Integration Competition Sample Hints

Vishal Gupta

Introduction

This document are just some hints for the problems, only look if you're stuck! I've ordered the definite integrals so that every now and then a technique is introduced and each time that happens, there will be an opportunity to use it. Because of that, it might be best to try do the questions in order. Each page will be for one question so you don't accidentally spoil yourself. There will be a few hints for most problems. The aim of the sample problems are to introduce/teach the main techniques which will be used in the competitions. Competitions likely will have more definite integrals than indefinite integrals as I feel like definite integrals can give a lot more variety in the techniques involved to evaluate them. DUTIS will be used as an abbreviation for differentiation under the integral sign:

$$\frac{d}{dt}\left(\int_a^b f(x,t)\mathrm{d}x\right) = \int_a^b \frac{\partial}{\partial t}\left(f(x,t)\right)\mathrm{d}x$$

After doing a problem, it might be a good idea to check the hints and solutions for that problem to learn any alternative approaches. For those who did the competition in 2020, the sample problems are the same as the ones then. Some hints have been updated to have some extra comments or hint at alternative solutions.

Question 1

Hint 1: Sorry if you've struggled for a while as you've never seen anything like this before! This is an integral which I think is best done with DUTIS.

To use DUTIS, parametrise something in your integral and differentiate - hopefully you get something which can be evaluated! Then just integrate what you get with respect to t and you'll need to find some way to evaluate the constant of integration which appears.

Hint 2: In this case the parameter is on the power of x, you want to have

$$I(t) = \int_0^1 \frac{x^t - 1}{\ln x} \mathrm{d}x$$

For the constant of integration you get, you want to consider a specific case which lets you avoid any serious calculation.

Hint 3: There is an alternative solution. We can write this integral as a double integral.

Hint 4: We can write the integral as

$$I = \int_0^1 \int_0^5 x^y \mathrm{d}y\mathrm{d}x$$

Now that it's a double integral, we can do something which we couldn't do while it was a single integral!

Question 2

Hint 1: Another DUTIS question, the typical example given of the power of this method. This time, you want to get rid of the x in the denominator. I would suggest bringing something in; parametrising the sine as $\sin(tx)$ leads you to a divergent integral. You want something which, upon differentiating, will remove the x and provide you with a convergent integral at the end of it. For convergence, you would want the oscillations of the sine to tend to 0 as $x \to \infty$ so attaching something which tends to 0 as $x \to \infty$ for the parametrisation will be helpful. Also keep in mind evaluating the constant at the end of all of this!

Hint 2: A parametrisation which can be used is bringing in e^{-tx}:

$$I(t) = \int_0^{\infty} e^{-tx} \frac{\sin x}{x} \mathrm{d}x$$

This now gives a convergent integral upon differentiating under the integral sign.

Hint 3: The constant of integration can be evaluated by letting $t \to \infty$.

Question 3

Hint 1: The property to use here is the reflection property of integrals, also known as the King's property. I prefer the name reflection as it's suggestive of what's going on, a reflection in a line which preserves the area.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$$

This can be proved by a substitution or by considering a reflection in the line $x = \dfrac{a+b}{2}$. Some of you may remember this from STEP, STEP 2 2010 Q4. For those who haven't done it, that problem can give some more practice of this technique. A helpful idea to keep in mind is that trigonometry generally goes well with the reflection property.

Hint 2: After the reflection identity substitution, add it to the original integral. Adding and subtracting integrals can be a useful technique.

Question 4

Hint 1: The substitution $u = \frac{1}{x}$ goes well with both the limits, 0 and ∞, and $\ln x$. I consider it similar to the reflection property but for the limits 0 and ∞ as it also reflects the bounds.

Question 5

Hint 1: Complete the square on the denominator and substitute. Some of you might recognise the integral now from STEP 1 or a past Putnam problem. It's known as Serret's integral.

Hint 2: The substitution $x = \tan u$ goes well with $x^2 + 1$ in the denominator.

Hint 3: The reflection property will finish the integral now.

Question 6

Hint 1: This integral I think is best done by writing $\ln(1-x)$ as a Maclaurin series and then swapping the order of summation and integration. Don't worry about whether this is allowed or not... If you're interested though, there are theorems about when you can swap the order of limits and conditions like uniform convergence. Don't worry about the final form of the answer being an infinite series. Again, some of you may recognise this technique from STEP, STEP 2 2018 Q3 in particular. For those who haven't done it, it can serve as good practice of this technique; it also yields to DUTIS too. That question also leads to a generalisation of Question 1 here.

Question 7

Hint 1: Integrate by parts. It will lead to a recurrence relation.

Question 8

Hint 1: This is best done by using infinite series. You'll need to modify it a little before you get a convergent series of integrals though.

Hint 2: Multiply the top and bottom by e^{-x} and then the denominator can be written as a geometric series.

Hint 3: After a substitution you can write it as the product of the gamma function and a series, the series being the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Question 9

Hint 1: Split it into two integrals. The floor function rounds down to the integer below x so in the range $k < x \leq k+1, \lfloor x \rfloor = k$. The integral can be written in the form

$$\int_1^\infty f(x)\mathrm{d}x = \sum_{k=1}^{\infty} \int_k^{k+1} f(x)\mathrm{d}x$$

to make use of this. This technique appeared on STEP too, 2018 S3 Q8.

Hint 2: The summation you get can be written in terms of the Riemann Zeta function.

Question 10

Hint 1: This can be done by splitting the integral in two and using integration by parts on the first and a substitution on the second but there is a better way. The first function is odd so its integral is 0. The second is quite standard for A level.

Hint 2: An alternative way to do the second integral is to recognise that the integral is essentially the area of a semicircle. The source of this problem is here. So now, as your reward completing this problem, you'll have access to the wifi at Nanjing University of Aeronautics and Astronautics!

Question 11

Hint 1: Multiply top and bottom by $\cos x$. Consider the sum and difference of a related integral.

Hint 2: The related integral is $\displaystyle\int \frac{\sin x}{\sin x + \cos x}\mathrm{d}x$

Question 12

Hint 1: When you differentiate a square root you get a root in the denominator as $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$. Trig functions can often be written in terms of other ones so perhaps do some algebra.

Question 13

Hint 1: Divide top and bottom by e^x

Question 14

Hint 1: First, substitute $u = \ln x$

Hint 2: Divide top and bottom by $\cos^2 x$.

Question 15

Hint 1: Sum of two cubes and partial fractions. This integral is quite a slog and quite messy, sorry!

Question 16

Hint 1: This integral is best done with the Weierstrass substitution, $t = \tan\left(\frac{x}{2}\right)$ which is useful for turning integrals involving sines and cosines into ones which are in terms of rational functions. This substitution is very useful for both definite and indefinite integrals.

Question 17

Hint 1: Write as e to the power of something.

Question 18

Hint 1: Multiply the top and bottom by $1 + \sin x$. Difference of two squares with trig identities is a good technique.

Question 19

Hint 1: Substitute $u = x^2$, now is there any function such that $1 + u$ and $1 - u$ are something squared?

Hint 2: Think about double angle identities.

Question 20

Hint 1: Substitute $u = e^x$. It will become a rational function integral.

Hint 2: After substituting $u = \tan v$, write $\tan^3 v = \tan v(\sec^2 v - 1)$.