0) = P(TT) = ¼$

$P(X = 1) = 2/ 4$

$P(X =2) = ¼$

Properties of Random Variable

- If X is a random variable, then aX is also a random variable. Here ‘a’ is a constant.
- If X and Y are random variables then $X + Y, X - Y, aX + bY, aX - bY$ are also random variable. Here ‘a’ and ‘b’ are constants
- If X and Y are random variables then $XY, X/Y$ are also random variables
- If X is a random variable then X^2, X^{-1}, etc. are also random variables.

Types of Random Variable

- Discrete Random Variable
- Continuous Random Variable

Discrete Random Variable

- A random variable is said to be a discrete random variable if it takes only integer values.
- It is a real valued function defined on a discrete sample space.
- For examples:

i. Number of printing mistakes in a page of a book
ii. Number of students enrolled in NCIT
iii. Number of patients admitted in a hospital
iv. Number of defective items, etc.

Probability Function (Probability Mass Function) of Discrete Random Variable

Let X be a discrete random variable taking values $x_1, x_2, x_3, \dots \dots \dots, x_n$ with probabilities $p_i = P(X = x_i) = P(x_i)$, $i = 1, 2, 3, \dots \dots \dots, n$. Then the probability function $p_i = P(X = x_i)$ is called probability mass function (pmf) if it satisfies the following conditions:

i. $P(x_i) \geq 0$, for all $i = 1, 2, \dots \dots, n$

ii. $\sum_{i=1}^{n} P(x_i) = 1$

Example 1

Consider a random experiment of tossing a coin. Let S = {H, T}. Consider the random variable X defined on S by

$$X = Number\ of\ heads$$

Then X is a discrete random variable since it can take only two values i.e. X = 0 and X = 1, with P(X = 0) = P(X = 1) = $\frac{1}{2}$.

Since P(X = 0) > 0 and P(X = 1) > 0, such that P(X = 0) + P(X = 1) = 1, thus X is a discrete random variable.

Example 2

- In tossing a fair coin twice, then S = {HH, HT, TH, TT}. Consider the random variable X defined on S by

$$X = Number\ of\ heads$$

Then X is a discrete random variable since it takes three values i.e.

X = 0, X = 1 and X = 2, with probabilities P(X=0) = 1/4 , P(X=1) = 1/2 and P(X=2) = 1/4 .

Continue..

Since $P(X=0)$, $P(X=1)$ and $P(X=2) > 0$ such that

$P(X=0) + P(X=1) + P(X=2) = 1$, thus X is called discrete random variable.

Example 3

- A fair coin is tossed three times. If X is the number of heads, give the probability distribution of X.

Solution:

Let p and q denotes the probabilities of getting head and tail in a single toss respectively. Then p = q = 1/2 . Here, the sample space S have

$2^3 = 8$ outcomes. That is

Continue…

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Here, {X = 0} = {TTT} and hence P(X=0) = 1/8.

Similarly, P(X=1), P(X=2) and P(X=3) can be computed. Thus the probability distribution of X is given by

X = x	0	1	2	3
P(X = x)	1/8	3/8	3/8	1/8

Example 4

- A lot of 10 items contains 3 defectives from which a sample of 4 items is drawn without replacement. Let X be the random variable being the number of defective items in the sample. Find

i. The probability distribution of X

ii. $P(X < 1)$ and

iii. $P(0 < X < 2)$.

Solution:

Clearly, X takes the values 0, 1, 2 and 3.

We find the probabilities P(X = i), where i = 0, 1, 2, 3

Now, P(X = 0) = P(no defective) = $\frac{{}^{7}C_4}{{}^{10}C_4} = \frac{1}{6}$

P(X = 1) = P(1 defective) = $\frac{{}^{3}C_1 . {}^{7}C_3}{{}^{10}C_4} = \frac{1}{2}$

Continue…

P(X = 2) = P(2 defectives) = $\frac{{}^{3}C_{2} . {}^{7}C_{2}}{{}^{10}C_{4}} = \frac{3}{10}$

And P(X = 3) = P(3 defectives) = $\frac{{}^{3}C_{3} . {}^{7}C_{1}}{{}^{10}C_{4}} = \frac{1}{30}$

Hence the probability distribution of X is given by

X = x	0	1	2	3
P(X = x)	1/6	1/2	3/10	1/30

Continue…

Remaining parts

$P(X < 1) = P(X = 0) = 1/6$

And $P(0 < X < 2) = P(X = 1) = 1/2$.

Discrete Distribution Function

A function of a discrete random variable X is said to have probability distribution function $F(x_i)$ if it is defined as

$$F(x_i) = P(X \le x_i) = \sum_{i=1}^{x} P(x_i)$$

- It is also known as cumulative probability function.

Properties of Discrete Distribution Function

1. $0 \leq F(x_i) \leq 1, i = 0, 1, 2, \ldots \ldots.$

2. $F(-\infty) = \lim_{x \to -\infty} F(x_i) = 0$

3. $F(+\infty) = \lim_{x \to +\infty} F(x_i) = 1$

4. $P(a < X \leq b) = F(b) - F(a)$

5. $F(x_i) < F(x_j)$, if $x_i < x_j$

Example 5

- If X is the random variable representing the number of heads in two tosses of a fair coin. Find the distribution function of X.

Solution:

Here, in two tosses of a fair coin, the sample space is given by

$S = \{HH, HT, TH, TT\}$, then $X = 0, 1, 2$

$P(X = 0) = P(TT) = 1/4$

Continue…

P(X = 1) = P(HT, TH) = 1/2

And P(X = 2) = P(HH) = 1/4

Hence, the distribution function of X is given by

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{1}{4} & if\ 0 \leq x < 1 \\ \frac{3}{4} & if\ 1 \leq x < 2 \\ 1 & if\ x \geq 2 \end{cases}$$

$X = x$	0	1	2
$P(X = x)$	1/4	1/2	1/4
$F(x)$	1/4	3/4	1

Example 6 (Practice)

- If a random variable X takes the values 1, 2, 3, 4 such that

$2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$.

Then find the probability distribution of X.

Solution:

Let $P(X = 3) = k$. From the given data, it follows that

$P(X = 1) = k/2$, $P(X = 2) = k/3$ and $P(X = 4) = k/5$

Continue…

Since total probability is 1, thus we must have

P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1

or, $\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow \frac{61k}{30} = 1 \Rightarrow k = \frac{30}{61}$

Hence, the probability distribution is given by

X = x	1	2	3	4
P(X = x)	15/61	10/61	30/61	6/61

Example 7 (Practice)

- A random variable X has the following probability distribution:

X = x	0	1	2	3	4	5	6
P(X = x)	c	2c	3c	4c	3c	2c	c

Find

i. The value of c.

ii. $P(X < 2)$, $P(X \geq 2)$, $P(0 < X < 4)$

iii. $P(0 < X < 4 / X \geq 2)$

$$P(0 < X < 4 \,/\, X \geq 2)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(0 < X < 4 \,/\, X \geq 2) = \frac{P(0 < X < 4 \cap X \geq 2)}{P(X \geq 2)} = \frac{P X=2)+P(X=3)}{1-P(X<2)}$$

A = {1,2,3}

B = {2,3,4,5,6}

A ∩ B = {2,3}

Continue…

iv. The smallest value of x for which **$P(X \leq x)$** > ½

v. The distribution function of X.

Example 8

A random variable X has the following probability distribution:

$X = x$	0	1	2	3	4	5	6
$P(x)$	k	$3k$	$2k$	$5k$	$7k$	$4k$	$8k$

a. Determine the value of k.

b. Find $P(X > 3)$, $P(X \leq 4)$ and $P(1 < X < 5)$

c. Find the distribution function of a random variable X.

Solution:

a. Since the total probability is 1, then we have

$\sum_{i=0}^{6} P(x_i) = 1$, sum of all probabilities

Or, k + 3k + 2k + 5k + 7k + 4k + 8k = 1

Or, 30k = 1

Or, k = 1 / 30

b. P(X > 3) = P(X = 4) +P(X = 5) + P(X = 6)

= 7 / 30 + 4 / 30 + 8 / 30

= 19 / 30

Continue..

$P(X \leq 4) = P(X = 0) + P(X = 1) + P(X=2) + P(X=3) + P(X=4)$

$= 1 / 30 + 3 / 30 + 2 / 30 + 5 / 30 + 7 / 30$

$= 18 / 30$

And $P(1 < X < 5) = P(X = 2) + P(X = 3) + P(X =4)$

$= 3 / 30 + 2 / 30 + 5 /30$

$= 10 / 30$

c. $F(4) = P(X \leq 4) = 18 /30$

d. The distribution function of X

X = x	P(x)	x. P(x)	$x^2.P(x)$	F(x)
0	1/30	0	0	1/30
1	3/30	3/30	3/30	4/30
2	2/30	4/30	8/30	6/30
3	5/30	15/30	45/30	11/30
4	7/30	28/30	112/30	18/30
5	4/30	20/30	100/30	22/30
6	8/30	48/30	288/30	1
		$\sum x.P(x) = 118/30$	$\sum x^2.P(x) = 556/30$	

Find E(X) and Var(X)

E(X) = $\sum x \,.\, P(x)$

= 118/30

$E(x^2) = \sum x^2 \,.\, P(x) = \frac{556}{30}$

$Var(X) = V(X) = \boldsymbol{E(x^2)} - [\boldsymbol{E(x)}]^2$

= 556/30 – (118/30)^2

= 18.53333 - 15.4711 = 3.062

Continuous Random Variable

- A random variable is said to be continuous random variable if it takes all possible values within a certain interval.
- Continuous random variable is associated with all the possible values of real line.

For examples:

1. Amount of rainfall in rainy season
2. Height of an individual
3. Temperature recorded in a particular day
4. Weight of an individual, etc.

Probability Function (Probability Density Function) of Continuous Random Variable

The function $f(x)$ is called probability density function (pdf) if it satisfies the following conditions:

i. $f(x) \geq 0$ for all x

ii. $\int_{-\infty}^{\infty} f(x)dx = 1, \quad -\infty \leq x \leq \infty$

iii. $P(E) = P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$, $P(E)$ is the probability of an event where X falls in the interval (a, b).

Continuous Probability Distribution Function

If X is a continuous random variable with pdf $f(x)$ then the function

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx \; , -\infty < x < \infty$$

is called probability distribution function or simply distribution function or cumulative probability function of a continuous random variable X.

Properties of Continuous Distribution Function

1. $0 \leq F(x_i) \leq 1, i = 0, 1, 2, \dots \dots \dots$

2. $F'(x) = f(x) \geq 0$ where $f(x)$ is a pdf.

3. $F(-\infty) = \lim_{x \to -\infty} F(x_i) = 0$

4. $F(+\infty) = \lim_{x \to +\infty} F(x_i) = 1$

5. $P(a \leq X \leq b) = F(b) - F(a)$

6. $F(x_i) < F(x_j)$, if $x_i < x_j$

Example 9

A continuous random variable X has **probability density function**

$$f(x) = kx^3\,, \qquad 0 < x < 1$$

i. Determine k

ii. Find P(X > 0.4)

iii. Find P(0.15 < X < 0.75)

iv. Find P(X ≤ 0.6) = 0.1296

Solution

Given pdf is

$$f(x) = kx^3 \,, \qquad 0 < x < 1$$

i. Since f(x) is a pdf, so

$\int_0^1 f(x)dx = 1$, total probability = 1

$\Rightarrow \int_0^1 kx^3 dx = 1$

$\int_0^1 kx^3 dx = 1$

k = 4

Therefore the pdf becomes $f(x) = 4x^3 \,,\ 0 < x < 1$

ii. P(X > 0.4) = $\int_{0.4}^{1} f(x)dx$

$$= \int_{0.4}^{1} 4x^3 dx$$

= 0.9744

iii. P(0.15 < X < 0.75) = $\int_{0.15}^{0.75} f(x)dx$

$$= \int_{0.15}^{0.75} 4x^3 dx$$

= 0.315

Example 10

A continuous random variable X is distributed at random between the range 0 to 2 has a probability function :

$$f(x) = \begin{cases} \dfrac{3}{4}x(2-x)\,; 0 < x < 2 \\ \quad 0 \qquad ; \; otherwise \end{cases}$$

i. Check whether $f(x)$ is pdf or not.

ii. Find P(X < 1)

iii. P(X > 1.5)

iv. P(0.5 < X < 1.5)

Solution:

Given probability function is

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) \; ; 0 < x < 2 \\ \quad 0 \qquad ; \; otherwise \end{cases}$$

We know that, total probability is 1, so

$\int_{-\infty}^{\infty} f(x)dx = 1$

For this, $\int_0^2 f(x)dx = \int_0^2 \frac{3}{4}x(2-x)dx$

$$= \frac{3}{4}\int_0^2 (2x - x^2)dx$$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

= 1.

Hence, the given function f(x) is pdf.

Example 11

The length of time (in minutes) that a certain lady speaks on the phone is found to be random phenomenon, with pdf

$$f(x) = \begin{cases} Ae^{-x/5} \; ; \; x \geq 0 \\ 0 \; ; otherwise \end{cases}$$

i. Find the value of A

ii. What is the probability that she will talk over the phone is:

a. More than 10 minutes. $P(X > 10) = ?$

b. Less than 5 minutes. $P(X < 5) = ?$

c. Between 5 and 10 minutes. $P(5 < X < 10) = ?$

Expectation (Mean)

Let X be a random variable, then the expectation of the random variable X is denoted by E(X) and is defined as

$$E(X) = \sum_{i=1}^{n} x_i P(x_i) \ , for\ discrete\ r.v.X$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx \ , for\ continuous\ r.v.X$$

Properties:

- $E(c) = c$, where c is any constant
- $E(aX) = a\,E(X)$, where a is any constant
- $E(aX + b) = a\,E(X) + b$, where a and b are any constants.

Example 12

A coin is tossed until a head appears. What is the expectation of number of tosses required?

Solution:

Let H denotes head and T denotes tail on tossing a coin. Let X be the number of tossed required to get first head, then the head can occur in following ways:

Outcome	No. of Toss (X = x)	P(X)	xP(x)
H	1	½	½
TH	2	¼	2/4
TTH	3	1/8	3/8
TTTH	4	1/16	4/16
:	:	:	:

Continue…

Now, $E(X) = \sum xP(x) = {}^1/_2 + {}^2/_4 + {}^3/_8 + {}^4/_{16} + \; \dots\dots\dots\dots \qquad (i)$

Let S = $\sum xP(x) = {}^1/_2 + {}^2/_4 + {}^3/_8 + {}^4/_{16} + \; \dots\dots\dots \qquad (ii)$

Also $\frac{S}{2} = {}^1/_4 + {}^2/_8 + {}^3/_{16} + {}^4/_{32} + \; \dots\dots\dots\dots \qquad (iii)$

Subtract (iii) from (ii), we get

$$S - \frac{S}{2} = \frac{1}{2} + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{3}{8} - \frac{2}{8}\right) + \left(\frac{4}{16} - \frac{3}{16}\right) + \; \dots\dots\dots\dots$$

Continue..

$$\Rightarrow \frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ \dots\dots\dots$$ (This series is geometric series)

$$\Rightarrow \frac{S}{2} = \frac{1/2}{1-1/2}$$

Here, S = *a* / (1 - *r*) since r < 1 and 'a' is the first term of series (Formula)

$$\Rightarrow \frac{S}{2} = \frac{1/2}{1/2}$$

$$\Rightarrow S = 2$$

Continue…

Now, substituting this S = 2 in equation (i) , we get

$\therefore E(X) = 2$

Hence, the expected number of tosses required until a head appears is 2

Variance

Let X be a random variable with expectation E(X). Then the variance of a random variable X is defined as:

$$Var(X) or\, V(X) = E[X - E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

Properties:

- V(a) = 0, where a is any constant
- V(aX + b) = a^2 V(X) , where a and b are any constants.
- If V(X) = 0 $\Rightarrow$ X = μ almost everywhere, where μ is the mean of the distribution

Covariance

Let X and Y be two random variables. Then the covariance between X and Y measures the simultaneous variation between X and Y, denoted by $\mathrm{Cov}(\mathrm{X},\mathrm{Y})$ and is given by

$$\mathrm{Cov}(\mathrm{X},\mathrm{Y}) = \mathrm{E}\{[\mathrm{X}-\mathrm{E}(\mathrm{X})][\mathrm{Y}-\mathrm{E}(\mathrm{Y})]\}$$

$$= E(XY) - E(X)E(Y)$$

If X and Y are independent random variable; $\mathrm{Cov}(\mathrm{X},\mathrm{Y}) = 0$

Example 13

- A random variable X has the following probability distribution:

$X = x$	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	2k	0.3	3k

Find:

i. The value of k.

ii. $P(X < 2)$ and $P(-2 < X < 2)$

iii. Cumulative distribution of X.

Continue…

iv. The expected value of X. That is E(X)

v. The variance of X. That is V(X)

Example 14

Find the mean and variance of the random variable with probability density function

$$f(x) = xe^{-x}\,; x \geq 0$$

Solution:

Let X be a random variable with pdf

$$f(x) = xe^{-x}\ ; x \geq 0$$

Now. The mean of random variable X is given by

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$= \int_{0}^{\infty} x.xe^{-x}dx$$

$$= \int_{0}^{\infty} x^2e^{-x}dx$$

Continue…

$$\boldsymbol{\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx}$$

$$= \int_0^\infty x^{3-1} e^{-x} dx$$

$$= \Gamma 3$$

$$= 2$$

$$\Gamma n = (n-1)!$$

Continue…

Also, $E(X^2) = \int_0^\infty x^2 f(x)dx$

$$= \int_0^\infty x^2 . xe^{-x} dx$$

$$= \int_0^\infty x^{4-1} e^{-x} dx$$

$$= \Gamma 4$$

$$= 3! = 6$$

Now, the variance of X is given by

$$\text{var}(\mathrm{X}) = E(X^2) - [E(X)]^2$$

$$= 6 - 4$$

$$= 2.$$

Thus, the mean and variance of random variable X with given pdf is 2.

Example 15

The probability density function of a continuous random variable X is given by

$$f(x) = Ax(1-x)^2\,; \quad 0 \leq x \leq 1$$

Find E(X) and Var(X).

Solution:

Let X be a continuous random variable with pdf

$$f(x) = Ax(1-x)^2\ ; \quad 0 \le x \le 1$$

Since f(x) is a pdf, so

$$\int_{-\infty}^{\infty} \mathrm{f(x)dx} = 1,$$

$$or, \int_{0}^{1} f(x)dx = 1$$

Continue…

$$or, \int_0^1 Ax(1-x)^2 dx = 1$$

$$or, A \int_0^1 x^{2-1}(1-x)^{3-1} dx = 1$$

$$\boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx}$$

$$or, A.\beta(2,3) = 1$$

Continue…

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$or, A.\frac{\Gamma 2 \, \Gamma 3}{\Gamma 5} = 1$$

$$or, A.1.2 = 24$$

Or, A = 12

Therefore, the given pdf becomes

$$f(x) = 12x(1-x)^2 \, ; \quad 0 \le x \le 1$$

Continue…

Now, the mean of X is given by

E(X) = $\int_0^1 xf(x)dx$

Also, E(X^2) = $\int_0^1 x^2\, f(x)dx$

Therefore, V(X) = E(X^2) - [E(X)]^2

Moments of a Random Variable

- Moments are the arithmetic mean of the different powers of the deviations of the given observations from any chosen value.
- Let X be a random variable. Let k be a positive integer and c be a constant. Then the moment of order k or the k^{th} moment of X about the point c is defined as $E[(X-c)]^k$. Where k = 1, 2, 3, 4, ………

Moments About The Origin

- Let X be a random variable. The moment of order k or the k^{th} moment of X about the origin, denoted by μ_k', is defined as $E(X^k)$. That is, $\mu_k' = E(X^k)$, k = 1, 2, 3, 4, ………
- The moments μ_k' are also called raw moments.
- If X is discrete RV with probability mass function (PMF) P(x), then

$$\mu_k' = E(X^k) = \sum_i x_i^k \, P(x_i)$$

Continue…

- If X is continuous RV with probability density function $f(x)$, then

$$\mu'_k = E\left(X^k\right) = \int_{-\infty}^{\infty} x^k f(x)dx$$

<u>**Remarks:**</u>

- $\mu'_0 = 1$
- $\mu'_1 = E(X) = \mu$, the mean of X

Moments about the Mean

- Let X be a random variable. The moment of order k or the k^{th} moment of X about the mean μ, denoted by μ_k, is defined as $E[(X-\mu)^k]$.

That is, μ_k = $E[(X-\mu)^k]$, k = 1, 2, 3, 4, ………

- The moments μ_k are also called central moments.
- If X is discrete RV with probability mass function (PMF) P(x), then

$$\mu_k = E[(X-\mu)^k] = \sum_i (x-\mu)^k \, P(x_i)$$

Continue….

- If X is continuous RV with probability density function $f(x)$, then

$$\mu_k = E[(X-\mu)^k] = \int_{-\infty}^{\infty} (x-\mu)^k f(x)dx$$

Remarks:

- $\mu_0 = 1$
- $\mu_1 = 0$
- $\mu_2 = V(X) = \sigma^2$, the variance of X

Relation between Raw moments and Central Moments

$$\mu_0 = \mu'_0 = 1$$

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - \mu'^2_1$$

$$\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$

Moment Generating Function

- The moment generating function (MGF) of a random variable X is denoted by $M_X(t)$ and is defined as

$$M_X(t) = E(e^{tx}), \qquad t \in \mathbb{R}$$

If X is discrete RV with PMF $P(x)$, then its MGF is given by

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} P(x)$$

Continue….

- If X is continuous RV with PDF $f(x)$, then its MGF is given by

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

- Let X be a random variable with MGF, $M_X(t)$. For any positive integer r, the r^{th} raw moment of X is given by

$$\mu_r' = \frac{d^r}{dt^r} M_X(t)|t = 0$$

Example 16

- Let X have the exponential density function

$$f(x) = \begin{cases} \theta e^{-\theta x}, & if\ x \geq 0 \\ 0\ , & otherwise \end{cases}$$

i. Find the MGF of X.

ii. Determine E(X) and V(X)

Solution:

Given pdf is

$$f(x) = \begin{cases} \theta e^{-\theta x}, & if\ x \geq 0 \\ 0\,, & otherwise \end{cases}$$

The MGF of X is given by

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\, f(x)dx$$

Continue…

$$M_X(t) = \int_0^\infty e^{tx}\, \theta e^{-\theta x} dx$$

$$= \theta \int_0^\infty e^{-(\theta - t)x}\, dx$$

$$= \theta \left[\frac{-e^{-(\theta - t)x}}{\theta - t} \right]_0^\infty$$

$$= \frac{\theta}{\theta - t} [-0 + 1]$$

Continue….

$$= \frac{\theta}{\theta - t}$$

$\therefore M_X(t) = \theta(\theta - t)^{-1}$ which exist for $t < \theta$

For mean and variance

We know, $\mu_r' = \frac{d^r}{dt^r} M_X(t) | t = 0$

Then, $\mu_1' = \frac{d}{dt}[\theta(\theta \ - t)^{-1}] | t = 0$

Continue…

$$\mu_1' = \theta \frac{d}{dt}[(\theta \ - t)^{-1}]|t = 0$$

$$\mu_1' = \ \theta[(\theta \ - t)^{-2}]|t = 0$$

$$\mu_1' = \theta[(\theta \ - 0)^{-2}]$$

$$\mu_1' = \theta[\theta^{-2}]$$

$$\mu_1' = \frac{1}{\theta}$$

$$\therefore mean = E(X) = \frac{1}{\theta}$$

Continue…

Also, $\mu_2' = \frac{d^2}{dt^2} M_X(t)|t = 0$

$$\mu_2' = \frac{d^2}{dt^2}[\theta(\theta \ - t)^{-1}]|t = 0$$

$$\mu_2' = \frac{d}{dt}\left\{\frac{d}{dt}[\theta(\theta \ - t)^{-1}]\right\}|t = 0$$

$$\mu_2' = \frac{d}{dt}\{\theta[(\theta \ - t)^{-2}]\}|t = 0$$

$$\mu_2' = 2\theta[(\theta \ - t)^{-3}]|t = 0$$

$$\mu_2' = 2\theta[(\theta \ - 0)^{-3}] = \frac{2}{\theta^2}$$

Continue….

Now, the variance of X is given by

$$V(X) = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$$

$$\therefore V(X) = \frac{1}{\theta^2}$$

Example 17

- Let X have the Poisson distribution with PMF

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad for\ x = 0, 1, 2, \dots \dots \dots$$

i. Find the MGF of X.

ii. Find E(X) and V(X)

Solution:

Given PMF is

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad for\; x = 0, 1, 2, \dots \dots \dots$$

Then the MGF of X is given by

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx}\, P(x)$$

$$M_X(t) = \sum_x e^{tx} \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Continue….

$$= e^{-\lambda} \sum_{x} e^{tx} \cdot \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$\therefore M_X(t) = e^{-\lambda(1-e^t)}$$

Series Expansion of e^x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots$$

We The Computer Guys

Continue…

For mean and variance

We know, $\mu_r' = \frac{d^r}{dt^r} M_X(t)|t = 0$

Then, $\mu_1' = \frac{d}{dt}\left[e^{-\lambda\left(1-e^t\right)}\right]|t = 0$

$$\mu_1' = \frac{d}{dt}\left[e^{-\lambda}e^{\lambda e^t}\right]|t = 0$$

$$\mu_1' = e^{-\lambda}.\frac{d}{dt}\left[e^{\lambda e^t}\right]|t = 0$$

$$\mu_1' = e^{-\lambda}\left[e^{\lambda e^t}.\lambda.e^t\right]|t = 0$$

Continue….

$\therefore \mu_1' = \lambda$

Therefore, the mean of X is

Mean = E(X) = λ

Also, $\mu_2' = \frac{d^2}{dt^2} M_X(t) | t = 0$

$$= \frac{d^2}{dt^2}\left[e^{-\lambda} e^{\lambda e^t}\right] | t = 0$$

Continue…

$$\mu_2' = \frac{d}{dt}\left\{\frac{d}{dt}\left[e^{-\lambda}e^{\lambda e^t}\right]\right\} |t = 0$$

$$= \frac{d}{dt}\{e^{-\lambda}[e^{\lambda e^t}.\lambda.e^t]\}|t = 0$$

$$= [e^{-\lambda}e^{\lambda e^t}.\lambda^2.e^{2t} + e^{-\lambda}e^{\lambda e^t}\lambda e^t] \,|t = 0$$

$$\therefore \mu_2' = \lambda^2 + \lambda$$

Now, the variance of X is given by

$$V(X) = \mu_2' - (\mu_1')^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

Example 18

- A perfect coin is tossed twice. Find the moment generating function of the number of heads. Hence, find the mean and variance.

Solution:

Let X be the random variable representing the number of heads in tossing a fair coin twice. Then the probability distribution of X is given by

Continue…

X = x	0	1	2
P(X = x)	1/4	1/2	1/4

Continue…

Now, the MGF of X is given by

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} P(x)$$

$$= \left[e^{t.0}.\frac{1}{4}\right] + \left[e^{t.1}.\frac{1}{2}\right] + \left[e^{t.2}.\frac{1}{4}\right]$$

$$= \frac{1}{4}[1 + 2e^t + e^{2t}]$$

$$\therefore M_X(t) = \frac{1}{4}(1 + e^t)^2$$

Characteristic Function

The MGF for all distributions does not exist, thus characteristic function is used to find the moments of the distribution since it exists for all distributions.

Let X be any random variable, then the characteristic function of X is denoted by $\phi_X(t)$ and is defined as

$$\phi_X(t) = E\left[e^{itx}\right], \qquad t \in \mathbb{R}$$

Continue….

If X is discrete RV with PMF $P(x)$, then its characteristic function is given by

$$\phi_X(t) = E\left(e^{itx}\right) = \sum_x e^{itx}\, P(x)$$

If X is continuous RV with PDF $f(x)$, then its Characteristic function is given by

$$\phi_X(t) = E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx}\, f(x) dx$$

Continue…

The raw moments of r^{th} order is given by

$$\mu_r' = \frac{1}{i^r}\left[\frac{d^r}{dt^r}\phi_X(t)\right]\lfloor t = 0$$

Chebyshev’s Inequality

Let X be a random variable with mean μ and finite variance σ^2. Then for any real number K > 0,

$$P[|X-\mu| \geq K\sigma] \leq \frac{1}{K^2}$$

Alternatively, for any real number K > 0,

$$P[|X-\mu| < K\sigma] \geq 1-\frac{1}{K^2}$$

Proof:

Let X be a continuous random variable with pdf $f(x)$. For any real number K > 0, such that $|X-\mu| \geq K\sigma$.

By definition, we have

$$\sigma^2 = E|X-\mu|^2 = \int_{-\infty}^{\infty} |x-\mu|^2 f(x)dx$$

i.e.

$$\sigma^2 = \int_{|X-\mu| \geq K\sigma} |x-\mu|^2 f(x)dx + \int_{|X-\mu| < K\sigma} |x-\mu|^2 f(x)dx$$

Continue….

$$i.e.\ \ \sigma^2 \geq \int_{|X-\mu|\geq K\sigma} |x-\mu|^2 f(x)dx$$

$$i.e.\ \ \sigma^2 \geq \int_{|X-\mu|\geq K\sigma} K^2 \sigma^2 f(x)dx$$

$$i.e. \frac{1}{K^2} \geq \int_{|X-\mu|\geq K\sigma} f(x)dx$$

Continue…

$$i.e. \frac{1}{K^2} \geq P[|X - \mu| \geq K\sigma]$$

$$\therefore P[|X - \mu| \geq K\sigma] \leq \frac{1}{K^2}$$

Equivalently, we can write, for any real number K > 0,

$$P[|X - \mu| < K\sigma] \geq 1 - \frac{1}{K^2}$$

Corollary

- Let X be a random variable with mean μ and finite variance σ^2. Then for any real number C > 0,

$$\boldsymbol{P[|X-\mu| \geq C] \leq \frac{\sigma^2}{C^2}}$$

Alternatively, for any real number C > 0,

$$P[|X-\mu| < C] \geq 1-\frac{\sigma^2}{C^2}$$

Where $\boldsymbol{K\sigma}$ **= C.**

Example

- A random variable X has a mean 10 and a variance 4, and an unknown probability distribution. Find the value of C such that

$P[|X - 10| \geq C] \leq 0.04.$

Solution:

Given that, $\mu = 10$ and $\sigma^2 = 4$. By Chebyshev's inequality, we have

$$P[|X - \mu| \geq C] \leq \frac{\sigma^2}{C^2}$$

Continue…

Then, $P[|X - 10| \geq C] \leq \frac{4}{C^2}$

Also we have,

$$P[|X - 10| \geq C] \leq 0.04$$

On comparing, we get

$\frac{4}{C^2} = 0.04$

$\Rightarrow C^2 = 100$

Thus, C = 10.

Example

- A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$. Find

i. P(- 4 < X < 20)

ii. $P(|X - 8| \geq 6)$

Solution:

i. Given that, $\mu = 8$ and $\sigma^2 = 9$. By Chebyshev's inequality, we have

$$\boldsymbol{P[|X - \mu| < C] \geq 1 - \frac{\sigma^2}{C^2}}$$

Continue…

Thus, $P(-4 < X < 20) = P[|X-8| < 12] \geq 1 - \frac{9}{12^2} = 1 - \frac{1}{16} = \frac{15}{16}$

Hence, $P(-4 < X < 20) = \frac{15}{16}$

ii. Again by Chebyshev's inequality, we have

$$P[|X-\mu| \geq C] \leq \frac{\sigma^2}{C^2}$$

Thus, $\mathrm{P}(|X-8| \geq 6) \leq \frac{9}{6^2} = \frac{1}{4}$

Hence, $\mathrm{P}(|X-8| \geq 6) \leq \frac{1}{4} = 0.25$

Example

- A random variable X has a mean 12, and variance 9. Using Chebyshev’s inequality, find

i. $P(6 < X < 18)$

ii. $P(3 < X < 21)$

Example

- Let X is a random variable with mean of 11 and variance of 9. Using Chebyshev’s inequality, find the following:

i. A lower bound for $P\ (6 < X < 16)$.

ii. The value of c such that $P\ (|X-11| \geq C) \leq 0.09$.

Example

- Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50. If the variance of a week's production is known to be 25, then what is the probability of the productivity that will be between 40 and 60?

Example

- Use Chebyshev's inequality to show that, if X is the number scored in a throw of a fair die, $P[|X - 3.5| > 2.5] < 0.47$.

Solution:

X = 1, 2, 3, 4, 5, 6

P(x) = 1/6

Mean = E(X) = $\sum x\, P(x)$ = 1 * 1/6 + 2 * 1/6 + …… + 6 * 1/6 = 7/2

- $E(X^2) = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + \ \ldots\ldots\ldots + 36 \times \frac{1}{6} =$
- V(X) = 2.956
- $P(|X - 3.5| > C) < \frac{2.956}{C^2}$
- C = 2.5
- $P(|X - 3.5| > 2.5) < \frac{2.956}{6.25} = 0.47$

Theoretical Distribution

Discrete Probability Distribution

- Binomial Distribution
- Poisson Distribution

Binomial Distribution

Conditions:

1. The number of trials (sample size), n, are finite and fixed.
2. The outcomes of the trial are dichotomous or binary (only two possible outcomes) i.e. success or failure.
3. The occurrence of an event is called success while the non – occurrence an event is called failure. The probability of success is usually denoted by p and the probability of failure is denoted by q such that $p + q = 1$.

Continue…

4. The probability of success in each trial remains constant.
5. The trials are independent. That is the outcomes of one trial does not affect the outcome of another trials.

Probability Function (PMF)

Let X be a discrete random variable and following binomial distribution with parameters n and p, then its probability mass function (PMF) is given by

$P(X = x) or\ P(x) = \binom{n}{x} p^x q^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}$; x = 0, 1, 2, …..,n

Where, x = no. of successes

p = probability of success

such that p + q = 1

or, q = 1- p, probability of failure.

$\binom{n}{x}$ = notation of combination

Properties..

1. Mean of the binomial distribution is np

i.e. $E(X) = np$

2. Variance of binomial distribution is npq

i.e. $Var(X) = npq = np(1 – p)$

3. Variance of binomial distribution is less than mean of the distribution.

Example

- Determine the binomial distribution for which the mean is 4 and variance is 3.

Solution:

Given, mean = np = 4 …… (1)

And variance = npq = 3 ……..(2)

Let $X \sim B(n, p) =?$

Continue..

Dividing equation 2 by 1, we get

$$\frac{npq}{np} = \frac{3}{4}$$

or, $q = \frac{3}{4}$

Therefore, $p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

Continue…

Substituting this value in equation 1, we get

$n * ¼ = 4$

or, $n = 16$

Hence, the required binomial distribution is $X \sim B(16, 1/4)$.

Example

For a binomial distribution with n = 4 and p = 0.45. Find

- $P(X = 0)$
- $P(X = 2)$
- $P(X \leq 2)$
- $P(X > 2)$

Solution:

Let $X \sim B(4, 0.45)$. Then

We have, n = 4 and p = 0.45

Continue…

Such that p + q = 1

i.e. q = 1- p = 1- 0.45 = 0.55

Since $P(X = x) = \binom{n}{x} p^x q^{n-x}$

Now, P(X = 0) = $\binom{4}{0}(0.45)^0(0.55)^{4-0}$ = 1 * 1 * $(0.55)^4$ = 0.092

Continue…

- $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$
- $P(X > 2) = P(X=3) + P(X = 4)$

Example

The incidence of occupational disease in an industry is such that the worker have 20% chance of suffering from it. What is the probability that out of six workers 4 or more will contract the disease?

Solution: Given, $n = 6$,

Probability of suffering from occupational disease, $p = 0.20$

Such that $q = 1- p = 0.80$

Continue…

P(X ≥ 4) = ?

Since $X \sim B(6, 0.20)$

We have, $P(X = x) = \binom{n}{x} p^x q^{n-x}$

P(X ≥ 4) = P(X=4) + P(X=5) + P(X=6)

$$= \binom{6}{4}(0.20)^4(0.80)^{6-4} + \binom{6}{5}(0.20)^5(0.80)^{6-5} + \binom{6}{6}(0.20)^6(0.80)^{6-6}$$

Example…

The bank of Kathmandu has recently starts a new credit program. Customers meeting certain credit requirements can obtained a credit card accepted by participating area merchants that carries a discount. Past numbers show that 20% of all applicants for this card are rejected. If 10 applicants are selected , what is the probability that

a. Exactly 4 will be rejected?

b. None of them are rejected?

c. At least two are rejected?

d. Less than three rejected?

Solution:

Given, n = 10

Probability of rejection for credit card, p = 0.20

Such that q = 1 – p = 0.80

Let X be the number of rejection in the application of credit card. Then, we have $X \sim B(10\,, 0.20)$

Therefore, $P(X = x) = \binom{n}{x} p^x q^{n-x}$

- P(X=4) = ?

$P(X = 4) = \binom{10}{4}(0.20)^4(0.80)^{10-4}$

Continue…

- P(X=0)?

$$P(X = 0) = \binom{10}{0}(0.20)^0(0.80)^{10-0}$$

- P(X $\geq$ 2) = 1 – P(X < 2)

  = 1 – [P(X=0) + P(X=1)]

- P(X < 3) = P(X=0) + P(X=1) + P(X=2)

Example….

The probability of a bomb hitting a target is 1/5. Two bombs are enough to destroy a bridge. If six bombs are aimed at bridge, find the probability that the bridge is destroyed.

Solution:

Given, $n = 6$, $p = 1/5$ and $q = 4/5$

$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)]$

Example

Out of 1000 families of 3 children each how many families would you expect to have two boys and one girl, assuming that boys and girls are equally likely?

Example

- If the probability of recovery from a certain disease is 0.2 and 10 people came down with the disease, what is the probability that, at most, 3 of them will recover?

Poisson Distribution:

Gamma Distribution

<u>Gamma Function (Properties)</u>

1. $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$
2. $\Gamma n = (n-1)\Gamma(n-1)$
3. $\Gamma n = (n-1)!$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
5. $\int_0^\infty e^{-\beta x} x^{\alpha-1} dx = \frac{\Gamma\alpha}{\beta^\alpha}$
6. $\Gamma 1 = 1$

Probability density function of Gamma Distribution

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \;;x > 0, \alpha\ and\ \beta > 0$$

Here, $\alpha\ and\ \beta$ are two parameters of Gamma distribution.

Sometimes, there is single parameter gamma distribution with α only (i.e. $\beta = 1$) and its pdf is given by

$$f(x) = \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x} \;;x > 0, \alpha > 0$$

Mean and Variance of Gamma Distribution with two parameters $\alpha \text{ and } \beta$

$$Mean = E(X) = \alpha\beta$$

And, $Variance = Var(X) = \alpha\beta^2$

Example:

1. Suppose that the life time of a certain kind of computer is random variable X having gamma distribution with α = 2 and β = 6. Find

a. Mean life time of computers.

b. The probability that such a computer will last more than 10 years. P(X > 10) = $\int_{10}^{\infty} f(x)dx$

<u>Solution:</u>

Let X be the life time of a certain kind of computer having gamma distribution with α = 2 and β = 6. Then, its pdf is given by

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \; ; x > 0$$

Continue…

or, $f(x) = \frac{1}{6^2\Gamma 2} x^{2-1} e^{-x/6}$

or, $f(x) = \frac{1}{36} x e^{-x/6}$ since $\Gamma n = (n-1)!$

Now,

b) The probability that such a computer will last more than 10 years is given by

$$P(X > 10) = 1 - P(X \leq 10)$$

$$= 1 - \int_0^{10} f(x)dx = 1 - \int_0^{10} \frac{1}{36} x e^{-x/6} dx$$

Integrate yourself using Product Rule.

Practice

The daily consumption of milk in Kathmandu city in excess 2,00,000 liters, is approximately distributed as gamma distribution with α = 2 and $\beta = 10^5$. The city has a daily stock of milk of 3,00,000 liters. What is the probability that the stock is insufficient on a particular day?

Solution:

Let X be the daily consumption of milk.

Let Y = X – 200000 has gamma distribution with α = 2 and $\beta = 10^5$.

Now the pdf is given by

$$f(y) = \frac{1}{\beta^\alpha \Gamma\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}} \ ; y > 0$$

$$f(y) = \frac{1}{(10^5)^2\Gamma 2} y^{2-1} e^{-\frac{y}{10^5}}$$

P(X > 300000) = P (Y+200000 > 300000) = P(Y>100000)

= 1 – P(X ≤ 100000)

= 1 - $\int_0^{100000} f(y)dy$

= 1- $\int_0^{100000} \frac{1}{(10^5)^2\Gamma 2} y^{2-1} e^{-\frac{y}{10^5}} dy$

= ?

Beta Distribution

<u>Beta Function (Properties)</u>

1. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$
2. $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

Pdf of Beta Distribution with parameter α and β.

$$f(x) = \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\ ;\ \ 0 < x < 1\ and\ \alpha,\beta > 0$$

Here, α and β are the parameters of the Beta distribution.

<u>Mean and Variance of Beta Distribution</u>

$Mean = E(X) = \frac{\alpha}{\alpha+\beta}$ and

$$Variance = Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Example

1. If the annual proportion of erroneous income tax returns filed with the Inland Revenue Department (IRD) can be looked upon as a random variable having beta distribution with $\alpha = 2 \; and \; \beta = 3.$ What is the probability that in any given year there will be fewer than 10% erroneous returns? $P(X < 0.10) = ?$

Solution :

Let X be the erroneous income tax returns having bets distribution with $\alpha = 2 \; and \; \beta = 3.$ then its pdf is

$$f(x) = \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \; 0 < x < 1$$

$$f(x) = \frac{1}{\beta(2,3)} x^{2-1}(1-x)^{3-1}$$

$$= \frac{\Gamma 5}{\Gamma 2 \Gamma 3} x(1-x)^2 \quad \text{since}, \beta(\alpha,\beta) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}$$

$$f(x) = \frac{24}{2} x\,(1-x)^2$$

P(X < 0.10) = $\int_0^{0.10} f(x)dx$

= $\int_0^{0.10} \frac{24}{2} x\,(1-x)^2 dx$

= ?

Exponential Distribution

Exponential distribution is a special case of Gamma distribution. When $\alpha = 1 \; and \; \frac{1}{\beta} = \lambda$, then the gamma distribution became

$$f(x) = \lambda e^{-\lambda x} \; ; \;\; x > 0 \; and \; \lambda > 0$$

Which is the pdf of exponential distribution with parameter λ.

Mean and Variance of exponential distribution

$mean = E(X) = \frac{1}{\lambda}$, and

$Variance = Var(X) = \frac{1}{\lambda^2}$

Example:

The life time of mechanical assembly in a vibration test is exponential distributed with a mean of 400hrs. Then

a. What is the probability that an assembly on test fails in less than 100hrs?
b. What is the probability that operates for more than 500 hrs before failure?
c. If an assembly has been on test for 400 hrs without failure, then what is the probability of a failure in the next 100 hrs?

Solution:

let X be the life time of mechanical assembly having exponential distribution with mean of 400 hrs.

Here, we have $E(X) = \frac{1}{\lambda} = 400$

$\lambda = 0.0025$

Now, the pdf is given by

$f(x) = \lambda e^{-\lambda x}$

$f(x) = 0.0025e^{-0.0025x}$

i. P(X < 100) = $\int_0^{100} f(x)dx$

$= \int_0^{100} 0.0025e^{-0.0025x}dx$

= ?

ii. P(X > 500) = $\int_{500}^{\infty} f(x)dx$

= 1 – P (X ≤ 500)

iii. Lack of memory property of exponential distribution

P (400 < X < 500/X>400) =P (X < 100) =

The time between arrivals of taxi at a busy intersection is exponentially distributed with a mean of 10 minutes.

a. What is the probability that you wait longer than one hour for a taxi?
b. Suppose you have already been waiting for one hour for a taxi, what is the probability that one arrives within the next 10 minutes?
c. Determine ‘x’ such that the probability that you wait less than ‘x’ minute is 0.10.

a. $P(X > 60) = ?$

b. $P(60 < X < 70 / X > 60) = P(X < 10) = ?$

$P(X < x) = 0.10$

Find x = ?

$P(X<x) = F(x) = 0.10$

Or, $1 - e^{-\lambda x} = 0.10$

Or, $1 - e^{-0.10x} = 0.10$

Or, $0.90 = 0.1e^{-0.10x}$

$or, e^{-0.10x} = 0.9$

X = 1.05

$$F(x) = \int_0^x f(x)dx$$

$$= \int_0^x \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]$$

$$= - \left[e^{-\lambda x}\right]_0^x$$

$$= 1 - e^{-\lambda x}$$

Bivariate Distribution

- Distribution of two – dimensional random variable is called bivariate distribution
- That is, the probability distribution involving two random variables is called bivariate distribution.
- For example, the distribution of the random variables associated with the height and weight of groups, heights of father and son for a group, etc.

Joint Probability Mass Function of (X, Y)

Let X and Y be two discrete random variables defined on the sample space S, then the probability function

$p(x_i, y_j)$ or $P(X = x, Y = y)$; $i = 1, 2, 3, \ldots\ldots$ and $j = 1, 2, 3, \ldots\ldots\ldots$

is called joint probability mass function if it satisfies the following conditions:

i. $p(x_i, y_j) \geq 0$; for all $(x_i, y_j) \in S$

ii. $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p(x_i, y_j) = 1$

Where the summations are taken over all possible values (x_i, y_j) of (X, Y)

Marginal Probability Mass Function of X and Y

The marginal probability mass function of X and Y respectively are defined by

$$p(x) = \sum_{y} p(x, y) \qquad or\ P(X = x) = \sum_{y} P(X = x, Y = y)$$

and

$$p(y) = \sum_{x} p(x, y) \qquad or\ P(Y = y) = \sum_{x} P(X = x, Y = y)$$

Conditional Probability Mass Function

- The conditional probability mass function of X given Y = y is denoted by $P(X = x/Y = y)$ or $p(x/y)$ and is defined by

$$P(X = x/Y = y) = \frac{P(X=x \cap Y=y)}{P(Y=y)} = \frac{P(X=x, Y=y)}{P(Y=y)}$$

provided that $P(Y = y) > 0$

- Similarly, the conditional probability mass function of Y given X = x is denoted by $P(Y = y/X = x)$ or $p(y/x)$ and is defined by

$$P(Y = y/X = x) = \frac{P(X=x \cap Y=y)}{P(X=x)} = \frac{P(X=x, Y=y)}{P(X=x)}$$

provided that $P(X = x) > 0$

Condition for Independence of X and Y

Two random variables X and Y are said to be independent if

$$P(X = x, Y = y) = P(X = x)\,.\,P(Y = y)$$

Or, $P(X = x/Y = y) = P(X = x)$

Joint Probability Distribution Function (Discrete)

Let P(x, y) be joint pmf of discrete random variables X and Y. The joint probability distribution function of two random variables X and Y is denoted F(x, y) and is defined as

$$F(x,y) = P(X \le x, Y \le y) = \sum_{x} \sum_{y} P(x,y)$$

Example 8

Find the marginal probability distributions of X and Y for the following joint probability distribution of two discrete random variables X and Y.

X \ Y	1	2
0	0	1/8
1	3/8	0
2	3/8	0
3	0	1/8

$P(X=0, Y=1) = 0$ $P(X=1, Y=1) = 3/8$

Solution:

The marginal probability mass function of X is given by

$P(X = x) = \sum_y P(X = x, Y = y)$,

which is the corresponding row totals in the given table of joint distribution of (X, Y). Then

$$P(X = 0) = \sum_{y=1,2} P(X = 0, Y = y)$$
$$= P(X = 0, Y = 1) + P(X = 0, Y = 2)$$
$$= 0 + \frac{1}{8} = \frac{1}{8}$$

Continue..

Similarly,

$$P(X=1)=\sum_{y=1,2} P(X=1,Y=y)$$

$$=P(X=1,Y=1)+P(X=1,Y=2)$$

$$=\frac{3}{8}+0=\frac{3}{8}$$

$$P(X=2)=\sum_{y=1,2} P(X=2,Y=y)$$

$$=P(X=2,Y=1)+P(X=2,Y=2)$$

$$=\frac{3}{8}+0=\frac{3}{8}$$

Continue..

$$P(X = 3) = \sum_{y=1,2} P(X = 3, Y = y)$$

$$= P(X = 3, Y = 1) + P(X = 3, Y = 2)$$

$$= 0 + \frac{1}{8} = \frac{1}{8}$$

Thus, the marginal probability of X is

X = x	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

Continue..

Similarly, the marginal probability mass function of Y is given by

$$P(Y = y) = \sum_{x} P(X = x, Y = y)$$

which is the corresponding column totals in the given table of joint distribution of (X, Y). Thus,

$P(Y = 1) = \sum_{x=0}^{3} P(X = x, Y = 1)$

$$= P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1) + P(X = 3, Y = 1)$$

$$= 0 + \frac{3}{8} + \frac{3}{8} + 0 = \frac{6}{8}$$

Continue…

$$P(Y = 2) = \sum_{x=0}^{3} P(X = x, Y = 2)$$

$$= P(X = 0, Y = 2) + P(X = 1, Y = 2) + P(X = 2, Y = 2) + P(X = 3, Y = 2)$$

$$= \frac{1}{8} + 0 + 0 + \frac{1}{8} = \frac{2}{8}$$

Thus, the marginal probability mass function of Y is given by

Y = y	1	2
P(y)	6/8	2/8

Example 9

Suppose X and Y have the following joint probability mass function

Y \ X	2	3
1	0.1	0.15
2	0.2	0.3
3	0.1	0.15

i. Find the marginal probability distribution of X and Y.
ii. The conditional probability of X given that Y = 2
iii. The conditional probability of Y given X = 3
iv. Find F(2, 2) and F(3, 2)
v. Are X and Y independent?

Solution

i. The marginal probability mass function of X is given by

$$P(X=x)=\sum_{y} P(X=x,Y=y)$$

which is the corresponding column totals in the given table of joint distribution of (X, Y). Thus,

$$P(X=2)=\sum_{y=1}^{3} P(X=2,Y=y)$$

$$= P(X=2,Y=1)+P(X=2,Y=2)+P(X=2,Y=3)$$

$$= 0.1+0.2+0.1=0.4$$

Continue..

$$P(X=3)=\sum_{y=1}^{3}P(X=3,Y=y)$$

$$=P(X=3,Y=1)+P(X=3,Y=2)+P(X=3,Y=3)$$

$$=0.15+0.3+0.15=0.6$$

Thus, the marginal probability mass function of X is given by

X = x	2	3
P(x)	0.4	0.6

Continue..

Similarly, the marginal probability mass function of Y is given by

$$P(Y = y) = \sum_{x} P(X = x, Y = y)$$

which is the corresponding row totals in the given table of joint distribution of (X, Y). Thus,

$$P(Y = 1) = \sum_{x=2}^{3} P(X = x, Y = 1)$$

$$= P(X = 2, Y = 1) + P(X = 3, Y = 1)$$

$$= 0.1 + 0.15 = 0.25$$

Continue..

$$P(Y = 2) = \sum_{x=2}^{3} P(X = x, Y = 2)$$

$$= P(X = 2, Y = 2) + P(X = 3, Y = 2)$$

$$= 0.2 + 0.3 = 0.5$$

$$P(Y = 3) = \sum_{x=2}^{3} P(X = x, Y = 1)$$

$$= P(X = 2, Y = 3) + P(X = 3, Y = 3)$$

$$= 0.1 + 0.15 = 0.25$$

Continue..

Thus, the marginal probability mass function of Y is given by

Y = y	1	2	3
P(y)	0.25	0.5	0.25

Continue…

ii. The conditional probability function of X given that Y = 2 is given by

$$P(X = x/Y = 2) = \frac{P(X=x,Y=2)}{P(Y=2)} \; ; x = 2, 3 \; and \; P(Y = 2) > 0$$

For x = 2,

$$P(X = 2/Y = 2) = \frac{P(X=2,Y=2)}{P(Y=2)} = \frac{0.2}{0.5} = 0.4$$

And For x = 3

$$P(X = 3/Y = 2) = \frac{P(X=3,Y=2)}{P(Y=2)} = \frac{0.3}{0.5} = 0.6$$

Continue..

iii. The conditional probability function of Y given that X = 3 is given by

$$P(Y = y/X = 3) = \frac{P(X=3, Y=y)}{P(X=3)} \; ; x = 1, 2, 3 \; and \; P(X = 3) > 0$$

For y = 1,

$$P(Y = 1/X = 3) = \frac{P(X=3, Y=1)}{P(X=3)} = \frac{0.15}{0.6} = 0.25$$

For y = 2

$$P(Y = 2/X = 3) = \frac{P(X=3, Y=2)}{P(X=3)} = \frac{0.3}{0.6} = 0.5$$

Continue..

For y = 3,

$$\mathrm{P}(\mathrm{Y}=3/\mathrm{X}=3)=\frac{\mathrm{P}(\mathrm{X}=3,\mathrm{Y}=3)}{\mathrm{P}(\mathrm{X}=3)}=\frac{0.15}{0.6}=0.25$$

iv. $\mathrm{F}(2,2)=\mathrm{P}(\mathrm{X}\leq 2,\mathrm{Y}\leq 2)=\sum_{\mathrm{x}\leq 2}\sum_{\mathrm{y}\leq 2}\mathrm{P}(\mathrm{X}=\mathrm{x},\mathrm{Y}=\mathrm{y})$

$$=\sum_{x\leq 2}[P(X=x,Y=1)+P(X=x,Y=2)]$$

$$=P(X=2,Y=1)+P(X=2,Y=2)$$

$$=0.1+0.2=0.3$$

Continue..

Similarly, F(3, 2) {You can solve this}

v. For independency,

We have, $P(X = x, Y = y) = P(X = x) \, . \, P(Y = y)$

Here, P(X = 2, Y = 1) = 0.1

Similarly, P(X = 2) = 0.4

and P(Y = 1) = 0.25

Now, P(X = 2) . P(Y = 1) = 0.4 × 0.25 = 0.1

∴ P(X = 2, Y = 1) = P(X = 2).P(Y = 1)

Hence, X and Y are independent

Continue..

Also, we can calculate $E(X)$, $E(Y)$, $Var(X)$, $Var(Y)$ and $Cov(X, Y)$

Joint Probability Density Function of (X, Y)

The probability function of (X, Y), denoted by $f(x, y)$, is said to be joint probability density function of (X, Y) if it satisfies the following properties:

i. $f(x, y) \geq 0$ for all x and y

ii. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$

For convenient simplification

$$\int_{x} \left[\int_{y} f(x, y)dy \right] dx = \int_{y} \left[\int_{x} f(x, y)dx \right] dy = 1$$

Example 10

Let (X, Y) be two random variables with the joint probability density function

$$f(x,y) = e^{-(x+y)};\ \ 0 \leq x \leq \infty \text{ and } 0 \leq y \leq \infty$$

Then, find

i. $P(X < 1 \cap Y < 1) = P(X < 1, Y < 1)$

ii. $P(X < y)$

iii. $P(X + Y < 1)$

The given joint pdf is

$$\mathrm{f(x,y)} = \mathrm{e}^{-(\mathrm{x+y})};\ \ 0 \le \mathrm{x} \le \infty \text{ and } 0 \le \mathrm{y} \le \infty$$

Here,

$$\mathrm{P(X < 1, Y < 1)} = \int_0^1 \int_0^1 f(x,y)dxdy$$

$$= \int_0^1 \int_0^1 \mathrm{e}^{-(\mathrm{x+y})} dxdy$$

$$= \int_0^1 e^{-x} \left[\int_0^1 e^{-y} dy\right] dx$$

$$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^1 dx$$

$$= (1 - e^{-1})^2$$

Marginal Probability Density Function

Let (X, Y) be two continuous random variable with joint probability density function $f(x, y)$. Then the marginal probability density function of X, denoted by $f(x)$, is given by

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

Similarly, the marginal probability density function of Y, denoted by $f(y)$, is given by

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Example 11

The joint probability density function of two random variables X and Y is given by

$$f(x, y) = e^{-(x+y)}; \ \ 0 \le x \le \infty \text{ and } 0 \le y \le \infty$$

Find the marginal probability density function of X and Y.

Given, the joint pdf of X and Y is

$$\mathrm{f(x,y) = e^{-(x+y)}};\ \ 0 \le \mathrm{x} \le \infty \text{ and } 0 \le \mathrm{y} \le \infty$$

Now, the marginal probability density function of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f(x) = \int_{0}^{\infty} \mathrm{e}^{-(\mathrm{x+y})} dy$$

$$= \int_{0}^{\infty} e^{-x}.e^{-y} dy$$

$$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = e^{-x}$$

Similarly, the marginal probability density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$f(y) = \int_{0}^{\infty} \mathrm{e}^{-(\mathrm{x+y})} dx$$

$$= \int_{0}^{\infty} e^{-x}.e^{-y} dy$$

$$= e^{-y} \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = e^{-y}$$

Conditional Probability Density Function

Let (X, Y) be two continuous random variable with joint probability density function $f(x, y)$. Then the conditional probability density function X given Y = y, denoted by $f\left({}^{x}/_{y}\right)$, is defined by

$$f\left({}^{x}/_{y}\right) = \frac{f(x, y)}{f(y)} \; ; \; provided\ that\ f(y) > 0$$

Similarly, the conditional probability density function of Y given X = x, denoted by $f\left({}^{y}/_{x}\right)$, is defined by

$$f\left({}^{y}/_{x}\right) = \frac{f(x, y)}{f(x)} \; ; \; provided\ that\ f(x) > 0$$

Example 12

If the joint probability density function of X and Y is,

$$f(x, y) = \frac{1}{8}(6 - x - y); \quad 0 < x < 2; 2 < y < 4$$

Find

i. $P(X + Y < 3)$;

ii. $\mathbf{P\left(x < \frac{3}{2} / y < \frac{5}{2}\right)}$

Given, the joint pdf of X and Y is

$$f(x,y) = \frac{1}{8}(6 - x - y); \quad 0 < x < 2; 2 < y < 4$$

1. $P(X + Y < 3) = \int_0^1 \int_2^{3-x} f(x,y)dxdy$

$$= \int_0^1 \int_2^{3-x} \frac{1}{8}(6 - x - y)dxdy$$

$$= \int_0^1 \frac{1}{8}\left[\int_2^{3-x}(6 - x - y)dy\right]dx$$

$$= \int_0^1 \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx = 5 / 24$$

2. $P\left(X<\frac{3}{2} \Big/ Y<\frac{5}{2}\right) = \frac{P\left(X<\frac{3}{2}, Y<\frac{5}{2}\right)}{P\left(Y<\frac{5}{2}\right)}$

Here, $P\left(X < \frac{3}{2}, Y < \frac{5}{2}\right) = \int_0^{\frac{3}{2}} \int_2^{\frac{5}{2}} f(x, y)\,dxdy$

$$= \int_0^{\frac{3}{2}} \int_2^{\frac{5}{2}} \frac{1}{8}(6 - x - y)\,dxdy$$

= 9/32

$$P\left(Y < \frac{5}{2}\right) = \int_{-\infty}^{\frac{5}{2}} f(y)dy = \int_{2}^{\frac{5}{2}} f(y)dy$$

Here, $f(y) = \int_0^2 f(x,y)dx$

$$= \int_0^2 \frac{1}{18}(6 - x - y)dx$$

$$= \frac{1}{18}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{18}(12 - 2 - 2y)$$

$$= \frac{5}{9} - \frac{y}{9} = \frac{1}{9}(5 - y)$$

Now, $P\left(Y < \frac{5}{2}\right) = \int_{-\infty}^{\frac{5}{2}} f(y)dy = \int_{2}^{\frac{5}{2}} f(y)dy$

$$= \int_{2}^{\frac{5}{2}} \frac{1}{9}(5 - y)dy$$

$$= \frac{1}{9}\left[5y - \frac{y^2}{2}\right]_{2}^{\frac{5}{2}}$$

= 11/32

$$\text{Hence } P\left(X<\tfrac{3}{2} \middle/ Y<\tfrac{5}{2}\right) = \frac{P\left(X<\frac{3}{2}, Y<\frac{5}{2}\right)}{P\left(Y<\frac{5}{2}\right)} = \frac{\frac{9}{32}}{\frac{11}{32}} = \frac{9}{11}$$

Example 13

Two random variables X and Y have the following probability density function

$$f(x, y) = k(4 - x - y);\ \ 0 < x < 2\ \ and\ 0 < y < 2$$

i. Find the conditional probability density function of X given Y = y.

ii. Find the conditional probability density function of Y given X = x.

Solution:

Given joint pdf is

$$f(x, y) = k(4 - x - y); \;\; 0 < x < 2 \;\; and \; 0 < y < 2$$

Since, $f(x, y)$ is a joint pdf

Then, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$

or, $\int_0^2 \int_0^2 k(4 - x - y)\, dxdy = 1$

or, $\int_0^2 k \left[4y - xy - \frac{y^2}{2}\right]_0^2 dx = 1$

or, $\int_0^2 k(6 - 2x) dx = 1$

$\therefore k = \frac{1}{8}$

Therefore, the given joint pdf becomes

$$f(x,y) = \frac{1}{8}(4 - x - y); \;\; 0 < x < 2 \;\; and \; 0 < y < 2$$

i. The marginal probability density function of X is given by

$$f(x) = \int_0^2 f(x,y)\,dy$$

$$= \int_0^2 \frac{1}{8}(4 - x - y)\,dy$$

$$= \left[\frac{1}{8}(4y - xy - \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{8}(8 - 2x - 2) = \frac{3-x}{4}$$

Similarly, the marginal probability density function of Y is given by

$$f(y) = \int_0^2 f(x, y)\, dx$$

$$= \int_0^2 \frac{1}{8}(4 - x - y)\, dx$$

$$= \left[\frac{1}{8}(4x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}(8 - 2 - 2y) = \frac{3-y}{4}$$

i. The conditional probability density function of X given by Y = y is given by,

$$f\left({}^{x}/_{y}\right) = \frac{f(x,y)}{f(y)}$$

$$= \frac{\frac{1}{8}(4-x-y)}{\frac{3-y}{4}}$$

$$= \frac{4-x-y}{2(3-y)}$$

Continue..

ii. The conditional probability density function of Y given X = x is obtained as

$$f\left({}^{y}/_{x}\right) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{1}{8}(4-x-y)}{\frac{3-x}{4}}$$

$$= \frac{4-x-y}{2(3-x)}$$

Solution:

Here, the joint probability density function of X and Y is given by

$$f(x,y) = k(4 - x - y);\;\; 0 < x < 2 \;\; and \; 0 < y < 2$$

Since, $f(x,y)$ is a joint pdf, then we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

or, $\int_0^2\int_0^2 k(4 - x - y)dxdy = 1$

or, $\int_0^2\left[\int_0^2 k(4 - x - y)dy\right]dx = 1$

or, $\int_0^2 k\left[4y - xy - \frac{y^2}{2}\right]_0^2 dx = 1$

or, $\int_0^2 k(6-2x)dx = 1$

K =1/8

Therefore, the given joint pdf becomes

$$f(x,y) = \frac{1}{8}(4-x-y);\ \ 0 < x < 2\ \ and\ 0 < y < 2$$

Now, the marginal probability density function of X is given by

$f(x) = \int_y\ f(x,y)dy$

$= \int_0^2 \frac{1}{8}(4-x-y)dy$

$$= \frac{1}{8}\left[4y - xy - \frac{y^2}{2}\right]_0^2$$

$$= \frac{(3-x)}{4}$$

Similarly, the marginal probability density function of Y is given by

$$f(y) = \int_x f(x,y)dx$$

$$= \int_0^2 \frac{1}{8}(4 - x - y)dx$$

$$= \frac{3-y}{4}$$

i. The conditional probability of X given Y = y is given by

$$f({}^{x}/_{y}) = \frac{f(x,y)}{f(y)}$$

$$= \frac{\frac{1}{8}(4-x-y)}{\frac{3-y}{4}}$$

$$= \frac{(4-x-y)}{2(3-y)}$$

ii. The conditional probability of Y given X = x is given by

$$f({}^{y}/_{x}) = \frac{f(x,y)}{f(x)}$$

Joint Probability Distribution Function

If $f(x, y)$ is joint probability density function of continuous random variables X and Y, then the joint probability distribution function, denoted by F(x, y), is defined as:

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) dx dy$$

Condition for Independence of X and Y

Two continuous random variables X and Y is said to be independent if

$$f(x, y) = f(x). f(y)$$

Where $f(x, y)$ is the joint probability density function of X and Y

$f(x)$ is the marginal probability density function of X

and $f(y)$ is the marginal probability function of Y

Marginal Probability Distribution Function

Two continuous random variables X and Y has joint probability density function $f(x, y)$, then the Joint probability distribution of X and Y is denoted by F(x, y). Therefore the marginal probability distribution function of X is given by

$$F(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x.y) dx\, dy$$

Similarly, the marginal probability distribution function of Y is given by

$$F(y) = P(Y \leq) = P(X \leq \infty, Y \leq y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x.y) dx\, dy$$

Example 14

Suppose that the joint probability function of random variables X and Y is given by

$$f(x, y) = 4xy \quad ; \quad 0 < x < 1 \ \ and \ \ 0 < y < 1$$

a. Verify the given joint probability function is a joint pdf.
b. Find P(X < 0.5, Y < 0.8)
c. Find the marginal probability distribution of X and Y.
d. Find the conditional distribution of X given Y
e. Is X and Y are independent?
f. Find E(X), E(Y), V(X), V(Y) and Cov (X, Y)

Solution:

Given joint probability function of X and Y is

$$f(x,y) = 4xy \quad ; \quad 0 < x < 1 \ \ and \ \ 0 < y < 1$$

i. $f(x,y)$ is said to be Joint pdf if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

So,

$$\int_{0}^{1}\int_{0}^{1} 4xydxdy = \int_{0}^{1}\left[\frac{4xy^2}{2}\right]_0^1 dx$$

$$= \int_{0}^{1} 2xdx = \left[\frac{2x^2}{2}\right]_0^1 = 1$$

Hence, the given joint probability function f(x,y) is a joint pdf.

ii. $P(X < 0.5, Y < 0.8) = \int_0^{0.5}\int_0^{0.8} f(x,y)dxdy$

$$= \int_0^{0.5}\int_0^{0.8} 4xy\,dxdy$$

$$= 0.16$$

iii. The marginal probability density function of X is given by

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 4xy\,dy = 2x$$

Similarly, the marginal probability density function of Y is given by

$$f(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 4xy\,dx = 2y$$

iv. The conditional probability density function of X given Y is obtained as

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$f(x/y) = \frac{4xy}{2y} = 2x$$

v. Here, Joint pdf of X and Y is

$f(x, y) = 4xy$

Also, the marginal pdf of X is

$f(x) = 2x$

And the marginal pdf of Y is

$f(y) = 2y$

Now, $f(x).f(y) = 2x\,.\,2y = 4xy = f(x, y)$

Hence, X and Y are independent.

vi. E(X) = $\int_0^1 x \,.\, f(x)dx$

$$= \int_0^1 x.\,2x\,dx$$

$$= \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

Here, $E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2.\,2x\ \ dx$

$$= \left[\frac{2x^4}{4}\right]_0^1 = \frac{2}{4}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{4} - \frac{4}{9} = \frac{2}{36}$$

Example 15

The joint probability density function of two random variables is given by

$$f(x, y) = Axy\,; \quad 0 < x < 1, and\ \ 0 < y < x$$

i. Find the value of A
ii. Find the marginal density function of X and Y
iii. Find the conditional density function of Y given X = x and conditional density function of X given Y = y.
iv. Check for independence of X and Y

Solution:

Given joint pdf is

$$f(x, y) = Axy\,; \quad 0 < x < 1, and\;\; 0 < y < x$$

i. Since $f(x, y)$ is joint pdf, so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dxdy = 1$$

or, $\int_0^1 \int_0^x Axy\, dxdy = 1$

$$or, \int_0^1 A\, x \left[\frac{y^2}{2}\right]_0^x dx = 1$$

$$or, \int_0^1 A\, x. \frac{x^2}{2} dx = 1$$

$$or, \frac{A}{2}\int_0^1 x^3\, dx = 1$$

$$or, \frac{A}{2}\left[\frac{x^4}{4}\right]_0^1 = 1$$

$$\therefore A = 8$$

Therefore, the given joint pdf becomes

$$f(x, y) = 8xy\,; \quad 0 < x < 1, and\;\; 0 < y < x$$

ii. The marginal probability density function of X is given by

$$f(x) = \int_0^x f(x,y)dy$$
$$= \int_0^x 8xydy$$
$$= 8x\left[\frac{y^2}{2}\right]_0^x$$
$$= 4x^3$$

Similarly, The marginal probability density function of Y is given by

$$f(y) = \int_y^1 f(x,y)dx$$
$$= \int_y^1 8xydx$$
$$= 8y\left[\frac{x^2}{2}\right]_y^1$$
$$= 4y(1-y^2)$$

iii. The conditional probability density function of Y given X = x is obtain as

$$f({}^{y}/_{x}) = \frac{f(x,y)}{f(x)}$$

$$= \frac{8xy}{4x^3}$$

$$= \frac{2y}{x^2}$$

Similarly, the conditional probability density function of X given Y = y is obtain as:

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{8xy}{4y(1-y^2)}$$

$$= \frac{2x}{(1-y^2)}$$

iv. Condition for independence of X and Y is

$$f(x,y) = f(x).f(y)$$

Now, f(x).f(y) = $4x^3.\ 4y(1-y^2) = 16\ x^3\ y(1-y^2) \neq f(x,y)$

Hence, X and Y are not independent.

Example 16

Joint probability distribution of random variables X and Y is given by

$$f(x,y) = k(6 - x - y); \;\; 0 < x < 2\,, and \;\; 2 < y < 4$$

Find:

i. The constant k
ii. Marginal density function of X and Y
iii. E(X) and V(X)
iv. Are X and Y independent?

Example 17

If two random variables have the joint probability density function

$$f(x, y) = \frac{2}{3}(x + 2y)\ ; \;\; 0 < x < 1, and\ 0 < y < 1$$

i. Find marginal density function of X and Y
ii. Find the conditional density of X given that Y = y
iii. Are X and Y independent?

Example 18

Suppose that the random variables X and Y have joint pdf

$$f(x, y) = \begin{cases} kx(x - y); & 0 < x < 2\,, and \quad -x < y < x \\ 0 & ; \quad otherwise \end{cases}$$

i. Find the value of constant k
ii. Find the marginal probability density function of X.
iii. Find the conditional probability distribution of Y given X = x