TMUA Algebra and Functions

Syllabus

Laws of indices; surds; quadratic functions, graphs, discriminant, completing the square; solving equations; simultaneous equations; inequalities; polynomials (factorising, Factor & Remainder Theorem); $\sqrt{x}$ and modulus function.

1a) Solve the equation $2^{x+2} = 4\sqrt{2}$ $= 2^2 . 2^{1/2}$

$x + 2 = 2\frac{1}{2}$

$x = \frac{1}{2}$

b) Solve the equation $\dfrac{27^a}{3^{a-1}} = 3\sqrt{3}$ $\quad \dfrac{3^{3a}}{3^{a-1}} = 3^{3/2}$

$2a + 1 = 3/2$

$2a = 1/2$

$a = 1/4$

c) Solve the equation $4^x - 2^{x+2} = 32$

$2^{2x} - 4.2^x - 32 = 0$

$(2^x - 8)(2^x + 4) = 0$

$2^x = 8 \qquad 2^x = -4$

$x = 3 \qquad$ no solutions

d) How many real solutions does the following equation have $8^x + 4 = 4^x + 2^{x+2}$

$(2^x)^3 - (2^x)^2 - 4(2^x) + 4 = 0$

$(2^x - 1)((2^x)^2 - 4) = 0$

$2^x = 1 \qquad 2^x = 2 \qquad 2^x = -2$

$x = 0 \qquad x = 1 \qquad$ no solutions

2 solutions

e) Find the values of k such that the equation $9^x - 3^{x+1} = k$ has one or more real solutions.

$(3^x)^2 - 3(3^x) - k = 0$

$b^2 - 4ac = 9 + 4k \geq 0$

$k \geq -9/4$

2a) Simplify the following expression giving your answer in the form $a + b\sqrt{3}$ $\quad \dfrac{(2\sqrt{3} - 1)(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})}$

$= \dfrac{4\sqrt{3} - 2 + 6 - \sqrt{3}}{4 - 3} = 4 + 3\sqrt{3}$

b) The area of a triangle is $(3 + \sqrt{3})$ cm² . Given that the base is $\sqrt{3}$ cm, find the height as a surd.

$\frac{1}{2} \times \sqrt{3} \times h = 3 + \sqrt{3} \qquad h = \dfrac{6 + 2\sqrt{3}}{\sqrt{3}} = \dfrac{6\sqrt{3} + 6}{3} = 2\sqrt{3} + 2$

c) What positive integer does this expression simplify to $\dfrac{1 + \sqrt{7}}{3 - \sqrt{7}} - \dfrac{8 - \sqrt{7}}{\sqrt{7} - 2}$

$\dfrac{(1 + \sqrt{7})(\sqrt{7} - 2) - (8 - \sqrt{7})(3 - \sqrt{7})}{5\sqrt{7} - 13}$

$= \dfrac{5 - \sqrt{7} - (31 - 11\sqrt{7})}{5\sqrt{7} - 13}$

$\dfrac{10\sqrt{7} - 26}{5\sqrt{7} - 13} = 2$

d) Write this expression as a single fraction in its simplest form $\dfrac{1}{x - \sqrt{y}} + \dfrac{1}{x + \sqrt{y}}$

$\dfrac{x + \sqrt{y} + x - \sqrt{y}}{x^2 - y} = \dfrac{2x}{x^2 - y}$

3a) The quadratic equation $x^2 + ax + b = 0$ where a and b are constants, is satisfied by $x = -2$ and $x = 5$. Find the values of a and b.

$(x+2)(x-5) = x^2 - 3x - 10$

$a = -3$

$b = -10$

b) $f(x) = ax^2 + bx + c$ where a, b and c are non-zero constants, or $f(x) = k(x-2)^2 - 6$ $k = 4$

Given $f(-1) = f(5) = 30$ and the minimum value of $f(x)$ is -6, solve the equation $f(x) = 3$

$a - b + c = 30$

$25a + 5b + c = 30$

$4a + 2b + c = -6$

$21a + 3b = 36$

$12a + 3b = 0$

$f'(x) = 2ax + b = 0$

$x = \frac{-b}{2a}$

$9a = 36$

$a = 4$ $b = -16$ $c = 10$

$24a + 6b = 0$

$4a = -b$

$2 = \frac{-b}{2a}$

$4x^2 - 16x + 7 = 0$

$(2x-1)(2x-7) = 0$

$x = 1/2, 7/2$

c) Solve the equation $x - \frac{14}{x} = 6\sqrt{2}$ giving your answers in the form $p\sqrt{2}$

$x^2 - 6\sqrt{2}x - 14 = 0$

$(x - 3\sqrt{2})^2 - 18 - 14 = 0$

$x - 3\sqrt{2} = \pm 4\sqrt{2}$

$x = 7\sqrt{2}, -\sqrt{2}$

or $x = \frac{6\sqrt{2} \pm \sqrt{72 + 56}}{2}$

$= \frac{6\sqrt{2} \pm 8\sqrt{2}}{2} = 3\sqrt{2} \pm 4\sqrt{2}$

$= 7\sqrt{2}, -\sqrt{2}$

d) Solve the equation $\sqrt{3}\left(x + \frac{6}{x}\right) = 9$ giving your answers in the form $p\sqrt{3}$

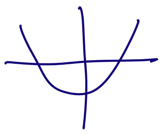
$x^2 + 6 = 3\sqrt{3}x$

$x^2 - 3\sqrt{3}x + 6 = 0$

$x = \frac{3\sqrt{3} \pm \sqrt{27 - 24}}{2}$

$= \frac{1}{2}(3\sqrt{3} \pm \sqrt{3}) = 2\sqrt{3}, \sqrt{3}$

e) Solve the inequality $x^4 < 8x^2 + 9$



$x^4 - 8x^2 - 9 < 0$

$(x^2 - 9)(x^2 + 1) < 0$

$x^2 + 1 > 0$ all x $\Rightarrow$ $x^2 - 9 < 0$

$(x-3)(x+3) < 0$

$-3 < x < 3$

f) Given that $f(x) = x^2 + 10x + 27$ find k, such that the graph of $f(x) - k$ touches the x-axis

$f(x) = (x+5)^2 + 2$

$k = 2$

g) A quadratic curve meets the coordinate axis at (-2,0), (4,0), and (0,-20).

Find the equation of the curve.

$y = a(x+2)(x-4) = a(x^2 - 2x - 8)$

$-8a = -20$ $a = \frac{5}{2}$

$y = \frac{5}{2}(x^2 - 2x - 8) = \frac{5}{2}x^2 - 5x - 20$

h) A quadratic curve meets the coordinate axis at (2,0), (6,0), and (0,3).

Find the minimum point of the curve.

$x = 4$

$y = a(x-2)(x-6)$

$12a = 3 \quad a = \frac{1}{4}$

$y = \frac{1}{4}(x-2)(x-6)$

$= \frac{1}{4}(2)(-2) = -1$

$(4,-1)$

i) Find the constant k, such that the quadratic curves with equations

$y = k(2x^2 + 1)$ and $y = x^2 - 2x$ touch each other

$2kx^2 + k = x^2 - 2x$

$x^2(2k-1) + 2x + k = 0$

$\Delta = 0 \quad 4 - 4(2k-1)k = 0$

$4 - 8k^2 + 4k = 0$

$2k^2 - k - 1 = 0$

$(2k+1)(k-1) = 0$

$k = -\frac{1}{2}, 1$

j) Find the range of values of the constant k, such that the curve C with equation

$y = 4x^2 - 7x + 11$ and straight line L with equation $y = 5x + k$ do not intersect

$4x^2 - 7x + 11 = 5x + k$

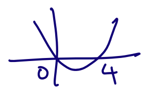
$4x^2 - 12x + 11 - k = 0$

$\Delta < 0 \quad 144 - 16(11-k) < 0$

$9 - 11 + k < 0$

$k < 2$

k) The straight line L crosses the y-axis at (0,-1). The curve with equation $y = x^2 + 2x$ does not intersect with L. Determine the range of possible values of the gradient of L.



$y = mx - 1$

$x^2 + 2x = mx - 1$

$x^2 + (2-m)x + 1 = 0$

$\Delta < 0 \quad 4 - 4m + m^2 - 4 < 0$

$m(m-4) < 0$

$0 < m < 4$

l) Given that f(n) is a square number for all values of n, find the possible values of the constant k.

$f(n) = n^2 - 2kn + k + 12, \quad n \in \mathbb{N}$

$= (n-k)^2 - k^2 + k + 12$

$k^2 - k - 12 = 0$

$(k-4)(k+3) = 0$

OR $\Delta = 0 \quad 4k^2 - 4(k+12) = 0$

$k^2 - k - 12 = 0$

$k = 4, k = -3$

m) The roots of $2x^2 - 7x + c = 0$ where c is a constant, differ by 3. Find the value of c.

$(x-\alpha)(x-\alpha-3) = 0$

$x^2 - 2\alpha x + \alpha^2 - 3x + 3\alpha = 0$

$2x^2 - (4\alpha + 6)x + (2\alpha^2 + 6\alpha) = 0$

$4\alpha + 6 = 7 \quad \alpha = \frac{1}{4}$

$c = \frac{2}{16} + \frac{3}{2} = \frac{1}{8} + \frac{12}{8} = \frac{13}{8}$

OR $x = \frac{7 \pm \sqrt{49-8c}}{4}$

$\frac{2\sqrt{49-8c}}{4} = 3$

$49 - 8c = 36$

$8c = 13 \quad c = \frac{13}{8}$

n) The roots of $2x^2 + 5x + c = 0$ where c is a constant, differ by 2. Find the value of c.

$(x-\alpha)(x-\alpha-2) = 0$

$x^2 - 2\alpha x - 2x + \alpha^2 + 2\alpha = 0$

$2x^2 - (4\alpha + 4)x + 2\alpha^2 + 4\alpha = 0$

$4\alpha + 4 = -5 \quad \alpha = -\frac{9}{4}$

$c = \frac{81}{8} - 9 = \frac{81-72}{8} = \frac{9}{8}$

$x = \frac{-5 \pm \sqrt{25-8c}}{4}$

$\frac{2\sqrt{25-8c}}{4} = 2$

$25 - 8c = 16$

$8c = 9$

$c = 9/8$

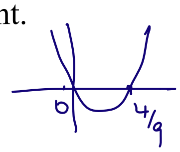
4a) Given that the equation below has two distinct real roots,

$x^2 + 3ax + a = 0$ determine the range of values of a, where a is a constant.

$\Delta > 0$ $\quad 9a^2 - 4a > 0$

$a(9a-4) > 0$

$a < 0, \quad a > \frac{4}{9}$



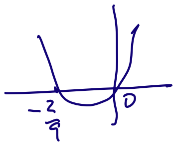
b) Given that the equation below has two distinct real roots,

$x^2 + 6mx - 2m = 0$ determine the range of values of m, where m is a constant.

$\Delta > 0$ $\quad 36m^2 + 8m > 0$

$m(9m+2) > 0$

$m < -\frac{2}{9}, \quad m > 0$



c) Given that the equation below has no real roots, $\Delta < 0$

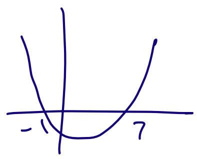
$x^2 + (k-1)x + (k+2) = 0$ determine the range of values of k, where k is a constant.

$k^2 - 2k + 1 - 4k - 8 < 0$

$k^2 - 6k - 7 < 0$

$(k-7)(k+1) < 0$

$-1 < k < 7$



d) Given that the equation below has two different real roots, $\Delta > 0$

$2x^2 + (3k-1)x + (3k^2 - 1) = 0$ determine the range of values of k, where k is a constant.

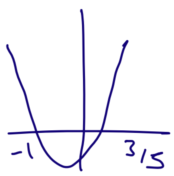
$9k^2 - 6k + 1 - 24k^2 + 8 > 0$

$-15k^2 - 6k + 9 > 0$

$5k^2 + 2k - 3 < 0$

$(5k-3)(k+1) < 0$

$-1 < k < 3/5$



e) Given that the equation below has no real roots, $\Delta < 0$

$kx^2 - x + (3k-1) = 0$ determine the range of values of k, where k is a non-zero constant.

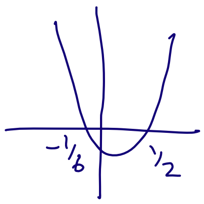
$1 - 4k(3k-1) < 0$

$1 - 12k^2 + 4k < 0$

$12k^2 - 4k - 1 > 0$

$(6k+1)(2k-1) > 0$

$k < -\frac{1}{6}, \quad k > \frac{1}{2}$



f) Given that the equation below has two distinct real roots, $\Delta > 0$

$mx^2 + (2m-3)x + 2m + 1 = 0$ determine the range of values of the non-zero constant m.

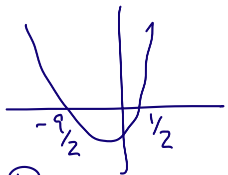
$4m^2 - 12m + 9 - 4m(2m+1) > 0$

$-4m^2 - 16m + 9 > 0$

$4m^2 + 16m - 9 < 0$

$(2m-1)(2m+9) < 0$

1×4
2×2
1×9
3×3



$-\frac{9}{2} < m < \frac{1}{2} \quad m \neq 0$

5a) The polynomial $x^3 + 4x^2 + 7x + a$ where a is a constant, has a factor of $(x + 2)$.

Find the value of a.

$f(-2) = 0 \qquad -8 + 16 - 14 + a = 0$

$a = 6$

b) $f(x) = ax^3 - x^2 - 5x + b$ where a and b are constants.

When f(x) is divided by $(x - 2)$ the remainder is 36

When f(x) is divided by $(x + 2)$ the remainder is 40. Find the value of a and the value of b.

$f(2) = 36 \qquad 8a - 4 - 10 + b = 36 \qquad 8a = 36 + 14 - 42$

$f(-2) = 40 \qquad -8a - 4 + 10 + b = 40 \qquad a = 1$

$-8 + 2b = 76$

$b = 42$

c) $f(x) = px^3 - 32x^2 - 10x + q$ where p and q are constants.

When f(x) is divided by $(x - 2)$ the remainder is exactly the same as when it is divided by $(2x + 3)$. Find the value of p.

$f(2) = f\left(-\frac{3}{2}\right) \qquad 8p - 128 - 20 + q = -p\left(\frac{27}{8}\right) - 32\left(\frac{9}{4}\right) + 15 + q$

$8p + \frac{27}{8}p = 148 - 72 + 15$

$\frac{91}{8}p = 91 \qquad p = 8$

d) $f(x) = 6x^2 + x + 7$

The remainder when f(x) is divided by $(x - a)$ is the same as the remainder when f(x) is divided by $(x + 2a)$, where a is a non-zero constant. Find the value of a.

$f(a) = f(-2a) \qquad 6a^2 + a + 7 = 24a^2 - 2a + 7$

$0 = 18a^2 - 3a$

$0 = 6a^2 - a$

$0 = a(6a - 1) \qquad a = 1/6$

e) $g(x) = x^3 + kx^2 - x + 12$

The remainder when g(x) is divided by $(x - 4)$ is 8 times the remainder when g(x) is divided by $(x - 1)$, where k is a constant. Find the value of k.

$g(4) = 8g(1) \qquad 64 + 16k - 4 + 12 = 8(1 + k - 1 + 12)$

$16k + 72 = 8(k + 12)$

$2k + 9 = k + 12$

$k = 3$

f) $f(x) = ax^2 + bx + c$ where *a, b* and *c* are non-zero constants

When f(x) is divided by $(x-1)$ the remainder is 1. $f(1) = 1$ $a + b + c = 1$ ①

When f(x) is divided by $(x-2)$, the remainder is 2. $f(2) = 2$ $4a + 2b + c = 2$ ②

When f(x) is divided by $(x+2)$, the remainder is 70. Find the values of *a, b* and *c.*

$f(-2) = 70$ $4a - 2b + c = 70$ ③

② - ③ : $4b = -68$ $\quad b = -17$

② - ① $3a + b = 1$ $\quad 3a = 18$ $\quad a = 6$

① : $6 - 17 + c = 1$

$c = 12$

g) $f(x) = 2x^2 + 9x - 5$ Find *k* such that when f(*x*) is divided by $(2x - k)$ the remainder is 13.

$f\left(\frac{k}{2}\right) = 13$ $\quad 2\left(\frac{k^2}{4}\right) + \frac{9k}{2} - 5 = 13$

$k^2 + 9k - 36 = 0$

$(k+12)(k-3) = 0$

$k = -12, 3$

h) $f(x) = x^3 + (a+2)x^2 - 2x + b$ where *a* and *b* are non-zero constants, and $a > 0$.

Given that $(x-2)$ and $(x+a)$ are factors of f(*x*), find the values of *a* and *b*.

$f(2) = 8 + 4(a+2) - 4 + b = 0$

$f(-a) = -a^3 + a^2(a+2) + 2a + b = 0$

$2a^2 + 2a + b = 0$

$12 + 4a + b = 0$

$2a^2 + 2a = 12 + 4a$

$a^2 - a - 6 = 0$

$(a-3)(a+2) = 0$

$a = 3, \cancel{-2}$

$b = -12 - 12$

$= -24$

i) When the polynomial $p(x) = x^2 - 2ax + a^4$ is divided by $(x+b)$ the remainder is 1

The polynomial $q(x) = bx^2 + x + 1$ has $(ax - 1)$ as a factor. Find the possible value(s) of *b*

$p(-b) = 1$ $\quad b^2 + 2ab + a^4 = 1$

$q\left(\frac{1}{a}\right) = 0$ $\quad \frac{b}{a^2} + \frac{1}{a} + 1 = 0$

$b + a + a^2 = 0$

$b = -a - a^2$

$a^2 + 2a^3 + a^4 - 2a^2 - 2a^3 + a^4 - 1 = 0$

$2a^4 - a^2 - 1 = 0$

$(2a^2 + 1)(a^2 - 1) = 0$

$a = \pm 1$ $\quad b = -1 - 1 = -2$

$b = 1 - 1 = 0$

j) Find the remainder when $1 + 3x + 5x^2 + 7x^3 + \ldots + 99x^{49}$ is divided by $(x-1)$.

$f(1) = 1 + 3 + 5 + 7 + \ldots + 97 + 99$

$= 25 \times 100$

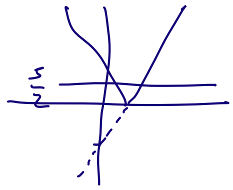
$= 2500$

6a) Solve the equation $|3x + 2| = 1$

$3x + 2 = 1$
$3x = -1$
$x = -\frac{1}{3}$

$3x + 2 = -1$
$3x = -3$
$x = -1$

b) Solve the inequality $12 - 2|2x - 3| > 7$



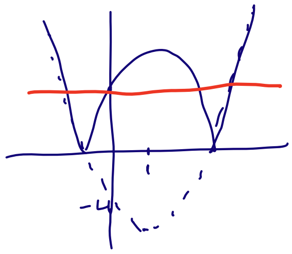
$2|2x - 3| < 5$

$2x - 3 = \frac{5}{2}$
$x = \frac{11}{4}$

$2x - 3 = -\frac{5}{2}$
$x = \frac{1}{4}$

$\frac{1}{4} < x < \frac{11}{4}$

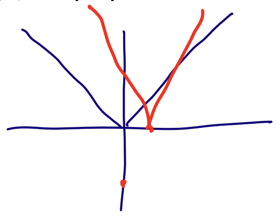
c) Solve the equation $|x^2 - 2x - 4| = 4$



$x^2 - 2x - 4 = 4$
$x^2 - 2x - 8 = 0$
$(x - 4)(x + 2) = 0$
$x = 4, \; x = -2$

$x^2 - 2x - 4 = -4$
$x(x - 2) = 0$
$x = 0 \quad x = 2$

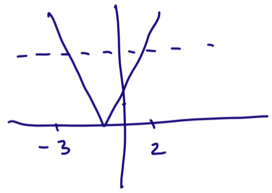
d) Solve the equation $f(x) = g(x)$ where $f(x) = |2x - 4|$ and $g(x) = |x|$ for $x \in \mathbb{R}$



$2x - 4 = x$
$x = 4$

$2x - 4 = -x$
$x = \frac{4}{3}$

e) Solve the inequality $fg(x) > 12$ where $f(x) = x + 4$ and $g(x) = |2x + 1| + 3$ for $x \in \mathbb{R}$



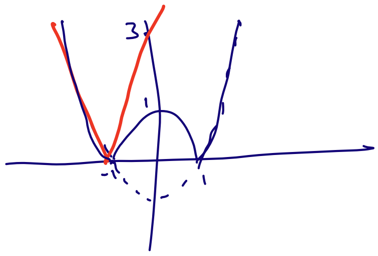
$|2x + 1| + 7 > 12$
$|2x + 1| > 5$

$2x + 1 = 5$
$x = 2$

$2x + 1 = -5$
$x = -3$

$x < -3, \; x > 2$

f) Solve the equation $|x^2 - 1| = |3x + 3|$

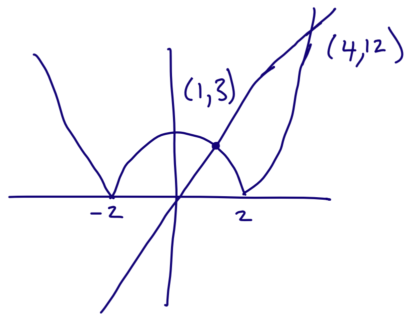


$x^2 - 1 = 3x + 3$
$x^2 - 3x - 4 = 0$
$(x - 4)(x + 1) = 0$
$x = 4, -1$

$x^2 - 1 = -3x - 3$
$x^2 + 3x + 2 = 0$
$(x + 1)(x + 2) = 0$
$x = -1, -2$

$x = -2, -1, 4$

g) Find the set of values of x for which $|x^2 - 4| < 3x$



$x^2 - 4 = 3x$
$x^2 - 3x - 4 = 0$
$(x - 4)(x + 1) = 0$
$x = 4 \quad x = -1$
✗

$x^2 - 4 = -3x$
$x^2 + 3x - 4 = 0$
$(x + 4)(x - 1) = 0$
$x = -4 \quad x = 1$
✗

$1 < x < 4$

7a) The function f(n) is defined for positive integers n by

$f(1) = 5$ and for $n > 1$, $f(n+1) = 3f(n) - 1$ if $f(n)$ is odd and

$f(n+1) = \dfrac{f(n)}{2}$ if $f(n)$ is even

a) Find $f(99)$ $= 20$

b) How many numbers n in the interval $1 \le n \le 50$ satisfy $f(n) \le 12$ $\frac{3}{5} \times 50 = 30$

$f(1) = 5$ $f(5) = 10$

$f(2) = 14$ $f(6) = 5$

$f(3) = 7$ ...

$f(4) = 20$ repeats every 5

b) The function f(n) is defined for positive integers n by

$f(1) = 2$ and for $n \ge 1$, $f(n+1) = 5f(n) + 1$ if $f(n)$ is odd and

$f(n+1) = \dfrac{1}{2}f(n)$ if $f(n)$ is even

a) Find $f(100)$ $= 1$

b) Find the value of $\displaystyle\sum_{r=1}^{50} f(r)$

$= (7 \times 40) + 2$

$= 282$

$f(2) = \frac{1}{2}(2) = 1$

$f(4) = \frac{1}{2}(6) = 3$

$f(6) = \frac{1}{2}(16) = 8$

$f(8) = \frac{1}{2}(4) = 2$

$f(1) = 2$

$f(3) = 6$

$f(5) = 16$

$f(7) = 4$

and repeat

c) The function f(n) is defined for positive integers n by

$f(1) = 4$ and for $n \ge 1$, $f(n+1) = \dfrac{1}{2}(f(n) + 3)$ if $f(n)$ is even and

$f(n+1) = 2f(n) + 3$ otherwise

Find the value of $f(99) + f(100)$

$f(1) = 4$

$f(3) = 10$

$f(5) = 16$

$f(7) = 22$

n^{th} term $3n + \frac{1}{2}$

$f(100) = 150\frac{1}{2}$

$f(2) = \frac{1}{2}(7) = \frac{7}{2}$

$f(4) = \frac{13}{2}$

$f(6) = \frac{19}{2}$

n^{th} term $6n - 2$

$f(99) = 6 \times 50 - 2 = 298$

$448\frac{1}{2}$

d) The function f is defined such that $f(mn) = \begin{cases} f(m)f(n) & \text{if mn is a multiple of 5} \\ mn & \text{if mn is not a multiple of 5} \end{cases}$

Given that $f(25) + f(9) - f(30) = 0$ find the value of $f(5)$

$f(25) = [f(5)]^2$ $f(9) = 9$ $f(30) = f(5) \times f(6) = 6f(5)$

$y = f(5)$ $y^2 - 6y + 9 = 0$

$(y-3)^2 = 0$

$f(5) = 3$