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Concepts

Newton and suchlike

There are certain ideas in mechanics without which it is difficult to get anywhere. This is mechanics, not rigours-R-us, so let's say displacement is the difference in position of something over a period of time. (It's a vector quantity, as opposed to distance.) Velocity is the rate of change of displacement; acceleration is the rate of change of velocity. Mass is… well, mass. Wikipedia defines it helpfully as 'a property of a physical body'. I suppose this will do. Anyway, momentum is the product of (scalar) mass with (vector) velocity. Probably enough definitions for now.

(I probably ought to note - while I shall try to introduce a lot of ideas from basic principles, in practice I will assume that you're reasonably familiar with the basic A-Level content. It is not so much that you need extra content to tackle STEP mechanics if you're comfortable with the A-Level modules, but there are certain ideas and ways of thinking about things that crop up in STEP a fair amount compared to A-Level, which is what I will try to focus on rather than what I might presumptuously call 'the basics'.)

Isaac Newton, on the whole, was a rather top bloke. A slightly impressive number of things are named after him - his law of cooling, his law of restitution that we will meet later, his law of gravitation that we probably will if I remember - though foremost among these were his three laws of motion, normally stated along these lines:

1. In an inertial frame, objects will move at a constant velocity unless acted on by a resultant force.
2. In an inertial frame, the sum of the forces on an object is equal to its rate of change of momentum.
3. Every action has an equal and opposite reaction.

The idea of an inertial frame is not to be worried about - it basically means that the overall 'laboratory' is moving at a constant velocity. (Notable examples of when we don't have the luxury of an inertial frame are dodgy relative motion questions where you have relative accelerations all about the place, which I shall come onto later.)

Each of these looks a bit abstruse; their application generally becomes clear with practice. It is often said today that the purpose of the first law - which would generally seem to be merely a specific case of the second - is really just a definition of an inertial frame. The second law can be written, via the product rule, as

$$F = dp/dt = m\,dv/dt + v\,dm/dt,$$

where F is force and p is momentum (both of which are, in general, vectors, but I here refer to as scalars for simplicity). Since the mass is often constant, this can often be written in the more famous form

$$F = m\,dv/dt = ma$$

which is more than occasionally useful. Another key idea in this regard is impulse, which is simply the vector change in momentum of the body,

$$\int_0^{p'} dp = \int_0^{t'} F dt.$$

The total change in momentum of a system over a period of time can hence be determined if the force on the system is known. This is often the starting point for variable mass problems, of which more later. Its more specific brother, that the momentum in an inertial frame is conserved, is often essential for collision problems.

Newton’s third law basically states that if an object A exerts a force F on another object B, then object A experiences a force, magnitude F, in the opposite direction, due to object B. There is a common misconception here which, although not tremendously important in most mathematical contexts, has always bothered me a fair bit. Many books state that if a teapot, say, rest on a table, the book is ‘pulled down’ by the force of its weight. While this is not in itself wrong, they often say that it thus experiences an equal and opposite force from the table. Again, it certainly does experience such a reaction force from the table. But this is **not** its “Newton’s third law pair”. After all, the two forces referred to by Newton must exist between the same pair of objects. The real such pair is the gravitational force exerted on the Earth by the book. As I said, a small distinction but I suppose it helps to have these things clear.

Energy

The idea of energy also has a very helpful Wikipedia definition: “A property of objects.” The thing is, there isn’t really that much more to it than that. At some point, someone noticed that this quantity was conserved, and thus started to work stuff out about this remarkable quantity. (Well, in fairness there’s a bit more to the story than that, but there are better places to discuss it.)

One form of energy that is often encountered is kinetic energy, which objects have on account of their having a mass and a speed. It is given by $T = 0.5mv^2$. Many other energies you may encounter are potential energies. These can often be interpreted as the ‘work done’ by some force,

$$W = \int_0^{x'} F dx.$$

For example, near the surface of the Earth, the gravitational force acting on a body is *mg,* where *g* is the freefall acceleration 9.8ms^-2. As such, the energy input required to raise an object of mass *m* a distance *x'* is given by

$$V = \int_0^{x'} mg\,dx = mgx'.$$

This is the gravitational potential energy of a body. (It is often helpful to set up an arbitrary height at which GPE is 'zero'.)
To give another example, the work done against a constant frictional force F after moving through a distance d is given by Fd.

The thing that makes this idea of energy worthwhile is that the bastard is conserved, provided there are no external forces acting on a system. If I drop a teapot of mass m from rest, and it falls a distance H, then the energy it loses due to the fall in GPE is *mgH.* This must be equal to the kinetic energy it gains, since there are no other energy changes going on, so

$$mgH = 0.5mv^2,$$

and thus

$$v = \sqrt{2gH}$$

where v is the final speed of the teapot. The conservation of energy is a very valuable tool in many types of problem. Another important concept is that of power - energy transfer per unit time - which is instantaneously equal to Fv.

Models

As sang Kraftwerk, models often look good - without them we would be left impotent. The real world is almost always far too complicated to do anything interesting with without shoving it into a computer program, so we adopt certain conventions in order to make problems doable. Most of these modelling assumptions are fairly clear, but I shall explain the awkward ones.

Common types of mechanical object include:

- Particles: point-like masses which allow us to ignore effects due to rotation and air resistance.
- Rods: An object with two dimensions infinitely small compared to the other. It doesn't bend.
- Laminae: An object with one dimension infinitely small compared to the others. It also doesn't bend.
- Beads: A particle that we can get onto a string or wire.
- Pegs: A fixed point on which we can rest something.

We can describe these as:

- Uniform: the mass is evenly distributed through the body.

- Light: possessing essentially zero mass.
- Smooth: creating no frictional resistances.

Other things will crop up occasionally - see the opening sentence of STEP III 2012, Q11 for a good example of excessive modelling - but normally it is either explained or obvious.

General Tips

Probably upwards of four in five mechanics problems benefit from a quick sketch or more detailed diagram. They range from mere clarifications about the directions of forces (common in dynamics) to absolutely positively essential godsends (anything with more than about three forces, especially when there's more than one object, or the forces go in funny directions). Remember there's no limits on the amount of paper you can use, so feel free to use diagrams going up to about half a page or more. I myself got into a bad habit of using idiotically tiny diagrams; don't do this. When you start adding forces to it, you will probably go insane with writing magnitudes on top of magnitudes, angles on top of angles, and so on. It doesn't need to be too artistic - you just need to make distinct objects, the directions and magnitudes of forces, any important velocities and accelerations (being clear to distinguish these from forces), and sometimes masses. It may prove relevant to have a 'before' and 'after' sketch, especially in collisions.

A handy technique in heavily algebraic work is dimensional analysis. In cases where they don't really give you numbers, common in STEP due to the lack of a calculator, it comes into its own. The process generally involves working out the units of important variables at some point, then comparing these with the final answer to see if it's viable. For example, when above I worked out the speed of the teapot, I obtained

$$\sqrt{2gH}.$$

Now H has units of metres, g (being an acceleration) has units of ms^-2, so this expression has units of (m* ms^-2)^0.5 = ms^-1, a speed. This suggests that the answer may well be right (but it does not ensure it, especially when constant factors are involved). While the inability of working out constant factors means this is rarely viable as a means of getting to a final answer directly, it can be useful to check if any silly errors have crept into calculations during involved working. (Problem 1 below shows how one can determine the form of an answer using this technique.)

Problems 1.1

1a) The Earth's mass is taken and compressed into a particle, because we can do these things in physics. The acceleration of another particle when it is a distance *x* from the centre of the new point mass is of the form

$$a = kx^{-2}$$

where k is a constant. Show that

$$k = gR^2,$$

where R is the initial Earth radius.

b) Verify that the time taken by the particle to fall from a distance *r* to the new point mass is plausibly of the form

$$T = C\left(\frac{r^3}{gR^2}\right)^{1/2}$$

where C is a dimensionless constant. Why might this not be unique (though in fact it is correct, as we shall show in a later problem)?

2) Show that Newton’s Third Law is consistent with the conservation of momentum for closed systems.

3) What can one say about the forces acting upon a light particle in a physical model?

4) Show that, instantaneously, P=Fv.

Kinematics and Dynamics

Constant acceleration

Kinematics is largely a story of differential equations getting worse. We shall start with some easy ones before ratcheting it up a bit.

Constant acceleration is a nice, simple model that is, well, what it says on the tin. It is often applicable when consider objects moving under freefall, since near the surface of the Earth the acceleration due to gravity can be taken as constant. Let's start with

$$dv/dt = a,$$

where a is constant. Integrating both sides,

$$\int_u^v dv = \int_0^t a dt$$
$$v - u = at \text{ (1)}$$

We can further derive:

$$dx/dt = u + at$$

$$\int_0^s dx = \int_0^t u + at\, dt$$
$$s = ut + \frac{1}{2}at^2 \text{ (2)}$$

I probably ought to say that it's generally bad practice to put the variable you're integrating with respect to as a limit of the integral. But this is mechanics - it very rarely matters.

In any case, we now have the time-dependence of both speed v and distance s. (These apply equally well if a, v, u and s are all vectors. Though t probably shouldn't be a vector. That would be quite bad in this context.)

Further algebra shuffling (which you may wish to try yourself) gives the equations

$$s = \frac{(u+v)t}{2}$$
$$v^2 = u^2 + 2as$$
$$s = vt - \frac{1}{2}at^2.$$

That's about all there is of much interest to say about constant acceleration.

Non-constant acceleration

Differential equations can generally be determined using the differential forms for acceleration:

$$acceleration = dv/dt = dv/dx * dx/dt = vdv/dx$$

depending on whether the acceleration is given in terms of displacement or time. Beyond this, almost everything is just either integration or the use of first- and second-order differential equation techniques, which I don't intend to go into in much detail here. Notice how, through F=ma, these techniques can easily be extended to problems where forces are given rather than accelerations directly.

Problems 2.1

1) A train starts from a station. The force exerted by the engine is initially constant, F. After the train reaches speed u, the engine works at constant rate P, such that P=Fu. The mass of the engine and train together is M. Find:

a) The time taken, t, to reach speed v>u;

b) The distance travelled, x, in reaching this speed u.

[Adapted from STEP II, 1993, Q13]

2) A particle of unit mass is projected vertically in a medium in which it faces a resistance of magnitude k times its velocity squared. If its initial velocity is u, find:

a) The speed of the particle after rising a distance s,

b) The maximum height above the point of projection reached by the particle.

c) Does your result in part (a) hold on the downward path?

[Adapted from STEP I, 1997, Q11]

3) Revisit problem 1.1.1b above. Show that C = $\frac{\pi}{2\sqrt{2}}$.

Relative motion

There are only really a couple of formulae in relative velocity. The first is:

$$Velocity\ of\ A\ relative\ to\ B = Velocity\ of\ A - Velocity\ of\ B,$$

or equivalently, the velocity of A assuming that B is stationary.

Much of the time, all that is required beyond this is geometry with vector diagrams. There are a few key ideas. The first is that effective velocities can often be found via adding vectors according to the standard geometrical rules, which can make a lot of problems more tractable provided you're comfortable with sine and cosine rule and suchlike. (To be fair, this sort of geometrical stuff doesn't come up tremendously often on STEP. Algebraic vector work is somewhat more common, but I shall briefly put this down for completeness.)
Another important idea is that when using these vector triangles, working can often be simplified by bringing one of the two moving bodies to rest. If a constant velocity is added to the motion of two bodies, their relative motion is unaffected, provided that one is taken as stationary. If two things need to collide, bring one to rest and make the relative velocity of the other point towards the one at rest. (This makes little sense, I'm sure; Edexcel M4 does a somewhat better job of it than I do.)

Alternatively you may need to find the closest approach of two things. Either approach it geometrically, which may work in STEP though not too often, or look at it vectorially. The relative displacement of two things is easy enough to find given their two position vectors, and from here it is required to minimise the magnitude of this relative displacement. In practice, square roots mean it is often a lot easier to minimise the magnitude squared. Another useful result is that when two things are at their closest approach, the dot product of their relative velocity and relative displacement is zero.

When things don't have constant speed, and forces are involved, it becomes appreciably more complicated. The neat idea of *Galilean relativity*, which basically says that as we have noted, physics works as usual as long as your lab has a constant velocity, ensures that adding and subtracting velocities doesn't much affect the physical laws at work - after all, 'bringing an object to rest' is essential choosing a convenient frame of reference to work in. However, accelerating frames are not inertial frames, unfortunately. To correct for this problems may be simplified by introducing a fictitious force.

For example, suppose you have a stationary block. You sit there with the block, nothing happens, and all is well. You wander off at a constant velocity, the block seems to move away at a constant velocity and as such there is no reason for a force to act on it, and all is mostly well. However, should you wander off with acceleration **a**, the block will seem to have acceleration **-a**: and as such a force will seem to act upon it. While it is not 'real' in the sense that any forces will be acting on yourself, not the block, it often proves helpful to introduce the idea of this *fictitious force*, which in this case is equal to -m**a**.

Frames of reference are rather easy to get muddled up in the heat of an exam, but choosing the right one can often simplify things. Just make sure you remember to change things as appropriate, or things can go quite horribly awry.

Problems 2.2

1) Show that, at closest approach, the dot product of relative velocity and relative displacement is zero.
2) Point B is a distance d due south of point A on a horizontal plane. Particle P is at rest at B at t = 0, when it begins to move with constant acceleration a in a straight line with fixed bearing β . Particle Q is projected from point A at t = 0 and moves in a straight line with constant speed v . Show that if the direction of projection of Q can be chosen so that Q strikes P, then $v^2 \geq ad(1 - cos\beta)$.

[Adapted from STEP III 2003, Q11]

3) A particle of mass *m* is held at rest on a plane face of a wedge of mass M; this wedge is free to move on a smooth horizontal surface, and this face is inclined at angle αto the horizontal. The particle is then released. Find the acceleration of the particle relative to a stationary observer.

Projectiles

The projectile is a stalwart of STEP mechanics; it's complicated enough such that there's a lot of interesting and new stuff that can be done with it, but it's sufficiently simple that the equations don't end up utterly intractable (though some may disagree). As such, they turn up really rather frequently, in all sorts of different guises.

Projectiles, at their core, consist of throwing something and seeing what happens. The key to doing stuff with them is realising that since the weight of the particle only acts vertically downwards, we can separate their motion (normally, at any rate) into vertical and horizontal components and look at each individually. Horizontally, no forces act upon the particle (provided no air resistance, which is not always true in STEP scenarios), and so it generally has a constant horizontal component of speed. Vertically, it has constant free-fall acceleration g, so we can use the constant acceleration equations that we derived above.

We can write thus write down

$x = utcos\alpha\,,\ y = h + utsin\alpha - \frac{1}{2}gt^2$

where h is the height of the point of projection, t is the time, u is the initial speed, and αis the angle of projection. We can rearrange the former, and substitute into the latter, to obtain the useful form

$y = h + xtan\alpha - \frac{gx^2}{2u^2cos^2\alpha}.$

The important thing about this is that it is not time-dependent. It is thus the cartesian curve described by the trajectory of our original projectile, which can often come in useful. It may also prove useful to note that since $\frac{1}{cos^2\alpha} = sec^2\alpha = 1 + tan^2\alpha$, we can consider this as a quadratic not just in x, but also in $tan\alpha$.

Problems 2.3

1) Derive expressions for the range, maximum height, and minimal speed of a particle projected at speed u, angle α from level ground. (These may be useful to commit to memory.)
2) Show that, if $\alpha$is the initial angle of projection, $\theta$is the angle of elevation of the projectile from the initial point of projection, and $\phi$is the angle that the velocity of the projectile makes with the horizontal, $tan\alpha \ + \ tan\varphi \ = \ 2tan\theta$ at all points on the trajectory.
3) A cannon is fixed at the bottom of a plane inclined at angle β to the horizontal. A ball is fired with initial speed u. Show that the maximum distance from the cannon that the ball can reach up the plane is $\frac{u^2}{g(1+sin\beta)}$.

[Adapted from STEP III 1995 Q10]

4) A shell explodes on the surface of horizontal ground. Earth is scattered in all directions with varying velocities. Show that particles of earth with initial speed v landing a distance r from the centre of explosion will do so at times t given by

$$\frac{1}{2}g^2t^2 \ = \ v^2 \ \pm \sqrt{v^4 - g^2r^2}$$

Find an expression in terms of v, r and g for the greatest height reached by such particles.

[STEP I, 1998, Q10]

<u>Elastic stuff</u>

Springs and strings don't seem to go very well anywhere, but mostly it is two formulae you need to know. Strings and springs (that single letter is often really quite important) have two main characteristics as far as we're concerned: a *natural length l* and a *modulus of elasticity* λ*.* The former is, if you like, the length it goes to if there are no external forces acting upon it; the latter is a measure of the 'stiffness' of the spring or string (the higher $\lambda$is, the stiffer it will be). The distinction between these two species is that a string does not exert a force if it is compressed such that its length is below its natural length - it simply goes slack, and has zero tension. A spring, on the other hand, seeks to return to its natural length. A compression by a distance x, in a spring, will produce the same magnitude force as an extension by x - it is simply in the opposite direction, and instead of tension it is often referred to as thrust (handily meaning we can use the letter T in both instances). Hopefully this will tally with common sense. The size of this tension (or thrust) is given by

$$T \ = \ \frac{\lambda x}{l}.$$

This gives rise to a stored *elastic potential energy*

$$E \ = \ \frac{\lambda x^2}{2l}.$$

<u>Problems 2.4</u>

1) Prove, by considering the work done on the string or spring, the above expression for E, stating in each case the values of x for which it is valid.

2) A smooth cylinder with circular cross-section of radius a is held with its axis horizontal. A light elastic band of unstretched length 2πa and modulus of elasticity λ is wrapped round the circumference of the cylinder, so that it forms a circle in a plane perpendicular to the axis of the cylinder. A particle of mass m is then attached to the rubber band at its lowest point and released from rest.

(i) Given that the particle falls to a distance 2a below the below the axis of the cylinder, but no further, show that

$$\lambda = \frac{9\pi mg}{(3\sqrt{3}-\pi)^2}.$$

(ii) Given instead that the particle reaches its maximum speed at a distance 2a below the axis of the cylinder, find a similar expression for λ .

[STEP I, 2001, Q10]

Circular Motion

Like poor lecturers, objects in mechanics often go around in circles. It is hence useful for us to have a bit of a think about what exactly happens when they do so - not just for their own sake, but also because it has a fairly neat conceptual linkage with simple harmonic motion, which we will soon meet.

The first thing to note is that when something goes in a circle, even if it goes at constant speed, it must be accelerating - for it is always changing direction at any rate. This acceleration can be investigated, and formulae derived, by a number of methods. The most common is probably that based on geometrical considerations, as can be found in a number of places online, but I'll go for a slightly different one based on complex numbers (a surprisingly powerful tool at all levels of mechanics). If we let the real axis on an argand diagram be the x-axis, and the imaginary axis be the y-axis (as is natural), then we can consider a particle moving around the origin in a circle of constant radius R. When the angle its radius vector makes with the real axis is θ, then we can consider its position to be

$$z = Re^{i\theta}.$$

Differentiating - and remembering that i can in most circumstances be regarded as just another constant for calculus - we get:

$$z' = R(i\frac{d\theta}{dt})(e^{i\theta}) = i\omega z,$$

where $\omega$is the first time derivative of θ. Note that multiplying a complex number by i rotates it anticlockwise by 90 degrees, so the velocity of our particle is always perpendicular to its radial vector. To get the acceleration, we differentiate once more, getting to

$$z'' = (i\omega')(z) + (i\omega)(z') = i\alpha z - \omega^2 z = R(i\alpha - \omega^2)e^{i\theta},$$

where αdenotes the second time derivative of θ. So we can read off the radial component of the acceleration as being directed towards the origin, and of magnitude $\omega^2 R,$ while the component perpendicular to the radial vector is of magnitude $R\alpha$ in the direction of increasing θ. When the particle moves at constant angular speed, then we can recover its constant speed as $v = \omega R$. Between these formulae and their application, most of circular motion can be sorted - but remain aware of where the circle and its centre actually are. It sounds stupid but the centripetal force can often end up misplaced.

The key idea is that if something is going around in a circle, there must be some force acting upon it that provides a resultant force towards the centre of the circle. If a herring is going about on a record turntable, there is a centripetal force, friction, in effect 'pushing' it towards the middle. If a ball is rolling around the inside of a smooth bowl in a circle, then the reaction force of the bowl upon the coin is doing the very same. Never make the fatal error of imagining some magical 'centripetal force' appearing out of nowhere - it will always be in the guise of one of the forces you are familiar with.

This will be followed up in the section on rotational dynamics, which I may eventually write.

Problems 2.5

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Simple Harmonic Motion

The definition of simple harmonic motion (SHM), taken alone, is not tremendously enlightening. If an object executes SHM, it has an acceleration proportional to, and oppositely directed to, its displacement from its centre of oscillations, or in mathematical terms

$$x'' = -\omega^2 x$$

where x is the displacement from the centre of oscillations and ωis the angular frequency of the oscillations. This turns out to be surprisingly inter-related with the ωwe have met from circular motion, which I will discuss in due course. We denote by A the amplitude (maximum displacement) of the oscillations, and by T the period of the oscillations.

Problems 2.6

Using the definition of SHM, derive the following formulae, which will likely prove useful:

$v^2 = \omega^2(A^2 - x^2)$, $x = Asin(\omega t + \varepsilon)$, $T = \frac{2\pi}{\omega}$.

In the second formula above, εdenotes the phase angle, in other words how far through the oscillation the stopwatch starts. If $\epsilon = 0$, the particle starts at the centre of oscillations; if $\varepsilon = \pi/2$, it starts at one end of the oscillations. Both of these special cases are useful, though the former is perhaps preferable. (Note that the latter case reduces to $x = Acos\omega t$).

This concept recurs often throughout physics, all the way down to the harmonic oscillator of quantum mechanics; we will generally be looking at somewhat more quotidian situations, alas, but STEP can sometimes throw an edgy spanner in the works. The most common situation is perhaps spring and string systems. (Here, the distinction is important: while spring systems will generally oscillate with SHM throughout the period, there is the possibility in a string system that a string goes slack partway in, buggering up the equations. This will be shown in an exercise below.)

When trying to get simple harmonic motion out of a system, it is best to examine it from the perspective of the equilibrium position. Analyse it like any other system: look at an extension 'x' (or other variable as appropriate) from the equilibrium position, work out the resultant force on the thing, use F=ma and pray to God that you end up with an equation of the form above.

However, many systems don't execute perfect SHM, but mere *approximate* SHM, normally for small amplitude oscillations. Perhaps the best example of this is a pendulum - while the bob moves with almost perfect SHM if you don't shift it too much, move it too far and the period goes a bit squiffy. As a general rule, the force equation will end up ugly in such circumstances, and you will often wish to use either a first-order (linear) binomial expansion, or a small-angle trig approximation (often $\sin\theta \approx \theta$).

A final technique that is occasionally of use is that of differentiating the energy equation. Suppose you have a spring oscillating horizontally with a mass on the end. The total energy of the system at a given time will be

$$0.5mv^2 + \frac{\lambda x^2}{2l} = E.$$

Differentiating both sides, and recalling that the total energy is conserved, this becomes

$$mva + \frac{\lambda x v}{l} = 0$$

Or

$$a = -\frac{\lambda}{ml}x,$$

Giving us the required SHM equation. Of course, this could easily be obtained from the force equation this time but this is not always the case.

<u>Problems 2.7</u>

1) The string AP has a natural length of 1·5metres and modulus of elasticity equal to 5g newtons. The end A is attached to the ceiling of a room of height 2·5 metres and a particle of mass 0·5 kg is attached to the end P. The end P is released from rest at a point 0·5 metres above the floor and vertically below A. Show that the string becomes slack, but that P does not reach the ceiling.

   Show also that while the string is in tension, P executes simple harmonic motion, and that the time in seconds that elapses from the instant when P is released to the instant when P first returns to its original position is

   $$\sqrt{\frac{8}{3g}} + \sqrt{\frac{3}{5g}}(\pi - arccos(3/7)).$$

[STEP II, 2000, Q11]

Collisions

‘Collisions’, so far as step is concerned, is throwing a few snooker balls at each other and seeing what happens. Naturally, this completely trivialises the area, which comes up with a fair regularity on STEP at all three levels. Ultimately, it boils down to two concepts: the conservation of momentum, touched on in the first section, and ‘Newton’s Experimental Law’.

To briefly recap, the principle of the conservation of momentum states that both before and after a collision - or anything else for that matter - assuming that no external forces act on the bodies involved, the total (vector) momentum must be the same before and after. Indeed, it follows as a result of Newton’s third law - see if you can see why. It is important to remember that this principle always holds, regardless of whether the collision is elastic, inelastic or anything in between.

Not content with only have three laws to his name, Newton came up with an expansion pack in 1687, for a collision of two bodies:

$$\frac{Relative\ velocity\ after\ collision}{Relative\ velocity\ before\ collision} = e,$$

where *e* is known as the *coefficient of restitution.* This is constant for any given pair of objects. This varies from e=0 (perfectly inelastic), in which all kinetic energy is lost, all the way up to e=1 (perfectly elastic), in which all kinetic energy is conserved. In reality, the value will be somewhere in between.

Problems 3.1

1. Show that, for a collision of two small smooth spheres of masses m and M, the conservation of kinetic energy implies that e=1, using the above definition.

This is enough to deal with almost all problems where two bodies collide in a straight line. However, they can also collide obliquely. The key here is to remember that you can sort of imagine the collision taking place only along the line between the objects, passing through the point of contact. Parallel to this line, once the velocities involved have been resolved into components, the two principles outlined above can be applied as ever. Perpendicular to this line - since no force is exerted in this direction - the velocity of either body cannot change. Note that this is not necessarily true for particles, which can go off how they like so long as momentum (and energy as appropriate) is conserved.

Problems 3.2

1. A particle P of mass m collides with a stationary sphere Q of mass 4m. Immediately prior to the collision, particle P has speed u. After the collision, P travels off at a speed v at an angle of θ to its initial direction, while Q travels off at a speed w at an angle of ϕ to its initial direction, with them both travelling away from each other. Show that, if kinetic energy is conserved in the collision, $3\theta \approx -8\phi$.

Statics

Please be patient; statics will only take a moment. And possibly resolving forces a few times.

Quite seriously, I've seen statics described as a matter of taking moments once and resolving twice, which seems to be a bit of a mantra. (Incidentally, taking moments twice and resolving once is often fine, but more on that later.) Perhaps statics is not quite the most interesting aspect of mechanics, but I assure you there is more guile to it than that recipe.

Perhaps the most important idea in statics is that of *equilibrium*, which quite naturally means that a system has no tendency to change. It must have no tendency to accelerate in one direction, nor may it tend to rotate at an increasing rate. However, it is perfectly permissible to say that a body moving at a constant velocity is in equilibrium, or that something spinning at a constant rate is in equilibrium. This is perhaps a bit unintuitive, but it follows from Newton's Laws (and their analogous rotational counterparts): since the laws of physics are the same in all inertial frames, there is no such thing as 'absolute motion' or being truly at rest. A common thought experiment is that a statics experiment taking place in a sealed train carriage moving along a straight track with constant speed will give exactly the same results as one conducted in a 'stationary' laboratory. As such, physically it makes little sense to deem the latter equilibrium superior to the former, since for all practical purposes they are indistinguishable. Our planet is moving with respect to the Sun, which is moving with respect to the galaxy, and so on.

In any case, I've probably just confused matters. The important thing to take away is that if an object is in equilibrium, it is neither rotationally nor linearly accelerating. As such, there is no resultant force, nor a resultant moment, on it.

This 'moment' is defined as the force concerned multiplied by the distance from the force's line of action to a certain point, or equivalently in vector notation as **r x F**, where r denotes the position vector of any point on the force's line of action with respect to the point about which moments are taken. It can broadly be considered as the rotational analogue of force, and as such, the resultant sum of the moments acting on a body must be zero, when taken about any point on the body, for the body to be in equilibrium.

A lot of the art to statics comes from cannily deciding where to take moments about, and in which direction to resolve forces. Since, respectively, any point and any direction can be chosen, it is often sensible to choose those which eliminate irrelevant forces in order to simplify the resulting equations. (While the system of equations will always be equivalent, some forms are better suited to not killing yourself than others.)

Friction

There are a number of models that can be used for friction. The default at this level is that for any pair of objects, there is a value called the coefficient of friction, normally denoted μor λ, for which the relationship

$Maximum\ frictional\ force\ =\ \mu\ *\ Reaction\ force\ between\ the\ objects$
holds.

There's a little subtlety in this. Often a static situation will be described as 'limiting equilibrium', which means that the frictional force is at its maximum as defined above. However, if this isn't explicitly stated, the frictional force may take any value up to and including this maximum value, which will need to be determined via other methods. This allows one to derive inequalities, which questions sometimes centre around.

Let's have a look at how this might all come together in practice, in a fairly simple case - the inclined plane, angled at θto the horizontal, and with a particle of mass m upon it, with coefficient of friction μ between the two pertinent objects. What is the condition for the particle to stay put?

We first choose directions to resolve forces in. In this case, since friction acts parallel to the plane, and the reaction is perpendicular to it, we have two directions that seem like good choices. So, parallel and perpendicular to the plane:

$mgsin\theta\ =\ F, mgcos\theta\ =\ R$

where the letters have their natural meanings. And since F/R≤μ, this becomes
$\mu\ \geq\ tan\theta$, which is pleasantly neat. Note also that this is a lower bound for μ, which is what we expect - it makes no sense to have a maximum level of friction if we want the particle to stay where it is. Simple checks like this can be useful, as I have mentioned.

Intriguingly, the angle of friction, arctanμ, comes in useful more often than you might expect. If a system is in limiting equilibrium, this is the angle between the normal to some surface and the total force acting on the object due to this surface, which can provide a quick way to solve certain problems in combination with the idea of a triangle of forces.

The Potential Test

Considering the total potential energy of a system gives us an equivalent, but in practice rather different, method for investigating equilibria. I shall provide a slightly dodgy, but I hope fairly intuitive reason of why this works.

We know that the force associated with a potential U_i is $\frac{-dU_i}{dx}$, and that a key condition for equilibrium is $\Sigma F_i =\ 0$. Equivalently we can say $\frac{d}{dx}\Sigma U_i =\ 0$, or $\frac{dU}{dx}\ =\ 0$- that is, the total potential energy is at a stationary point. If it is a minimum, the system is in a *stable* equilibrium (it will tend to return to this equilibrium if slightly displaced). If it is a maximum, the system is in *unstable* equilibrium.

This can sometimes be an easier test to work with in certain situations. Note that the variable x can be assigned to anything that fully describes the system - this is often an angle or a distance. The expression for the total potential will generally have a constant at the end of it due to the ambiguity of the zero potential, but this doesn't tremendously matter (why?).

Lami's Theorem

This essentially states that if three forces are acting on a body in equilibrium, they all have to meet at the same point. Think about it - any pair of forces must meet at some point. Taking moments about that point, the third force must not provide a resultant moment as otherwise it wouldn't be in rotational equilibrium. As a result, this third force must also pass through this point. This can be combined with the idea of a triangle of forces to link the angles involved and the magnitudes involved with the sine rule.

Problems 4.1

1. A small lamp of mass m is at the end A of a light rod AB of length 2a attached at B to a vertical wall in such a way that the rod can rotate freely about B in a vertical plane perpendicular to the wall. A spring CD of natural length a and modulus of elasticity λ is joined to the rod at its midpoint C and to the wall at D, a above B. Show that if $\lambda > 4mg$the lamp can hang in equilibrium away from the wall, and find the angle DBA. (Attempt this using both the 'traditional' forces and moments method and the potential energy method.)

   [STEP I, 1993, Q9]

Centres of mass

The centre of mass can be defined in a number of ways - for example, it is the point at which a force, when applied to a body, will not cause rotation; perhaps more intuitively it is the point at which the weighted sum of the relative position vectors of the distributed masses sum to zero.

This latter definition implies

$$\sum_i m_i(r_i - R) = 0,$$

Where R is the location of the centre of mass. This can be rearranged to show that

$$R = \frac{1}{M}\sum_i m_i r_i$$

where M is the sum of all the masses. We can intuitively extend this for continuous mass distributions using

$$R = \frac{1}{M}\int r dm.$$

A useful property of the centre of mass - which you should seek to prove using the above definitions - is that when taking moments about an axis, the sum of the individual moments of all the individual masses is equal to the equivalent moment produced by the mass of the body about its centre of mass. This provides a way to find the centre of mass of a body about a specific axis. This can be extended to uniform (and potentially even non-uniform) laminae and solids, by imagining it as a sum of separate strips, of infinitesimal width, perpendicular to the axis in question. This leads to an integral, and thence to victory.

Another property of the centre of mass is that an object, when dangled from a string attached at any point, always hangs such that the centre of mass lies directly beneath the point of attachment. It may be worth considering this in terms of the resultant moment about the centre of mass.

A brief note on toppling, slipping and sliding, which sometimes crop up. Toppling occurs when the centre of mass of an object is just further than above the lowest point of the body - this sounds awkward, I must redraft it, but a little experimentation should give you the picture. Slipping occurs when the resultant force parallel to a surface exceeds the maximum frictional force opposing it, and the object moves off. Sliding is much the same as slipping, except with the connotation of an inclined plane.