Curve Sketching Questions

Waleed

Note

The questions in this document have been compiled from various sources both published and unpublished. I take no credit for any of these questions as none are originally mine.

The purpose of these questions is to get students to be comfortable with sketching curves and graphs, and ultimately to prepare for Oxbridge interviews for those applying for courses such as Maths, Computer Science, and JMC, which have maths-heavy entry requirements.

The questions have been laid out in no particular order and may vary in difficulty.

1. Sketch the graphs of $\sin(x^2)$ and $\sin^2(x)$. What shape is the first graph near the origin?

2. Define $\cosh x$ and $\sinh x$. Sketch both on the same set of axes. Where do both graphs cut the axes? What is the gradient of $\sinh x$ at $x = 0$?

3. (a) Sketch the graphs of:
   i. $|x|^2 + |y|^2 = 1$
   ii. $|x| + |y| = 1$
   iii. $|x|^5 + |y|^5 = 1$

   (b) Hence sketch the graph of $|x|^{100} + |y|^{100} = 1$

4. Sketch the graph of $y = e^{\frac{1}{x}}$

   Hint: What is the gradient close to 0 in the second quadrant?

5. (a) Find $f'(x)$ when $f(x) = x^x$

   (b) Hence sketch the graph of $f(x)$ for $x > 0$. (Consider the graph below $y = 1$)

   (c) Sketch the graph of $y = x^{\frac{1}{x}}$

6. Sketch the graph of
$$y = \frac{(x-3)(x-2)}{(x+2)(x-1)}$$

7. Sketch the graph of
$$y = \frac{1}{(1-x)^2}$$

8. Imagine an octagon with unit edges, centred at the origin. What are the coordinates of its corners?

9. Sketch the graph of $y = x^2 + 3x^3$

10. Sketch the graph of
$$y = \frac{\ln x}{x}$$
and hence find all natural solutions of the equation $a^b = b^a$.
(Problem 84 of TBO Problem Solving Booklet)

11. Sketch the graph of
$$y = \frac{1+x}{x-1}$$

12. Sketch the graph of
$$y = \frac{1}{(x+1)(x-1)}$$

13. Sketch the graph of $y = x^n e^{-x}$ for:
    (a) n is even
    (b) n is odd

14. Sketch the graph of
$$y = \frac{x^2+1}{x+1}$$

15. Sketch $f(x) = x^3 + Ax^2 + B$ for:
    (a) $A > 0$, $B > 0$
    (b) $A < 0$, $B > 0$

16. On the same set of axes:
    (a) sketch the graphs of:
        i. $y = \ln x$
        ii. $y = \log_{10} x$
        iii. $y = \log_2 x$

(b) Why do they look the way they do?

(c) Now sketch $y = \log_x e$ on a new set of axes and reason what you do.

(d) Hence compute

$$\lim_{x\to\infty}(1+\frac{1}{n})^n$$

17. Sketch the graph of $x(x^2+y^2) = 2ay^2$, $a > 0$

18. (a) Sketch the graph of $x^2 - ny^2 = 0$, $n \in \mathbb{Z}$

(b) Find all natural solution pairs (x, y) in the case $n = 9$.

(c) Find all natural solution pairs (x, y) in the case $n = 10$.

(Problem 34 of TBO Problem Solving Booklet)

19. Sketch the graph of $y = x \ln x$

20. Sketch the graphs of:

(a)

$$y = \frac{\sin x}{x}$$

(b)

$$y = \frac{\sin x}{x-1}$$

21. Sketch the graphs of:

(a)

$$y = \sin\frac{1}{x}$$

(b)

$$y = \cos\frac{1}{x}$$

22. Sketch the graphs of:

(a)

$$y = \frac{x+\sin x}{x-\sin x}$$

(b)

$$y = \cos(x+|x|),\ -2\pi < x < 2\pi$$

(c)

$$y = \sqrt{x^3-x}$$

(d)

$$y^2 = x^3 - x$$

(e)
$$y = \frac{x^4 - 7x^2 + 12}{x^4 - 4x^2 + 4}$$

(f)
$$y = \frac{x^2 + 1}{x^2 - 1}$$

(Problems 88-92 of TBO Problem Solving Booklet)

23. Sketch the graphs of the following and comment on their derivatives:

    (a) $y = |x^2 - 1|$

    (b) $y = x^{\frac{1}{3}}$

    (c) $y = x^{\frac{2}{3}}$

    (d) $y = x^2 - x^4$

    (e) $y^2 = x^2 - x^4$

    (Problems 93-94 of TBO Problem Solving Booklet)

24. Sketch the graph of $y = e^{-x^2} - e^{-3x^2}$

    (Problem 95 of TBO Problem Solving Booklet)

25. (a) By sketching the appropriate graphs, find all solutions to the equation $x - 1 = (e - 1)\ln(x)$.

    (b) Hence sketch the graph with equation $y = e^x - x^e$.

    (Problem 96 of TBO Problem Solving Booklet)

26. Write
$$\frac{3e^x - e^{-x}}{e^x + e^{-x}}$$
in the form
$$a + \frac{b}{e^{2x} + 1}$$
and hence sketch the graph of
$$y = \frac{3e^x - e^{-x}}{e^x + e^{-x}}$$

    (Problem 97 of TBO Problem Solving Booklet)

27. Sketch the graphs of:

    (a) $x^{2n} + y^{2n} = 1$ for $n = 2, 4$ and explain what happens to the graph as $n \to \infty$.

    (b) $|3x^2 + y^2 - 12| = |x^2 - y^2 + |$

(c) $y = \sqrt{1 - x^2} + \sqrt{4 - x^2}$

(d) $1 = |x| + |y|$

(e) $1 = |x| - |y|$

(f) $1 = |y - x|$

(Problems 98-101 of TBO Problem Solving Booklet)

28. (a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky,\ x > 0$$

subject to the initial condition $x = 0, y = 1$ and $k > 0$.

(b) Sketch the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky(1 - \frac{y}{M})$$

where M is a large constant and the same initial conditions apply (without directly finding y).

(Problem 104 of TBO Problem Solving Booklet)

29. Sketch the graph of

$$y = \frac{\ln x}{x - 1},$$
$$x > 0$$

For values of y near $x = 1$, it may help to treat $y(1)$ as the limit associated with a derivative.

(Problem 1314 of Towards Higher Mathematics by Earl)