TMUA Coordinate Geometry

Syllabus

Equation of a straight line; parallel and perpendicular lines; equation of a circle; circle theorems.

1a) Find the coordinates of the point lying between A (2,3) and B (8, − 3) which divides the line segment AB in the ratio 1:2.

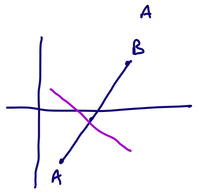
$2 \to 8 \quad +6 \quad \frac{1}{3} = 2 \quad 2+2 = 4$

$3 \to -3 \quad -6 \quad \frac{1}{3} = -2 \quad 3-2 = 1$

$\underline{(4,1)}$

b) Find the x-coordinate of the point where the perpendicular bisector of the line segment joining the points (2,-6) and (5,4) cuts the x-axis.

A B



Midpoint $\left(\frac{7}{2}, -1\right)$

Gradient $= \frac{10}{3}$

$\perp^r$ gradient $= -\frac{3}{10}$

$y + 1 = -\frac{3}{10}\left(x - \frac{7}{2}\right)$

$y = 0$

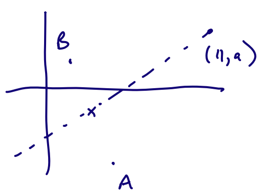
$-\frac{10}{3} = x - \frac{7}{2}$

$x = \frac{21}{6} - \frac{20}{6} = \underline{\frac{1}{6}}$

$\left(10y + 10 = -3x + \frac{21}{2} \quad 10y = -3x + \frac{1}{2}\right)$ final equation not required

A B

c) The perpendicular bisector of the line segment joining the points (3,-5) and (1,1) passes through the point with coordinates (11,*a*). Find the value of *a*.



Midpoint $(2,-2)$

Gradient $= \frac{6}{-2} = -3$

$\perp^r$ gradient $= \frac{1}{3}$

$y + 2 = \frac{1}{3}(x - 2)$

$x = 11$

$y + 2 = \frac{1}{3}(9)$

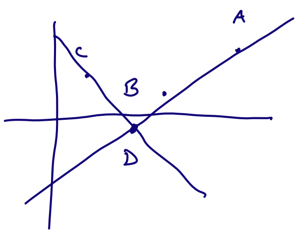
$y = 1$

$\underline{a = 1}$

d) The straight line L_1 passes through points A (13,5) and B (9,2) and D.

The straight line L_2 passes through points C (2,3) and D and is perpendicular to L_1

Find the coordinates of *D*.



gradient AB $= \frac{3}{4}$

gradient CD $= -\frac{4}{3}$

AB: $y - 2 = \frac{3}{4}(x - 9)$

CD: $y - 3 = -\frac{4}{3}(x - 2)$

$\frac{3}{4}x - \frac{27}{4} + 2 = -\frac{4}{3}x + \frac{8}{3} + 3$

$x\left(\frac{9}{12} + \frac{16}{12}\right) = \frac{32}{12} + \frac{81}{12} + \frac{12}{12}$

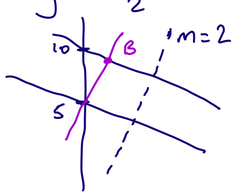
$25x = 125$

$\underline{x = 5} \quad \underline{y = -1}$

e) Find the shortest distance between the parallel lines with equations:

$x + 2y = 10$ and $x + 2y = 20$

$y = -\frac{1}{2}x + 5 \qquad y = -\frac{1}{2}x + 10$



$y = 2x + 5 \quad x + 2y = 20$

$x + 4x + 10 = 20$

$x = 2 \quad y = 9$

B $(2, 9)$

$(0, 5)$

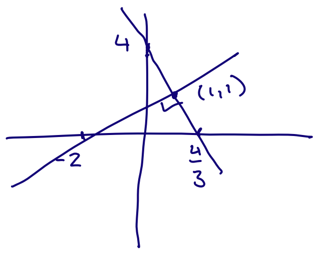
Distance $= \sqrt{4^2 + 2^2}$

$= \sqrt{20}$

$= \underline{2\sqrt{5}}$

f) A line L has equation $y = 4 - 3x$. A second line is perpendicular to L and passes through (-2,0).

Find the area of the region enclosed by the two lines and the x-axis.



gradient $= \frac{1}{3}$ $(-2,0)$

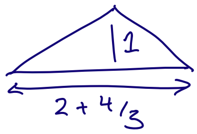
$y = \frac{1}{3}(x+2)$

$y = \frac{x}{3} + \frac{2}{3}$

$4 - 3x = \frac{1}{3}x + \frac{2}{3}$

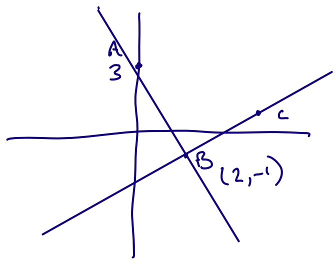
$\frac{10}{3} = \frac{10}{3}x$ $\quad x = 1 \quad y = 1$

Area $= \frac{1}{2} \times 1 \times \frac{10}{3} = \frac{5}{3}$



g) The points A, B and C have coordinates (0,3) and $(2, -1)$ and $(k,1)$ respectively.

AB and BC are perpendicular. Find the area of the triangle ABC.



$|AB| = \sqrt{4^2+2^2} = \sqrt{20}$

grad $AB = -2$

grad $BC = 1/2$

BC: $y + 1 = \frac{1}{2}(x-2)$

$y = \frac{x}{2} - 2$

$y = 1$, $x = 6$ $\quad c(6,1)$

$|BC| = \sqrt{4^2+2^2} = \sqrt{20}$

Area $= \frac{1}{2} \times \sqrt{20} \times \sqrt{20}$

$= 10$

h) The straight line L passes through points (2,5) and $(-2,3)$ and meets the coordinate axes at P and Q. Find the area of a square with side PQ.

gradient $L = \frac{2}{4} = \frac{1}{2}$

$y - 5 = \frac{1}{2}(x-2)$

$y = \frac{1}{2}x + 4$

$P(0,4)$ $\quad Q(-8,0)$

$|PQ|^2 = 4^2 + 8^2$

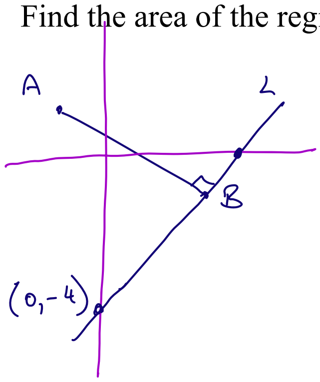
$= 16 + 64$

$= 80$

i) The points A and B have coordinates $(-1,4)$ and $(3,-2)$ respectively.

A line L is perpendicular to AB and passes through B

Find the area of the region enclosed by L and the coordinate axes



gradient $AB = \frac{6}{-4} = -\frac{3}{2}$

gradient $L = 2/3$

$y + 2 = \frac{2}{3}(x-3)$

$y = \frac{2}{3}x - 4$

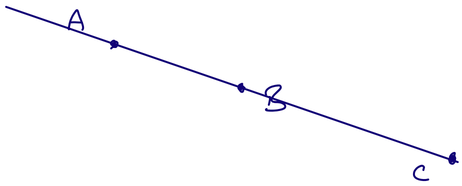
$(0,-4)$ $(6,0)$

Area $= \frac{1}{2} \times 4 \times 6$

$= 12$

j) The points A and B have coordinates $(-4,5)$ and $(0,4)$ respectively.

The point C lies on the straight line through A and B such that the distance AB is the same as the distance BC. Find the coordinates of C.



A B C

$(-4,5)$

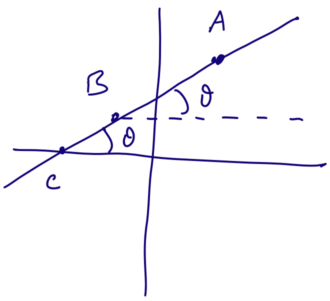
$(0,4)$

$(4,3)$

k) The points A and B have coordinates $(1,4\sqrt{3})$ and $(-3+\sqrt{3},3)$ respectively.

The straight line L through A and B meets the x-axis at C.

Calculate the acute angle between L and the x-axis



$$\tan\theta = \frac{4\sqrt{3}-3}{1+3-\sqrt{3}} = \frac{4\sqrt{3}-3}{4-\sqrt{3}}$$

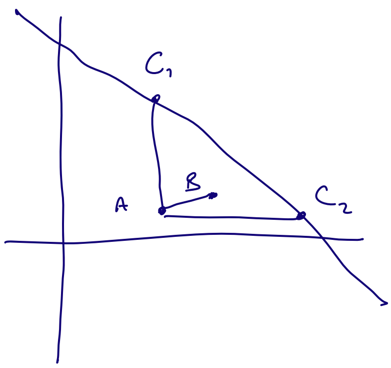
$$= \frac{\sqrt{3}(4-\sqrt{3})}{4-\sqrt{3}} = \sqrt{3}$$

$$\theta = 60^\circ$$

l) The points A and B have coordinates $(8,2)$ and $(11,3)$ respectively.

The point C lies on the straight line with equation $x + y = 14$ $\quad C(a,\ 14-a)$

Given that the distance AC is twice as large as the distance AB, find the two possible sets of coordinates of C.



$$|AB| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$|AC| = 2\sqrt{10} \qquad AC^2 = 40$$

$$40 = (a-8)^2 + (12-a)^2$$

$$40 = a^2 - 16a + 64 + 144 - 24a + a^2$$

$$2a^2 - 40a + 168 = 0$$

$$a^2 - 20a + 84 = 0$$

$$(a-6)(a-14) = 0$$

$$a = 6,\ 14$$

$(6,8)$

$(14,0)$

2a) The straight line segment joining the points (6,-3) and (14,9) is a diameter of a circle. What is the equation of the circle?

Midpoint $(10, 3)$

Diameter $= \sqrt{12^2+8^2}$

$= \sqrt{144+64}$

$= \sqrt{208}$

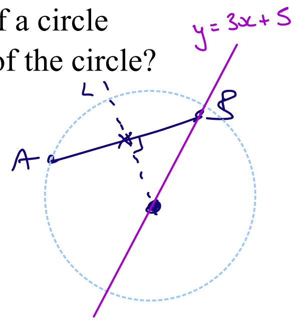
$= 2\sqrt{52}$

Radius $= \sqrt{52}$

$r^2 = 52$

$(x-10)^2 + (y-3)^2 = 52$

b) The straight line segment joining the points $(-4,3)$ and $(0,5)$ is a chord of a circle with centre on the line with equation $y = 3x + 5$. What is the equation of the circle?



gradient AB $= 1/2$

midpoint AB $= (-2, 4)$

gradient L $= -2$

$y - 4 = -2(x+2)$

$y = -2x$

$-2x = 3x+5$

$-5 = 5x$

$x = -1, \ y = 2$ centre $(-1, 2)$

$r^2 = 3^2 + 1^2 = 10$

$(x+1)^2 + (y-2)^2 = 10$

c) Find the equation of the tangent to the circle $x^2 + y^2 - 8x - 14y + 40 = 0$ at the point (8,4)

$(x-4)^2 - 16 + (y-7)^2 - 49 + 40 = 0$

$(x-4)^2 + (y-7)^2 = 25$

centre $(4,7)$ radius 5

gradient radius to $(8,4)$ $= -\frac{3}{4}$

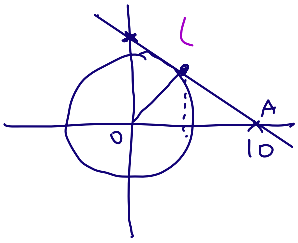
gradient tangent $= 4/3$

$y - 4 = \frac{4}{3}(x-8)$

$3y - 12 = 4x - 32$

$4x - 3y = 20$

d) A tangent to the circle $x^2 + y^2 = 36$ passes through the point (10,0) and crosses the positive y-axis. What is the coordinate of the point where the tangent meets the y-axis?



$y = mx + c$

$0 = 10m + c$

$c = -10m$

$y = m(x-10)$

$y^2 = m^2(x^2 - 20x + 100)$

$x^2 + m^2(x^2 - 20x + 100) = 36$

$x^2(1+m^2) - 20m^2x + (100m^2 - 36) = 0$

$\Delta = 0 \quad 400m^4 - 4(1+m^2)(100m^2 - 36) = 0$

$400m^4 - 400m^2 - 400m^4 + 144 + 144m^2 = 0$

$-100m^2 + 36 + 36m^2 = 0$

$64m^2 = 36 \quad m^2 = \frac{9}{16} \quad m = \pm\frac{3}{4}$

$c = \frac{15}{2} \quad m = -\frac{3}{4} \quad y = -\frac{3}{4}x + \frac{15}{2} \quad \left(0, \frac{15}{2}\right)$

e) Find the radius of the circle with equation $2x^2 + 2y^2 + 12x - 4y + 13 = 0$

$x^2 + y^2 + 6x - 2y + \frac{13}{2} = 0$

$(x+3)^2 - 9 + (y-1)^2 - 1 + \frac{13}{2} = 0$

$(x+3)^2 + (y-1)^2 = \frac{7}{2}$

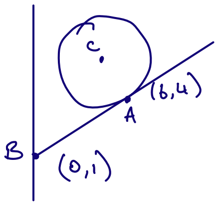
radius $= \sqrt{\frac{7}{2}}$

f) A circle has equation $x^2 + y^2 - 10x - 12y + 56 = 0$ and C is the centre of the circle.

The tangent to the circle at A (6,4) meets the y-axis at B. Find the area of triangle ABC.

$(x-5)^2 - 25 + (y-6)^2 - 36 + 56 = 0$

$(x-5)^2 + (y-6)^2 = 5$ $\quad (5,6)$



grad $AC = -2$

grad $AB = \frac{1}{2}$

$y - 4 = \frac{1}{2}(x-6)$

$y = \frac{1}{2}x + 1$

$|AB| = \sqrt{6^2 + 3^2} = \sqrt{45}$

$|AC| = \sqrt{5}$

Area $ABC = \frac{1}{2} \times \sqrt{9 \times 5} \times \sqrt{5}$

$= \frac{15}{2}$

g) A circle has centre $(8,k)$ where k is a constant.

The straight line with equation $y = 3x - 12$ is tangent to the circle at (5,3).

Find the equation of the circle.

grad radius $= \frac{k-3}{3} = -\frac{1}{3}$ $\quad k = 2$ $\quad$ Centre $(8,2)$

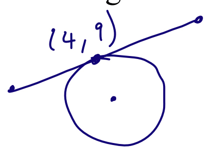
$|r|^2 = 3^2 + 1^2 = 10$

$(x-8)^2 + (y-2)^2 = 10$

h) A circle has centre (5,6).

The straight line which passes through (1,8) and (10,11) is a tangent to the circle.

Find the radius of the circle.



gradient of tangent $= \frac{3}{9} = \frac{1}{3}$

gradient of radius $= -3$

$y - 6 = -3(x-5)$

$y = -3x + 21$

$y - 8 = \frac{1}{3}(x-1)$

$y = \frac{1}{3}x + \frac{23}{3}$

$-3x + 21 = \frac{1}{3}x + \frac{23}{3}$

$-9x + 63 = x + 23$

$10x = 40 \quad x = 4$

$y = 9$

$r = \sqrt{1^2 + 3^2}$

$= \sqrt{10}$

i) A circle has equation $x^2 + y^2 + 2x - 4y + 1 = 0$.

The straight line with equation $y = mx$ is a tangent to the circle.

Find the difference in the possible values of m.

$(x+1)^2 + (y-2)^2 = 4$ $\quad y = mx$

$(x+1)^2 + (mx-2)^2 = 4$

$x^2 + 2x + 1 + m^2x^2 - 4mx = 0$

$(1+m^2)x^2 + (2-4m)x + 1 = 0$

$\Delta = 0 \quad (2-4m)^2 - 4(1+m^2) = 0$

$1 - 4m + 4m^2 - 1 - m^2 = 0$

$3m^2 - 4m = 0$

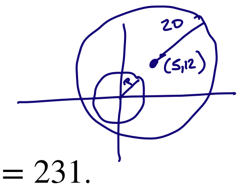
$m(3m-4) = 0$

$m = 0, \ ^4/_3$

$\frac{4}{3}$

j) A circle has centre at the origin and radius R.

$x^2+y^2=R^2$



The circle fits wholly inside the circle with equation $x^2+y^2-10x-24y=231$.

Find the range of possible value of R.

$(x-5)^2+(y-12)^2-25-144=231$

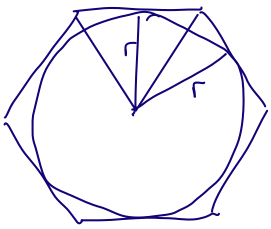
$(x-5)^2+(y-12)^2=20^2$

Distance $0 \to$ centre $=\sqrt{5^2+12^2}=13$

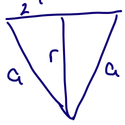
$R<20-13 \qquad R<7$

k) A circle is drawn inside a regular hexagon so that the circle touches each side of the hexagon.

What fraction of the hexagon is covered by the circle?



Circle: πr^2



$\frac{a^2}{4}+r^2=a^2$

$r^2=\frac{3a^2}{4} \qquad a^2=\frac{4r^2}{3}$

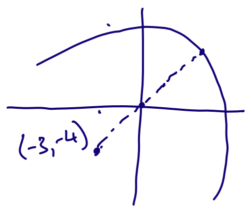
Fraction: $\frac{\pi r^2}{2\sqrt{3}r^2}$

$\frac{\pi}{2\sqrt{3}}=\frac{\sqrt{3}\pi}{6}$

Hexagon: $6\times\frac{1}{2}a^2\frac{\sqrt{3}}{2}=\cancel{6}^{2}\times\frac{\cancel{4}}{\cancel{3}}r^2\times\frac{\sqrt{3}}{\cancel{4}}=2\sqrt{3}r^2$

l) Find the shortest distance between the circle $x^2+y^2+6x+8y=75$ and the origin.

$(x+3)^2+(y+4)^2=100$



gradient $=\frac{4}{3}$

$y=\frac{4}{3}x$

$x^2+\frac{16}{9}x^2+6x+\frac{32}{3}x=75$

$\frac{25x^2}{9}+\frac{50x}{3}-75=0$

$x^2+6x-27=0$

$(x-3)(x+6)=0$

$x=3 \quad y=4$

$(3,4)$

Distance $=\sqrt{3^2+4^2}$

$=5$

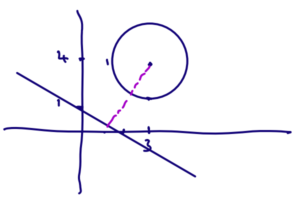
m) Find the shortest distance between the line $x+2y=2$

and the circle $x^2+y^2-6x-8y+21=0$

$y=-\frac{1}{2}x+1$

grad $=-\frac{1}{2}$

$(x-3)^2+(y-4)^2=4$



$\perp$ gradient $=2$

$(3,4)$

$y-4=2(x-3)$

$y=2x-2$

$2x-2=-\frac{1}{2}x+1$

$\frac{5}{2}x=3 \qquad x=\frac{6}{5} \qquad y=\frac{2}{5}$

Distance line to centre

$=\sqrt{\left(\frac{18}{5}\right)^2+\left(\frac{9}{5}\right)^2}$

$=\frac{9}{5}\sqrt{4+1}=\frac{9}{5}\sqrt{5}$

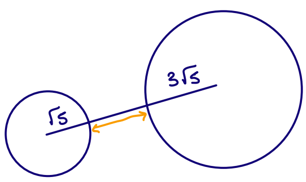
Distance line to circle

$=\frac{9}{5}\sqrt{5}-2$

n) Find the shortest distance between the two circle with equations:

$(x-5)^2+(y-9)^2=45$ and $(x+1)^2+(y+3)^2=5$

$(5,9) \quad r=3\sqrt{5} \qquad (-1,-3) \quad r=\sqrt{5}$



distance between centres $=\sqrt{6^2+12^2}$

$=6\sqrt{5}$

distance between circles

$=6\sqrt{5}-3\sqrt{5}-\sqrt{5}$

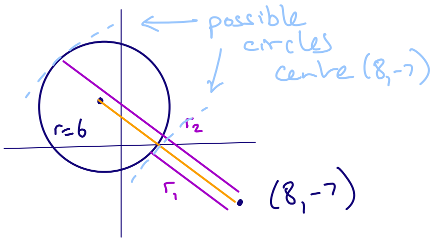
$=2\sqrt{5}$

o) The two circles with equations below have exactly one point in common.

$(x + 1)^2 + (y - 5)^2 = 36$ and $(x - 8)^2 + (y + 7)^2 = r^2$

Find the two possible values of r

Centre $(-1, 5)$ $r = 6$
Centre $(8, -7)$ radius r



Distance between centres $= \sqrt{9^2 + 12^2} = 15$

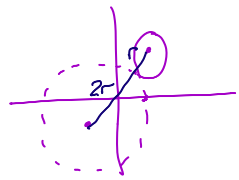
$r_1 = 15 - 6 = 9$

$r_2 = 15 + 6 = 21$

p) The two circles with equations below have exactly one point in common.

$(x + r)^2 + (y + r)^2 = 4r^2$ and $(x - r)^2 + (y - 2)^2 = r^2$

Find the value of r



Distance between centres $= 3r$ $(-r, -r)$ $(r, 2)$

$(r + 2)^2 + (2r)^2 = 9r^2$

$r^2 + 4r + 4 + 4r^2 = 9r^2$

$4r^2 - 4r - 4 = 0$

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1 + 4}}{2}$ $r > 0$

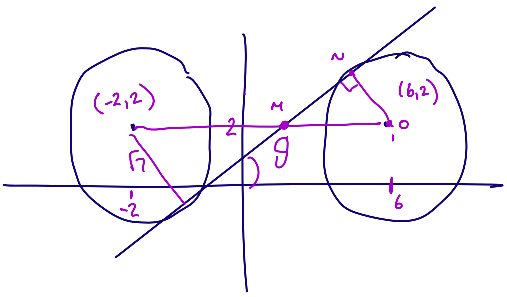
$r = \frac{1 + \sqrt{5}}{2}$

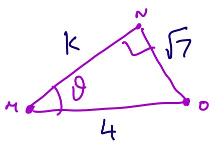
q) Circle C_1 has equation $(x + 2)^2 + (y - 2)^2 = 7$ $(-2, 2)$ $r = \sqrt{7}$

Circle C_2 has equation $(x - 6)^2 + (y - 2)^2 = 7$ $(6, 2)$

The straight line L is a tangent to both circles and has a positive gradient.

The angle between L and the x-axis is θ. Find $cos\theta$





Distance between centres $= 8$

$OM = 4$

$k^2 = 16 - 7 = 9$

$k = 3$

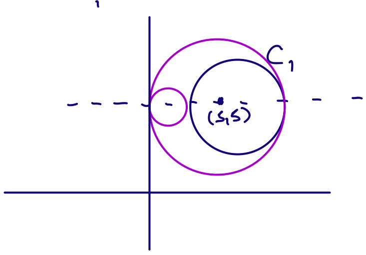
$\cos\theta = \frac{3}{4}$

r) Circle C_1 has equation $x^2 + y^2 - 10x - 10y + 41 = 0$

Circle C_2 has centre $(k,5)$ and touches both C_1 and the y-axis

Find the difference between the two possible values of k.

C_1 $(x - 5)^2 + (y - 5)^2 = 9$ $r = 3$



small circle $d = 2$, $r = 1$ $k = 1$

large circle $d = 8$, $r = 4$ $k = 4$

Difference $= 3$