Section A: Pure Mathematics

1 **(i)** Use the substitution $x = \dfrac{1}{1-u}$, where $0 < u < 1$, to find in terms of x the integral

$$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x \qquad \text{(where } x > 1\text{)}.$$

(ii) Find in terms of x the integral

$$\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x \qquad \text{(where } x > 2\text{)}.$$

(iii) Show that

$$\int_2^\infty \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \frac{1}{3}\pi$$

2 The curves C_1 and C_2 both satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kxy - y}{x - kxy}$$

where $k = \ln 2$.

All points on C_1 have positive x and y co-ordinates and C_1 passes through $(1, 1)$. All points on C_2 have negative x and y co-ordinates and C_2 passes through (-1,-1)

(i) Show that the equation of C_1 can be written as $(x-y)^2 = (x+y)^2 - 2^{x+y}$.

Determine a similar result for curve C_2.

Hence show that $y = x$ is a line of symmetry of each curve.

(ii) Sketch on the same axes the curves $y = x^2$ and $y = 2^x$, for $x \geqslant 0$. Hence show that C_1 lies between the lines $x + y = 2$ and $x + y = 4$.

Sketch curve C_1.

(iii) Sketch curve C_2.

3 A sequence $u_1, u_2, \ldots, u_n$ of positive real numbers is said to be unimodal if there is a value k such that

$$u_1 \leqslant u_2 \leqslant \ldots \leqslant u_k$$

and

$$u_k \geqslant u_{k+1} \geqslant \ldots \geqslant u_n$$

So the sequences $1, 2, 3, 2, 1$; $1, 2, 3, 4, 5$; $1, 1, 3, 3, 2$ and $2, 2, 2, 2, 2$ are all unimodal, but $1, 2, 1, 3, 1$ is not.

A sequence $u_1, u_2, \ldots, u_n$ of positive real numbers is said to have property L if $u_{r-1}u_{r+1} \leqslant u_r^2$ for all r with $2 \leqslant r \leqslant n-1$.

(i) Show that, in any sequence of positive real numbers with property L

$$u_{r-1} \geqslant u_r \Longrightarrow u_r \geqslant u_{r+1}.$$

Prove that any sequence of positive real numbers with property L is unimodal.

(ii) A sequence $u_1, u_2, \ldots, u_n$ of real numbers satisfies $u_r = 2\alpha u_{r-1} - \alpha^2 u_{r-2}$ for $3 \leqslant r \leqslant n$ where α is a positive real constant. Prove that, for $2 \leqslant r \leqslant n$

$$u_r - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$$

and, for $2 \leqslant r \leqslant n-1$,

$$u_r^2 - u_{r-1}u_{r+1} = (u_r - \alpha u_{r-1})^2.$$

Hence show that the sequence consists of positive terms and is unimodal, provided $u_2 > \alpha u_1 > 0$

In the case $u_1 = 1$ and $u_2 = 2$, prove by induction that $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$.

Let $\alpha = 1 - \dfrac{1}{N}$, where N is an integer with $2 \leqslant N \leqslant n$.

In the case $u_1 = 1$ and $u_2 = 2$, prove that u_r is largest when $r = N$.

4 **(i)** Given that a, b and c are the lengths of the sides of a triangle, explain why $c < a + b$ $a < b + c$ and $b < a + c$.

(ii) Use a diagram to show that the converse of the result in part (i) also holds: if a, b and c are positive numbers such that $c < a + b, a < b + c$ and $b < c + a$ then it is possible to construct a triangle with sides of length a, b and c.

(iii) When a, b and c are the lengths of the sides of a triangle, determine in each case whether the following sets of three lengths can

- always
- sometimes but not always
- never

form the sides of a triangle. Prove your claims.

(a) $a + 1,\ b + 1,\ c + 1.$

(b) $\dfrac{a}{b},\ \dfrac{b}{c},\ \dfrac{c}{a}.$

(c) $|a - b|,\ |b - c|,\ |c - a|.$

(d) $a^2 + bc,\ b^2 + ca,\ c^2 + ab.$

(iv) Let f be a function defined on the positive real numbers and such that, whenever $x > y > 0$,

$$f(x) > f(y) > 0 \quad \text{but} \quad \frac{f(x)}{x} < \frac{f(y)}{y}.$$

Show that, whenever a, b and c are the lengths of the sides of a triangle, then $f(a), f(b)$ and $f(c)$ can also be the lengths of the sides of a triangle.

5 If x is a positive integer, the value of the function $\mathrm{d}(x)$ is the sum of the digits of x in base 10. For example, $\mathrm{d}(249) = 2 + 4 + 9 = 15$.

An n-digit positive integer x is written in the form $\displaystyle\sum_{r=0}^{n-1} a_r \times 10^r$, where $0 \leqslant a_r \leqslant 9$ for all $0 \leqslant r \leqslant n - 1$ and $a_{n-1} > 0$.

(i) Prove that $x - \mathrm{d}(x)$ is non-negative and divisible by 9.

(ii) Prove that $x - 44\,\mathrm{d}(x)$ is a multiple of 9 if and only if x is a multiple of 9.

Suppose that $x = 44\,\mathrm{d}(x)$. Show that if x has n digits, then $x \leqslant 396n$ and $x \geqslant 10^{n-1}$ and hence that $n \leqslant 4$.

Find a value of x for which $x = 44\,\mathrm{d}(x)$. Show that there are no further values of x satisfying this equation.

(iii) Find a value of x for which $x = 107\mathrm{d}(\mathrm{d}(x))$. Show that there are no further values of x satisfying this equation.

6 A 2×2 matrix $\mathbf{M}$ is real if it can be written as $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real. In this case, the *trace* of matrix $\mathbf{M}$ is defined to be $\operatorname{tr}(\mathbf{M}) = a + d$ and $\det(\mathbf{M})$ is the determinant of matrix $\mathbf{M}$. In this question, $\mathbf{M}$ is a real 2×2 matrix.

(i) Prove that
$$\operatorname{tr}\left(\mathbf{M}^2\right) = \operatorname{tr}(\mathbf{M})^2 - 2\det(\mathbf{M}).$$

(ii) Prove that
$$\mathrm{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm\mathbf{I} \iff \operatorname{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1.$$

and that
$$\mathbf{M}^2 = -\mathbf{I} \iff \operatorname{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part (ii) to prove that
$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm\mathbf{I}$$

Find a necessary and sufficient condition on $\det(\mathbf{M})$ and $\operatorname{tr}(\mathbf{M})$ so that $\mathbf{M}^4 = -\mathbf{I}$.

(iv) Give an example of a matrix M for which $\mathrm{M}^8 = \mathrm{I}$, but which does not represent a rotation or reflection. [Note that the matrices $\pm$ I are both rotations.]

7 In this question, $w = \dfrac{2}{z-2}$.

(i) Let z be the complex number $3 + t\,\mathrm{i}$, where $t \in \mathbb{R}$. Show that $|w - 1|$ is independent of t. Hence show that, if z is a complex number on the line $\operatorname{Re}(z) = 3$ in the Argand diagram, then w lies on a circle in the Argand diagram with centre 1.

Let V be the line $\operatorname{Re}(z) = p$, where p is a real constant not equal to 2. Show that, if z lies on V, then w lies on a circle whose centre and radius you should give in terms of p. For which z on V is $\operatorname{Im}(w) > 0$?

(ii) Let H be the line $\operatorname{Im}(z) = q$, where q is a non-zero real constant. Show that, if z lies on H, then w lies on a circle whose centre and radius you should give in terms of q For which z on H is $\operatorname{Re}(w) > 0$?

8 In this question, $f(x)$ is a quartic polynomial where the coefficient of x^4 is equal to 1, and which has four real roots, $0, a, b$ and c, where $0 < a < b < c$.

$F(x)$ is defined by $F(x) = \displaystyle\int_0^x f(t)\ \mathrm{d}t$.

The area enclosed by the curve $y = f(x)$ and the x-axis between 0 and a is equal to that between b and c, and half that between a and b.

(i) Sketch the curve $y = F(x)$, showing the x co-ordinates of its turning points.

Explain why $F(x)$ must have the form $F(x) = \frac{1}{5}x^2(x-c)^2(x-h)$, where $0 < h < c$.

Find, in factorised form, an expression for $F(x) + F(c-x)$ in terms of c, h and x.

(ii) If $0 \leqslant x \leqslant c$, explain why $F(b) + F(x) \geqslant 0$ and why $F(b) + F(x) > 0$ if $x \neq a$ Hence show that $c - b = a$ or $c > 2h$.

By considering also $F(a) + F(x)$, show that $c = a + b$ and that $c = 2h$

(iii) Find an expression for $f(x)$ in terms of c and x only.

Show that the points of inflection on $y = f(x)$ lie on the x-axis.

Section B: Mechanics

9 Point A is a distance h above ground level and point N is directly below A at ground level. Point B is also at ground level, a distance d horizontally from N. The angle of elevation of A from B is β. A particle is projected horizontally from A, with initial speed V. A second particle is projected from B with speed U at an acute angle θ above the horizontal. The horizontal components of the velocities of the two particles are in opposite directions. The two particles are projected simultaneously, in the vertical plane through A, N and B.

Given that the two particles collide, show that

$$d\sin\theta - h\cos\theta = \frac{Vh}{U}$$

(i) $\theta > \beta$;

(ii) $U\sin\theta \geq \sqrt{\dfrac{gh}{2}}$;

(iii) $\dfrac{U}{V} > \sin\beta$.

Show that the particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

10 A particle P of mass m moves freely and without friction on a wire circle of radius a, whose axis is horizontal. The highest point of the circle is H, the lowest point of the circle is L and angle $PHL = \theta$. A light spring of modulus of elasticity λ is attached to P and to H natural length of the spring is l, which is less than the diameter of the circle.

(i) Show that, if there is an cquilibrium position of the particle at $\theta = \alpha$, where $\alpha > 0$ then $\cos\alpha = \dfrac{\lambda l}{2(a\lambda - mgl)}$.

Show also that there will only be such an equilibrium position if $\lambda > \dfrac{2mgl}{2a - l}$.

When the particle is at the lowest point L of the circular wire, it has speed u.

(ii) Show that, if the particle comes to rest before reaching H, it does so when $\theta = \beta$ where $\cos\beta$ satisfies

$$(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$$

where $\cos\alpha = \dfrac{\lambda l}{2(a\lambda - mgl)}$

Show also that this will only occur if $u^2 < \dfrac{2a\lambda}{m}(2 - \sec\alpha)$.

Section C: Probability and Statistics

11 A coin is tossed repeatedly. The probability that a head appears is p and the probability that a tail appears is $q = 1 - p$.

(i) A and B play a game. The game ends if two successive heads appear, in which case A wins, or if two successive tails appear, in which case B wins.

Show that the probability that the game never ends is 0.

Given that the first toss is a head, show that the probability that A wins is $\dfrac{p}{1-pq}$.

Find and simplify an expression for the probability that A wins.

(ii) A and B play another game. The game ends if three successive heads appear, in which case A wins, or if three successive tails appear, in which case A wins. Show that

$$\mathrm{P}(\text{A wins} \mid \text{the first toss is a head}) = p^2 + (q + pq)\mathrm{P}(\text{A wins} \mid \text{the first toss is a tail})$$

and give a similar result for P(A wins |the first toss is a tail).

Show that

$$P(\text{A wins}) = \frac{p^2\left(1-q^3\right)}{1-\left(1-p^2\right)\left(1-q^2\right)}$$

(iii) A and B play a third game. The game ends if a successive heads appear, in which case A wins, or if b successive tails appear, in which case B wins, where a and b are integers greater than 1 Find the probability that A wins this game. Verify that your result agrees with part (i) when $a = b = 2$

12 The score shown on a biased n -sided die is represented by the random variable X which has distribution $\mathrm{P}(X = i) = \dfrac{1}{n} + \varepsilon_i$ for $i = 1, 2, \ldots, n$, where not all the ε_i are equal to 0.

(i) Find the probability that, when the die is rolled twice, the same score is shown on both rolls. Hence determine whether it is more likely for a fair die or a biased die to show the same score on two successive rolls.

(ii) Use part (i) to prove that, for any set of n positive numbers $x_i (i = 1, 2, \ldots, n)$

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1} x_i x_j \leqslant \frac{n-1}{2n}\left(\sum_{i=1}^{n} x_i\right)^2.$$

(iii) Determine, with justification, whether it is more likely for a fair die or a biased die to show the same score on three successive rolls.