THE UK UNIVERSITY INTEGRATION BEE 2024/2025

$$\int$$

ROUND ONE SOLUTIONS

Problems by:
Alfie Jones
Camille Mau
Finn Hogan
Sharvil Kesarwani
Vishal Gupta
Yuepeng Alex Wang

Solutions by:
Camille Mau (main author)
Finn Hogan
Sharvil Kesarwani
Vishal Gupta

1

$$\fint_{-1}^{1} e^{in} \, \mathrm{d}n, \qquad \text{where} \fint_{a}^{b} f(x) \, \mathrm{d}x = \frac{1}{b-a} \int_{a}^{b} f(x) \, \mathrm{d}x$$

Solution. This *fein* problem evaluates as follows:

$$\fint_{-1}^{1} e^{in} \, \mathrm{d}n = \frac{1}{2} \int_{-1}^{1} e^{in} \, \mathrm{d}n = \frac{e^{i} - e^{-i}}{2i} = \sin(1).$$

□

2

$$\int (1 + 2024x^{2024})e^{x^{2024}} \, dx$$

Solution. The trick here is to spot that the product rule for derivatives is hiding in the expression. The answer is $xe^{x^{2024}} + C$.

Alternatively, apply integration by parts to the first component of the integral:

$$\begin{aligned}
\int (1 + 2024x^{2024})e^{x^{2024}} \, dx &= \int 1 \cdot e^{x^{2024}} \, dx + \int 2024x^{2024}e^{x^{2024}} \, dx \\
&= xe^{x^{2024}} - \int x \cdot 2024x^{2023}e^{x^{2024}} \, dx + \int 2024x^{2024}e^{x^{2024}} \, dx \\
&= xe^{x^{2024}} + C.
\end{aligned}$$

□

3

$$\int \frac{\ln(x) + 1}{x^x + x^{-x}} \, dx$$

Solution. An unusual substitution problem. After some messing around hopefully we spot that $\frac{d}{dx} x^x = (\ln(x) + 1)\, x^x$. This motivates the following:

$$\int \frac{\ln(x) + 1}{x^x + x^{-x}} \, dx = \int \frac{x^x(\ln(x) + 1)}{(x^x)^2 + 1} \, dx = \tan^{-1}(x^x) + C.$$

□

4

$$\int \sin^2 + \cos^2 \, \mathrm{ds}$$

Solution. Knowledge of five (or six) letters of the English alphabet is an absolute must in this problem.

$$\int \sin^2 + \cos^2 \, \mathrm{ds} = \frac{\mathrm{s}^2\mathrm{in}^2}{2} + \frac{\mathrm{cos}^3}{3} + C.$$

$\square$

5

$$\int_{\frac{1}{4}}^{4} \frac{\tan^{-1}(x)}{x}\, \mathrm{d}x$$

Solution. The bounds imply the substitution $u = \dfrac{1}{x}$. Indeed, we get

$$I := \int_{\frac{1}{4}}^{4} \frac{\tan^{-1}(x)}{x}\, \mathrm{d}x = \int_{\frac{1}{4}}^{4} \frac{\tan^{-1}\left(\frac{1}{u}\right)}{u}\, \mathrm{d}u$$

Adding both gives

$$2I = \int_{\frac{1}{4}}^{4} \frac{\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)}{x}\, \mathrm{d}x = \frac{\pi}{2}\int_{\frac{1}{4}}^{4} \frac{1}{x}\, \mathrm{d}x = \pi \ln(4),$$

where we have used the identity $\tan^{-1}(x) + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ for $x > 0$. Hence $I = \pi \ln(2)$. $\square$

6

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{e^x + 1}\,\mathrm{d}x$$

Solution. Reflecting across the y-axis quickly yields the solution. Take $x = -u$. Then we have

$$I := \int_{-\infty}^{\infty} \frac{e^{-x^2}}{e^x + 1}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{e^u e^{-u^2}}{e^u + 1}\,\mathrm{d}u,$$

so

$$2I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi},$$

from which $I = \dfrac{\sqrt{\pi}}{2}$. □

7

$$\int_0^1 \sqrt[3]{x} \left\lfloor \frac{1}{\sqrt[3]{x}} \right\rfloor \, \mathrm{d}x$$

Solution. Gotta split the bounds like a watermelon on a summer beach in Japan! You can do it directly, but for avoidance of confusion we'll transform it by a substitution first.

$$\begin{aligned}
\int_0^1 \sqrt[3]{x} \left\lfloor \frac{1}{\sqrt[3]{x}} \right\rfloor \, \mathrm{d}x &= \int_1^\infty \frac{3}{y^5} \lfloor y \rfloor \, \mathrm{d}y \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{3n}{y^5} \, \mathrm{d}y \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \left(\frac{n}{n^4} - \frac{n}{(n+1)^4} \right) \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= \frac{3}{4} \zeta(4) \\
&= \frac{\pi^4}{120}.
\end{aligned}$$

Note that for full credit, specific knowledge of $\zeta(4)$ was not required and the penultimate line was accepted as a valid final answer. □

8

$$\int_0^\infty \frac{1}{(x+1)(\ln(x)^2+1)}\,\mathrm{d}x$$

Solution. Our friend the $x = \dfrac{1}{u}$ sub works beautifully here. We have

$$\begin{aligned}
I &= \int_0^\infty \frac{1}{(x+1)(\ln(x)^2+1)}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{u(u+1)(\ln(u)^2+1)}\,\mathrm{d}u \\
&= \int_0^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)\frac{1}{(\ln(u)^2+1)}\,\mathrm{d}u \\
&= \int_0^\infty \frac{1}{x(\ln(x)^2+1)}\,\mathrm{d}x - I,
\end{aligned}$$

thus with $y = \ln(x)$ we have

$$I = \frac{1}{2}\int_0^\infty \frac{1}{x(\ln(x)^2+1)}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{y^2+1}\,\mathrm{d}y = \frac{1}{2}\left[\tan^{-1}(y)\right]_{-\infty}^{\infty} = \frac{\pi}{2}.$$

The following brilliant solution was suggested by Sharvil Kesarwani. Let

$$I(t) = \int_0^\infty \frac{1}{(x+t)(\ln(x)^2+1)}\,\mathrm{d}x.$$

Then observe that

$$\begin{aligned}
tI(t) + \frac{1}{t}I\left(\frac{1}{t}\right) &= \pi \\
\implies I(1) &= \frac{\pi}{2}.
\end{aligned}$$

□

9

$$\int_0^{2\pi} \frac{1}{1+\sqrt{1-\sin^2(x)}}\,\mathrm{d}x$$

Solution. A slightly evil problem. In principle this problem is simple, but we must be careful due to the range of the square root function.

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+\sqrt{1-\sin^2(x)}}\,\mathrm{d}x &= \int_0^{2\pi} \frac{1}{1+|\cos(x)|}\,\mathrm{d}x \\
&= 4\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos(x)}\,\mathrm{d}x \\
&= 4\int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\,\mathrm{d}x \\
&= 2\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right)\,\mathrm{d}x \\
&= 4\left[\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}} \\
&= 4.
\end{aligned}$$

Note that the absolute values here are necessary as the integral won't converge at all without them. □

10

$$\int_0^\infty e^{-t^2}\cos(2xt)\ \mathrm{d}t$$

Solution. A classic DUTIS problem. Let $I(x) = \displaystyle\int_0^\infty e^{-t^2}\cos(2xt)\ \mathrm{d}t$, then

$$\begin{aligned} I'(x) &= \int_0^\infty 2te^{-t^2}\sin(2xt)\ \mathrm{d}t \\ &= [e^{-t^2}\sin(2xt)]_0^\infty - 2x\int_0^\infty e^{-t^2}\cos(2xt)\ \mathrm{d}t \\ &= 2xI(x). \end{aligned}$$

Solving this differential equation gives $I(x) = Ce^{-x^2}$. We have

$$C = I(0) = \int_0^\infty e^{-t^2}\ \mathrm{d}t = \frac{\sqrt{\pi}}{2},$$

so $I(x) = \dfrac{\sqrt{\pi}}{2}e^{-x^2}$.

Those with experience in complex analysis and contour integrals may have opted for the following:

$$\begin{aligned} \int_0^\infty e^{-t^2}\cos(2xt)\ \mathrm{d}t &= \operatorname{Re}\int_0^\infty e^{-t^2+2ixt}\ \mathrm{d}t \\ &= \operatorname{Re}\int_0^\infty e^{-(t-ix)^2-x^2}\ \mathrm{d}t \\ &= e^{-x^2}\operatorname{Re}\int_0^\infty e^{-t^2}\ \mathrm{d}t \\ &= \frac{\sqrt{\pi}}{2}e^{-x^2}. \end{aligned}$$

Of course, some further justification is needed when the substitution $t - ix \to t$ is implemented but we will leave this as an exercise to the enthusiastic reader. □

11

$$\int_0^\infty \frac{\mathrm{Si}(x)\sin(x)}{x}\,\mathrm{d}x, \qquad \text{where } \mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,\mathrm{d}t \text{ is the sine integral}$$

Solution. This is just an unusual appearance of the inverse chain rule.

$$\int_0^\infty \frac{\mathrm{Si}(x)\sin(x)}{x}\,\mathrm{d}x = \frac{1}{2}[\mathrm{Si}(x)^2]_0^\infty = \frac{1}{2}\left(\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}x\right)^2 = \frac{\pi^2}{8}.$$

□

12

$$\int_{-\infty}^{\infty} \frac{1}{(e^x + e^{-x} + 2)^2} \, \mathrm{d}x$$

Solution. We use hyperbolic functions and write

$$I := \int_{-\infty}^{\infty} \frac{1}{(e^x + e^{-x} + 2)^2} \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{4} \frac{1}{(\cosh(x) + 1)^2} \, \mathrm{d}x.$$

Now we use the Weierstraß substitution (it exists for hyperbolic trig!). We have $t = \tanh\left(\frac{x}{2}\right)$, so $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1-t^2}$ and $\cosh(x) = \frac{1+t^2}{1-t^2}$. Hence, we have

$$\frac{1}{4} \int_{-1}^{1} \frac{1}{\left(\frac{1+t^2}{1-t^2} + 1\right)^2} \frac{2}{1-t^2} \, \mathrm{d}t = \frac{1}{8} \int_{-1}^{1} 1 - t^2 \, \mathrm{d}t = \frac{1}{8} \left[\left(t - \frac{1}{3} t^3 \right) \right]_{-1}^{1} = \frac{1}{6}.$$

With some knowledge of hyperbolic trig identities, we can also have the following succinct method:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{4} \frac{1}{(\cosh(x) + 1)^2} \, \mathrm{d}x &= \frac{1}{16} \int_{-\infty}^{\infty} \frac{1}{\left(\cosh^2\left(\frac{x}{2}\right)\right)^2} \, \mathrm{d}x \\
&= \frac{1}{16} \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{x}{2}\right) \, \mathrm{d}x \\
&= \frac{1}{16} \int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{x}{2}\right) \left(1 - \tanh^2\left(\frac{x}{2}\right)\right) \, \mathrm{d}x \\
&= \frac{1}{8} \int_{-1}^{1} 1 - t^2 \, \mathrm{d}t \\
&= \frac{1}{6},
\end{aligned}$$

where the last transformation follows from the substitution $\tanh\left(\frac{x}{2}\right) \to t$.

Here's a more boring (and probably easier) way suggested by Camille. Let $u = e^x$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(e^x + e^{-x} + 2)^2} \, \mathrm{d}x &= \int_{0}^{\infty} \frac{u^{-1}}{(u + u^{-1} + 2)^2} \, \mathrm{d}u \\
&= \int_{0}^{\infty} \frac{u}{(u+1)^4} \, \mathrm{d}u \\
&= \int_{0}^{\infty} \frac{1}{(u+1)^3} - \frac{1}{(u+1)^4} \, \mathrm{d}u \\
&= \frac{1}{2} - \frac{1}{3} \\
&= \frac{1}{6}.
\end{aligned}$$

Instead of partial fractions, one can also recognise the integral form of the beta function $B(2,2)$.

□

13

$$\int_0^\infty \frac{\sin(x) + \sin\left(\frac{1}{x}\right)}{x(1+x^2)}\,\mathrm{d}x$$

Solution. The substitution is quite obvious from the form of the integral. Split up the integral and apply $u = \dfrac{1}{x}$ to get

$$\begin{aligned}
\int_0^\infty \frac{\sin(x) + \sin\left(\frac{1}{x}\right)}{x(1+x^2)}\,\mathrm{d}x &= \int_0^\infty \frac{\sin(x)}{x(1+x^2)}\,\mathrm{d}x + \int_0^\infty \frac{\sin\left(\frac{1}{x}\right)}{x(1+x^2)}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\sin(x)}{x(1+x^2)}\,\mathrm{d}x + \int_0^\infty \frac{u^2\sin(u)}{u(1+u^2)}\,\mathrm{d}u \\
&= \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x \\
&= \frac{\pi}{2}.
\end{aligned}$$

□

14

$$\int_0^1 \frac{\tan^{-1}(x^n)}{x}\,\mathrm{d}x, \qquad \text{where } n \geq 1 \text{ is an integer}$$

Solution. Just apply the Taylor series directly. We get

$$\begin{aligned}\int_0^1 \frac{\tan^{-1}(x^n)}{x}\,\mathrm{d}x &= \sum_{k=0}^{\infty}\int_0^1 \frac{(-1)^k x^{(2k+1)n-1}}{(2k+1)}\,\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 n} \\ &= \frac{G}{n}.\end{aligned}$$

We could also make it nicer first. Let $y = x^n$. Then,

$$\begin{aligned}\int_0^1 \frac{\tan^{-1}(x^n)}{x}\,\mathrm{d}x &= \frac{1}{n}\int_0^1 \frac{\tan^{-1}(y)}{y}\,\mathrm{d}y \\ &= \frac{1}{n}\sum_{k=0}^{\infty}\int_0^1 \frac{(-1)^k y^{2k}}{(2k+1)}\,\mathrm{d}y \\ &= \frac{1}{n}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \\ &= \frac{G}{n}.\end{aligned}$$

□

15

$$\int_{-1}^{1} \frac{\sin(\sqrt{x})}{\sinh(\sqrt{x}) + \sin(\sqrt{x})}\, \mathrm{d}x$$

Solution. We spam the identity $\sinh(x) = -i\sin(ix)$. The substitution $x = -u$ gives

$$\begin{aligned}
I &:= \int_{-1}^{1} \frac{\sin(\sqrt{x})}{\sinh(\sqrt{x}) + \sin(\sqrt{x})}\, \mathrm{d}x \\
&= \int_{-1}^{1} \frac{\sin(i\sqrt{u})}{\sinh(i\sqrt{u}) + \sin(i\sqrt{u})}\, \mathrm{d}u \\
&= \int_{-1}^{1} \frac{i\sinh(\sqrt{u})}{-i\sin(-\sqrt{u}) + i\sinh(\sqrt{u})}\, \mathrm{d}u \\
&= \int_{-1}^{1} \frac{\sinh(\sqrt{u})}{\sinh(\sqrt{u}) + \sin(\sqrt{u})}\, \mathrm{d}u.
\end{aligned}$$

Hence $2I = \int_{-1}^{1} 1\, \mathrm{d}x = 2$, so $I = 1$. □

16

$$\int_0^\infty \frac{e^{-px^2} - e^{-qx^2}}{x^2}\, \mathrm{d}x, \qquad \text{where } p, q > 0$$

Solution. The easiest way to do this is to express it as a double integral. Treating x^2 as a *coefficient* instead of a *variable* means the denominator disappears by turning the integrand into an integral. We have

$$\begin{aligned}
\int_0^\infty \frac{e^{-px^2} - e^{-qx^2}}{x^2}\, \mathrm{d}x &= \int_0^\infty \int_q^p -e^{-tx^2}\, \mathrm{d}t\, \mathrm{d}x \\
&= \int_p^q \int_0^\infty e^{-tx^2}\, \mathrm{d}x\, \mathrm{d}t \\
&= \int_p^q \frac{1}{\sqrt{t}} \int_0^\infty e^{-y^2}\, \mathrm{d}y\, \mathrm{d}t \\
&= \frac{\sqrt{\pi}}{2} \int_p^q \frac{1}{\sqrt{t}}\, \mathrm{d}t \\
&= \frac{\sqrt{\pi}}{2} [2\sqrt{t}]_p^q \\
&= \sqrt{\pi}(\sqrt{q} - \sqrt{p}).
\end{aligned}$$

□

17

$$\int_0^{\frac{\pi}{4}+1} \tan\left(x - \tan\left(x - \ldots\right)\right)\, \mathrm{d}x$$

Solution. An inverse integral problem. Let $y = \tan\left(x - \tan\left(x - \ldots\right)\right) = \tan\left(x - y\right)$, then $x = \tan^{-1}(y) + y$. Then we have

$$\begin{aligned}
\int_0^{\frac{\pi}{4}+1} \tan\left(x - \tan\left(x - \ldots\right)\right)\, \mathrm{d}x &= \frac{\pi}{4} + 1 - \int_0^1 \tan^{-1}(y)\, \mathrm{d}y + \int_0^1 y\, \mathrm{d}y \\
&= \frac{\pi}{4} + \frac{1}{2} - \left[y \tan^{-1}(y)\right]_0^1 + \int_0^1 \frac{y}{1+y^2}\, \mathrm{d}y \\
&= \frac{1}{2} + \frac{1}{2}\left[\ln(1+y^2)\right]_0^1 \\
&= \frac{1}{2}\left(1 + \ln(2)\right).
\end{aligned}$$

Alternatively, we can carry out the substitution $x = \tan^{-1}(y) + y$ with $\mathrm{d}x = \left(\frac{1}{1+y^2} + 1\right)\mathrm{d}y$:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}+1} \tan\left(x - \tan\left(x - \ldots\right)\right)\, \mathrm{d}x &= \int_0^1 y\left(\frac{1}{1+y^2} + 1\right)\mathrm{d}y \\
&= \frac{1}{2}\left[y^2\right]_0^1 + \frac{1}{2}\left[\ln(1+y^2)\right]_0^1 \\
&= \frac{1}{2}(1 + \ln(2)).
\end{aligned}$$

□

18

$$\int_0^{\frac{\pi}{2}} \ln(\sin(x))\, \mathrm{d}x$$

Solution. As the famous saying goes...

You never know what premium integration is until you have tried your hand at the log-sine integral.

(Long Beach Seafood)

If you've studied any text on integral tricks ever, you've likely seen this one, it's pretty much *the* standard example. Start with a reflection. Substitute $u = \frac{\pi}{2} - x$, and we get

$$I := \int_0^{\frac{\pi}{2}} \ln(\sin(x))\, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(\frac{\pi}{2} - u\right)\right)\, \mathrm{d}u = \int_0^{\frac{\pi}{2}} \ln(\cos(x))\, \mathrm{d}u,$$

and thus

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln\left(\sin(x)\cos(x)\right)\, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2x)\right)\, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\right)\, \mathrm{d}x + \frac{1}{2}\int_0^{\pi} \ln\left(\sin(x)\right)\, \mathrm{d}x \\ &= -\frac{\pi}{2}\ln(2) + \frac{1}{2}\int_0^{\pi} \ln\left(\sin(x)\right)\, \mathrm{d}x. \end{aligned}$$

For this new integral, split the bounds, and we get

$$\begin{aligned} \int_0^{\pi} \ln\left(\sin(x)\right)\, \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \ln\left(\sin(x)\right)\, \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \ln\left(\sin(x)\right)\, \mathrm{d}x \\ &= I + \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(u + \frac{\pi}{2}\right)\right)\, \mathrm{d}u \\ &= I + \int_0^{\frac{\pi}{2}} \ln\left(\cos(u)\right)\, \mathrm{d}u \\ &= 2I. \end{aligned}$$

Thus $2I = -\frac{\pi}{2}\ln(2) + I$, and $I = -\frac{\pi}{2}\ln(2)$. □

19

$$\int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{1+\tan(x)}{\sqrt{2}}\right)\,\mathrm{d}x$$

Solution. The substitution here is $u = \dfrac{\pi}{4} - x$. This is hard to spot but we see the harmonic form present itself after a bit of manipulation:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{1+\tan(x)}{\sqrt{2}}\right)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{\sin(x)+\cos(x)}{\sqrt{2}\cos(x)}\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{\cos\left(x-\frac{\pi}{4}\right)}{\cos(x)}\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos(x)}\right)\,\mathrm{d}x.
\end{aligned}$$

Now the substitution appears. We have

$$I := \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos(x)}\right)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{\cos(u)}{\cos\left(\frac{\pi}{4}-u\right)}\right)\,\mathrm{d}u,$$

and thus

$$2I = \int_0^{\frac{\pi}{4}} \frac{\pi}{2}\,\mathrm{d}x = \frac{\pi^2}{8},$$

so $I = \dfrac{\pi^2}{16}$. □

20

$$\int_0^{100} x^{\{x\}-1}(\ln(x^x) + \{x\})\, \mathrm{d}x$$

Solution. A very pretty bound-splitting problem. Everything simplifies very nicely.

$$\begin{aligned}
\int_0^{100} x^{\{x\}-1}(\ln(x^x) + \{x\})\, \mathrm{d}x &= \sum_{n=0}^{99} \int_0^1 (x+n)^{x-1}((x+n)\ln(x+n) + x)\, \mathrm{d}x \\
&= \sum_{n=0}^{99} \left[(x+n)^x\right]_0^1 \\
&= \sum_{n=0}^{99} n \\
&= 4950.
\end{aligned}$$

□

21

$$\int_1^{2024} \frac{e^{x^x}}{\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{2}{x}\cdot\frac{3}{x}\cdot\frac{5}{x}\cdot\frac{8}{x}\cdot\frac{13}{x}\cdot\frac{21}{x}\cdot\ldots}\,\mathrm{d}x$$

Solution. Traditionally, Problem 21 is the founder's favourite problem. Despite how intimidating the problem looks, the denominator is infinite. So we are just integrating 0. So the answer is 0.

$\square$

22

$$\int_a^b \frac{\tan^{-1}\left(\frac{x}{a}\right)+\tan^{-1}\left(\frac{x}{b}\right)}{x}\,\mathrm{d}x, \qquad \text{where } a,b>0$$

Solution. A hard-to-spot use of the identity $\tan^{-1}(x)+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$. We use the border flip $y=\frac{ab}{x}$. Thus, we get

$$I := \int_a^b \frac{\tan^{-1}\left(\frac{x}{a}\right)+\tan^{-1}\left(\frac{x}{b}\right)}{x}\,\mathrm{d}x = \int_a^b \frac{\tan^{-1}\left(\frac{b}{y}\right)+\tan^{-1}\left(\frac{a}{y}\right)}{y}\,\mathrm{d}y.$$

Then,

$$\begin{aligned}
I+I &= \int_a^b \frac{\left(\tan^{-1}\left(\frac{x}{a}\right)+\tan^{-1}\left(\frac{a}{x}\right)\right)+\left(\tan^{-1}\left(\frac{x}{b}\right)+\tan^{-1}\left(\frac{b}{x}\right)\right)}{x}\,\mathrm{d}x \\
&= \int_a^b \frac{\frac{\pi}{2}+\frac{\pi}{2}}{x}\,\mathrm{d}x \\
&= \pi\ln\left(\frac{b}{a}\right),
\end{aligned}$$

so $I=\frac{\pi}{2}\ln\left(\frac{b}{a}\right)$. □

23

$$\int_0^\infty \frac{\sin(x)\sin(2x)}{x}\,\mathrm{d}x$$

Solution. Applying the product to sum formula, we have

$$\int_0^\infty \frac{\sin(x)\sin(2x)}{x}\,\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\cos(x)-\cos(3x)}{x}\,\mathrm{d}x.$$

From here, there are four approaches which can be taken that we know of:

1. DUTIS (or Laplace transform);
2. Carefully handling limits;
3. Extended version of the Frullani integral;
4. Pretending the usual Frullani works (but can be formally justified by functional analysis).

The motivation behind the DUTIS method is similar to the Dirichlet integral $\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x$ – the trick lies in clearing the denominator x. Parametrising so that the integrand is $\frac{\sin(tx)}{x}$ does not work as the resulting integral does not converge. Instead, an e^{-tx} clears the denominator while also decaying the oscillations as $x \to \infty$ and (fortunately) gives a doable integral. In the same manner, let

$$I(t) := \int_0^\infty \frac{\cos(x)-\cos(3x)}{x}e^{-tx}\,\mathrm{d}x.$$

Then,

$$I'(t) = \int_0^\infty (\cos(3x)-\cos(x))e^{-tx}\,\mathrm{d}x = \cdots = \frac{t}{t^2+9} - \frac{t}{t^2+1}.$$

(We skipped the evaluation because it's standard.) It follows that

$$I(t) = \frac{\ln(t^2+9)-\ln(t^2+1)}{2} + C = \frac{1}{2}\ln\left(1+\frac{8}{t^2+1}\right) + C.$$

Since $\lim_{t\to\infty} I(t) = 0$, it follows $C = 0$. Then the required value is

$$I(0) = \frac{1}{2}\ln(9) = \ln(3).$$

Therefore the original integral is $\frac{\ln(3)}{2}$.

Now that we're done getting the answer, it would be remiss of us to not mention that taking the limit at the end like that works often, but not *always*. The implicit unsaid step is

$$\lim_{t\to\infty}\int_0^\infty \frac{\cos(x)-\cos(3x)}{x}e^{-tx}\,\mathrm{d}x = \int_0^\infty \lim_{t\to\infty}\frac{\cos(x)-\cos(3x)}{x}e^{-tx}\,\mathrm{d}x.$$

For those who are curious, the rigorous justification here uses the dominated convergence theorem together with the convergence of the original integral (it can be proven without evaluating it).

Sadly, Frullani's formula can't be applied directly, but the integral can be worked out with the same ideas:

$$\begin{aligned}\int_0^\infty \frac{\cos(x)-\cos(3x)}{x}\,\mathrm{d}x &= \lim_{\substack{a\to 0^+\\ b\to\infty}}\left(\int_a^b \frac{\cos(x)}{x} - \int_{3a}^{3b}\frac{\cos(x)}{x}\right)\\ &= \lim_{b\to\infty}\int_{3b}^{b}\frac{\cos(x)}{x} - \lim_{a\to 0^+}\int_{3a}^{a}\frac{\cos(x)}{x}.\end{aligned}$$

The first can be estimated by parts as

$$\begin{aligned}\lim_{b\to\infty}\int_{3b}^{b}\frac{\cos(x)}{x} &= \lim_{b\to\infty}\left(\frac{\sin(b)}{b}-\frac{\sin(3b)}{3b}+\int_{3b}^{b}\frac{\sin(x)}{x^2}\,\mathrm{d}x\right)\\ &= \lim_{b\to\infty}\int_{3b}^{b}\frac{\sin(x)}{x^2}\,\mathrm{d}x.\end{aligned}$$

This final limit is 0 since $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2}\,\mathrm{d}x$ converges. The second can be estimated as

$$\lim_{a\to 0^+}\int_{3a}^{a}\frac{\cos(x)}{x} = \lim_{a\to 0^+}\int_{3a}^{a}\frac{1}{x}\,\mathrm{d}x + \sum_{n=1}^{\infty}\lim_{a\to 0^+}\int_{3a}^{a}\frac{(-1)^n x^{2n-1}}{(2n)!}\,\mathrm{d}x = \ln\left(\frac{1}{3}\right).$$

Therefore the answer is $\dfrac{\ln(3)}{2}$.

This solution uses the *graduate* Frullani integral (due to Agnew, 1951):

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,\mathrm{d}x = \left(\lim_{x\to\infty}\frac{1}{x}\int_1^x \frac{f(t)}{t}\,\mathrm{d}t - \lim_{x\to 0} x\int_x^1 \frac{f(t)}{t^2}\,\mathrm{d}t\right)\ln\left(\frac{b}{a}\right).$$

We have

$$\lim_{x\to\infty}\frac{1}{x}\int_1^x\frac{\cos(t)}{t}\,\mathrm{d}t = 0,$$

which follows from the estimate

$$\frac{1}{x}\left|\int_1^x \frac{\cos(t)}{t}\,\mathrm{d}t\right| \le \frac{1}{x}\int_1^x\frac{1}{t}\,\mathrm{d}t = \frac{\ln(x)}{x}, \qquad x\ge 1,$$

and

$$\begin{aligned}\lim_{x\to 0} x\int_x^1 \frac{\cos(t)}{t^2}\,\mathrm{d}t &= \lim_{x\to 0} x\sum_{n=0}^{\infty}\int_x^1 \frac{(-1)^n t^{2n-2}}{(2n)!}\,\mathrm{d}t\\ &= \lim_{x\to 0} x\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!(2n-1)} - \frac{(-1)^n x^{2n-1}}{(2n)!(2n-1)}\right)\\ &= -\lim_{x\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!(2n-1)}\\ &= -1.\end{aligned}$$

Thus the answer is $\dfrac{\ln(3)}{2}$.

The standard Frullani integral

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x = (f(\infty) - f(0)) \ln\left(\frac{b}{a}\right)$$

doesn't work here as $f(\infty)$ isn't defined. If we just pretend it's 0 though (i.e the average value of $\cos x$ on a period), it works. Justification for this comes from functional analysis, specifically the convergence $\sin(nx) \overset{*}{\rightharpoonup} 0$ in the weak-$*$ topology. Details are left to the the interested reader. $\square$

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$$\int_{-1}^{1} \frac{\ln((1+x)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^{2^{2024}}))}{x}\,\mathrm{d}x$$

Solution. This expression is quite daunting, but there is an elegant trick. We can multiply by $1-x$ to force the difference of squares formula to appear. Then we get a nice collapsing product:

$$\begin{aligned}
&\int_{-1}^{1} \frac{\ln((1+x)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^{2^{2024}}))}{x}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{\ln((1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^{2^{2024}})) - \ln(1-x)}{x}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{\ln(1-x^{2^{2025}}) - \ln(1-x)}{x}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{\ln(1-x^{2^{2025}})}{x}\,\mathrm{d}x - \int_{-1}^{1} \frac{\ln(1-x)}{x}\,\mathrm{d}x.
\end{aligned}$$

The first integral is 0 because the integrand is odd. The second integral is evaluated by applying a Taylor expansion, as

$$\begin{aligned}
-\int_{-1}^{1} \frac{\ln(1-x)}{x}\,\mathrm{d}x &= \sum_{n=1}^{\infty} \int_{-1}^{1} \frac{x^{n-1}}{n}\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{[x^n]_{-1}^{1}}{n^2} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \\
&= 2\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{4n^2}\right) \\
&= 2\left(\frac{\pi^2}{6} - \frac{\pi^2}{24}\right) \\
&= \frac{\pi^2}{4}.
\end{aligned}$$

□

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$$\int_0^1 \sin(\pi x)\ln(\Gamma(x))\,\mathrm{d}x$$

Solution. Knowledge of Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ immediately suggests using King's rule.

$$\begin{aligned} I &:= \int_0^1 \sin(\pi x)\ln(\Gamma(x))\,\mathrm{d}x \\ &= \int_0^1 \sin(\pi - \pi x)\ln(\Gamma(1-x))\,\mathrm{d}x \\ &= \int_0^1 \sin(\pi x)\ln(\Gamma(1-x))\,\mathrm{d}x. \end{aligned}$$

Then we have

$$\begin{aligned} 2I &= \int_0^1 \sin(\pi x)\ln(\Gamma(x)\Gamma(1-x))\,\mathrm{d}x \\ &= \int_0^1 \sin(\pi x)\ln\left(\frac{\pi}{\sin(\pi x)}\right)\,\mathrm{d}x \\ &= \frac{\ln(\pi)}{\pi}\int_0^\pi \sin(u)\,\mathrm{d}u - \frac{1}{\pi}\int_0^\pi \sin(u)\ln(\sin(u))\,\mathrm{d}u \\ &= \frac{2\ln(\pi)}{\pi} - \frac{1}{\pi}\left([-\cos(u)\ln(\sin(u))]_0^\pi + \int_0^\pi \frac{\cos^2(u)}{\sin(u)}\,\mathrm{d}u\right) \\ &= \frac{2\ln(\pi)}{\pi} - \frac{1}{\pi}\left[-\cos(u)\ln(\sin(u)) + \ln\left(\tan\left(\frac{u}{2}\right)\right)\right]_0^\pi + \frac{1}{\pi}[\cos(u)]_0^\pi \\ &= \frac{2\ln(\pi)+2}{\pi} - \frac{1}{\pi}\left[\ln\left(\tan\left(\frac{u}{2}\right)\right) - \cos(u)\ln(\sin(u))\right]_0^\pi. \end{aligned}$$

This is improper, so take limits:

$$\begin{aligned} &\lim_{u\to 0^+}\left(\ln\left(\tan\left(\frac{u}{2}\right)\right) - \cos(u)\ln(\sin(u))\right) \\ &= \lim_{u\to 0^+}\left(\ln\left(\sin\left(\frac{u}{2}\right)\right) - \ln\left(\cos\left(\frac{u}{2}\right)\right) - \ln(2)\cos(u)\right. \\ &\qquad \left. - \cos(u)\ln\left(\sin\left(\frac{u}{2}\right)\right) - \cos(u)\ln\left(\cos\left(\frac{u}{2}\right)\right)\right) \\ &= -\ln(2) + \lim_{u\to 0^+}\left((1-\cos(u))\ln\left(\sin\left(\frac{u}{2}\right)\right)\right) \\ &= -\ln(2), \end{aligned}$$

where we have applied L'Hôpital's rule to evaluate the last limit. Similarly, we have

$$\begin{aligned} &\lim_{u\to \pi^-}\left(\ln\left(\tan\left(\frac{u}{2}\right)\right) - \cos(u)\ln(\sin(u))\right) \\ &= \lim_{u\to \pi^-}\left(\ln\left(\sin\left(\frac{u}{2}\right)\right) - \ln\left(\cos\left(\frac{u}{2}\right)\right) - \ln(2)\cos(u)\right. \\ &\qquad \left. - \cos(u)\ln\left(\sin\left(\frac{u}{2}\right)\right) - \cos(u)\ln\left(\cos\left(\frac{u}{2}\right)\right)\right) \\ &= \ln(2) - \lim_{u\to \pi^-}\left((\cos(u)+1)\ln\left(\cos\left(\frac{u}{2}\right)\right)\right) \\ &= \ln(2). \end{aligned}$$

Combining everything, we obtain

$$I = \frac{\ln(\pi) - \ln(2) + 1}{\pi}.$$

A faster and incredibly elegant way to compute the improper integral is as follows. Instead of using $\tan\left(\frac{u}{2}\right)$ form of the antiderivative, we use the $\ln(|\csc(u) - \cot(u)|)$ form. Then we have

$$[-\cos(u)\ln(\sin(u)) + \ln\left(\csc(u) - \cot(u)\right)]_0^\pi = \left[\ln\left(\frac{1-\cos(u)}{\sin^{1+\cos(u)}(u)}\right)\right]_0^\pi.$$

Using series expansion to take limits, we have

$$\lim_{u\to 0^+} \ln\left(\frac{1-\cos(u)}{\sin^{1+\cos(u)}(u)}\right) = \lim_{u\to 0^+} \ln\left(\frac{\frac{u^2}{2} + O(u^4)}{u^2 + O(u^3)}\right) = -\ln(2),$$

and

$$\lim_{u\to \pi^-} \ln\left(\frac{1-\cos(u)}{\sin^{1+\cos(u)}(u)}\right) = \lim_{u\to 0^+} \ln\left(\frac{1+\cos(u)}{\sin^{1-\cos(u)}(u)}\right) = \lim_{u\to 0^+} \ln\left(\frac{2+O(u^2)}{1+O(u)}\right) = \ln(2).$$

$\square$

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$$\max_{k\in\mathbb{R}}\left(\int_0^{\sin^2(k)}\sin^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x+\int_0^{\cos^2(k)}\cos^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x\right)$$

Solution. The way we recommend is by exploiting the Fundamental Theorem of Calculus. Let

$$f(k)=\int_0^{\sin^2(k)}\sin^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x+\int_0^{\cos^2(k)}\cos^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x,$$

then observing $\sin^{-1}(|\sin(k)|)=\cos^{-1}(|\cos(k)|)$, we have

$$f'(k)=2\sin(x)\cos(x)\sin^{-1}(|\sin k|)-2\sin(x)\cos(x)\cos^{-1}(|\cos(k)|)=0.$$

The derivative vanishes so f must be constant, so the max is irrelevant. We may pick the most convenient value to evaluate f; we suggest $k=\dfrac{\pi}{4}$ to exploit a trigonometric identity. Thus, we have

$$f\left(\frac{\pi}{4}\right)=\int_0^{\frac{1}{2}}\sin^{-1}\left(\sqrt{x}\right)+\cos^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x=\int_0^{\frac{1}{2}}\frac{\pi}{2}\,\mathrm{d}x=\frac{\pi}{4}.$$

The way we *don't* recommend is to bash the antiderivatives directly. It works, but you might bash the desk in frustration. We have, over $[0,1]$,

$$\begin{aligned}\int\sin^{-1}\left(\sqrt{x}\right)\,\mathrm{d}x&=\int 2u\sin^{-1}(u)\,\mathrm{d}u\\&=x\sin^{-1}\left(\sqrt{x}\right)-\int\frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u\\&=x\sin^{-1}\left(\sqrt{x}\right)-\int\frac{\sin^2(t)\cos(t)}{|\cos(t)|}\,\mathrm{d}t\\&=x\sin^{-1}\left(\sqrt{x}\right)-\int\sin^2(t)\,\mathrm{d}t\\&=x\sin^{-1}\left(\sqrt{x}\right)-\frac{t}{2}+\frac{\sin(t)\cos(t)}{2}+C\\&=x\sin^{-1}\left(\sqrt{x}\right)-\frac{\sin^{-1}\left(\sqrt{x}\right)}{2}+\frac{\sqrt{x(1-x)}}{2}+C.\end{aligned}$$

Similarly,

$$\begin{aligned}\int\cos^{-1}(\sqrt{x})\,\mathrm{d}x&=\int 2u\cos^{-1}(u)\,\mathrm{d}u\\&=x\cos^{-1}\left(\sqrt{x}\right)+\int\frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u\\&=x\cos^{-1}\left(\sqrt{x}\right)+\int\frac{\sin^2(t)\cos(t)}{|\cos(t)|}\,\mathrm{d}t\\&=x\cos^{-1}\left(\sqrt{x}\right)+\int\sin^2(t)\,\mathrm{d}t\\&=x\cos^{-1}\left(\sqrt{x}\right)+\frac{t}{2}-\frac{\sin(t)\cos(t)}{2}+C\\&=x\cos^{-1}\left(\sqrt{x}\right)+\frac{\sin^{-1}\left(\sqrt{x}\right)}{2}-\frac{\sqrt{x(1-x)}}{2}+C.\end{aligned}$$

Then we have

$$\begin{aligned}
f(k) &= \sin^2(k)\sin^{-1}(|\sin(k)|) - \frac{\sin^{-1}(|\sin(k)|)}{2} + \frac{\sqrt{\sin^2(k)\left(1-\sin^2(k)\right)}}{2} \\
&\quad + \cos^2(k)\cos^{-1}(|\cos(k)|) + \frac{\sin^{-1}(|\cos(k)|)}{2} - \frac{\sqrt{\cos^2(k)\,(1-\cos^2(k))}}{2} \\
&= \left(\sin^2(k)+\cos^2(k)\right)\sin^{-1}(|\sin(k)|) - \frac{\sin^{-1}(|\sin(k)|)}{2} + \frac{\pi}{4} - \frac{\cos^{-1}(|\cos(k)|)}{2} \\
&= \frac{\pi}{4},
\end{aligned}$$

where we used the fact that $\sin^{-1}(|\sin(k)|) = \cos^{-1}(|\cos(k)|)$. If you got here and are wondering if you did something terrible because k has vanished, it is our pleasure to confirm that this integral is working as intended. $\square$

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$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}\sqrt[x]{e}}\,\mathrm{d}x$$

Solution. Let $u = \sqrt{x}$. Then we have

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}\sqrt[x]{e}}\,\mathrm{d}x = 2\int_0^\infty e^{-u^2-\frac{1}{u^2}}\,\mathrm{d}u = \int_{-\infty}^\infty e^{-\left(u-\frac{1}{u}\right)^2-2}\,\mathrm{d}u = \frac{1}{e^2}\int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{e^2},$$

where we have applied Glasser's master theorem (Cauchy-Schlömilch transformation) and the Gaussian integral.

Glasser's master theorem isn't required knowledge (though you really should look it up, it's quite cool) for this bee. That's okay, there is another solution via elementary techniques. Like in the previous solution, we perform some initial manipulations to put it into a more familiar form:

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}\sqrt[x]{e}}\mathrm{d}x = 2\int_0^\infty e^{-u^2-\frac{1}{u^2}}\,\mathrm{d}u = \frac{1}{e^2}\int_{-\infty}^\infty e^{-\left(u-\frac{1}{u}\right)^2}\,\mathrm{d}u.$$

Now consider the form

$$I = \int_0^\infty e^{-\left(u-\frac{1}{u}\right)^2-2}\,\mathrm{d}u,$$

and the required answer is $2I$. If we could substitute $v = u - \frac{1}{u}$, we would have a Gaussian. The problem here is that this generates an extra factor from $\mathrm{d}v = \left(1+\frac{1}{u^2}\right)\mathrm{d}u$. To handle this we simply apply the $u = \frac{1}{x}$ substitution. This gives us

$$\int_0^\infty e^{-\left(u-\frac{1}{u}\right)^2-2}\,\mathrm{d}u = \int_0^\infty \frac{1}{u^2}e^{-\left(u-\frac{1}{u}\right)^2-2}\,\mathrm{d}u.$$

Adding these two expressions together gives

$$2I = \int_0^\infty \left(1+\frac{1}{u^2}\right)e^{-\left(u-\frac{1}{u}\right)^2-2}\,\mathrm{d}u = \int_{-\infty}^\infty e^{-x^2-2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{e^2}.$$

We note that an alternative way to get the $\frac{1}{u^2}$ factor is via DUTIS. □

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$$\int_0^1 \frac{\sin^{-1}\left(x^2\right)+\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Solution. We perform integration by parts on one component. Split the integral and specify a square substitution on a single side.

$$\begin{aligned}
&\int_0^1 \frac{\sin^{-1}\left(x^2\right)+\sin^{-1}\left(\sqrt{x}\right)}{\sqrt{1-x^2}}\,\mathrm{d}x\\
&=\int_0^1 \frac{\sin^{-1}\left(x^2\right)}{\sqrt{1-x^2}}\,\mathrm{d}x+\int_0^1 \frac{2u\sin^{-1}(u)}{\sqrt{1-u^4}}\,\mathrm{d}u\\
&=\int_0^1 \frac{\sin^{-1}\left(x^2\right)}{\sqrt{1-x^2}}\,\mathrm{d}x+\left[\sin^{-1}(u)\sin^{-1}\left(u^2\right)\right]_0^1-\int_0^1 \frac{\sin^{-1}\left(u^2\right)}{\sqrt{1-u^2}}\,\mathrm{d}u\\
&=\left[\sin^{-1}(u)\sin^{-1}\left(u^2\right)\right]_0^1\\
&=\frac{\pi^2}{4}.
\end{aligned}$$

In fact, in the same way one can prove the more general form

$$\int_a^b f(g(x))f'(x)+f(g^{-1}(x))f'(x)\,\mathrm{d}x=f(b)^2-f(a)^2,$$

with $g:[a,b]\to[a,b]$ an increasing differentiable bijection. □

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$$\int_{\frac{1}{\ln(2)}}^{2024} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt[n]{2^{kx}}kx^4}{nkx^3+1}\,\mathrm{d}x$$

Solution. A very hard problem under timed conditions. We first transform the integrand into a more amenable form, as

$$\int_{\frac{1}{\ln(2)}}^{2024} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt[n]{2^{kx}}kx^4}{nkx^3+1}\,\mathrm{d}x = \int_{\frac{1}{\ln(2)}}^{2024} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{x2^{\frac{k}{n}x}}{n+\frac{1}{kx^3}}\,\mathrm{d}x.$$

Observe that the expression resembles a Riemann sum on $[0,x]$: if we can throw out the $\frac{1}{kx^3}$ term, we get exactly that. For $x \in \left[\frac{1}{\ln(2)}, 2024\right]$ (in fact it generalises to $x > 0$) we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{\frac{k}{n}x}x}{n+\frac{1}{kx^3}}\,\mathrm{d}x \le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{\frac{k}{n}x}x}{n}\,\mathrm{d}x = \int_0^x 2^u\,\mathrm{d}u$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{\frac{k}{n}x}x}{n+\frac{1}{kx^3}}\,\mathrm{d}x \ge \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{\frac{k}{n}x}x}{n+1}\,\mathrm{d}x \ge \lim_{n\to\infty} \frac{n}{n+1} \sum_{k=1}^{n} \frac{2^{\frac{k}{n}x}x}{n}\,\mathrm{d}x = \int_0^x 2^u\,\mathrm{d}u,$$

so the integral is

$$\int_{\frac{1}{\log(2)}}^{2024} \int_0^x 2^u\,\mathrm{d}u\,\mathrm{d}x = \frac{1}{\ln(2)} \int_{\frac{1}{\log(2)}}^{2024} 2^x - 1\,\mathrm{d}x = \frac{1}{\ln(2)} \left(\frac{2^{2024}-e+1}{\ln(2)} - 2024 \right).$$

□

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$$\int_0^{\frac{\pi}{2}} \cosh^{-1}(\sin(x) + \cos(x))\ \mathrm{d}x$$

Solution. This amazing problem requires a mad parametrisation for DUTIS. First write

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cosh^{-1}(\sin(x) + \cos(x))\ \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \cosh^{-1}\left(\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)\right)\ \mathrm{d}x \\ &= 2\int_0^{\frac{\pi}{4}} \cosh^{-1}\left(\sqrt{2}\cos(x)\right)\ \mathrm{d}x.\end{aligned}$$

Let

$$I(t) = \int_0^{\sec^{-1}(t)} \cosh^{-1}\left(t\cos(x)\right)\ \mathrm{d}x.$$

This DUTIS can be motivated by trying to create a DUTIS that integrates over the whole interval over which the integrand is defined over the reals. We have

$$\begin{aligned}I'(t) &= \int_0^{\sec^{-1}(t)} \frac{\cos(x)}{\sqrt{t^2\cos^2(x) - 1}}\ \mathrm{d}x \\ &= \int_0^{\sqrt{t^2-1}} \frac{1}{t\sqrt{(t^2-1) - u^2}}\ \mathrm{d}u \\ &= \frac{1}{t}\left[\sin^{-1}\left(\frac{u}{\sqrt{t^2-1}}\right)\right]_0^{\sqrt{t^2-1}} \\ &= \frac{\pi}{2t},\end{aligned}$$

where we have used the substitution $u = t\sin(x)$. Note that the secondary DUTIS term coming from the parameter in the upper bound evaluates to just 0. Hence $I(t) = \frac{\pi}{2}\ln|t| + C$. From $I(1) = 0$ we have $C = 0$. Hence $I(t) = \frac{\pi}{2}\ln|t|$, and in particular

$$I\left(\sqrt{2}\right) = \frac{\pi}{4}\ln(2).$$

Thus the original integral evaluates to $\frac{\pi}{2}\ln(2)$. □