20 YEARS OXBRIDGE APPLICATIONS

TEST FOR MATHEMATICS FOR UNIVERSITY ADMISSION (TMUA)

MOCK TEST 5 ANSWERS

SECTION 1
MATHEMATICAL KNOWLEDGE

1 E.

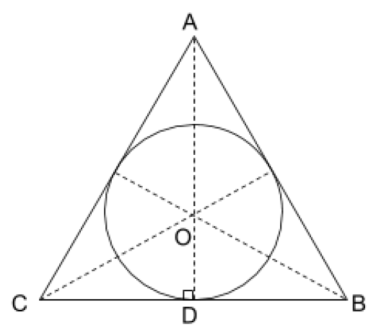
Direct calculation:
$$\int_0^1 \frac{3x^2 - 2\sqrt{x}}{\sqrt[3]{x^4}} dx = \int_0^1 3x^{\frac{2}{3}} - 2x^{\frac{5}{6}} dx = \frac{9}{5} - 12 = -\frac{51}{3}\,.$$

2 A.

Since each wall can be painted in three possible ways, there are 3^4 ways in which the room can be painted. The number of ways in which the room can be monochromatic is 3 because once we choose the colour of the first wall, the colours of the remaining walls are automatically determined. So the probability of the room being monochromatic (all walls having the same colour) is $\frac{3}{3^4} = \frac{1}{27}$.

3 D.

Consider the following diagram:



The length of OD is 3, and the angle COD IS $60°$. Therefore, the length CD is $3\tan 60° = 3\sqrt{3}$. It follows that the area of triangle COD is $\frac{1}{2} \times 3 \times 3\sqrt{3} = \frac{9\sqrt{3}}{2}$. The area of triangle ABC is six times the area of triangle COD. Hence, the area of triangle ABC is $27\sqrt{3}$.

4 E.

After reducing each of a, b, c, d and e by 20%, the given expression becomes
$$\frac{0.8a - 0.8b}{(0.8c)(0.8d)} - \frac{(0.8b)^2 + (0.8c)^2}{(0.8d)^3 + (0.8a)(0.8e)^2} = \frac{1}{0.8}\left(\frac{a-b}{cd} - \frac{b^2+c^2}{d^3+ae^2}\right).$$
The percentage change is therefore $\frac{1}{0.8} \times 100 = 1.25\%$, which is a 25% increase.

5 B.

The inequality $|x+1| \leq 6$ means that $-6 \leq x+1 \leq 6$, which implies that $-7 \leq x \leq 5$. Similarly, $|y+4| \leq 5$ implies that $-9 \leq y \leq 1$. Therefore, the greatest value of xy is $(-7)(-9) = 63$.

6 B.

As x varies over the real numbers, so too does $8x - 7$. Therefore, $0 \leq \cos^2(8x-7) \leq 1$, so that $-6 \leq 9\cos^2(8x-7) - 6 \leq 3$. The maximum value attained by $\left(9\cos^2(8x-7) - 6\right)^2$ is therefore 36.

7 D.

Since h is inversely proportional to r^3, there is a constant k such that $h = \frac{k}{r^3}$. So the volume of the cylinder is

$$\pi r^2 h = \pi r^2 \left(\frac{k}{r^3}\right) = \frac{k\pi}{r}\,.$$

When $r = 1$, we therefore have $3\pi = k\pi$, which implies $k = 3$. When the volume equals π, we find that $r = 3$. Therefore, the height is equal to $\frac{k}{r^3} = \frac{1}{9}$.

8 A.

A reflection in the line $y = 1$ is the same as a reflection in the x-axis, followed by a translation of two units in the positive y-direction. After this first transformation, the graph has equation $y = 2 - \cos(x)$. The translation of $\frac{\pi}{2}$ in the negative x-direction transforms this graph to the one with equation

$$y = 2 - \cos\left(x + \frac{\pi}{2}\right) = 2 - (-\sin(x)) = 2 + \sin(x).$$

9 B.

The greatest common divisor of two numbers is the product of all their common prime factors. Therefore, 23 is the only prime which divides both x and y. Now the prime factorization of 138 (the least common multiple) is $2 \cdot 3 \cdot 23$. Since $x < y$, we have two possibilities: either $x = 23$ and $y = 23 \cdot 2 \cdot 3$, or $x = 23 \cdot 2$ and $y = 23 \cdot 3$. Since y must be odd, we deduce that $x = 23 \cdot 2 = 46$ and $y = 23 \cdot 3 = 69$. We conclude that $x + y = 115$.

10 D.

First note that the value of **A** is $\frac{1}{2}$. Since $\sqrt[3]{28} > \sqrt[3]{27} = 3$, we may eliminate **B**. Since $\frac{\pi}{4} < 1$, we see that $2^{-\frac{\pi}{4}} > 2^{-1} = \frac{1}{2}$. This eliminates **C**. The value of **E** is $\frac{1}{\sqrt{3}}$, which is greater than $\frac{1}{2}$, and so **E** may also be eliminated. Finally, the value of **D** is $\frac{19}{40}$, so we may eliminate **A** to conclude that **D** is the smallest.

11 B.

The total weight of the rowers is: $72 + 74 + 75 + 76 + 79 + 80 + 83 + 85 = 624$ kg. The mean weight is therefore 78 kg. The mean weight after a 2.5% increase in all weights is therefore: $78 \times 1.025 = 79.95$ kg.

12 D.

If n is even, then $x_n = 1 + (-1) + 1 = 1$, and if n is odd then $x_n = (-1) + 1 + (-1) = -1$. Thus:

$$\sum_{n=0}^{100} x_n = (51 \times 1) + (50 \times -1) = 1\,.$$

13 E.

Note that

$$\frac{2n+38}{n+1} = \frac{2n+2}{n+1} + \frac{36}{n+1} = 2 + \frac{36}{n+1}\,.$$

Therefore, the given fraction is an integer if and only if $\frac{36}{n+1}$ is an integer. This is the case precisely when $(n+1)$ divides 36, so the possible values of $(n+1)$ are $1, 2, 3, 4, 6, 9, 12$ and 18. Note that $(n+1)$ cannot take negative values, since n is positive. Therefore, there are 8 values of n for which the given fraction is an integer.

14 D.

There are $6 \times 6 = 36$ possible outcomes. Of these, the following pairs satisfy the condition that the outcome of the second dice is strictly greater than the outcome of the first one:

$$\begin{array}{ccccc} (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,3) & (2,4) & (2,5) & (2,6) & (3,4) \\ (3,5) & (3,6) & (4,5) & (4,6) & (5,6) \end{array}$$

Therefore, the required probability is $\frac{15}{36} = \frac{5}{12}$.

15 E.

Direct calculation shows that **A** is true. Similarly, **B** is true by direct calculation, of by applying **A** twice. Similarly, **C** follows from **B** by cancelling the middle terms. **D** is also true: to see this one can use **A** to split each fraction and then cancel the middle terms (as in **B** and **C**). By process of elimination, we see that **E** is false. This can also be seen by comparing both sides: $\frac{9}{10}$ on the left is close to 1, but the right hand side is very small by comparison.

16 A.

We need $x \geq 0$ in order for the square root to exist. The inequality can be rewritten as $x - \sqrt{x} - 2 \geq 0$. Viewing the left hand side as a quadratic in $\sqrt{x}$, we have $(\sqrt{x} - 2)(\sqrt{x} + 1) \geq 0$. The second factor is always positive, so the inequality is satisfied as long as $\sqrt{x} - 2 \geq 0$, that is: $x \geq 4$.

17 D.

Rotation anticlockwise by 45° maps $(1, 0)$ to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. This point then maps to $\left(\frac{1}{\sqrt{2}} + 1, \frac{1}{\sqrt{2}}\right)$ upon translation by $(1, 0)$ and this last point is mapped to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + 1\right)$ upon reflection in the line $y = x$.

18 C

Direct calculation:

$$\frac{4}{3} \cdot \frac{6}{4} \cdot \frac{8}{5} \cdot \frac{10}{6} \cdot \ldots \cdot \frac{2(n+1)}{n+2} = 2^n \left(\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{(n+1)}{n+2}\right) = 2^n \left(\frac{2}{n+2}\right) = \frac{2^{n+1}}{n+2}$$

19 B

The x-intercepts of the intersection points are solutions to the equation:

$$x^2 + ax + b = x + 1.$$

Substituting $x = 2$ and $x = 4$ into this equation gives two equations in a and b, which can be solved simultaneously to give $a = -5$ and $b = 9$.

20 B.

The graph of the derivative is defined by the equation $y = \cos(x) - x \sin(x)$. Setting $x = 0$ allows us to eliminate **A** and **C**. In order to choose between **C** and **D**, we differentiate the equation of the derivative twice, and observe that the graph of the derivative must have a turning point at $x = 0$, which must be a local maximum. This eliminates **D** and identifies **B** as the correct graph.

SECTION 2
MATHEMATICAL THINKING

21 E.

First observe that the common ratio for the geometric sequence is:

$$\frac{3x+2}{2x+4}=\frac{x^2-11}{3x+2}$$

Cross-multiplying and simplifying gives the cubic equation $2x^3-5x^2-34x-46$. We are given that $x-6$ is a factor of this equation, so $g(x)=2x^2+7x+8$, by polynomial long division. The discriminant of $g(x)$ is therefore $7^2-4(2)(8)=49-64=-15$.

22 C.

Writing all numbers as powers of 2 allows us to re-express the given equation as:

$$(2^x)^3+2(2^x)^2-2^x-2=0.$$

The left hand side is a cubic in 2^x, which can be factorized to give $(2^x-1)(2^x+1)(2^x+2)=0$. Since 2^x cannot be negative, this implies that $2^x=1$, which implies that $x=0$ is the only solution. The sum of all solutions is therefore 0.

23 B.

Solving the equations of the line and curve simultaneously gives two points of intersection:

$$(x_1,y_1)=\left(\frac{5+\sqrt{5}}{2},5+\sqrt{5}\right)\text{ and }(x_2,y_2)=\left(\frac{5-\sqrt{5}}{2},5-\sqrt{5}\right).$$

The right-angled triangle for which the line segment joining (x_1,y_1) and (x_2,y_2) is the hypotenuse has base $x_2-x_1=\sqrt{5}$ and height $y_2-y_1=2\sqrt{5}$. The area of the triangle is therefore equal to 5.

24 B.

A general term in the binomial expansion looks like:

$$\left((-1)^{4-k}\cdot 2^r\cdot 3^{s-r}\cdot {}^5C_r\cdot {}^4C_k\right)x^{r+k},$$

where $0\leq r\leq 5$ and $0\leq k\leq 4$. There are three possibilities for when $r+k=2$. When $r=2,k=0$, the coefficient is 1080. When $r=k=1$, the coefficient is -3240 and when $r=0,k=2$, the coefficient is 1458. Therefore, the coefficient of x^2 in the expansion is $1080-3240+1458=-702$.

25 E.

If $p=5$, then $p+2=7$. This supports the conjecture, and is therefore not a counterexample. $p=15$ is odd, but is not prime, so it cannot be a counterexample to the conjecture. Finally, $p=19$ is an odd prime but $p+2=21$, which is not prime. So this is a valid counterexample to the conjecture.

26 B.

Applying $\log_2$ to the inequalities $3>2$ and $3<4$ shows that **A** and **B** are both true. Now since $x>1$ (as shown just now), we see that $x^2+x^3>1+1=2$, and since $x<2$ (also shown just now), we see that $x^2+x^3<4+8=12$. So **D** is true. For **E**, we see that

$$\left(2^{\frac{x}{2}}-1\right)\left(2^{\frac{x}{2}}+1\right)=2^x-1=2^{\log_2(3)}-1=3-1=2,$$

and also

$$3^{\frac{1}{x}} = 3^{\frac{1}{\log_2(3)}} = 3^{\log_3(2)} = 2.$$

So **E** is also true. Therefore, by process of elimination, **B** must be false. This can also be seen by observing that $2^x + 3^x = 3 + 3^x > 3 + 3 = 6$.

27 C.

Direct computation:

$$\frac{(x+1)^2+1}{x^4+4} = \frac{x^2+2x+2}{(x+2)^2-(2x)^2} = \frac{x^2+2x+2}{(x^2-2x+2)(x^2+2x+2)} = \frac{1}{x^2-2x+2}$$

28 C.

If $a = b = 1$, then $a^2 = b^2 = 1$, so **I** is correct. It then follows that $0 = a^2 - b^2 = (a-b)(a+b)$, so **II** is also correct. However, $a - b = 0$, so cancelling it from both sides is not allowed. Therefore **III** (and only **III**) is incorrect.

29 A.

Completing the square on the right hand side gives

$$\sin(x) = (x-\pi)^2 + 1.$$

The right hand side attains a minimum value of 1, which is the maximum value that sin can attain. So equality can only hold when both sides equal 1. The right hand side equals 1 only when $x = \pi$. However, $\sin(\pi) = 0$. So there is no value of x for which both sides are equal. Therefore the equation has no solutions.

30 E.

The equation of the first circle can be expanded to give $x^2 + y^2 - 8x - 14y + 40 = 0$. Subtracting this from the equation of the second circle gives $3x + 5y = 30$. This equation is satisfied by any points (x, y) that lie on both circles. In other words, the two intersection points of the circle lie on this line.

31 C.

First, if n is even, then $n = 2k$ for some positive integer k, so $n^2 = 4k$ and n^2 is even. So 'n is even' is sufficient for $(\star)$. Conversely, suppose n^2 is even. If n were odd (say, $n = 2k+1$ for some non-negative integer k) then $n^2 = 2(2k^2+2k)+1$, so n^2 would be odd. This is a contradiction. So n must be even. Therefore 'n is even' is also necessary for $(\star)$.

32 D.

For $n \geq 3$, we have $3^{a_n} = na_n$, from which it follows (by taking $\log_3$ on both sides) that $a_n - \log_3(a_n) = \log_3(n)$. Therefore, for $n \geq 3$:

$$-3^{a_n - \log_3(a_n)} = -3^{\log_3(n)} = -n, \quad \text{so} \quad 3^{-3^{a_n - \log_3(a_n)}} = 3^{-n}.$$

Therefore the given sum is geometric with first term 3^{-3} and common ratio $\frac{1}{3}$. It equals $\frac{1}{18}$.

33 E.

A is true, by direct computation. Then, since each I_n is non-negative, we have $I_{n-1} \leq I_{n-1} + I_n = \frac{1}{n}$. So **B** is true. Next, for $0 \leq x \leq 1$, we have $x^{n-1} \geq x^n$. Dividing both sides by $x + 1 \geq 0$ and integrating gives $I_{n-1} \geq I_n$, which shows that **C** is true. Finally, by combining **A** and **C**, we have $2I_{n-1} \geq I_n + I_{n-1} = \frac{1}{n}$, which implies that $I_n \geq \frac{1}{2n}$. This shows that **D** is true. So, by process of elimination, **E** must be false. We can also see this by replacing n by $n+1$ in **D**, which gives $I_n \geq \frac{1}{2(n+1)}$.

34 B.

The correct implication is **II** $\Rightarrow$ **I**, but not conversely (since **I** implies $x \geq 2$ **or** $x \leq -2$).

35 E.

By the quadratic formula, we have

$$7 = \frac{-13 + \sqrt{13^2 - 4(3)(-c)}}{2(3)} - \frac{-13 - \sqrt{13^2 - 4(3)(-c)}}{2(3)},$$

which simplifies to $169 + 12c = 441$. Therefore $c = \frac{68}{3}$.

36 C.

We are given that $f(0) = 8$, which implies that $c = 8$. Differentiating gives $f'(x) = 3x^2 + 2ax + b$ and there is exactly one stationary point. Therefore, the discriminant of $f'(x)$ is zero. It follows from this that $b = \frac{a^2}{3}$. The stationary point occurs when $x = 1$, therefore $3 + 2a + b = 0$. Substituting $b = \frac{a^2}{3}$ into this equation and solving gives $a = -3$. This implies $b = 3$. Therefore $a + b + c = 8$.

37 C.

Let n be the number of terms in the given list. The common difference of the arithmetic sequence is 4, so $x = 1 + 4(n-1)$ and

$$325 = \frac{n(2 + 4(n-1))}{2},$$

which implies that $650 = n(4n - 2)$. Solving this quadratic and noting that n is positive gives $n = 13$, so that $x = 49$. The product of the digits of x is therefore 36.

38 B.

The given equation is a quadratic in x. It has two distinct real equations precisely when the discriminant is positive, i.e. $\frac{1}{a^2} + 4a > 0$. This is equivalent to saying that $1 + 4a^3 > 0$, i.e. $a > 4^{-\frac{1}{3}}$.

39 H.

First note that **III** is true, by raising both sides of $(x-1)^{\frac{1}{3}} = x$ to the power 3. Next, **I** is true: the two functions are inverses to one another, so their graphs are symmetric about $y = x$. Now using **I**, statement **II** tells us that if the two graphs intersect then they intersect on the line $y = x$. So the solution also satisfies $x^3 + 1 = x$, which can be put in the form of **II**. So **II** is also true.

40 D.

Selecting two points x, y at random in $[0,1]$ is equivalent to selecting a point (x, y) at random in the square $[0,1] \times [0,1]$. The area of this square is 1, while the area of the region within the square which satisfies the inequality $x^2 + y < 1$ is

$$\int_0^1 1 - x^2 \, dx = \frac{2}{3}.$$

The required probability is therefore $\frac{2}{3}$.

END OF SOLUTIONS