TMUA Logs and Exponentials

Syllabus
Graph of a^x for positive values of a; laws of logarithms; solving log equations and simultaneous equations.

1. Simplify the following expressions, giving the answer as a single logarithm:

a) $log_3 7 + log_3 5$ $\qquad \log_3(7\times 5) = \log_3 35$

b) $log_5 24 - log_5 6$ $\qquad \log_5\left(\frac{24}{6}\right) = \log_5 4$

c) $3log_5 2 + log_5 6$ $\qquad \log_5(2^3\times 6) = \log_5 48$

d) $1 + 2log_n 3 + log_n 4$ $\qquad \log_n(n\times 3^2\times 4) = \log_n 36n$

e) $2log_4 3 + log_4 5 - log_4 0.5$ $\qquad \log_4\left(3^2\times 5 \div \frac{1}{2}\right) = \log_4 90$

2. Simplify the following expressions, giving the answer as a single number:

a) $log_8 25 + log_8 10 - 3log_8 5$ $\qquad \log_8(25\times 10 \div 125) = \log_8 2 = \frac{1}{3}$

b) $log_6 4 + log_6 9$ $\qquad \log_6(4\times 9) = \log_6 36 = 2$

c) $log_2 5 + log_2 1.6$ $\qquad \log_2(5\times 1.6) = \log_2 8 = 3$

d) $log_2(\frac{5}{2}) + log_2(\frac{4}{3}) - log_2(\frac{5}{3})$ $\qquad \log_2\left(\frac{5}{2}\times\frac{4}{3}\times\frac{3}{5}\right) = \log_2 2 = 1$

e) $\frac{1}{3}log_{1.5}(\frac{8}{27}) + \frac{1}{2}log_{1.5}(\frac{4}{9})$ $\qquad \log_{3/2}\left(\frac{2}{3}\times\frac{2}{3}\right) = -\log_{3/2}\left(\frac{3}{2}\right)^2 = -2$

f) $log_a(a^2) - 4log_a(\frac{1}{a})$ $\qquad 2 - \log_a(a^{-4}) = 2 + 4 = 6$

3. Solve the following equations leaving your answer in terms of logarithms base 10 (*log*):

a) $2^x = 3^{x+1}$

$x \log 2 = (x+1) \log 3$

$x(\log 2 - \log 3) = \log 3$

$x = \dfrac{\log 3}{\log 2 - \log 3} = \dfrac{\log 3}{\log \frac{2}{3}}$

b) $3^{y-1} = 2^{2y}$

$(y-1)\log 3 = 2y \log 2$

$y(\log 3 - 2\log 2) = \log 3$

$y = \dfrac{\log 3}{\log 3 - 2\log 2} = \dfrac{\log 3}{\log \frac{3}{4}}$

c) $2^{x+3} = 6^{x-1}$

$(x+3)\log 2 = (x-1)\log 6$

$\log 6 + \log 8 = x(\log 6 - \log 2)$

$\dfrac{\log 48}{\log 3} = x$

d) $8^{4-3y} = 7^y$

$(4-3y)\log 8 = y \log 7$

$4\log 8 = y(\log 7 + 3\log 8)$

$y = \dfrac{4\log 8}{\log 7 + 3\log 8}$

e) $2^{2x} - 2^x - 6 = 0$

$y = 2^x$ $\quad y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3 \quad y = -2$ ×

$2^x = 3$

$x = \dfrac{\log 3}{\log 2}$ or $x = \log_2 3$

f) $4^y - 3(2^y) - 10 = 0$

$x = 2^y$ $\quad x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x = 5 \quad x = -2$ ×

$2^y = 5$

$y = \dfrac{\log 5}{\log 2}$ or $y = \log_2 5$

g) $3^{2y+1} - 11(3^y) - 4 = 0$

$x = 3^y$ $\quad 3x^2 - 11x - 4 = 0$

$(3x+1)(x-4) = 0$

$x = -\frac{1}{3} \quad x = 4$

$3^y = 4$

$y = \dfrac{\log 4}{\log 3}$ or $y = \log_3 4$

h) $x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$

$y = 8^{\log_2 x} = (2^{\log_2 x})^3 = x^3$

$\log_2 y = \log_2 x \ \log_2 8$

$9^{\log_3 x} = (3^{\log_3 x})^2 = x^2$

$4^{\log_2 x} = (2^{\log_2 x})^2 = x^2$

$\log_{\frac{1}{2}}(\frac{1}{4}) = 2$

$x = x^3 - 2x^2 + 2$

$x^3 - 2x^2 - x + 2 = 0$

$(x-1)(x^2 - x - 2) = 0$

$(x-1)(x-2)(x+1) = 0$

$\underline{x = 1 \quad x = 2} \quad x = -1$ ×

4. Solve the following equations.

a) $log_2(x+1) - log_2 x = log_2 3$

$$\frac{x+1}{x} = 3$$
$$x+1 = 3x$$
$$x = 1/2$$

b) $log_a y = log_a 3 + log_a(2y-1)$

$$y = 6y - 3$$
$$5y = 3$$
$$y = 3/5$$

c) $log_5(4w+3) - log_5(w-1) = 2$

$$\frac{4w+3}{w-1} = 5^2$$
$$4w+3 = 25w - 25$$
$$21w = 28$$
$$w = 4/3$$

d) $log_3(4x+1) - log_3(x-1) = 2$

$$\frac{4x+1}{x-1} = 3^2$$
$$4x+1 = 9x - 9$$
$$10 = 5x$$
$$x = 2$$

e) $log_2(3y+4) - log_2 y = 3$

$$\frac{3y+4}{y} = 2^3$$
$$3y+4 = 8y$$
$$5y = 4$$
$$y = 4/5$$

f) $log_2(4z+4) = 6$

$$4z+4 = 2^6$$
$$4z = 60$$
$$z = 15$$

g) $log_2(x^2+4x+3) = 4 + log_2(x^2+x)$

$$\frac{x^2+4x+3}{x^2+x} = 16$$
$$x^2+4x+3 = 16x^2+16x$$
$$15x^2+12x-3 = 0$$
$$5x^2+4x-1 = 0$$
$$(5x-1)(x+1) = 0$$
$$x = 1/5 \qquad x = -1$$
$$\times$$

5. Find the difference between the solutions of the following equations:

a) $2^{2x} - 8.2^{x} + 15 = 0$

$y^2 - 8y + 15 = 0$
$(y-3)(y-5) = 0$
$y = 3 \quad y = 5$

$2^x = 3$
$x = \frac{\log 3}{\log 2}$

$2^x = 5$
$x = \frac{\log 5}{\log 2}$

$\frac{\log 5 - \log 3}{\log 2}$
$\frac{\log 5/3}{\log 2}$

b) $4^{2x} + 12 = 2^{2x+3}$

$y = 2^{2x}$ $\quad y^2 - 8y + 12 = 0$
$y = 2, 6$

$2^{2x} = 2$
$2x = 1$
$x = \frac{1}{2}$

$2^{2x} = 6$
$2x = \frac{\log 6}{\log 2}$
$x = \frac{\log 6}{2\log 2}$

$\frac{\log 6}{2\log 2} - \frac{1}{2}$

$\frac{\log 6 - \log 2}{2\log 2} = \frac{\log 3}{\log 4}$

c) $3^x - (\sqrt{3})^{x+4} + 20 = 0$

$y = (\sqrt{3})^x$ $\quad y^2 - 9y + 20 = 0$
$y = 4, 5$

$3^{\frac{1}{2}x} = 4$ $\quad 5$
$\frac{1}{2}x = \frac{\log 4}{\log 3}$ $\quad \frac{\log 5}{\log 3}$

$x = \frac{2\log 4}{\log 3}$ $\quad \frac{2\log 5}{\log 3}$ $\quad \frac{2\log(5/4)}{\log 3}$

d) $2log_a x = log_a 18 + log_a(x-4)$

$x^2 = 18(x-4)$
$x^2 - 18x + 72 = 0$

$(x-6)(x-12) = 0$
$x = 6, 12$

e) $2log_a y - log_a(5y-24) = log_a 4$

$\frac{y^2}{5y-24} = 4$
$y^2 - 20y + 96 = 0$

$(y-8)(y-12) = 0$
$y = 8, 12$

6. Given that $y = log_2 x$ write each expression in terms of y

a) $log_2 x^4$ $\quad 4y$

b) $log_2(8x^2) = \log_2 8 + 2\log_2 x = 3 + 2y$

c) $log_4 x$ $\quad \frac{1}{2}y$

c)
$2^y = x$
$2^{2y} = x^2$
$4^y = x^2$
$4^{\frac{1}{2}y} = x$

d) $log_2(\frac{1}{2}x) = \log_2 \frac{1}{2} + \log_2 x = y - 1$

7. Given that $p = log_a 4$ and $q = log_a 5$ write each expression in terms of p and q

a) $log_a 100 = \log_a(4 \times 5^2) = p + 2q$

b) $log_a 0.4 = \log_a\left(\frac{4}{2 \times 5}\right) = p - q - \log_a 4^{1/2} = p - q - \frac{1}{2}p = \frac{1}{2}p - q$

OR $= \log_a\left(\frac{2}{5}\right) = \log_a 4^{1/2} - \log_a 5 = \frac{1}{2}p - q$

c) $log_a 3.2 = \log_a\left(\frac{16}{5}\right) = 2p - q$

d) $log_a 80a^2 = \log_a 80 + \log_a a^2 = \log_a(4^2 \times 5) + 2$
$= 2p + q + 2$

8. Rearrange the equation to make x the subject.

a) $y = -\frac{1}{2}log_{10}(10 - x)$

$$-2y = \log_{10}(10 - x)$$
$$10^{-2y} = 10 - x$$
$$x = 10 - 10^{-2y}$$
$$x = 10 - \frac{1}{10^{2y}}$$

b) $y = a^x b^{2x} c^{3x}$

$$\log y = x \log a + x \log b^2 + x \log c^3$$
$$= x(\log ab^2c^3)$$
$$x = \frac{\log y}{\log ab^2c^3}$$

c) $y = log_3 8 - 3log_3 x$

$$3\log_3 x = \log_3 8 - y$$
$$\log_3 x = \log_3 2 - \tfrac{1}{3}y$$
$$x = 3^{\wedge}\log_3 2 - \tfrac{1}{3}y$$
$$= \frac{3^{\log_3 2}}{3^{\frac{1}{3}y}} = \frac{2}{3^{\frac{1}{3}y}}$$

d) $2 + log_a b + 3log_a x = 2log_a(a^2 x)$

$2 = \log_a a^2$

$$\log_a a^2bx^3 = \log_a a^4x^2$$
$$a^2bx^3 = a^4x^2$$

$x \neq 0$
$a, b \neq 0$

$$x = \frac{a^2}{b}$$

9. Which is the largest of the following:

a)	$log_2 4$ 2	$log_4 2$ $\frac{1}{2}$	$log_3 5$ $1-2$	$log_8 2$ $\frac{1}{3}$

b)	$log_2 3$ $1-2$ $> \frac{3}{2}$	$log_4 8$ $\frac{3}{2}$	$log_3 2$ < 1	$log_5 10$ $1-2$ $< \frac{3}{2}$

$\log_2 3$ $\quad \frac{3}{2}$
$3 \quad 2^{3/2}$
$3 > \sqrt{8}$
so $\log_2 3 > \frac{3}{2}$

$\log_5 10$ $\quad \frac{3}{2}$
$10 \quad 5^{3/2}$
$10 < \sqrt{125}$
so $\log_5 10 < \frac{3}{2}$

Which is the smallest of the following:

c)	$L = log_{10}\pi$ $L < 1$	$\sqrt{log_{10}\pi}$ $= \sqrt{L}$ $L < \sqrt{L} < 1$	$\left(\frac{1}{log_{10}\pi}\right)^3$ $L^{-3} > 1 > L$	$\frac{1}{log_{10}\sqrt{\pi}}$ $= \frac{1}{\frac{1}{2}\log_{10}\pi} = \frac{2}{L} > 2$

10. Solve the following simultaneous equations

a) $log_3(xy^2) = 1$ $\qquad (log_3 x)(log_3 y) = -3$

$\log_3 x + 2\log_3 y = 1$

$X + 2Y = 1 \qquad XY = -3$

$-\frac{3}{Y} + 2Y = 1$

$2Y^2 - Y - 3 = 0$

$(2Y-3)(Y+1) = 0$

$Y = \frac{3}{2} \qquad Y = -1$

$X = -2 \qquad X = 3$

$\log_3 y = \frac{3}{2}, \; -1$

$y = 3^{3/2}, \; \frac{1}{3}$

$\log_3 x = -2 \quad 3$

$x = \frac{1}{9}, \; 27$

b) $2^x + 3(2^y) = 3$ $\qquad 2^{2x} - 9(2^{2y}) = 6$

$a = 2^x$
$b = 2^y$

$a + 3b = 3 \qquad a = 3 - 3b$

$a^2 - 9b^2 = 6$

$9 - 18b + 9b^2 - 9b^2 = 6$

$18b = 3 \quad b = \frac{1}{6} \quad a = \frac{5}{2}$

$2^x = \frac{5}{2} \qquad 2^y = \frac{1}{6}$

$x = \log_2 \frac{5}{2} \qquad y = \log_2 \frac{1}{6}$

c) $2^{3x} = 8^{y+3}$ $\qquad 4^{x+1} = \frac{16^{y+1}}{8^{y+3}}$

$a^3 = 8^3 b^3$

$a = 8b$

$4a^2 = \frac{16b^4}{8^3 b^3}$

$4a^2 = \frac{2b}{64}$

$128a^2 = b$

$\frac{a}{8} = 128a^2$

$a = \frac{1}{2^3 . 2^7} = \frac{1}{2^{10}} \qquad x = -10$

$b = \frac{1}{2^{13}} \qquad y = -13$

d) $log_y x = 3$ $\qquad log_3 x = 1 + log_3 y$

$y^3 = x$

$3\log_3 y = \log_3 x$

$3\log_3 y = 1 + \log_3 y$

$2\log_3 y = 1$

$\log_3 y = \frac{1}{2}$

$y = 3^{1/2} = \sqrt{3}$

$x = 3^{3/2} = \sqrt{27}$

11. Find the solution of the following equations:

a) $log_\pi(\underbrace{log_2(log_7 x)}_{=1}) = 0$

$\log_2(\log_7 x) = 1$

$\log_7 x = 2$

$x = 7^2 = 49$

b) $log_{99}(\underbrace{log_2(log_3 x)}_{=1}) = 0$

$\log_2(\log_3 x) = 1$

$\log_3 x = 2$

$x = 3^2 = 9$

c) $log_a x = log_{a^2}(x + 20)$

$\log_{a^2}(x+20) = y$

$a^{2y} = x + 20$

$a^y = (x+20)^{1/2}$

$\log_a (x+20)^{1/2} = y$

$\log_a x = \log_a (x+20)^{1/2}$

$x^2 = x + 20$

$x^2 - x - 20 = 0$

$(x-5)(x+4) = 0$

$x > 0 \implies x = 5$

12. The numbers a, b and c are each greater than or equal to 1.

a) The logarithms below are all to the same base. What is the base?

$log(ab^2c) = 6$

$log(a^2bc^4) = 9$

$log(a^5b^7c^5) = 25$

$X = \log a$
$Y = \log b$
$Z = \log c$

$\log ab^2c = 6$
$X + 2Y + Z = 6$
$2X + Y + 4Z = 9$
$5X + 7Y + 5Z = 25$
$5X + 10Y + 5Z = 30$

$\log a^2bc^4 = 9$
$2X + 4Y + 2Z = 12$
$3Y - 2Z = 3$
$3Y = 5$
$Y = \frac{5}{3}$ $Z = 1$ $X = \frac{5}{3}$

$\log a^5b^7c^5 = 25$

$\log c = 1$
$\Rightarrow$ base c

b) The logarithms below are all to the same base. What is the base?

$log(a^2b^3c^5) = 21$

$log(a^3b^6c^{15}) = 51$

$log(a^5b^4c^{10}) = 37$

$2x + 3y + 5z = 21$
$3x + 6y + 15z = 51$
$5x + 4y + 10z = 37$

$x = \log a$
$y = \log b$
$z = \log c$

$4x + 6y + 10z = 42$
$3x + 6y + 15z = 51$
$-x + 5z = 9$

$6x + 12y + 30z = 102$
$15x + 12y + 30z = 111$
$9x = 9$

$x = 1$ $z = 2$ $y = 3$

base a

c) The logarithms below are all to the same base. What is the base?

$log(\frac{ac^3}{b}) = 5$

$log(abc^6) = 12$

$log(\frac{a^3c^2}{b^2}) = 10$

$x = \log a$
$y = \log b$
$z = \log c$

$x - y + 3z = 5$
$x + y + 6z = 12$
$3x - 2y + 2z = 10$

$2x + 9z = 17$
$5x + 14z = 34$

$10x + 45z = 85$
$10x + 28z = 68$
$17z = 17$
$z = 1$

$x = 4$
$y = 2$
$z = 1$

base c