20 YEARS OXBRIDGE APPLICATIONS

# TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION (TMUA)

MOCK EXAM 2
SOLUTIONS

# SECTION 1
# MATHEMATICAL KNOWLEDGE

**1 B.**

Options **C** and **D** are ruled out since their $y$-intercept is $(0, 1)$. Completing the square on the remaining options, and reading off the turning point shows that the correct option is $x^2 + 2x - 1$.

**2 D.**

Note that $256 = 2^8$. Moreover, for $n \geq 2$ we have:

$$x_n = \frac{x_{n-1}}{2} = \frac{x_{n-2}}{2^2} = \ldots = \frac{x_1}{2^{n-1}}\,.$$

Therefore, we conclude that $x_{16} = 2^{-15} \cdot 2^8 = 2^{-7} = \frac{1}{128}$.

**3 E.**

Direct calculation:

$$\int_1^4 \frac{2x^2 - \sqrt{x}}{x\sqrt{x}} dx = \int_1^4 2x^{\frac{1}{2}} - \frac{1}{x} dx = \left[\frac{4}{3}x^{\frac{3}{2}} - \ln x\right]_1^4 = \frac{28}{3} - 2\ln 2\,.$$

**4 B.**

The second equation gives us $y = a - x$. Substituting this into the second equation gives $x^2 + 2x - 1 - (a - x)^2 = 4$, so that

$$2(1 + a)x = a^2 + 5.$$

We conclude that a solution exists for all real values of $a$ except $a = -1$.

**5 E.**

A quick sketch of the graph of $y = x \sin 2x$ shows that it intersects the $x$-axis 7 times in the range $0 \leq x \leq 3\pi$.

**6 E.**

Taking all terms to the right and factorizing gives:

$$0 < \frac{1}{2x} + \frac{3x^2 + 2}{x + 1} - 2 = \frac{6x^3 - 4x^2 + x + 1}{2x(x + 1)} = \frac{(3x + 1)(2x^2 - 2x + 1)}{2x(x + 1)} =: f(x).$$

Note that $x$ might be negative and the inequality might be flipped by cross-multiplying. The quadratic $2x^2 - 2x + 1$ has a negative discriminant, and so does not factor over the reals. Therefore, we need to determine whether $f(x)$ is positive or negative in each of the intervals $(-\infty, -1), (-1, -\frac{1}{3}), (-\frac{1}{3}, 0)$ and $(0, \infty)$. The most systematic way to do this is to make a table: We see from the table that $f(x) > 0$ precisely

| | $x$ | $x+1$ | $3x+1$ | $2x^2-2x+1$ | $f(x)$ |
|---|---|---|---|---|---|
| $x < -1$ | $-$ | $-$ | $-$ | $+$ | $-$ |
| $-1 < x < -\frac{1}{3}$ | $-$ | $+$ | $-$ | $+$ | $+$ |
| $-\frac{1}{3} < x < 0$ | $-$ | $+$ | $+$ | $+$ | $-$ |
| $x > 0$ | $+$ | $+$ | $+$ | $+$ | $+$ |

when $-1 < x < -\frac{1}{3}$ and $x > 0$.

**7 C.**

The cubic factorizes as $(x-2)(x^2+x+1)$. The discriminant of the quadratic is $-3$, so it has no real roots. Therefore the quadratic has exactly one real root.

**8 D.**

Let £$a$ be the price of the book in Shop $A$ and let £$b$ be the price of the book in Shop $B$. We have that $a = 0.8b$. During the summer sale, the price of the book in Shop $B$ is $0.7b$, and the price of the book in Shop $A$ is $a-3$. Now since the price in Shop $A$ is always 80% of the price in Shop $B$, we have: $a-3 = 0.8(0.7b)$, so that:

$$0.8b - 3 = 0.8(0.7b)\,.$$

Therefore, $0.24b = 3$, and we conclude that $b = 12.5$.

**9 B.**

After choosing a pencil from the box, we are left with $4n-1$ pencils in the box, of which $3n$ are of a different colour to the one just chosen. Therefore, the probability that the second pencil chosen is of a different colour to the first one is $\frac{3n}{4n-1}$.

**10 D.**

First note that since $2 < \pi$, we can take $\log_2$ on both sides to get $1 < \log_2 \pi$ and also $\log_\pi$ on both sides to get $\log_\pi 2 < 1$. Therefore $\log_\pi 2 < \log_2 \pi$, so that $e^{\log_\pi(2)} < e^{\log_2(\pi)}$. This eliminates **A**. Similar reasoning applied to the inequality $e < \pi$ tells us that $2^{\log_\pi(e)} < 2 < 2^{\log_e(\pi)}$. This eliminates **B** and **E**. It remains to compare $2^{\log_e \pi}$ and $e^{\log_2 \pi}$. Notice that $2 < e$, so the same reasoning again gives $1 < \log_e \pi < \log_2 \pi$. Combining this with $2 < e$ shows that $2^{\log_e \pi} < e^{\log_2 \pi}$, so that $e^{\log_2 \pi}$ is the largest.

**11 A.**

We make the replacements $v \mapsto 1.2v$, $w \mapsto 3w$, $x \mapsto 0.8x$ and $y \mapsto 0.8y$, so that

$$z \mapsto \frac{3w(0.8x + 2\times 0.8y)^3}{(1.2v)^2} = \frac{3(0.8)^3}{(1.2)^2} \times \frac{w(x+2y)^3}{v^2} = \frac{16}{15}\cdot z = (1.0667)z$$

Thus $z$ increases by 6.67%.

**12 C.**

Squaring both sides of the equation and rearranging gives $x^2 + x = 2x\sqrt{2x-1}$. Squaring both sides again and factorizing gives

$$x^2(x-5)(x-1) = 0.$$

So the possible solutions are $0, 1$ and 5. However, note that 0 cannot be a solution to the original equation, which therefore only has roots 1 and 5. The product of the roots is therefore 5.

**13 A.**

The exterior angles of a regular polygon with $n$ sides is equal to $\frac{360}{n}$. From the given information

$$\frac{360}{n} = \frac{360}{n+2} + 2\,.$$

Therefore $n^2 + 2n - 360 = 0$, which implies that $n = -20$ or $n = 18$. Since the number of sides of a regular $n$-gon must be positive, we conclude that $n = 18$.

**14 E.**

There are 10 numbers from 1 to 100 (inclusive) that end in 1. Therefore there are $10 \cdot 10 = 100$ numbers from 1 to 1000 ending in 1. The required probability is therefore $\frac{100}{1000} = \frac{1}{10}$.

**15 D.**

The triangle has sides of length $x, 1+x$ and hypotenuse $2x$, for some $x$. By Pythagoras' Theorem:

$$x^2 + (x+1)^2 = (2x)^2,$$

So that $2x^2 - 2x - 1 = 0$. Solving this quadratic and discarding the negative root tells us that $x = \frac{1+\sqrt{3}}{2}$. The area of the triangle is therefore equal to

$$\frac{1}{2} \cdot \left(\frac{1+\sqrt{3}}{2}\right) \cdot \left(\frac{3+\sqrt{3}}{2}\right) = \frac{3+2\sqrt{3}}{4}\,.$$

**16 E.**

**A** is false as 2 is a counterexample. **B** is false as 3 is a counterexample. **C** is false as $p = 5, q = 3$ is a counterexample. **D** is false as 19 is a counterexample. By process of elimination, **E** must be true. To see this directly, note that if two such primes existed then they would both be odd and their sum would be even, and in particular would never equal 99.

**17 A.**

Direct calculation using standard values of trigonometric functions:

$$1 + \frac{3\tan(30°)}{\sin(60°) + 3\cos(60°)} + \frac{5}{\cos(45°) + \sqrt{3}} = 1 + (\sqrt{3} - 1) + (2\sqrt{3} - 2) = 3\sqrt{3} - \sqrt{2}\,.$$

**18 E.**

The total age of all people in Group $A$ is $10a$ and the total age of all people in Group $B$ is $10x^2$. After exchanging people, Group $A$ has a total age of $10x - x + x^2 = x^2 + 9x$ and Group $B$ has a total age of $10x^2 - x^2 + x = 9x^2 + x$. We have that $4(9x + x^2) = 9x^2 + x$, so that $5x^2 - 35x = 0$. Since $x$ is non-zero, we conclude that $x = 7$.

**19 B.**

If we take $x$ to be any positive integer less than 5, then the given expression equals zero and is not negative. This means that **A** and **E** can be ruled out. Now suppose $x > 5$. Then the value inside each bracket is negative. For the entire expression to be negative, there must be an odd number of negative terms. Consider the case when $n$ is a multiple of 3, and let $n = 6$. Then all the brackets, and hence the entire expression, will be positive. This counterexample allows us to eliminate **C**. A similar argument shows that $n$ cannot be even, eliminating **D**. By process of elimination we see that the correct option must be **B**.

**20 C.**

We can eliminate **A** by noting that $y$ takes negative values and the graph does not show any. We can eliminate **B** by noting that $y$ does not have any vertical asymptotes. Note that $y$ is an even function, so its graph must be symmetric about the $y$-axis. Thus, we can eliminate **D**. Finally, the value of $y$ at $x = 0$ is 1, not 0, so we can eliminate **E**. By process of elimination, we conclude that the correct graph must be **C**.

## SECTION 2
## MATHEMATICAL THINKING

**21 E.**

The expansion is a sum of terms of the form

$$3 \cdot {}^5C_r \cdot 2^r \cdot x^{2r-3} \quad \text{with } 0 \leq r \leq 5.$$

This expression is a term in $x^5$ when $2r - 3 = 5$, i.e. r=4. So the coefficient of $x^5$ in the expansion is $3 \cdot {}^5C_4 \cdot 2^4 = 3 \cdot 5 \cdot 16 = 240$.

**22 E.**

We can expand the numerator and simplify to arrive at

$$y = x^2 + 6x^{\frac{3}{2}} + 12x + 8x^{\frac{1}{2}}.$$

Differentiating this gives $\frac{dy}{dx} = 2x + 9x^{\frac{1}{2}} + 12 + 4x^{-\frac{1}{2}}$.

**23 D.**

If $r$ denotes the common ratio, then $7\sqrt{2}r^3 = \frac{7}{2}$, which implies that $r = \frac{1}{\sqrt{2}}$. Substituting this into the formula for the sum to infinity of a geometric sequence gives $\frac{14}{-1+\sqrt{2}}$.

**24 B.**

Writing each of the numbers $6, 3, 18, 8$ and 12 as a product of primes and collecting powers shows that the given expression is equal to

$$\frac{2^{2n+m} \times 3^{6n+2m}}{2^{2n+11m} \times 3^{n+m}}.$$

This further simplifies to $2^{-10m} \times 3^{3n}$. This is an integer whenever $n > 0$ and $m < 0$.

**25 C.**

$N = 101$ is prime and $N = 3002$ does not have three digits, so they cannot be counterexamples to the conjecture. On the other hand, 205 is a positive integer with three digits, and the sum of its digits is a prime number. Also, 205 is not prime, as it is divisible by 5. So it is a valid counterexample.

**26 D.**

For the given range of values of $x$, we have both $\log_4 x$ and $\log_7 x$ are negative, $x^2$ is less than 1 and $\sin x$ is (always) less than 1. Therefore, since $e^x > 1$ for $0 < x < 1$, it must be the largest on this domain.

**27 C.**

The sum of digits of all numbers from 0 to 999 inclusive is a sum of the numbers $0, 1, \ldots, 9$. We must determine how many time each number appears in this sum. Note that for any integer $k$ in $0, 1, \ldots, 9$ there are 100 numbers from 0 to 999 with hundreds digit $k$, 100 numbers with tens digit $k$ and 100 with units digit $k$. So $k$ appears in the sum 300 times. The required sum of digits is therefore:

$$300 \cdot (1 + 2 + 3 + \ldots + 9) = 300 \cdot 45,$$

and this equals 13500.

**28 C.**

If $a = b$, then $a^2 = ab$, which implies that $a^2 + a^2 = a^2 + ab$, so that $2a^2 = a^2 + ab$. Therefore **I** is correct. Subtracting $2ab$ from both sides now shows that **II** is also correct. However, $a^2 - ab = 0$, so cancelling it from both sides is not allowed. Therefore **III** (and only **III**) is incorrect.

**29 E.**

Direct computation shows that $u_1 = 3$, $u_2 = 0$, $u_3 = 1.5$ and $u_4 = 3$. Thus the sequence is periodic with period 3. The sum of the first 99 terms is $33 \cdot (3 + 0 + 1.5) = 148.5$ and therefore the sum of the first 100 terms is $148.5 + 3 = 151.5$.

**30 B.**

The original cube has surface area equal to $6 \cdot 9^2$ cm$^2$. Subdividing the cube creates $3^3 = 27$ smaller cubes, each with sides of length 3cm. The surface area of each of these smaller cubes is $6 \cdot 3^2$ cm$^2$. Therefore

$$\frac{\text{total surface area of smaller cubes}}{\text{surface area of original cube}} = \frac{6 \cdot 3^2 \cdot 3^3}{6 \cdot 9^2} = 3\,.$$

The total surface area has therefore increased by a factor of 3.

**31 C.**

If $c = 1$ then the line $y = mx + c$ passes through the point $(0, 1)$. A unique solution to the equation exists precisely when the line is tangent to the curve of $y = e^x$ at the point $(1, 0)$. This happens precisely when $m$ equals the gradient of $y = e^x$ at $(0, 1)$, i.e. $m = 1$.

**32 A.**

Let $A > 0$ denote the common area. The idea is to calculate the perimeter of each shape in terms of $A$. First consider the triangle, whose side length we denote $l$. Subdividing the triangle into two right-angled triangles and using Pythagoras shows that the height of the triangle is $\frac{l\sqrt{3}}{2}$, so that the area $A$ equals $\frac{l^2\sqrt{3}}{4}$. Therefore, $l = 2\sqrt{\frac{A}{\sqrt{3}}}$, which implies that

$$p = \frac{6}{\sqrt[4]{3}}\sqrt{A}\,.$$

Next consider the square, whose side length we denote $x$. We have: $A = x^2$, so that $x = \sqrt{A}$. Therefore

$$q = 4\sqrt{A}\,.$$

Finally for the circle, whose radius we denote $\rho$, we have $A = \pi\rho^2$, so that $\rho = \sqrt{\frac{A}{\pi}}$. We obtain

$$r = 2\sqrt{\pi}\sqrt{A}\,.$$

In order to compare $p, q$ and $r$, it is enough to compare $\frac{6}{\sqrt[4]{3}}, 4$ and $2\sqrt{\pi}$. Since $\pi < 4$, we must have $r < q$. Moreover, $\frac{6}{\sqrt[4]{3}} > 4$ if and only if $81 > 48$ (which is true). So $p > q$. We conclude that $p > q > r$.

**33 A.**

The equation can be rewritten as $m(m + n) = 6(m + 3p)$. The left hand side must therefore be divisible by 6, so at least one of $m$ or $m + n$ must be divisible by 3. Setting $m = n = p = 0$ gives a counterexample for **B**. Setting $m = 6, n = 3, p = 1$ gives a counterexample for **C**. Setting $m = p = 1$ and $n = 23$ gives a counterexample for **D**. Finally, since **A** is true, **E** must be false.

**34** **B.**

The correct implication is **I** $\Rightarrow$ **II**, but not conversely (since **I** only refers to the positive square root).

**35** **B.**

As $x$ ranges over all real numbers, so too does $8x-7$. Therefore $0 \leq \cos^2(8x-7) \leq 1$, so that $-6 \leq 9\cos^2(8x-7)-6 \leq 3$. We conclude that the maximum value attained by $\left(9\cos^2(8x-7)-6\right)^2$ is $(-6)^2 = 36$.

**36** **C.**

We note that $\sin^2(x) \geq \frac{1}{4}$ when $\sin(x) \geq \frac{1}{2}$ and $\sin(x) \leq \frac{1}{2}$. These correspond to the intervals $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ and $\frac{7\pi}{6} \leq x \leq \frac{11\pi}{6}$. We also have that $\cos(x) \geq \frac{1}{2}$ for $0 \leq x \leq \frac{\pi}{3}$ and for $\frac{5\pi}{3} \leq x \leq 2\pi$. Therefore, the intervals on which both conditions are met are $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$ and $\frac{5\pi}{3} \leq x \leq \frac{11\pi}{6}$. The total length of the intervals is $\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+\left(\frac{11\pi}{6}-\frac{5\pi}{3}\right) = \frac{\pi}{3}$.

**37** **D.**

Direct calculation: $f(n+2) = f((n+1)+1) = 1 - f(n+1)^2 = 1 - \left(1-f(n)^2\right) = 2f(n)^2 - f(n)^4$.

**38** **C.**

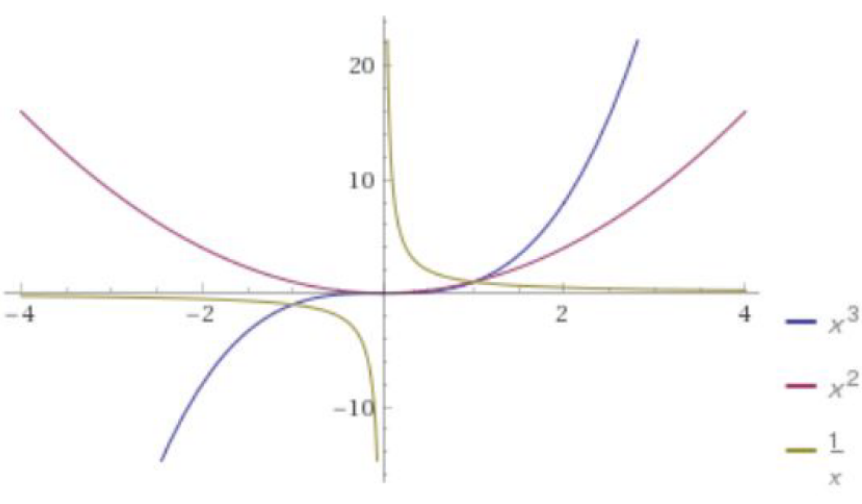
The shortest distance between the circle and the point is the distance between he centre of the circle and the point minus the radius. The circle has centre $(2,-3)$ and radius 3, so the shortest distance is $\sqrt{(2-6)^2+(-3-1)^2} - 3 = -3 + 4\sqrt{2}$.

**39** **B.**

This can be done by trial and error. We see that **C** and **D** violate the second inequality, while **A** and **E** violate the first.

**40** **D.**

Plotting the curves on the same set of axes gives the following graph:




The graphs of $y = \frac{1}{x}$ and $y = x^2$ intersect at $(1,1)$. The graphs of $y = \frac{1}{x}$ and $y = x^3$ intersect at $(0,0)$ and $(1,1)$. We see from the diagram that the graphs subdivide the plane into 8 regions.

**END OF SOLUTIONS**