Section A: Pure Mathematics

1 For non-negative integers a and b, let

$$\mathrm{I}(a,b) = \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \, \mathrm{d}x.$$

(i) Show that for positive integers a and b,

$$\mathrm{I}(a,b) = \frac{a}{a+b}\mathrm{I}(a-1,b-1).$$

(ii) Prove by induction on n that for non-negative integers n and m,

$$\int_0^{\frac{\pi}{2}} \cos^n x \cos(n+2m+1)x \, \mathrm{d}x = (-1)^m \frac{2^n n!(2m)!(n+m)!}{m!(2n+2m+1)!}.$$

2 The curve C has equation $\sinh x + \sinh y = 2k$, where k is a positive constant.

(i) Show that the curve C has no stationary points and that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ at the point (x, y) on the curve if and only if

$$1 + \sinh x \sinh y = 0$$

Find the coordinates of the inflection points of the curve C, leaving your answer in terms of inverse hyperbolic functions.

(ii) Show if (x, y) lies on C and on the line $x + y = a$, then

$$\mathrm{e}^{2x}\left(1 - \mathrm{e}^{-a}\right) - 4k\mathrm{e}^{x} + (\mathrm{e}^{a} - 1) = 0$$

and prove that $1 < \cosh a \leqslant 2k^2 + 1$.

(iii) Sketch the curve C.

3 Given distinct points A and B in the complex plane, the point G_{AB} is defined as the centroid of the triangle ABK, where the point K is the image of B under rotation about A through a clockwise angle of $\frac{1}{3}\pi$.

Note: if the points P, Q and R are represented in the complex plane by p, q and r, the centroid of triangle PQR is defined to be the point represented by $\frac{1}{3}(p+q+r)$.

(i) If A, B and G_{AB} are represented in the complex plane by a, b and g_{ab}, show that

$$g_{ab} = \frac{1}{\sqrt{3}}\left(\omega a + \omega^* b\right),$$

where $\omega = \mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$.

(ii) The quadrilateral Q_1 has vertices A, B, C and D, in that order, and the quadrilateral Q_2 has vertices G_{AB}, G_{BC}, G_{CD} and G_{DA}, in that order. Using the result in part (i), show that Q_1 is a parallelogram if and only if Q_2 is a parallelogram.

(iii) The triangle T_1 has vertices A, B and C and the triangle T_2 has vertices G_{AB}, G_{BC} and G_{CA}. Using the result in part (i), show that T_2 is always equilateral.

4 The plane Π has equation $\mathbf{r}\cdot\mathbf{n} = 0$ where $\mathbf{n}$ is a unit vector. Let P be a point with position vector $\mathbf{x}$ which does not lie on the plane Π. Show that the point Q with position vector $\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}$ lies on Π and that PQ is perpendicular to Π.

(i) Let transformation T be a reflection in the plane $ax+by+cz = 0$, where $a^2+b^2+c^2 = 1$. Show that the image of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ under T is $\begin{pmatrix} b^2+c^2-a^2 \\ -2ab \\ -2ac \end{pmatrix}$, and find the images of $\mathbf{j}$ and $\mathbf{k}$ under T.

Write down the matrix M which represents transformation T

(ii) The matrix

$$\begin{pmatrix} 0.64 & 0.48 & 0.6 \\ 0.48 & 0.36 & -0.8 \\ 0.6 & -0.8 & 0 \end{pmatrix}$$

represents a reflection in a plane. Find the cartesian equation of the plane.

(iii) The matrix $\mathbf{N}$ represents a rotation through angle π about the line through the origin parallel to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $a^2+b^2+c^2 = 1$. Find the matrix $\mathbf{N}$.

(iv) Identify the single transformation which is represented by the matrix $\mathbf{NM}$.

5 Show that for positive integer $n, x^n - y^n = (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1}$.

(i) Let F be defined by

$$\mathrm{F}(x) = \frac{1}{x^n(x-k)} \qquad \text{for } x \neq 0$$

where n is a positive integer and $k \neq 0$.

(a) Given that

$$\mathrm{F}(x) = \frac{A}{x-k} + \frac{\mathrm{f}(x)}{x^n},$$

where A is a constant and $\mathrm{f}(x)$ is a polynomial, show that

$$\mathrm{f}(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right).$$

Deduce that

$$\mathrm{F}(x) = \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}.$$

(b) By differentiating $x^n\mathrm{F}(x)$, prove that

$$\frac{1}{x^n(x-k)^2} = \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}.$$

(ii) Hence evaluate the limit of

$$\int_2^N \frac{1}{x^3(x-1)^2}\,\mathrm{d}x$$

as $N \to \infty$, justifying your answer.

6 **(i)** Sketch the curve $y = \cos x + \sqrt{\cos 2x}$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$.

(ii) The equation of curve C_1 in polar co-ordinates is

$$r = \cos\theta + \sqrt{\cos 2\theta} \qquad -\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi.$$

Sketch the curve C_1.

(iii) The equation of curve C_2 in polar co-ordinates is

$$r^2 - 2r\cos\theta + \sin^2\theta = 0 \qquad -\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi.$$

Find the value of r when $\theta = \pm\frac{1}{4}\pi$.

Show that, when r is small, $r \approx \frac{1}{2}\theta^2$.

Sketch the curve C_2, indicating clearly the behaviour of the curve near $r = 0$ and near $\theta = \pm\frac{1}{4}\pi$.

Show that the area enclosed by curve C_2 and above the line $\theta = 0$ is $\frac{\pi}{2\sqrt{2}}$.

7 **(i)** Given that the variables x, y and u are connected by the differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{f}(x)u = \mathrm{h}(x) \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = u$$

show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\mathrm{g}(x) + \mathrm{f}(x))\frac{\mathrm{d}y}{\mathrm{d}x} + \big(\mathrm{g}'(x) + \mathrm{f}(x)\mathrm{g}(x)\big)\, y = \mathrm{h}(x). \qquad (1)$$

(ii) Given that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(1 + \frac{4}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2}{x} + \frac{2}{x^2}\right) y = 4x + 12 \qquad (2)$$

can be written in the same form as (1), find a first order differential equation which is satisfied by $\mathrm{g}(x)$.

If $g(x) = kx^n$, find a possible value of n and the corresponding value of k.

Hence find a solution of (2) with $y = 5$ and $\frac{dy}{dx} = -3$ at $x = 1$.

8 A sequence u_k, for integer $k \geqslant 1$, is defined as follows.

$$\begin{aligned} u_1 &= 1 \\ u_{2k} &= u_k \text{ for } k \geqslant 1 \\ u_{2k+1} &= u_k + u_{k+1} \text{ for } k \geqslant 1 \end{aligned}$$

(i) Show that, for every pair of consecutive terms of this sequence, except the first pair, the term with odd subscript is larger than the term with even subscript.

(ii) Suppose that two consecutive terms in this sequence have a common factor greater than one. Show that there are then two consecutive terms earlier in the sequence which have the same common factor. Deduce that any two consecutive terms in this sequence are co-prime (do not have a common factor greater than one).

(iii) Prove that it is not possible for two positive integers to appear consecutively in the same order in two different places in the sequence.

(iv) Suppose that a and b are two co-prime positive integers which do not occur consecutively in the sequence with b following a. If $a > b$, show that $a - b$ and b are two co-prime positive integers which do not occur consecutively in the sequence with b following $a - b$, and whose sum is smaller than $a + b$. Find a similar result for $a < b$

(v) For each integer $n \geqslant 1$, define the function f from the positive integers to the positive rational numbers by $\mathrm{f}(n) = \dfrac{u_n}{u_{n+1}}$. Show that the range of f is all the positive rational numbers, and that f has an inverse.

Section B: Mechanics

9 Two inclined planes Π_1 and Π_2 meet in a horizontal line at the lowest points of both planes and lie on either side of this line. Π_1 and Π_2 make angles of α and β, respectively, to the horizontal, where $0 < \beta < \alpha < \frac{1}{2}\pi$.

A uniform rigid rod PQ of mass m rests with P lying on Π_1 and Q lying on Π_2 so that the rod lies in a vertical plane perpendicular to Π_1 and Π_2 with P higher than Q.

(i) It is given that both planes are smooth and that the rod makes an angle θ with the horizontal. Show that $2\tan\theta = \cot\beta - \cot\alpha$.

(ii) It is given instead that Π_1 is smooth, that Π_2 is rough with coefficient of friction μ and that the rod makes an angle ϕ with the horizontal. Given that the rod is in limiting equilibrium, with P about to slip down the plane Π_1, show that

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta}$$

where θ is the angle satisfying $2\tan\theta = \cot\beta - \cot\alpha$.

10 A light elastic spring AB, of natural length a and modulus of elasticity kmg, hangs vertically with one end A attached to a fixed point. A particle of mass m is attached to the other end B. The particle is held at rest so that $AB > a$ and is released.

Find the equation of motion of the particle and deduce that the particle oscillates vertically.

If the period of oscillation is $\dfrac{2\pi}{\Omega}$, show that $kg = a\Omega^2$.

Suppose instead that the particle, still attached to B, lies on a horizontal platform which performs simple harmonic motion vertically with amplitude b and period $\dfrac{2\pi}{\omega}$.

At the lowest point of its oscillation, the platform is a distance h below A.

Let x be the distance of the particle above the lowest point of the oscillation of the platform. When the particle is in contact with the platform, show that the upward force on the particle from the platform is

$$mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

Given that $\omega < \Omega$, show that, if the particle remains in contact with the platform throughout its motion,

$$h \leqslant a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$$

Find the corresponding inequality if $\omega > \Omega$.

Hence show that, if the particle remains in contact with the platform throughout its motion, it is necessary that

$$h \leqslant a\left(1 + \frac{1}{k}\right) + b$$

whatever the value of ω.

Section C: Probability and Statistics

11 The continuous random variable X is uniformly distributed on $[a, b]$ where $0 < a < b$.

(i) Let f be a function defined for all $x \in [a, b]$

- with $\mathrm{f}(a) = b$ and $\mathrm{f}(b) = a$,
- which is strictly decreasing on $[a, b]$,
- for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all $x \in [a, b]$.

The random variable Y is defined by $Y = \mathrm{f}(X)$. Show that

$$\mathrm{P}(Y \leqslant y) = \frac{b - \mathrm{f}(y)}{b - a} \quad \text{for } y \in [a, b].$$

Find the probability density function for Y and hence show that

$$\mathrm{E}\left(Y^2\right) = -ab + \int_a^b \frac{2x\mathrm{f}(x)}{b - a}\,\mathrm{d}x.$$

(ii) The random variable Z is defined by $\dfrac{1}{Z} + \dfrac{1}{X} = \dfrac{1}{c}$ where $\dfrac{1}{c} = \dfrac{1}{a} + \dfrac{1}{b}$. By finding the variance of Z, show that

$$\ln\left(\frac{b - c}{a - c}\right) < \frac{b - a}{c}$$

12 A and B both toss the same biased coin. The probability that the coin shows heads is p where $0 < p < 1$, and the probability that it shows tails is $q = 1 - p$.

Let X be the number of times A tosses the coin until it shows heads. Let Y be the number of times B tosses the coin until it shows heads.

(i) The random variable S is defined by $S = X + Y$ and the random variable T is the maximum of X and Y. Find an expression for $\mathrm{P}(S = s)$ and show that

$$\mathrm{P}(T = t) = pq^{t-1}\left(2 - q^{t-1} - q^t\right).$$

(ii) The random variable U is defined by $U = |X - Y|$, and the random variable W is the minimum of X and Y. Find expressions for $\mathrm{P}(U = u)$ and $\mathrm{P}(W = w)$.

(iii) Show that $\mathrm{P}(S = 2 \text{ and } T = 3) \neq \mathrm{P}(S = 2) \times \mathrm{P}(T = 3)$.

(iv) Show that U and W are independent, and show that no other pair of the four variables S, T, U and W are independent.