TMUA Proof

Syllabus

Logic of arguments (true/false, and/or, if, then, not) converse, contrapositive, necessary sufficient; mathematical proof (deduction, proof by cases, proof by contradiction, disproof by counterexample); identifying errors in proofs.

Logic of Arguments

1. Consider the following statement: "If it is my birthday, I will eat some cake"

What conclusion can I draw from each of the following statements:

A It is my birthday

B It is not my birthday

C I eat some cake

D I do not eat some cake

2. Consider the following statement: "If it rains the ground will get wet"

What conclusion can I draw from each of the following statements:

A The ground is wet

B The ground is not wet

C It is raining

D It is not raining

3. Consider the following statement: "If I am in Paris, then I am in France"

What conclusion can I draw from each of the following statements:

A I am in Paris

B I am in France

C I am in London

D I am at the Eiffel Tower

4. Consider the following statement: "If a shape is a square, then it is a quadrilateral"

What conclusion can I draw from each of the following statements:

A The shape is a square

B The shape is a quadrilateral

C The shape is not a quadrilateral

D The shape is a rhombus

The following statements are all equivalent:

If an animal is a zebra, **then** it has stripes

All zebras have stripes

Any zebra has stripes

Being a zebra **implies** having stripes

An animal is a zebra **only if** it has stripes

An animal has stripes **if** it is a zebra

An animal with **no** stripes is **not** a zebra

Having stripes is **necessary** for an animal to be a zebra

Being a zebra is a **sufficient** condition for an animal to have stripes

Equivalent Quantifiers - for all A/ every A / any A / if A

- for some A / there exists A / for at least one A / for most A

5. Rewrite the following true statements in the form **If… Then …**

a) The ground gets wet when it rains

b) All mammals have hair

c) I always go to bed when I am sick

d) A fruit is yellow if it is a banana

e) I am in Paris only if I am in France

6. Rewrite the following true mathematical statements in the form **If… Then …**

a) Any rectangle is a quadrilateral

b) Any triangle has 3 sides

c) The number 2 is the only even prime number

d) $x > 10$ if $x > 100$

e) $k < 1$ when $k^2 < 1$

f) $p^2 < p$ only if $p < 1$

TMUA Style Multiple Choice Practice 'if…then'

1. a and b are real numbers and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $a < b$ **then** $f(a) < f(b)$

B $a < b$ **only if** $f(a) < f(b)$

C $f(a) < f(b)$ is **sufficient** for $a < b$

D $f(a) < f(b)$ is **necessary** for $a < b$

2. Consider the four options below about a particular statement:

A The statement is true if $x^2 < 1$

B The statement is true if and only if $x^2 < 1$

C The statement is true if $x^2 < 4$

D The statement is true if and only if $x^2 < 4$

Given that exactly one of these options is correct, which one is it?

3. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $a > 0$ **then** $f(a) > 0$

B $a > 0$ **only if** $f(a) > 0$

C $a > 0$ is **sufficient** for $f(a) > 0$

D $a > 0$ is **necessary** for $f(a) > 0$

4. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $f(a) > 0$ **then** $|a| < 1$

B $f(a) > 0$ **if** $|a| < 1$

C $|a| < 1$ **only if** $f(a) > 0$

D $|a| < 1$ is **sufficient** for $f(a) > 0$

The following statements are all equivalent

If $a < b$ **then** $f(a) < f(b)$	If P then Q
$a < b$ **implies that** $f(a) < f(b)$	P implies Q
$a < b$ **only if** $f(a) < f(b)$	P only if Q
$a < b$ **is sufficient for** $f(a) < f(b)$	P is sufficient for Q
$f(a) < f(b)$ **if** $a < b$	Q if P
$f(a) < f(b)$ **is necessary for** $a < b$	Q is necessary for P
If $f(a) \geq f(b)$ **then** $a \geq b$	If 'not Q' then 'not P' (*contrapositive*)
$f(a) \geq f(b)$ **implies** $a \geq b$	'Not Q' implies 'not P'
$f(a) \geq f(b)$ **only if** $a \geq b$	'Not Q' only if 'not P'
$f(a) \geq f(b)$ **is sufficient for** $a \geq b$	'Not Q' is sufficient for 'not P'
$a \geq b$ **if** $f(a) \geq f(b)$	'Not P' if 'not Q'
$a \geq b$ **is necessary for** $f(a) \geq f(b)$	'Not P' is necessary for 'not Q'

The ***contrapositive*** is always logically equivalent to the original statement.

The **converse** of a statement is found by swapping the 'if' and 'then' parts of the statement, but does not always result in a true statement.

If I am in London, then I am in England	This is TRUE
If I am in England, then I am in London	Not necessarily true - I could be in Bristol

The **converse is true** when the original statement is an 'if and only if' statement

If I am in London, then I am in the capital of England	This is TRUE
If I am in the capital of England, then I am in London	This is TRUE

Note: The order of quantifiers in a statement is important.

For all positive real x, **there exists** a real y such that $y^2 = x$	TRUE (pick any x>0)
There exists a real y, such that for all positive real x, $y^2 = x$	FALSE (value of y changes with x)

1. Are the following statements true or false?

a)	$x > 5$	if	$x > 10$
b)	$x < 8$	only if	$x < 3$
c)	x is even	if and only if	$(x + 1)$ is odd
d)	$ab = ac$	if and only if	$b = c$
e)	$a^2 < a$	if	$a < 1$
f)	$a^2 < a$	only if	$a < 1$
g)	$a^2 < a$	if and only if	$-1 < a < 1$
h)	an even number is prime	if and only if	it is 2
i)	an odd number is prime	if and only if	it is 3
j)	a triangle is equilateral	if and only if	all it angles are 60°
k)	a triangle is isosceles	if	it is equilateral

2. Write the contrapositive of the following statements:

a) If I have enough money, I will go on holiday

b) If I pass my driving test, I will get my driving licence

c) Ben will not go to school only if he is sick

d) If you do not study, you will not do well in your exams

e) I wear a hat if it is sunny

3. Write the contrapositive of the following mathematical statements:

a) If an integer is not equal to 2, then it is not an even prime

b) If a shape has 4 sides, it is a quadrilateral

c) A number is even only if the square of the number is even

d) $f(a) > 0$ **if** $a > 0$

e) $a^2 < a$ is sufficient for $a < 1$

TMUA Style Multiple Choice Practice 'equivalence'

1. Given that exactly one of the following statements is true, which one is it?

A x is not an even prime **only if** $x = 2$

B **if** x is an even prime, **then** $x \neq 2$

C $x \neq 2$ is **sufficient** for x to be an even prime

D $x \neq 2$ is **necessary** for x to be an even prime

E $x = 2$ **if and only if** x is not an even prime

F x is not an even prime **only if** $x \neq 2$

2. f is a function and *a is* a real number.
Given that exactly one of the following statements is true, which one is it?

A $a \leq 0$ **only if** $f(a) \leq 0$

B $f(a) > 0$ **if** $a > 0$

C $f(a) > 0$ is **sufficient** for $a > 0$

D $f(a) \leq 0$ is **necessary** for $a \leq 0$

E **If** $f(a) > 0$ **then** $a > 0$

F $a > 0$ **if** $f(a) > 0$

3. f is a function and *a, b* are real numbers.
Given that exactly one of the following statements is true, which one is it?

A $f(a) \geq f(b)$ if and only if $a \geq b$

B $f(a) \geq f(b)$ only if $a < b$

C $f(a) < f(b)$ if $a \geq b$

D $a \geq b$ if $f(a) \geq f(b)$

E $a < b$ only if $f(a) \geq f(b)$

F $a < b$ only if $f(a) < f(b)$

Negation (denial not opposite)

Statement	**Negation**	
He is a doctor	He is not a doctor	
She is tall	She is not tall	(She is short would be incorrect)
A	*not A*	

1. I am hungry
2. They do their homework
3. It is not raining
4. The melon is not ripe

I have blue eyes **and** blond hair	Either I do not have blue eyes **or** I do not have blond hair (**or** I do not have either)
A and B	*not A or not B*

5. My socks are blue and stripy
6. I play hockey and basketball
7. I had lunch with Bill and Ben
8. It is not hot or sunny

I study English **or** German	I do not study English **and** I do not study German
A or B	*not A and not B / neither A nor B*

9. Jan drinks tea or coffee
10. The man is called Jim or John
11. The children eat apples or bananas
12. Neither my brother nor sister will help me

Statement	**Negation**
Everyone like pizza	**Not** everyone likes pizza / At least one person doesn't like pizza Some people don't like pizza There exists someone who doesn't like pizza
For all A, then B	*Not every A implies B / There exists A such that not B*

13. All vegetarians eat carrots
14. My teacher is always right
15. All dogs bark
16. Not every integer is odd

There is a prime number less than 2	There are no prime numbers less than 2 All prime numbers are greater than or equal to 2
There exists A such that B	*There is no A such that B* *For all A, not B*

17. Some boys like football
18. At least one square number is less than 3
19. There exist some birds who can not fly
20. There are no prime numbers that are even

If the sun shines, I will wear a hat	If the sun shines, I will not wear a hat
If A, then B	*If A, then not B* *A and not B*

21. If it is raining I will take an umbrella
22. I will receive a gold medal if I win
23. If $a < b$ then $f(a) < f(b)$
24. $f(a) > 0$ if $a > 0$

Negation of 'Nested' statements

A Consider the statement: The class can complete their homework online, if for **every** student in the class, the student has online access

The negation is: A class can **not** complete their homework online, if there is **at least** one student who does **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has online access

Then the statement becomes: P is true if **for every** Q, **there exists** R

The negation of this is: P is **not** true if **there exists** Q such that **not** R

B Consider the statement: The class can complete their homework online, if **for every** student in the class, the student **has** a friend who has online access

The negation is: A class can **not** complete their homework online if **there exists** a student in the class, **all** of whose friends do **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has a friend

S = friend has online access

Then the statement becomes: P is true if **for every** Q, **there exists** 'R such that S'

The negation of this is: P is **not** true if **there exists** Q such that **not** '**there exists** R such that S'

or: P is **not** true if **there exists** Q such that '**for all** R **not** S'

Write the negation of the following nested statements:

1) A set of integers P is the set of even numbers iff for any integer *n* in P, $\frac{n}{2}$ is also an integer.

2) A set of integers P is the set of square numbers iff for any integer *n* in P, there exists an integer k such that $k^2 = n$

Counter Examples

1) Find a counter example to the following statements:

a) All quadrilaterals with equal side length are squares

b) The square root of a number is always less than the number

c) If a three-dimensional solid has a circular base, then it is a cylinder

d) If n is an integer and n^2 is divisible by 4, then n is divisible by 4

e) If p is an odd prime then p+2 is also an odd prime

f) The sum of 2 numbers is always greater than both numbers

g) $10k^2 + 1$ is prime if k is an odd prime

h) For all real x, 5x > 4x

i) For all real x, $\sqrt{1 - sin^2 x} = cosx$

2) A set of five signs has a letter printed on the left and a number printed on the right

A 8 B 4 C 1 D 7 E 3

Which sign(s) provide a counterexample to the following statements:

a) Every card that has a vowel on the left has an even number on the right

b) Every card that has an even number on the right has a vowel on the left.

c) Every card that has a consonant on the left has a prime number on the right

d) Every card that has a prime number on the right has a vowel on the left

3) How many counter examples are there to the following statements:

a) All odd numbers between 2 and 20 are prime.

b) If n is a prime integer less than or equal to 10, then $n^2 + 2$ is also prime

c) A whole number n less than or equal to 50, is prime if it is 1 less or 5 less than a multiple of 6

Logic

1) On an island people either always tell the truth or always tell lies. You are approached by 2 people. Identify if they are truth-tellers or liars in the following situations.

a) The first person says “we both always tell lies”

b) The first person points at the second and says “he is a liar” and the second person says “neither of us are liars”

c) The first person says “we are both telling the truth” and the second one replies “he is lying”.

d) The first person says “at least one of us is lying”

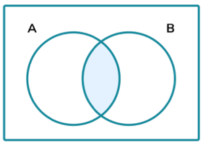
e) The first person says “exactly one of us is lying”, and the second replies “actually we’re both lying”

TMUA Proof and Logic Summary

Definitions

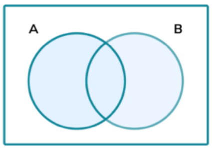
<u>**and**</u> A **and** B means A and B together ($A \cap B$)

For A **and** B to be true, **both A and B** must be true



<u>**or**</u> A **or** B means A or B or both ($A \cup B$)

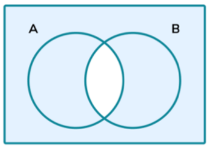
For A **or** B to be true, **either A or B or both** must be true



negation = <u>**not**</u>

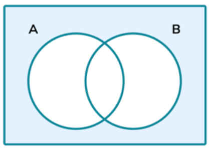
not (A and B) = not A or not B

not *(blue eyed* ***and*** *blonde) = not blue-eyed or not blond*
so could be one or the other but not both



not (A **or** B) = **not** A **and not** B

not (blue eyed ***or*** *blonde) = not blue-eyed and not blond*
so does not have either characteristic

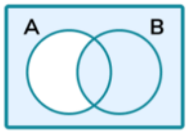


<u>**if, then**</u>

if A then B means if A is true, then B must be true (But if A is not true, then B could be true or false)

We can also write this in the following ways:

A implies B	$A \implies B$
B if A	A only if B
A is **sufficient** for B	B is **necessary** for A



The **converse** statement (swapping statements) is

‘if B then A’ but these are not always equivalent

The **contrapositive** statement (swapping and negating both statements) is

‘if not B then not A’ and this is an equivalent statement to the original.

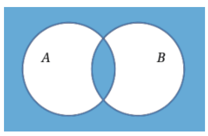
The **negation** of the statement is

‘if A then not B’ or ‘A and not B’

<u>**if and only if**</u>

A if and only if B. We can also write this in the following ways:

A implies B and B implies A	$A \iff B$
A if B and A only if B	A iff B
A is **sufficient** and **necessary** for B	



Quantifiers

These are equivalent:

- For every *A* / for any *A* / for all *A* / for each *A* / if *A*
- For some *A* / there exists *A* / for at least one *A*

The order of a combined statement is important.

For all positive real x, **there exists** a real y such that $y^2 = x$ TRUE (pick any $x>0$)

There exists a real y, such that for all positive real x, $y^2 = x$ FALSE (value of y changes with x)

Original statement	Negation
For all A, then B	*Not every A implies B* *There exists A such that not B*
Every integer is odd	*Not every integer is odd* *There exists an integer that is not odd*

Original statement	Negation
There exists A such that B	*There is no A such that B* *For all A, not B*
There exists a prime number less than 2	*There are no prime numbers less than 2* *All prime numbers are greater than or equal to 2*

<u>Example of negation of 'nested' statements</u>

Statement	*Negated Statement*
There exists P iff for every Q there exists R	*Not P = there exists Q such that 'not R'*
For any D there exists 'E such that F'	*There exists D such that NOT '**E such that F**'* *There exists D such that for all E not F*