MOCK TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

Paper 2

ANSWERS

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1. answer: **D**

If $(x^2+y^2-1)^2 = 1$ then either $x^2+y^2-1 = 1$ which is a circle of radius $\sqrt{2}$, or $x^2+y^2-1 = -1$ which is a single point (the origin). A sketch will therefore show a circle.

Alternatively, if a student were uncomfortable with square-rooting, $(x^2+y^2)^2 - 2(x^2+y^2) = 2 \implies (x^2+y^2)(x^2+y^2-2) = 0$ gives the same result.

2. answer: **E**

$\sin(x^{-1}) = 0$ means $x^{-1} = k\pi$ for each $k \in \mathbb{Z}$ so $x = (k\pi)^{-1}$ for all $k \neq 0$, of which there are infinitely many in $(0, 1)$ (in fact, for all positive k).

3. answer: **C**

Crucially, since 3 is prime, if it divides a product ab it must divide either a or b. Hence 3 divides n (1), and therefore n^3 is divisible by 27 (2). (3) is false e.g. let $n = 3$ and $d = 27$.

4. answer: **B**

Intuitively, since the leading coefficient is fixed, two points determine the quadratic. Algebraically, two points (p, q) and (r, s) yield a system of equations for a and b:

$$p^2 + ap + b = q; \qquad r^2 + ar + b = s$$

Subtracting one from the other gives $(p - r)(a + p + r) = q - s$, and since $p \neq r$ this gives a single solution.

5. answer: **D**

All of these functions are convex in at least one region. However, (C) is not differentiable so it does not belong to the set about which the claim is made. One might immediately spot that x^4 is twice-differentiates to $12x^2$ which is 0 at 0, but if not, the astute candidate will note that, if f is convex on (a, b), then $f''(x) = 0$ can only happen within (a, b) at a local minimum of f. Going through the list:

$x^2 + \cos x$ twice-differentiates to $2 - \cos x$ which is > 0 everywhere.

$x + \cos x$ differentiates to $1 - \sin x$ so zeroes at $x = \frac{\pi}{2} + 2k\pi$, which do not agree with the zeroes of $-\cos x$, the second derivative.

Finally, if one were unlucky to skip x^4 and arrive at $x^2(1 + x^2)^{-1}$, differentiating once gives a single minimum at the origin, however it is clearly not a turning point of the derivative, so the second derivative cannot be 0 there.

6. answer: **B**

Geometrically, in the a-b plane, the first is a line and the latter a circle centred at the origin. There are solutions if and only if the line intersects the circle, or in other words when the radius is $\geqslant$ the distance from the line to the origin.

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The former is $\sqrt{\mu}$ and the latter is $\frac{1}{2}\mu$ (since the closest point is $(\frac{1}{2}\mu, \frac{1}{2}\mu)$), squaring and equating gives $\mu = 2$ since $\mu > 0$ so $\mu > 2$ presents no solutions.

An alternative some students might try is a purely algebraic approach, though this turns out to be significantly trickier: for instance substituting $b = \mu - a$ into the second lines gives $\mu^2 - (2a+1)\mu + 2a^2 = 0$. Applying the quadratic formula, one then needs to find the range of the solutions for μ etc.

7. answer: **E**

This question tests understanding of the contrapositive of a statement, without which the candidate will be stuck. The natural approach should be to form strings of implications, which results in the following maximal string (up to contraposition):

$$(\beta) \leftarrow (\alpha) \leftarrow (\gamma) \leftarrow (\text{not } \delta) \leftarrow (\text{not } \beta)$$

Hence we have (not β) $\rightarrow$ (β), a contradiction.

An alternative is to go by elimination: (A) $(\alpha) \rightarrow (\beta)$ but (β) does not imply anything; (B) says $(\beta) \rightarrow (\alpha)$, yet as mentioned previously, (β) cannot lead anywhere; (C) same reason once again; (D) (not δ) is only implied by (not β), which is not implied by anything again for the same reason.

8. answer: **F**

This question tests the important skill in questioning a mathematical statement by trying to think of counterexamples if one suspects it is false:

(A) $z'(x)$ having three roots only means z has three turning points, all of which may occur above or below the x-axis. (B) is also false as a result.

(C) perhaps a little less obvious, z has two turning points if one of the roots of the cubic is a repeated root, which means this is a saddle point. This situation gives a maximum of two roots.

(D) The condition $(ac)^2 > 16bd$ is not equivalent to $a^2 > 4b$ and $c^2 > 4d$ (it necessary but not sufficient), so does not guarantee the quadratics each have two roots each.

(E) If, for instance, $z(x) = (x-\alpha)(x-\beta)q(x)$ for some $\alpha \neq \beta$ and some quadratic q with no real solutions, z is divisible by $x - \alpha$, $x - \beta$, $q(x)$ and $z(x)$ itself.

9. answer: **D**

With 5 distinct letters, there are $5! = 120$ permutations. To arrive at 20, this needs to scale down by a factor of 6. Since $3! = 6$, the desired condition (which is in fact both necessary and sufficient) is for the word to contain three identical letters.

10. answer: **A**

This questions tests the ability to dissect a proof meticulously. Unless the candidate makes an integration mistake, the only potentially troubling steps should be (1) or (4).

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(1) holds because, if each of these equalities hold, the claim is automatically verified. To make this clear, since the claim is meant to hold for all cubics: if the cubic were of the form ax^3, we would necessitate the first equality to be true. As a result, the second equality would have to hold as well, so the claim would be true for all cubics of the form $ax^3 + bx^2$. Similarly we get $ax^3 + bx^2 + cx$ and as a result, the general cubic form.

(4) related to the points in (1), if the values of α, β work with (1) (which they do since the integral is 0 and $\alpha = -\beta$), putting the cubic together term by term verifies the result. It is important to realise that the assumption of the existence of α and β does *not* question the validity of the values the student has found.

11. answer: **E**

This is an exercise in analysing a mathematical set-up.

One observation that should come about is that the answers, with one exception (answer (E)), only contain 1 or 2 as coefficients (up to sign). This suggests we need to think about how a, b, c, d affect coefficients and whether (E) can fit into this. The most efficient way would most likely be to work backwards, assuming (E) is *abcd*-composite. d quite clearly cannot be the last function to have been applied because we would have had a $\frac{3}{2}$, and c cannot either since we cannot have achieved a 3 in the denominator after 3 steps. Since b is present in the denominator of (E), the only last step possible would be a, but undoing a gives rise to a 4, which takes us even further away.

Alternatively, whilst a student might have trouble articulating their solution clearly, playing around with the functions quickly demystifies the idea of achieving anything other than 1s or 2s in the resulting expression.

The *wrong* way to do this question is to attempt to write each expression in terms of a, b, c, d (which, whilst possible, might be time consuming).

12. answer: **D**

If a and b are coprime, then all factors of b must come into 7, which has only two factors (1 and 7), forcing b to be 1 or 7.

13. answer: **A**

The function f looks like horizontal segments of length 1, each $\frac{1}{2}$ the distance from the x-axis as the previous, but alternating between negative and positive.

Since the first segment for $x > 0$ is positive, F (the cumulative area from 0) is positive for all x, despite f alternating in sign, which disproves (1). However, F is not an increasing function, since it decreases on every interval where f is negative. f is not increasing either, so this is *not* a counterexample to (3), but merely an example of the property described in (3) (which is false because it claims this property is *always* true). Finally, F does not satisfy (2), so it is not a counterexample there either.

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14. answer: **A**

This is just a matter of verifying each possible scenario. If it *is* raining in Paris, then (i) is true, and (ii) is true because it would begin with a false premise. If it is *not* raining in Paris, then (i) is true if and only if it is raining in New York, which agrees with (ii). [*this is sometimes used a definition for 'or' in propositional logic*]

15. answer: **A**

To get anywhere with this, it is necessary to look at the first terms. With very little work, $\xi_0 = 0$ and $\xi_1 = 1$. Then, successively, using the identities given we get that ξ_2, $\xi_3 \ldots$ are integers, so the answer is (A) or (B). The parity pattern might not be obvious to most candidates, so it is necessary to find the next couple terms to see whether (B) could hold (doing this through the identities of course, rather than the original definition) which is disproved by $\xi_5 = 5$. Some candidates might recognise this as the Fibonacci sequence, though this is of course not necessary.

16. answer: **F**

This tests the ability to dissect an argument, similarly to question 10. Here (2), and (3) are incorrect and based upon the same flaw, which is to neglect $a \leqslant -1$, and more precisely in the case of (3), just $a = -1$. Whilst this is a striking issue, it is also one that can be glossed over if one skims over the proof too quickly.

17. answer: **E**

To answer the question, it is necessary to integrate:

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$$

However, the integrand is not defined at 0, and has an asymptote there.

18. answer: **D**

This tests the ability to 'dissect' a mathematical definition. The definition is saying that, no matter how small a positive number ϵ we choose, there is a point in the sequence (e.g. x_N) beyond which all terms are $< \epsilon$ away from a.

The negation therefore is where, at any point in the sequence, some element beyond it will *not* fall into a given neighbourhood of a. In other words, there must be some distance ϵ, where regardless of which point we look at in the sequence (e.g. x_N), there will always be at least one point beyond it which is at a larger distance from a than ϵ. Translating into the language used for the definition, this is (D).

Note:

Whilst (C) and (E) will be more easily observed to be false, a candidate might be tempted to interpret this negation as answer (A) i.e. 'no matter what positive number ϵ we choose, and

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which point in the sequence we look at, there is at least some element beyond it which is at least ϵ away from a. Whilst this is sufficient (it implies the sequence is unbounded), it is not necessary, because a sequence can fail to converge but not satisfy (A) (e.g. $x_n = (-1)^n$, $a = 0$ and $\epsilon = 2$).

In a similar vein, one might be tempted by (B) i.e. 'no matter what positive number ϵ is chosen, there is no point in the sequence beyond which all terms are $< \epsilon$ away from a. Again, the same example shows that this is not necessary either.

19. answer: **A**

A neat logical argument, in which there are only two cases: if $\sqrt{2}^{\sqrt{2}}$ is rational the (A) is verified, if it is irrational then from (3), (A) is also verified.

As a result, (B) is false, and (D) is clearly false e.g. $2^{\frac{1}{2}}$. (A) can be proven constructively, and (C) similarly, using exponentials/logs, though this is of course not part of the question.

20. answer: **D**

Like question 11., this tests the ability to analyse a mathematical object.

Whilst upon first glance the candidate might feel like the definition is tricky, they should attempt to clear it up by writing out the first terms, which makes it a lot more approachable. It consists of the ordered natural numbers, interspersed with a slower-growing sequence of the same numbers, occurring at indices with powers of 2. e.g.

Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}	Φ_{11}
0	1	1	2	2	3	4	5	3	6	7	8

Staring at this, we can see that neither (i), (ii) or (iv) hold: for (i), let (A_n) be the even numbers, for (ii) and (iii) terms Φ_6, Φ_7, Φ_8 are counterexamples. Answer (G) is there purely to prevent a candidate from answering this by elimination only.

For (iii), the key is that Φ_{G_n} cannot truly 'dabble' in both of the subsequences outlined in the question. Indeed only two cases arise: either (G_n) contains only powers of 2, in which case Φ_{G_n} is a subsequence of Φ_{T_n} and is therefore increasing, or (G_n) contains at most one power of 2 (necessarily the first term if so), in which case, neglecting the first term (if a power of 2), it is a subsequence of Φ_{V_n}, which is also increasing.

(we can neglect the first term because it will always be smaller than the following one. If a candidate wished to convince themselves: if $G_0 = 2^q$ and $G_1 = 2^q r$ (r not being a power of 2) then, for instance, $G_1 > G_0 + 1$ (so $\Phi_{G_1} > \Phi_{G_0+1}$) but $\Phi_{G_0} < \Phi_{G_0+1}$ as well, hence $\Phi_{G_0} = q < \Phi_{G_1}$. Given the context of the examination, intuition would be entirely sufficient for this detail)

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