THE UK UNIVERSITY INTEGRATION BEE 2022/23

$$\int$$

Round One

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1

$$\int_0^1 \frac{1}{\sqrt{x-x^2}}\mathrm{d}x$$

Solution 1

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x-x^2}}\,\mathrm{d}x &= \int_0^1 \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x \\ &= \left[\arcsin\left(2\left(x-\frac{1}{2}\right)\right)\right]_0^1 \\ &= \frac{\pi}{2}-\left(\frac{\pi}{2}\right)=\pi\end{aligned}$$

Solution 2
We make use of one of the properties of Euler's Beta Function:

$$B\left(a,b\right)=\int_0^1 x^{a-1}\left(1-x\right)^{b-1}\,\mathrm{d}x=\frac{\Gamma\left(a\right)\Gamma\left(b\right)}{\Gamma\left(a+b\right)},$$

where Γ is the Gamma Function.

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{x-x^2}} &= \int_0^1 x^{\frac{1}{2}-1}\left(1-x\right)^{\frac{1}{2}-1}\,\mathrm{d}x \\ &= B\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1\right)} \\ &= \frac{\sqrt{\pi}\sqrt{\pi}}{1}=\pi\end{aligned}$$

2

$$\int_0^{100} \lceil x \rceil \lfloor x \rfloor \mathrm{d}x,$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the greatest integer less than x and the smallest integer greater than x, respectively.

$$\begin{aligned}
\int_0^{100} \lceil x \rceil \lfloor x \rfloor \, \mathrm{d}x &= \sum_{n=1}^{100} \int_{n-1}^{n} n\,(n-1)\ \mathrm{d}x \\
&= \sum_{n=1}^{100} n\,(n-1)\,(n-(n-1)) \\
&= \sum_{n=1}^{100} n^2 - n \\
\frac{(2 \cdot 100 + 1)\,(100 + 1\,(100))}{6} - \frac{100\,(100+1)}{2} & \\
&= 67 \cdot 101 \cdot 50 - 50 \cdot 101 = 66 \cdot 101 \cdot 50 \\
&= 6666 \cdot 50 \\
&= 333300
\end{aligned}$$

3

$$\int_0^{\pi} \cos(x + \cos(x))\mathrm{d}x$$

$$\begin{aligned}\int_0^{\pi} \cos\left(x + \cos\left(x\right)\right)\,\mathrm{d}x &\overset{u=\pi-x}{=} -\int_0^{\pi} \cos\left(x + \cos\left(x\right)\right)\,\mathrm{d}x \\ &\implies \int_0^{\pi} \cos\left(x + \cos\left(x\right)\right)\,\mathrm{d}x = 0\end{aligned}$$

4

$$\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}\mathrm{d}x$$

We begin by finding an alternate expression for the integrand

$$y = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}} = \sqrt{x+y}$$

Rearranging the above equation to make y the subject yields

$$y = \sqrt{x+\frac{1}{4}}+\frac{1}{2}$$

Thus, we may substitute this expression into the question

$$\begin{aligned}
\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}\mathrm{d}x &= \int_0^1 \sqrt{x+\frac{1}{4}}+\frac{1}{2}\,\mathrm{d}x \\
&= \left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{\frac{3}{2}}+\frac{1}{2}x\right]_0^1 \\
&= \frac{2}{3}\sqrt{\frac{5}{4}}^3+\frac{1}{2}-\frac{2}{3}\sqrt{\frac{1}{4}} \\
&= \frac{\sqrt{5}^3}{12}+\frac{5}{12} \\
&= \frac{5}{12}\left(\sqrt{5}+1\right)
\end{aligned}$$

5

$$\int_0^1 \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \mathrm{d}x &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{x} - k \,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} [\log x]_{\frac{1}{k+1}}^{\frac{1}{k}} - k\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log(k+1) - \log(k) - \frac{1}{k+1} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \log(n+1) - \sum_{k=1}^{n} \frac{1}{k+1} \\
&= \lim_{n\to\infty} \log(n+1) + 1 - \sum_{k=1}^{n} \frac{1}{n} \\
&= 1 - \gamma
\end{aligned}$$

6

$$\int_0^1 \frac{\arctan x + \operatorname{arccot} x}{x^2+1} \mathrm{d}x$$

$$\begin{aligned}\int_0^1 \frac{\arctan x + \operatorname{arccot} x}{x^2+1} \, \mathrm{d}x &= \frac{\pi}{2} \int_0^1 \frac{1}{x^2+1} \, \mathrm{d}x \\ &= \frac{\pi^2}{8}\end{aligned}$$

The first equality uses the formula $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$, which can easily be shown by considering a right triangle with perpendicular sides $1, x$.

7

$$\int_0^{\frac{\pi}{2}} x \prod_{i=1}^{\infty} \cos\left(\frac{x}{2^i}\right) \mathrm{d}x$$

Consider the following identity $\cos(x) = \dfrac{\sin(2x)}{2\sin(x)}$. Now we will evaluate the product

$$\begin{aligned} x \prod_{i=1}^{\infty} \cos\left(\frac{x}{2^i}\right) &= \lim_{n\to\infty} \prod_{i=1}^{n} \cos\left(\frac{x}{2^i}\right) \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} \end{aligned}$$

8

$$\int_0^{\frac{\pi}{4}} \log(\cot x - 1) \mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \log\left(\cot x - 1\right)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \log\left(1 - \tan x\right)\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \log\left(\tan x\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \log\left(1 - \tan x\right)\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{\pi}{4} - x\right)\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \log\left(1 - \tan x\right)\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \log\left(\frac{1 - \tan x}{1 + \tan x}\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan x\right)\,\mathrm{d}x
\end{aligned}$$

Let $I = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan x\right)\,\mathrm{d}x$, then

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan x\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \log 2\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \log\left(1 + \tan x\right)\,\mathrm{d}x \\
&= \frac{\pi}{4}\log 2 - I
\end{aligned}$$

Hence, $I = \frac{\pi}{8}\log 2$

9

$$\int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(b\sin x)}{\sin x} \mathrm{d}x$$

Differentiate both sides with respect to b.

$$\begin{aligned}
I'(b) &= \int_0^{\frac{\pi}{2}} \frac{1}{1+b^2\sin^2 x}\,\mathrm{d}x, \text{ I refuse to use the Weierstrass substitution.} \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{(1+b^2)\sin^2 x + \cos^2 x}\,\mathrm{d}x\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1+b^2)\tan^2 x + 1}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{1+b^2}}\left[\arctan\left(\frac{\tan x}{\sqrt{1+b^2}}\right)\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2\sqrt{1+b^2}}
\end{aligned}$$

Now we may integrate to obtain $I(b)$:

$$I(b) - I(0) = I(b) = \int_0^b I'(t)\frac{\pi}{2}\int_0^b \frac{\mathrm{d}t}{\sqrt{1+t^2}} = \frac{\pi}{2}\,\mathrm{arsinh}\, b.$$

10

$$\int_0^\infty \frac{x^3}{e^x+1}\mathrm{d}x$$

$$\begin{aligned}\int_0^\infty \frac{x^3}{e^x+1}\,\mathrm{d}x &= \int_0^\infty \frac{x^3e^{-x}}{1+e^{-x}}\,\mathrm{d}x \\ &= \int_0^\infty x^3e^{-x}\sum_{n=0}^{\infty}(-1)^n e^{-nx}\,\mathrm{d}x \\ &\text{where we have used the infinite geometric series expansion} \\ &= \sum_{n=0}^{\infty}(-1)^n\int_0^\infty x^3e^{-(1+n)x}\,\mathrm{d}x \\ &\text{We interchange limits thanks to the dominated convergence theorem.} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(4)}{(1+n)^4} \\ &= \eta(4)\Gamma(4) = \frac{7}{8}\zeta(4)\Gamma(4) = \frac{21}{4}\zeta(4) = \frac{7\pi^4}{120}.\end{aligned}$$

Where η, ζ, and Γ are the Dirichlet Eta, Riemann Zeta, and Gamma functions respectively. The relation between Dirichlet Eta and the Zeta function can be derived by considering odd and even terms of the series':

$$\eta(s) = \left(1-2^{1-s}\right)\zeta\left(s\right).$$

11

$$\int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} \binom{2n}{n} x^n \mathrm{d}x$$

Notice that the integrand resembles a Maclaurin series expansion. Recall the Maclaurin series expansion of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where $f^(n)(0)$ denotes the n$^{\text{th}}$ derivative of f evaluated at $x = 0$.
We will force the integrand into the above form and try to find a closed form for the sum by inspection. Consider the following...

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \times \frac{x^n}{n!}.$$

Thus, we may deduce that we are looking for f such that $f^{(n)}(0) = \dfrac{(2n)!}{n!}$. We can construct the following recurrence relation, $f^{(n)}(0) = \dfrac{(2n)(2n-1)}{n} f^{(n-1)}(0) = 2(2n-1)f^{(n-1)}(0)$. At this point we have enough to make the following educated guess...

$$f(x) = \frac{1}{\sqrt{1-4x}}.$$

Finally, we can do the integration:

$$\sum_0^{\frac{1}{4}} \sum_{n=0}^{\infty} \binom{2n}{n} x^n \,\mathrm{d}x = \int_0^{\frac{1}{4}} \frac{1}{1-4x} \,\mathrm{d}x = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{4x}} \,\mathrm{d}x = \left[\sqrt{x}\right]_0^{\frac{1}{4}} = \frac{1}{2}$$

12

$$\int_0^\infty \cos(x^2)\mathrm{d}x$$

We will provide a solution by Laplace Transforms.

$$I(t) = \int_0^\infty \cos\left(tx^2\right)\,\mathrm{d}x$$

$$\begin{aligned}\mathscr{L}\left\{I(t)\right\} &= \int_0^\infty \int_0^\infty \cos\left(tx^2\right)\,\mathrm{d}x e^{-st}\mathrm{d}t \\ &= \int_0^\infty \int_0^\infty \cos\left(tx^2\right) e^{-st}\,\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty \frac{s}{s^2+x^4}\,\mathrm{d}x\end{aligned}$$

This integral can be computed by using partial fraction decomposition on the factorisation, $x^4+s^2 = \left(x^2-\sqrt{2s}x+1\right)\left(x^2+\sqrt{2s}x+1\right)$.

However, we will leave that as an exercise.

$$\mathscr{L}\left\{I\left(t\right)\right\} = \frac{\pi}{2\sqrt{2}}.$$

For those familiar with Laplace Transforms, you will see that

$$I(t) = \mathscr{L}^{-1}\left\{\frac{\pi}{2\sqrt{2s}}\right\} = \sqrt{\frac{\pi}{8t}}$$

Thus, $\int_0^\infty \cos\left(x^2\right)\,\mathrm{d}x = I(1) = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

13

$$\int_0^\infty \frac{\log x}{1-x^2} \mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \frac{\log x}{1-x^2}\,\mathrm{d}x &= \int_0^1 \frac{\log x}{1-x^2}\,\mathrm{d}x + \int_1^\infty \frac{\log x}{1-x^2}\,\mathrm{d}x \\
&= 2\int_0^1 \frac{\log x}{1-x^2}\,\mathrm{d}x \\
&\text{where we have used the substitution } u = \frac{1}{x} \text{ on the second integral} \\
&= 2\int_0^1 \log x \sum_{n=0}^\infty x^{2n}\,\mathrm{d}x \\
&= 2\int_0^1 \log x \sum_{n=0}^\infty \int_0^1 x^{2n} \log x\,\mathrm{d}x \\
&= 2\sum_{n=0}^\infty -\frac{1}{(2n+1)^2} = -\frac{\pi^2}{4}.
\end{aligned}$$

14

$$\int_0^{\frac{\pi}{2}} \frac{\log(\sin x)}{\cos^2 x} \mathrm{d}x$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\log(\sin x)}{\cos^2 x}\,\mathrm{d}x &\overset{\text{IBP}}{=} \left[\tan x \log(\sin x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \tan x \cot x \,\mathrm{d}x \\ &= -\int_0^{\frac{\pi}{2}} 1\,\mathrm{d}x = -\frac{\pi}{2}\end{aligned}$$

15

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x(1+\cos x)}{(1+\cos x+\sin x)^2}\mathrm{d}x$$

We may re-express the denominator to arrive at the solution

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,(1+\cos x)}{(1+\cos x+\sin x)^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,(1+\cos x)}{2\,(1+\cos x)\,(1+\sin x)}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,(1-\sin x)}{1-\sin^2 x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\sin x\,\mathrm{d}x = \frac{\pi}{4}-\frac{1}{2} = \frac{1}{4}\,(\pi-2)\,.
\end{aligned}$$

16

$$\int_1^\infty \frac{x - \lfloor x \rfloor}{x^2} \, \mathrm{d}x$$

By use of the $u = \dfrac{1}{x}$ substitution we can show this integral is equivalent to problem 5. Therefore, the answer is $1 - \gamma$.

17

$$\int_{-\infty}^{\infty} \frac{\cos t}{\cosh t} \mathrm{d}t$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos t}{\cosh t} \, \mathrm{d}t &= \int_{0}^{\infty} \frac{\cos t}{e^{t} + e^{-t}} \, \mathrm{d}t \\
&= 4 \int_{0}^{\infty} \frac{\cos(t) \, e^{-t}}{1 + e^{-2t}} \, \mathrm{d}t \\
&= 4 \int_{0}^{\infty} \cos(t) \, e^{-t} \sum_{n=0}^{\infty} (-1)^{n} e^{-2nt} \, \mathrm{d}t \\
&= 4 \sum_{n=0}^{\infty} (-1)^{n} \int_{0}^{\infty} \cos(t) \, e^{-(2n+1)t} \, \mathrm{d}t \\
&= 4 \sum_{n=0}^{\infty} (-1)^{n} \frac{2n+1}{(2n+1)^{2} + 1} = \pi \operatorname{sech}\left(\frac{\pi}{2}\right)
\end{aligned}$$

It's pretty tough to recognise that series as sech - you can use contour integration to avoid it. The original integral can also be done with a rectangular contour.

18

$$\int_0^\infty \frac{\log(x+1)}{x^2+1}\mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \frac{\log(x+1)}{x^2+1}\,\mathrm{d}x &\overset{x=\tan\theta}{=} \int_0^{\frac{\pi}{2}} \log(\tan\theta+1)\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \log(\sin\theta+\cos\theta)-\log(\cos\theta)\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \log\left(\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\right)\ \mathrm{d}\theta+\frac{\pi}{2}\log 2
\end{aligned}$$

where we have used the well known $\int_0^{\frac{\pi}{2}} \log\cos\theta\,\mathrm{d}\theta = -\frac{\pi}{2}\log 2$,

$$\begin{aligned}
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\sin\left(\theta+\frac{\pi}{2}\right)\right)\ \mathrm{d}\theta+\frac{3\pi}{4}\log 2 \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\cos\theta\,\mathrm{d}\theta+\frac{3\pi}{4}\log 2 \\
&= 2\int_0^{\frac{\pi}{4}} \log\cos\theta\,\mathrm{d}\theta+\frac{3\pi}{4}\log 2
\end{aligned}$$

Let $A=\int_0^{\frac{\pi}{4}}\log\cos\theta\,\mathrm{d}\theta$ and $B=\int_0^{\frac{\pi}{4}}\log\sin\theta\,\mathrm{d}\theta$. Then,

$$\begin{aligned}
A+B &= \int_0^{\frac{\pi}{4}} \log(\sin\theta\cos\theta)\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \log\sin 2\theta\,\mathrm{d}\theta-\frac{\pi}{4}\log 2\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin\theta\,\mathrm{d}\theta-\frac{\pi}{4}\log 2 = -\frac{\pi}{4}\log 2-\frac{\pi}{4}\log 2 = -\frac{\pi}{2}\log 2 \\
B-A &= \int_0^{\frac{\pi}{4}} \log\tan\theta\,\mathrm{d}\theta \\
&= \int_0^1 \frac{\log x}{1+x^2}\,\mathrm{d}x = -G
\end{aligned}$$

where the method used here is largely the same as in Question 13.

Hence, we have

$$\begin{aligned}
\int_0^\infty \frac{\log(x+1)}{x^2+1}\,\mathrm{d}x &= 2A+\frac{3\pi}{4}\log 2 \\
&= 2\times\frac{(A+B)-(B-A)}{2}+\frac{3\pi}{4}\log 2 \\
&= \frac{\pi}{4}\log 2+G.
\end{aligned}$$

19

$$\int_0^{\pi} \sec x \log(1 + 3\cos x)\,\mathrm{d}x$$

Let $I(a) = \displaystyle\int_0^{\pi} \sec x \log\left(1 + a\cos x\right)\,\mathrm{d}x$ and differentiate with respect to a.

$$\begin{aligned}
I'(a) &= \int_0^{\pi} \frac{\mathrm{d}x}{1 + a\cos x} \\
&\overset{t=\tan\frac{x}{2}}{=} \int_0^{\infty} \frac{1}{1 + a\frac{1-t^2}{1+t^2}} \times \frac{2\,\mathrm{d}t}{1+t^2} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{1 + a + (1-a)\,t^2}\,\mathrm{d}t \\
&= \frac{2}{1-a}\int_0^{\infty} \frac{\mathrm{d}t}{\frac{1+a}{1-a} + t^2}\,\mathrm{d}t \\
&= \frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}\left[\arctan\left(\sqrt{\frac{1+a}{1-a}}t\right)\right]_0^{\infty} \\
&= \frac{\pi}{\sqrt{1-a^2}}.
\end{aligned}$$

We may now finish the problem.

$$\begin{aligned}
\int_0^{\pi} \sec x \log(1 + 3\cos x)\,\mathrm{d}x &= I\left(\frac{1}{3}\right) \\
&= I\left(\frac{1}{3}\right) - I(0) \\
&= \int_0^{\frac{1}{3}} I'(t)\,\mathrm{d}t \\
&= \int_0^{\frac{1}{3}} \frac{\pi}{\sqrt{1-t^2}}\,\mathrm{d}t = \pi\arcsin\left(\frac{1}{3}\right).
\end{aligned}$$

20

$$\int_0^1 \frac{\log(1+x+x^2)}{x} \mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{\log\left(1+x+x^2\right)}{x}\,\mathrm{d}x &= \int_0^1 \log\left(\frac{1-x^3}{1-x}\right) \times \frac{\mathrm{d}x}{x} \\
&= \int_0^1 \frac{\log\left(1-x^3\right)}{x}\,\mathrm{d}x - \int_0^1 \frac{\log\left(1-x\right)}{x}\,\mathrm{d}x \\
&= -\int_0^1 \sum_{n=1}^{\infty} \frac{x^{3n-1}}{n}\,\mathrm{d}x + \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \left(-\frac{1}{3n^2} + \frac{1}{n^2}\right) \\
&= -\frac{\pi^2}{18} + \frac{\pi^2}{6} = \frac{\pi^2}{9}
\end{aligned}$$

21

$$\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\log x+1}}{x^2} \mathrm{d}x$$

$$\begin{aligned}
\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\log x+1}}{x^2}\,\mathrm{d}x \overset{u=\log x+1}{=} & \int_0^{\infty} \frac{\sqrt{u}}{\left(e^{u-1}\right)^2} \times e^{u-1}\,\mathrm{d}u \\
= & \quad e\int_0^{\infty} \sqrt{u}e^{-u}\,\mathrm{d}u \\
= & \quad e\frac{\sqrt{\pi}}{2}
\end{aligned}$$

The last equality may be derived by using the recurrence property of Γ and the substitution $t = u^2$.
Traditionally, problem 21 is my favourite problem because there's a clever idea involved. Here, the alternative solution is a pretty clever idea. Write $1 = \log e$ and then

$$\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\log x+1}}{x^2}\,\mathrm{d}x = \int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\log(xe)}}{x^2}\,\mathrm{d}x$$

The lower bound being $\frac{1}{e}$ is the motivation for this; now we can substitute $u = ex$ and then $t = \log u$, we get

$$\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\log(xe)}}{x^2}\,\mathrm{d}x = e\int_1^{\infty} \frac{\sqrt{\log u}}{u^2}\,\mathrm{d}u = e\int_0^{\infty} \sqrt{t}e^{-t}\,\mathrm{d}t = \frac{e\sqrt{\pi}}{2}.$$

22

$$\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)$$

$$\begin{aligned}
\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)\,\mathrm{d}x &= \int_0^\infty \log\left(\frac{1+e^{-x}}{1-e^{-x}}\right)\,\mathrm{d}x \\
&= \int_0^\infty \log\left(1+e^{-x}\right)\,\mathrm{d}x - \int_0^\infty \log\left(1-e^{-x}\right)\,\mathrm{d}x \\
&= \int_0^\infty \sum_1^\infty \frac{(-1)^{n+1}e^{-nx}}{n}\,\mathrm{d}x + \int_0^\infty \sum_1^\infty \frac{e^{-nx}}{n}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{12} + \frac{\pi^2}{6} \\
&= \frac{\pi^2}{4}
\end{aligned}$$

23

$$\int_0^1 \sqrt[4]{\frac{1}{x} - 1}\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \sqrt[4]{\frac{1}{x} - 1}\,\mathrm{d}x = B\left(\frac{5}{4}, \frac{3}{4}\right) \\
&= \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}+\frac{3}{4}\right)} \\
&= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{4\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

This can also be done by recognising the inverse of the integrand is $\frac{1}{x^4+1}$ which can be done via partial fractions & Sophie Germain identity, the beta function, or contour integration.

24

$$\int_0^{2\pi} e^{\cos x}\cos(nx - \sin x)\mathrm{d}x \text{ for } n \in \mathbf{N}$$

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos\left(nx - \sin x\right)\,\mathrm{d}x = \Re\left(\int_0^{2\pi} e^{\cos x}e^{i(nx-\sin x)}\,\mathrm{d}x\right) \\
&= \Re\left(\int_0^{2\pi} e^{inx+\cos x - i\sin x}\,\mathrm{d}x\right) \\
&= \Re\left(\int_0^{2\pi} e^{inx}e^{e^{-ix}}\,\mathrm{d}x\right) \\
&= \Re\left(\int_0^{2\pi} e^{inx}\sum_{k=0}^{\infty}\frac{e^{-ikx}}{k!}\,\mathrm{d}x\right) \\
&= \Re\left(\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^{2\pi} e^{i(n-k)x}\,\mathrm{d}x\right) \\
&= \Re\left(\sum_{k=0}^{\infty}\frac{1}{k!}\times\left(2\pi\delta_{n,k}\right)\right) = \frac{2\pi}{n!}.
\end{aligned}$$

25

$$\int_0^\infty \frac{\log x \sin x}{x} \mathrm{d}x$$

We will make use of the following formula for $\log x$ (by Frullani):

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \,\mathrm{d}t$$

$$\begin{aligned}
\int_0^\infty \frac{\log x \sin x}{x} \,\mathrm{d}x &= \int_0^\infty \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \times \frac{\sin x}{x} \,\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \frac{1}{t} \int_0^\infty \frac{\sin x}{x} \left(e^{-t} - e^{-xt}\right) \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \frac{1}{t} \left(\frac{\pi}{2} e^{-t - \frac{\pi}{2} + \arctan t}\right) \,\mathrm{d}t \\
&= \left[\log t \left(\frac{\pi}{2} e^{-t} - \frac{\pi}{2} + \arctan t\right)\right]_0^\infty - \int_0^\infty \log t \left(-\frac{\pi}{2} e^{-t} + \frac{1}{1+t^2}\right) \,\mathrm{d}t \\
&= \frac{\pi}{2} \int_0^\infty \log(t)\, e^{-t} \,\mathrm{d}t = \frac{-\gamma\pi}{2}
\end{aligned}$$

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$$\int_0^1 \frac{x-1}{(x+1)\log x} \mathrm{d}x$$

Let $I(a) = \displaystyle\int_0^1 \frac{x^a - 1}{(x+1)\log x}\,\mathrm{d}x$ and differentiate with respect to a.

$$\begin{aligned}
I'(a) &= \int_0^1 \frac{x^a}{x+1}\,\mathrm{d}x \\
&= \int_0^1 x^a \sum_{n=0}^{\infty} (-1)^n x^n \,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{a+n}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{a+n+1}
\end{aligned}$$

Now, we find the original integral.

$$\begin{aligned}
\int_0^1 \frac{x-1}{(x+1)\log x}\,\mathrm{d}x &= I(1) \\
&= I(1) - I(0) \\
&= \int_0^1 I'(t)\,\mathrm{d}t \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{t+n+1}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\log\left(t+n+1\right)\right]_0^1 \\
&= \sum_{n=0}^{\infty} (-1)^n \log\left(n+2\right) - (-1)^n \log\left(n+1\right) \\
&= \sum_{n=0}^{\infty} \log\left(2n+2\right) - \log\left(2n+3\right) + \log\left(2n+2\right) - \log\left(2n+1\right) \\
&= \log\left(\prod_{n=0}^{\infty}\right) \frac{\left(2n+2\right)\left(2n+2\right)}{\left(2n+3\right)\left(2n+1\right)} = \log\left(\frac{\pi}{2}\right)
\end{aligned}$$

where in the last line, we have made use of the Wallis Product.

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$$\int_0^1 \frac{\sin(\log x) - \log x}{\log^2 x} \mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{\sin(\log x) - \log x}{\log^2 x} \, \mathrm{d}x &\stackrel{u=-\log x}{=} \int_0^\infty \frac{u - \sin u}{u^2} e^{-u} \, \mathrm{d}u \\
&= \;=\; \int_0^\infty t \int_0^\infty (u - \sin u) \, e^{-(1+t)u} \, \mathrm{d}u\mathrm{d}t \\
&= \;=\; \int_0^\infty t \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^2} + 1 \right) \, \mathrm{d}t \\
&= \;=\; \left[\log(1+t) + \frac{1}{1+t} - \frac{1}{2} \log\left((1+t)^2 + 1 \right) + \arctan 1 + t \right]_0^\infty \\
&= \;=\; \left[\log\left(\frac{1+t}{\sqrt{(1+t)^2 + 1}} \right) + \frac{1}{1+t} + \arctan(1+t) \right]_0^\infty \\
&= \frac{1}{2} \log 2 + \frac{\pi}{4} - 1
\end{aligned}$$

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$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \log\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) \mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-x^2}} \log\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) \mathrm{d}x \overset{x=\cos\theta}{=} & \int_0^{\frac{\pi}{2}} \log\left(\frac{\sqrt{1+\cos\theta}+1}{\sqrt{1+\cos\theta}-1}\right) \\
= & \int_0^{\frac{\pi}{2}} \log\left(\frac{\left(\sqrt{1+\cos\theta}+1\right)}{\left(\sqrt{1+\cos\theta}-1\right)\left(\sqrt{1+\cos\theta}+1\right)}\right) \mathrm{d}\theta \\
= & \ 2\int_0^{\frac{\pi}{2}} \log\left(\sqrt{1+\cos\theta}+1\right) \mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \log\cos\theta \,\mathrm{d}\theta \\
= & \ 2\int_0^{\frac{\pi}{2}} \log\left(1+\sqrt{2}\cos\left(\frac{\theta}{2}\right)\right) \mathrm{d}\theta + \frac{\pi}{2}\log 2 \\
\overset{\varphi=\frac{\theta}{2}}{=} & \ 4\int_0^{\frac{\pi}{4}} \log\left(1+\sqrt{2}\cos\varphi\right) \mathrm{d}\theta + \frac{\pi}{2}\log 2 \\
= & \ 4\int_0^{\frac{\pi}{4}} \log\left(\frac{1}{\sqrt{2}}+\cos\varphi\right) \mathrm{d}\varphi + \pi\log 2 \\
= & \ 4\int_0^{\frac{\pi}{4}} \log\left(\cos\left(\frac{\pi}{4}\right)+\cos\varphi\right) \mathrm{d}\varphi + \pi\log 2 \\
= & \ 4\int_0^{\frac{\pi}{4}} \log\left(\cos\left(\frac{\varphi+\frac{\pi}{4}}{2}\right)\cos\left(\frac{\varphi-\frac{\pi}{4}}{2}\right)\right) \mathrm{d}\varphi \\
& +2\pi\log 2 \\
= & \ 2\pi\log 2 + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\left(\cos\left(\frac{\varphi}{2}\right)\right) \mathrm{d}t \\
& +4\int_{-\frac{\pi}{4}}^{0} \log\left(\cos\left(\frac{\varphi}{2}\right)\right) \mathrm{d}\varphi \\
= & \ 2\pi\log 2 + \int_0^{\frac{\pi}{2}} \log\left(\cos\left(\frac{\varphi}{2}\right)\right) \mathrm{d}\varphi \\
= & \ 2\pi\log 2 + 4\int_0^{\frac{\pi}{2}} -\log 2 + 4\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\cos(n\varphi)\ \mathrm{d}\varphi
\end{aligned}$$

where we have used the log cosine Fourier series

$$= \ 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 4G$$

This problem can be solved without knowledge of the log cosine Fourier series in the same way as Q18.

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$$\int_0^{\frac{1}{2}} \log(\sqrt{1-x} - \sqrt{x})\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \log\left(\sqrt{1-x} - \sqrt{x}\right)\,\mathrm{d}x \overset{x=\sin^2\theta}{=} & \int_0^{\frac{\pi}{4}} \log\left(\cos\theta - \sin\theta\right) \times 2\sin\theta\cos\theta\,\mathrm{d}\theta \\
\overset{\text{IBP}}{=} & -\frac{1}{2}\left[\cos 2\theta \log\left(\cos\theta - \sin\theta\right)\right]_0^{\frac{\pi}{4}} \\
& +\frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta\left(-\sin\theta - \cos\theta\right)}{\cos\theta - \sin\theta}\,\mathrm{d}\theta \\
= & -\frac{1}{2}{}_0^{\frac{\pi}{4}} \left(\sin\theta + \cos\theta\right)^2\,\mathrm{d}\theta \\
= & -\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin^2\left(\theta + \frac{\pi}{4}\right)\,\mathrm{d}\theta \\
= & -\frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos\left(2\theta - \frac{\pi}{2}\right)\,\mathrm{d}\theta \\
= & -\left(\frac{\pi}{8} + \frac{1}{4}\right) = -\frac{\pi + 2}{8}
\end{aligned}$$

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$$\int_0^{\infty} \frac{\arctan x \log(1+x^2)}{x(a^2+x^2)} \mathrm{d}x$$

Unfortunately, we have yet to think of a solution that does not make use of complex analysis. Using the modulus argument form we have $1 + ix = \sqrt{1+x^2} e^{\arctan x}$. Then, we may rewrite:

$$\arctan x = \frac{i}{2} \left(\log\left(1 - ix\right) - \log\left(1 + ix\right) \right)$$

$$\log\left(1 + x^2\right) = \log\left(1 + ix\right) + \log\left(1 - ix\right).$$

Thus, we get the following...

$$\begin{aligned} &\int_0^{\infty} \frac{\arctan x \log(1+x^2)}{x(a^2+x^2)} \, \mathrm{d}x \\ &\qquad = \frac{i}{4} \int_{-\infty}^{\infty} \frac{\left(\log\left(1-ix\right) - \log\left(1+ix\right)\right)\left(\log\left(1+ix\right) + \log\left(1-ix\right)\right)}{x\left(a^2+x^2\right)} \, \mathrm{d}x \end{aligned}$$

We have also used the "even"ness of the function to extend the limits of integration.

$$= \frac{i}{4} \left(\int_{-\infty}^{\infty} \frac{\log^2\left(1-ix\right)}{x\left(a^2+x^2\right)} \, \mathrm{d}x - \int_{-\infty}^{\infty} \frac{\log^2\left(1+ix\right)}{x\left(a^2+x^2\right)} \right) \mathrm{d}x$$

To perform integration, we close the contour in the complex half-plane by the half-circle of the radius $R \to \infty$: for the first term - in the upper half-plane (counter-clockwise); for the second - in the lower (clockwise). In both cases, logarithms do not have branch points in the designated closed contours. It is straightforward to show that the integrals along the big half-circles $\to 0$. $z = 0$ is a removable singular point; therefore

$$\begin{aligned} \int_0^{\infty} \frac{\arctan x \log(1+x^2)}{x(a^2+x^2)} \mathrm{d}x &= 2\pi i \times \frac{i}{4} \operatorname*{Res}_{z=ia} \frac{\log^2\left(1-ix\right)}{x\left(a^2+x^2\right)} - 2\pi i \left(-\frac{i}{4}\right) \operatorname*{Res}_{z=-ia} \frac{\log^2\left(1+ix\right)}{x\left(a^2+x^2\right)} \\ &= \frac{\pi \log^2\left(1+a\right)}{2a^2} \end{aligned}$$

References