THE UK UNIVERSITY INTEGRATION BEE 2020

$$\int$$

Virtual Bee

By: Vishal Gupta

Solutions By: Finn Hogan

Sponsored by

Jane Street

Integrals involve many tricks; however, many of these tricks appear quite frequently. For the sake of time and clarity there will be some abbreviations in the solutions. The full list of these abbreviations is below...

The Weierstass Substitution $t = \tan\left(\frac{x}{2}\right)$: $\overset{\text{W}}{=}$

$$\sin x = \frac{2t}{1-t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}$$

The King Property for Integrals $u = a + b - x$: $\overset{\text{K}}{=}$

$$\int_a^b f(x)\,\mathrm{d}x \overset{\text{W}}{=} \int_a^b f(a+b-x)\,\mathrm{d}x.$$

The Log Cosine and Log Sine Integrals: $\overset{\text{G}}{=}$

$$\int_0^{\frac{\pi}{2}} \log\sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \log\cos x\,\mathrm{d}x \overset{\text{G}}{=}= -\frac{\pi}{2}\ln 2$$

Integration by parts: $\overset{\text{IBP}}{=}$

$$\int u\,\mathrm{d}v \overset{\text{IBP}}{=} uv - \int v\,\mathrm{d}u$$

1

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\,\mathrm{d}x$$

$$\begin{aligned}
I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\,\mathrm{d}x &\overset{\mathrm{K}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{-x}}\,\mathrm{d}x \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos x}{1+e^x}\,\mathrm{d}x = I \\
I &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+e^x)\cos x}{1+e^x}\,\mathrm{d}x \\
&= \frac{1}{2}\sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1
\end{aligned}$$

2

$$\int_0^\infty \frac{\mathrm{d}x}{\left(x+\frac{1}{x}\right)^2}$$

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{\left(x+\frac{1}{x}\right)^2} &= \int_0^\infty \frac{x^2\mathrm{d}x}{\left(x^2+1\right)^2} \\
&= \int_0^\infty \frac{1}{1+x^2} - \frac{1}{\left(1+x^2\right)^2}\,\mathrm{d}x \\
&\overset{x=\tan\theta}{=} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^2\theta} - \frac{\sec^2\theta}{\sec^4\theta}\,\mathrm{d}\theta \\
&= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

3

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\sqrt[4]{(1+x)^2\,(1-x)^6}}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\sqrt[4]{(1+x)^2\,(1-x)^6}}\,\mathrm{d}x &\overset{x=\cos 2\theta}{=} \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{\left(1+\sqrt{3}\right)\cdot -2\sin 2\theta}{\sqrt[4]{4\cos^2\theta\cdot 64\sin^6\theta}}\,\mathrm{d}\theta \\
&= \frac{1+\sqrt{3}}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2\theta\,\mathrm{d}\theta \\
&= -\frac{1+\sqrt{3}}{4}\cot\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 2
\end{aligned}$$

4

$$\int_0^\infty \frac{\cos(\pi x) - \cos(ex)}{x}\,\mathrm{d}x$$

$$\begin{aligned}
I(a) &= \int_0^\infty \frac{\cos(\pi x) - \cos(ex)}{x}\,e^{-ax}\,\mathrm{d}x \\
I'(a) &= \int_0^\infty \cos(ex)\,e^{-ax} - \cos(\pi x)\,e^{-ax}\,\mathrm{d}x \\
&= \frac{a}{a^2+e^2} - \frac{a}{a^2+\pi^2} \\
I(\infty) - I(a) &= \int_a^\infty \frac{a}{a^2+e^2} - \frac{a}{a^2+\pi^2}\,\mathrm{d}a \\
&= \frac{1}{2}\ln\left|\frac{a^2+e^2}{a^2+\pi^2}\right|\Bigg|_a^\infty \\
&= \frac{1}{2}\ln\left|\frac{a^2+\pi^2}{a^2+e^2}\right|
\end{aligned}$$

But then we can see that $I(\infty) = 0$ and therefore we have that

$$\begin{aligned}
I(a) &= \frac{1}{2}\ln\left|\frac{a^2+e^2}{a^2+\pi^2}\right| \\
I(0) &= \frac{1}{2}\ln\left(\frac{e}{\pi}\right)
\end{aligned}$$

5

$$\int \sqrt{\tan x}\,\mathrm{d}x$$

Begin with the substitution $u = \sqrt{\tan x}$:

$$\int \sqrt{\tan x}\,\mathrm{d}x = \int \frac{2u^2}{1+u^4}\,du$$

To find the antiderivative of $\frac{2u^2}{1+u^4}$, we'll use partial fraction decomposition. The integral is:

$$\int \frac{2u^2}{1+u^4}\,du$$

First, factorize the denominator:

$$1+u^4 = (u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)$$

Now, decompose the fraction:

$$\frac{u^2}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)} = \frac{Au+B}{u^2+\sqrt{2}u+1} + \frac{Cu+D}{u^2-\sqrt{2}u+1}$$

Solving for A, B, C, and D, we find:

$$A = \frac{\sqrt{2}}{4}, \quad B = 0, \quad C = -\frac{\sqrt{2}}{4}, \quad D = 0$$

So, the integral becomes:

$$\int \left(\frac{\sqrt{2}u}{4(u^2+\sqrt{2}u+1)} - \frac{\sqrt{2}u}{4(u^2-\sqrt{2}u+1)} \right) du$$

After some messing around with reverse chain rule of $\ln(\cdot)$ and $\arctan(\cdot)$ expressions we arrive at

$$\frac{\sqrt{2}}{8}\ln\left(\frac{u^2+\sqrt{2}u+1}{u^2-\sqrt{2}u+1}\right) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}u+1) - \frac{\sqrt{2}}{4}\arctan(\sqrt{2}u-1) + C$$

where C is the constant of integration. Finally, we undo our original substitution to arrive at the answer...

$$\frac{\sqrt{2}}{8}\ln\left(\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2\tan x}+1) - \frac{\sqrt{2}}{4}\arctan(\sqrt{2\tan x}-1) + C$$

6

$$\int_0^{\frac{\pi}{2}} \cot x \ln\left(\sec x\right)\, \mathrm{d}x$$

We begin by rewriting the expression:

$$\int_0^{\frac{\pi}{2}} \cot x \ln\left(\sec x\right)\, \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cot x \ln\left(1+\tan^2 x\right)\, \mathrm{d}x$$

This new form motivates the following generalised integral:

$$I(a) = \int_0^{\frac{\pi}{2}} \cot x \ln\left(1+a\tan^2 x\right)\, \mathrm{d}x.$$

Now, we can apply Leibniz's Rule and evaluate $I'(a)$.

$$\begin{aligned} I'(a) &= \int_0^{\frac{\pi}{2}} \frac{\tan x}{1+a\tan^2 x}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\cos^2 x + a\sin^2 x}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{1+(a-1)\sin^2 x}\,\mathrm{d}x \\ &= \frac{1}{2(a-1)}\ln\left|1+(a-1)\sin^2 x\right|\Big|_0^{\frac{\pi}{2}} = \frac{\ln a}{2(a-1)}. \end{aligned}$$

Next, we undo the derivative by integrating both sides:

$$I(1) - I(0) = \int_0^1 \frac{\ln a}{2(a-1)}\,\mathrm{d}a.$$

Now we can evaluate this integral by consulting our favourite, geometric series:

$$\begin{aligned} \int_0^1 \frac{\ln a}{2(a-1)}\,\mathrm{d}a &= \frac{1}{4}\int_0^1 -\ln a \sum_{n=0}^{\infty} a^n\,\mathrm{d}a \\ &\overset{t=-\ln a}{=} \frac{1}{4}\sum_{n=0}^{\infty}\int_0^{\infty} te^{-(n+1)t} \\ &\frac{1}{4}\sum_{n=0}^{\infty}\frac{\Gamma(2)}{(n+1)^2} = \frac{\pi^2}{24}. \end{aligned}$$

7

$$\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}\,\mathrm{d}x &\overset{u=x^{\frac{1}{6}}}{=} \int \frac{6u^2}{1+u^2}\,\mathrm{d}u \\
&= \int 6 - \frac{6}{1+u^2}\,\mathrm{d}u \\
&= 6u - 6\arctan(u) + C \\
&= 6x^{\frac{1}{6}} - 6\arctan\left(x^{\frac{1}{6}}\right) + C.
\end{aligned}$$

8

$$\int_0^1 \ln(x)\sin(\ln(x))\,\,\mathrm{d}x$$

We employ a little trick to evaluate this integral. First, notice that

$$\int_0^1 \ln x \sin(\ln x)\,\,\mathrm{d}x = \Im\left[\int_0^1 \ln x \times x^i\,\mathrm{d}x\right]$$

Here $\Im$ denotes the imaginary part. Now, consider the following:

$$I(a) = \int_0^1 x^a\,\mathrm{d}x = \frac{1}{a+1}.$$

Then differentiating both sides with respect to a gives us:

$$I'(a) = \int_0^1 \ln x \times x^a\,\mathrm{d}a = -\frac{1}{(a+1)^2}.$$

Thus, our answer is given by

$$\Im\left[I'(i)\right] = \Im\left[\left(-\frac{1}{(1+i)^2}\right)\right] = \frac{1}{2}$$

9

$$\int \frac{\mathrm{d}x}{x^2 - x\sqrt{x^2-1}}$$

Fairly tricky substitution problem

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2 - x\sqrt{x^2-1}} &\overset{x=\sec\theta}{=} \int \frac{\sec\theta\tan\theta\mathrm{d}\theta}{\sec^2\theta - \sec\theta\tan\theta} \\
&= \int \frac{\sin\theta}{1-\sin\theta}\,\mathrm{d}\theta \\
&= \int -1 + \frac{1}{1-\sin\theta\,\mathrm{d}\theta} \\
&= -\theta + \int \sec^2\theta + \sec\theta\tan\theta\,\mathrm{d}\theta \\
&= -\theta + \tan\theta + \sec\theta + C = -\operatorname{arcsec} x + \sqrt{x^2-1} + x + C.
\end{aligned}$$

10

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,\mathrm{d}x}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$$

The integrand is quite intimidating but we see that the King Property leads to some nice simplifications.

$$\begin{aligned}
I &= \int_2^4 \frac{\sqrt{\ln(9-x)}\,\mathrm{d}x}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}} \\
&\overset{\text{K}}{=} \int_2^4 \frac{\sqrt{\ln(x+3)}\,\mathrm{d}x}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}} = I \\
I &= \frac{1}{2}\int_2^4 \frac{\sqrt{\ln(x+3)}+\sqrt{\ln(x+3)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}\,\mathrm{d}x \\
&= \frac{1}{2}\int_2^4 \mathrm{d}x = 1
\end{aligned}$$

11

$$\int \frac{\left(3x^{10}+2x^{8}-2\right)\sqrt[4]{x^{10}+x^{8}+1}}{x^{6}}\,\mathrm{d}x$$

A hard substitution problem.

$$\begin{aligned}\int \frac{\left(3x^{10}+2x^{8}-2\right)\sqrt[4]{x^{10}+x^{8}+1}}{x^{6}}\,\mathrm{d}x &= \int \left(3x^{5}+2x^{3}-2x^{-5}\right)\sqrt[4]{x^{6}+x^{4}+x^{-4}}\,\mathrm{d}x \\ &= \frac{2}{5}\left(x^{6}+x^{4}+x^{-4}\right)^{\frac{5}{4}}+C\end{aligned}$$

12

$$\int_0^\infty \frac{\sin^2 x}{x^2\,(x^2+1)}\,\mathrm{d}x$$

This is a famous tricky DUTIS problem.

$$\begin{aligned}
I(a) &= \int_0^\infty \frac{\sin^2(ax)}{x^2\,(x^2+1)}\,\mathrm{d}x \\
I'(a) &= \int_0^\infty \frac{2\sin(ax)\cos(ax)}{x\,(x^2+1)} \\
&= \int_0^\infty \frac{\sin(2ax)}{x\,(x^2+1)}\,\mathrm{d}x \\
I''(a) &= \int_0^\infty \frac{2\cos(2ax)}{x^2+1}\,\mathrm{d}x \\
I'''(a) &= -4\int_0^\infty \frac{x\sin(2ax)}{x^2+1}\,\mathrm{d}x \\
&= -4\int_0^\infty \frac{x^2\sin(2ax)}{x\,(x^2+1)}\,\mathrm{d}x \\
&= -4\int_0^\infty \frac{\sin(2ax)}{x}\,\mathrm{d}x + 4\int_0^\infty \frac{\sin(2ax)}{x\,(x^2+1)}\,\mathrm{d}x \\
&= -2\pi + 4I'(a).
\end{aligned}$$

We therefore get a second-order ordinary differential equation. The solution is

$$I'(a) = Ae^{2a} + Be^{-2a} + \frac{\pi}{2}.$$

This is an IVP: $I'(0) = 0$, $I''(0) = \int_0^\infty \frac{2}{x^2+1} = \pi$. Plugging these in we get that

$$I'(a) = -\frac{\pi}{2}e^{-2a} + \frac{\pi}{2}$$

Integrating both sides:

$$I(1) - I(0) = \int_0^1 -\frac{\pi}{2}e^{-2a} + \frac{\pi}{2}\,\mathrm{d}a = \frac{\pi}{4}e^{-2} + \frac{\pi}{4}.$$

This is the answer to the original problem as $I(0) = 0$.

13

$$\int \sqrt{1+e^x}\,\mathrm{d}x$$

Annoying substitution spam.

$$\begin{aligned}\int \sqrt{1+e^x}\,\mathrm{d}x &= \int \frac{\sqrt{1+e^x}}{e^x}\,e^x\,\mathrm{d}x \\ &\overset{u=\sqrt{1+e^x}}{=} \int \frac{u}{u^2-1}\,2u\,\mathrm{d}u \\ &= 2u - 2\operatorname{artanh} u + C = 2\sqrt{1+e^x} - \operatorname{artanh}\left(\sqrt{1+e^x}\right) + C\end{aligned}$$

14

$$\int_0^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,\mathrm{d}x$$

$$\int_0^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,\mathrm{d}x \overset{x=\tan\theta}{=} \int_0^{\frac{\pi}{2}} \ln\left(\sec^2\theta\right)\,\mathrm{d}\theta \overset{\mathrm{G}}{=} \pi\ln 2$$

15

$$\int \frac{1}{\sqrt{1+e^{2x}}}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{1}{\sqrt{1+e^{2x}}}\,\mathrm{d}x &= \int \frac{e^{-x}}{\sqrt{1+e^{-2x}}}\,\mathrm{d}x \\
&\overset{u=e^{-x}}{=} -\int \frac{\mathrm{d}u}{\sqrt{1+u^2}} \\
&= -\operatorname{arsinh}(u) + C \\
&= -\operatorname{arsinh}\left(e^{-x}\right) + C
\end{aligned}$$

16

$$\int_0^{\pi} \frac{\ln(1+k\cos x)}{\cos x}\,\mathrm{d}x \text{ where } 0<k<1.$$

$$\begin{aligned}
I(k) &= \int_0^{\pi} \frac{\ln(1+k\cos x)}{\cos x}\,\mathrm{d}x \\
I'(k) &= \int_0^{\pi} \frac{\mathrm{d}x}{1+k\cos x}\,\mathrm{d}x \\
&\overset{\text{W}}{=} \int_0^{\infty} \frac{2\,\mathrm{d}t}{1+t^2+k(1-t^2)} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{1+k+(1-k)t^2} \\
&= \frac{2}{\sqrt{1-k^2}} \arctan\left(\sqrt{\frac{1-k}{1+k}}t\right)\Bigg|_0^{\infty} = \frac{\pi}{\sqrt{1-k^2}}
\end{aligned}$$

Integrating both sides:

$$I(k)-I(0) = \int_0^k \frac{\pi}{\sqrt{1-k^2}}\,\mathrm{d}k = \pi\arcsin(k)$$

17

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt{x}}\,\mathrm{d}x$$

$$\begin{aligned}\int_0^\infty \frac{\ln(1+x)}{x\sqrt{x}}\,\mathrm{d}x &\overset{\text{IBP}}{=} -2\ln(1+x)\,x^{-\frac{1}{2}}\Big|_0^\infty + 2\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\,(1+x)}\\ &= 0 + 4\arctan\sqrt{x}\Big|_0^\infty = 2\pi\end{aligned}$$

18

$$\int \frac{\mathrm{d}x}{2+2\sin x+\cos x}$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{2+2\sin x+\cos x} &\overset{\mathrm{W}}{=} \int \frac{2}{2\left(1+t^2\right)+2\left(2t\right)+\left(1-t^2\right)}\,\mathrm{d}t \\
&= \int \frac{2\,\mathrm{d}t}{\left(t+1\right)\left(t+3\right)} \\
&= \int \frac{1}{t+1}-\frac{1}{t+3}\,\mathrm{d}t \\
&= \ln\left|\frac{t+1}{t+3}\right|+C \\
&= \ln\left|\frac{\tan\left(\frac{x}{2}\right)+1}{\tan\left(\frac{x}{2}\right)+3}\right|+C
\end{aligned}$$

19

$$\int_0^1 x^{-x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 x^{-x}\,\mathrm{d}x &= \int_0^1 e^{-x\ln x}\,\mathrm{d}x \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n \left(-\ln x\right)^n\, \mathrm{d}x \\
&\overset{u=-\ln x}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} e^{-(1+n)u} u^n\,\mathrm{d}u \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \times \frac{\Gamma n+1}{(n+1)^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^n}
\end{aligned}$$

20

$$\int_0^{\frac{\pi}{2}} \left(\ln\left(\tan\theta\right)\right)^2 \, \mathrm{d}\theta$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \left(\ln\left(\tan\theta\right)\right)^2 \, \mathrm{d}\theta &\stackrel{u=\tan x}{=} \int_0^{\infty} \frac{\ln^2 u}{1+u^2} \, \mathrm{d}u \\
&\stackrel{\mathrm{K}}{=} 2\int_0^{1} \frac{\ln^2 u}{1+u^2} \, \mathrm{d}u \\
&\stackrel{\mathrm{IBP}}{=} 2\ln^2 x \arctan x \Big|_0^1 - 4\int_0^1 \frac{\ln x \arctan x}{x} \, \mathrm{d}x \\
&= 4\int_0^1 -\ln x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \, \mathrm{d}x \\
&= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^{\infty} u e^{-(2n+1)} \, \mathrm{d}u \\
&= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \Gamma\left(2\right) = \frac{\pi^3}{8}
\end{aligned}$$

21

$$\int_0^\pi \ln\left(1 - 2\pi\cos x + \pi^2\right)$$

If

$$I(a) = \int_0^\pi \ln\left(1 - 2a\cos x + a^2\right)\,\mathrm{d}x,$$

then

$$I'(a) = \int_0^\pi \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,\mathrm{d}x, \tag{1}$$

$$I'(a) = \frac{1}{a}\int_0^\pi \frac{2a^2 - 2a\cos x}{1 - 2a\cos x + a^2}\,\mathrm{d}x \tag{2}$$

$$= \frac{1}{a}\int_0^\pi \frac{1 - 1 + a^2 + a^2 - 2a\cos x}{1 - 2a\cos x + a^2}\,\mathrm{d}x \tag{3}$$

$$= \frac{\pi}{a} + \frac{1}{a}\int_0^\pi \frac{a^2 - 1}{1 - 2a\cos x + a^2}\,\mathrm{d}x \tag{4}$$

$$I'(a) \overset{\text{W}}{=} \frac{\pi}{a} + \frac{2}{a}\int_0^\infty \frac{a^2 - 1}{(1 + a^2)(1 + t^2) - 2a(1 - t^2)}\,\mathrm{d}t \tag{5}$$

$$I'(a) = \frac{\pi}{a} + \frac{2}{a}\int_0^\infty \frac{a^2 - 1}{(1 - a)^2 + (1 + a)^2 t^2}\,\mathrm{d}t \tag{6}$$

$$I'(a) = \frac{\pi}{a} + \frac{\pi}{a}\operatorname{sgn}(a^2 - 1), \tag{7}$$

$$I'(a) = \frac{\pi}{a} + \frac{\pi}{a}\operatorname{sgn}(a^2 - 1),$$

so for $a > 1$,

$$I'(a) = \frac{2\pi}{a},$$

Integrating both sides

$$I(\pi) - I(1) = \int_1^\pi \frac{2\pi}{a}\,\mathrm{d}a = 2\pi\ln\pi$$

$$I(1) = \int_0^\pi \ln(2 - 2\cos x)\,\mathrm{d}x \overset{\text{K}}{=} \tag{8}$$

$$I(1) = \frac{1}{2}\int_0^\pi \ln(4\sin^2 x)\,\mathrm{d}x \tag{9}$$

$$= \frac{\pi}{2}\ln 4 + \int_0^\pi \ln(\sin x)\,\mathrm{d}x \tag{10}$$

$$= \pi\ln 2 + 2\int_0^{\pi/2} \ln(\sin x)\,\mathrm{d}x \overset{\text{G}}{=} 0 \tag{11}$$

Thus,

$$I(\pi) = 2\pi\ln\pi.$$

22

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} \, dx$$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} \, dx \\
&\overset{\mathrm{K}}{=} \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x + \cos x} \, dx = I \\
I &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 x - \sin x \cos x + \cos^2 x \, dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 - \frac{1}{2} \sin 2x \, dx \\
&= \frac{\pi}{4} - \frac{1}{4}
\end{aligned}$$

23

$$\int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor\,\mathrm{d}x &= \int_1^\infty \frac{\{x\}\lfloor x\rfloor}{x^3}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{(x-n)\,n}{x^3}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_n^{n+1} nx^{-2} - n^2x^{-3}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty -nx^{-1}\Big|_n^{n+1} + \frac{n^2}{2}x^{-2}\Big|_n^{n+1} \\
&= \sum_{n=1}^\infty 1 - \frac{n}{n+1} - \frac{1}{2} + \frac{n^2}{2\,(n+1)^2} \\
&= \sum_{n=1}^\infty \frac{1}{2\,(n+1)^2} = \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) = \frac{\pi^2}{12} - \frac{1}{2}
\end{aligned}$$

24

$$\int \sqrt{x-\sqrt{x+\sqrt{x-\sqrt{x+\dots}}}}\,\mathrm{d}x$$

Begin by defining A and B as follows

$$A = \sqrt{x-\sqrt{x+\sqrt{x-\sqrt{x+\dots}}}} \qquad B = \sqrt{x+\sqrt{x-\sqrt{x+\sqrt{x-\dots}}}}$$

$$A = \sqrt{x-B} \qquad B = \sqrt{x+A}$$

$$\begin{aligned} B^2 - A^2 &= A + B \\ B - A &= 1 \\ B &= 1 + A \end{aligned}$$

$$\begin{aligned} A &= \sqrt{x-1-A} \\ A^2 + A + 1 - x &= 0 \\ A &= -\frac{1}{2} + \frac{1}{2}\sqrt{4x-3} \end{aligned}$$

Hence,

$$\begin{aligned} \int \sqrt{x-\sqrt{x+\sqrt{x-\sqrt{x+\dots}}}}\,\mathrm{d}x &= \int -\frac{1}{2} + \frac{1}{2}\sqrt{4x-3}\,\mathrm{d}x \\ &= -\frac{1}{2}x + \frac{1}{12}\left(4x-3\right)^{\frac{3}{2}} + C \end{aligned}$$

25

$$\int_0^\pi \ln\left(a+b\cos x\right)\,\mathrm{d}x \text{ where } a>b$$

$$\begin{aligned}
I(a,b) &= \int_0^\pi \ln\left(a+b\cos x\right)\,\mathrm{d}x \\
\frac{\partial}{\partial a} I(a,b) &= \int_0^\pi \frac{\mathrm{d}x}{a+b\cos x} \\
&\overset{\text{W}}{=} \int_0^\infty \frac{2\,\mathrm{d}t}{a+b+(a-b)\,t^2} \\
&= \frac{2}{\sqrt{a^2-b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}}t\right)\Bigg|_0^\infty \\
&= \frac{\pi}{\sqrt{a^2-b^2}}
\end{aligned}$$

Integrating both sides with respect to a:

$$I(a,b) = \int \frac{\pi}{\sqrt{a^2-b^2}}\,\mathrm{d}a = \pi \operatorname{arcosh}\left(\frac{a}{b}\right) + C$$

$$\begin{aligned}
I\left(b,b\right) &= \pi \ln b + \int_0^\pi \ln\left(1+\cos x\right)\,\mathrm{d}x \\
&\overset{\text{K}}{=} \pi \ln b + \int_0^\pi \ln\left(1-\cos x\right)\,\mathrm{d}x \\
&= \pi \ln b + \int_0^\pi \ln\left(\sin x\right)\,\mathrm{d}x \\
&= \pi \ln b + 2\int_0^{\frac{\pi}{2}} \ln\left(\sin x\right)\,\mathrm{d}x \\
&\overset{\text{G}}{=} \pi \ln b - \pi \ln 2 = \pi \ln\left(\frac{b}{2}\right)
\end{aligned}$$

$$I(b,b) = \operatorname{arcosh}\left(\frac{b}{b}\right) + C = C = \pi \ln\left(\frac{b}{2}\right)$$

Thus,

$$I(a,b) = \pi \operatorname{arcosh}\left(\frac{a}{b}\right) + \pi \ln\left(\frac{b}{2}\right).$$

26

$$\int \cos(\ln x)\ \mathrm{d}x$$

Quite similar to earlier

$$\begin{aligned}
\int \cos(\ln x)\ \mathrm{d}x &= \Re\left[\int x^i\,\mathrm{d}x\right] \\
&= \Re\left[\frac{1}{1+i}x^{1+i} + C\right] \\
&= \Re\left[\frac{1-i}{2}x\left(\cos(\ln x) + i\sin(\ln x)\right) + C\right] \\
&= \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C
\end{aligned}$$

27

$$\int_0^\infty \frac{\arctan x}{1+x} \, \frac{\mathrm{d}x}{\sqrt{x}}$$

$$\begin{aligned}
I &= \int_0^\infty \frac{\arctan x}{1+x} \, \frac{\mathrm{d}x}{\sqrt{x}} \\
&\overset{u=\frac{1}{x}}{=} \int_0^\infty \frac{\arctan \frac{1}{x}}{1+x} \, \frac{\mathrm{d}x}{\sqrt{x}} = I \\
I &= \frac{\pi}{4} \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}\,(1+x)} \\
&= \frac{\pi}{2} \arctan\left(\sqrt{x}\right)\Big|_0^\infty \\
&= \frac{\pi^2}{4}
\end{aligned}$$

28

$$\int_0^1 \ln(x)\ln(1-x)\,\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \log x \log(1-x)\,\,\mathrm{d}x &= \int_0^1 -\log x \sum_{n=1}^{\infty} \frac{x^n}{n}\,\mathrm{d}x \\
&\overset{u=-\log x}{=} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-(n+1)u} u\,\mathrm{d}u \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \times \frac{\Gamma(2)}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \\
&= 1 - \left(\frac{\pi^2}{6} - 1\right) = 2 - \frac{\pi^2}{6}
\end{aligned}$$

29

$$\int \frac{x\,\ln\left(x+\sqrt{x^2+1}\right)}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{x\,\ln\left(x+\sqrt{x^2+1}\right)}{\sqrt{1+x^2}}\,\mathrm{d}x &\overset{x=\sinh\theta}{\int} \theta\sinh\theta\,\mathrm{d}\theta \\ &= \theta\cosh\theta - \sinh\theta + C \\ &= \sqrt{1+x^2}\,\operatorname{arsinh}(x) - x + C\end{aligned}$$

30

$$\int_0^{2\pi} \frac{1}{\sin^4(x)+\cos^4(x)}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}x}{\sin^4 x+\cos^4 x} &= 8\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{\sin^4 x+\cos^4 x}\\
&= 8\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}\\
&= 16\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{2-\sin^2(2x)}\\
&= 16\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{2\cos^2(2x)+\sin^2(2x)}\\
&= 16\int_0^{\frac{\pi}{4}} \frac{\sec^2(2x)}{2+\tan^2(2x)}\,\mathrm{d}x\\
&= 4\sqrt{2}\arctan\frac{\tan(2x)}{\sqrt{2}}\Bigg|_0^{\frac{\pi}{4}}\\
&= 2\sqrt{2}\pi
\end{aligned}$$

The first line can be shown using the King Property. However, you can also completely avoid changing the bound right up to the point where you are plugging numbers into the anti derivative and instead opt to split up the bound at each of its jump discontinuities instead.

31

$$\int \left[1 + \sin \frac{x}{2}\right] \, \mathrm{d}x$$

$$\begin{aligned}\int \left[1 + \sin \frac{x}{2}\right] \, \mathrm{d}x &= \int \left[\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right) + 2\sin\frac{x}{4}\cos\left(x\right)4\right] \, \mathrm{d}x \\ &= \int \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right) \, \mathrm{d}x \\ &= 4\sin\left(\frac{x}{4}\right) - 4\cos\left(\frac{x}{4}\right) + C\end{aligned}$$

32

$$\int_0^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x &= \frac{1}{2}\int_{-1}^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x \\
&\overset{u=1+x}{=} \frac{1}{2}\int_0^2 \ln(u)\ln(2-u)\ \mathrm{d}u \\
&\overset{u=2x}{=} \int_0^1 \ln(2x)\ln(2-2x)\ \mathrm{d}x \\
&= \int_0^1 (\ln x + \ln 2)(\ln(1-x) + \ln 2)\ \mathrm{d}x \\
&= \int_0^1 \ln^2 2 + \ln 2 \ln x + \ln 2 \ln(1-x) + \ln x \ln(1-x)\ \mathrm{d}x \\
&= \ln^2 2 - 2\ln 2 + 2 - \frac{\pi^2}{6}
\end{aligned}$$

33

$$\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)\,\mathrm{d}x &= \int_0^1 \arctan\left(\frac{x+(1-x)}{1-x\,(1-x)}\right)\,\mathrm{d}x \\
&= \int_0^1 \arctan(x) + \arctan(1-x)\,\mathrm{d}x \\
&= 2\int_0^1 \arctan(x)\,\mathrm{d}x \\
&\overset{\text{IBP}}{=} 2\,x\arctan(x)\Big|_0^1 - \int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{2} - \ln\left(1+x^2\right)\Big|_0^1 = \frac{\pi}{2} - \ln 2
\end{aligned}$$

34

$$\int e^{x+e^x}\,\mathrm{d}x$$

$$\begin{aligned}\int e^{x+e^x}\,\mathrm{d}x &= \int e^x \times e^{e^x}\,\mathrm{d}x \\ &= e^{e^x} + C\end{aligned}$$

35

$$\int_0^{\pi} \arctan\left(3^{\cos x}\right)\, \mathrm{d}x$$

$$\begin{aligned} I &= \int_0^{\pi} \arctan\left(3^{\cos x}\right)\, \mathrm{d}x \\ &\overset{\mathrm{K}}{=} \int_0^{\pi} \arctan\left(3^{-\cos x}\right)\, \mathrm{d}x = I \\ I &= \frac{1}{2}\int_0^{\pi} \arctan\left(3^{\cos x}\right) + \arctan\left(3^{-\cos x}\right)\, \mathrm{d}x \\ &= \frac{1}{2}\int_0^{\pi} \frac{\pi}{2}\, \mathrm{d}x \\ &= \frac{\pi^2}{4} \end{aligned}$$

36

$$\int \frac{x}{x^4 + 4} \, \mathrm{d}x$$

$$\int \frac{x}{x^4 + 4} \, \mathrm{d}x = \frac{1}{4} \arctan \left(\frac{x^2}{2} \right) + C$$

37

$$\int \frac{\mathrm{d}x}{\sqrt{x-1}+\sqrt{(x-1)^3}}$$

$$\begin{aligned}\frac{\mathrm{d}x}{\sqrt{x-1}+\sqrt{(x-1)^3}} &= \int \frac{1}{\sqrt{x-1}} \times \frac{1}{1+(x-1)}\,\mathrm{d}x \\ &= 2\arctan\left(\sqrt{x-1}\right)+C\end{aligned}$$

38

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} \,\mathrm{d}x$$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin 2x} \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin x}{2 - \sin 2x} \,\mathrm{d}x = I \\
I &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{2 - \sin 2x} \,\mathrm{d}x \\
&\overset{u=\sin x-\cos x}{=} \frac{1}{2} \int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2} \\
&= \frac{\pi}{4}
\end{aligned}$$

39

$$\int_0^1 \ln\left(\frac{2+x}{2-x}\right) \frac{\mathrm{d}x}{x\sqrt{1-x^2}}$$

$$\int_0^1 \ln\left(\frac{2+x}{2-x}\right) \frac{\mathrm{d}x}{x\sqrt{1-x^2}} = \int_0^1 \ln\left(\frac{1+\frac{x}{2}}{1-\frac{x}{2}}\right) \frac{\mathrm{d}x}{x\sqrt{1-x^2}}$$

$$\begin{aligned}
I(a) &= \int_0^1 \ln\left(\frac{1+ax}{1-ax}\right) \frac{\mathrm{d}x}{x\sqrt{1-x^2}} \\
I'(a) &= \int_0^1 \frac{2\,\mathrm{d}x}{\sqrt{1-x^2}\,(1-a^2x^2)} \\
&\overset{x=\sin\theta}{=} \int_0^{\frac{\pi}{2}} \frac{2\,\mathrm{d}\theta}{1-a^2\sin^2\theta} \\
&= \int_0^{\frac{\pi}{2}} \frac{2\,\mathrm{d}x}{(1-a^2)\sin^2 x + \cos^2 x} \\
&= \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x\,\mathrm{d}x}{(1-a^2)\tan^2 x + 1} \\
&= \frac{2}{\sqrt{1-a^2}} \arctan\left(\sqrt{1-a^2}\tan x\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{\sqrt{1-a^2}}
\end{aligned}$$

Integrating both sides:

$$\begin{aligned}
I\left(\frac{1}{2}\right) - I(0) &= \int_0^{\frac{1}{2}} \frac{\pi}{\sqrt{1-a^2}}\,\mathrm{d}a \\
&= \pi \arcsin(a)\Big|_0^{\frac{1}{2}} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Since, $I(0) = 0$ we see that:

$$\int_0^1 \ln\left(\frac{2+x}{2-x}\right) \frac{\mathrm{d}x}{x\sqrt{1-x^2}} = \frac{\pi^2}{6}$$

40

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{\sec^2 x - 2}{\tan x - 1}\right)\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln\left(\frac{\sec^2 x - 2}{\tan x - 1}\right)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\tan^2 x - 1}{\tan x - 1}\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\tan x + 1\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\sin x + \cos x\right)\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln\left(\cos x\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\cos\left(\frac{\pi}{4} - x\right)\right) - \ln\left(\cos x\right) + \frac{1}{2}\ln 2\,\mathrm{d}x \\
&= \frac{\pi}{8}\ln 2
\end{aligned}$$

References