Please attempt questions 1–10.

Some may be unfamiliar in style and require more thought, and on some you might get stuck. If you can't do a question, don't panic – write down your ideas, then we can discuss them and work towards a solution in the supervision. You are welcome to mail me, but you should tell me what you've tried.

However, please don't look up any solutions. I can help you if I get to see your attempts, but it's fairly useless if I just get to see someone else's answer!

1. By considering $(r+1)^3 - r^3$, derive the formula $\displaystyle\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$.

2. Use induction to prove that, for every $n \geqslant 1$,

   (a) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$;

   (b) $n^3 + 5n$ is divisible by 6;

   (c) $2^{n+2} + 3^{2n+1}$ is divisible by 7.

3. Give alternative solutions to question 2, as follows:

   (a) for 2(a), by using the formula in question 1;

   (b) for 2(b), by factorising some suitable expression;

   (c) for 2(c), by using modular arithmetic (or, if you haven't met modular arithmetic, using the result that $a - b$ divides $a^k - b^k$ for $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$).

4. *Theorem.* All sheep are the same colour.

   Proof. We will use induction to show that, for each n, any n sheep are the same colour. The base case, $n = 1$, is easy: any one sheep is the same colour as itself. Now suppose the result is true for any set of n sheep, but that we have $n+1$ sheep. Call them $s_1, \ldots, s_{n+1}$.

   By induction, we know sheep $s_1, \ldots, s_n$ are the same colour (because there are n of them), and that sheep $s_2, \ldots, s_{n+1}$ are the same colour (because there are n of them). Therefore, we deduce that sheep $s_1, \ldots, s_{n+1}$ are the same colour. So, by induction, all sheep are the same colour. $\square$

   But clearly not all sheep are the same colour. So where is the mistake?

5. The Fibonacci numbers are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

   Let F_n be the n^{th} Fibonacci number. By observing that

   $$F_{n+2} = F_{n+1} + F_n, \quad F_{n+3} = 2F_{n+1} + F_n, \quad F_{n+4} = 3F_{n+1} + 2F_n, \quad \ldots,$$

   guess a general formula for F_{n+k} in terms of F_{n+1} and F_n, and verify it by induction on k.

   Deduce that F_{kn} is a multiple of F_n for all $k \in \mathbb{N}$.

   Deduce also that $F_{n+1}^2 + F_n^2$ and $F_{n+2}^2 - F_n^2$ are Fibonacci numbers.

6. (a) Find all natural numbers n such that n divides $(n-1)!$.
   (b) Find the smallest natural number n such that $n!$ ends in (at least) 2015 zeroes.

7. (a) Prove that $\sqrt[3]{4}$ and $\log_3 4$ are irrational.
   (b) Show that if there are $m, n \in \mathbb{N}$ such that $\dfrac{m}{n} = \sqrt{11}$, then also $\dfrac{11n - 3m}{m - 3n} = \sqrt{11}$.

   By considering the size of $m - 3n$, deduce that $\sqrt{11}$ is irrational.

   Can you generalise this method? E.g., what fraction could we use for $\sqrt{111}$?

   (c) Let $\alpha, \beta, r \in \mathbb{R}$, with α, β irrational and r rational. Which, if any, of $r + \alpha$, $r\alpha$, $\alpha + \beta$, $\alpha\beta$, α^r, r^α and α^β must be irrational? Give proofs or counterexamples.

   If you give a counterexample involving irrational numbers, try to use numbers that you are able to prove are irrational.

8. Let A be a collection of $n + 1$ distinct integers chosen from the set $\{1, \ldots, 2n\}$. Prove that A must contain two numbers which are coprime, and also two numbers such that one divides the other.

9. You are asked to drive a lunar rover around the moon (which is just a circle in this question). There are (finitely many) fuel depots on the way, with the total amount of fuel stored in them enough to get around the moon exactly once. Show that there exists a depot from which you can start driving and travel the whole way around the moon, picking up fuel at each depot as you pass, without running out of fuel between depots.

10. There are six towns, such that between each pair of towns there is either a train or bus service (but not both). Prove that there are three towns that can be visited in a loop, going via no other towns, using only one mode of transport.

    Is the result still true if there are only five towns?

Additional questions

These questions are perhaps more difficult than the rest of the sheet – attempt them if they interest you, but not at the expense of other work.

11. The region in Question 10 has grown, and there are now eighteen towns. It is still the case that between each pair of towns there is either a train or bus service (but not both). Prove that there are four towns such that all six of their pairwise connections use the same mode of transport. *You might want to try twenty towns first.*

12. Is there a power of 7 that starts (in base 10) with the digits 2015...?

13. Let R be a rectangle which can be divided into smaller rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.

Please send any comments to me at glt1000@cam.ac.uk