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List of STEP Tricks

The following is a list of tricks, methods or relatively non-standard ways of approaching problems that are useful in STEP questions. Each trick is generally deduced from solutions to STEP questions so you are likely to fully understand the context of each 'trick' when you come to do the relevant question yourself. Some bullet points are direct requirements of the markschemes, these are in blue.

Graph Sketching

- State what is a vertical/horizontal asymptote and indicate what is approached when and from what direction e.g. 'as $x \to +\infty,\ y \to 0^+$' i.e. 'as x tends to positive infinity, y tends to 0 from above'.
- When differentiating to find stationary points, state 'We differentiate to find stationary points.' Always find the y-coordinates of stationary points you have found.
- If it is possible to easily (quickly) find the second derivative - do so, so that you can see where the function is concave $(f''(x) < 0)$ or convex $(f''(x) > 0)$.
- If two curves 'touch' at a point P, then (i) the two curves have the same coordinates at P AND (ii) the curves have the same gradients at P. If when equating the two curves the resulting equation is a quadratic, then the discriminant of this quadratic is zero (as the curves touch).
- Always consider if functions are ODD: $f(-x) = -f(x)$ or EVEN: $f(-x) = f(x)$ and state the relevant symmetry i.e. even implies symmetry about y-axis and odd implies symmetry about the origin.
- The period of $sin\,(ax+b)$ is $\frac{2\pi}{a}$, similarly for cosine, cosecant and secant.
- The period of $tan\,(ax+b)$ is $\frac{\pi}{a}$, similarly for cotangent.
- The period of a sum of trigonometric functions is the lowest common multiple of the periods of the individual functions.
- To sketch $y = f(ax+b)$, divide all x values by a and then subtract $\frac{b}{a}$ from all x values. To further sketch $y = c\,f(ax+b)+d$, sketch $y = f(ax+b)$ then multiply all y values by c and then add d to all y values.
- To compute limits use the fact that:

As $x \to \infty,\ x^x > x! > a^x > p(x) > log_a\, x$, where a is a positive constant and $p(x)$ is a polynomial of positive degree n. Hence we know that exponentials are larger than polynomials for large values of x, thus we can deduce that as $x \to \infty,\ \frac{x^2}{e^x} \to 0$, for example.

- Whenever you sketch a curve $y = p(x)$, where $p(x)$ is a polynomial of positive integer degree n, be able to deduce the sketch of $y^2 = p(x)$. In particular consider the symmetry and points where the tangent to the curve is vertical.

General Algebraic Manipulation

- $\tan y = x \Rightarrow y + k\pi = \arctan x$, where k is an integer to be determined such that $-\frac{\pi}{2} < y + k\pi < \frac{\pi}{2}$.
- For inequalities, consider if the function you are applying to both sides is increasing, strictly increasing, decreasing or strictly decreasing on the given interval:
- If $A < B$, then $f(A) < f(B)$, if f is strictly increasing on $[A, B]$.
- If $A < B$, then $f(A) \leq f(B)$, if f is increasing on $[A, B]$.
- If $A < B$, then $f(A) > f(B)$, if f is strictly decreasing on $[A, B]$.
- If $A < B$, then $f(A) \geq f(B)$, if f is decreasing on $[A, B]$.
- If you have to repeatedly write something out during a long algebraic simplification, use helpful notation such as: $t = \tan\theta$, $s = \sin\theta$, $c = \cos\theta$.
- To prove that a function f satisfies $f(x) > 0$ for all $x \in S$ where S is a subset of the real numbers, consider showing that $f(0) = 0, f'(x) > 0$ or $f(0) = f'(0) = 0, f''(x) > 0$ or a similar demonstration up to higher derivatives. From this it follows that $f(x) > 0$ for all $x \in S$.
- The discriminant of a quadratic equation imposes an 'if and only if' relationship.
- Use standard identities such as:
- $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1})$, where n is a positive integer.
- $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + ... - xy^{n-2} + y^{n-1})$**, where n is a positive odd integer.

**i.e. The signs alternate.

- Know the small angle approximations e.g. $\sin x \approx x$, for small x.

Solving Equations

- Spot if a certain equation resembles a trigonometric identity, doing this allows to express the roots in terms of trig functions and to find exact values e.g. if asked to solve $8x^4 - 8x^2 + 1 = 0$, if $x = \cos\theta$, then the equation becomes $8\cos^4\theta - 8\cos^2\theta + 1 = 0$ which is equivalent to $\cos 4\theta = 0$, solve this for θ and proceed to find the corresponding values of x.
- If you are solving an equation involving square roots then you will need to repeatedly square both sides. When you eventually obtain solutions to the resulting polynomial, check whether or not each of these satisfies the initial equation as it is likely that some solutions are spurious (arising from the squaring).

- Use substitutions to make solving equations much easier e.g. to solve $\frac{(x^2-x+1)^3}{(x^2-x)^2} = \frac{343}{36}$, let $t = x^2 - x$, then solve for values of t and obtain the corresponding values of x by solving the resulting quadratics (of course check that these values of x satisfy the original equation). This method is particularly useful with equations which feature many square roots.
- Know Vieta's Formula **,** in particular that if a quadratic $x^2 + ax + b$ has roots p and q, then $-a = p + q$ and $b = pq$.

Vectors

- Use the trick of taking the dot product of a vector with itself i.e $\mathbf{x.x} = OX^2$ - this is particularly useful if $\mathbf{x}$ is defined in terms of unknowns but the magnitude of $\mathbf{x}$ is known.
- Two triangles are similar if both triangles have the same three angles.
- 'Squaring' vectors is defined by $\overline{OA}^2 = \overline{OA}.\overline{OA}$. Hence the following is possible:

$(\lambda\,\mathbf{a} + \mu\,\mathbf{b})^2 = \lambda^2\,\mathbf{a.a} + 2\lambda\mu\,\mathbf{a.b} + \mu^2\,\mathbf{b.b} \equiv \lambda^2 a^2 + \mu^2 b^2 + 2\lambda\mu\,\mathbf{a.b}$. (Where λ and μ are scalar).

- You can equate vector coefficients i.e. If $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, then we have that $\lambda\,\mathbf{a} + \mu\,\mathbf{b} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} \quad \Leftrightarrow \quad \lambda = \alpha$ and $\mu = \beta$. (Where λ, μ, α and β are scalar).

Complex Numbers

- Use the fact that $|z|^2 = zz^*$.
- For complex transformations, it may be helpful to write $x = tan\,\frac{\theta}{2}$ in order to be able to consider the range of values for θ given a range of values for x and hence be able to deduce a range of values for y. With this you are more likely to sketch the correct segment(s) of a locus in the complex plane.
- Multiplying a complex number by $e^{i\theta}$ rotates is θ radians anticlockwise about O.

Trigonometry

- Get used to using the Sum-to-Product formulas e.g. $sin\,x + sin\,y = 2sin\,(\frac{x+y}{2})cos\,(\frac{x-y}{2})$. These are in the Edexcel formula booklet under C3 but are not actually in A Level Maths, however they are extremely useful and assumed knowledge in STEP.
- The general solutions to Trigonometric Equations are extremely useful and you should try and derive them from the graphs for yourself. They are as follows:

\- $sin\,x = sin\,y$, we have: $x = (-1)^n y + n\pi$, for all integers n*.

- $cos\, x = cos\, y,$ we have: $x = 2n\pi \pm y,$ for all integers n*.
- $tan\, x = tan\, y,$ we have: $x = y + n\pi,$ for all integers n*.

*for all integers n such that x is in the specified range of values e.g. $0 < x < 2\pi.$

- Know that: $arctan\, x + arctan\, \frac{1}{x} = \frac{\pi}{2}$, this can be proved by considering the sum of angles in a right-angled triangle with hypotenuse of length $\sqrt{1+x^2}$ and sides of length 1 and $x.$
- Know fundamental identities such as: $sin\, x = cos\,(x - \frac{\pi}{2})$ and be able to derive them either by the Unit Circle or by observing trigonometric graphs.

Proving Inequalities

- Utilise the Trivial Inequality: $x^2 \geq 0$ for all real numbers $x.$
- It is easier to compare two numbers to zero than it is to compare them to one another. Hence to prove which number is the bigger (of two numbers), consider their difference and deduce if it is positive or negative.
- You cannot subtract inequalities.
- To prove inequalities involving real numbers approach by contradiction e.g. to prove that $\sqrt{3} > \sqrt{2} - 1,$ suppose that $\sqrt{3} \leq \sqrt{2} - 1$, squaring both sides yields: $3 \leq 2 - 2\sqrt{2} + 1,$ hence $0 \leq -2\sqrt{2}$ which is clearly false and a contradiction. Hence $\sqrt{3} > \sqrt{2} - 1$.
- If a and b are positive real numbers, then $a - b$ has the same sign as $a^2 - b^2,\ a^4 - b^4,\ a^6 - b^6$... as $a^2 - b^2 = (a+b)(a-b)$, $a + b$ is clearly positive so $a - b$ and $a^2 - b^2$ have the same sign ($sgn(a^2 - b^2) = sgn(a - b)$). The result above follows inductively.
- Prove positivity by completing the square.
- To prove inequalities for series, try:
- Summing the first few terms - given that all terms in the series are positive.
- Using the Arithmetic Mean-Geometric Mean Inequality (the AM-GM) i.e. AM≥ GM.

Logic/Proof

- Have a good understanding of fundamental ways in which statements can be related:
- A is necessary for B means that B implies A i.e. $B \Rightarrow A$ — **If B is true then A is also true.**
- A is sufficient for B means that A implies B i.e. $A \Rightarrow B$ — **If A is true then B is also true.**
- A is necessary and sufficient for B means that A implies B and B implies A i.e. $A \Leftrightarrow B$ — A **is true if and only of B is true.**

- The converse of $A \Rightarrow B$ is $B \Rightarrow A$, **these statements are not logically equivalent.**
- The contrapositive of $A \Rightarrow B$ is $not\,(B) \Rightarrow not\,(A)$, these statements **are logically equivalent.**

- To prove an 'at least one' statement i.e. that at least one object from a set of objects satisfies some given conditions, use proof by contradiction i.e. assume that **no** objects satisfy the conditions and proceed to obtain a contradictory statement.
- To prove an 'at most one' statement i.e. that at most one object from a set of objects satisfies some given conditions, use proof by contradiction by considering the all possible pairs of objects i.e. assuming (separately) that two objects from the set of objects satisfy the given conditions and obtaining a contradictory statement for each possible pair, this shows that no two objects can be simultaneously true, hence no 3, 4, 5 etc. can be simultaneously true. The result follows by contradiction.

Approximation

- Any time something is being approximated think binomial, Maclaurin and Taylor series.
- In certain cases you may use approximations such as $\sqrt{2} \approx 1.41,\ \sqrt{5} \approx 2.24,\ \sqrt{3} \approx 1.73,\ e \approx 2.72,\ \pi \approx 3.14,\ \sqrt{10} \approx 3.16.$

Calculus

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$ $(\dagger)$
- $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx.$ $(\ddagger)$
- Learn to use the 'tangent half-angle' substitution: let $t = tan\,\frac{x}{2}$. The associated results for this are in the STEP formula booklet.
- If an integral features hyperbolic trigonometric functions, the substitution $u = e^x$ may be useful.
- It may be helpful to use the 'inverse limit' substitution for integrands containing periodic functions: if a definite integral has lower and upper limits a and b respectively, let $t = a + b - x$. With this you will obtain a second expression for the same definite integral (as the limits do not change), you can then use $(\dagger)$ and $(\ddagger)$ in turn in order to evaluate the integral, as by the following example:

We find $I = \int_0^{\frac{\pi}{4}} ln\,(1 + tan\,x)\,dx \quad (*)$

Let $t = \frac{\pi}{4} - x$, hence $dx = -dt,\ x = 0 \Rightarrow t = \frac{\pi}{4}$ and $x = \frac{\pi}{4} \Rightarrow t = 0$. Thus we can express the integral as:

$$I = -\int_{\frac{\pi}{4}}^{0} ln\,(1 + tan\,(\tfrac{\pi}{4} - t))\,dt = \int_0^{\frac{\pi}{4}} ln\,(1 + tan\,(\tfrac{\pi}{4} - t))\,dt,\ \text{by}\ (\dagger).$$

$$I = \int_0^{\frac{\pi}{4}} ln\,(1 + \tfrac{1 - tan\,t}{1 + tan\,t})\,dt = \int_0^{\frac{\pi}{4}} ln\,(\tfrac{2}{1 + tant\,t})\,dt = \int_0^{\frac{\pi}{4}} ln\,2 - ln\,(1 + tan\,t)\,dt$$

Hence we have a second expression for I, namely,

$$I = \int_0^{\frac{\pi}{4}} ln\,2 - ln\,(1 + tan\,x)\,dx \qquad (**)$$

Adding $(*)$ and $(**)$ we have:

$$2I = \int_0^{\frac{\pi}{4}} (ln\,(1 + tan\,t) + ln\,2 - ln\,(1 + tan\,x))\,dt,\ \text{by}\ (\ddagger).$$

Hence, $I = \frac{1}{2}\int_0^{\frac{\pi}{4}} ln\,2\,dx = \frac{\pi}{8} ln\,2.$

- Know the formulas for arc length in Cartesian, Parametric and Polar form - these are not in the STEP formula booklet.
- Be as familiar with hyperbolic functions as with regular trigonometric functions, in particular be able to rapidly derive the derivatives for inverse trigonometric and hyperbolic functions.
- Know L'hôpital's Rule as a way of checking limits but do not rely on it and use more fundamental methods if possible (e.g. Taylor series).
- Some common limits to know:

- As $x \to 0$, $xln\,x \to 0^+$. Also any relevant ones from: https://en.wikipedia.org/wiki/List_of_limits .