Differentiation

Syllabus

Derivative as gradient of tangent (rate of change, second order derivatives, notation); differentiation of x^n; application to tangents, normals, stationary points, strictly increasing and decreasing functions.

1. Differentiate the following expressions with respect to x

a) $\dfrac{2x+4x^2}{\sqrt{x}} = 2x^{1/2} + 4x^{3/2}$

$\frac{d}{dx} = x^{-1/2} + 6x^{1/2} = \frac{1}{\sqrt{x}} + 6\sqrt{x}$

b) $\dfrac{1-\sqrt{x}}{4x^3} = \frac{1}{4}x^{-3} - \frac{1}{4}x^{-5/2}$

$\frac{d}{dx} = -\frac{3}{4}x^{-4} + \frac{5}{8}x^{-7/2} = -\frac{3}{4x^4} + \frac{5}{8x^{7/2}}$

c) $2\sqrt{x}\left(\dfrac{5}{x}+x^2\right) = 10x^{-1/2} + 2x^{5/2}$

$\frac{d}{dx} = -5x^{-3/2} + 5x^{3/2} = \frac{-5}{x^{3/2}} + 5x^{3/2}$

d) $\dfrac{(3+2\sqrt{x})^2}{4x} = \frac{9+12x^{1/2}+4x}{4x} = \frac{9}{4}x^{-1} + 3x^{-1/2} + 1$

$\frac{d}{dx} = -\frac{9}{4}x^{-2} - \frac{3}{2}x^{-3/2} = -\frac{9}{4x^2} - \frac{3}{2x^{3/2}}$

e) $\dfrac{(2x-1)(x^2+4)}{2\sqrt[3]{x}} = \frac{2x^3-x^2+8x-4}{2x^{1/3}} = x^{8/3} - \frac{1}{2}x^{5/3} + 4x^{2/3} - 2x^{-1/3}$

$\frac{d}{dx} = \frac{8}{3}x^{5/3} - \frac{5}{6}x^{2/3} + \frac{8}{3}x^{-1/3} + \frac{2}{3}x^{-4/3}$

2. Find the equation of the tangent to the curve at the point given

a) $y = 2\sqrt{x} - \dfrac{6}{\sqrt{x}}$ where $x = 4$

$y = 1$

$y = 2x^{1/2} - 6x^{-1/2}$

$\dfrac{dy}{dx} = x^{-1/2} + 3x^{-3/2}$

$= \dfrac{1}{\sqrt{x}} + \dfrac{3}{\sqrt{x^3}}$

At $x = 4$ $\dfrac{dy}{dx} = \dfrac{1}{2} + \dfrac{3}{8} = \dfrac{7}{8}$

$y - 1 = \frac{7}{8}(x - 4)$

$8y - 8 = 7x - 28$

$\underline{8y - 7x + 20 = 0}$

b) $y = 3x^{\frac{3}{2}} - \dfrac{32}{x}$ where $x = 4$

$y = 24 - 8$
$= 16$

$\dfrac{dy}{dx} = \frac{9}{2}x^{1/2} + 32x^{-2}$

$\left.\dfrac{dy}{dx}\right|_4 = 9 + 2 = 11$

$y - 16 = 11(x - 4)$

$y - 16 = 11x - 44$

$\underline{y = 11x - 28}$

Find the equation of the normal to the curve at the point given

c) $y = x^2(x - 6) + \dfrac{5}{x} - 1$ where $x = 1$

$= x^3 - 6x^2 + 5x^{-1} - 1$

$y' = 3x^2 - 12x - 5x^{-2}$

$y = -5 + 5 - 1 = -1$

At $x = 1$ $y' = 3 - 12 - 5$
$= -14$

$m = \frac{1}{14}$

$y + 1 = \frac{1}{14}(x - 1)$

$14y + 14 = x - 1$

$14y - x + 15 = 0$

d) $y = 2x^2 - 4x^{\frac{3}{2}} - \dfrac{8}{x} - 1$ where $x = 4$

$y' = 4x - 6x^{1/2} + 8x^{-2}$

$y'(4) = 16 - 12 + \frac{1}{2} = \frac{9}{2}$

$m = -\frac{2}{9}$

$y = 32 - 32 - 2 - 1$
$= -3$

$y + 3 = -\frac{2}{9}(x - 4)$

$9y + 27 = -2x + 8$

$9y + 2x + 19 = 0$

e) The tangent to the curve $y = x^3 - x$ at the point P (1,0) meets the curve again at the point Q.

What is the distance PQ?

$y' = 3x^2 - 1$

$y'(1) = 2$

$y = 2(x - 1)$

$y = 2x - 2$

At P, Q

$x^3 - x = 2x - 2$

$x^3 - 3x + 2 = 0$

$(x - 1)(x^2 + x - 2) = 0$

$(x - 1)^2(x + 2) = 0$

At Q $x = -2$ $y = -6$ $(-2, -6)$

$PQ = \sqrt{3^2 + 6^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$

f) The normal to the curve $y = (x - 1)(x^2 + 4x + 5)$ at the point where $x = -1$ meets the coordinate axes at the points P and Q.

$y = (-2)(2) = -4$

What is the area of triangle OPQ, where O is the origin?

$y = x^3 + 3x^2 + x - 5$

$y' = 3x^2 + 6x + 1$

$y'(-1) = 3 - 6 + 1 = -2$

$m = 1/2$

$y + 4 = \frac{1}{2}(x + 1)$

$x = 0$ $y + 4 = \frac{1}{2}$
$y = -7/2$

$y = 0$ $4 = \frac{1}{2}(x + 1)$
$x = 7$

P, Q $(0, -\frac{7}{2})(7, 0)$

Area OPQ $= \frac{1}{2} \times \frac{7}{2} \times 7$

$= \dfrac{49}{4}$

3. Find the coordinates of the stationary point(s) of the following equations, and determine if they are maximums, minimums, or points of inflexion.

a) $y = x^3 - 3x^2 - 9x + 3$

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad y = 27 - 27 - 27 + 3$

$= -24$

$x = -1 \quad y = -1 - 3 + 9 + 3$

$= 8$

$(3, -24)$ min

$(-1, 8)$ max

$\frac{d^2y}{dx^2} = 6x - 6$

$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 18 - 6 > 0$ so minimum

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -6 - 6 < 0$ so maximum

b) $y = x^2 + \frac{16}{x}$

$y' = 2x - 16x^{-2} = 0$

$y'' = 2 + 32x^{-3}$

$2x = \frac{16}{x^2}$

$x^3 = 8$

$x = 2$

$y = 4 + 8 = 12$

$y'' = 2 + 4 > 0$

so $(2, 12)$ is minimum

c) $y = 3x^4 + 16x^3 + 24x^2 + 3$

$y' = 12x^3 + 48x^2 + 48x = 0$

$x(x^2 + 4x + 4) = 0$

$x(x+2)^2 = 0$

$y'' = 36x^2 + 96x + 48$

$x = 0 \quad y = 3$

$x = -2 \quad y = 3(16) - 8(16) + (24)4 + 3$

$= -80 + 99 = 19$

At $x = 0 \quad y'' > 0$ minimum $(0, 3)$

$x = -2 \quad y'' = 36(4) - 96(2) + 48$

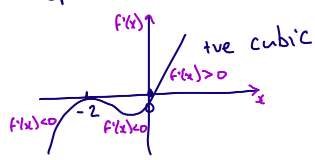
$= 12(12 - 16 + 4)$

$= 0$

Consider gradient function

$f'(x) = x(x+2)^2$

repeated root at -2



f(x) has point of inflexion at $(-2, 19)$

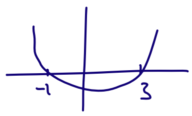
4 Find the range of values of x, for which y is a decreasing function

a) $y = x^3 - 3x^2 - 9x + 10$

$\frac{dy}{dx} = 3x^2 - 6x - 9 < 0$

$x^2 - 2x - 3 < 0$

$(x-3)(x+1) < 0$



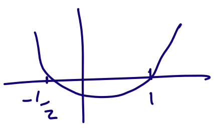
$-1 < x < 3$

b) $y = 6x + 3x^2 - 4x^3$

$\frac{dy}{dx} = 6 + 6x - 12x^2 < 0$

$2x^2 - x - 1 > 0$

$(2x+1)(x-1) > 0$



$x < -\frac{1}{2}, \quad x > 1$