Graph Sketching

Sketch the following graph:

$$y^2 = x^2 \pm 1$$

(Note: distinguish between the positive 1 and negative 1 curves).

Give the equation of its asymptotes.

Give the equation of a curve that touches this graph at 4 different points.

Sketch the following graph ($\alpha, \beta \in \mathbb{R}$) in the cases where $\beta = 0$ and $\beta \neq 0$ and give equations for the corresponding graphs in the previous part for the case where $\beta \neq 0$.

$$(\alpha - y - \beta x)(y - \alpha - \beta x) \pm 1 \ = 0$$

Algebra

For these following questions, use of Calculus is NOT permitted.

We define the symbol ε such that $\varepsilon^2 = 0, \varepsilon \neq 0$. A *dual number* is of the form $x + y\varepsilon\,; x, y \in \mathbb{R}$.

(i) Simplify $(a + b\varepsilon)(c + d\varepsilon)$ and $\frac{a+b\varepsilon}{c+d\varepsilon}$.

$$P(x) = \sum_{n=0}^{\infty} p_n x^n$$

(ii) By considering the dual number $a + b\varepsilon$ find an expression for $P'(a)$ in terms of P, a and ε, where $P'(a)$ represents the derivative of $P(x)$ evaluated at $x = a$.

Recall that the definition of the Taylor series of an analytic function f about a is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(iii) Find an expression for the derivative f' of any analytic function f evaluated at a in terms of f, a and ε.

(iv) Find an expression for $\left.\left(\frac{\mathrm{d}g}{\mathrm{d}f} \times \frac{\mathrm{d}f}{\mathrm{d}x}\right)\right|_{x=a}$ in terms of f, g, a and ε.

Vectors

(i) Draw a diagram showing 2D position vectors $\vec{a}, \vec{b}$ and $\frac{(\vec{a} \cdot \vec{b})\vec{b}}{|\vec{b}|^2}$.

(ii) Use the Cauchy Schwarz inequality $|\vec{u}||\vec{v}| \geq |\vec{u} \cdot \vec{v}|$ to prove the triangle inequality $|\vec{u}| + |\vec{v}| \geq |\vec{u} + \vec{v}|$.

(iii) Identify the following 3D surfaces, where $\hat{u}$ is a fixed vector of unit length:

$$|\vec{r}| = \rho$$
$$\vec{r} \cdot \hat{u} = l$$
$$\vec{r} \cdot \hat{u} = m|\vec{r}|$$
$$|\vec{r} - (\vec{r} \cdot \hat{u})\hat{u}| = n$$

(iv) Solve the vector equation $\vec{a} \times \vec{b} + \lambda \vec{r} = \vec{c}$ for $\vec{r}$ given that $\lambda \neq 0$.

Vectors 2 (FM Required)

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero vectors in $\mathbb{R}^3$.

(a) State the geometric significance of the equation $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Assume that $\vec{a} \cdot (\vec{b} \times \vec{c}) > 0$.

(b) Prove that if $\vec{r}$ lies on the plane containing $\vec{a}$, $\vec{b}$ and $\vec{c}$, then:

$$\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

(c) Find necessary and sufficient conditions for $\vec{r}$ to lie in or on the trapezoid formed by the origin, and vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Calculus

Recall the definition of the derivative $\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

You may assume the mean value theorem; that for any continuous function f, there exists a value $c \in [a, b]$ such that

$$\int_{a}^{b} f(t)dt = f(c)(b-a)$$

(i) Let $F(x) = \int_0^x f(t)dt$, where f is continuous. Find $F'(x)$. [Hint: consider the bound on c].

(ii) Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation:

$$f(x) = \lambda(1+x^2)\left[1 + \int_0^x \frac{f(t)}{1+t^2} dt\right]$$

Sets

Sketch the curve $x^2 + y^2 = 65$ and give an integer point on this curve.
Write down the equation of a line that passes through this point in the form $y = m(x - x_0) + c$.

Find a quadratic in x in terms of m.
By factorizing your quadratic, find the set of all rational points on the curve.