Disclaimer: I am not affiliated with the Admissions Testing Service in any way and the topics covered may differ from those covered in the Test of Mathematics for University Admission (TMUA). For official TMUA resources go to the Admissions Testing Service's website: `https://www.admissionstesting.org`.

These questions are of my own making taking inspiration from TMUA papers and are intended as an additional free resource for test takers. But these questions may not necessarily reflect those found in the TMUA.

Solutions to the problems are not in this document. Answers can be checked with other TSR users or by tagging me.

Best wishes, Cryptokyo.

Problem 1. Evaluate the integral

$$\int_0^1 \frac{1-x}{1-\sqrt{x}}\mathrm{d}x$$

Problem 2. Rationalise the denominator of

$$\frac{1}{1+\sqrt{3}+\sqrt{5}}$$

Problem 3. Find the coefficient of x^6 in the expansion of

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)$$

Problem 4. A quadratic polynomial $f(x)$ has two real roots α and β. We are given that $\alpha\beta = 1$ and that $\alpha + \beta = \gamma$. What is the possible range of values of γ?

Problem 5. The polynomial $x^2 + bx + c$ has roots α and β. Find $\alpha^4 + \beta^4$ in terms of b and c.

Problem 6. Find $f'(2)$ for

$$f(x) = \frac{x^3 + 8x^2 + 11x - 20}{x^2 + 4x - 5}$$

Problem 7. Without using a calculator compute the exact value of $\sqrt[3]{19.683}$.

Inspired by the 1876 MIT Arithmetic Entrance Exam.

Problem 8. How many positive integers are there less than 1000 that are not divisible by 2 or 3?

Problem 9. For which values of a does the equation

$$3^{2x} - a \cdot 3^{x+1} + a^2 = 0$$

have two distinct real solutions?

I $a < 0$
II $a > 0$
III No such values of a
IV $a = 0$

Problem 10. The logarithm function $\log(x)$ is defined for $x > 0$. For which values of a does the equation

$$\log(x + a) = \log(x) + \log(a)$$

have a solution?

Problem 11. Find a counter-example to the statement *if p is prime then $2^p - 1$ is prime.*

Problem 12. We say a function is *multiplicatively bounded* **if and only if** $f(xy) \leq f(x)f(y)$ and $f(0) > 0$.

Is $f(x) \geq 1$ a **necessary** condition for a multiplicatively bounded function?

Problem 13. For this question we assume our given function $f(x)$ is differentiable and thus is (Riemann) integrable.

We say $f(x)$ is *odd* **if and only if** $f(-x) = -f(x)$.

Which of the following conditions are **necessary** for $f(x)$ to be an odd function?

I $f(x)$ is a constant function.
II $f(a - x) + f(a + x) = f(a)$, where a is any real constant.
III $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 0$, where a is some real constant.
IV $f'(x) > 0$.

Problem 14. How many solutions are there to the equation

$$\tan(2x - x\sin(x)) = 1$$

for $0 < x < \pi$?

Problem 15. A discrete random variable X takes non-negative values with distribution $\mathbb{P}(X = k) = ab^{-k}$, where $k \geq 0$ is an integer and $a > 0$ and $b > 1$ are real numbers. Find a in terms of b.

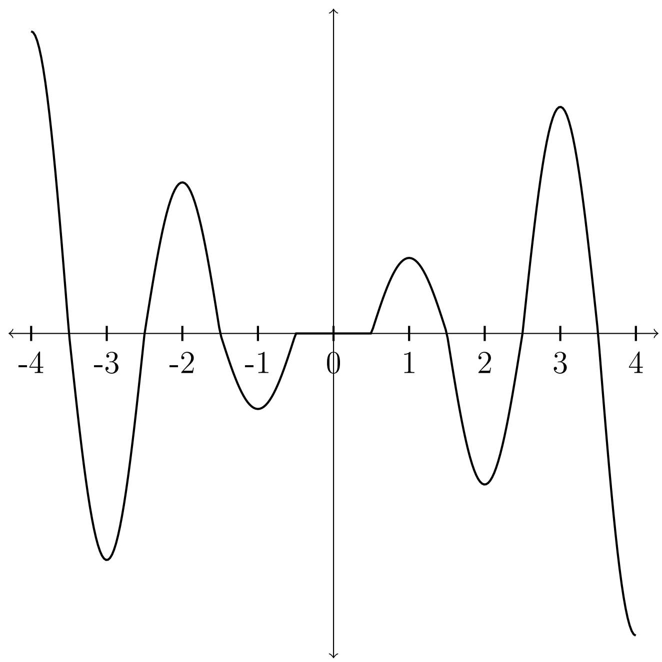
Problem 16. Find the value of

$$\log\left(\frac{1}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{9}{4}\right) + \log\left(\frac{16}{5}\right) + \cdots + \log\left(\frac{n^2}{n+1}\right)$$

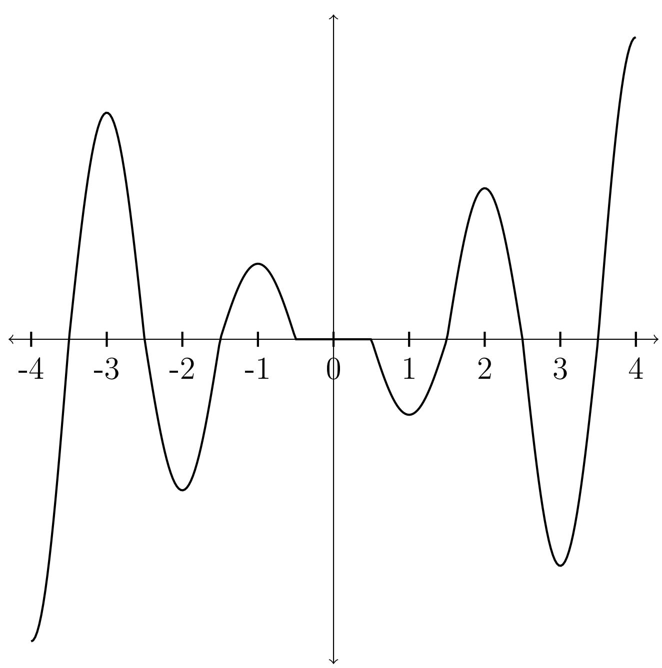
in the form $\log f(n)$ where $f(n)$ is some function of n.

Problem 17. We define the floor function $\lfloor x \rfloor$ to be equal to the largest integer greater than or equal to x. e.g. $\lfloor 2 \rfloor = 2$ and $\lfloor \frac{10}{3} \rfloor = 3$.

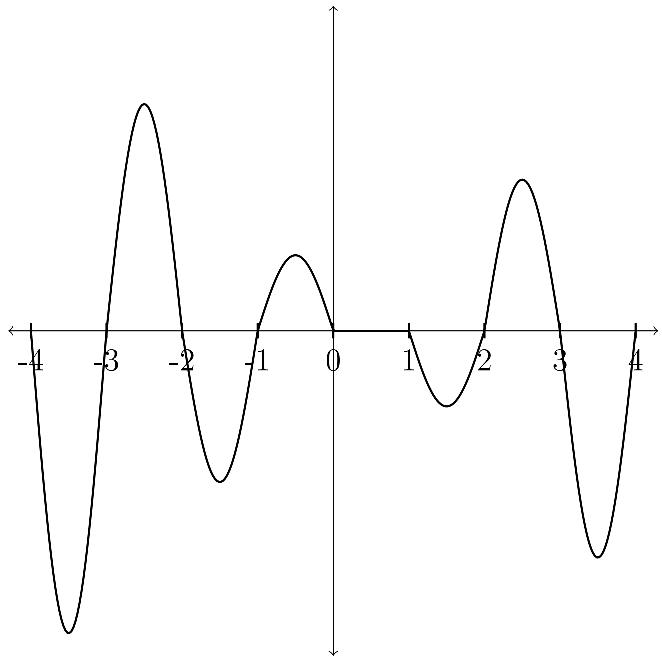
Which of the following is a graph of $f(x) = \lfloor x + \frac{1}{2} \rfloor \sin(\pi x + \frac{\pi}{2})$?



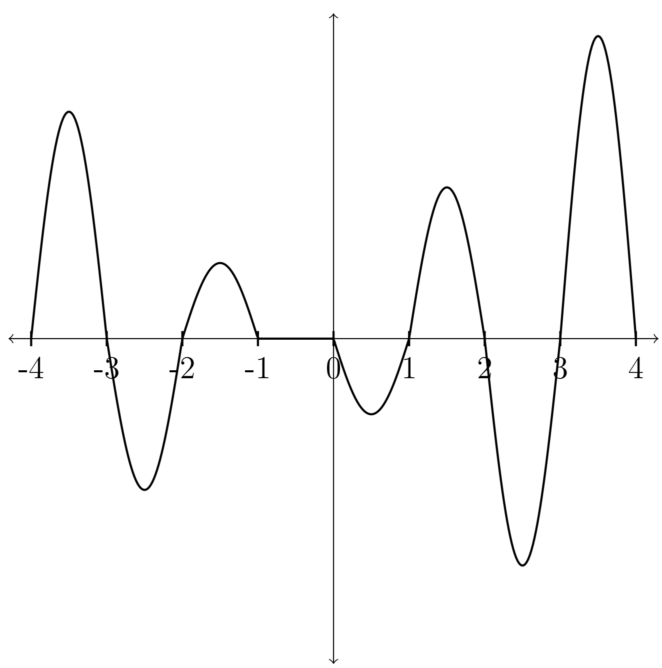
Graph 1



Graph 2



Graph 3



Graph 4

Problem 18. A sequence u_k is defined such that $u_1 = a$, $u_2 = b$ and $u_{n+1} = u_n - u_{n-1}$ for $n \geq 2$. Find the value of $\displaystyle\sum_{k=1}^{62} u_k$ in terms of a and b.

Problem 19. A sequence of functions $f_k(x)$ is defined such that $f_1(x) = \alpha + \beta x$ and

$$f_{n+1}(x) = \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}^2 x}\left[\int_0^x x f_n(x)\mathrm{d}x\right]$$

for $n \geq 1$. What is the value of $\sum_{k=1}^{\infty} f_k(1)$?

SOLUTIONS

Problem 1. Note that $1 - x = (1+\sqrt{x})(1-\sqrt{x})$ to find that the integral is $5/3$.

Problem 2. By removing the radical $\sqrt{3}+\sqrt{5}$ from the bottom you obtain

$$\frac{7 + 3\sqrt{3} - \sqrt{5} - 2\sqrt{15}}{11} \tag{1}$$

Problem 3. The coefficient is 4. Consider how many ways you can make the number 6 using the numbers 0,1,2,3,4,5,6 by adding using each at most once. This is related to the theory of partitions of numbers.

Problem 4. We need that $|\gamma| > 2$. Hint: polynomial discriminant

Problem 5. We now that $\alpha+\beta = -b$ and $\alpha\beta = c$ and $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = b^2 - 2c$. Now $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = (b^2-2c)^2 - 2c^2 = b^4 - 4b^2c + 2c^2$.

Problem 6. $f(x)$ is $x + 4$ in disguise so the derivative is $f'(2) = 1$.

Problem 7. The answer is 2.7 as the result must be between 2 and 3 and can have at most 1 decimal place as 19.683 has 3 decimal places. 7 is the only cube that has last digit 3.

Problem 8. There are 499 such numbers divisible by 2, 333 by 3 and 166 divisible by 6 thus the answer is $499 + 333 - 166 = 666$.

Problem 9. We must have $a > 0$.

Problem 10. This is really just $x + a = ax \Rightarrow x = \frac{a}{a-1}$. So we have $a > 1$.

Problem 11. This is a bit of a silly one and the smallest such p is $p = 11$.

Problem 12. We have $f(0) \leq f(0)f(x)$ so $f(x) \geq 1$. The condition is necessary.

Problem 13. The only necessary condition is condition III.

Problem 14. $f(x) = 2x - x\sin x$ is strictly increasing and $f(0) = 0$, $f(\pi) = 2\pi$. So there are two such solutions.

Problem 15. $a = 1 - 1/b$.

Problem 16. $\log \dfrac{n!}{n+1}$

Problem 17. Graph 2

Problem 18. The answer of the sum is b.

Problem 19. The answer is $4\alpha/3 + 2\beta$.