Various interview questions.

1 Sketch the graph of $y = \sin(e^{1/x})$.
Differentiate $\ln(\sin x)$ and sketch the curve of this function.
What problem arises with this?

2 Prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$, where a and b are positive real numbers.

3 Discuss the integral $\int_2^6 \dfrac{1}{x-3}\,\mathrm{d}x$.

4 What's wrong with this "proof" that $-1 = 1$?
$\int_{-1}^{1} \dfrac{dx}{1+x^2} = \dfrac{\pi}{2}$. Put $u = 1/x$: $\int_{-1}^{1} \dfrac{dx}{1+x^2} = -\int_{-1}^{1} \dfrac{du}{1+u^2} = -\dfrac{\pi}{2}$. Hence $\dfrac{\pi}{2} = -\dfrac{\pi}{2}$ so $1 = -1$.

5 What do you understand by e^x?
Why is it written in this form?
Prove that the series converges. What assumptions have you made? How would you prove them?
Prove that the derivative is e^x. What assumptions have you made? How would you prove them?

6 The random variable R has the distribution Geo(p).
Find the expectation of R. What assumptions have you made? How would you go about justifying them?

from Caius College, Cambridge, December 2007

7 What are the roots of $z^5 = 1$?

8 Prove that $\sum r^3 = \frac{1}{4} n^2(n+1)^2$ by induction.

9 Prove that $n^5 - n$ is divisible by 5 for all $n \in \mathrm{N}$.

10 Find $\int \ln x \,\mathrm{d}x$.

11 Prove that ${}^nC_r = {}^nC_{n-r}$.

12 Prove that ${}^{2n}C_n = [{}^nC_0]^2 + [{}^nC_1]^2 + [{}^nC_2]^2 + \ldots + [{}^nC_n]^2$.

13 What is the time taken for a particle to fall 1 m from rest?

14 Sketch the curve $y = \ln x / x$.

15 Which is greater, e^π or π^e?

from Trinity College, Cambridge, December 2007

16 $a_0 = 0$, $a_n = a_{n-1} + 3$ or $a_{n-1} - 2$ for $n \geq 1$. Find the possible values of a_{100}.

17 Sketch the graphs of $y = \dfrac{x^2(1+x)}{1-x}$ and $y^2 = \dfrac{x^2(1+x)}{1-x}$.

18 $I_n = \int_0^{2\pi} |\cos^n x + \sin^n x|\, dx$. Find I_1 and I_2.

19 $A = \sqrt{2007 + \sqrt{2007 + \sqrt{2007 + \ldots}}}$

$B = \sqrt{2007 - \sqrt{2007 - \sqrt{2007 - \ldots}}}$

Without explicitly calculating A or B find $A - B$ and AB.

20 Using suitable squares and circles, show that $2\sqrt{2} < \pi \leq 4$.
Then show that $3 < \pi \leq 2\sqrt{3}$.

21 Alice tosses 3 coins; Bob tosses 2 coins. Alice wins if she has more heads than Bob. If it is a tie, then they flip one more coin to decide the winner. Find the probability that Alice wins.

22 Find the fourth digit of 1001^{46}.

23 A matrix is "stochastic" if the sum of each row and column is 1 and all entries are non-negative. A and B are stochastic. Prove that AB is stochastic.

24 A finite set of points in the plane is given. Prove that there exists a unique circle with minimal radius that contains all the points.

25 A child at the top of a downward slope of angle θ to the horizontal throws a ball at α to the horizontal. Find α as a function of θ in order to maximise the distance travelled down the slope by the ball when it lands.

from Trinity College, Cambridge, December 2009

26 How many ways are there of making £10 out of 20p, 10p and 5p coins?

27 How many solutions are there to $\dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c} + \dfrac{1}{x-d} = 0$?
Show that the sum of these solutions is ¾ $(a + b + c + d)$.

28 Which is more likely: rolling two sixes at least once when two dice are rolled 100 times, or rolling 4 sixes at least once when rolling 4 dice 3600 times?

29 I have 4 fair dice and one that always lands on a 6. I roll two of the dice and obtain a double 6. If I roll them again, what is the probability that I roll another double 6?

30 If I have a finite set of complex numbers $\{z_1, z_2, z_3, \ldots\}$ such that z_i^* and $z_i z_j$ are in the set for any i, j, prove that $|z_i| = 1$ for each i.

31 Evaluate $\int_0^{\pi/4} t\cos^2 t\,\mathrm{d}t$.

32 Prove that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta) = 1 + \tan^2\theta$.

Given that $\mathrm{F}(x) = \int_0^{\infty} \dfrac{1}{x^2 + y^2}\,\mathrm{d}y$, find F($x$) in terms of x and plot it.

Given that $\mathrm{G}(x) = \int_0^{\infty} \dfrac{1}{x^4 + y^4}\,\mathrm{d}y$ plot G(x) on the same axes as F(x).

What is the form of G(x)?

33 A cylinder of length b, radius r and mass m is rolling down a hill in such a way that its central axis is moving at speed v. What is its kinetic energy?

for Dominic Yeo…

34 Evaluate 8^{2007} mod 13.

35 Explain why positive integer solutions to the equation $x^2 + y^2 + z^2 = w^2$ which have no common factor must be of the form (x, y, z; w) = (odd, even, even; odd) for some permutation of x, y, z.

Is the corresponding result true for Pythagorean quintuples? Pythagorean octuples?

36 Factorise $x^4 + 1$ in $Z_5[x]$.
Can $x^3 + 3x + 2$ in $Z_5[x]$ be factorised?
Show that $x^2 - 1$ has two different factorisations in $Z_5[x]$.

37 Read and explain Apostol pp 228 to 230.

For Comp Sci…

38 For reasons of aesthetic efficiency, bureaucrats in Brussels have decreed that the wagons of freight trains may in future be arranged only according to the following rules.

- All trains are drawn by a single engine.
- A train consisting of a single red wagon is legal.
- A legal train may be created by attaching to the rear of a legal train an identical train (apart from its engine).
- Any three adjacent red wagons may be removed.
- A white wagon may be attached to the rear of any train ending in a red wagon.

a) Show that a train consisting of (a) 40 and (b) 41 wagons is legal.
b) Is a light engine (i.e., a train consisting of an engine only) legal?
c) What all-red trains are legal?
d) Is the combination (40 reds/white/41 reds/white) legal?
e) What can be said about the number of red wagons in any legal train?

39 I have 12 coins all looking identical. 11 weigh the same as each other; the twelfth is counterfeit and is either lighter or heavier. I have a balance on which I can place as many coins as I like on each side and it will tell me which side (if any) is heavier. How many weighings do I need to determine which is the counterfeit coin and whether it is lighter or heavier? [Answer = 3]

40 Computers work in binary – why?
Is this the most efficient number base? Assume that the cost of storing a number is proportional to the product of how many digits that number contains multiplied by the complexity of the device. Assume further that the complexity of the device is proportional to the number of different states it needs to hold. How could you then cost the storage of an *x*-digit (base 10) number in base b?

[Answer, cost roughly proportional to b/ln(b), so although base 3 is slightly cheaper than base 2 or 4 (2.73 : 2.88) base 2 is preferred for the simplicity and therefore reliability of the switches.]

CompSci interview:

41 Sketch $y = \sin(x)/x$.

42 Prove by induction that $2^{(3n+1)} + 3^{(n+1)}$ is divisible by 5 for positive integers n.

43 Identify that an iterative algorithm produced the factorial of a number. (Check the largest number possible by comparing factorial and integer tables)

44 Draw the result of some code with this syntax:
<page><rect x = 0 y = 20 w = 100 h = 40 fill></rect><rect x = 0 y = 60 w = 100 h = 40><rect x = 0 y = 0 w = 100 h = 50></rect></rect></page>
produces two identical rectangles which fill the width of the starting page but start 20% and 60% down it respectively, and fill 40% of the height each. The top rectangle is shaded, and inside the bottom one there is another rectangle which extends across its entire width and down half its height.

Maths interview:

45 A stream of cars is moving past a point with speed v. Each car has length l and the distance between each car is a. What is the number of cars which will pass the point in time t?
The safe distance a is proportional to v^2: $a = kv^2$. What should v be to maximise the number of cars passing the point?

46 Sketch $1/(1 - x^2)$

47 Two graphs have equations $(x - u)^2 + y^2 = 1$ and $y = kx$.
For what values of u will the graphs always meet?
Under what conditions of u and k will they never meet?

Engineering

48 Derive the formula $s = ut + \frac{1}{2}at^2$.
What happens when a is not constant?

49 A particle of mass m is moved along a rough horizontal floor, coefficient of friction μ, by a constant force P inclined at α to the horizontal.
(i) The force P can act either above or below the horizontal.
Which provides the greater acceleration, and why?
(ii) What is the optimal value of α?
(iii) Is this always the case?

50 Sketch the graphs of $x^{50} + y^{50} = 1$ and $x^{3/2} + y^{3/2} = 1$.
What is the parametric equation of the latter?

From St John's, October 2007

51 Sketch the graph of $y = \sin(1/x)$.

52 The function * is defined by $\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ \dfrac{b+d}{2} \end{pmatrix}$.
Is this (a) commutative, (b) associative?

53 How many digits are there in a binary number greater than 1000, or 1000000?

54 Differentiate x^{x^x}.

55 Evaluate (a) $\displaystyle\int_a^b \frac{x}{\sqrt{x^2+5}}\,dx$ (b) $\displaystyle\int_a^b \frac{x}{\sqrt{x^4+5}}\,dx$

56 Define a sequence. Define the concept 'let $n \to \infty$'.

57 Prove that any prime number consisting entirely of 1s has a prime number of digits.

58 Teach me the ideas behind calculus (as a beginner).

59 What is the volume of a triangular-based pyramid?

60 The penultimate digit of a square number is 7. What numbers can the last digit be?

61 In a mathematics examination, all candidates are of different heights, and sit in a rectangular array. The tallest candidate in each row is chosen, and of these the shortest is called P. The shortest candidate in each column is chosen, and of these the tallest is called Q. Who is taller, P or Q?

62 Solve the equation $\log_2(x) = \log_2(x + 2) + 3$. Comment.

63 Find m in terms of $k > 0$ if the area enclosed by the curve $y = kx - x^2$ is exactly bisected by the line $y = mx$.

64 Sketch the graphs of $y = \log_{½}(x)$ and $y = \log_x(½)$.

65 Evaluate $\int_0^{\ln N} \lfloor e^t \rfloor dt$.

66 Find all solutions to the equation $2^x + 25^y = z^2$.

67 f is a real-valued function on the reals such that $f(x + y) = f(x) + f(y)$.
What can be said about f? What if $f(xy) = f(x)f(y)$ as well?

68 Find all solutions of $\sqrt{x(x+2)+1} + \sqrt{x(x-2)+1} = 2$.

69 Given n fair six-sided dice, find the probability that the smallest score is 3.
Find the expected largest throw for n dice.
What does this tend to as n tends to infinity?

70 Sketch the graphs of $y = x \cot x$ and $y = \sqrt{c^2 - x^2}$.
For what range of values is there no intersection?
How many solutions are there as c increases?

71 A circle C with centre O has unit radius. Two radii OA and OB are drawn, with $\angle AOB = \alpha$. A semicircle on AB as diameter is drawn, outside the circle. The area enclosed between C and the semicircle is denoted by A.
(i) Find A as a function of α.
(ii) What happens to A when $\alpha = \pi$?
(iii) Find α such that A is rational and non-zero.
(iv) Find the value of α that maximises A.

72 All positive integers can be written as the sum or difference of λ powers of 2, where λ is a minimum. For example, $7 = 2^3 - 2^0$. Prove that no adjacent or identical powers are necessary.

73 Three lines have gradients m_1, m_2 and m_3. Their intersections form an equilateral triangle. Find the value of $m_1m_2 + m_2m_3 + m_3m_1$.

74 (i) Some functions when reflected in the x-axis or in the y-axis map onto themselves. Give an example of such a function.
(ii) A function is said to "slide" if a translation in the x-direction maps it onto itself. Give an example of such a function. Prove that no polynomial function can slide.

75 A projectile is fired at 45° at speed v on horizontal ground. Another particle is later fired from the same point at speed $2v$ in order to collide with the first particle. Assuming no air resistance, find the latest time that the second particle can be fired.

76 Work out the parametric equations of a conical helix.

77 Solve the functional equation $f(x) = 1 + x \cdot \int_{-1}^{1} f(t^2)\,dt$

77(a) Solve the functional equation $f(x) = 1 + \int_{-1}^{x} f(t)\,dt$.

78 Sketch $\sin\left(\frac{1}{1-x^2}\right)$

79 Sketch the solution to $\frac{dy}{dx} = x + y^2$.

Trinity College NatSci (Biology)

1. An ice cube is floating in a glass of water which is full to the brim with water. As the cube melts, what happens to the water level? (Obviously quite a common question. I gave a qualitative explanation but when I asked them if there was a mathematical basis for this we did it in terms of Archimedes' principle which was fairly straightforward too) .

2. Carbon-13 has a natural occurrence of 1%, deuterium one of 0.01%. In an alkane then($C_2H_2N_{+2}$), what is the probability of there being K atoms of C-13 and L atoms of deuterium?

3. In a 50cm^3 box how many oranges can we pack in it without squashing any? (I gave two different packing methods though we developed on this in the interview to face-centred cubics and determined the packing fraction)

4. Sketch $y = (\ln(1 - x))/x$ (This is the one I didn't do).

Trinity Maths for Nat Sci

5 Differentiate sin (x°).

6 If you integrate $\sin x \cos x$ you can get either $\frac{1}{2}\sin^2 x$ or $-\frac{1}{2}\cos^2 x$. Reconcile these two answers.

Exeter Oxford *(Alexei Kalveks)*:

As soon as I entered I was sat down at a table with two interviewers, beginning straightaway with some problems.
They had a few sheets of questions, each sheet clearly had problems of the same type but of varying difficulty over the sheet.

1. Find $\int \frac{1}{x^2+a^2}\,\mathrm{d}x$ (using $\tan^{-1}$ I immediately wrote it down)

   and then prove it.

2. Integrate $1/((x^2 + x + 1)(x^2 + x + 2))\,\mathrm{d}x$
   (by completing the square then splitting into partial fractions in x^2)

   other pre-A level candidates were asked easier versions like $\int \frac{1}{(x+a)(x+b)}\,\mathrm{d}x$

3. Sketch $y^2 = (1 - x^2)(4x^2 - 1)$
   (by sketching first with y and x, then replacing x with x^2, and finally y with y^2)

Overall the interview took about 50 minutes and I thought it went fairly well.

Lady Margaret Hall:

Again, I was immediately sat with two interviewers and there was one problem sheet on the table which everyone worked through.
It was fairly simple, and I experienced rapid fire easy questions rather than harder more interesting ones.
Some of the questions:

Prove $N^3 - N$ is always divisible by 6 (by factorising)

Sketch $y = x + |x - 2|$

What does $y = \ln(x + |x - 2|)$ tend to as x tends to infinity?

Differentiate $\sin(x^3)$, then integrate $x^2 \sin(x^3)$

It is given that p(girl) = ½. If I have 3 children, what is the probability they are all girls?
If at least one of them is a girl what is the new probability?

If there is a fixed length of perimeter fence, what is the shape of the field with the greatest area, and the least (no proof required, just the answer)

I can’t really remember the others on the sheet, but we definitely finished it quite quickly, so they asked me some other made up questions like:

Solve $4x + y = 6$, $3x + 2y = 7$ (this must have been the absolute easiest!)
then $4x + 2y = 8$, $2x + y = 4$ (ditto)

Prove that a 3 digit number is divisible by 3 if its digits add up to a multiple of three. Can you derive a similar expression for what will be true of the 3 digits if the number is divisible by 7?

Express 7 as the difference of two squares. Express 47 as the difference of two squares (by factorising $a^2 - b^2$). Express 48 as the difference of two squares (comically at this point neither interviewer nor me noticed that this was just 49 – 1 which was a bit embarrassing as we'd found the factors of 48 (it has quite a few which is why the interviewer chose it) and run through the method before it became clear)

TRINITY COLLEGE
ADMISSIONS QUIZ (MATHEMATICS 2)
DECEMBER 1997.

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others rather hard. You are not expected to answer all of them, or necessarily to complete questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper with you to the interview afterwards.

1. In a tennis tournament there are $2n$ participants. In the first round of the tournament, each player plays exactly once, so there are n games. Show that the pairings for the first round can be arranged in exactly $(2n-1)!/2^{n-1}(n-1)!$ ways.

2. Let L_1 and L_2 be two lines in the plane, with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Suppose that they intersect at an acute angle θ. Show that
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

3. Calculate $\displaystyle\int_0^\pi (x\sin x)^2\,dx$.

4. Of the numbers 1, 2, 3, . . . , 6000, how many are not multiples of 2, 3 or 5?

5. There is a pile of 129 coins on a table, all unbiased except for one which has heads on both sides. Bob chooses a coin at random and tosses it eight times. The coin comes up heads every time. What is the probability that it will come up heads the ninth time as well?

6. A packing case is held on the side of a hill and given a kick down the hill. The hill makes an angle of θ to the horizontal, and the coefficient of friction between the packing case and the ground is μ. What relationship between μ and θ guarantees that the packing case eventually comes to rest? Let gravitational acceleration be g. If the relationship above is satisfied, what must the initial speed of the packing case be to ensure that the distance it goes before stopping is d?

7. Let $\binom{n}{r}$ stand for the number of subsets of size r taken from a set of size n. (This is the number of ways of choosing r objects from n if the order of choice does not matter. You may be more familiar with the notation ${}^nC_r$, in which case feel free to use it.)

   Every subset of the set $\{1, 2, \ldots, n\}$ either contains the element 1 or it doesn't. By considering these two possibilities, show that
$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$
By using a similar method, or otherwise, prove that
$$\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}.$$

8. One end of a rod of uniform density is attached to the ceiling in such a way that the rod can swing about freely with no resistance. The other end of the rod is held still so that it touches the ceiling as well. Then the second end is released. If the length of the rod is l metres and gravitational acceleration is g metres per second squared, how fast is the unattached end of the rod moving when the rod is first vertical?

9. Let M be a large real number. Explain briefly why there must be exactly one root w of the equation $Mx = e^x$ with $w > 1$. Why is $\log M$ a reasonable approximation to w?
Write $w = \log M + y$. Can you give an approximation to y, and hence improve on $\log M$ as an approximation to w?

10. Twenty balls are placed in an urn. Five are red, five green, five yellow and five blue. Three balls are drawn from the urn at random without replacement. Write down expressions for the probabilities of the following events. (You need not calculate their numerical values.)
(i) Exactly one of the balls drawn is red.
(ii) The three balls drawn have different colours.
(iii) The number of blue balls drawn is strictly greater than the number of yellow balls drawn.

For James Aaronson (Trinity Cambridge)

1 Two red, two blue and two green books are arranged at random on a shelf. Find the probability that two of the same colour books are next to one another.

2 Prove that $\int_0^1 \frac{dx}{1+x^4} > \frac{3}{4}$.

3 Sketch the graphs of:
(i) $y = \sin^3 x$
(ii) $y = \sin(x^3)$
(iii) $y^2 = \sin^3 x$

4 Let $p(x) = x^3 + x^2 - 1$.
a) Show that $p(x) = 0$ has precisely one real root.
b) Show that that root is irrational.

5 The probability that horse A wins is 0.4. The probability that horse B wins is 0.25. You win £10 per winning horse. You won £10.
(i) Given that the horses were in different races, find P(B won).
(ii) Given that the horses were in the same race, find P(B won).

6 Find $\min\{n\}$ such that for all sets $0, A, A_1, A_2, \ldots, A_n$ such that there necessarily exists i and j such that $\angle A_iOA_j < 90°$.

7 Complex numbers a, b, c, d e are given. Prove that

$$\text{“}\exists\, w, z \in \square \text{ s.t. } aw^2 + bwz + cz^2 = dw + ez = 0\text{”} \Leftrightarrow \text{“}\begin{vmatrix} a & b/2 & d \\ b/2 & c & e \\ d & e & 0 \end{vmatrix} = 0\text{”}$$

8 A segment of a quarter-circle is given. A is the area below S, B is the area to the left of S. Prove that $A + B$ is independent of the position of S depending only on its length.

9 Show that $\int_0^1 (-\ln x)^n \, dx = n!$.

10 Two balls, of masses m and M, with the ball of mass m above the ball of mass M, are dropped from height h, where h >> the radii of the balls. All collisions are elastic. Find the height to which the ball with mass m rises.