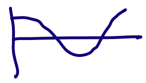
TMUA Trigonometry

Syllabus

Sine, cosine rule, area of triangle; radian measure; arcs and sectors; exact values; sin/cos/tan functions and graphs; basic trig identities (tan x=sin x/cos x and $\sin^2 x+\cos^2 x = 1$); solution of trig equations

1. Solve the following trigonometric equations in the range given



a) $\quad cos(2\theta + 30) = 0.5 \quad 0 \leq \theta \leq 360$

$30 \leq 2\theta + 30 \leq 750$

$2\theta + 30 = 60, 300, 420, 660$

$2\theta = 30 \quad 270 \quad 390 \quad 630$

$\theta = 15, 135, 195, 315$

b) $\quad tan(5\theta - 35) = \sqrt{3} \quad 0 \leq \theta \leq 90$

$-35 \leq 5\theta - 35 \leq 415$

$5\theta - 35 = 60 \quad 240$

$5\theta = 95 \quad 275$

$\theta = 19 \quad 55$

c) $\quad 2sin^2\theta + 3cos\theta = 0 \quad 0 \leq \theta \leq 360$

$2(1 - \cos^2\theta) + 3\cos\theta = 0$

$2 - 2\cos^2\theta + 3\cos\theta = 0$

$2\cos^2\theta - 3\cos\theta - 2 = 0$

$(2\cos\theta + 1)(\cos\theta - 2) = 0$

$\cos\theta = 2$ no solutions

$\cos\theta = -\frac{1}{2}$

$\theta = 120, 240$

d) $\quad sin(3\theta + 72) = cos48 \quad 0 \leq \theta \leq 180$

$72 \leq 3\theta + 72 \leq 612$

$\sin(3\theta + 72) = \sin(90 - 48)$

$3\theta + 72 = 42, 138, 402, 498$

$3\theta = \quad 66 \quad 330 \quad 426$

$\theta = \quad 22 \quad 110 \quad 142$

e) $\quad \dfrac{3 + sin^2\theta}{cos\theta - 2} = 3cos\theta \quad 0 \leq \theta \leq 360$

$3 + \sin^2\theta = 3\cos^2\theta - 6\cos\theta$

$3 + 1 - \cos^2\theta = 3\cos^2\theta - 6\cos\theta$

$4\cos^2\theta - 6\cos\theta - 4 = 0$

$2\cos^2\theta - 3\cos\theta - 2 = 0$

$(2\cos\theta + 1)(\cos\theta - 2) = 0$

$\cos\theta = -\frac{1}{2} \qquad \cos\theta = 2$

$\theta = 120, 240 \qquad$ no solutions

f) $\quad cos^2(2x) + \sqrt{3}sin(2x) - \dfrac{7}{4} = 0 \quad 0 \leq x \leq 360$

$\cos^2 2x + \sqrt{3}\sin 2x - \frac{7}{4} = 0$

$1 - \sin^2 2x + \sqrt{3}\sin 2x - \frac{7}{4} = 0$

$4\sin^2 2x - 4\sqrt{3}\sin 2x + 3 = 0$

$(2\sin 2x - \sqrt{3})(2\sin 2x - \sqrt{3}) = 0$

$\sin 2x = \frac{\sqrt{3}}{2}$

$2x = 60, 120, 420, 480$

$x = 30, 60, 210, 240$

$\dfrac{4\sqrt{3} \pm \sqrt{48 - 48}}{8} = \dfrac{\sqrt{3}}{2}$

2. Solve, in radians, the following trigonometric equations, giving your answers in terms of π .

a) $cos2x = cos\dfrac{2\pi}{5} \quad 0 \le x \le 2\pi$

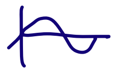
$0 \le 2x \le 4\pi$

$2x = \dfrac{2\pi}{5}, \dfrac{8\pi}{5}, \dfrac{12\pi}{5}, \dfrac{18\pi}{5}$

$x = \dfrac{\pi}{5}, \dfrac{4\pi}{5}, \dfrac{6\pi}{5}, \dfrac{9\pi}{5}$

b) $8sin(\dfrac{\pi}{3} - 2x) = 4 \quad 0 \le x \le 2\pi$

$-\dfrac{\pi}{3} \le 2x - \dfrac{\pi}{3} \le \dfrac{11\pi}{3}$



$\sin\left(\dfrac{\pi}{3} - 2x\right) = \dfrac{1}{2}$

$\sin\left(2x - \dfrac{\pi}{3}\right) = -\tfrac{1}{2}$

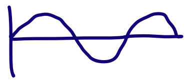
$2x - \dfrac{\pi}{3} = -\dfrac{\pi}{6} \quad \dfrac{7\pi}{6} \quad \dfrac{11\pi}{6} \quad \dfrac{19\pi}{6}$

$2x = \dfrac{\pi}{6} \quad \dfrac{3\pi}{2} \quad \dfrac{13\pi}{6} \quad \dfrac{7\pi}{2}$

$x = \dfrac{\pi}{12} \quad \dfrac{3\pi}{4} \quad \dfrac{13\pi}{12} \quad \dfrac{7\pi}{4}$

c) $sin^2\dfrac{3\theta}{2} = \dfrac{1}{2} \quad 0 \le \theta \le 2\pi$

$0 \le \dfrac{3\theta}{2} \le 3\pi$



$\sin\dfrac{3\theta}{2} = \pm\sqrt{\dfrac{1}{2}}$

$\dfrac{3\theta}{2} = \dfrac{\pi}{4} \quad \dfrac{3\pi}{4} \quad \dfrac{9\pi}{4} \quad \dfrac{11\pi}{4}$

$\theta = \dfrac{\pi}{6} \quad \dfrac{\pi}{2} \quad \dfrac{3\pi}{2} \quad \dfrac{11\pi}{6}$

$\dfrac{3\theta}{2} = \dfrac{5\pi}{4} \quad \dfrac{7\pi}{4}$

$\theta = \dfrac{5\pi}{6} \quad \dfrac{7\pi}{6}$

d) $2sin^2y - 5cosy + 1 = 0 \quad 0 \le y \le 2\pi$

$2 - 2\cos^2 y - 5\cos y + 1 = 0$

$2\cos^2 y + 5\cos y - 3 = 0$

$(2\cos y - 1)(\cos y + 3) = 0$

$\cos y = \dfrac{1}{2} \qquad \cos y = -3$

$y = \dfrac{\pi}{3} \quad \dfrac{5\pi}{3}$

e) $tan^4x - tan^2x = 6 \quad 0 \le x \le 2\pi$

$\tan^4 x - \tan^2 x - 6 = 0$

$(\tan^2 x - 3)(\tan^2 x + 2) = 0$

$\tan^2 x = 3$ no solutions

$\tan x = \pm\sqrt{3}$

$\tan x = \sqrt{3} \qquad \tan x = -\sqrt{3}$

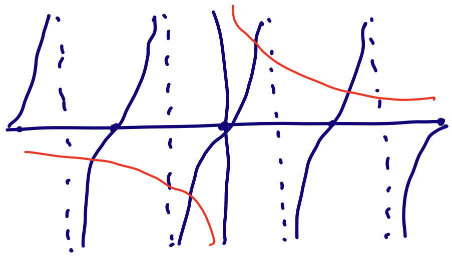
$x = \dfrac{\pi}{3} \quad \dfrac{4\pi}{3} \qquad x = \dfrac{2\pi}{3} \quad \dfrac{5\pi}{3}$

3. Find the number of solutions to the following equations:

a) $x\tan x = 2$ $\quad -2\pi \le x \le 2\pi$

$\tan x = \dfrac{2}{x}$

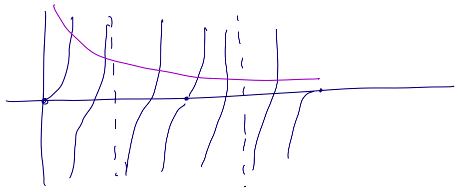
4 solutions



b) $x\sin 3x = \cos 3x$ $\quad 0 \le x \le 2\pi$

$\tan 3x = \dfrac{1}{x}$

6 solutions



c) $\sin^2\theta + 3\cos\theta = 3$ $\quad 0 \le \theta \le 4\pi$

$1 - \cos^2\theta + 3\cos\theta = 3$

$\cos^2\theta - 3\cos\theta + 2 = 0$

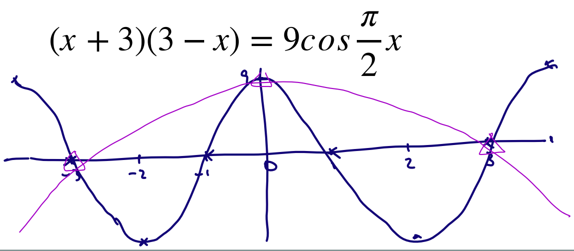
$(\cos\theta - 2)(\cos\theta - 1) = 0$

$\cos\theta = 2$ no solutions

$\cos\theta = 1 \quad \theta = 0, 2\pi, 4\pi$

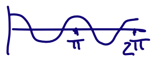
3 solutions

d) $(x+3)(3-x) = 9\cos\dfrac{\pi}{2}x$



3 solutions

4. Find the complete set of values for which:



a) $y \ge 0$ where $y = \tan x \cos 2x$ $\quad 0 \le x \le \pi$

	$0 - \pi/4$	$\pi/4 - \pi/2$	$\pi/2 - 3\pi/4$	$3\pi/4 - \pi$
$\tan x$	+	+	−	−
$\cos 2x$	+	−	−	+

$0 \le x \le \dfrac{\pi}{4}$

$\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{4}$

b) $y \ge 0$ where $y = (1 - 2\sin x)\cos x$ $\quad 0 \le x \le \pi$

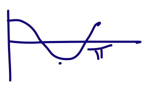
$\sin x = \dfrac{1}{2} \quad x = \dfrac{\pi}{6} \quad \dfrac{5\pi}{6}$

$0 - \pi/6$	$\pi/6 - \pi/3$	$\pi/3 - \pi/2$	$\pi/2 - 2\pi/3$	$2\pi/3 - 5\pi/6$	$5\pi/6 - \pi$
+	−	−	−	−	+
+	+	+	−	−	−

$0 \le x \le \pi/6$

$\dfrac{\pi}{2} \le x \le \dfrac{5\pi}{6}$

c) $y \le 0$ where $y = (1 + \cos 2x)\cos 2x$ $\quad 0 \le x \le \pi$



$\cos 2x = -1$

$2x = \pi$

$x = \dfrac{\pi}{2}$

	$0 - \dfrac{\pi}{4}$	$\dfrac{\pi}{4} - \dfrac{\pi}{2}$	$\dfrac{\pi}{2} - \dfrac{3\pi}{4}$	$\dfrac{3\pi}{4} - \pi$
$1 + \cos 2x$	+	+	+	+
$\cos 2x$	+	−	−	+

$\cos 2x = 0$

$2x = \dfrac{\pi}{2} \quad \dfrac{3\pi}{2}$

$\dfrac{\pi}{4} \le x \le \dfrac{3\pi}{4}$

d) $\tan x \le \sin 2x$ $\quad 0 \le x \le \pi$

$x = \dfrac{\pi}{4} \quad \tan x = 1$

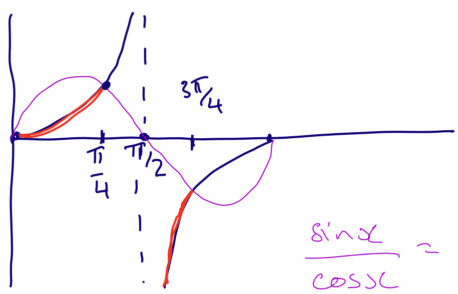
$\sin 2x = 1$

$x = \dfrac{3\pi}{4} \quad \tan x = -1$

$\sin 2x = -1$

$0 \le x \le \dfrac{\pi}{4}$

$\dfrac{\pi}{2} < x \le \dfrac{3\pi}{4}$



$\dfrac{\sin x}{\cos x} = 2\sin x\cos x$

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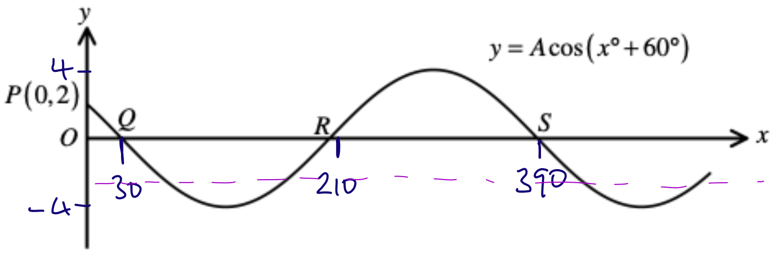
5. The figure shows part of the graph of the curve with equation $y = A\cos(x + 60)$



$2 = A\cos 60^\circ \quad A = 4$

$y = 4\cos(x + 60)$

The point P (0,2) lies on the curve. Find the value of A and the coordinates of Q, R, and S.

Find the coordinates of the points where the straight line with equation $y = -2\sqrt{3}$ meets this graph.

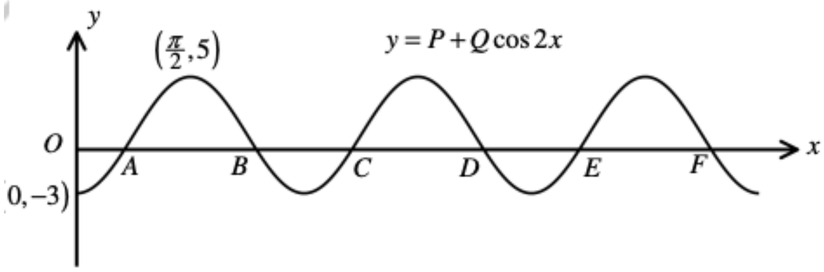
$4\cos(x+60) = -2\sqrt{3}$

$\cos(x+60) = -\frac{\sqrt{3}}{2}$

$x + 60 = 150 \quad 210 \quad 510 \quad 570$

$x = 90 \quad 150 \quad 450 \quad 510$

6. The figure shows part of the graph of the curve with equation $y = P + Q\cos 2x$ $\qquad x \geq 0$

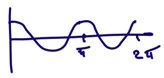


The points (0 , -3) and $(\frac{\pi}{2}, 5)$ lies on the curve. Find the value of P and Q

$y = 1 - 4\cos 2x$

Find the coordinates of the points where the straight line with equation $y = 3$ meets this graph for $0 \leq x \leq 2\pi$.

$0 \leq x \leq 2\pi$



$-3 = P + Q$

$5 = P - Q$

$P = 1$

$Q = -4$

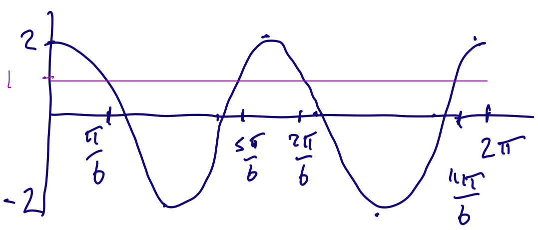
$3 = 1 - 4\cos 2x$

$\cos 2x = -\frac{1}{2}$

$2x = \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{8\pi}{3} \quad \frac{10\pi}{3}$

$x = \frac{\pi}{3} \quad \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{5\pi}{3}$

7. Sketch the graph of $f(x) = 2\cos 2x$ for $0 \leq x \leq 2\pi$ and hence solve $f(x) \leq 1$



$0 \leq x \leq 2\pi$

$\frac{\pi}{6} \leq x \leq \frac{5\pi}{6} \qquad \frac{7\pi}{6} \leq x \leq \frac{11\pi}{6}$

8. A curve has equation $y = A + k\sin x$, $0 \leq x \leq 2\pi$, where A and k are non-zero constants.

Given that the curve passes through $(\frac{\pi}{6}, 1)$ and $(\frac{7\pi}{6}, 5)$, find the minimum and maximum value of y.

$1 = A + \frac{1}{2}k$

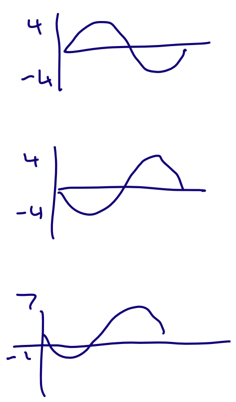
$5 = A - \frac{1}{2}k$

$A = 3 \quad k = -4$

$y = 3 - 4\sin x$

$\sin x = -1 \quad y = 7$

$\sin x = 1 \quad y = -1$



$y = 4\sin x$

$y = -4\sin x$

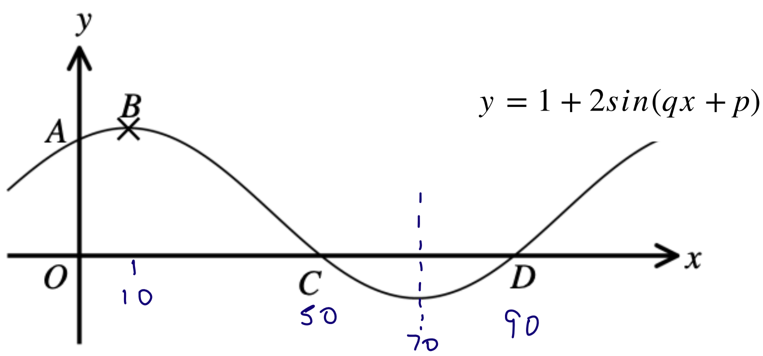
$y = 3 - 4\sin x$

9. The figure shows part of the graph of $y = 1 + 2\sin(qx + p)$,

where q and p are positive constants, with $0 < p < 90$, $0 < q < 5$

The graph crosses the y-axis at the point A $(0, 1 + \sqrt{3})$ and the x-axis at the points C (50,0) and D, and point B is a maximum point on the curve.

Find the values of p and q, and the coordinates of B and D



$y = 1 + 2\sin(3x + 60)$

$= 1 + 2\sin 3(x + 20)$

$1 + \sqrt{3} = 1 + 2\sin p$

$\sin p = \frac{\sqrt{3}}{2}$

$p = 60°$

$0 = 1 + 2\sin(50q + 60)$

$\sin(50q + 60) = -\frac{1}{2}$

$50q + 60 = 210$

$50q = 150$

$q = 3$

B (10, 3) D (90, 0)

10.

a) Find the greatest value of the function $f(x) = (2cos^2(4x+9) - 5)^2$ for $x \in \mathbb{R}$

$0 \le \cos^2(4x+9) \le 1$

$0 \le 2\cos^2(4x+9) \le 2$

$-5 \le 2\cos^2(4x+9) \le -3$

max = 25

b) Find the largest value achieved by $f(x) = 3cos^2x + 2sinx + 1$ for $x \in \mathbb{R}$

$3 - 3\sin^2 x + 2\sin x + 1$

$-3\sin^2 x + 2\sin x + 4$

$-3\left(\sin^2 x - \frac{2}{3}\sin x - \frac{4}{3}\right)$

$-3\left[\left(\sin x - \frac{1}{3}\right)^2 - \frac{1}{9} - \frac{4}{3}\right]$

$-3\left[\left(\sin x - \frac{1}{3}\right)^2 - \frac{13}{9}\right]$

$-3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3}$

Max at $\sin x = \frac{1}{3}$ $= \frac{13}{3}$

c) Find the largest value of $f(x) = (4sin^2x + 4cosx + 1)^2$ for $x \in \mathbb{R}$

$= (4 - 4\cos^2 x + 4\cos x + 1)^2$

$= (5 + 4\cos x - 4\cos^2 x)^2$

$= \left(-4\left[\cos^2 x - \cos x - \frac{5}{4}\right]\right)^2$

$= \left(-4\left[\left(\cos x - \frac{1}{2}\right)^2 - \frac{3}{2}\right]\right)^2$

$= \left(6 - 4\left(\cos x - \frac{1}{2}\right)^2\right)^2$

Max when $\cos x = \frac{1}{2}$ 36

d) Find the minimum value of the function $f(x) = 9cos^4x - 12cos^2x + 7$ for $x \in \mathbb{R}$

$= 9\left[\cos^4 x - \frac{4}{3}\cos^2 x + \frac{7}{9}\right]$

$= 9\left[\left(\cos^2 x - \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{7}{9}\right]$

$= 9\left(\cos^2 x - \frac{2}{3}\right)^2 + 3$

Min at $\cos^2 x = \frac{2}{3}$ 3

e) Find the maximum value of $9^{sinx} - 4(3^{sinx}) + \dfrac{13}{3}$ for $x \in \mathbb{R}$

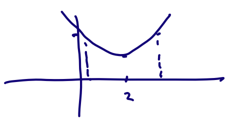
$y = 3^{\sin x}$

$-1 \le \sin x \le 1$

$\frac{1}{3} \le y \le 3$

$y^2 - 4y + \frac{13}{3}$

$(y-2)^2 + \frac{1}{3}$



$y = \frac{1}{3}$ $\frac{25}{9} + \frac{3}{9} = \frac{28}{9}$ ← Max

$y = 3$ $1 + \frac{1}{3} = \frac{4}{3}$

f) Find the minimum value of $f(x) = \dfrac{cosx + 4}{9 + 6cosx - sin^2x}$ for $x \in \mathbb{R}$

$f(x) = \dfrac{\cos x + 4}{9 + 6\cos x + \cos^2 x - 1}$

$= \dfrac{\cos x + 4}{\cos^2 x + 6\cos x + 8}$

$= \dfrac{\cos x + 4}{(\cos x + 4)(\cos x + 2)}$

$= \dfrac{1}{\cos x + 2}$

Min when $\cos x = 1$

$\frac{1}{3}$

11. Find the equation of the new curves after the following transformations:

a) The curve $y = cos x$ is stretched in the horizontal direction by a scale factor of ½, and the resulting curve is translated by 4 units in the positive y-direction.

$y = \cos 2x + 4$

b) The curve $y = sin x$ is stretched in the vertical direction by a scale factor of 2, followed by a translation by $\frac{\pi}{3}$ units in the positive x-direction.

Does order matter?

$y = 2\sin\left(x - \frac{\pi}{3}\right)$

c) The curve $y = tan x$ is stretched in the horizontal direction by a scale factor of 2, followed by a reflection in the x-axis, followed by a translation by 3 units in the positive y-direction.

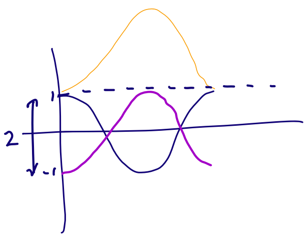
$y = 3 + \tan\left(-\frac{1}{2}x\right)$

d) The curve $y = sin x$ is reflected in the y-axis, translated by 2 units in the negative y-direction and then translated by $\frac{\pi}{4}$ units in the positive x-direction.

$y = \sin(-x) = -\sin x$

$y = -\sin\left(x - \frac{\pi}{4}\right) - 2$

e) The curve $y = cos x$ is reflected in the line $y = 1$

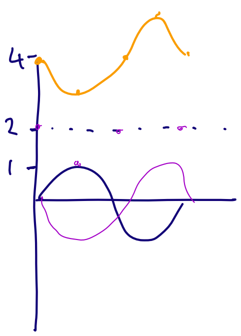


$y = \cos x$

Reflect in x-axis $\Rightarrow$ $y = -\cos x$

Move up 2 $\Rightarrow$ $y = 2 - \cos x$

f) The curve $y = sin x$ is reflected in the line $y = 2$ followed by a translation by $\frac{\pi}{3}$ units in the negative x-direction.

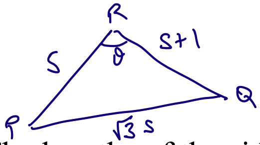


$y = \sin x$

Reflect in x-axis $\Rightarrow$ $y = -\sin x$

Move up 4 $\Rightarrow$ $y = 4 - \sin x$

Move $\frac{\pi}{3}$ left $\quad y = 4 - \sin\left(x + \frac{\pi}{3}\right)$



$$\cos\theta = \frac{s^2+s^2+2s+1-3s^2}{2s(s+1)} = \frac{-s^2+2s+1}{2s(s+1)} = -\frac{s^2-2s-1}{2s(s+1)}$$

12a) The lengths of the sides QR, RP and PQ in triangle PQR are $s+1$, s, and $\sqrt{3}s$

Find the full range of values of s that make angle PRQ an obtuse angle. $-1 \le \cos\theta \le 0$

$\cos\theta < 0$

$s^2 - 2s - 1 > 0$

$s = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$s > 1+\sqrt{2}$

$\cos\theta \ge -1$

$\frac{s^2-2s-1}{2s(s+1)} \le 1$

$s^2 - 2s - 1 \le 2s^2 + 2s$

$s^2 + 4s + 1 \ge 0$

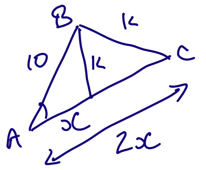
$s = \frac{-4 \pm \sqrt{16-4}}{2}$

$= -2 \pm \sqrt{3}$ both −ve $s > 0$

$\Rightarrow s > 1+\sqrt{2}$

b) A triangle ABC has AB=10cm, angle BAC = 30° and BC = k cm, where k is a constant.

Of the two possible triangles that could be drawn, the larger triangle has side AC twice as long as side AC in the smaller triangle. Find the value of k.



$k^2 = 10^2 + x^2 - 20x\left(\frac{\sqrt{3}}{2}\right)$

$= 100 + x^2 - 10\sqrt{3}x$

$k^2 = 10^2 + 4x^2 - 40x\left(\frac{\sqrt{3}}{2}\right)$

$= 100 + 4x^2 - 20\sqrt{3}x$

$x^2 - 10\sqrt{3}x = 4x^2 - 20\sqrt{3}x$

$10\sqrt{3}x = 3x^2$

$x = 0$

$x = \frac{10\sqrt{3}}{3}$

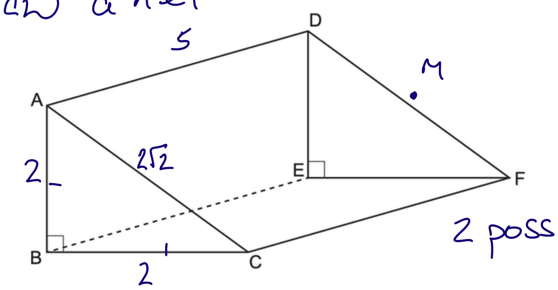
$k^2 = 100 + \frac{300}{9} - \frac{300}{3} = \frac{100}{3}$

$k = \frac{10\sqrt{3}}{3}$

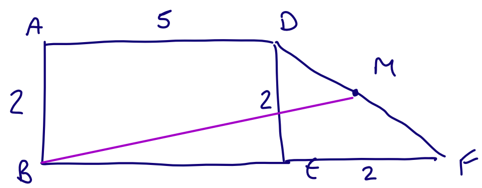
13. A solid right angled triangular prism, with base BEFC, has AB = BC = 2m, and AD =5m.

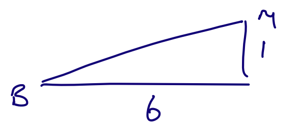
Find the shortest distance, in metres, along the outside of the prism from B to the midpoint of DF.

* Draw a net

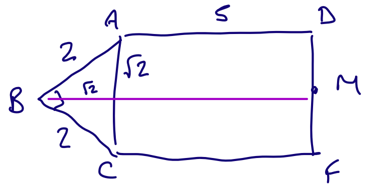


2 possibles





$BM = \sqrt{37}$



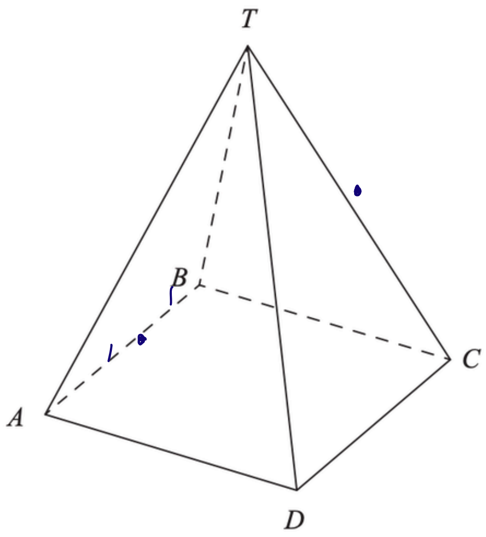
$BM = 5 + \sqrt{2}$

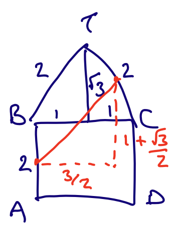
$\sqrt{37} < 5 + \sqrt{2}$

shortest

14. A square based pyramid, with base ABCD, and vertex T has all edges of length 2m.

Find the shortest distance, in metres, along the outer surface of the prism from the midpoint of AB to the midpoint of CT

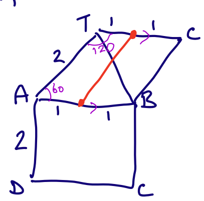




$\left(\frac{3}{2}\right)^2 + \left(1 + \frac{\sqrt{3}}{2}\right)^2$

$\frac{9}{4} + 1 + \sqrt{3} + \frac{3}{4} = 4 + \sqrt{3}$

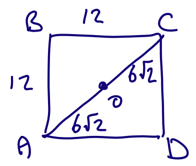
$CT = \sqrt{4+\sqrt{3}}$

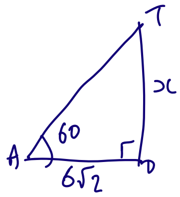


$CT = 2$ shortest distance

15. A square based pyramid has base ABCD, where all the sides of the square are 12cm in length. The diagonals of the square intersect at O, and the vertex of the pyramid is at T, directly above O. Each of the sloping edges of the pyramid makes an angle of 60° with the base.

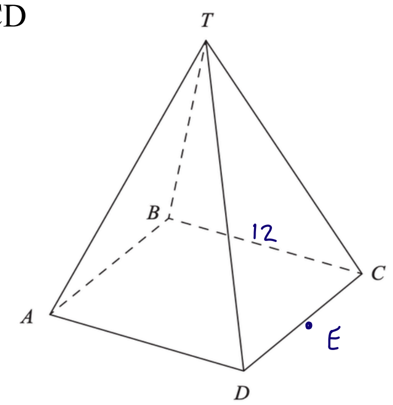
Find the tangent of the angle between the face CDT and the base ABCD

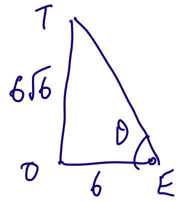




$\tan 60 = \dfrac{x}{6\sqrt{2}} = \sqrt{3}$

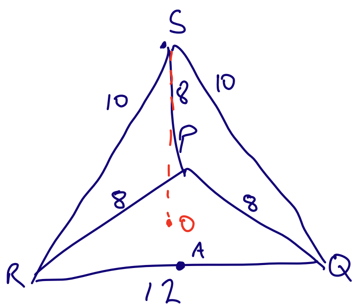
$x = 6\sqrt{6}$

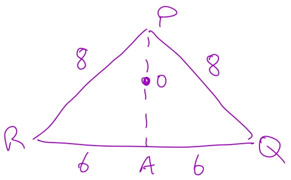




$\tan \theta = \dfrac{6\sqrt{6}}{6} = \sqrt{6}$

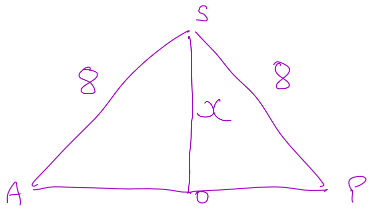
16. A triangular-based pyramid PQRS has horizontal base PQR where PQ=PR=8m and RQ=12m The vertex of the pyramid S lies directly above the level of PQR so that SQ=SR=10m and SP=8m. Find the shortest distance of S from the base PQR.





$PA = \sqrt{8^2 - 6^2}$

$= 2\sqrt{7}$



isosceles

$\Rightarrow AO = AP = \sqrt{7}$

$x = \sqrt{8^2 - (\sqrt{7})^2} = \sqrt{57}$