TMUA/MAT Graphs of Functions

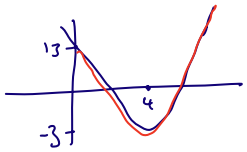
Syllabus

Sketch common functions; transformations of graphs; stationary points / increasing / decreasing functions; intersection with coordinate axes / number of roots; graphs and simultaneous equations.

1. Sketch each of the following functions and find the range for the given domains:

a) $f(x) = x^2 - 8x + 13 \qquad x \in \mathbb{R} \quad x > 0$

$= (x-4)^2 - 3$

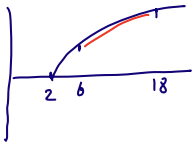


$f(x) > -3$

b) $f(x) = \sqrt{x-2} \qquad x \in \mathbb{R} \quad 6 < x < 18$

$x = 6 \quad f(x) = 2$

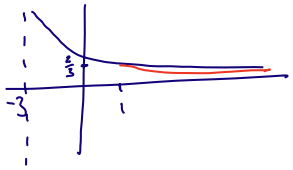
$x = 18 \quad f(x) = 4$



$2 < f(x) < 4$

c) $f(x) = \dfrac{2}{x+3} \qquad x \in \mathbb{R} \quad x \geq 1$

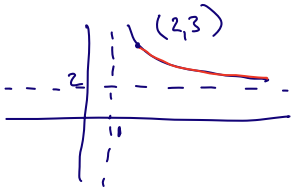
$x = 1 \quad f(x) = \frac{1}{2}$



$0 < f(x) \leq \frac{1}{2}$

d) $f(x) = \dfrac{1}{x-1} + 2 \qquad x \in \mathbb{R} \quad x > 2$

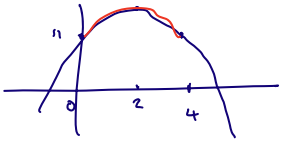
$x = 2 \quad f(x) = 3$



$2 < f(x) < 3$

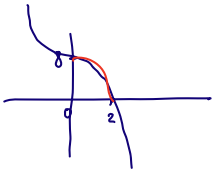
e) $f(x) = 15 - (x-2)^2 \qquad x \in \mathbb{R} \quad 0 \leq x \leq 4$

$f(0) = 11 \quad f(4) = 11$



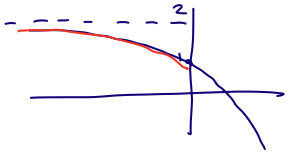
$11 \leq f(x) \leq 15$

f) $f(x) = 8 - x^3 \qquad x \in \mathbb{R} \quad 0 \leq x \leq 2$



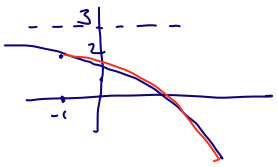
$0 \leq f(x) \leq 8$

g) $f(x) = 2 - e^x \qquad x \in \mathbb{R} \quad x \leq 0$



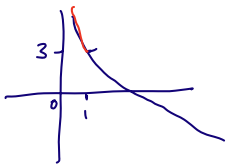
$1 \leq f(x) < 2$

h) $f(x) = 3 - e^{x+1} \qquad x \in \mathbb{R} \quad x \geq -1$



$f(x) \leq 2$

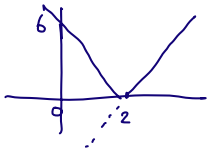
i) $f(x) = 3 - \ln x \qquad x \in \mathbb{R} \quad 0 < x < 1$



$f(x) > 3$

2. Sketch each of the following graphs, stating any values of x for which the function is not defined

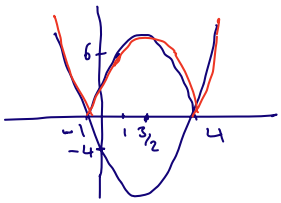
a) $y = |3x - 6|$



b) $y = |x^2 - 3x - 4|$

$(x - \frac{3}{2})^2 - \frac{9}{4} - 4$

$f(-1) = 0 \qquad f(4) = 0$

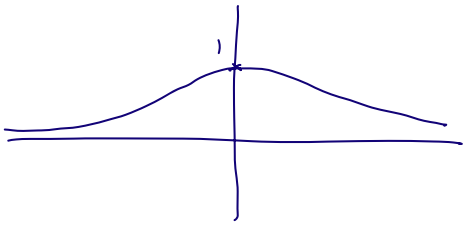


c) $y = \dfrac{1}{1 + x^2}$ Max when $(1+x^2)$ is min $(=1)$

As $x \to \pm\infty$ $y \to 0$

Even function

$(0, 1)$

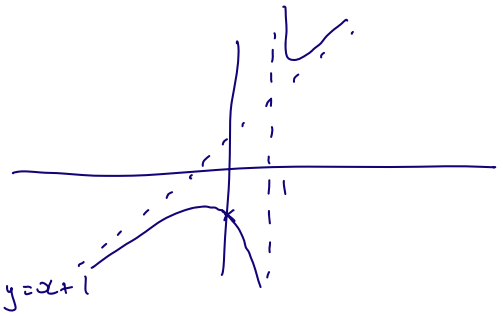


d) $y = \dfrac{x^2 + 1}{x - 1} = x + 1 + \dfrac{2}{x-1}$

Asymptote $x = 1$ Asymptote at $y = x + 1$

$(0, -1)$

Not defined for $x = 1$



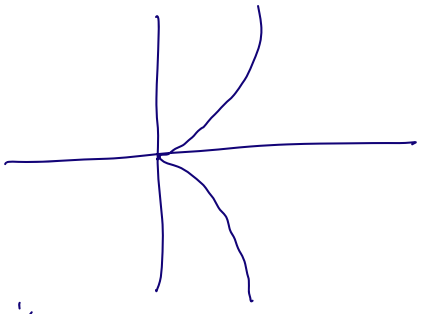
e) $y^2 = x^3$

$x \geq 0$ symmetry in x-axis

$(0, 0)$

Stat pt at $x = 0$

Not defined for $x < 0$

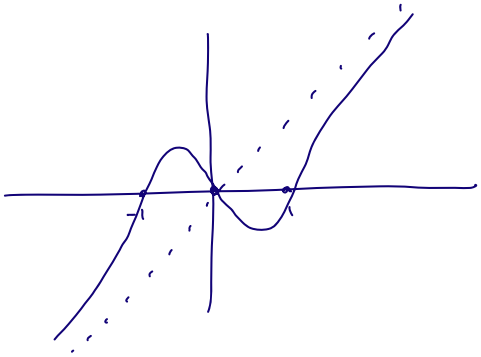


f) $y = \sqrt[3]{x^3 - x}$

odd function

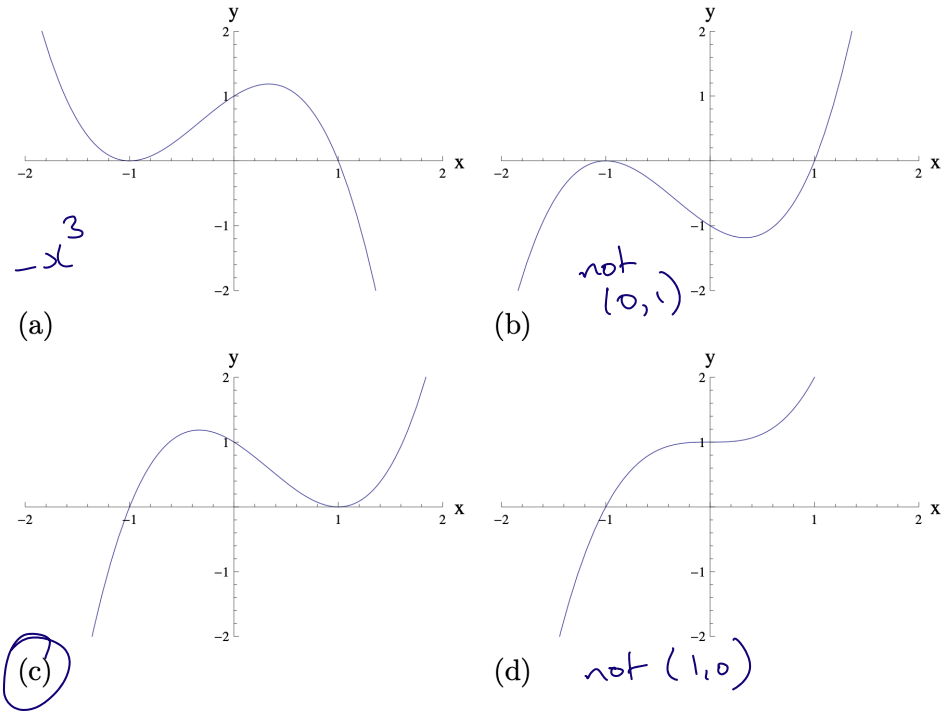
$y = 0 \quad x = 0 \quad x = \pm 1$

$x \to \infty \quad y \to x$



+ve cubic, (0,1), (1,0)

3a) A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?

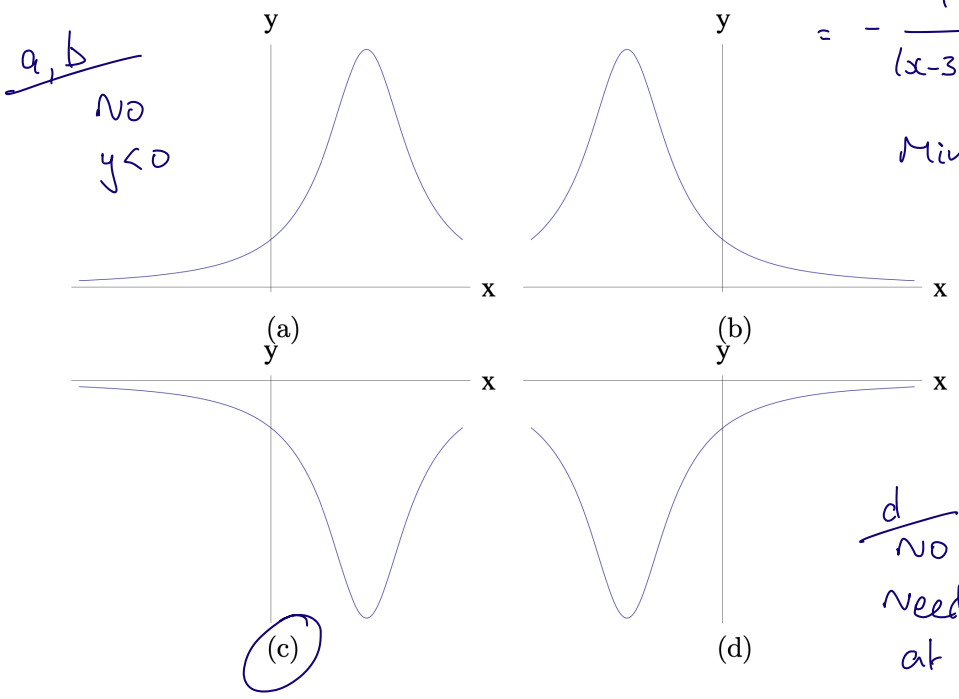


$-x^3$

not (0,1)

not (1,0)

b) Which of the following graphs is a sketch of $y = \dfrac{1}{6x - x^2 - 10}$ $= -\dfrac{1}{x^2 - 6x + 10}$



a, b

No

$y < 0$

$= -\dfrac{1}{(x-3)^2 + 1} < 0$

Min at $x = 3$

d

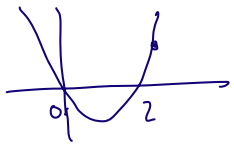
No

Need min at $x = 3$

4. Find the composite function $fg(x)$ and sketch this function.
State any values of x for which the function $fg(x)$ is not valid.

a) $f(x) = x^2 - 4$ $\quad$ $g(x) = 2x - 2$

$fg(x) = (2x-2)^2 - 4$
$= 4x^2 - 8x$
$= 4x(x-2)$



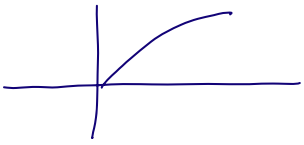
b) $f(x) = 2x^2 - 3$ $\quad$ $g(x) = \sqrt{x+4}$

$fg(x) = 2(x+4) - 3$
$= 2x + 5$

c) $f(x) = 2e^{\frac{1}{2}x}$ $\quad$ $g(x) = ln(4x)$

$fg(x) = 2e^{\frac{1}{2}\ln(4x)}$
$= 2(4x)^{1/2} = 4\sqrt{x}$

Not valid for $x < 0$



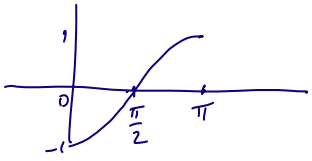
d) $f(x) = sin\,x$ $\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $\quad$ $g(x) = x - \frac{\pi}{2}$ $\quad x \ge 0$

Domain of $g(x)$
$x \ge 0 \Rightarrow g(x) \ge -\frac{\pi}{2}$

Domain of $f(x)$
$g(x) \le \frac{\pi}{2} \Rightarrow x \le \pi$

Combined domain $\quad 0 \le x \le \pi$

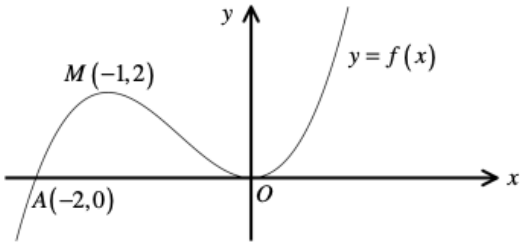
$fg(x) = \sin\left(x - \frac{\pi}{2}\right)$



Defined for
$0 \le x \le \pi$

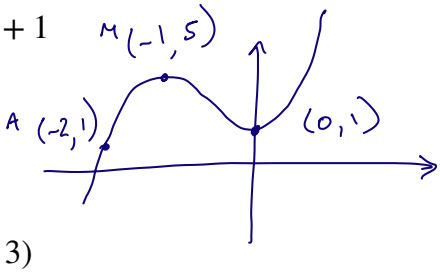
So range is $-1 \le fg(x) \le 1$

5. The figure shows the graph of the curve with equation $y = f(x)$

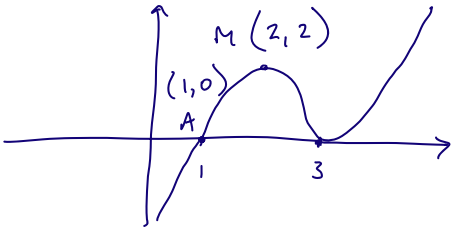


Sketch the graphs of the following functions and include the new coordinates of points A and M.

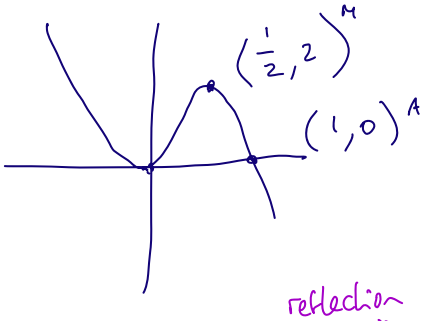
a) $y = 2f(x) + 1$



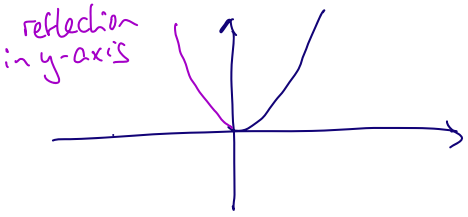
b) $y = f(x - 3)$



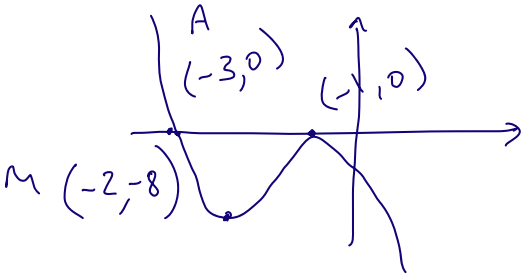
c) $y = f(-\frac{1}{2}x)$



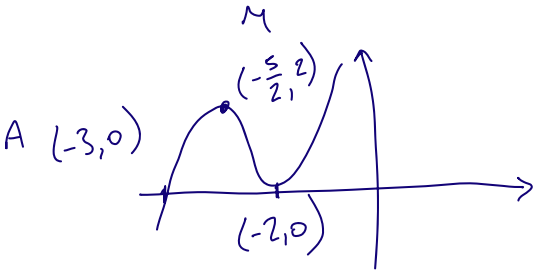
d) $y = f(|x|)$



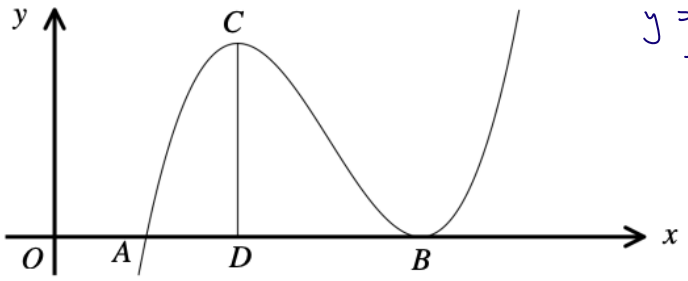
e) $y = -4f(x + 1)$



f) $y = f(2x + 4)$



6. The figure shows a cubic curve whose coefficient of x^3 is 1. The curve crosses the x-axis at A $(a,0)$ and touches the x-axis at B $(b,0)$ where a and b are positive constants such that $a < b$. The point C is a local maximum of the curve. Find the coordinate of D in terms of a and b.



$y = (x-a)(x-b)^2$

$= (x-a)(x^2 - 2bx + b^2)$

$= x^3 - ax^2 - 2bx^2 + 2abx + b^2x - ab^2$

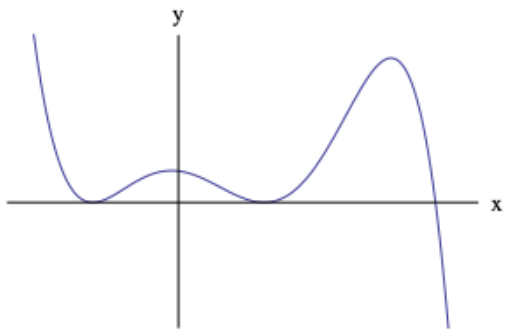
$\frac{dy}{dx} = 3x^2 - 2ax - 4bx + 2ab + b^2$

$= (x-b)(3x - b - 2a) = 0$

At C $3x = 2a+b$ $x = \frac{2a+b}{3}$

D $\left(\frac{2a+b}{3}, 0\right)$

7. Which one of the following equations could possibly be the graph below:



I $y = (3-x)^2(3+x)^2(1-x)$

II $y = -x^2(x-9)(x^2-3)$

III $y = (x-6)(x-2)^2(x+2)^2$

IV $y = (x^2-1)^2(3-x) = (x-1)^2(x+1)^2(3-x)$

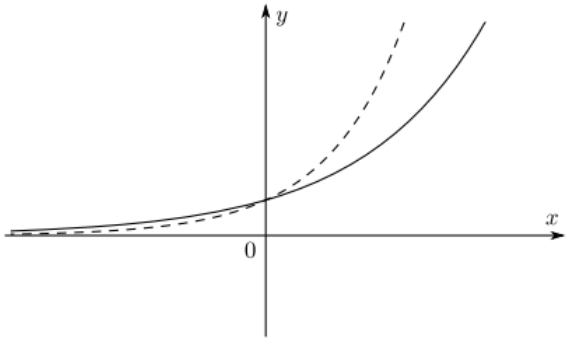
-ve quintic, 2 repeated roots

single root > repeated root

8. The graphs of two functions are shown.

$y = a^x$ is shown with a solid line where a is a positive real number.

$y = f(x)$ is shown with a dashed line



Which of the following could be true?

I $f(x) = b^x$ for some $b > a$ ✓

II $f(x) = b^x$ for some $b < a$ ✗

III $f(x) = a^{kx}$ for some $k > 1$ ✓

IV $f(x) = a^{kx}$ for some $k < 1$ ✗