MOCK TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

Paper 1

ANSWERS

Oxbridge Applications — 14 Waterloo Place — London SW1Y 4RS
Tel: +44 (0)207 499 2394 — Fax: 0207 4910 343
www.oxbridgeapplications.com
Oxbridge Applications is a trading name of Dukes Education Ltd, registered in England and Wales, number 03757054
Registered office: 14-16 Waterloo Place, London SW1Y 4AR

1. answer: **D**

It is sufficient to look at the circle, then add 1. The largest value $x+y$ can take, intuitively, is when the line from the centre to (x, y) forms a $\frac{\pi}{4}$ angle with the x-axis. One way to see this, if we picture starting at $(2, 0)$ and moving our way anti-clockwise around the circle till $(1, 1)$, $x+y$ is the same at both these ends but is at first increasing, hence the maximum must be reached 'in the middle' by symmetry.

Alternatively, we can parametrise with $x = 1 + \cos\vartheta$, $y = \sin\vartheta$ which gives $x + y = 1 + \sqrt{2}\sin 2\vartheta$, maximised at $\vartheta = \frac{\pi}{4}$, in agreement with the reasoning above.

2. answer: **A**

This is a disguised set of simultaneous equations: we have

$$\frac{a}{1-\mu} = \frac{2b}{1-\frac{1}{2}\mu}; \qquad \frac{a}{1-\mu} = \frac{3b}{1-\frac{1}{3}\mu}$$

Cross multiplying and scaling to isolate $b(1-\mu)$ in both cases we get

$$\frac{a}{2}\left(1-\frac{\mu}{3}\right) = \frac{a}{3}\left(1-\frac{\mu}{2}\right)$$

Cancelling a (since non-zero) and solving gives $\mu = \frac{5}{6}$ (< 1), which just requires $a = -b$.

3. answer: **H**

The value of $\tan\alpha$ determines α (since it is strictly increasing on $(0, \frac{\pi}{2})$ for instance). Knowing the hypothenuse, the angle then determines the area and the lengths of both other sides.

4. answer: **B**

Since $f'(x) = (1-x)^{-n}$ gives $f(x) = \frac{1}{n-1}(1-x)^{1-n}$, we get:

$$r(x) = \frac{1}{6(1-x)^2} + \frac{1}{2(1-x)} + Ax + B$$

for some constants A, B. Now, $r(0) = B + \frac{2}{3}$, $r(2) = -\frac{1}{3} + 2A + B$ so the first condition gives $A + B = 0$. Moreover, $r(\frac{4}{3}) = \frac{1}{3}A$ hence $A = 0$. Since this is not immediately in the form of the given answers, it is sensible to combine both fractions, which gives the answer.

5. answer: **C**

The ways in which the polynomial can have 4 real roots are:

(i) two of the p_i to have 2 roots and two have 0: the product of Δ_i will then contain 2 positive determinants and two negative so > 0.

(ii) at least one of the p_i has exactly 1 root, which makes the product of Δ_i equal to 0.

6. answer: **C**

The segment measuring the distance between the two is perpendicular to $y = x + 7$ and therefore lies on a line of the form $y = a - x$. Since this must go through $(1, -1)$, we have $a = 0$. This intersects $y = x + 7$ at $(-\frac{7}{2}, \frac{7}{2})$, so the distance between the line and the centre of the circle is:

$$\sqrt{\left(1 + \frac{7}{2}\right)^2 + \left(-1 - \frac{7}{2}\right)^2} = \sqrt{2}\left(1 + \frac{7}{2}\right) = \frac{9}{\sqrt{2}}$$

Subtracting the radius, $\sqrt{7}$, gives the answer.

7. answer: **E**

Since the integrals look similar, it is sensible to try and relate them: the second integrand is a translation of the first integrand to the left by 1 unit. Therefore the second integral is equal to the first. Then we have:

$$2\int_0^1 \frac{x+1}{\sqrt[3]{x}}\,\mathrm{d}x = 2\int_0^1 x^{\frac{2}{3}} + x^{-\frac{1}{3}}\,\mathrm{d}x = 2\left(\frac{3}{5}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{2}{3}}\right)\Bigg|_0^1 = \frac{21}{5}$$

8. answer: **C**

$\sin x = 0$ has solutions exactly when $x = k\pi$. However, $|\cos x| \leqslant 1$ so the only values which will give solutions are when $\cos x = 0$ which occurs twice in $[0, 2\pi]$.

9. answer: **D**

Successive differentiation of a polynomial of degree n reaches 0 precisely after $n + 1$ differentiations. The degree of the given polynomial is $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$. So:

$$\frac{1}{2}k(k+1) + 1 = 121 \implies k(k+1) = 240 \implies k = 15$$

10. answer: **D**

There are several ways of viewing this, one through the following steps: mirroring in $[1, 3]$ is equivalent to reflecting along $x = 0$ (giving the function $g(x) = f(-x)$) and then translating $1 + 3 = 4$ to the right (giving $g(x - 4) = f(4 - x)$).

11. answer: **F**

There are various ways of going about it, but the easiest is perhaps through examples:

a) 0 roots: $2x^2 + 1$ and $-x^2 + 1$; b) 1 root: $2x^2$ and $-x^2$; c) 2 roots: $2x^2 - 1$ and $-x^2 - 1$; d) infinitely many roots: x^2 and $-x^2$

Oxbridge Applications — 14 Waterloo Place — London SW1Y 4RS
Tel: +44 (0)207 499 2394 — Fax: 0207 4910 343
www.oxbridgeapplications.com

12. answer: **A**

Let $2^x = X$ and $2^y = Y$. Subtract the second line from the first:

$$X + Y^2 - (Y + X^2) = 0 \implies (X - Y)(X + Y + 1) = 0$$

$X + Y + 1 = 0$ cannot happen since $X, Y > 0$. If $X = Y$ we have $X^2 + X - 1 = 0$ which gives $X = \frac{1}{2}(\sqrt{5} - 1)$ (since $X > 0$). Hence $x = y = \log_2\left(\frac{1}{2}(\sqrt{5} - 1)\right) = \log_2(\sqrt{5} - 1) - 1$.

13. answer: **A**

Through the basic trigonometric formulas and/or thinking back to question 1., $\sin\vartheta + \cos\vartheta = \sqrt{2}\sin 2\vartheta$ is has maximum $\sqrt{2}$, but $\sqrt{2} < \frac{3}{2}$ (which one can see by squaring if one is not sure about the value of $\sqrt{2}$).

14. answer: **B**

By inspection, $x = -q$ is the unique root. The triangle ABC will be isosceles precisely when the tangent at A has gradient 1 (it cannot be -1 because the leading term of the cubic is positive). Differentiating gives $3x^3 + 2qx + q^2$ which at A (i.e. $x = -q$) is $2q^2$, so we get answer (B).

15. answer: **D**

Let $n > 1$ be the number of green apples. There are various ways of phrasing a solution, one being that the probability of picking 2 red and 2 green is $\binom{n}{2}\binom{2}{2}\binom{n+2}{4}^{-1}$ (picking 2 from n, 2 from 2, in a situation with $\binom{n+2}{4}$ possibilities), and the probability of choose neither red is $\binom{n}{4}\binom{2}{0}\binom{n+2}{4}^{-1}$ (picking 4 from n, none from 2). Equating the first to twice the second:

$$\frac{n(n-1)}{2!} = 2 \cdot \frac{n(n-1)(n-2)(n-3)}{4!} \implies (n-2)(n-3) = 6 \implies n = 5$$

16. answer: **G**

The area is:

$$\int_{-a}^{0} v(x) - w(x)\,\mathrm{d}x + \int_{0}^{a} w(x) - v(x)\,\mathrm{d}x$$

(1) is verified by:

$$\int_{-a}^{0} |v(x) - w(x)|\,\mathrm{d}x + \int_{0}^{a} |w(x) - v(x)|\,\mathrm{d}x = \int_{-a}^{a} |v(x) - w(x)|\,\mathrm{d}x$$

Since, from the graph, $v(-x) = -v(x)$ and $w(-x) = -w(x)$ (i.e. v, w are odd), (2) and (3) are also verified.

Oxbridge Applications — 14 Waterloo Place — London SW1Y 4RS
Tel: +44 (0)207 499 2394 — Fax: 0207 4910 343
www.oxbridgeapplications.com

17. answer: **B**

It is probably easiest to do this by writing $v_A = 2\pi/t_A$ and $v_B = 2\pi/t_B$ as their velocities. Their relative velocities is then $v_A - v_B$, and they meet after the initial distance $2\pi - d$ is covered at that speed, so the time is:

$$(2\pi - d)(v_A - v_B)^{-1} = (2\pi - d)\left(\frac{2\pi}{t_A} - \frac{2\pi}{t_B}\right)^{-1} = \left(1 - \frac{d}{2\pi}\right)\left(\frac{1}{t_A} - \frac{1}{t_B}\right)^{-1} = \left(1 - \frac{d}{2\pi}\right)\frac{t_A t_B}{t_B - t_A}$$

18. answer: **G**

Write $o_n = a + dn$. The equality becomes:

$$\frac{1}{2}(a+dn) = \int_0^{a+dn} x - n \,\mathrm{d}x = \frac{1}{2}x^2 - nx\Big|_0^{a+dn} = \frac{1}{2}(a+dn)^2 - n(a+dn)$$

Cancelling $a + dn$ (since non-zero) gives $\frac{1}{2} = \frac{1}{2}(a+dn) - n$. Hence $a = 1$ and $d = 2$, so $o_n = 2n + 1$ i.e. the odd positive integers, and therefore $o_{100} = 201$.

19. answer: **C**

The new line ℓ necessarily intersects all others (since it is not parallel) and at distinct points (since no 3 meet). These n intersections divide ℓ into $n+1$ portions, each of which divides an original region into 2. Hence we get an additional $n+1$ regions, so $n+r+1$ in total.

20. answer: **A**

This tests a proper understanding of the chain rule, which says $\big(f(u(x))\big)' = u'(x)f'(u(x))$.

This means $f_2'(x) = f'(x)f'(f(x))$, and generally $f_k'(x) = f_{k-1}'(x)f'(f_{k-1}(x))$. Hence a factor common to all $f_k'(x)$ is $f'(x)$. Noting that $f'(1) = 0$ we have $f_k'(1) = 0$ for all $k \geqslant 1$.

Oxbridge Applications — 14 Waterloo Place — London SW1Y 4RS
Tel: +44 (0)207 499 2394 — Fax: 0207 4910 343
www.oxbridgeapplications.com