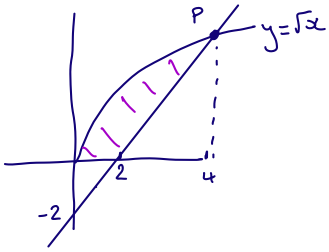
TMUA Practice - Integration

1) The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis is:

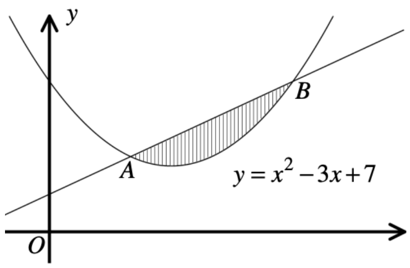
A 2 B $\frac{5}{2}$ C 3 D $\frac{10}{3}$ E $\frac{16}{3}$



At P $\sqrt{x} = x - 2$
$x = x^2 - 4x + 4$
$x^2 - 5x + 4 = 0$
$(x-1)(x-4) = 0$
$P(4, 2)$

Area $= \int_0^4 x^{1/2}\,dx - \frac{1}{2} \times 2 \times 2$
$= \left[\frac{2}{3}x^{3/2}\right]_0^4 - 2$
$= \frac{16}{3} - 2 = \frac{10}{3}$

2) The graph shows a quadratic curve with equation $y = x^2 - 3x + 7$ and a straight line $y = x + 4$. What is the value of the shaded area ?



At A, B $x^2 - 3x + 7 = x + 4$
$x^2 - 4x + 3 = 0$
$(x-1)(x-3) = 0$
$A(1, 5)$ $B(3, 7)$

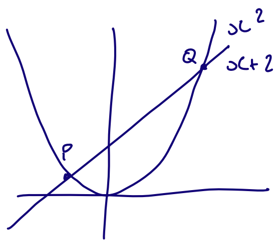
Area $= \int_1^3 x + 4 - x^2 + 3x - 7\,dx$
$= \int_1^3 -x^2 + 4x - 3\,dx$

A $\frac{1}{3}$ B $\frac{4}{3}$ C $\frac{7}{3}$ D $\frac{32}{3}$ E 12

Area $= \left[-\frac{1}{3}x^3 + 2x^2 - 3x\right]_1^3$
$= (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3\right) = \frac{4}{3}$

3) The area of the region bounded by the curves $y = x^2$, $y = x + 2$ is:

A $\frac{9}{2}$ B $\frac{7}{3}$ C $\frac{7}{2}$ D $\frac{9}{4}$ E $\frac{11}{2}$



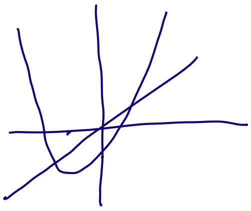
At P, Q $x^2 = x + 2$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2$ $x = -1$

Area $= \int_{-1}^2 x + 2 - x^2\,dx$
$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^2$
$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$
$= 6 - 3 + \frac{3}{2} = \frac{9}{2}$

4) Find the area of the finite region between the curves with equations

$$y = x^2 + x - 1 \qquad \text{and} \qquad y = x$$

A $\frac{2}{3}$ B 1 C $\frac{4}{3}$ D $\frac{5}{3}$ E 2



$x^2 + x - 1 = x$

$x^2 = 1$

$x = \pm 1$

$\text{Area} = \int_{-1}^{1} 1 - x^2 \, dx = \left[x - \frac{1}{3}x^3 \right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$

$= 2 - \frac{2}{3} = \frac{4}{3}$

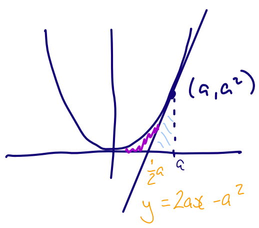
5) A line is tangent to the parabola $y = x^2$ at the point (a, a^2) where $a > 0$.

The area of the finite region bounded by the parabola, the tangent line and the x-axis equals:

$\frac{dy}{dx} = 2x \quad \text{grad} = 2a \quad y - a^2 = 2a(x - a) \quad y = 2ax - a^2$

$y = 0 \quad x = \frac{1}{2}a$

A $\frac{a^2}{3}$ B $\frac{2a^2}{3}$ C $\frac{a^3}{12}$ D $\frac{5a^3}{6}$ E $\frac{a^4}{10}$



$\int_0^a x^2 \, dx - \frac{1}{2} \times \frac{1}{2}a \times a^2$

$\left[\frac{1}{3}x^3 \right]_0^a - \frac{1}{4}a^3$

$\frac{1}{3}a^3 - \frac{1}{4}a^3 = \frac{1}{12}a^3$

6) The area of the finite region between the parabolas with equations

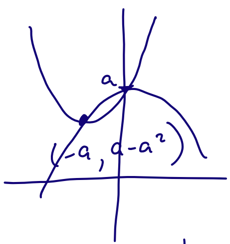
$$y = x^2 + 2ax + a \qquad \text{and} \qquad y = a - x^2 \qquad \text{equals } 9.$$

The possible values of a are:

$y = (x + a)^2 + a - a^2$

A $a = 1$ B $a = \pm 3$ C $a = -3$ D $a = \pm 1$ E $a = 3$

$a > 0$



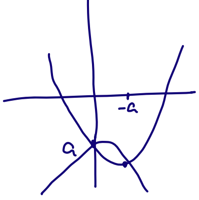
Intersection

$x^2 + 2ax + a = a - x^2$

$2x^2 + 2ax = 0$

$x = 0 \quad x = -a$

$a < 0$



$\int_0^{-a} a - x^2 - x^2 - 2ax - a \, dx$

$\left[-\frac{2}{3}x^3 - ax^2 \right]_0^{-a} = 9$

$\frac{2}{3}a^3 - a^3 = 9$

$-\frac{1}{3}a^3 = 9 \qquad a^3 = -27$

$\int_{-a}^{0} a - x^2 - x^2 - 2ax - a \, dx = \pm 9$

$\int_{-a}^{0} -2x^2 - 2ax \, dx = \pm 9$

$\left[-\frac{2}{3}x^3 - ax^2 \right]_{-a}^{0} = \pm 9$

$-\left(\frac{2}{3}a^3 - a^3 \right) = \pm 9$

$\frac{1}{3}a^3 = \pm 9$

$a = \pm 3$

$\boxed{\text{Area is 9 so integral can be } \pm 9}$

7) Find the area of the finite region between the curves with equations

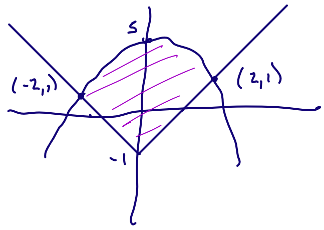
$$y = 5 - x^2 \quad \text{and} \quad y = |x| - 1$$

A $\frac{19}{3}$ B $\frac{22}{3}$ C $\frac{25}{3}$ D $\frac{28}{3}$ E $\frac{44}{3}$

$x > 0$: $x - 1 = 5 - x^2$
$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x = 2$

$x < 0$: $-x - 1 = 5 - x^2$
$x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$
$x = -2$

$$\int_{-2}^{0} 6 - x^2 + x \, dx + \int_0^2 6 - x^2 - x \, dx$$
$$= \left[6x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_{-2}^{0} + \left[6x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^2$$
$$= 0 - \left(-12 + \tfrac{8}{3} + 2\right) + \left(12 - \tfrac{8}{3} - 2\right)$$
$$= 2 \times \tfrac{22}{3} = \tfrac{44}{3}$$



8) Evaluate the following integral $\int_{-1}^{1} 2(x + |x|) - 7x|x| \; dx$

$-1 < x < 0$
$|x| = -x$

$0 < x < 1$
$|x| = x$

A 2 B $\frac{7}{3}$ C $\frac{5}{2}$ D 4 E $\frac{9}{2}$

$$= \int_{-1}^{0} 2(x - x) + 7x^2 \, dx + \int_0^1 2(x + x) - 7x^2 \, dx$$
$$= \left[\tfrac{7}{3}x^3\right]_{-1}^{0} + \left[2x^2 - \tfrac{7}{3}x^3\right]_0^1$$
$$= 0 - -\tfrac{7}{3} + 2 - \tfrac{7}{3} - 0$$
$$= 2$$

9) The positive number k satisfies $\int_0^k (\sqrt{x} + x^2) \; dx = 5$ for which value of k?

A $k = (\sqrt{21} - 1)^{\frac{1}{3}}$

B $k = \sqrt{3}$

C $k = 3^{\frac{2}{3}}$

D $k = (\sqrt{6} - 1)^{\frac{2}{3}}$

E $k = 5^{\frac{2}{3}}$

$$\int_0^k x^{1/2} + x^2 \, dx = 5$$
$$\left[\tfrac{2}{3}x^{3/2} + \tfrac{1}{3}x^3\right]_0^k = 5$$
$$2k^{3/2} + k^3 = 15$$
$$k^3 + 2k^{3/2} - 15 = 0$$
$$(k^{3/2} + 5)(k^{3/2} - 3) = 0$$
$k > 0$ $\quad k^{3/2} = 3$
$k = 3^{2/3}$

10) Let $f(x) = \int_{-x}^{x} \frac{1}{2}t^2 \; dt \qquad g(x) = \int_{0}^{1} x^2 t \; dt$

Which of the following statements is true?

$f(x) = \left[\frac{1}{6}t^3 \right]_{-x}^{x} = \frac{1}{3}x^3$

$g(x) = \left[\frac{1}{2}x^2 t^2 \right]_0^1 = \frac{1}{2}x^2$

A $gf(A) > fg(A)$ for all $A > 0$

B $gf(A) < fg(A)$ for all $A > 0$

C $gf(A) = fg(A)$ for all $A > 0$

D $gf(A) > fg(A)$ for $A > 1$ and $gf(A) < fg(A)$ for $A < 1$

E $gf(A) < fg(A)$ for $A > 1$ and $gf(A) > fg(A)$ for $A < 1$

$gf(x) = \frac{1}{2}\left(\frac{1}{3}x^3\right)^2 = \frac{1}{18}x^6$

$fg(x) = \frac{1}{3}\left(\frac{1}{2}x^2\right)^3 = \frac{1}{24}x^6$

$\frac{1}{18}x^6 > \frac{1}{24}x^6$ for all x

11) Find the minimum value of the function $f(t)$ where $f(t) \equiv \int_0^1 (x-t)^2 + t^2 \; dx \quad t \geq 0$

A 0 B $\frac{5}{24}$ C $\frac{1}{4}$ D $\frac{1}{3}$ E $\frac{7}{12}$

$f(t) = \int_0^1 x^2 - 2tx + 2t^2 \; dx$

$= \left[\frac{1}{3}x^3 - tx^2 + 2t^2 x \right]_0^1$

$= \frac{1}{3} - t + 2t^2$

$= 2t^2 - t + \frac{1}{3}$

At min $f(t)$ $\frac{df}{dt} = 4t - 1 = 0$

$t = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{8} - \frac{1}{4} + \frac{1}{3}$

$= \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$

12) The trapezium rule approximation using four trapezia for:

$$\int_0^6 |x(x-3)(x-6)| \; dx$$

A $\frac{3^5}{2^3}$ B $\frac{3^5}{4}$ C $\frac{3^3}{2^5}$ D 6 E $\left(\frac{3}{2}\right)^5$

x	0	1.5	3	4.5	6
y	0	$\frac{81}{8}$	0	$81/8$	0

$x = \frac{3}{2}$ $\quad \frac{3}{2}\left(-\frac{3}{2}\right)\left(-\frac{9}{2}\right) = \frac{81}{8}$

$y = \frac{81}{8}$

$x = \frac{9}{2}$ $\quad \frac{9}{2}\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) = -\frac{81}{8}$

$y = \frac{81}{8}$

$= \frac{1}{2} \times \frac{3}{2}\left(0 + 0 + 2\left(\frac{81}{8} + \frac{81}{8}\right)\right) = \frac{3}{4}\left(\frac{81}{2}\right) = \frac{243}{8} = \frac{3^5}{2^3}$

10) Let $f(x) = \int_0^1 (xt)^2 \; dt$ $\quad x^2t^2$ $\qquad g(x) = \int_0^x t^2 \; dt$

Which of the following statements is true?

$f(x) = \left[\frac{1}{3}x^2t^3\right]_0^1 = \frac{1}{3}x^2$

$g(x) = \left[\frac{1}{3}t^3\right]_0^x = \frac{1}{3}x^3$

A $\quad gf(A) > fg(A)$ for all $A > 0$

B $\quad gf(A) < fg(A)$ for all $A > 0$

C $\quad gf(A) = fg(A)$ for all $A > 0$

D $\quad gf(A) > fg(A)$ for $A > 1$ and $gf(A) < fg(A)$ for $A < 1$

E $\quad gf(A) < fg(A)$ for $A > 1$ and $gf(A) > fg(A)$ for $A < 1$

$gf(x) = \frac{1}{3}\left(\frac{1}{3}x^2\right)^3 = \frac{1}{81}x^6$

$fg(x) = \frac{1}{3}\left(\frac{1}{3}x^3\right)^2 = \frac{1}{27}x^6$

$\frac{1}{81}x^6 < \frac{1}{27}x^6$ for all x

11) Find the minimum value of the function $f(t)$ where $f(t) \equiv \int_0^1 (x-t)^2 + t^2 \; dx \quad t \geq 0$

A $\; 0$ $\qquad$ B $\; \frac{5}{24}$ $\qquad$ C $\; \frac{1}{4}$ $\qquad$ D $\; \frac{1}{3}$ $\qquad$ E $\; \frac{7}{12}$

$f(t) = \int_0^1 x^2 - 2tx + 2t^2 \; dx$

$= \left[\frac{1}{3}x^3 - tx^2 + 2t^2x\right]_0^1$

$= \frac{1}{3} - t + 2t^2$

$= 2t^2 - t + \frac{1}{3}$

At min $f(t)$ $\quad \frac{df}{dt} = 4t - 1 = 0$

$t = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{8} - \frac{1}{4} + \frac{1}{3}$

$= \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$

12) The trapezium rule approximation using four trapezia for:

$$\int_0^6 |x(x-3)(x-6)| \; dx$$

A $\; \frac{3^5}{2^3}$ $\qquad$ B $\; \frac{3^5}{4}$ $\qquad$ C $\; \frac{3^3}{2^5}$ $\qquad$ D $\; 6$ $\qquad$ E $\; \left(\frac{3}{2}\right)^5$

x	0	1.5	3	4.5	6
y	0	$\frac{81}{8}$	0	$81/8$	0

$x = \frac{3}{2}$ $\quad \frac{3}{2}\left(-\frac{3}{2}\right)\left(-\frac{9}{2}\right) = \frac{81}{8}$ $\quad y = \frac{81}{8}$

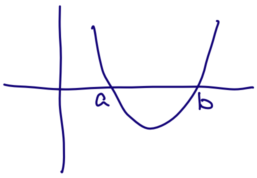
$x = \frac{9}{2}$ $\quad \frac{9}{2}\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) = -\frac{81}{8}$ $\quad y = \frac{81}{8}$

$= \frac{1}{2} \times \frac{3}{2}\left(0 + 0 + 2\left(\frac{81}{8} + \frac{81}{8}\right)\right) = \frac{3}{4}\left(\frac{81}{2}\right) = \frac{243}{8} = \frac{3^5}{2^3}$

13) Find the area of the finite region between the curve with equation

$y = (x - a)(x - b)$ where $0 < a < b$ and the x-axis

A $\frac{1}{3}(b-a)^3$ B $\frac{1}{6}(b-a)^3$ C $\frac{1}{2}(b+a)^2$ D $\frac{1}{3}(b+a)^2$ E $\frac{1}{2}(b+a)^3$



Area $= \left| \int_a^b x^2 - (a+b)x + ab \, dx \right|$

$= \left| \left[\frac{1}{3}x^3 - \frac{1}{2}(a+b)x^2 + abx \right]_a^b \right|$

$= \left| \left(\frac{1}{3}b^3 - \frac{1}{2}(a+b)b^2 + ab^2\right) - \left(\frac{1}{3}a^3 - \frac{1}{2}(a+b)a^2 + a^2b\right) \right|$

$= \left| \frac{1}{6}\left(2b^3 - 3ab^2 - 3b^3 + 6ab^2 - 2a^3 + 3a^3 + 3a^2b - 6a^2b\right) \right|$

$= \left| \frac{1}{6}\left(a^3 - 3a^2b + 3ab^2 - b^3\right) \right| = \frac{1}{6}\left|(a-b)^3\right| = \frac{1}{6}(b-a)^3$ as $b > a$

14) The function $f(x)$ is such that $f(x) + 4f(-x) \equiv 1 + x^2 \int_{-1}^{1} f(u)\ du$

Determine the value of $\int_{-1}^{1} f(x)\ dx = k$ $w = -x$ $k = -\int_{1}^{-1} f(-w)\,dw = \int_{-1}^{1} f(-w)\,dw$

A $\frac{6}{13}$ B $\frac{5}{6}$ C 2 D $\frac{5}{2}$ E $\frac{25}{9}$

$\int_{-1}^{1} f(x) + 4f(-x)\ dx = \int_{-1}^{1} 1 + kx^2\ dx$

$5k = \left[x + \frac{1}{3}kx^3\right]_{-1}^{1} = \left(1 + \frac{1}{3}k\right) - \left(-1 - \frac{1}{3}k\right)$

$5k = 2 + \frac{2}{3}k$

$\frac{13}{3}k = 2$ $k = \frac{6}{13}$

15) Place the following integrals in order of size from smallest to largest.

$K = \int_1^4 log_4\sqrt{x}\ \ dx$ $L = \int_1^4 log_4 x\ \ dx$ $M = \int_1^4 \sqrt{log_4 x}\ \ dx$

A K < L < M

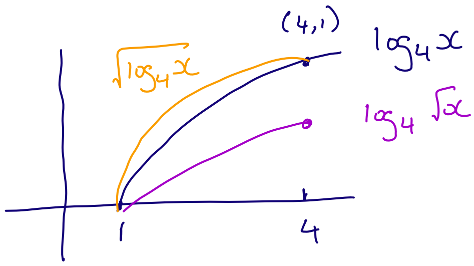
B K < M < L

C L < M < K

D L < K < M

E M < K < L

F M < L < K



$\int_1^4 \log_4\sqrt{x} < \int_1^4 \log_4 x < \int_1^4 \sqrt{\log_4 x}$