THE UK UNIVERSITY INTEGRATION BEE 2023/24

$$\int$$

Round 1 Worked Answers

By: Vishal Gupta, Alfie Jones, Sharvil Kesarwani, Yuepeng Alex Yang

Answers worked and document created by Finn Hogan.

Sponsored by

Jane Street

1

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

If we set $y = \sqrt{1-x^2}$, we see that the integrand is simply a half circle with radius 1 centred at the origin: $x^2 + y^2 = 1$.
The integral represents the area under the curve so the answer is $\dfrac{\pi}{2}$.

2

$$\int \frac{1}{1-\sin(x)} + \frac{1}{1+\sin(x)}\,\mathrm{d}x$$

Not a particularly interesting trigonometric integral.

$$\begin{aligned}\int \frac{1}{1-\sin(x)} + \frac{1}{1+\sin(x)}\,\mathrm{d}x &= \int \frac{2}{1-\sin^2 x}\,\mathrm{d}x \\ &= 2\int \frac{1}{\cos^2 x}\,\mathrm{d}x \\ &= 2\int \sec^2 x\,\mathrm{d}x \\ &= 2\tan(x) + C\end{aligned}$$

3

$$\int_0^\infty 4^{-\lfloor x \rfloor}\,\mathrm{d}x$$

To evaluate problems of this form, it is usually best to split up the bounds.

$$\begin{aligned}\int_0^\infty 4^{-\lfloor x \rfloor}\,\mathrm{d}x &= \sum_{n=0}^{\infty}\int_n^{n+1} 4^{-n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} 4^{-n} \\ &= \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3}\end{aligned}$$

The evaluation of this sum is brought to us by our good friend, the geometric series.

4

$$\int 2x \, \mathrm{d}X$$

This question required advanced knowledge of capital letters.

$$\int 2x \, \mathrm{d}X = 2xX + C$$

5

$$\int_{1_{\mathbb{Q}}(e+\pi)}^{1_{\mathbb{Q}}(e\pi)} \cos\left(\pi x\right)\, \mathrm{d}x, \text{ where } 1_{\mathbb{Q}}\left(x\right) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}.$$

Such a deep and profound problem. It's unknown whether both $e + \pi$ or $e * \pi$ are irrational - only that they can't both be rational. So the bounds are either 0 to 1, 1 to 1 or 1 to 0. But since

$$\int_0^1 \cos(\pi x)\mathrm{d}x = 0$$

the integral is 0 in any of these cases.

6

$$\int_{-\infty}^{\infty} e^{-x^2+4x+1}\,\mathrm{d}x$$

After a quick rearrangement, we are lead to the solution.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2+4x+1}\,\mathrm{d}x &= \int_{-\infty}^{\infty} e^{-(x-2)^2+5}\,\mathrm{d}x \\ &= e^5 \int_{-\infty}^{\infty} e^{-(x-2)^2}\,\mathrm{d}x \\ &= e^5 \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = e^5\sqrt{\pi}.\end{aligned}$$

7

$$\int_{-1}^{1} \frac{1}{3^x+1}\,\mathrm{d}x$$

The trick to this problem is the substitution $u = -x$. Enforcing this gives us:

$$\begin{aligned}
I = \int_{-1}^{1} \frac{1}{3^x+1}\,\mathrm{d}x = \int_{-1}^{1} \frac{1}{3^{-u}+1}\,\mathrm{d}u = I \\
I + I &= \int_{-1}^{1} \frac{1}{3^x+1}\,\mathrm{d}x + \int_{-1}^{1} \frac{1}{3^{-x}+1}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{3^x+1}\,\mathrm{d}x + \int_{-1}^{1} \frac{3^x}{3^x+1}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{3^x+1}{3^x+1}\,\mathrm{d}x \\
&= \int_{-1}^{1} 1\,\mathrm{d}x = 2 \\
I &= 1.
\end{aligned}$$

8

$$\int_0^1 \sqrt{2^x\sqrt{4^x\sqrt{8^x\sqrt{16^x\sqrt{\dots}}}}}\,\mathrm{d}x$$

For this problem, the idea is to look for a series in the exponents.

$$\begin{aligned}
\int_0^1 \sqrt{2^x\sqrt{4^x\sqrt{8^x\sqrt{16^x\sqrt{\dots}}}}}\,\mathrm{d}x &= \int_0^1 \sqrt{2^x\sqrt{2^{2x}\sqrt{2^{3x}\sqrt{2^{4x}\sqrt{\dots}}}}}\,\mathrm{d}x \\
&= \int_0^1 \sqrt{2^x} \times \sqrt{\sqrt{2^{2x}}} \times \sqrt{\sqrt{\sqrt{2^{3x}}}} \times \sqrt{\sqrt{\sqrt{\sqrt{2^{4x}}}}} \times \dots\,\mathrm{d}x \\
&= \int_0^1 2^{\frac{x}{2}} \times 2^{\frac{2x}{4}} \times 2^{\frac{3x}{8}} \times 2^{\frac{4x}{16}} \times \dots\,\mathrm{d}x \\
&= \int_0^1 2^{\sum_{n=1}^{\infty} \frac{nx}{2^n}}\,\mathrm{d}x
\end{aligned}$$

With a bit of effort and reverse engineering, you may notice that the summation looks like the derivative of the geometric series evaluated at a certain point. Specifically, this thinking leads us to seek out the following summation:

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n, \qquad |x| < 1.$$

Thus, plugging $x = \frac{1}{2}$, our integrand becomes:

$$\begin{aligned}
&= \int_0^1 2^{2x}\,\mathrm{d}x \\
&= \int_0^1 4^x\,\mathrm{d}x \\
&= \frac{1}{\ln(4)}\, 4^x\Big|_0^1 \\
&= \frac{3}{2\ln(2)}.
\end{aligned}$$

9

$$\int_0^\infty (-\{x\})^{\lfloor x \rfloor}\,\mathrm{d}x, \text{ where } \{x\} \stackrel{\text{def}}{=} x - \lfloor x \rfloor.$$

Similar idea to Question 3, we will split up the intervals of integration.

$$\begin{aligned}
\int_0^\infty (-\{x\})^{\lfloor x \rfloor}\,\mathrm{d}x &= \sum_{n=0}^{\infty} (-1)^n \int_n^{n+1} \{x\}^n\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^1 \{x\}^n\,\mathrm{d}x \\
&\text{we justify the previous step by noticing that } \{x\} = \{x+n\}, \text{ where } n \in \mathbb{N}. \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^n\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln(2).
\end{aligned}$$

Where the last line is evaluated using the power series expansion of $\ln(1+x)$.

10

$$\int_0^1 \frac{\sin(a\ln(x))}{\ln(x)}\,\mathrm{d}x$$

This is a classic DUTIS problem. Define the integral as $I(a)$.

$$\begin{aligned}
I(a) = \int_0^1 \frac{\sin(a\ln(x))}{\ln(x)}\,\mathrm{d}x &\overset{u=-\ln(x)}{=} \int_0^\infty \frac{e^{-u}\sin(au)}{u}\,\mathrm{d}u \\
I'(a) &= \int_0^\infty e^{-u}\cos(au)\,\mathrm{d}u \\
&\text{after some somewhat tedious integration by parts we get} \\
&= \frac{1}{1+a^2} \\
I(a) &= I(a) - I(0) \\
&\overset{\text{FTC}}{=} \int_0^a \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \tan^{-1}(a)
\end{aligned}$$

11

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x)\, \mathrm{d}x$$

Not much needs to be said about this one

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x)\, \mathrm{d}x &= \ln(\sec x + \tan x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \ln\left(2+\sqrt{3}\right) - \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\ &= \ln\left(\frac{2}{\sqrt{3}} + 1\right).\end{aligned}$$

12

$$\int_{420}^{1672} \frac{\sqrt{\log(2023-x)}}{\sqrt{\log(2023-x)}+\sqrt{\log(69+x)}}\,\mathrm{d}x$$

This problem is obliterated by the King Property for Integrals:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x.$$

$$\begin{aligned}
I &= \int_{420}^{1672} \frac{\sqrt{\log(2023-x)}}{\sqrt{\log(2023-x)}+\sqrt{\log(69+x)}}\,\mathrm{d}x \\
&= \int_{420}^{1672} \frac{\sqrt{\log(69+x)}}{\sqrt{\log(2023-x)}+\sqrt{\log(69+x)}}\,\mathrm{d}x = I \\
I+I &= \int_{420}^{1672} \frac{\sqrt{\log(2023-x)}+\sqrt{\log(69+x)}}{\sqrt{\log(2023-x)}+\sqrt{\log(69+x)}}\,\mathrm{d}x \\
&= \int_{420}^{1672} 1\,\mathrm{d}x \\
&= 1672-420 = 1252 \\
I &= 626.
\end{aligned}$$

13

$$\int_{-1}^{0} 5\left(x^6 + x\right)^4 \, \mathrm{d}x$$

A little factorisation and the question falls apart.

$$\begin{aligned}
\int_{-1}^{0} 5\left(x^6 + x\right)^4 \, \mathrm{d}x &= \int_{-1}^{0} 5x^4 \left(x^5 + 1\right)^4 \, \mathrm{d}x \\
&= \frac{1}{5} \left(x^5 + 1\right)^5 \Big|_{-1}^{0} \\
&= \frac{1}{5}.
\end{aligned}$$

14

$$\int_0^1 (\ln(\ln(x)))^{\frac{\ln(\ln(x))}{\ln(\ln(\ln(x)))}}\,\mathrm{d}x$$

The manipulation required here is not the easiest to spot. It is very similar to the following MIT Integration Bee Qualifying problem.

$$\int x^{\frac{1}{\ln(x)}}\,\mathrm{d}x.$$

The trick with both of these problems is to replace the base of the exponent with e.

$$\begin{aligned}
\int_0^1 (\ln(\ln(x)))^{\frac{\ln(\ln(x))}{\ln(\ln(\ln(x)))}}\,\mathrm{d}x &= \int_0^1 \left(e^{\ln(\ln(\ln(x)))}\right)^{\frac{\ln(\ln(x))}{\ln(\ln(\ln(x)))}}\,\mathrm{d}x \\
&= \int_0^1 e^{\ln(\ln(\ln(x)))\times\frac{\ln(\ln(x))}{\ln(\ln(\ln(x)))}}\,\mathrm{d}x \\
&= \int_0^1 e^{\ln(\ln(x))}\,\mathrm{d}x \\
&= \int_0^1 \ln x\,\mathrm{d}x \\
&= x\ln(x) - x\Big|_0^1 \\
&= -1
\end{aligned}$$

To evaluate the limit at 0, we used *powers beat logs*.

15

$$\int_1^\infty \frac{1 + 2x \ln(2)}{x\sqrt{x4^x - 1}}\,\mathrm{d}x$$

This is was a very hard substitution problem. The substitution is motivated by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x4^x = (1 + 2x\ln(2))4^x.$$

In fact, we can do slightly better; by substituting $u = x4^x - 1$. This gives:

$$\begin{aligned}
\int_1^\infty \frac{1 + 2x \ln(2)}{x\sqrt{x4^x - 1}}\,\mathrm{d}x &= \int_1^\infty \frac{(1 + 2x\ln(2))\,4^x}{x4^x\sqrt{x4^x - 1}}\,\mathrm{d}x \\
&= \int_3^\infty \frac{1}{(u+1)\sqrt{u}}\,\mathrm{d}u \\
&\overset{u=t^2}{=} 2\int_{\sqrt{3}}^\infty \frac{1}{t^2+1}\,\mathrm{d}t \\
&= 2\tan^{-1}(t)\Big|_{\sqrt{3}}^{\infty} \\
&= 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

16

$$\int \frac{\mathrm{d}x}{x^{23}+x}$$

We will solve this question in greater generality.

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^n+x} &= \int \frac{x^{-n}}{1+x^{1-n}}\,\mathrm{d}x \\ &= \frac{1}{1-n}\ln\left|1+x^{1-n}\right|+C\end{aligned}$$

In the case $n=23$ we get that

$$\frac{1}{1-n}\ln\left|1+x^{1-n}\right|+C=-\frac{1}{22}\ln\left|1+x^{-22}\right|+C.$$

This can be rearranged to give

$$\ln(x)-\frac{1}{22}\ln\left(x^{22}+1\right)+C$$

17

$$\int_1^\infty \frac{e^{\sec^{-1}\left(\sqrt{x}\right)}}{x\sqrt{x}}\,\mathrm{d}x$$

It seems that the sensible thing to do here is substitution $u = \sec^{-1}\left(\sqrt{x}\right)$.

$$\begin{aligned}
\int_1^\infty \frac{e^{\sec^{-1}\left(\sqrt{x}\right)}}{x\sqrt{x}}\,\mathrm{d}x &= 2\int_0^{\frac{\pi}{2}} \frac{e^{-u}}{\sec^2 u}\sec u\tan u\,\mathrm{d}u \\
&= 2\int_0^{\frac{\pi}{2}} e^u \sin u\,\mathrm{d}u \\
\text{after some more tedious integration by parts...} \quad &= e^u\left(\sin u - \cos u\right)\Big|_0^{\frac{\pi}{2}} \\
&= e^{\frac{\pi}{2}} + 1
\end{aligned}$$

18

$$\int_0^1 \frac{\mathrm{d}x}{\Gamma(x)^2 + \pi\csc(\pi x)}$$

Knowledge of the identity $\Gamma(x)\,\Gamma(1-x) = \pi\csc(\pi x)$ hints at the use of the King's Property substitution. Messing around with the expressions after this leads to a solution.

$$\begin{aligned}
I &= \int_0^1 \frac{\mathrm{d}x}{\Gamma(x)^2 + \pi\csc(\pi x)} \\
&= \int_0^1 \frac{\mathrm{d}x}{\Gamma(1-x)^2 + \pi\csc(\pi x)} = I \\
I + I &= \int_0^1 \frac{\Gamma(1-x)^2 + \pi\csc(\pi x) + \Gamma(x)^2 + \pi\csc(\pi x)}{\left(\Gamma(x)^2 + \pi\csc(\pi x)\right)\left(\Gamma(1-x)^2 + \pi\csc(\pi x)\right)}\,\mathrm{d}x \\
&= \int_0^1 \frac{(\Gamma(1-x) + \Gamma(x))^2}{\Gamma(x)\,\Gamma(1-x)\,(\Gamma(x) + \Gamma(1-x))^2}\,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{\Gamma(x)\,\Gamma(1-x)} \\
&= \frac{1}{\pi}\int_0^1 \sin(\pi x)\,\mathrm{d}x \\
&= \frac{2}{\pi^2} \\
I &= \frac{1}{\pi^2}.
\end{aligned}$$

19

$$\int_0^\infty \frac{1}{x^4+4}\,\mathrm{d}x$$

There are many ways to solve this problem. The fastest I can think of is spamming identities with the Beta and Gamma functions. In particular, I'll use the following identities

$$B(a,b) = \int_0^\infty \frac{z^{a-1}}{(1+z)^{a+b}}\,\mathrm{d}z \qquad\qquad \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

to arrive at a solution.

$$\begin{aligned}
\int_0^\infty \frac{1}{x^4+4}\,\mathrm{d}x &\overset{u=\sqrt{2}x}{=} \frac{1}{2\sqrt{2}}\int_0^\infty \frac{1}{x^4+1}\,\mathrm{d}x \\
&\overset{u=x^4}{=} \frac{1}{16\sqrt{2}}\int_0^\infty \frac{x^{\frac{1}{4}-1}}{1+x}\,\mathrm{d}x \\
&= \frac{1}{8\sqrt{2}}B\left(\frac{1}{4},\frac{3}{4}\right) \\
&= \frac{1}{8\sqrt{2}} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} \\
&= \frac{1}{8\sqrt{2}} \times \frac{\pi}{\sin\left(\pi \times \frac{1}{4}\right)} \\
&= \frac{\pi}{8\sqrt{2}} \times \sqrt{2} \\
&= \frac{\pi}{8}
\end{aligned}$$

20

$$\int_{\frac{1}{4}}^{\frac{3}{4}} f\left(f\left(x\right)\right)\,\mathrm{d}x,\ \text{where } f(x) = \frac{4^x}{2+4^x}.$$

It was hard to decided whether this was my favourite problem or not. I felt that the trick was very surprising and hard to spot. Notice that the function f satisfies the following...

$$f(1-x) = 1 - f(x)$$

then the problem falls apart quickly. Applying the King Property for integrals...

$$\begin{aligned}
I &= \int_{\frac{1}{4}}^{\frac{3}{4}} f\left(f\left(x\right)\right)\,\mathrm{d}x \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} f\left(f\left(1-x\right)\right)\,\mathrm{d}x \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} f\left(1-f\left(x\right)\right)\,\mathrm{d}x \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} 1-f\left(f\left(x\right)\right)\,\mathrm{d}x = I \\
I+I &= \int_{\frac{1}{4}}^{\frac{3}{4}} 1\,\mathrm{d}x \\
I &= \frac{1}{4}.
\end{aligned}$$

21

$$\int \frac{2023^x}{2023^x + 2024^x}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{2023^x}{2023^x + 2024^x}\,\mathrm{d}x &= \int \frac{\left(\frac{2023}{2024}\right)^x}{1 + \left(\frac{2023}{2024}\right)^x}\,\mathrm{d}x \\
&= \frac{1}{\ln\left(\frac{2023}{2024}\right)} \ln\left(1 + \left(\frac{2023}{2024}\right)^x\right) + C \\
&= x - \frac{\ln\left(1 + \left(\frac{2024}{2023}\right)^x\right)}{\ln\left(\frac{2024}{2023}\right)} + C
\end{aligned}$$

22

$$\int_0^\infty \frac{x + \sin(x)}{\sqrt{e^x}}\, dx$$

This is a somewhat standard integration by parts question.

$$\begin{aligned}\int_0^\infty \frac{x + \sin(x)}{\sqrt{e^x}}\, dx &= \int_0^\infty xe^{-\frac{x}{2}}\, dx + \int_0^\infty \sin(x)\, e^{-\frac{x}{2}}\, dx \\ &= 4 + \frac{4}{5} = \frac{24}{5}.\end{aligned}$$

Knowing your Laplace transforms would save you from the tedious calculations.

23

$$\int_0^\infty \sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{3x}\right)\,\mathrm{d}x$$

This is one of the very famous Borwein Integrals. There is a very good video by 3Blue1Brown on this set of integrals. The video details a solution using the convolution theorem for Fourier transforms. Unfortunately, this is not in our syllabus so we'll provide a method using DUTIS.

$$\begin{aligned}
I(a) &= \int_0^\infty \sin\left(\frac{1}{x}\right)\sin\left(\frac{a}{x}\right)\,\mathrm{d}x \\
&\overset{u=\frac{1}{x}}{=} \int_0^\infty \frac{\sin(x)\sin(ax)}{x^2}\,\mathrm{d}x \\
I'(a) &= \int_0^\infty \frac{\sin(x)\cos(ax)}{x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \frac{\sin((a+1)x)+\sin((a-1)x)}{x}\,\mathrm{d}x \\
&= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

The above is the famous Dirichlet integral, the canonical DUTIS problem. Integrating this expression, we get:

$$\int_0^\infty \sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{3x}\right) = I\left(\frac{1}{3}\right) - I(0) = \int_0^{\frac{1}{3}} \frac{\pi}{2}\,\mathrm{d}a = \frac{\pi}{6}.$$

A nice alternative solution by ritwin is as follows. Using the product-to-sum formula, the integral is equivalent to

$$I = \frac{1}{2}\int_0^\infty \left(\cos\frac{2}{3x} - \cos\frac{4}{3x}\right).$$

Taking inspiration from this form, we consider the integral

$$J(\alpha) = \int_0^\infty \left(1-\cos\frac{\alpha}{x}\right)\mathrm{d}x.$$

Substituting $x = \frac{\alpha}{u}$, after simplification we are left with

$$J(\alpha) = \alpha\int_0^\infty \frac{\sin^2 u}{u^2}\mathrm{d}u = \alpha\cdot\frac{\pi}{2}.$$

Hence our final answer is

$$I = J(4/3) - J(2/3) = \boxed{\frac{\pi}{6}}.$$

24

$$\int_0^\infty \frac{1}{x^4 - x^2 + 1}\,\mathrm{d}x$$

We use the substitution $u = \dfrac{1}{x}$:

$$\begin{aligned}
I &= \int_0^\infty \frac{1}{x^4 - x^2 + 1}\,\mathrm{d}x \\
&= \int_0^\infty \frac{x^2}{x^4 - x^2 + 1}\,\mathrm{d}x = I \\
I + I &= \int_0^\infty \frac{x^2 + 1}{x^4 - x^2 + 1}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1 + x^{-2}}{x^2 - 1 + x^{-2}}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1 + x^{-2}}{\left(x - \frac{1}{x}\right)^2 + 1}\,\mathrm{d}x \\
&= \tan^{-1}\left(x - \frac{1}{x}\right)\Big|_0^\infty \\
&= \pi \\
I &= \frac{\pi}{2}.
\end{aligned}$$

25

$$\int_0^{e^{1+e}} \frac{W\left(W\left(x\right)\right)}{x}\,\mathrm{d}x, \text{ where } W(x) \text{ is the inverse function of } xe^x.$$

Use the substitution $u = W(x)$, twice.

$$\begin{aligned}
\int_0^{e^{1+e}} \frac{W\left(W\left(x\right)\right)}{x}\,\mathrm{d}x &= \int_0^{e} \frac{W(x)}{xe^x} \times \left(x+1\right)e^x\,\mathrm{d}x \\
&= \int_0^{e} W(x) \times \frac{1+x}{x}\,\mathrm{d}x \\
&= \int_0^{1} x \times \frac{1+xe^x}{xe^x} \times \left(1+x\right)e^x\,\mathrm{d}x \\
&= \int_0^{1} 1+x+xe^x+x^2e^x\,\mathrm{d}x \\
&= 1+\frac{1}{2}x^2+\left(x-1\right)e^x+\left(x^2-2x+2\right)e^x\Big|_0^1 \\
&= 1+\frac{1}{2}+1+e-2 = e+\frac{1}{2}
\end{aligned}$$

26

$$\int_0^\infty \frac{\tan^{-1}(x)}{x^{\frac{4}{3}}}\,\mathrm{d}x$$

Just as in question 19, we will be making use of the Gamma and Beta function identities. We start by integrating by parts:

$$\begin{aligned}
\int_0^\infty \frac{\tan^{-1}(x)}{x^{\frac{4}{3}}}\,\mathrm{d}x &= -3\,x^{-\frac{1}{3}}\tan^{-1}(x)\Big|_0^\infty + 3\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x^2}\,\mathrm{d}x \\
&\overset{u=x^2}{=} \frac{3}{2}\int_0^\infty \frac{-\frac{2}{3}}{1+x}\,\mathrm{d}x \\
&= \frac{3}{2}B\left(\frac{1}{3},\frac{2}{3}\right) \\
&= \frac{3}{2}\times\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} \\
&= \frac{3}{2}\times\frac{\pi}{\sin\left(\frac{\pi}{3}\right)} \\
&= \sqrt{3}\pi.
\end{aligned}$$

27

$$\int_0^1 \frac{x^3+x+1}{3x^2-3x+4}\,\mathrm{d}x$$

The key realisation in this problem is that the denominator does not change under the substitution $u = 1 - x$. Using this substitution on the numerator however, leads to some of the nicest cancellation of terms you'll ever see...

$$\begin{aligned}
I = \int_0^1 \frac{x^3+x+1}{3x^2-3x+4}\,\mathrm{d}x &= \int_0^1 \frac{(1-x)^3+(1-x)+1}{3\,(1-x)^2-3\,(1-x)+4}\,\mathrm{d}x \\
&= \int_0^1 \frac{3-4x+3x^2-x^3}{3x^2-3x+4}\,\mathrm{d}x = I \\
I + I &= \int_0^1 \frac{3x^2-3x+4}{3x^2-3x+4}\,\mathrm{d}x \\
&= \int_0^1 1\,\mathrm{d}x = 1 \\
I &= \frac{1}{2}.
\end{aligned}$$

28

$$\int_0^7 \left(\sqrt[3]{\sqrt{x^2+1}+x}-\sqrt[3]{\sqrt{x^2+1}-x}\right)\,\mathrm{d}x$$

Attempting to simplify the integrand we'd be tempted to cube the entire expression. In fact, the substitution $y=\sqrt[3]{\sqrt{x^2+1}+x}-\sqrt[3]{\sqrt{x^2+1}-x}$ works out quite well. Simplifying this expression is reminiscent of Cardano's method.

$$\begin{aligned}
y &= \sqrt[3]{\sqrt{x^2+1}+x}-\sqrt[3]{\sqrt{x^2+1}-x}\\
&= a-b\\
y^3 &= a^3-3a^2b+3ab^2-b^3\\
&= a^3-b^3-3ab\,(a-b)\\
&= a^3-b^3-3aby\\
&= \left(\sqrt{x^2+1}+x\right)-\left(\sqrt{x^2+1}-x\right)-3\sqrt[3]{\sqrt{x^2+1}+x}\times\sqrt[3]{\sqrt{x^2+1}-x}\times y\\
&= 2x-3\sqrt[3]{x^2+1-x^2}\times y\\
&= 2x-3y\\
x &= \frac{y^3+3y}{2}.
\end{aligned}$$

Now implementing the substitution is rather straightforward.

$$\begin{aligned}
\int_0^7 \left(\sqrt[3]{\sqrt{x^2+1}+x}-\sqrt[3]{\sqrt{x^2+1}-x}\right)\,\mathrm{d}x &= \frac{3}{2}\int_0^2 y\left(y^2+1\right)\,\mathrm{d}y\\
&= \frac{3}{2}\left(\frac{1}{4}y^4+\frac{1}{2}y^2\right)\Bigg|_0^2\\
&= 9.
\end{aligned}$$

29

$$\int_0^1 \frac{\ln\left(x+x^{-1}\right)}{x+x^{-1}}\,\mathrm{d}x$$

This problem looks quite intimidating. However, we can break it down into two seperate easier problems.

$$\begin{aligned}
\int_0^1 \frac{\ln\left(x+\frac{1}{x}\right)}{x+\frac{1}{x}}\,\mathrm{d}x &= \int_0^1 \frac{x\left(\ln\left(x^2+1\right)-\ln\left(x\right)\right)}{x^2+1}\,\mathrm{d}x \\
&= \int_0^1 \frac{x\ln\left(x^2+1\right)}{x^2+1}\,\mathrm{d}x - \int_0^1 \frac{x\ln\left(x\right)}{x^2+1}\,\mathrm{d}x \\
&= \frac{1}{4}\ln^2\left(x^2+1\right)\Big|_0^1 - \frac{1}{2}\ln\left(x^2+1\right)\ln\left(x\right)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\ln\left(1+x^2\right)}{x}\,\mathrm{d}x \\
&= \frac{1}{4}\ln^2\left(2\right) + \frac{1}{2}\int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^{2n}\,\mathrm{d}x \\
&= \frac{1}{4}\ln^2\left(2\right) + \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 x^{2n-1}\,\mathrm{d}x \\
&= \frac{1}{4}\ln^2\left(2\right) + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \\
&= \frac{1}{4}\ln^2\left(2\right) + \frac{\pi^2}{48}.
\end{aligned}$$

30

$$\int_0^1 \frac{1}{1-x}\sqrt{\frac{\left\{\frac{1}{x}\right\}}{1-\left\{\frac{1}{x}\right\}}}\,\mathrm{d}x$$

This is my favourite problem in the Bee. We make use of a nice trick we made use of earlier: $\{x\} = \{x+1\}$. We'll see that the scary part of the integral plays no part in its evaluation. I like to think of this integral is almost a telescoping summation.

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\sqrt{\frac{\left\{\frac{1}{x}\right\}}{1-\left\{\frac{1}{x}\right\}}}\,\mathrm{d}x &= \int_1^\infty \frac{1}{x\,(x-1)}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_1^\infty \left(\frac{1}{x-1}-\frac{1}{x}\right)\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_1^\infty \frac{1}{x-1}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x - \int_1^\infty \frac{1}{x}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_1^\infty \frac{1}{x-1}\sqrt{\frac{\{x-1\}}{1-\{x-1\}}}\,\mathrm{d}x - \int_1^\infty \frac{1}{x}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{x}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x - \int_1^\infty \frac{1}{x}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\sqrt{\frac{\{x\}}{1-\{x\}}}\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1-x}} \\
&= 2\arcsin\left(\sqrt{x}\right)\Big|_0^1 \\
&= \pi.
\end{aligned}$$

References