TMUA Proof

Syllabus

Logic of arguments (true/false, and/or, if, then, not) converse, contrapositive, necessary sufficient; quantifiers; mathematical proof (deduction, proof by cases, proof by contradiction, disproof by counterexample); identifying errors in proofs.

Logic of Arguments

1a. Consider the following statement: "If it is my birthday, I will eat some cake"

What conclusion can I draw from each of the following statements: birthday $\Rightarrow$ cake

i) It is my birthday — I will eat cake

ii) It is not my birthday — No conclusion - might eat cake / might not

iii) I eat some cake — No conclusion - might eat cake on another day so could be my birthday but might not be

iv) I do not eat some cake — It is not my birthday (because if it was my birthday I would eat cake)

1b. Consider the following statement: "If it rains the ground will get wet"

What conclusion can I draw from each of the following statements: rain $\Rightarrow$ wet

i) The ground is wet — No conclusion - might be wet from rain or from a sprinkler

ii) The ground is not wet — It has not been raining

iii) It is raining — The ground will get wet

iv) It is not raining — No conclusion - ground might be wet or dry

1c. Consider the following statement: "If I am in Paris, then I am in France"

What conclusion can I draw from each of the following statements:

i) I am in Paris — I am in France

ii) I am in France — no conclusion - might be in Paris or in Lille...

iii) I am in London = I am not in France $\Rightarrow$ I am not in Paris

iv) I am at the Eiffel Tower $\Rightarrow$ I am in Paris $\Rightarrow$ I am in France

1d. Consider the following statement: "If a shape is a square, then it is a quadrilateral"

What conclusion can I draw from each of the following statements:

i) The shape is a square — it is a quadrilateral

ii) The shape is a quadrilateral — no conclusion - might be a square or a parallelogram...

iii) The shape is not a quadrilateral — it is not a square

iv) The shape is a rhombus — no conclusion

An if/then statement is called a **conditional statement**.

'**if**' is the hypothesis, and '**then**' is the conclusion.

It tells us what happens if the hypothesis is true, but doesn't tell us anything about what happens if the hypothesis is false.

Example: **If** it is raining **then** I will wear a coat.

This tells me that if it rains, I will wear a coat, but if it is not raining I may or may not wear a coat.

The **converse** of a statement swaps the hypothesis and conclusion and is *not always true*.

If I wear a coat, then it is raining (eg I might wear a coat when it is cold but not raining).

However the original statement does tell me something when I am not wearing a coat. In this case it can't be raining, because if it was I would wear a coat which is a contradiction. Therefore I can say:

If I am not wearing a coat, then it is not raining.

This is called the ***contrapositive*** and is always logically equivalent to the original statement. It is obtained by negating both hypothesis and conclusion and swapping them.

2. Write the contrapositive of the following statements:

a) If I have enough money, I will go on holiday

I won't go on holiday if I don't have enough money

b) If I pass my driving test, I will get my driving licence

I won't get my driving licence if I don't pass my test

c) Ben will not go to school only if he is sick

(no school $\Rightarrow$ sick) If Ben is not sick, he will go to school

d) If you do not study, you will not do well in your exams

If you do well in your exams, then you did study

e) I wear a hat if it is sunny

(= If sunny, then hat) If I don't wear a hat, then it's not sunny

3. Write the contrapositive of the following mathematical statements:

a) If an integer is not equal to 2, then it is not an even prime

If even prime, then equal to 2

b) If a shape has 4 sides, it is a quadrilateral

If not a quadrilateral, then doesn't have 4 sides

c) A number is even only if the square of the number is even

If square is not even, then number not even OR If square is odd, then number is odd

d) $f(a) > 0$ **if** $a > 0$

(= If $a > 0$ then $f(a) > 0$) If $f(a) \leq 0$ then $a \leq 0$

e) $a^2 < a$ is sufficient for $a < 1$

(= If $a^2 < a$ then $a < 1$) If $a \geq 1$ then $a^2 \geq a$

The following statements are all equivalent:

If an animal is a zebra, **then** it has stripes

All zebras have stripes

Any zebra has stripes

Being a zebra **implies** having stripes

An animal is a zebra **only if** it has stripes

An animal has stripes **if** it is a zebra

An animal with **no** stripes is **not** a zebra

Being a zebra is a **sufficient** condition for an animal to have stripes

Having stripes is **necessary** for an animal to be a zebra

Sufficient If P is sufficient for Q, then Q must happen if P happens, so we have '**if P then Q**'

Necessary If P is necessary for Q, then Q can't happen without P, so we have '**if Q then P**'

4. Rewrite the following true statements in the form **If… Then …**

a) The ground gets wet when it rains If it rains, then the ground gets wet

b) All mammals have hair If it is a mammal, then it has hair

c) I always go to bed when I am sick If I am sick, then I go to bed

d) A fruit is yellow if it is a banana If it's a banana, then it is yellow

e) I am in Paris only if I am in France If I'm in Paris, then I'm in France

5. Rewrite the following true mathematical statements in the form **If… Then …**

a) Any rectangle is a quadrilateral If it's a rectangle, then it's a quadrilateral

b) Any triangle has 3 sides If it's a triangle, then it has 3 sides

c) The number 2 is the only even prime number If it's an even prime number, then it's 2
If it's 2, then it's an even prime number

d) $x > 10$ if $x > 100$ If $x > 100$, then $x > 10$

e) $k < 1$ when $k^2 < 1$ If $k^2 < 1$, then $k < 1$ (converse is not true)

f) $p^2 < p$ only if $p < 1$ If $p^2 < p$, then $p < 1$ (converse is not true)

The following statements are all equivalent

If $a < b$ **then** $f(a) < f(b)$	If P then Q
$a < b$ **implies that** $f(a) < f(b)$	P implies Q
$a < b$ **only if** $f(a) < f(b)$	P only if Q
$a < b$ **is sufficient for** $f(a) < f(b)$	P is sufficient for Q
$f(a) < f(b)$ **if** $a < b$	Q if P
$f(a) < f(b)$ **is necessary for** $a < b$	Q is necessary for P
If $f(a) \geq f(b)$ **then** $a \geq b$	If 'not Q' then 'not P' (*contrapositive*)
$f(a) \geq f(b)$ **implies** $a \geq b$	'Not Q' implies 'not P'
$f(a) \geq f(b)$ **only if** $a \geq b$	'Not Q' only if 'not P'
$f(a) \geq f(b)$ **is sufficient for** $a \geq b$	'Not Q' is sufficient for 'not P'
$a \geq b$ **if** $f(a) \geq f(b)$	'Not P' if 'not Q'
$a \geq b$ **is necessary for** $f(a) \geq f(b)$	'Not P' is necessary for 'not Q'

The **contrapositive** is always logically equivalent to the original statement. It is obtained by negating both 'if' and 'then' parts of the statement and swapping them.

The **converse** of a statement is found by swapping the 'if' and 'then' parts of the statement, but does not always result in a true statement.

If I am in London, then I am in England	This is TRUE
If I am in England, then I am in London	Not necessarily true - I could be in Bristol

The **converse is true** when the original statement is an 'if and only if' statement

If I am in London, then I am in the capital of England	This is TRUE
If I am in the capital of England, then I am in London	This is TRUE

Therefore I am in London **if and only if** I am in the capital of England

How to answer: Is P sufficient for Q?

Look for a counter example - an instance where P is true but Q is not true

Or look for proof that P implies Q (if P then Q is true)

How to answer: Is P necessary for Q?

Look for a counter example - an instance where Q is true but not P

Or look for proof that Q implies P

6. Complete the statements with one of the following:

$\Leftarrow$ $\Rightarrow$ $\Leftrightarrow$ $\not\Leftrightarrow$

'necessary' 'sufficient' 'necessary and sufficient' 'not necessary and not sufficient'

N S N & S not N & not S

a) being a bird is a <u>N & S</u> condition for having feathers

(only birds have feathers and all birds have feathers)

b) being a robin is a <u>S</u> condition for being a bird (robin $\Rightarrow$ bird)

c) having 2 legs is a <u>N</u> condition for being a bird (bird $\Rightarrow$ 2 legs)

d) having feathers is a <u>not N / not S</u> condition for being able to fly

(penguins can't fly / bats don't have feathers)

e) being an odd number is a <u>N</u> condition for being a prime number greater than 10

prime > 10 $\Rightarrow$ odd

f) being greater then 20 is a <u>S</u> condition for being a greater than 10

$x > 20 \Rightarrow x > 10$

$x > 10 \not\Rightarrow x > 20$ eg $x = 12$

g) being a rectangle is a <u>N</u> condition for being a square

square $\Rightarrow$ rectangle

h) $x^2 = 1$ is a <u>N</u> condition for $x = 1$

$x = 1 \Rightarrow x^2 = 1$

$x^2 = 1 \not\Rightarrow x = 1$ as x could be -1

i) $x^2 < 5$ is a <u>S</u> condition for $x^2 < 10$

$x^2 < 5 \Rightarrow x^2 < 10$

$x^2 < 10 \not\Rightarrow x^2 < 5$ eg $x^2 = 6$

j) $x^2 < 1$ is a <u>N & S</u> condition for $-1 < x < 1$

k) $ab < ac$ is a <u>not N & not S</u> condition for $b < c$

not N eg $a = 0$

not S eg $a = -1$

7. Are the following statements true or false? Find counterexample

a)	$x > 5$	if	$x > 10$	True. If $x > 10$ then $x > 5$
b)	$x < 8$	only if	$x < 3$	False eg $x = 5$ (Converse is true)
c)	x is even	if and only if	$(x + 1)$ is odd	True
d)	$ab = ac$	✓ if and only ✗ if	$b = c$	'only if' is False $a = 0$ | 'If' is True $b = c \Rightarrow ab = ac$ so False
e)	$a^2 < a$	if	$a < 1$	False $a = -2$
f)	$a^2 < a$	only if	$a < 1$	True (This is converse of (e))
g)	$a^2 < a$ $a(a-1) < 0$, $0 < a < 1$	if and only if	$-1 < a < 1$	False $a = -\frac{1}{2}$
h)	an even number is prime	if and only if	it is 2	True
i)	an odd number is prime	if and only if	it is 3	False eg 7
j)	a triangle is equilateral	if and only if	all it angles are 60°	True
k)	a triangle is isosceles	if	it is equilateral	True

TMUA Style Multiple Choice Practice ‘if…then’ and equivalence

1. a and b are real numbers and f is a function.
Given that exactly one of the following statements is true, which one is it?

Let $P = a < b$
$Q = f(a) < f(b)$

A **If** $a < b$ **then** $f(a) < f(b)$ — If P then Q

B $a < b$ **only if** $f(a) < f(b)$ — P only if Q = If P then Q

C $f(a) < f(b)$ is **sufficient** for $a < b$ — Q is sufficient for P = If Q then P

D $f(a) < f(b)$ is **necessary** for $a < b$ — Q is necessary for P = If P then Q

2. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $\overset{P}{a > 0}$ **then** $\overset{Q}{f(a) > 0}$ — If P then Q

B $a > 0$ **only if** $f(a) > 0$ — P only if Q = If P then Q

C $a > 0$ is **sufficient** for $f(a) > 0$ — P sufficient for Q = If P then Q

D $a > 0$ is **necessary** for $f(a) > 0$ — P necessary for Q = If Q then P

3. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $\overset{P}{f(a) > 0}$ **then** $\overset{Q}{|a| < 1}$ — If P then Q

B $f(a) > 0$ **if** $|a| < 1$ — P if Q = if Q then P

C $|a| < 1$ **only if** $f(a) > 0$ — Q only if P = if Q then P

D $|a| < 1$ is **sufficient** for $f(a) > 0$ — Q sufficient for P = if Q then P

4. Consider the four options below about a particular statement:

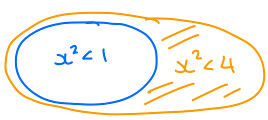
A The statement is true if $x^2 < 1$ — If $x^2 < 1$ then statement is true

B The statement is true if and only if $x^2 < 1$ — $\Rightarrow$ B + A true ✗

C The statement is true if $x^2 < 4$ — If $x^2 < 4$ then statement is true $\Rightarrow$ A also true (*)

D The statement is true if and only if $x^2 < 4$ — $\Rightarrow$ D + C true ✗

Given that exactly one of these options is correct, which one is it?



(*)
C: If inside then statement is true
$\Rightarrow$ A is true as is enclosed by
A: If inside then statement is true
$\Rightarrow$ C may not be true if in

P: even prime   Q: $x = 2$

5. Given that exactly one of the following statements is true, which one is it?

A x is not an even prime **only if** $x = 2$ not P only if Q = If not P then Q = If not Q then P

B **if** x is an even prime, **then** $x \neq 2$ if P then not Q

C $x \neq 2$ is **sufficient** for x to be an even prime   not Q sufficient for P = if not Q then P

D $x \neq 2$ is **necessary** for x to be an even prime   not Q necessary for P = if P then not Q

E $x = 2$ **if and only if** x is not an even prime   if not P then Q / if Q then not P

F x is not an even prime **only if** $x \neq 2$ not P only if not Q = if not P then not Q = if Q then P

6. f is a function and a *is* a real number.   P: $a \le 0$ Q: $f(a) \le 0$
Given that exactly one of the following statements is true, which one is it?

A $a \le 0$ **only if** $f(a) \le 0$ P only if Q : if P then Q

B $f(a) > 0$ **if** $a > 0$ not Q if not P : if not P then not Q : if Q then P

C $f(a) > 0$ is **sufficient** for $a > 0$ not Q sufficient for not P : if not Q then not P : if P then Q

D $f(a) \le 0$ is **necessary** for $a \le 0$ Q necessary for P : if P then Q

E **If** $f(a) > 0$ **then** $a > 0$ if not Q then not P : if P then Q

F $a > 0$ **if** $f(a) > 0$ not P if not Q : if not Q then not P : if P then Q

7. f is a function and a, b are real numbers.
Given that exactly one of the following statements is true, which one is it?

A $f(a) \ge f(b)$ if and only if $a \ge b$ If $f(a) \geq f(b)$ then $a \geq b$ / If $a \geq b$ then $f(a) \geq f(b)$ * same

B $f(a) \ge f(b)$ only if $a < b$ If $f(a) \geq f(b)$ then $a < b$

C $f(a) < f(b)$ if $a \ge b$ If $a \geq b$ then $f(a) < f(b)$ contrapositive

D $a \ge b$ if $f(a) \ge f(b)$ If $f(a) \geq f(b)$ then $a \geq b$ *

E $a < b$ only if $f(a) \ge f(b)$ If $a < b$ then $f(a) \geq f(b)$ contrapositive

F $a < b$ only if $f(a) < f(b)$ If $a < b$ then $f(a) < f(b)$

$P \Rightarrow Q$
not $Q \Rightarrow$ not P

A, D equivalent
B, C   D, F   contrapositive

Quantifiers

The words 'all', 'some, 'none' are examples of quantifiers. These tell us how many instances satisfy the statement, and a statement containing one or more of these words is called a **quantified statement**.

In English there are many ways to write these statements:

All: All / Every / Each / Any / If

All even numbers are divisible by 2

For all even numbers x, x is divisible by 2

For each/every even number x, x is divisible by 2

If x is an even number, then x is divisible by 2

Some: Some / At least one / There exists

Some even numbers are prime

For some even number x, x is prime

There exists an even number x, such that x is prime

There is at least one even number x, for which x is prime

None: No / Not any / There does not exist / There are no

No real square numbers are negative

There are not any real square numbers that are negative

There does not exist a real square number x for which x is negative

There is no real square number x such that x is negative

Consider the order of quantifiers:

When the same type of quantifier is used, the order does not matter:

For all odd numbers x and **all** even numbers y the sum of x and y is odd

For all even numbers y and **all** odd numbers x the sum of x and y is odd

However, with different quantifiers, the order changes the meaning of the statement:

For all positive real x, **there exists** a real y such that $y^2 = x$

This is TRUE as we can choose any positive value for x and find a value of y that makes the equation true by calculating $y = \sqrt{x}$

There exists a real y, such that for all positive real x, $y^2 = x$

However this is FALSE as there is not a single value of y that makes the equation true; the value of y that we need changes with our choice of x

Negation (denial not opposite)

The negation of a statement is achieved by placing 'not' in front of the statement. In reality there are often multiple ways of phrasing this in English. Be careful not to infer too much from a negation, for example 'not hot' does not mean cold - it just means not hot (eg it could be warm).

Statement	**Negation**
He is a doctor	Not 'He is a doctor' = He is not a doctor
She is tall	Not 'She is tall' = She is not tall (She is short would be incorrect)

1. I am hungry — I am not hungry
2. They do their homework — They don't do their homework
3. It is not raining — It is raining
4. The melon is not ripe — The melon is ripe

To negate *(A and B)* we use *not (A and B)* which is the same as *not A or not B*

I have blue eyes **and** blond hair — Either I do not have blue eyes **or** I do not have blond hair (**or** I do not have either)

5. My socks are blue and stripy — Either not blue or not stripy
6. I play hockey and basketball — Don't play both – either not hockey or not basketball
7. I had lunch with Bill and Ben — Didn't have lunch with both / Both didn't have lunch with me
8. Neither my brother nor sister will help me — Either my brother or my sister will help me

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To negate *(A or B)* we use *not (A or B)* which is the same as *not A and not B / neither A nor B*

I study English **or** German — I do not study English **and** I do not study German

9. Jan drinks tea or coffee — Jan doesn't drink tea and doesn't drink coffee
10. The man is called Jim or John — not called Jim and not called John
11. The children eat apples or bananas — The children eat neither apples nor bananas
12. It is not hot or sunny — It is hot and sunny

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*Note how the words '**and**' & '**or**' swap when we negate a statement*

To negate *(for all A, then B)* we use *not (for all A, then B)* which is the same as *not every A implies B / there exists A such that not B*

Statement	**Negation**
Everyone like pizza	**Not** everyone likes pizza / At least one person doesn't like pizza / Some people don't like pizza / There exists someone who doesn't like pizza

13. All vegetarians eat carrots — Some vegetarians don't eat carrots

14. My teacher is always right — My teacher is sometimes wrong

15. All dogs bark — Some dogs don't bark

16. There are no prime numbers that are even (FALSE Statement) — At least one prime number is even (TRUE Statement)

To negate *(there exists A such that B)* we use *not (there exists A such that B)* which is the same as *there is no A such that B / for all A, not B*

There is a prime number less than 2 — There are no prime numbers less than 2. All prime numbers are greater than or equal to 2

17. Some boys like football — No boys like football / All boys don't like football

18. At least one square number is less than 3 — No square numbers are less than 3

19. There exist some birds who can not fly — All birds can fly

20. Not every integer is odd — All integers are odd

To negate *(if A, then B)* we use *not (if A, then B)* which is the same as *if A, then not B / A and not B*

If the sun shines, I will wear a hat — If the sun shines, I will not wear a hat

21. If it is raining I will take an umbrella — If it is raining, I won't take an umbrella

22. I will receive a gold medal if I win — If I win, I won't receive a gold medal

23. If $a < b$ then $f(a) < f(b)$ — If $a<b$ then $f(a) \geq f(b)$

24. $f(a) > 0$ if $a > 0$ (if $a>0$ then $f(a)>0$) — if $a>0$ then $f(a) \leq 0$

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*Note how the words '**for all**' & '**there exists**' swap when we negate a statement*

Negation of 'Nested' statements

A Consider the statement:	The class can complete their homework online, if for **every** student in the class, the student has online access
The negation is:	A class can **not** complete their homework online, if there is **at least** one student who does **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has online access

Then the statement becomes: P is true if **for every** Q, **there exists** R

The negation of this is: P is **not** true if **there exists** Q such that **not** R

B Consider the statement:	The class can complete their homework online, if **for every** student in the class, the student **has** a friend who has online access
The negation is:	A class can **not** complete their homework online if **there exists** a student in the class, **all** of whose friends do **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has a friend

S = friend has online access

Then the statement becomes: P is true if **for every** Q, **there exists** 'R such that S'

The negation of this is: P is **not** true if **there exists** Q such that **not** '**there exists** R such that S'

or: P is **not** true if **there exists** Q such that '**for all** R **not** S'

Write the negation of the following nested statements:

1) A set of integers P is the set of even numbers iff for any $\overbrace{\text{integer } n \text{ in P}}^{B}$, $\overbrace{\frac{n}{2} \text{ is also an integer}}^{C}$.

P iff for any B then C

not P if there exists B such that not C

P not set of even numbers if there exists an integer n in P such that $\frac{n}{2}$ is not an integer

2) A set of integers P is the set of square numbers iff for any $\underbrace{\text{integer } n \text{ in P}}_{B}$, there exists $\underbrace{\text{an integer k}}_{C}$ such that $\underbrace{k^2 = n}_{D}$

P iff for any B there exists C such that D

not P if there exists B such that not 'exists C such that D'

———————— " ———————— all C not D

P not set of square numbers if there exists an integer n in P such that for all integers k, $k^2 \neq n$

Counter Examples

A counter example is one example which disproves a statement. It proves a statement is not true.

1) Find a counter example to the following statements:

a) All quadrilaterals with equal side length are squares   Rhombus

b) The square root of a number is always less than the number $\sqrt{\frac{1}{4}} = \frac{1}{2}$

c) If a three-dimensional solid has a circular base, then it is a cylinder   Cone

d) If n is an integer and n^2 is divisible by 4, then n is divisible by 4 $n = 6$

e) If p is an odd prime then $p+2$ is also an odd prime $p = 7$

f) The sum of 2 numbers is always greater than both numbers $(-2) + (-6)$

g) $10k^2 + 1$ is prime if k is an odd prime $k = 3$ $10(3)^2 + 1 = 91 = 7 \times 13$

h) For all real x, 5x > 4x   any $x < 0$

i) For all real x, $\sqrt{1 - sin^2x} = cosx$ $x = 180$ $(1 = -1)$

2) A set of five signs has a letter printed on the left and a number printed on the right

A 8   B 4   C 1   D 7   E 3

Which sign(s) provide a counterexample to the following statements:

a) Every card that has a vowel on the left has an even number on the right   E3

b) Every card that has an even number on the right has a vowel on the left.   B4

c) Every card that has a consonant on the left has a prime number on the right   B4   C1

d) Every card that has a prime number on the right has a vowel on the left   D7

3) How many counter examples are there to the following statements:

a) All odd numbers between 2 and 20 are prime.   3 5 7 9 11 13 15 17 19

2 counterexamples (odd but not prime)

b) If n is a prime integer less than or equal to 10, then $n^2 + 2$ is also prime

$n =$ 2 3 5 7

$n^2+2 =$ 6 11 27 51

3 counterexamples (n prime < 10, n^2+2 not prime)

c) A whole number n less than or equal to 50, is prime if it is 1 less or 5 less than a multiple of 6

1 less or 5 less : 1 5 7 11 13 17 19 23 25 29 31 35 37 41 43 47 49

4 counterexample (1 less or 5 less, not prime)

4) Prove the following statements, or find a counter example to disprove them:

a) For all real x, $3x < 3^x$

False : when $x = 1$ $3 < 3$ is not true

b) For all real x, if $ax = bx$ then $a = b$

False : when $x = 0$ a, b can be different

c) If a positive integer p has remainder 1 when divided by 3, then p^3 also has remainder 1 when divided by 3

True : $p = 1, 4, 7 \ldots$ $p^3 = 1, 64, 343, \ldots$

$p = 3n+1$ $p^3 = 27n^3 + 9n^2 + 3n + 1$

$= 3(9n^3 + 3n^2 + n) + 1$

d) For all integers p and q, if $p < q$, then $p^2 < q^2$

False : $p = -5$, $q = 1$

e) For all integers n, $9n^2 + 24n$ is not prime

True : $9n^2 + 24n = 3(3n^2 + 8n)$ has factor 3 for all n

f) If a positive integer p is prime, then $2p + 1$ is also prime

False : $p =$ 2 3 5 7 $p = 7$

$2p+1 =$ 5 7 11 15

g) For consecutive even integers p and q, $p^3 - q^3$ is a multiple of 8

True: $p = 2k+2$ $q = 2k$

$p^3 - q^3 = 2^3(k+1)^3 - 2^3k^3$ which is a multiple of $2^3 = 8$

h) For all integers n, if n is prime, then $(-1)^n = -1$

False : $n = 2$ $(-1)^2 = 1$

i) $4^n + 3^{n-2} + 3$ is divisible by 5 for all integers $n \geq 2$

False : $n = 2$ $16 + 1 + 3$ ✓ $\underline{n = 4}$ $256 + 9 + 3$ ✗ (last digit not 0 or 5)

$n = 3$ $64 + 3 + 3$ ✓

j) If p and q are irrational, such that $p \neq q$, then $p + q$ is irrational

False : $p = +\sqrt{2}$ $q = -\sqrt{2}$ $p+q = 0$ rational

k) If p is rational and q is irrational, then $log_p q$ is irrational

False : $p = 2$ $q = \sqrt{2}$ $\log_2 \sqrt{2} = \frac{1}{2}\log_2 2 = \frac{1}{2}$ rational

Logic

1) On an island people either always tell the truth or always tell lies. You are approached by 2 people. Identify if they are truth-tellers or liars in the following situations.

a) The first person says “we both always tell lies”

A

If A truth ⇒ both tell lies × contradiction

∴ A liar ⇒ not both tell lies ⇒ B truth

b) The first person points at the second and says “he is a liar” and the second person says “neither of us are liars”

If A truth ⇒ B is liar ⇒ at least one liar

If A liar ⇒ B truth ⇒ no liars × contradiction

c) The first person says “we are both telling the truth” and the second one replies “he is lying”.

If A truth ⇒ B truth ⇒ A liar × contradiction

A liar ⇒ at least one liar

If B liar ⇒ A truth × contradiction   If B truth ⇒ A liar

d) The first person says “at least one of us is lying”

If A truth ⇒ B liar

If A liar ⇒ no liars × contradiction

e) The first person says “exactly one of us is lying”, and the second replies “actually we’re both lying”

If A truth ⇒ B liar ⇒ at least one truth

If A liar ⇒ either no liars or both liars ⇒ both liars

but B liar ⇒ at least one truth × contradiction