TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION
Mock Paper 3 ANSWERS

1: d

Using the quadratic formula will help solve this question but the question can be made easier to answer if (c) and (d) are changed to

(c′) $x^2 + 5x - 2$

(d′) $x^2 + 5x - 7$

by dividing through by 3 and 2 respectively. The roots are unaffected.

Noticing that the first two coefficients are the same for all 4 possible answers the task is made easier just by considering the discriminant $(b^2 - 4ac)$. The largest discriminant will mean the largest root. The discriminants are

(a) 29

(b) 21

(c′) 33

(d′) 53

(e) 52

Hence the answer is (d)

2: a

The quadratic on the right can be written as

$$x^2 - 2\pi x + \pi^2 + 1 = (x - \pi)^2 + 1$$

Hence this quadratic has a minimum value of 1 at $x = \pi$

The maximum possible value of $sin x$ is 1 and for any solution to exist it must equal 1 at $x = \pi$.

$sin\pi = 0$ hence this equation has no solutions.

This could be very easily seen by sketching both functions, and although this would not constitute a proof, there are no marks awarded for working in this section.

3: c

$$N_n = \frac{4}{3}.\frac{6}{4}.\frac{8}{5}.\frac{10}{6}\ldots\frac{2(n+1)}{n+2}$$

$$= 2^n\frac{2}{3}.\frac{3}{4}.\frac{4}{5}\ldots.\frac{n+1}{n+2} = 2^n\frac{2}{n+2} = \frac{2^{n+1}}{n+2}$$

4: d

$e^{\log_\pi 2}$ can be eliminated since $2 < \pi$ and therefore $\log_\pi 2 < 1 < \log_2 \pi$.

(d) is therefore larger than (a).

By the same reasoning, $2^{\log_\pi e}$ can be eliminated since $e < \pi$ and therefore $\log_\pi e < 1 < \log_e \pi$.

(c) is therefore larger than (b).

So $e^{\log_2 \pi}$ and $2^{\log_e \pi}$ are left.

Since $e > 2$ we can eliminate (e) and we have that

$$\log_2 \pi > \log_e \pi \; (> 1)$$

Using this fact and again that $e > 2$, we see that (d) is therefore larger than (c).

5: d

$$\int_a^c f(x)dx - \int_b^d f(x)dx = \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right) - \left(\int_b^c f(x)dx + \int_c^d f(x)dx\right)$$
$$= \int_a^b f(x)dx - \int_c^d f(x)dx$$

6: c

Consider e^{-x^2}

This function has a maximum value of 1 at $x = 0$ and then approaches zero in both the positive and negative direction as x grows 'larger' (larger in the absolute value sense). However it never actually achieves the value of 0.

This in turn means that the function $\sin(\left(e^{-x^2} + 1\right)\frac{\pi}{2})$

is 0 at $x = 0$ and approaches $\sin\left(\frac{\pi}{2}\right) = 1$ from below as x grows large in both negative and positive direction.

Note that it approaches 'from below' because any slight deviation from $\frac{\pi}{2}$ has a sine that is smaller than 1 (the maximum value of sin(x) is 1).

This means that

$$3\sin\left(\left(e^{-x^2} + 1\right)\frac{\pi}{2}\right) - 2$$

approaches, but never reaches, $3 \times 1 - 2 = 1$ as x grows larger in both directions of the x-axis. Therefore the answer is (c).

7: b

Let's start by labelling the people 1 to n. Person 1 shakes all n-1 other people's hands. Having done that person 1 leaves the room. Person 2 then shakes all n-2 other people's hands and then leaves the room. Repeat this until person n-1 and n are left to make the final handshake. Using the formula for arithmetic series, the total number of handshakes is therefore

$$(n-1)+(n-2)+(n-3)+\cdots+3+2+1=\frac{(n-1)}{2}$$

Note that if the formula for arithmetic series is not known, it can be easily worked out by considering the cases where *n* is even and *n* is odd separately, and doing the addition in pairs: $(n-1)$ with 1, $(n-2)$ with 2, etc.

8: c

$2-\sqrt{2}$ is an irrational number, otherwise this would imply $\sqrt{2}$ is rational.

Therefore $\sqrt{2}+\left(2-\sqrt{2}\right)=2$ is the sum or two irrational numbers which equals the rational number 2. This is therefore a counterexample to the statement, 'an irrational number plus an irrational number is an irrational number'.

9: d

First find the point(s) of intersection and then establish the gradients at these points

$$ln(x) = ln(3) + 3x - x^2$$

The $ln(3)$ hints at a possible point of intersect at $x = 3$ which is the case. The gradients of each side are:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(3) + 3x - x^2) = 3 - 2x$$

Evaluated at $x = 3$ the gradients are 1/3 and -3 respectively.

The product of the gradients is -1 at $x = 3$ hence the graphs of each function intersect perpendicularly there.

10: c

The question is made easier by recognising that

$$y = \frac{1}{2}\sin(2x)$$

The maximum value that y can therefore obtain is ½. Hence this rules out answer (b) and therefore (d).

The second derivative of $\frac{1}{2}\sin(2x)$ is $-2\sin(2x)$ so $\frac{1}{2}\sin(2x)$ satisfies the differential equation of (c).

So the answer is (c). To clarify that (a) is not satisfied we see that

$$\frac{dy}{dx} = \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

It may be quicker to note that (a) and (b) are not satisfied, thus ruling out (d) and leaving (c) as the correct answer.

11: a

When $x = y$, $x^n - y^n = x^n - x^n = 0$. Hence $(x - y)$ is a factor of $x^n - y^n$ for all *n*.

12: b

The equation can be rearranged to a quadratic equation and then solved,

$$(10^2)^x = 10^x + 110$$

$$10^{2x} - 10^x - 110 = 0$$

$$(10^x - 11)(10^x + 10) = 0$$

$$10^x = -10 \; or \; 11$$

10^x cannot be negative as *x* is real, hence

$$10^x = 11$$

$$= \log_{10} 11$$

So the original equation has only one (real) solution.

13: b

Let

$$x = (\frac{b-a}{d-c})\,(u-c) + a$$

so

$$\frac{dx}{du} = \frac{b-a}{d-c}$$

$$u = \frac{d-c}{b-a}(x-a) + c$$

When $x = a, u = \frac{d-c}{b-a}(a-a) + c = c$ and when $x = b, u = \frac{d-c}{b-a}(b-a) + c = d - c + c = d$

Hence using integration by substitution

$$\int_a^b f(x)dx = \int_c^d f\,(\frac{b-a}{d-c}(u-c) + a)\,\frac{dx}{du}du$$

$$= \int_c^d f\,((\frac{b-a}{d-c})\,(u-c) + a)\,(\frac{b-a}{d-c})\,du$$

$$= (\frac{b-a}{d-c})\int_c^d f\,(\frac{b-a}{d-c}(u-c) + a)\,du$$

u is a dummy variable which can simply be changed back to an x.

14: b

a) $6^{76} \times \frac{7^{75}}{3\times21^2} = \frac{2^{76}\times3^{76}\times7^{75}}{3\times3^2\times7^2} = 2^{76} \times 3^{73} \times 7^{73}$

b) $\frac{42^{75}}{196} = \frac{(2\times3\times7)^{75}}{2^2\times7^2} = 2^{73} \times 3^{75} \times 7^{73}$

c) $\frac{1764^{37}}{7} = \frac{(42^2)^{37}}{7} = \frac{42^{74}}{7} = \frac{(2\times3\times7)^{74}}{7} = 2^{74} \times 3^{74} \times 7^{73}$

d) $7 \times 42^{73} = 7(2 \times 3 \times 7)^{73} = 2^{73} \times 3^{73} \times 7^{74}$

e) $1764^{35} = (42)^{2x35}$

a) is $8 \times 2^{73} \times 3^{73} \times 7^{73}$

b) is $9 \times 2^{73} \times 3^{73} \times 7^{73}$

c) is $6 \times 2^{73} \times 3^{73} \times 7^{73}$

d) is $7 \times 2^{73} \times 3^{73} \times 7^{73}$

Hence b) is the largest

15: d

$x^2 - 1$ is zero for $x = -1$ or 1 and $x^2 + 1$ is never zero hence $\frac{x^2-1}{x^2+1}$

is zero at $x = -1$ and $x = 1$.
Hence graphs (a) and (b) can be eliminated. The given function is only zero when $x^2 - 1 = 0$ ie only has two zeros whereas graph (c) has 4 zeros. Hence the answer is (d)

16: b

Imagine the jar is inside the box (completely). If the penny is in the jar it is therefore in the box.

a) The penny can be in the box without being in the jar.

b) If the penny is in the jar but not in box the jar must be outside the box. This contradicts the scenario the given statement implies.

c) The penny can be in the box without being in the jar.

d) The penny can be not in the box and also not be in the jar.

17: b

Inspecting the equation will reveal that $x = 1$ is a root. This means $(x - 1)$ is a factor of the polynomial and

$$x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = (x - 1)(ax^4 + bx^3 + cx^2 + dx + e)$$

for some values *a*, *b*, *c*, *d*, *e* to be determined.

Working through the powers of *x* in descending order will allow the values of *a* to *e* to be established:

$$x^5: 1 = \mathrm{a}$$

$$x^4: -1 = b - 1 \; so \; b \; = 0$$

$$x^3: 2 = c$$

$$x^2: -2 = d - 2 \; so \; d \; = 0$$

$$x: 1 = e$$

The final term constant term is consistent with $-1 = -1 \times 1$

So we have

$$x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = (x - 1)(x^4 + 2x^2 + 1)$$

Using the common quadratic $(y^2 + 2y - 1) = (y + 1)^2$ with $y = x^2$, it should be easily recognised that the newly found factor is

$$(x^4 + 2x^2 + 1) = (x^2 + 1)^2$$

So $(x^2 + 1)$ is a factor.

[Note this has determinant of -4 hence cannot be factored further]

18. d

An answer can be quickly obtained if it is spotted that

$$\frac{2u + 2v}{uv} = 2\left(\frac{1}{u} + \frac{1}{v}\right)$$

Therefore the sum of the reciprocals is

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{2}\frac{20}{6} = \frac{5}{3} = 1\frac{2}{3}$$

19: b

In factorials, the key is to count the factors of 2 and 5 since a factor of 10 is required to give a number a 0 on the end.

101! has 20 numbers that have a factor of 5, namely, 5, 10, 15….,95 ,100. Of those 25,50,75,100 have another factor 5 so there are 20+4=24 factors of five. Factors of 2 far exceed this hence the number 101! has 24 zeros on the end.

$((10^3)^3)^3 = 10^{3^3} = 10^{27}$, so $((10^3)^3)^3$ has 27 zeros and eliminates option (e).

$2^{25} \times 3^{38} \times 5^{48} \times 7^{13}$ – the number of zeros on the end is the minimum of the power of 2 and the power of 5, hence 25.

$112!$ can be worked out along the same lines as for 101! i.e. 5, 10, 15, …, 105, 110 have a factor of 5 and 25, 50, 75, 100 have another factor of 5. Hence 112! Has 22+4=26 zeros. (Indeed, since $101 < 112$, we can immediately discount (a) as an option and just do the calculation for (d) $112!$)

Therefore $((10^3)^3)^3$ ends in the most zeros.

20. b

At least two approaches here. Either carry out the division on the left-hand side to get

$\frac{2(x-1)+6(x-1)}{x-1} = 2x + 6$

which gives

$$2x + 6 = x - 1$$

$$x = -7$$

So one solution.

If the approach of multiplying through by $(x - 1)$ is chosen then care needs to be taken when dealing with the solutions of the subsequent quadratic,

$$x^2 + 6x - 7 = 0$$

This quadratic yields the solutions $x = -7 \text{ } or \text{ } 1$. The $x = 1$ result comes from the multiplication by $(x - 1)$, which is zero when $x = 1$. $x = 1$ is not a solution of the original equation. Hence there is only one solution, namely $x = -7$

21: a

We have, by the binomial theorem,

$$\left(x+\frac{2}{x^2}\right)^6 = x^6\left(\frac{2}{x^2}\right)^0 + 6x^5\left(\frac{2}{x^2}\right)^1 + 15x^4\left(\frac{2}{x^2}\right)^2 + 20x^3\left(\frac{2}{x^2}\right)^3 + 15x^2\left(\frac{2}{x^2}\right)^4 + 6x\left(\frac{2}{x^2}\right)^5 + \left(\frac{2}{x^2}\right)^6$$

We also have that

$$\left(3-\frac{5}{x^2}\right)\left(x+\frac{2}{x^2}\right)^6 = 3\left(x+\frac{2}{x^2}\right)^6 - \frac{5}{x^2}\left(x+\frac{2}{x^2}\right)^6$$

So the coefficient of x is $3(0) - 5(6.2) = -60$

22: e

We know that f'(x) = 3αx^2 + 2x, and that f(-1/2) = ¾

α– 1 = 0, so α = 4/3.

23: b

$$4r + 8r^2 + 16r^3 + 32r^4 + \cdots = 4r(1 + 2r + (2r)^2 + (2r)^3 + (2r)^4 + \cdots)$$

This is a geometric series and so converges if $|2r| < 1$, that is,

$$-\frac{1}{2} < r < \frac{1}{2}$$

Multiplying the series by $4r$ does not affect this.

24: b

To describe the concavity, we must examine the second derivative.

$$f(x) = \frac{1}{4}(-\sqrt{3}x^2 - \sin(2x))$$

$$f'(x) = \frac{-\sqrt{3}}{2}x - \frac{\cos(2x)}{2}$$

$$f''(x) = \frac{-\sqrt{3}}{2} + \sin(2x)$$

Consider $f''(x) = 0$, that is, $\sin(2x) = \frac{\sqrt{3}}{2}$.

For $0 \le x \le 2\pi$, $2x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, and so for $0 \le x \le \pi$, $x = \frac{\pi}{6}$ or $\frac{\pi}{3}$

- For $0 \le x < \frac{\pi}{6}$, $f''(x) \le 0$, that is, $f(x)$ is concave down
- For $\frac{\pi}{6} < x < \frac{\pi}{3}$, $f''(x) \ge 0$, that is, $f(x)$ is concave up
- For $\frac{\pi}{3} < x \le \pi$, $f''(x) \le 0$, that is, $f(x)$ is concave down

25: c

This can be easily simplified:

$$\ln 1 + \ln e + \ln e^2 + \cdots + \ln e^n = 1 + 2 + 3 + \cdots + n = \frac{n(1+n)}{2}$$

26: e

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{2(x+h)^3 - 2x^3}{h} = \lim_{h\to 0}\frac{2x^3 + 6x^2h + 6xh + 2h^3 - 2x^3}{h} = \lim_{h\to 0}(6x^2 + 6xh + 2h^2) = 6x^2$$

The original function could have been $y = 2x^3 + 1$

27: c

One approach to this problem is to work out the derivative and plug in initial values. The derivative of $y = xcos(x)$ is $y = \cos(x) - xsin(x)$

$x = 0$ is a good place to start and eliminates (a) and (b).

Considering the sign of the graph at various values allows you to choose between (c) and (d); alternatively, the second derivative allows us to decide whether there is a maximum or minimum at $x = 0$.

The second derivative of $y = xcos(x)$ is $-sin(x) - \sin(x) - xcos(x) = -2\sin(x) - xcos(x)$ and this is the first derivative of our desired graph. We differentiate again to give $-2\cos(x) + xsin(x) - \cos(x) = xsin(x) - 3\cos(x)$. At $x = 0$, this is negative. Hence our graph should show a local maximum at $x = 0$. So (c) is the correct graph.

(c) represents the graph of $y = cos(x) - xsin(x)$, the derivative of $y = xcos(x)$

(a) represents the graph of $y = xcos(x)$. It may also be eliminated from consideration by noting that it is an odd function.

(b) represents the graph of $y = xsin(x)$

(d) represents the graph of $y = cos(x) + xsin(x)$

28. c

(a) is the converse of the original statement and is not necessarily true

(b) is the inverse of the original statement and is not necessarily true

(c) is the contrapositive of the original statement so is true if and only if the original statementis true

(d) is the original statement with the hypothesis negated, this is not necessarily true

29 d

So $$x^4 + a + x^{-4} \geq k$$

So $$(x^2 + x^{-2})^2 - 2 + a \geq k$$

So $$(x^2 + x^{-2})^2 \geq k + 2 - a$$

$$(x^2 + x^{-2}) \geq \sqrt{k + 2 - a}$$

So $\sqrt{k + 2 - a}$ is the minimum positive value of $x^2 + x^{-2}$

30. b

$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$ holds in the special case where $b^2 = 4ac$, that is, the discriminant of the quadratic equals zero, and the equation has one repeated root. Hence, the answer is b. It does not necessarily follow that $c = 0$ and (c) and (d) describe the alternatives to a repeated root.

31: c

$y = |x^3 - x|$ is equivalent to $y = \sqrt{(x^3 - x)^2}$ and $y = |(x-1)x(x+1)|$, therefore there are roots at $(-1,0)$, $(0,0)$, $(1,0)$. So the answer is c.

32: a

$\left[\frac{d}{dx}(x^3 \sin(x))^7\right] = 7x^{21} sin^6(x)\cos(x) + 21x^{20} sin^7(x) - 7x^{13} sin^6(x)\cos(x) - 7x^9 \sin(x)$ has degree 21.

$7x^{18}(\cos(x)\left(xsin^2(x)\right)^3) + x^2\sin(x)) = x^2\sin(x) - 7x^{21}sin^6(x)\cos(x)$ has degree 21.

However, the first term is cancelled and so the highest power is 20.

33: d

Though it is tempting to times out the brackets that is unnecessary here. A systematic approach is required to identify all those terms that combine to make x^6. Write out each bracket and move along each combination, keeping track of all the terms separately.

34: b

$$(3.6)^5 = 3^5(1.2)^5 = 243(1+0.2)^5 = 243(1 + 5(0.2) + 10(0.2)^2 + 10(0.2)^3 + 5(0.2)^4 + 1(0.2)^5)$$
$$= 243(1 + 1 + 0.4 + 0.08 + 0.008 + 0.00032) = 604.66176)$$

35: a

(a) As $log(x)$ cannot take negative values, there is no root at $x = -3$. We do have a root at $x = 1$, as the graph shows. The other graphs can be eliminated as shown and this is the correct function.

(b) $(x + 3)$ is not raised to the power of 2 though the more revealing aspect of this equation is the two negative roots which will not be plotted on a real-valued graph.

(c) has a negative coefficient of x so is a reflection in the y axis.

(d) The function does not have a real-valued plot at $x = -1$ and $x = -3$, and only takes positive y values at $x = 1$.

36: b

The number of maps between a set of size M and a set of size N is N^M as every element in the set of size M has N choices.

37: d

$f\left(\frac{-b}{2a}\right) = 0$

Then we see that $\frac{-b}{2a}$ is a root, and necessarily a repeated root and the discriminant of the quadratic equation is zero.

38: b

By dividing the two given identities, we can find $\tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$

This must be applied twice, using $\beta = 2\alpha$ followed by $\beta = \alpha$, to give $\frac{3\tan(\alpha) - tan^3(\alpha)}{1 - 3tan^2(\alpha)}$

39: a

2^{1000} is not a multiple of 10 so we must have some power of 10 that is strictly larger than it, say 10^m.

[10^m contains $m + 1$ digits.]

For $k < m$, assume $2^{1000} = 10^k$

$k = 1000log2 \approx 301.03 \ldots$

Therefore $m - 1 = 301$ and $m = 302$. So 2^{1000} has 302 digits.

40: d

We have $5^x = 2$

So $5 = 2^{\frac{1}{x}}$

So $log_2 5 = log_2(2^{\frac{1}{x}}) = \frac{1}{x}$

Denote $y = log_8 25$

So $8^y = 25$

$5^{3xy} = 5^2$

$y = \frac{2}{3x}$

So $log_8 25 = \frac{2}{3x}$