1) If $f(x)$ satisfies $f(x) = f(x^2)$ for all real x, which of the following must be true:

A $f(4) = f(2)f(2)$

B $f(16) - f(-2) = 0$

C $f(0) = 0$

D $f(-2) + f(4) = 0$

$f(2) = f(2^2) = f(4)$
$f(4) = f(16)$
$f(-2) = f(4)$
$f(0) = f(0)$

Look for counterexamples

2) If $a < 0$ and $b < c$, which of the following must be true?

A $ab < c$ × $a=-1$, $b=-2$, $c=1$

B $ac > b$ × $a=-1$, $b=2$, $c=3$

C $ab > 0$ × $a=-1$, $b=2$, $c=3$

D $ac < 0$ × $a=-1$, $b=-2$, $c=-1$

E $ab > ac$

3) Given that p and q are integers, and that three of the following statements are true, which is the false statement?

A pq is even $\Rightarrow$ at least one of p, q is even

B $p + q$ is even $\Rightarrow$ p, q are both even or both odd

C $2p + q^2$ is odd $\Rightarrow$ q is odd

D $p^2 + 2q$ is odd $\Rightarrow$ p is odd

A+B $\Rightarrow$ p, q both even
$\Rightarrow$ C+D not true
So A or B must be false
$\Rightarrow$ C+D are true
$\Rightarrow$ p, q both odd
$\Rightarrow$ A is false

4) Consider the following conjecture:

"If N is a positive integer that consists of the digit 2 followed by an even number of 1 digits, then N is a prime number"

Here are three numbers:

I $N = 21$ (which equals 3x7) not even number of 1's

II $N = 211$ (a prime number) not counterexample

III $N = 21111$ (which equals 227 x 93) ✓

Which of these provide(s) a counterexample to the original conjecture?

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

5) Which of the following is a necessary and sufficient condition for

$$\sum_{k=1}^{n} tan(\frac{k\pi}{3}) = 0$$

$\tan\frac{\pi}{3}$ $\tan\frac{2\pi}{3}$ $\tan\pi$ ...

$\sqrt{3}$ $-\sqrt{3}$ 0 $\sqrt{3}$ $-\sqrt{3}$ 0

A n = 3

B n is a multiple of 3

C n is a multiple of 6

D n is a multiple of 3 or n is 1 less than a multiple of 3

E n is a multiple of 6 or n is 1 less than a multiple of 6

multiple of 3 or 1 less than multiple of 3

6) The real numbers *a, b, c, d* satisfy

$0 < a + b < c + d$ and $0 < a + c < b + d$

Which of the following must be true:

Either prove statement is true for all values

Or find a single counterexample to prove false

I $a < d$ ✓ $2a+b+c < 2d+b+c$

II $b < c$ × $a=1$ $b=6$ $c=5$ $d=9$

III $a + b + c + d > 0$ ✓

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

7) The arithmetic mean of five consecutive integers is an odd integer. Does $A \Rightarrow$ I / II / III ?

A

Which of the following must be true?

I The largest of the integers is even $\times$ $n+2$ is odd

II The sum of the integers is odd ✓

III The difference between the largest and smallest of the integers is even. ✓

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

$$\frac{n-2+n-1+n+n+1+n+2}{5} = n \quad n \text{ is odd}$$

o e o e o

o − o = e

8) Consider the following statement: Need $f(f(x)) = x$ but $f(x) \neq x$

If $f(f(x)) = x$ **then** $f(x) = x$

Which function provides a counterexample:

A $f(x) = 1$ B $f(x) = x$ C $f(x) = \dfrac{1}{x}$ D $f(x) = x^2$

$f(f(x)) = 1 \neq x$ $\times$

$\times$

$ff(x) = \frac{1}{1/x} = x$ ✓

$f(f(x)) = x^4 \neq x$

9) If *a, b* and *c* are integers, consider the statement $\dfrac{ab^2}{c}$ is a positive even integer (*)

Which of the following is a necessary but not sufficient condition for (*) to be true $* \Rightarrow$ I, II, III ?

I ab is even ✓ II $ab > 0$ III c is even

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

Consider the contrapositive

$* \Rightarrow$ I : not I $\Rightarrow$ not *

ab odd $\Rightarrow$ a, b odd

$\Rightarrow ab^2$ odd $\Rightarrow$ c odd $\Rightarrow$ * odd

$ab < 0$ $a > 0$ $b < 0$ $ab^2 > 0$

$a = 2$ $b = -1$ $c = 1$

$\frac{ab^2}{c} = 2$ not necessary

$a = 2$ $b = -1$ $c = 1$

$\frac{ab^2}{c} = 2$ not necessary

10) For any real numbers a, b, and c where $a \geq b$, which of these three statements **must** be true?

I $\quad -b \geq -a$ ✓ $\times$ by -1

II $\quad a^2 + b^2 \geq 2ab$ ✓ $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$

III $\quad ac \geq bc$ no if c negative

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

remainder < divisor

11) If x, a and b are positive integers such that when x is divided by a the remainder is b and when x is divided by b the remainder is $(a-2)$, then which of the following must be true?

$x = pa + b \qquad x = qb + a - 2$

$b < a \qquad a - 2 < b$

$a < b + 2$

A $\quad a$ is even

B $\quad b = a - 1$

C $\quad x + b$ is divisible by a

D $\quad a + 2 = b + 1$

$b < a < b + 2$

if a, b are integers there's only one integer between b and $b+2$

$a = b + 1 \quad b = a - 1$

12) Consider the following statement about positive integers a, b, c and d

If a divides $(b + c)$ and a divides $(c + d)$, then a divides $(b + d)$

Which of the following provide(s) a counterexample to the original conjecture?

I $\quad a = 3, b = 4, c = 5, d = 8$ a not $| \, c+d$

II $\quad a = 5, b = 3, c = 2, d = 8$ ✓

III $\quad a = 2, b = 3, c = 5, d = 7$ example not counterexample

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

13) If $\left|\frac{x}{4}\right| > 1$ which of the following must be true? which of these is necessary for A?

A $\Rightarrow$ I II III

I/II/III is necessary for A

I $\quad x > 4$ sufficient $\qquad$ II $\quad x \neq 4$ true $\qquad$ III $\quad x < -4$ sufficient

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

Solving inequality: $x > 0$, $|x| = x$, $\frac{x}{4} > 1$, $x > 4$ $\qquad$ $x < 0$, $|x| = -x$, $-\frac{x}{4} > 1$, $x < -4$

so $x < -4$ or $x > 4$

14) Consider the statement:.

(*) A whole number n is prime if it 1 less or 5 less than a multiple of 8.

How many counterexamples to (*) are there in the range $0 < n < 50$

A $\quad$ 2

B $\quad$ 3

C $\quad$ 4

D $\quad$ 5

E $\quad$ 6

3 11 19 27 35 43

7 15 23 31 39 47

15) A cubic polynomial is given by $f(x) = x^3 + bx^2 + cx + d$ where b, c and d are constants.

Two of its factors are $(x - 1)$ and $(x + 1)$

Which of the following statements, taken independently, is/are **necessarily** true?

I $\quad$ If $f(0) = k$ then $f(k) = 0$

II $\quad f(x) = x^3 - x$

III $\quad$ The graph of f(x) is symmetrical in the y-axis.

A $\quad$ none of them

B $\quad$ I only

C $\quad$ II only

D $\quad$ III only

E $\quad$ I and II only

F $\quad$ II and III only

G $\quad$ I and III only

H $\quad$ I, II and III

$f(1) = 1 + b + c + d = 0$

$f(-1) = -1 + b - c + d = 0$

$2 + 2c = 0 \quad c = -1$

$b = -d$

$f(x) = x^3 + bx^2 - x - b$

1. $f(0) = k \quad -b = k$

$f(k) = k^3 - k^3 - k + k = 0$ ✓

2. sufficient, but not necessary ×

$b = 0$

3. $f(-x) = -x^3 + bx^2 + x - b \neq f(x)$ ×

16) Consider the following statements about the polynomial $p(x)$ where a is a constant

I $\quad p(a) = 0$ and $p'(a) = 0$.

II $\quad p'(a) = 0$ and $p''(a) > 0$

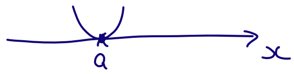
III $\quad p''(a) < 0$ and $p(a) > 0$

Which of these statements are sufficient for $p(x)$ to have a local maximum point at $x = a$?

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

Does I, II, III $\Rightarrow$ max at $x = a$

I : root + stat point could be minimum ✗



II : stat. point convex must be minimum point ✗

III Concave, point above x-axis not necessarily a stat. point ✗

Does I II III $\Rightarrow$ (*)

17) Which of the following conditions is sufficient but not necessary for $\frac{x}{|x|} < x$

I $\quad x > 1$ $\qquad$ II $\quad x > -1$ $\qquad$ III $\quad |x| < 1$

or $-1 < x < 1$

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

I $\quad x > 1 \quad |x| = x$

$\frac{x}{|x|} = 1$

$\Rightarrow x > \frac{x}{|x|}$ (*)

(not necessary eg $x = -\frac{1}{2}$) ✓

II Counterexample $x = \frac{1}{2}$

III Counterexample $x = \frac{1}{2}$

Consider $x > 0$ $\quad |x| = x \quad \frac{x}{x} < x \quad x > 1$

Consider $x < 0$ $\quad |x| = -x \quad \frac{x}{-x} < x \quad -1 < x < 0$

18) Consider the statement: $\qquad f(x) > x$ for all real values of $x > 1$

Which one of the following is a negation of this statement?

A $\quad f(x) \le x$ for all real values of $x \le 1$
B $\quad f(x) \le x$ for all real values of $x > 1$
C $\quad f(x) \le x$ for at least one real value of $x \le 1$
D $\quad f(x) \le x$ for at least one real value of $x > 1$
E $\quad f(x) > x$ for at least one real value of $x \le 1$
F $\quad f(x) > x$ for at least one real value of $x > 1$
G $\quad f(x) > x$ for no real values of $x \le 1$
H $\quad f(x) \le x$ for no real values of $x > 1$

Statement

For all $x > 1$ then $f(x) > x$

Negation

There exists $x > 1$ such that NOT $f(x) > x$

There exists $x > 1$ such that $f(x) \le x$

19) A set of cards has a single letter or number on each side.

Five of these cards are laid on the table so that only one side of each card is visible.

The cards show the following: Q K 3 6 8

Dan states that all cards with an Q on one side have an even number on the reverse.

Which cards do you need to turn over in order to check his statement? If Q then even

A card Q only — Q ✓ check for even number

B cards Q and 3 — K × irrelevant

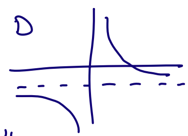
C cards Q and 6 and 8 — 3 ✓ check for Q - this could be a contradiction

D cards K and 3 — 6 × can be Q or not Q

E all of the cards — 8 × can be Q or not Q

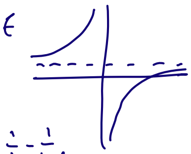
20) Consider the following statement:

D



$\frac{4}{x}-1$

E



$\frac{1}{4}-\frac{1}{4x}$

If $f'(x) > 0$ *for all real* x *then* $f(x+1) > f(x)$ *for all real* x

Which function provides a counterexample:

Need $f'(x) > 0$ but $f(x+1) \leq f(x)$ for some x

A $f(x) = 4^x$ B $f(x) = 4x^2 + 1$ C $f(x) = 4x^3$

D $f(x) = \dfrac{4-x}{x}$ E $f(x) = \dfrac{x-1}{4x}$

$f'(x) > 0$ (+ve gradient) : A ✓ B × $(x<0)$ C × $(x=0)$ D × E ✓

$f(x+1) > f(x)$ (strictly increasing) ✓

eg $\overset{\times}{-\frac{1}{2}}, \frac{1}{2}$

21) x and y are non-zero real numbers. Consider the three statements below:

I $x > y$ **if** $\dfrac{x}{y} > 1$ — If $\frac{x}{y} > 1$ then $x > y$

II $\dfrac{x}{y} > 1$ **if and only if** $\dfrac{y}{x} < 1$ — $\Rightarrow$ only if yes — $\Leftarrow$ if no

III **If** $xy < 1$ **then** both $x < 1$ and $y < 1$

Which of these statements, taken independently, is/are true?

A none of them

B I only

C II only

D III only

E I and II only

F II and III only

G I and III only

H I, II and III

I $x=-4$, $y=-2$ $\frac{x}{y} = 2$ but $x < y$ ×

II $\frac{x}{y} > 1 \Rightarrow x, y$ both +ve or both -ve $\Downarrow$ $\frac{y}{x} < 1$ ✓ $x < y$, $1 > \frac{y}{x}$ If $\frac{y}{x} < 1$ can have y -ve and x +ve so $\frac{x}{y} \not> 1$ ×

III $x = \frac{1}{12}$, $y = 3$ ×

22) Jill, Kate and Lara each wear a hat. There are three hats: one black, one red, and one blue.

It is known that:

1) If Jill wears black, then Kate wears blue.

2) If Jill wears red, then Lara wears blue.

3) If Kate does not wear red, then Lara wears black.

J	Black	Red	Blue	
K	Blue	Black	Red	Black
L	Red	Blue	Black	Black
	3) x	3) x	✓	3) x

this combination is the only one that works even though no statements apply

What is the colour of the hat Kate is wearing?

A black

B red

C blue

D there is insufficient information to answer the question

Need $f(x) > 0$ for $x \geq 0$ but $f'(x) \leq 0$ for some $x \geq 0$

23) Consider the following statement:

If $f(x) > 0$ for all $x \geq 0$, **then** $f'(x) > 0$ for all $x \geq 0$

A/B $(x \pm \frac{3}{2})^2 + \frac{7}{4} > 0$

Which function provides a counterexample:

A $f(x) = x^2 + 3x + 4$ > 0

B $f(x) = x^2 - 3x + 4$ > 0

C $f(x) = x^2 + 3x - 4$ $f(0) < 0$

D $f(x) = x^2 - 3x - 4$ $f(0) < 0$

	A	B	C	D
$f(x) > 0$	✓	✓	x	x
$f'(x) > 0$	✓	x	✓	x

A: $f'(x) = 2x + 3$
≥ 0 for $x \geq 0$

B: $f'(x) = 2x - 3$
$f'(0) = -3$

A is sufficient for I/II/III

A need $A \Rightarrow$ I, II, III I/II/III is necessary for A

24) If $f(x) = \dfrac{x}{x+1}$ for all integers $x \neq -1$, which of the following must be true?

I $f(x+1) > f(x)$ True

II $f(x) > 0$ No when $x = 0$

III $f(x) \neq 0$ No when $x = 0$

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

I $f(x+1) = \dfrac{x+1}{x+2}$

Consider $f(x+1) - f(x)$ - is this > 0 ?

$$= \frac{x+1}{x+2} - \frac{x}{x+1} = \frac{(x+1)^2 - (x^2+2x)}{(x+2)(x+1)} = \frac{1}{(x+2)(x+1)} > 0$$

all $x \in \mathbb{Z}$
$x \neq -1, -2$

25) A set P of integers is called a *closed set under addition* **if and only if** for any integer a in set P, *there exists* an integer k in P, such that k is the sum of a and b *for all* integers b *which are in* P.

Q R S

Which of the following is true **if and only if** P is **not** *closed under addition*?

A There exists an integer a in P, such that for any integer k in P, and any integer b in P, k is not the sum of a and b.

B There exists an integer a in P, and an integer k in P, for which there is no integer b in P, such that k is the sum of a and b.

C There exists an integer a in P, such that for any integer k in P, there is no integer b in P, such that k is the sum of a and b.

D There exists an integer a in P, such that for any integer k in P, there is an integer b in P, such that k is not the sum of a and b.

E For any integer a in P, there exists an integer k in P, and an integer b in P, such that k is not the sum of a and b.

F For any integer a in P, there exists an integer k in P, and an integer b in P, such that k is the sum of a and b.

G For any integer a in P, there exists an integer k in P, such that for any integer b in P, k is not the sum of a and b.

H For any integer a in P, and any integer k in P, there is no integer b in P, such that k is the sum of a and b.

statement

for any Q there exists R such that S

negation

there exists Q such that not 'there exists R such that S'

" such that for all R not S

rules out EFGH

rules out B

not S = k is not sum of a and b for some b in P

rules out A, C