TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS WITH PHYSICS 2)

DECEMBER 1999.

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others rather hard. You are not expected to answer all of them, or necessarily to complete questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper with you to the interview afterwards.

1. Sketch the curve defined by the equation

$$y = \sqrt{|x|} + \frac{1}{x^2 - 1}.$$

Sketch the three dimensional surface defined by

$$x^2 + y^2 - z^2 = 1.$$

2(a). A plane cuts through a sphere of radius r at distance a from its centre. Find the volumes of the two pieces of the sphere on either side.
(b) Two spheres (of radii r and a, with $r < 2a$) meet in such a way that the centre of the one of radius r lies on the surface of the one of radius a. Find the volume of the intersection.

3. Each week, the chance of winning a prize with your Premium Bond is 1 in 14,000. Mr. Optimist buys 14,000 Premium Bonds and claims he is on to a certain winner. Explain to him why that is not quite the case, and find out roughly what his chances are of getting a winner in a given week.

Hint: if you are unfamiliar with how to approximate the expression you come up with, try taking the logarithm of something and using the Taylor series for log(1+h)

4. (a) Simplify the expression $(m + n)^2 - (m - n)^2$.
(b) A Pythagorean triple is a trio of integers a, b and c with $a^2+b^2 = c^2$. It is a fundamental triple if a, b and c have no common factor. Find a systematic way of constructing infinitely many distinct fundamental Pythagorean triples. Explain why your method works.

5. Little Johnny is a biologist tracking the movements of deer with a radio homing device. He has two listening posts, located at points $(X, 0)$ and $(0, Y)$ in the plane, and periodically they give simultaneous, accurate bearings (θ_n, ϕ_n) (let's say, anticlockwise from the X axis) from the listening posts to the deer. Johnny wants to calculate the deer's positions (x_n, y_n) in cartesian coordinates. He has hitherto done this by spreading the map of the area on his kitchen table and using rulers, protractors and Stuff.

Devise a couple of trignometric formulae so he can do the whole thing on his computer.

Note: this problem was given to one of your interviewers, Dr. Read, in real life. Thousands of points had been done by his predecessor using the kitchen table method, when "Johnny" took over the project and the bright idea of consulting a mathematician occurred to him.

6. Fifteen men are placed on a Dead Man's Chest in a rectangular pattern, with each man distant a from his neighbours, thus:

The average weight of the men is w, and the heaviest man weighs no more than $2w$. Find the maximum possible horizontal distance from the centre of the rectangle to the centre of mass of the fifteen men (you are allowed to have some of the men as pixies, with zero weight, if it helps... but negative weights are not allowed). Show how the men should be placed so as to achieve this, and explain why your solution is the best.

7. Consider the following simplified model of a skater falling over: Two equal masses m are attached to opposite ends of a light, straight rod of length 2. One mass rests at $(0, 0)$ on a *frictionless* horizontal surface and the other is balanced, rather precariously, vertically above it at $(0, 2)$. The system is disturbed slightly. Find the (x, y) coordinates of the two masses when the angle of the rod to the vertical is θ.

The lower mass loses contact with the frictionless surface when the horizontal velocity of the lower mass is a maximum. Why? What is the value of $\cos\theta$ when this happens?

8. A rope is wrapped M whole turns round a cylindrical post, the two ends of the rope going in opposite directions. The coefficient of friction between rope and post is 0.25. It is desired that by pulling with a force of $1N$ on one end of the rope, I can prevent the rope from moving away from me even if a force of $10^6 N$ is applied to the other end. How large does M have to be? (Note that to 3 significant figures, $\log_e(10^6) = 13.8$).

Hint: Let the tension in the rope decline like $T(\theta)$ with the angle θ round the post. Investigate $T(\theta + \delta\theta) - T(\theta)$.

9. Consider a mass m at position $x(t)$ on a rough horizontal table attached to $x = 0$ by a spring that exerts a force $-kx$. The force f due to friction between the table and the mass is given by

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \le f \le F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0 \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves then the maximum distance from the origin decreases by $2F/k$ per half cycle.

10. Suppose $m^3 = n^4 - 4$ where m and n are both integers.

(a) Show that m cannot be even if n is odd.
(b) Show that m and n cannot both be even.
(c) By considering the prime factors of m, $n^2 - 2$ and $n^2 + 2$ show that, in fact, there are no integer solutions to $m^3 = n^4 - 4$.