THE UK UNIVERSITY INTEGRATION BEE 2021/22

$$\int$$

Round 1 Solutions

By: Finn Hogan, Vishal Gupta and Alfie Jones

Sponsored by

Jane Street

Integrals involve many tricks; however, some of these tricks appear quite frequently. For the sake of time and clarity there will be some abbreviations in the solutions. The full list of these abbreviations is below...

The Weierstass Substitution $t = \tan\left(\frac{x}{2}\right)$: $\overset{\text{W}}{=}$

$$\sin x = \frac{2t}{1-t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}$$

The King Property for Integrals $u = a + b - x$: $\overset{\text{K}}{=}$

$$\int_a^b f(x)\ \mathrm{d}x \overset{\text{K}}{=} \int_a^b f(a+b-x)\ \mathrm{d}x.$$

The Log Cosine and Log Sine Integrals: $\overset{\text{G}}{=}$

$$\int_0^{\frac{\pi}{2}} \log\sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \log\cos x\,\mathrm{d}x \overset{\text{G}}{=\!=} -\frac{\pi}{2}\ln 2$$

Integration by parts: $\overset{\text{IBP}}{=}$

$$\int u\,\mathrm{d}v \overset{\text{IBP}}{=} uv - \int v\,\mathrm{d}u$$

1

$$\int \sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\ldots}}}}\,\mathrm{d}x$$

We can separate the radicals and use our rules of indices.

$$\begin{aligned}
\int \sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\ldots}}}}\,\mathrm{d}x &= \int x^{\frac{1}{2}} \times x^{\frac{1}{2}\times\frac{1}{3}} \times x^{\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}} \times x^{\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}} \times \ldots\,\mathrm{d}x \\
&= \int x^{\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots}\,\mathrm{d}x \\
&= \int x^{e-2}\,\mathrm{d}x \\
&= \frac{1}{e-1}x^{e-1} + C
\end{aligned}$$

2

$$\int_0^{2\pi} \cos^{420}(x)\ \mathrm{d}x$$

Using Cauchy's Integral Formula from Complex Analysis is far faster but, sadly, out of syllabus. We will calculate this using the Beta Function.

$$\begin{aligned}
\int_0^{2\pi} \cos^{420}(x)\ \mathrm{d}x &= 4\int_0^{\frac{\pi}{2}} \cos^{420}(x)\ \mathrm{d}x \\
&= 2B\left(\frac{1}{2}, 210+\frac{1}{2}\right) \\
&= 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(210+\frac{1}{2}\right)}{\Gamma(211)} \\
&= 2\frac{\sqrt{\pi}\times 2^{1-420}\sqrt{\pi}\Gamma(420)}{\Gamma(211)\Gamma(210)} && \text{Legendre's Duplication Formula} \\
&= \frac{2\pi \times 420!}{2^{420}(210!)^2} = \frac{\pi}{2^{419}}\binom{420}{210}
\end{aligned}$$

3

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}\,\mathrm{d}\theta$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}\,\mathrm{d}\theta &= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\left(1+\sqrt{\tan\theta}\right)^5}\,\mathrm{d}\theta \\
&\overset{u^2=\tan\theta}{=} 2\int_0^{\infty} \frac{x\,\mathrm{d}x}{(1+x)^5} \\
&= 2\int_0^{\infty} (1+x)^{-4}-(1+x)^{-5}\,\mathrm{d}x \\
&= -\frac{2}{3}\,(1+x)^{-3}\Big|_0^{\infty} + \frac{1}{2}\,(1+x)^{-4}\Big|_0^{\infty} = \frac{2}{3}-\frac{1}{2} = \frac{1}{6}
\end{aligned}$$

4

$$\int_0^1 \frac{\sin(\ln x)}{\ln x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{\sin(\ln x)}{\ln x}\,\mathrm{d}x &= \frac{1}{2i}\int_0^1 \frac{x^i - x^{-i}}{\ln x}\,\mathrm{d}x \\
&= \frac{1}{2i}\int_0^1 \int_{-i}^{i} x^a \,\mathrm{d}a\mathrm{d}x \\
&= \frac{1}{2i}\int_{-i}^{i} \frac{1}{1+a}\,\mathrm{d}a \\
&= \frac{1}{2i}\,\ln(1+a)\Big|_{-i}^{i} \\
&= \frac{1}{2i}\ln\left(\frac{1+i}{1-i}\right) \\
&= \arctan(1) = \frac{\pi}{4}
\end{aligned}$$

5

$$\int_0^4 \frac{\ln(x)}{\sqrt{4x-x^2}}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^4 \frac{\ln(x)}{\sqrt{4x-x^2}}\,\mathrm{d}x &= \int_0^4 \frac{\ln(2+(x-2))}{\sqrt{4-(x-2)^2}}\,\mathrm{d}x \\
&\overset{x-2=2\sin\theta}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(2+2\sin\theta)\ \mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln 2 + \ln(1+\sin\theta)\ \mathrm{d}\theta \\
&\overset{\text{K}}{=} \pi\ln 2 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(1-\sin\theta)\ \mathrm{d}\theta \\
&= \pi\ln 2 + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\left(1-\sin^2\theta\right)\ \mathrm{d}\theta \\
&= \pi\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\cos\theta)\ \mathrm{d}\theta \\
&\overset{\text{G}}{=} \pi\ln 2 - \pi\ln 2 = 0
\end{aligned}$$

6

$$\int_0^\infty \left(\frac{\ln x}{1+x}\right)^2 \mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \left(\frac{\ln(x)}{1+x}\right)^2 \mathrm{d}x &= \int_0^1 \left(\frac{\ln(x)}{1+x}\right)^2 \mathrm{d}x + \int_1^\infty \left(\frac{\ln(x)}{1+x}\right)^2 \mathrm{d}x \\
&\overset{u=\frac{1}{x}}{=} 2\int_1^\infty \left(\frac{\ln(x)}{1+x}\right)^2 \mathrm{d}x \\
&\overset{\text{IBP}}{=} -\frac{2\ln^2(x)}{1+x} + 4\int_1^\infty \frac{\ln x}{x(1+x)}\,\mathrm{d}x \\
&\overset{u=\frac{1}{x}}{=} 4\int_0^1 \frac{-\ln x}{1+x}\,\mathrm{d}x \\
&= 4\int_0^1 -\ln x \sum_{n=0}^\infty (-1)^n x^n \,\mathrm{d}x \\
&\overset{u=-\ln x}{=} 4\sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(n+1)u} u \,\mathrm{d}u \\
&= 4\sum_{n=0}^\infty \frac{(-1)^n \Gamma(2)}{(n+1)^2} \\
&= 4 \times \frac{\pi^2}{12} = \frac{\pi^2}{3}
\end{aligned}$$

7

$$\int_0^\infty \frac{\ln(x)}{x^2+2x+2}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \frac{\ln(x)}{x^2+2x+2}\,\mathrm{d}x &\overset{u=\frac{2}{x}}{=} \int_0^\infty \frac{\ln\left(\frac{2}{x}\right)}{x^2+2x+2}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\ln 2 - \ln(x)}{x^2+2x+2}\,\mathrm{d}x \\
&= \frac{\ln 2}{2}\int_0^\infty \frac{\mathrm{d}x}{x^2+2x+2} \\
&= \frac{\ln 2}{2}\arctan(1+x)\Big|_0^\infty = \frac{\pi}{8}\ln 2
\end{aligned}$$

8

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\left(x^3 + \frac{1}{x^3}\right)^2}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\left(x^3 + \frac{1}{x^3}\right)^2} &= 2\int_0^{\infty} \frac{x^6 \,\mathrm{d}x}{\left(x^6+1\right)^2} \\
&\overset{u=x^6}{=} \frac{1}{3}\int_0^{\infty} \frac{u^{\frac{1}{6}}}{\left(u+1\right)^2}\,\mathrm{d}u \\
&= \frac{1}{3} B\left(\frac{7}{6}, \frac{5}{6}\right) \\
&= \frac{1}{3}\frac{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(2\right)} \\
&= \frac{1}{18}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) \\
&= \frac{1}{18}\frac{\pi}{\sin\left(\frac{\pi}{6}\right)} = \frac{\pi}{9}
\end{aligned}$$

9

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,\mathrm{d}x$$

$$\begin{aligned}
\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,\mathrm{d}x &= \int \frac{x-1}{\sqrt{x}\,(x+1)\sqrt{x^2+x+1}} \\
&\overset{u=\sqrt{x}}{=} 2\int \frac{u^2-1}{(u^2+1)\sqrt{u^4+u^2+1}}\,\mathrm{d}u \\
&= 2\int \frac{1-u^{-2}}{(u+u^{-1})\sqrt{u^2+1+u^{-2}}}\,\mathrm{d}u \\
&= 2\int \frac{1-u^{-2}}{(u+u^{-1})\sqrt{(u+u^{-1})^2-1}}\,\mathrm{d}u \\
&\overset{u+u^{-1}=\sec\theta}{=} 2\int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,\mathrm{d}\theta \\
&= 2\theta + C \\
&= 2\,\mathrm{arcsec}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) + C
\end{aligned}$$

10

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{2ix}{e^{ix}-e^{-ix}}\,\mathrm{d}x \\
&= 2i\int_0^{\frac{\pi}{2}} \frac{x\,e^{-ix}}{1-e^{-2ix}}\,\mathrm{d}x \\
&= 2i\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} x\,e^{-ix(1+2n)}\,\mathrm{d}x \\
&= 2i\sum_{n=0}^{\infty} \left.\frac{x\,e^{-ix(1+2n)}}{-i\,(1+2n)}\right|_0^{\frac{\pi}{2}} - \left.\frac{e^{-ix(1+2n)}}{(-i\,(1+2n))^2}\right|_0^{\frac{\pi}{2}} \\
&= \pi i\sum_{n=0}^{\infty}\frac{(-1)^n}{1+2n} + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^2} - 2i\sum_{n=0}^{\infty}\frac{1}{(1+2n)^2} \\
&= \pi i\times\frac{\pi}{4} + 2\times G - 2i\times\frac{\pi^2}{8} = 2G
\end{aligned}$$

where G is Catalan's Constant.

11

$$\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} \mathrm{d}x$$

$$\begin{aligned}
\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} \mathrm{d}x &\overset{u=\frac{1}{x}}{=} \int_0^1 (-\ln u)^{2011}\, u^{2009}\, \mathrm{d}u \\
&\overset{t=-\ln u}{=} \int_0^\infty t^{2011} e^{-2010t}\, \mathrm{d}t \\
&= \frac{\Gamma\left(2012\right)}{2010^{2012}} = \frac{2011!}{2010^{2012}}
\end{aligned}$$

12

$$\int_0^\infty \frac{\sin x}{x^n}\,\mathrm{d}x\,(0 < n < 2)$$

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x^n}\,\mathrm{d}x &= \frac{1}{2i}\lim_{\varepsilon\to0^+}\int_\varepsilon^\infty \left(e^{ix}-e^{-ix}\right)x^{-n}\,\mathrm{d}x \\
&= \frac{1}{2i}\lim_{\varepsilon\to0^+}\int_\varepsilon^\infty e^{ix}x^{-n}\,\mathrm{d}x - \int_\varepsilon^\infty e^{-ix}x^{-n}\,\mathrm{d}x \\
&= \frac{1}{2i}\lim_{\varepsilon\to0^+}\int_\varepsilon^\infty e^{-u}i^{1-n}u^{-n}\,\mathrm{d}u - \int_\varepsilon^\infty e^{-u}i^{n-1}u^{-n}\,\mathrm{d}u \\
&= \lim_{\varepsilon\to0^+}\sin\left(\frac{(1-n)\,\pi}{2}\right)\int_\varepsilon^\infty e^{-n}u^{-n}\,\mathrm{d}u \\
&= \sin\left(\frac{(1-n)\,\pi}{2}\right)\Gamma\left(1-n\right).
\end{aligned}$$

13

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^2}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin(2x)}\right)^2}\,\mathrm{d}x &\overset{\text{K}}{=} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin(2x)}\right)^2}\,\mathrm{d}x \\
I+I &= \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\left(1+\sqrt{\sin(2x)}\right)}\,\mathrm{d}x \\
I &\overset{u=\sin x-\cos x}{=} \frac{1}{2}\int_{-1}^{1} \frac{1}{\left(1-\sqrt{1-u^2}\right)^2}\,\mathrm{d}u \\
&= \int_0^1 \frac{1}{\left(1-\sqrt{1-u^2}\right)^2}\,\mathrm{d}u \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\cos x)^2}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} - \frac{1}{(1+\cos x)^2}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2\left(\frac{x}{2}\right)} - \frac{1}{\left(2\cos^2\left(\frac{x}{2}\right)\right)^2}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sec^2\left(\frac{x}{2}\right) - \frac{1}{4}\sec^4\left(\frac{x}{2}\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{4}\sec^2\left(\frac{x}{2}\right) - \frac{1}{4}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)\,\mathrm{d}x \\
&= \frac{1}{2}\tan\left(\frac{x}{2}\right) - \frac{1}{6}\tan^3\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}
\end{aligned}$$

14

$$\int_0^{\frac{\pi}{2}} \ln\left(7997\sin^2\theta + 7945\cos^2\theta\right)\,\mathrm{d}\theta$$

We begin by evaluating

$$\text{Let } I(a) = \int_0^{\pi} \ln\left(1 + a^2 - 2a\cos x\right)\,\mathrm{d}x$$

$$\begin{aligned}
I'(a) &= \int_0^{\pi} \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,\mathrm{d}x \\
&= \frac{1}{a}\int_0^{\pi} \frac{1 - 2a\cos x + a^2 - 1 + a^2}{1 - 2a\cos x + a^2}\,\mathrm{d}x \\
&= \frac{\pi}{a} + \frac{1}{a}\int_0^{\pi} \frac{a^2 - 1}{1 - 2a\cos x + a^2}\,\mathrm{d}x \\
&\overset{\underline{\text{W}}}{=} \frac{\pi}{a} + \frac{2}{a}\int_0^{\infty} \frac{a^2 - 1}{(1-a)^2 + (1+a)^2 t^2}\,\mathrm{d}t \\
&= \frac{\pi}{a} + \frac{\pi}{a}\operatorname{sgn}\left(\frac{1+a}{1-a}\right) \\
&= \begin{cases} \dfrac{2\pi}{a} & \text{if } a < 1 \\ 0 & \text{if } a > 1. \end{cases}
\end{aligned}$$

We use this result to evaluate a more general expression.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln\left(\alpha\sin^2 x + \beta\cos^2 x\right)\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \ln\left(\alpha + (\beta + \alpha)\cos^2 x\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\alpha + (\beta - \alpha)\left(\frac{1 + \cos(2x)}{2}\right)\right)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\cos(2x)\right)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} \ln\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\cos x\right)\,\mathrm{d}x
\end{aligned}$$

Now solving the system of equations

$$\frac{\alpha+\beta}{2} = k\left(1 + r^2\right), \quad \frac{\alpha-\beta}{2} = 2kr$$

yields

$$k = \frac{1}{4}\left(\sqrt{\alpha} - \sqrt{\beta}\right)^2, \quad r = \frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}}$$

Finally, we get

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi} \ln\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\cos x\right)\,\mathrm{d}x &= \frac{1}{2}\int_0^{\pi} \ln\left(\frac{1}{4}\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2\right)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_0^{\pi} \ln\left(1 + r^2 - 2r\cos x\right)\,\mathrm{d}x \\
&= \pi\ln\left(\frac{\sqrt{\alpha} + \sqrt{\beta}}{2}\right)
\end{aligned}$$

$$\text{Therefore, the integral equals } \pi\ln\left(\frac{\sqrt{7997} + \sqrt{7945}}{2}\right).$$

15

$$\int \frac{\mathrm{d}x}{\csc x + 1}$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\csc x + 1} &= \int \frac{\sin x}{1 + \sin x}\,\mathrm{d}x \\
&= \int \frac{\sin x\,(1 - \sin x)}{1 - \sin^2 x}\,\mathrm{d}x \\
&= \int \tan x \sec x - \tan^2 x\,\mathrm{d}x \\
&= \int \tan x \sec x - \sec^2 x + 1\,\mathrm{d}x \\
&= \sec x - \tan x + x + C
\end{aligned}$$

16

$$\int_0^{\frac{\pi}{2}} \frac{\{\tan(x)\}}{\tan(x)}\,\mathrm{d}x, \text{ where } \{x\} \text{ is the fractional part of } x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\{\tan x\}}{\tan x}\,\mathrm{d}x &\overset{u=\tan x}{=} \int_0^{\infty} \frac{\{u\}}{u} \times \frac{1}{1+u^2}\,\mathrm{d}u \\
&= \frac{u-\lfloor u \rfloor}{u} \times \frac{1}{1+u^2}\,\mathrm{d}u \\
&= \int_0^{\infty} \frac{1}{1+u^2}\,\mathrm{d}u - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n}{u} \times \frac{1}{1+u^2}\,\mathrm{d}u \\
&= \frac{\pi}{2} - \sum_{n=1}^{\infty} n \int_n^{n+1} \frac{1}{u} - \frac{u}{1+u^2}\,\mathrm{d}u \\
&= \frac{\pi}{2} - \frac{1}{2}\sum_{n=1}^{\infty} n \ln\left(\frac{u^2}{1+u^2}\right)\Bigg|_n^{n+1} \\
&= \frac{\pi}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \Big(2n\ln(n+1) - 2n\ln(n) - n\ln\left(1+(1+n)^2\right) + n\ln\left(1+n^2\right)\Big) \\
&= \frac{\pi}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \Big(2(n+1)\ln(n+1) - 2n\ln(n) - 2\ln(n+1) \\
&\qquad - (n+1)\ln\left(1+(1+n)^2\right) + n\ln\left(1+n^2\right) + \ln\left(1+(1+n)^2\right)\Big) \\
&= \frac{\pi}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\sum_{n=2}^{\infty} \ln\left(\frac{1+n^2}{n^2}\right) \\
&= \frac{\pi}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2}\right)\right) = \frac{\pi}{2} - \frac{1}{2}\ln\left(\frac{\sinh \pi}{2\pi}\right).
\end{aligned}$$

The last product comes from

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right), \quad \text{with } x = i\pi.$$

Then we have

$$\frac{\sinh \pi}{2\pi} = \frac{1}{2} \times \frac{\sin(i\pi)}{i\pi} = \frac{1}{2}\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\left(1+\frac{1}{n^2}\right)$$

17

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2+\sin x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2+\sin x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + 2\sin^2 x - \sin^2 x - 4\sin x + 4\sin x + 8 - 8}{2+\sin x}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \sin^2 x - 2\sin x + 4 - \frac{8}{2+\sin x}\,\mathrm{d}x \\
&\stackrel{\mathrm{W}}{=} \frac{\pi}{4} - 2 + 2\pi - 8\int_0^1 \frac{\mathrm{d}t}{1+t+t^2} \\
&= \frac{9\pi}{4} - 2 - 8\int_0^1 \frac{\mathrm{d}t}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \frac{9\pi}{4} - 2 - \frac{16}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)\Bigg|_0^1 \\
&= \frac{9\pi}{4} - 2 - \frac{8\pi}{3\sqrt{3}} \\
&= \frac{\pi\left(81 - 32\sqrt{3}\right)}{36} - 2
\end{aligned}$$

18

$$\int \sqrt{\frac{1}{x} - 1}\,\mathrm{d}x$$

$$\begin{aligned}
\int \sqrt{\frac{1}{x} - 1}\,\mathrm{d}x &= \int \frac{\sqrt{1-x}}{\sqrt{x}}\,\mathrm{d}x \\
&\overset{u=\sqrt{x}}{=} 2\int \sqrt{1-u^2}\,\mathrm{d}u \\
&\overset{u=\sin\theta}{=} 2\int \cos^2\theta\,\mathrm{d}\theta \\
&= \int 1 + \cos 2\theta\,\mathrm{d}\theta \\
&= \theta + \frac{1}{2}\sin 2\theta + C \\
&= \arcsin\left(\sqrt{x}\right) + \sqrt{x}\sqrt{1-x} + C
\end{aligned}$$

19

$$\int_0^{2\pi} e^{3\cos\theta} \cos(3\sin\theta)\ \mathrm{d}\theta$$

$$\begin{aligned}\int_0^{2\pi} e^{3\cos\theta} \cos(3\sin\theta)\ \mathrm{d}\theta = \Re \int_0^{2\pi} e^{3e^{i\theta}}\, \mathrm{d}\theta \\ = \Re \int_0^{2\pi} \sum_{n=0}^{\infty} 3^n e^{in\theta}\, \mathrm{d}\theta \\ = \Re \sum_{n=0}^{\infty} 2\pi\delta_{0,n} 3^n = 2\pi\end{aligned}$$

20

$$\int \sqrt{x^2 - 1} \, \mathrm{d}x$$

$$\begin{aligned}
\int \sqrt{x^2 - 1} \, \mathrm{d}x &\stackrel{x=\cosh\theta}{=} \int \sinh^2 \theta \, \mathrm{d}\theta \\
&= \frac{1}{2} \int \cosh 2\theta - 1 \, \mathrm{d}\theta \\
&= \frac{1}{2} \sinh\theta \cosh\theta - \frac{1}{2}\theta + C \\
&= \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \operatorname{arcosh} x + C
\end{aligned}$$

21

$$\int_0^a \frac{x\,\mathrm{d}x}{\cos(x)\cos(a-x)}$$

$$\begin{aligned}
\int_0^a \frac{x}{\cos(x)\cos(a-x)}\,\mathrm{d}x &\overset{\mathrm{K}}{=} \frac{a}{2}\int_0^a \frac{\mathrm{d}x}{\cos(x)\cos(a-x)}\,\mathrm{d}x \\
&= \frac{a}{2\sin a}\int_0^a \frac{\sin(a-x+x)}{\cos(x)\cos(a-x)}\,\mathrm{d}x \\
&= \frac{a}{2\sin a}\int_0^a \frac{\sin(a-x)\cos(x)+\cos(a-x)\sin(x)}{\cos(x)\cos(a-x)}\,\mathrm{d}x \\
&= \frac{a}{2\sin a}\int_0^a \tan(a-x)+\tan(x)\ \mathrm{d}x \\
&= \frac{a}{\sin a}\int_0^a \tan x\,\mathrm{d}x \\
&= \frac{a}{\sin a}\ln|\sec x|\Big|_0^a \\
&= \frac{a\ln|\sec a|}{\sin a}.
\end{aligned}$$

22

$$\int_0^1 \frac{\arctan x}{1+x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{\arctan x}{1+x}\,\mathrm{d}x & \\
&\overset{\text{IBP}}{=} \arctan x \ln(1+x)\Big|_0^1 - \int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x \\
&\overset{x=\tan\theta}{=} \frac{\pi}{2}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta \\
\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta &\overset{\text{K}}{=} \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln 2 - \ln(1+\tan\theta)\ \mathrm{d}\theta
\end{aligned}$$

Thus, we have

$$\int_0^1 \frac{\arctan x}{1+x}\,\mathrm{d}x = \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 2 = \frac{\pi}{8}\ln 2.$$

23

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)}{\sin x}\, \mathrm{d}x$$

We begin by defining

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{\ln\left(1 - a\sin^2 x\right)}{\sin x}\, \mathrm{d}x$$

$$\begin{aligned}
I'(a) &= -\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 - a\sin^2 x}\, \mathrm{d}x \\
&= -\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 - a + a\cos^2 x}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{a}\sqrt{1-a}} \arctan\left(\sqrt{\frac{a}{1-a}}\cos x\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= -\frac{1}{\sqrt{a}\sqrt{1-a}} \arctan\left(\sqrt{\frac{a}{1-a}}\right) \\
&= -\frac{\arcsin\sqrt{a}}{\sqrt{a}\sqrt{1-a}}
\end{aligned}$$

Notice that

$$I(1) = \int_0^{\frac{\pi}{2}} \frac{\ln\left(1 - \sin^2 x\right)}{\sin x}\, \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \quad \text{and} \quad I(0) = 0$$

Therefore,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)}{\sin x}\, \mathrm{d}x &= \frac{1}{2}\left(I(1) - I(0)\right) \\
&= -\frac{1}{2}\int_0^1 \frac{\arcsin\left(\sqrt{a}\right)}{\sqrt{a}\sqrt{1-a}}\, \mathrm{d}a \\
&= -\frac{1}{2}\arcsin^2\left(\sqrt{a}\right)\Big|_0^1 \\
&= -\frac{\pi^2}{8}
\end{aligned}$$

24

$$\int \frac{x \ \arcsin x}{\sqrt{1-x^2}} \, \mathrm{d}x$$

$$\begin{aligned}\int \frac{x \arcsin x}{\sqrt{1-x^2}} \, \mathrm{d}x &\overset{x=\sin\theta}{=} \int \theta \sin\theta \, \mathrm{d}\theta \\ &\overset{\text{IBP}}{=} -\theta\cos\theta + \sin\theta + C \\ &= -\sqrt{1-x^2}\arcsin x + x + C\end{aligned}$$

25

$$\int_0^\infty \frac{\arctan(x)}{x\left(\ln(x)^2+1\right)}\,\mathrm{d}x$$

$$\begin{aligned}\int_0^\infty \frac{\arctan x}{x\left(\ln^2(x)+1\right)}\,\mathrm{d}x &\overset{u=\frac{1}{x}}{=} \frac{\pi}{4}\int_0^\infty \frac{1}{x\left(\ln^2 x+1\right)}\,\mathrm{d}x \\ &= \frac{\pi}{4}\arctan(\ln x)\Big|_0^\infty \\ &= \frac{\pi^2}{4}\end{aligned}$$

26

$$\int_1^e \frac{x - \ln x + 1}{x\,(x+1)^2 + x\ln^2 x}\,\mathrm{d}x$$

$$\begin{aligned}
\int_1^e \frac{x - \ln x + 1}{x\,(x+1)^2 + x\ln^2 x}\,\mathrm{d}x &= \int_1^e \frac{1}{1+\left(\frac{\ln x}{1+x}\right)^2} \times \left(\frac{(x+1)\times\frac{1}{x} - \ln x}{(x+1)^2}\right)\,\mathrm{d}x \\
&= \arctan\left(\frac{\ln x}{1+x}\right)\bigg|_1^e \\
&= \arctan\left(\frac{1}{1+e}\right)
\end{aligned}$$

27

$$\int_0^\infty \frac{\cos(ax)}{x^2+b^2}\,\mathrm{d}x$$

Let the integral be $I(a)$.

$$\begin{aligned}
I'(a) &= -\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}\,\mathrm{d}x \\
&= -\int_0^\infty \frac{x^2\sin(ax)}{x(x^2+b^2)}\,\mathrm{d}x \\
&= -\int_0^\infty \frac{\sin(ax)}{x}\,\mathrm{d}x + b^2\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,\mathrm{d}x \\
&= -\frac{\pi}{2} + b^2\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,\mathrm{d}x \\
I''(a) &= b^2\int_0^\infty \frac{\cos(ax)}{x^2+b^2}\,\mathrm{d}x
\end{aligned}$$

Thus, we have the following differential equation

$$I''(a) - b^2 I(a) = 0 \quad I(0) = \frac{\pi}{2b},\ I'(0) = 0.$$

This has the solution $I(a) = \dfrac{\pi}{2be^{ab}}$.

28

$$\int_1^\infty \frac{\mathrm{d}x}{x + x^{\sqrt{2}}}$$

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x + x^{\sqrt{2}}} &= \int_1^\infty \frac{x^{-\sqrt{2}}}{x^{1-\sqrt{2}} + 1}\,\mathrm{d}x \\
&= \frac{1}{1-\sqrt{2}} \ln\left|x^{1-\sqrt{2}} + 1\right|\Big|_1^\infty \\
&= -\frac{1}{1-\sqrt{2}} \ln \\
&= \left(\sqrt{2} + 1\right) \ln 2.
\end{aligned}$$

29

$$\int_0^\infty \lfloor x \rfloor e^{-x} \, \mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \lfloor x \rfloor e^{-x} \, \mathrm{d}x &= \sum_{n=1}^{\infty} \int_n^{n+1} n e^{-x} \, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} n \left(e^{-n} - e^{-n+1} \right) \\
&= \sum_{n=1}^{\infty} n e^{-n} - (n+1) \, e^{-(n+1)} + e^{-(n+1)} \\
&= e^{-1} + \sum_{n=1}^{\infty} e^{-(n+1)} \\
&= e^{-1} + \frac{e^{-2}}{1 - e^{-1}} \\
&= \frac{1}{e-1}
\end{aligned}$$

30

$$\int_0^1 \frac{\arctan\left(x^2\right)}{1+x^2}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^1 \frac{\arctan x^2}{1+x^2}\,\mathrm{d}x &\overset{\text{IBP}}{=} \arctan(x)\arctan\left(x^2\right)\Big|_0^1 - \int_0^1 \frac{2x\arctan x}{1+x^4}\,\mathrm{d}x \\
&= \frac{\pi^2}{16} - \int_0^1 \frac{2x\arctan x}{1+x^4}\,\mathrm{d}x \\
I &= \int_0^1 \frac{2x\arctan x}{1+x^4}\,\mathrm{d}x \\
&= \int_0^1 \int_0^1 \frac{2x^2}{\left(1+t^2x^2\right)\left(1+x^4\right)}\,\mathrm{d}t\mathrm{d}x
\end{aligned}$$

Making use of the following identity

$$x^2\left(1+t^4\right) + t^2\left(1+x^4\right) = x^2\left(1+x^2t^2\right) + t^2\left(1+x^2t^2\right)$$

we get that the above integral is equal to

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{2t^2}{\left(1+t^4\right)\left(1+x^4\right)} + \frac{2x^2}{\left(1+t^4\right)\left(1+x^4\right)} - \frac{2t^2}{\left(1+t^2x^2\right)\left(1+x^4\right)}\,\mathrm{d}t\mathrm{d}x \\
&= 4\int_0^1 \frac{t^2}{1+t^4}\,\mathrm{d}t \int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x - I \\
I &= 2\int_0^1 \frac{x^2}{1+x^4}\,\mathrm{d}x \int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x
\end{aligned}$$

We shall skip the evaluation of the two above integrals however, they can be solved using partial fractions and substitutions.

$$I = 2\left(\frac{1}{4\sqrt{2}}\left(\pi + \ln\left(3-2\sqrt{2}\right)\right)\right) \times \frac{1}{4\sqrt{2}}\left(\pi - \ln\left(3-2\sqrt{2}\right)\right) = \frac{\pi^2}{16} - \frac{1}{16}\ln^2\left(3-2\sqrt{2}\right)$$

Thus, the final answer is

$$\int_0^1 \frac{\arctan\left(x^2\right)}{1+x^2}\,\mathrm{d}x = \frac{1}{16}\ln^2\left(3-2\sqrt{2}\right)$$

31

$$\int_0^1 \frac{\mathrm{d}x}{1+\left\lfloor \frac{1}{x} \right\rfloor}$$

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{1+\left\lfloor \frac{1}{x} \right\rfloor} &= \int_1^\infty \frac{\mathrm{d}x}{x^2\left(1+\lfloor x \rfloor\right)} \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\mathrm{d}x}{x^2\left(1+n\right)} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{1}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n}-\frac{1}{n+1}-\frac{1}{(n+1)^2} \\
&= 2-\frac{\pi^2}{6}.
\end{aligned}$$

32

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx$$

$$\begin{aligned}
\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx &= \int_0^\infty \int_a^b \frac{\sin tx}{x}\,dt dx \\
&= \int_a^b \frac{\pi}{2}\,dt \\
&= \frac{\pi}{2}(b-a)
\end{aligned}$$

33

$$\int_0^{2\pi} e^{\cos x} \cos(\sin x) \cos(5x) \, \mathrm{d}x$$

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x} \cos(\sin x) \cos(5x) \, \mathrm{d}x &= \Re \int_0^{2\pi} e^{e^{ix}} \cos(5x) \, \mathrm{d}x \\
&= \Re \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{e^{inx}}{n!} \cos(5x) \, \mathrm{d}x \\
&= \Re \sum_{n=0}^{\infty} \pi \delta_{5,n} \frac{1}{n!} = \frac{\pi}{120}.
\end{aligned}$$

34

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^2(x)}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^2 x} = \tanh x \Big|_{-\infty}^{\infty} = 2$$

35

$\int_0^e W(x)\ \mathrm{d}x,$ where W is the Lambert W function, the solution to $W(x)e^{W(x)} = x$

$$\begin{aligned}\int_0^e \mathrm{W}(x)\,\mathrm{d}x &\overset{u=\mathrm{W}(x)}{=} \int_0^1 u\,(u+1)\,e^u\,\mathrm{d}u \\ &= \left(u^2 - u + 1\right)e^u\Big|_0^1 = e - 1\end{aligned}$$

36

$$\int_0^1 \frac{\arctan^2 x}{x}\, \mathrm{d}x$$

$$\begin{aligned}
&\int_0^1 \frac{\arctan^2 x}{x}\, \mathrm{d}x \\
&\qquad = \ln(x)\arctan^2(x)\Big|_0^1 - \int_0^1 \frac{2\arctan(x)\ln(x)}{1+x^2}\, \mathrm{d}x \\
&\qquad = -\int_0^{\frac{\pi}{4}} 2\theta \ln(\tan\theta)\ \mathrm{d}\theta \\
&\qquad = \int_0^{\frac{\pi}{4}} 4\theta \sum_{k=1}^{\infty} \frac{\cos(2(2k-1))}{2k-1}\, \mathrm{d}\theta \\
&\qquad = 4\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\frac{\pi}{4}} \theta \cos((4k-2)\theta)\ \mathrm{d}\theta \\
&\qquad = 4\sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{\theta}{4k-2}\sin((4k-2)\theta) + \frac{1}{(4k-2)^2}\cos((4k-2)\theta) \right)\Bigg|_0^{\frac{\pi}{4}} \\
&\qquad = \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \\
&\qquad = \frac{\pi G}{2} + \frac{7}{8}\zeta(3)
\end{aligned}$$

37

$$\int_0^{\frac{\pi}{2}} \cos x^{\sin x^{\cos x}} - \sin x^{\cos x^{\sin x}} \, \mathrm{d}x$$

This integral is immediately 0 by the King Property.

38

$$\int_3^5 \ln\Gamma\left(x\right)\,\mathrm{d}x$$

The result follows from Raabe's Formula:

$$\int_n^{n+1} \ln\left(\Gamma\left(x\right)\right)\,\mathrm{d}x = \frac{1}{2}\ln\left(2\pi\right) + n\ln\left(n\right) - n$$

$$\int_3^5 \ln\left(\Gamma\left(x\right)\right)\,\mathrm{d}x = \ln\left(2\pi\right) + 3\ln 3 - 3 + 4\ln 4 - 4 = \ln\left(2\pi\right) + 3\ln 3 + 8\ln 2 - 7$$

39

$$\int \frac{\mathrm{d}x}{\sin^4(x) + \cos^4(x)}$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sin^4 x + \cos^4 x} &= \int \frac{\mathrm{d}x}{\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x} \\
&= \int \frac{2}{2 - \sin^2(2x)}\,\mathrm{d}x \\
&= \int \frac{2}{2\cos^2(2x) + \sin^2(2x)}\,\mathrm{d}x \\
&= \int \frac{2\sec^2(2x)}{2 + \tan^2(2x)}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan(2x)}{\sqrt{2}}\right) + C
\end{aligned}$$

40

$$\int_0^\infty \frac{\cos(\ln x)}{(1+x)^2}\,\mathrm{d}x$$

$$\begin{aligned}
\int_0^\infty \frac{\cos(\ln x)}{(1+x)^2}\,\mathrm{d}x &= \Re \int_0^\infty \frac{x^2}{(1+x)^2}\,\mathrm{d}x \\
&= B(1-i,\, 1+i) \\
&= \frac{\Gamma(1-i)\,\Gamma(1+i)}{\Gamma(2)} \\
&= i\Gamma(1-i)\,\Gamma(i) \\
&= \frac{i\pi}{\sin(i\pi)} \\
&= \frac{\pi}{\sinh(\pi)}
\end{aligned}$$

References