TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS 4)

DECEMBER 1999.

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some should be easy and others rather hard. You are not expected to answer all of them, or necessarily to complete questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring the question paper with you to the interview afterwards.

1. Each month, the chance of winning a prize with your Premium Bond is 1 in 14,000. Mr. Optimist buys 14,000 Premium Bonds and claims he is on to a certain winner. Explain to him why that is not quite the case, and find out roughly what his chances are of getting a winner in a given month.

Hint: if you are unfamiliar with how to approximate the expression you come up with, try taking the logarithm of something and using the Taylor series for log(1+h)

2. Let A denote the matrix $\begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$. Find the matrix A^{1000}. *Hint: it's a good idea to find a recurrence relation between the powers of A.*

3. Let $\mathbb{Z}[\sqrt{2}]$ denote the collection of all real numbers r of form $r = a + b\sqrt{2}$, $a, b \in \mathbb{Z}$. For such an r, define $N(r) = a^2 - 2b^2$; and show that $N(rs) = N(r)N(s)$. Hence show that there are infinitely many pairs of integers a,b with $a^2 - 2b^2 = 1$, and infinitely many pairs such that $a^2 - 2b^2 = -1$.

4. Sketch the curve defined by the equation

$$y^2 = \frac{1}{x-1} + \frac{2}{(x-3)^2}$$

Sketch the three dimensional surface defined by

$$x^2 - y^2 - z^2 = 1.$$

5. Little Johnny is a biologist tracking the movements of deer with a radio homing device. He has two listening posts, located at points $(X, 0)$ and $(0, Y)$ in the plane, and periodically they give simultaneous, accurate bearings (θ_n, ϕ_n) (let's say, anticlockwise from the X axis) from the listening posts to the deer. Johnny wants to calculate the deer's positions (x_n, y_n) in cartesian coordinates. He has hitherto done this by spreading the map of the area on his kitchen table and using rulers, protractors and Stuff.

Devise a couple of trignometric formulae so he can do the whole thing on his computer.

Note: this problem was given to one of your interviewers, Dr. Read, in real life. Thousands of points had been done by his predecessor using the kitchen table method, when

"Johnny" took over the project and the bright idea of consulting a mathematician occurred to him.

6. I can divide the number 11 up in a number of ways, eg

$$11 = 9 + 2$$

$$11 = 3.1 + 4.5 + 3.2 + 0.2$$

but what I want is the way of splitting it up so that the product of all the numbers I split it into is as large as possible. Eg for $11 = 9 + 2$ the product concerned is $9 \times 2 = 18$. Can you find a general way of solving this puzzle so that I could have started with numbers other than 11?

7. A plane cuts through a sphere of radius r at distance a from its centre. Find the volumes of the two pieces of the sphere on either side.

8. Candidates are being interviewed for admission to University to read mathematics. Most candidates are considered Normal and for these, the interviewers simply toss an unbiased coin to decide whether to give the candidate a place. With probability 0.1, however, a candidate is considered to *have a very funny face*, and is therefore admitted without more ado. With probability 0.1 again, a candidate is considered to *have a very boring face*, and in that case he's always rejected. Naturally, nobody's face can be both very funny and very boring.

George goes up for interview and is eventually admitted. What is the probability that he had a very funny face? Would your answer be different if you knew there were no boring faces that year?

9. A rope is wrapped M whole turns round a cylindrical post, the two ends of the rope going in opposite directions. The coefficient of friction between rope and post is 0.25. It is desired that by pulling with a force of $1N$ on one end of the rope, I can prevent the rope from moving away from me even if a force of $10^6 N$ is applied to the other end. How large does M have to be? (Note that to 3 significant figures, $\log_e(10^6) = 13.8$).

Hint: Let the tension in the rope decline like $T(\theta)$ with the angle θ round the pole. Investigate $T(\theta + \delta\theta) - T(\theta)$.

10. A uniform cylindrical log rolls down a slope at angle 45 degrees to the horizontal, under the watchful eye of Mr. Laurel. At the same time, his friend Mr. Hardy decides it would be nice to slide down the same slope on a slippery newspaper. Assuming both start at the top with zero speed, and that friction plays a negligible role in the second case, which will get to the bottom faster, the man who slides or the log that rolls? What is the ratio between the two speeds as they get to the bottom?