TMUA Integration

Syllabus

Definite integration vs finding area; definite and indefinite integrals of x^n; Fundamental Theorem of Calculus (differentiation-integration link); combining integrals; trapezium rule; differential equations.

1. Integrate the following expressions with respect to x

a) $\displaystyle\int \frac{2+5x}{3x^3}\,dx$

b) $\displaystyle\int \frac{\sqrt{x}(4-x)}{2x^2}\,dx$

c) $\displaystyle\int \frac{1}{\sqrt{x}}\left(\frac{2}{x}-\frac{3}{4x^2}\right)dx$

d) $\displaystyle\int \frac{2x^3+\sqrt{x^3}}{\sqrt{x}}\,dx$

e) $\displaystyle\int \frac{(1+\sqrt{x})(3-\sqrt{x})}{x^4}\,dx$

2. Evaluate the following integrals

a) $$\int_1^2 x^4 + 3 - \frac{2}{5x^2} \; dx$$

b) $$\int_1^4 \frac{x^3 + 2\sqrt{x}}{x} \; dx$$

c) $$\int_0^4 (x^{1/2} - 3)^2 \; dx$$

d) $$\int_1^5 3\sqrt{x} - \frac{1}{\sqrt{x}} \; dx$$

e) $$\int_0^1 \frac{15(2x+1)^2}{2\sqrt{x}} \; dx$$

f) $$\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}} \; dx$$

g) $$\int_{-2}^3 |x|(1-x) \; dx$$

3. a) Find an equation for y, given $y = 3$ when $x = 1$

$$\frac{dy}{dx} = 6x^2 - 4x \qquad x \in \mathbb{R}$$

b) Find an equation for y, given $y = 5$ when $x = 1$

$$\frac{dy}{dx} = 4 + \frac{1}{x^2} \qquad x \neq 0$$

c) The point (-1 , -1) lies on the curve C whose gradient function is given by

$$\frac{dy}{dx} = \frac{5x^3 - 6}{x^3} \qquad x \neq 0$$ Find an equation for C

d) Given that $\displaystyle\int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}} \; dx = k\sqrt{3}$ where k is a constant. Find k

e) $\displaystyle\int_1^3 f(x) \; dx = \frac{4}{3}$ where $f(x) = 2x^2 + 3x + k$ where k is a constant. Find k

f) The points (0 , -3) and (2 , 7) lie on the curve *C* whose gradient function is given by

$f'(x) = 3x^2 - 4x + k$ where *k* is a constant. Find an equation for *C*

g) $f'(x) = 3 - \dfrac{6}{x^2}$ $x \neq 0$ Find the value of $f(2)$ given that $2f(1) - f(3) = 3$

h) A quadratic curve *C* passes through (*a* , *b*) and (*2a* , *2b*), where *a* and *b* are constants.

The gradient at any given point on *C* is given by $\dfrac{dy}{dx} = 2x - 6$

Find an equation for *C,* in terms of *a* .

i) It is given that $\dfrac{dP}{dt} = \dfrac{15\pi(t-1)}{(1+\sqrt{t})}$ for $t \geq 1$ and P = 3 when t = 1

Find the value of P when t = 4.

j) For a positive number a, let $I(a) = \int_0^a (4 - 2^{x^2}) \ dx$

Then $\frac{dI}{da} = 0$ when a equals

I $\frac{1+\sqrt{5}}{2}$ II $\sqrt{2}$ III $\frac{\sqrt{5}-1}{2}$ IV 1

k) Find the smallest value of $I(a) = \int_0^1 (x^2 - a)^2 \ dx$ as a varies

I $\frac{3}{20}$ II $\frac{4}{45}$ III $\frac{7}{13}$ IV 1

4. a) Find the exact area of the region bounded by the curve with equation

$y = (3 - x)(x + 1) \quad x \in \mathbb{R}$ and the x-axis

b) Find the exact area of the region bounded by the curve with equation

$f(x) = x^2 - 2x + 2 \quad x \in \mathbb{R}$ the x-axis , and the lines $x = 1$ and $x = 4$

c) Find the exact area of the region bounded by the curve with equation

$\sqrt{x} + \sqrt{y} = 1 \quad x \in \mathbb{R}, \quad 0 \leq x \leq 1$ and the coordinate axes.

d) Find the exact area of the region bounded by the curve with equation

$y = (x + 1)(x - 2)(x - 4) \quad x \in \mathbb{R}$ and the x-axis.

e) Find the exact area of the region enclosed by the curve with equation

$y = x^2 - 1$ the x-axis, and the lines $x = -2$ and $x = 2$

f) Find the exact area of the region enclosed by the curve with equation

$y = x^3 + 3x^2 - 4x$ and the x-axis

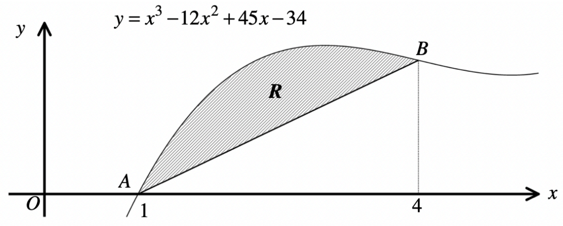
g) Find the exact area of the region enclosed by the curve with equation

$y = |x| - 1$ and the line $y = \dfrac{1}{2}x$

5. a) The figure shows the curve with equation $y = x^3 - 12x^2 + 45x - 34$

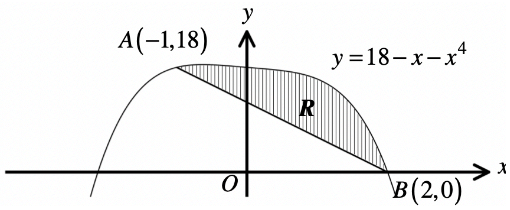
where A has coordinate (1,0) and B has coordinate (4,18)

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



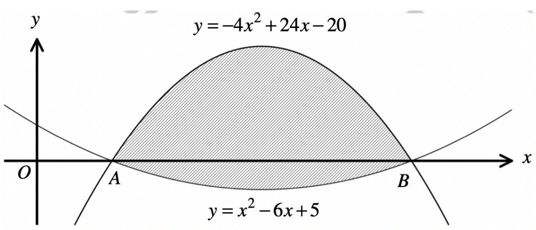
b) The figure shows the curve with equation $y = 18 - x - x^4$

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



c) The figure shows the curves with equations $y = -4x^2 + 24x - 20$ and $y = x^2 - 6x + 5$

Find the area of the shaded region bounded by the two curves.



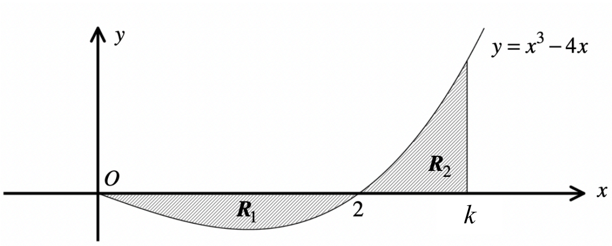
d) The exact area of the region bounded by the curve with equation $y = x^3$ and $y = mx$ is 6, where m is a positive constant. Find the value of m.

e) Find the area of the region bounded by the curve with equation $y = x^3 - x^2$ and the line $y = 2x$ from $x = -1$ to $x = 2$

f) Find the area between the curves with equations $y = p\sqrt{x}$ and $x = p\sqrt{y}$, where p is a positive constant

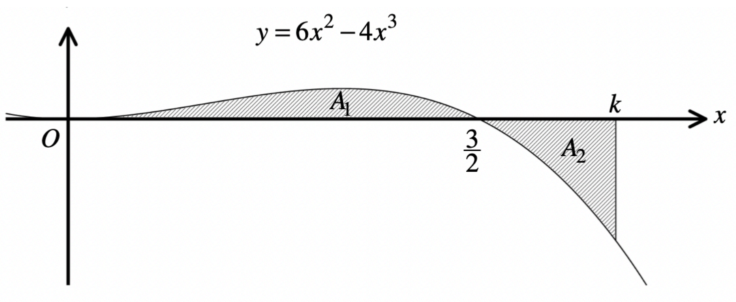
g) The figure shows the cubic curve with equation $y = x^3 - 4x, \quad x \geq 0$

Find the value of k that makes the area of R_1 equal to the area of R_2.



h) The figure shows the graph of the curve with equation $y = 6x^2 - 4x^3, \quad x \in \mathbb{R}$

The point $(k, 0)$ is such that the area of A_1 equal to the area of A_2. Determine the value of k

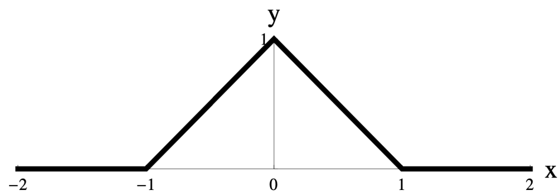


i) Consider the two functions

$f(x) = a - x^2$ and $g(x) = x^4 - a$

Find the values of $a > 0$ for which the area of the region bounded by the x-axis and the curve $y = f(x)$ is bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$.

j) A graph of the function $y = f(x)$ is sketched on the axes below:



Find the value of $\displaystyle\int_{-1}^{1} f(x^2 - 1)\ dx$

k) Let $A = \left(\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cosx\ dx\right) \text{ x } \left(\displaystyle\int_{\pi}^{2\pi} sinx\ dx\right) \text{ x } \left(\displaystyle\int_{0}^{\frac{\pi}{8}} \frac{1}{cos3x}\ dx\right)$

Which of the following is true:

I A = 0 II A > 0 III A < 0 IV A is not defined

6a) The polynomial function f(x) is such that f(x) > 0 for all values of x.

Given that $\int_2^4 f(x)\ dx = A$ and $\int_4^6 f(x)\ dx = B$

find the following integrals in terms of A and B

$\int_2^6 f(x)\ dx$

$\int_2^4 f(x) + 1\ \ dx$

$\int_1^3 f(2x) + 3\ \ dx$

$\int_0^2 f(x+2)\ dx$

$\int_{-6}^{-4} f(-x)\ dx$

$\int_{-2}^2 f(x+4) + 2\ \ dx$

$\int_6^8 f(x-2)\ dx$

$\int_2^3 f(2x)\ dx$

$\int_3^7 5f(x-1)\ dx$

b) A curve has equation $f(x) = x(x-a)(x-b)(c-x)$ with $0 < a < b < c$

You are given that $\int_0^c f(x)\ dx = 0$ $\int_0^b f(x)\ dx = -1$ $\int_a^c f(x)\ dx = -4$

Find the total area enclosed by the curve and the x-axis for $0 \le x \le c$

c) It is given that for all real numbers x the function $f(x)$ satisfies:

$$7 + 2f(x) = f(-x) + 4\left(\int_{-1}^1 f(t)\ dt\right)$$

Find the value of $\int_{-1}^1 f(x)\ dx$

d) Given that $2\int_0^1 f(x)\ dx + 3\int_1^2 f(x)\ dx = 16$ and $\int_0^1 f(x+1)\ dx = 8$

Find the value of $\int_0^2 f(x)\ dx$

e) The function $f(x)$ satisfies the condition that $f(-x) = f(x)$

You are given that $\int_{-4}^{4} f(x)\ dx = 6$ $\int_{-1}^{4} f(x)\ dx = 2$ $\int_{-5}^{-1} f(x)\ dx = 7$

Find $\int_0^5 f(x)\ dx$

f) The function $f(x)$ is such that $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$

You are given that $\int_{-4}^{4} f(x)\ dx = 4$ $\int_{-4}^{4} |f(x)|\ dx = 10$

Find $\int_{-4}^{0} f(|x|)\ dx$

g) Given that $\int_0^4 (f(x))^2\ dx + \int_0^4 f(x)\ dx = \int_0^1 f(x)\ dx$

Are the following statements true or false, or is there insufficient information?

i) $\int_0^4 f(x)\ dx \leq \int_0^1 f(x)\ dx$

ii) $\int_0^1 f(x)\ dx \geq 0$

iii) $f(x) \leq 0$ for some x with $1 \leq x \leq 4$

7a) Use the trapezium rule with five equally spaced ordinates (four strips) to find the value of

$$\int_0^4 \frac{2^x}{x+2}\ dx$$

Use your answer to estimate the value of the following integrals:

$$\int_0^4 \frac{2^x}{x+2} + 3\ dx \qquad\qquad \int_0^4 \frac{2^{x+3}}{x+2}\ dx$$

b) A curve has equation $y = f(x) = -x^2 + 8$

State whether the trapezium rule gives an overestimate or underestimate for the total area under the following curves between $x = 0$ and $x = 1$

i) $y = f(x)$

ii) $y = f(x-1)$

iii) $y = 10 - f(x)$

iv) the curve $y = f(x-1)$ reflected in the line $y = 5$

c) The function f is such that $0 < f(x) < 1$ for $0 \le x \le 1$

The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ and this produces an underestimate.

If the trapezium rule is used to estimate the following integrals with the same number of equal intervals, does it give an overestimate or underestimate

i) $\int_0^1 3f(x)\ dx$

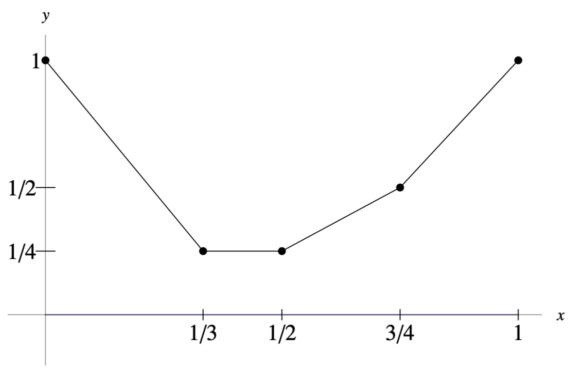
ii) $\int_0^1 f(x) + 1\ dx$

iii) $\int_{-1}^0 f(x+1)\ dx$

iv) $\int_{-1}^0 f(-x)\ dx$

v) $\int_0^1 1 - f(x)\ dx$

d) The graph of a function f(x), is drawn below for $0 \le x \le 1$



The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ by dividing $0 \le x \le 1$ into n equal intervals

The estimate calculated will equal the actual integral when

I n is a multiple of 4 | II n is a multiple of 6

III n is a multiple of 8 | IV n is a multiple of 12

e) The trapezium rule is used to estimate $\int_0^1 2^x \; dx$

by dividing the interval $0 \le x \le 1$ into N equal subintervals, and the answer achieved is:

I $\quad \dfrac{1}{2N}\left\{1 + \dfrac{1}{2^{\frac{1}{N}} + 1}\right\}$ $\qquad$ II) $\quad \dfrac{1}{2N}\left\{1 + \dfrac{2}{2^{\frac{1}{N}} - 1}\right\}$

III) $\quad \dfrac{1}{N}\left\{1 - \dfrac{1}{2^{\frac{1}{N}} - 1}\right\}$ $\qquad$ IV) $\quad \dfrac{1}{2N}\left\{\dfrac{5}{2^{\frac{1}{N}} + 1} - 1\right\}$

f) For a function f(x), the trapezium rule with 3 ordinates (2 strips) gives an estimate of **6**

for the following definite integral $\int_0^4 f(x) \; dx$

With 5 ordinates (4 strips) the estimate is **6.2**

What would the trapezium rule estimate be with 2 ordinates (1 strip) of $\int_1^3 f(x) \; dx$