TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS 2)

DECEMBER 1999.

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others rather hard. You are not expected to answer all of them, or necessarily to complete questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper with you to the interview afterwards.

1. Sketch the curve defined by the equation

$$y = \sqrt{|x|} + \frac{1}{x^2 - 1}.$$

Sketch the three dimensional surface defined by

$$x^2 + y^2 - z^2 = 1.$$

2(a). A plane cuts through a sphere of radius r at distance a from its centre. Find the volumes of the two pieces of the sphere on either side.
(b) Two spheres (of radii r and a, with $r < 2a$) meet in such a way that the centre of the one of radius r lies on the surface of the one of radius a. Find the volume of the intersection.

3. Each week, the chance of winning a prize with your Premium Bond is 1 in 14,000. Mr. Optimist buys 14,000 Premium Bonds and claims he is on to a certain winner. Explain to him why that is not quite the case, and find out roughly what his chances are of getting a winner in a given week.

Hint: if you are unfamiliar with how to approximate the expression you come up with, try taking the logarithm of something and using the Taylor series for log(1+h)

4. (a) Simplify the expression $(m+n)^2 - (m-n)^2$.
(b) A Pythagorean triple is a trio of integers a, b and c with $a^2+b^2 = c^2$. It is a fundamental triple if a, b and c have no common factor. Find a systematic way of constructing infinitely many distinct fundamental Pythagorean triples. Explain why your method works.

5. Little Johnny is a biologist tracking the movements of deer with a radio homing device. He has two listening posts, located at points $(X, 0)$ and $(0, Y)$ in the plane, and periodically they give simultaneous, accurate bearings (θ_n, ϕ_n) (let's say, anticlockwise from the X axis) from the listening posts to the deer. Johnny wants to calculate the deer's positions (x_n, y_n) in cartesian coordinates. He has hitherto done this by spreading the map of the area on his kitchen table and using rulers, protractors and Stuff.

Devise a couple of trignometric formulae so he can do the whole thing on his computer.

Note: this problem was given to one of your interviewers, Dr. Read, in real life. Thousands of points had been done by his predecessor using the kitchen table method, when "Johnny" took over the project and the bright idea of consulting a mathematician occurred to him.

6. A honeycomb consists of a large regular lattice of hexagonal cells. Starting from a cell in the middle of the honeycomb, a bee wanders around in the following manner: Each second it can either stay where it is or climb into one of the 6 neighbouring cells.

How many possibilities are there for the location of the bee after 1 second, 2 seconds, and 10 seconds? How do your answers change if the bee is not allowed to rest but must change cells every second?

7. An unwise person defines a "positive" matrix to be one such that all its eigenvalues are positive. Find two 2 by 2 matrices A and B such that A and B are "positive", but $A+B$ isn't. Is it possible for $A+B$ to be "negative"?

8. Sketch the regions of the complex plane defined by (a)

$$\mathrm{Re}\,\frac{1+z}{1-z} < 0$$

and (b)

$$\mathrm{Re}\,\log\,\frac{1+z}{1-z} < 0.$$

9. A spiral is defined in polar coordinates by the equation $r = e^{k\theta}$, where $k > 0$. A second spiral is defined by $r = e^{k(\theta+\theta_0)}$. Find the area between the two spirals, for $\theta = -\infty$ to 0, i.e. the area of the region $e^{k\theta} < r < e^{k(\theta+\theta_0)}$, $\theta = -\infty$ to 0.

10. In a computer graphics program it is desired to check if two lines in three dimensions meet, or at least nearly meet. To this effect it is necessary to calculate the distance between the lines at the place where they come closest to one another. Discuss how this can be done.

11. [for Maths with Computer science candidates] An optimistic programmer writes code that is supposed to sort N numbers into order. It works by repeating the following two steps

(a) Select a random number (k) in the range 1 to N and exchange the two value at positions 1 and k;

(b) Check the resulting arrangement and see if it is now in the correct order.

The program stops if step (b) discovers that the numbers are indeed sorted. Are there any initial arrangements of input data where this method can never manage to sort things, even if the random numbers turn out to be "lucky"? If so, find an example of such an arrangement and show why it causes trouble. If there are no impossible initial situations, what is the smallest number of passes (ie uses of stages (a) and (b) above) that are guaranteed to be sufficient to get data sorted (again assuming that the random numbers conspire to avoid waste)?

If the numbers start of in exactly the opposite order from the one they must end up in show how the method might manage to get them sorted using no more than around $2N$ passes.

12. [for Maths with Computer science candidates] Five suspicious interviewers need to come to a joint decision about whether to make a conditional offer to a certain candidate. Four of them are stable, reliable, truthful and self-consistent, but one (and nobody knows which) is prepared to do **anything** that could possibly disrupt the decision making process. Of course if the group can detect which one has been lying then they can discount all further statements from that quarter and do well. All communication between the five committee members is by phone — and each phone call links only two members (but those two will always know who they are talking to: impersonation is not possible). Phone calls can not be tape recorded, so frequently people will report onwards what they heard from other committee members — or maybe what they **say** they heard!

The group will count it a success if at least all the reliable members end up with the same understanding of their joint decision, and if this agreement is in line with any majority view of those four. If, at first, the four reliable members were split two for and two against the applicant it does not matter which way the decision goes, provided all four end up voting together at the full College meeting and all four report the same way to the relevant enquiry from a certain Head Master. The joker counts it a success if either no decision is come to, or if all the others think they have agreed but they end up contradicting each other in public.

Is there a strategy that reliable and logical (mathematical?) admissions officers can use to come to an agreement, or is there a way that the joker can retain at least some chance of wreaking havoc?
[Note that this problem may have practical applications when multiple collaborating but potentially unreliable computers take joint control of a factory, aeroplane or set of telephone switchboards.]

13. [for Maths with Physics candidates] Consider a mass m at position $x(t)$ on a rough horizontal table attached to $x = 0$ by a spring that exerts a force $-kx$. The force f due to friction between the table and the mass is given by

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \leq f \leq F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0 \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves then the maximum distance from the origin decreases by $2F/k$ per half cycle.

14. [for Maths with Physics candidates] A rope is wrapped M whole turns round a cylindrical post, the two ends of the rope going in opposite directions. The coefficient of friction between rope and post is 0.25. It is desired that by pulling with a force of $1N$ on one end of the rope, I can prevent the rope from moving away from me even if a force of $10^6 N$ is applied to the other end. How large does M have to be? (Note that to 3 significant figures, $\log_e(10^6) = 13.8$).

Hint: Let the tension in the rope decline like $T(\theta)$ *with the angle* θ *round the post. Investigate* $T(\theta + \delta\theta) - T(\theta)$.