TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS WITH PHYSICS 3)

DECEMBER 1999.

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others rather hard. You are not expected to answer all of them, or necessarily to complete questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper with you to the interview afterwards.

1. Sketch the locus of the centres of circles which touch two fixed and unequal circles. Find the angle between the asymptotes.

2. Let A denote the matrix $\begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$. Find the matrix A^{1000}. *Hint: it's a good idea to find a recurrence relation between the powers of A.*

3. Let $\mathbb{Z}[\sqrt{2}]$ denote the collection of all real numbers r of form $r = a + b\sqrt{2}$, $a, b \in \mathbb{Z}$. For such an r, define $N(r) = a^2 - 2b^2$; and show that $N(rs) = N(r)N(s)$. Hence show that there are infinitely many pairs of integers a,b with $a^2 - 2b^2 = 1$, and infinitely many pairs such that $a^2 - 2b^2 = -1$. How about pairs such that $a^2 - 3b^2 = \pm 1$?

4. The parallelogram P1 has vertices $(0,0), (0,1), (1,1)$ and $(1,2)$. The parallelogram P2 has vertices $(0,0), (6,0), (1,1)$ and $(7,1)$. Find a 2 by 2 matrix A that takes P1 onto P2 when it acts on $\mathbb{R}^2$. Do you think there is any matrix B such that B^2 takes P1 onto P2? Without doing detailed calculations, explain why (or why not).

5. Sketch the curve defined by the equation

$$y^2 = \frac{1}{x-1} + \frac{2}{(x-3)^2}$$

Sketch the three dimensional surface defined by

$$x^2 - y^2 - z^2 = 1.$$

6. Little Johnny is a biologist tracking the movements of deer with a radio homing device. He has two listening posts, located at points $(X, 0)$ and $(0, Y)$ in the plane, and periodically they give simultaneous, accurate bearings (θ_n, ϕ_n) (let's say, anticlockwise from the X axis) from the listening posts to the deer. Johnny wants to calculate the deer's positions (x_n, y_n) in cartesian coordinates. He has hitherto done this by spreading the map of the area on his kitchen table and using rulers, protractors and Stuff.

Devise a couple of trignometric formulae so he can do the whole thing on his computer.

Note: this problem was given to one of your interviewers, Dr. Read, in real life. Thousands of points had been done by his predecessor using the kitchen table method, when

"Johnny" took over the project and the bright idea of consulting a mathematician occurred to him.

7. A politically correct serial killer murders her victims with a lethal overdose of Fair-Shares-For-All Coffee. She serves 6 cups of coffee on a tray, 4 of which have a dose x grammes each of the deadly powder, but the other two contain $2x$ each. A dose of $4x$ or more is instantly fatal.

Fred Innocent drinks two cups of the coffee at random and survives. What is the probability that he will survive if he takes a third?

8. Consider the following simplified model of a skater falling over: Two equal masses m are attached to opposite ends of a light, straight rod of length 2. One mass rests at $(0, 0)$ on a *frictionless* horizontal surface and the other is balanced, rather precariously, vertically above it at $(0, 2)$. The system is disturbed slightly. Find the (x, y) coordinates of the two masses when the angle of the rod to the vertical is θ.

The lower mass loses contact with the frictionless surface when the horizontal velocity of the lower mass is a maximum. Why? What is the value of $\cos\theta$ when this happens?

9. A rope is wrapped M whole turns round a cylindrical post, the two ends of the rope going in opposite directions. The coefficient of friction between rope and post is 0.25. It is desired that by pulling with a force of $1N$ on one end of the rope, I can prevent the rope from moving away from me even if a force of $10^6 N$ is applied to the other end. How large does M have to be? (Note that to 3 significant figures, $\log_e(10^6) = 13.8$).

Hint: Let the tension in the rope decline like $T(\theta)$ with the angle θ round the post. Investigate $T(\theta + \delta\theta) - T(\theta)$.

10. Consider a mass m at position $x(t)$ on a rough horizontal table attached to $x = 0$ by a spring that exerts a force $-kx$. The force f due to friction between the table and the mass is given by

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \le f \le F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0 \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves then the maximum distance from the origin decreases by $2F/k$ per half cycle.