TMUA Practice - Sequences and Binomial

1) The first three terms of an arithmetic series are $(m + 1)$, $(m^2 + m)$ and $(3m^2 - m - 4)$, where m is a positive constant. Find the 21st term of the series.

(a) 0 (b) 172 (c) 164 (d) 84 (e) 64

$m^2+m-m-1 = 3m^2-m-4-m^2-m$

$0 = m^2-2m-3$

$(m-3)(m+1)=0$

$m=3$

$a=4$

$d=8$

$4, 12, 20$

$u_{21} = 4+20(8)$

$= 164$

2) The sum to infinity of a geometric series is 3 times as large as its first term, and the third term of the same series is 40. Find the first term of the series.

(a) 90 (b) $\frac{2}{3}$ (c) $\frac{40}{3}$ (d) 360 (e) $\frac{125}{2}$

$S_\infty = \frac{a}{1-r} = 3a$

$u_3 = ar^2 = 40$

$a = 3a - 3ar$

$3r = 2$

$r = 2/3$

$a\left(\frac{4}{9}\right) = 40$

$a = 90$

3) A geometric series G , whose first term is a and common ratio is r , has a sum to infinity of 128. A geometric series G′ , with first term a and common ratio $3r$ has a sum to infinity of 384 . Find the first term of these series.

1152
128
1024

(a) 90 (b) 48 (c) 60 (d) 96 (e) 32

$\frac{a}{1-r} = 128$

$a = 128 - 128r$

$\frac{a}{1-3r} = 384$

$a = 384 - 1152r$

$128 - 128r = 384 - 1152r$

$1024r = 256$

$r = 1/4$

$\frac{a}{3/4} = 128$

$a = 128 \times \frac{3}{4}$

$= 96$

4) The 2nd, 3rd and 9th terms of an arithmetic progression are three consecutive terms of a geometric progression. Find the common ratio of the geometric progression.

(a) $\frac{5}{4}$ (b) $-\frac{5}{4}$ (c) 4 (d) -2 (e) 6

$a+d, \ a+2d, \ a+8d$

$r = \frac{a+2d}{a+d} = \frac{a+8d}{a+2d}$

$a^2+4ad+4d^2 = a^2+9ad+8d^2$

$4d^2+5ad = 0$

$4d = -5a$

$d = -\frac{5}{4}a$

$r = \frac{a-\frac{5}{2}a}{a-\frac{5}{4}a}$

$= \frac{1-5/2}{1-5/4}$

$= \frac{3/2}{1/4} = 6$

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5) Three numbers A, B, C are the first three terms of a geometric progression. Given that A, 2B, C are in arithmetic progression determine the common ratio of the geometric progression.

(a) ± 2 (b) $2 \pm \sqrt{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$ (e) $1 \pm \sqrt{3}$

$r = \frac{B}{A} = \frac{C}{B}$ $\quad 2B - A = C - 2B$ $\quad 4B = A + C$

$B^2 = AC$

$B^2 = A(4B - A) = 4AB - A^2$

$(B^2 - 4AB + A^2) = 0$

$\frac{B^2}{A^2} - \frac{4B}{A} + 1 = 0$

$\frac{B}{A} = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$

or $B = \frac{4A \pm \sqrt{16A^2 - 4A^2}}{2}$

$= 2A \pm \sqrt{3}A$

$A = \frac{4B \pm \sqrt{16B^2 - 4B^2}}{2}$

$= 2B \pm \sqrt{3}B$

6) A sequence (p_n) has first term $p_1 = k^2$ and subsequent terms defined by $p_{n+1} = kp_n$ for $n \geq 1$. What is the product of the first 12 terms of the sequence?

(a) k^{13} (b) $12 + k^{13}$ (c) k^{90} (d) k^{91} (e) $12k^{13}$

$p_1 = k^2$

$p_2 = k^3$

$p_3 = k^4$

$p_{12} = k^{13}$

$k^{(2+3+\dots+13)}$

$k^{(6 \times 15)} = k^{90}$

7) The sequence (a_n) where $n \geq 0$, is defined by $a_0 = \frac{1}{2}$ and $a_n = \sum_{r=0}^{n-1} a_r$ for $n \geq 1$

Find the sum $\sum_{r=0}^{\infty} \frac{1}{a_r}$

$a_1 = a_0 = \frac{1}{2}$

$a_2 = a_0 + a_1 = \frac{1}{2} + \frac{1}{2} = 1$

$a_3 = \frac{1}{2} + \frac{1}{2} + 1 = 2$

$a_4 = 4$

(a) 3 (b) 5 (c) $\frac{8}{3}$ (d) 6 (e) $\frac{32}{5}$

$\sum_{0}^{\infty} 2 + 2 + 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \dots}_{1}$

8) The sequence (x_n) is defined by $x_{n+1} = \frac{x_n}{x_{n-1}}$ for $n \geq 2$ with $x_1 = 6$ and $x_2 = 3$

What is the value of x_{2023} ?

(a) 2 (b) 6 (c) $\frac{1}{3}$ (d) 3 (e) $\frac{3}{2}$

$x_3 = \frac{3}{6}$ $\quad x_4 = \frac{1/2}{3} = \frac{1}{6}$ $\quad x_5 = \frac{1/6}{1/2} = \frac{1}{3}$ $\quad x_6 = \frac{1/3}{1/6} = 2$ $\quad x_7 = \frac{2}{1/3} = 6$

$x_1 = x_7$ period 6

$2023 \div 6 = 337 \text{ r } 1$

$x_{2023} = x_1 = 6$

$a = 1 \quad r = 1/6$ $\qquad\qquad$ $a = \frac{1}{2} \quad r = \frac{1}{6}$

$1 + \frac{1}{6} + \frac{1}{36} + \ldots \quad S_n = \frac{1(1-(1/6)^n)}{5/6}$ $\qquad$ $\frac{1}{2} + \frac{1}{12} + \frac{1}{72}$ $\qquad$ $S_n = \frac{1}{2}\frac{(1-(1/6)^n)}{5/6}$

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9) What is the sum of the first $2n$ terms of the following series:

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \frac{1}{72} + \ldots$$

(a) $\frac{1}{6^n}$ (b) $\frac{9}{5}(1 - \frac{1}{6^n})$ (c) $\frac{7}{2^n}$ (d) $\frac{1}{5}(1 + \frac{1}{6^n})$ (e) $\frac{1}{2^n} + \frac{1}{3^n}$

$$S_{2n} = \frac{6}{5}\left[1 - \frac{1}{6^n} + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{6^n}\right)\right] = \frac{6}{5}\left[\frac{3}{2} - \frac{3}{2}\left(\frac{1}{6^n}\right)\right] = \frac{9}{5}\left[1 - \frac{1}{6^n}\right]$$

10) A sequence (u_n) is defined by $u_n = (-1)^{n+1}n$ for $n \geq 1$. Let $w_n = \sum_{r=1}^{n} u_r$

For which value of n is $w_n = 500$ $\quad$ n odd

(a) 999 (b) 500 (c) 499 (d) 1000 (e) 501

$u_1 = 1$

$u_2 = -2$

$u_3 = 3$

$u_{n-2} = n-2$

$u_{n-1} = -(n-1)$

$u_n = n$

$$w_n = \underbrace{1-2}_{-1} + \underbrace{3-4}_{-1} + 5 - 6 + \ldots + \underbrace{u_{n-2} + u_{n-1}}_{-1} + u_n = 500$$

$$= (-1)\left(\frac{n-1}{2}\right) + n = 500$$

$$-\frac{1}{2}n + \frac{1}{2} + n = 500 \qquad -n + 1 + 2n = 1000$$

$$n = 999$$

11) The sequence (a_n) is defined by $a_{n+2} = \frac{a_{n+1}}{a_n}$ for $n \geq 1$ with $a_1 = x$ and $a_2 = y$

What is the period of this sequence?

(a) 4 (b) 5 (c) 6 (d) 8 (e) the sequence is not periodic

$\frac{y}{x}$ $\quad$ $\frac{y/x}{y} = \frac{1}{x}$ $\quad$ $\frac{1/x}{y/x} = \frac{1}{y}$ $\quad$ $\frac{1/y}{1/x} = \frac{x}{y}$ $\quad$ $\frac{x/y}{1/y} = x$

a_3 $\quad$ a_4 $\quad$ a_5 $\quad$ a_6 $\quad$ $a_7 = a_1$

12) For what value(s) of k does the sequence $a_{n+1} = \frac{k(a_n + 2)}{a_n}$ with $a_1 = 2$, have period 3?

(a) -2 (b) 1 (c) -2, or 1 (d) 2 (e) all even values of k

$a_2 = \frac{4k}{2} = 2k$ $\qquad$ $a_3 = \frac{k(2k+2)}{2k}$ $\qquad$ $a_4 = \frac{k(k+3)}{k+1} = 2$

$= k + 1$

$k^2 + 3k = 2k + 2$

$k = -2 \quad 2, -4, -1, 2$

$k^2 + k - 2 = 0$

$k = 1 \quad 2, 2, 2, 2$

$(k+2)(k-1) = 0$

13) What is the coefficient of x^2 in the expansion of $(2-x^2)\left[(1+2x+3x^2)^6-(1+2x^3)^4\right]$

(a) 18 (b) 36 (c) 120 (d) 156 (e) 3^6

$(2-x^2)\left[1+6(2x+3x^2)+15(2x+3x^2)^2+\ldots-(1+8x^3+\ldots\right.$

x^2: $2(18+60)-1(1-1)$ 156

14) Find the coefficient of x^2 in the expansion of $(3x^2-x+1)^7$

(a) 21 (b) 42 (c) -21 (d) 10 (e) -3

$(1+3x^2-x)^7$

$1+7(3x^2-x)+21(3x^2-x)^2+\ldots$

$21 + 21 = 42$

15) Find the coefficient of x in the series expansion of $(1+\frac{2}{x})^2(1+\frac{x}{2})^7$

(a) 42 (b) 21 (c) $\frac{35}{2}$ (d) $\frac{7}{2}$ (e) 1

$\left(1+\frac{4}{x}+\frac{4}{x^2}\right)\left(1+\frac{7x}{2}+\frac{21}{4}x^2+\binom{7}{3}\left(\frac{x}{2}\right)^3\right.$

$7\times\cancel{6}\times5$ $\frac{35}{8}x^3$

$\frac{7}{2}+21+\frac{35}{2}=42$

16) Find the coefficient of x^5 in the series expansion of $(1-x)^5(1+x)^6$

(a) 1 (b) 5 (c) 10 (d) 15 (e) 30

$(1+x)(1-x^2)^5$ $\leftarrow$ only even powers of x need x^4

$\binom{5}{2}(-x^2)^2$

$10x^4$ 10