TMUA Proof

Syllabus

Logic of arguments (true/false, and/or, if, then, not) converse, contrapositive, necessary sufficient; mathematical proof (deduction, proof by cases, proof by contradiction, disproof by counterexample); identifying errors in proofs.

Logic of Arguments

1. Consider the following statement: "If it is my birthday, I will eat some cake"

What conclusion can I draw from each of the following statements: birthday $\Rightarrow$ cake

x A It is my birthday — I will eat cake

B It is not my birthday — No conclusion - might eat cake / might not

C I eat some cake — No conclusion - might eat cake on another day so could be my birthday but might not be

x D I do not eat some cake — It is not my birthday (because if it was my birthday I would eat cake)

2. Consider the following statement: "If it rains the ground will get wet"

What conclusion can I draw from each of the following statements: rain $\Rightarrow$ wet

A The ground is wet — No conclusion - might be wet from rain or from a sprinkler

x B The ground is not wet — It has not been raining

x C It is raining — The ground will get wet

D It is not raining — No conclusion - ground might be wet or dry

3. Consider the following statement: "If I am in Paris, then I am in France"

What conclusion can I draw from each of the following statements:

x A I am in Paris — I am in France

B I am in France — no conclusion - might be in Paris or in Lille...

x C I am in London = I am not in France $\Rightarrow$ I am not in Paris

D I am at the Eiffel Tower $\Rightarrow$ I am in Paris $\Rightarrow$ I am in France

4. Consider the following statement: "If a shape is a square, then it is a quadrilateral"

What conclusion can I draw from each of the following statements:

x A The shape is a square — it is a quadrilateral

B The shape is a quadrilateral — no conclusion - might be a square or a parallelogram...

x C The shape is not a quadrilateral — it is not a square

D The shape is a rhombus — no conclusion

The following statements are all equivalent:

If an animal is a zebra, **then** it has stripes

All zebras have stripes

Any zebra has stripes

Being a zebra **implies** having stripes

An animal is a zebra **only if** it has stripes

An animal has stripes **if** it is a zebra

An animal with **no** stripes is **not** a zebra

Having stripes is **necessary** for an animal to be a zebra

Being a zebra is a **sufficient** condition for an animal to have stripes

Equivalent Quantifiers - for all A/ every A / any A / if A

- for some A / there exists A / for at least one A / for most A

5. Rewrite the following true statements in the form **If… Then …**

a) The ground gets wet when it rains — If it rains, then the ground gets wet

b) All mammals have hair — If it is a mammal, then it has hair

c) I always go to bed when I am sick — If I am sick, then I go to bed

d) A fruit is yellow if it is a banana — If it's a banana, then it is yellow

e) I am in Paris only if I am in France — If I'm in Paris, then I'm in France

6. Rewrite the following true mathematical statements in the form **If… Then …**

a) Any rectangle is a quadrilateral — If it's a rectangle, then it's a quadrilateral

b) Any triangle has 3 sides — If it's a triangle, then it has 3 sides

c) The number 2 is the only even prime number — If it's an even prime number, then it's 2
If it's 2, then it's an even prime number

d) $x > 10$ if $x > 100$ — If $x > 100$, then $x > 10$

e) $k < 1$ when $k^2 < 1$ — If $k^2 < 1$, then $k < 1$ (converse is not true)

f) $p^2 < p$ only if $p < 1$ — If $p^2 < p$, then $p < 1$ (converse is not true)

TMUA Style Multiple Choice Practice 'if…then'

1. a and b are real numbers and f is a function.
Given that exactly one of the following statements is true, which one is it?

Let P = $a<b$
Q = $f(a)<f(b)$

A **If** $a<b$ **then** $f(a)<f(b)$
B $a<b$ **only if** $f(a)<f(b)$
C $f(a)<f(b)$ is **sufficient** for $a<b$
D $f(a)<f(b)$ is **necessary** for $a<b$

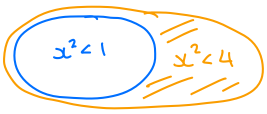
If P then Q
P only if Q = If P then Q
Q is sufficient for P = If Q then P
Q is necessary for P = If P then Q

2. Consider the four options below about a particular statement:

doesn't tell us anything about $x^2=3$ so C not necessarily true

A The statement is true if $x^2<1$ If $x^2<1$ then statement is true
-1 0 1
B The statement is true if and only if $x^2<1$ ⇒ B + A true ✗
C The statement is true if $x^2<4$ If $x^2<4$ then statement is true
-2 0 2 C ⇒ A
⇒ if $x^2<1$ statement is true
D The statement is true if and only if $x^2<4$ ⇒ D + C true ✗

Given that exactly one of these options is correct, which one is it?



C: If inside ◯ then statement is true
⇒ A is true
A: If inside ◯ then statement is true
⇒ C may not be true if in ◯

3. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $a>0$ **then** $f(a)>0$ (P, Q)
B $a>0$ **only if** $f(a)>0$
C $a>0$ is **sufficient** for $f(a)>0$
D $a>0$ is **necessary** for $f(a)>0$

If P then Q
P only if Q = If P then Q
P sufficient for Q = If P then Q
P necessary for Q = If Q then P

4. a is a real number and f is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $f(a)>0$ **then** $|a|<1$ (P, Q)
B $f(a)>0$ **if** $|a|<1$
C $|a|<1$ **only if** $f(a)>0$
D $|a|<1$ is **sufficient** for $f(a)>0$

If P then Q
P if Q = if Q then P
Q only if P = if Q then P
Q sufficient for P = if Q then P

The following statements are all equivalent

If $a < b$ **then** $f(a) < f(b)$	If P then Q
$a < b$ **implies that** $f(a) < f(b)$	P implies Q
$a < b$ **only if** $f(a) < f(b)$	P only if Q
$a < b$ **is sufficient for** $f(a) < f(b)$	P is sufficient for Q
$f(a) < f(b)$ **if** $a < b$	Q if P
$f(a) < f(b)$ **is necessary for** $a < b$	Q is necessary for P
If $f(a) \geq f(b)$ **then** $a \geq b$	If 'not Q' then 'not P' (*contrapositive*)
$f(a) \geq f(b)$ **implies** $a \geq b$	'Not Q' implies 'not P'
$f(a) \geq f(b)$ **only if** $a \geq b$	'Not Q' only if 'not P'
$f(a) \geq f(b)$ **is sufficient for** $a \geq b$	'Not Q' is sufficient for 'not P'
$a \geq b$ **if** $f(a) \geq f(b)$	'Not P' if 'not Q'
$a \geq b$ **is necessary for** $f(a) \geq f(b)$	'Not P' is necessary for 'not Q'

The ***contrapositive*** is always logically equivalent to the original statement.

The **converse** of a statement is found by swapping the 'if' and 'then' parts of the statement, but does not always result in a true statement.

If I am in London, then I am in England	This is TRUE
If I am in England, then I am in London	Not necessarily true - I could be in Bristol

The **converse is true** when the original statement is an 'if and only if' statement

If I am in London, then I am in the capital of England	This is TRUE
If I am in the capital of England, then I am in London	This is TRUE

Note: The order of quantifiers in a statement is important.

For all positive real x, **there exists** a real y such that $y^2 = x$	TRUE (pick any x>0)
There exists a real y, such that for all positive real x, $y^2 = x$	FALSE (value of y changes with x)

Prove — Find counterexample

1. Are the following statements true or false?

a)	$x > 5$	if	$x > 10$	True
b)	$x < 8$	only if	$x < 3$	False eg $x=5$ Converse is true
c)	x is even	if and only if	$(x + 1)$ is odd	True
d)	$ab = ac$	if and only if	$b = c$	False $a=0$ But $b=c \Rightarrow ab=ac$
e)	$a^2 < a$	if	$a < 1$	False $a=-2$
f)	$a^2 < a$	only if	$a < 1$	True
g)	$a^2 < a$ $a(a-1)<0$ $0<a<1$	if and only if	$-1 < a < 1$	False $a=-\frac{1}{2}$
h)	an even number is prime	if and only if	it is 2	True
i)	an odd number is prime	if and only if	it is 3	False eg 7
j)	a triangle is equilateral	if and only if	all it angles are 60°	True
k)	a triangle is isosceles	if	it is equilateral	True

2. Write the contrapositive of the following statements:

a) If I have enough money, I will go on holiday — I won't go on holiday if I don't have enough money

b) If I pass my driving test, I will get my driving licence — I won't get my driving licence if I don't pass my test

c) Ben will not go to school only if he is sick — If Ben is not sick, he will go to school
(no school $\Rightarrow$ sick)

d) If you do not study, you will not do well in your exams — If you do well in your exams, then you did study

e) I wear a hat if it is sunny — If I don't wear a hat, then it's not sunny
(= If sunny, then hat)

3. Write the contrapositive of the following mathematical statements:

a) If an integer is not equal to 2, then it is not an even prime — If even prime, then equal to 2

b) If a shape has 4 sides, it is a quadrilateral — If not a quadrilateral, then doesn't have 4 sides

c) A number is even only if the square of the number is even — If square is not even, then number not even OR If square is odd, then number is odd

d) $f(a) > 0$ **if** $a > 0$ (= If $a>0$ then $f(a)>0$) so If $f(a) \leq 0$ then $a \leq 0$

e) $a^2 < a$ is sufficient for $a < 1$ — If $a \geq 1$ then $a^2 \geq a$
(= If $a^2<a$ then $a<1$)

TMUA Style Multiple Choice Practice 'equivalence'

P: even prime Q: $x = 2$

1. Given that exactly one of the following statements is true, which one is it?

A x is not an even prime **only if** $x = 2$ — not P only if Q = If not P then Q = If not Q then P

B **if** x is an even prime, **then** $x \neq 2$ — if P then not Q

C $x \neq 2$ is **sufficient** for x to be an even prime — not Q sufficient for P = if not Q then P

D $x \neq 2$ is **necessary** for x to be an even prime — not Q necessary for P = if P then not Q

E $x = 2$ **if and only if** x is not an even prime — if not P then Q; if Q then not P

F x is not an even prime **only if** $x \neq 2$ — not P only if not Q = if not P then not Q = if Q then P

2. f is a function and a *is* a real number. P: $a \leq 0$ Q: $f(a) \leq 0$
Given that exactly one of the following statements is true, which one is it?

A $a \leq 0$ **only if** $f(a) \leq 0$ — P only if Q : if P then Q

B $f(a) > 0$ **if** $a > 0$ — not Q if not P : if not P then not Q : if Q then P

C $f(a) > 0$ is **sufficient** for $a > 0$ — not Q sufficient for not P : if not Q then not P : if P then Q

D $f(a) \leq 0$ is **necessary** for $a \leq 0$ — Q necessary for P : if P then Q

E **If** $f(a) > 0$ **then** $a > 0$ — if not Q then not P : if P then Q

F $a > 0$ **if** $f(a) > 0$ — not P if not Q : if not Q then not P : if P then Q

3. f is a function and a, b are real numbers.
Given that exactly one of the following statements is true, which one is it?

A $f(a) \geq f(b)$ if and only if $a \geq b$ — If $f(a) \geq f(b)$ then $a \geq b$; If $a \geq b$ then $f(a) \geq f(b)$ * same

B $f(a) \geq f(b)$ only if $a < b$ — If $f(a) \geq f(b)$ then $a < b$

C $f(a) < f(b)$ if $a \geq b$ — If $a \geq b$ then $f(a) < f(b)$ — contrapositive

D $a \geq b$ if $f(a) \geq f(b)$ — If $f(a) \geq f(b)$ then $a \geq b$ *

E $a < b$ only if $f(a) \geq f(b)$ — If $a < b$ then $f(a) \geq f(b)$ — contrapositive

F $a < b$ only if $f(a) < f(b)$ — If $a < b$ then $f(a) < f(b)$

$P \Rightarrow Q$

not $Q \Rightarrow$ not P

A, D equivalent

B, C D, F contrapositive

Negation (denial not opposite)

Statement	Negation	
He is a doctor	He is not a doctor	
She is tall	She is not tall	(She is short would be incorrect)
A	*not A*	

1. I am hungry — I am not hungry
2. They do their homework — They don't do their homework
3. It is not raining — It is raining
4. The melon is not ripe — The melon is ripe

I have blue eyes **and** blond hair	Either I do not have blue eyes **or** I do not have blond hair (**or** I do not have either)
A and B	*not A or not B*

5. My socks are blue and stripy — Either not blue or not stripy
6. I play hockey and basketball — Don't play both – either not hockey or not basketball
7. I had lunch with Bill and Ben — Didn't have lunch with both / Both didn't have lunch with me
8. It is not hot or sunny — It is hot and sunny

I study English **or** German	I do not study English **and** I do not study German
A or B	*not A and not B / neither A nor B*

9. Jan drinks tea or coffee — Jan doesn't drink tea and doesn't drink coffee
10. The man is called Jim or John — not called Jim and not called John
11. The children eat apples or bananas — The children eat neither apples nor bananas
12. Neither my brother nor sister will help me — Either my brother or my sister will help me

Statement	Negation
Everyone like pizza	**Not** everyone likes pizza / At least one person doesn't like pizza Some people don't like pizza There exists someone who doesn't like pizza
For all A, then B	*Not every A implies B / There exists A such that not B*
13. All vegetarians eat carrots	Some vegetarians don't eat carrots
14. My teacher is always right	My teacher is sometimes wrong
15. All dogs bark	Some dogs don't bark
16. Not every integer is odd	All integers are odd

There is a prime number less than 2	There are no prime numbers less than 2 All prime numbers are greater than or equal to 2
There exists A such that B	*There is no A such that B* *For all A, not B*
17. Some boys like football	No boys like football / All boys don't like football
18. At least one square number is less than 3	No square numbers are less than 3
19. There exist some birds who can not fly	All birds can fly
20. There are no prime numbers that are even	At least one prime number is even

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If the sun shines, I will wear a hat	If the sun shines, I will not wear a hat
If A, then B	*If A, then not B* *A and not B*
21. If it is raining I will take an umbrella	If it is raining, I won't take an umbrella
22. I will receive a gold medal if I win	If I win, I won't receive a gold medal
23. If $a < b$ then $f(a) < f(b)$	If $a < b$ then $f(a) \geq f(b)$
24. $f(a) > 0$ if $a > 0$ (if $a > 0$ then $f(a) > 0$)	if $a > 0$ then $f(a) \leq 0$

Negation of ‘Nested’ statements

A Consider the statement: The class can complete their homework online, if for **every** student in the class, the student has online access

The negation is: A class can **not** complete their homework online, if there is **at least** one student who does **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has online access

Then the statement becomes: P is true if **for every** Q, **there exists** R

The negation of this is: P is **not** true if **there exists** Q such that **not** R

B Consider the statement: The class can complete their homework online, if **for every** student in the class, the student **has** a friend who has online access

The negation is: A class can **not** complete their homework online if **there exists** a student in the class, **all** of whose friends do **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has a friend

S = friend has online access

Then the statement becomes: P is true if **for every** Q, **there exists** ‘R such that S’

The negation of this is: P is **not** true if **there exists** Q such that **not** ‘**there exists** R such that S’

or: P is **not** true if **there exists** Q such that ‘**for all** R **not** S’

Write the negation of the following nested statements:

1) A set of integers P is the set of even numbers iff for any $\overbrace{\text{integer } n \text{ in P}}^{B}$, $\overbrace{\frac{n}{2} \text{ is also an integer}}^{C}$.

P iff for any B then C

not P if there exists B such that not C

P not set of even numbers if there exists an integer n in P such that $\frac{n}{2}$ is not an integer

2) A set of integers P is the set of square numbers iff for any $\underbrace{\text{integer } n \text{ in P}}_{B}$, there exists $\underbrace{\text{an integer k}}_{C}$ such that $\underbrace{k^2 = n}_{D}$

P iff for any B there exists C such that D

not P if there exists B such that not ‘exists C such that D’

―――― " ―――― all C not D

P not set of square numbers if there exists an integer n in P such that for all integers k, $k^2 \neq n$

Counter Examples

1) Find a counter example to the following statements:

a) All quadrilaterals with equal side length are squares Rhombus

b) The square root of a number is always less than the number $\sqrt{\frac{1}{4}} = \frac{1}{2}$

c) If a three-dimensional solid has a circular base, then it is a cylinder Cone

d) If n is an integer and n^2 is divisible by 4, then n is divisible by 4 $n = 6$

e) If p is an odd prime then p+2 is also an odd prime $p = 7$

f) The sum of 2 numbers is always greater than both numbers $(-2) + (-6)$

g) $10k^2 + 1$ is prime if k is an odd prime $k = 3$ $10(3)^2 + 1 = 91 = 7 \times 13$

h) For all real x, 5x > 4x any $x < 0$

i) For all real x, $\sqrt{1 - sin^2 x} = cosx$ $x = 180$ $(1 = -1)$

2) A set of five signs has a letter printed on the left and a number printed on the right

A 8 B 4 C 1 D 7 E 3

Which sign(s) provide a counterexample to the following statements:

a) Every card that has a vowel on the left has an even number on the right E3

b) Every card that has an even number on the right has a vowel on the left. B4

c) Every card that has a consonant on the left has a prime number on the right B4 C1

d) Every card that has a prime number on the right has a vowel on the left D7

3) How many counter examples are there to the following statements:

a) All odd numbers between 2 and 20 are prime. 3 5 7 9 11 13 15 17 19

2 counterexamples (odd but not prime)

b) If n is a prime integer less than or equal to 10, then $n^2 + 2$ is also prime

$n =$ 2 3 5 7

$n^2 + 2 =$ 6 11 27 51

3 counterexamples (n prime < 10, n^2+2 not prime)

c) A whole number n less than or equal to 50, is prime if it is 1 less or 5 less than a multiple of 6

1 less or 5 less : 1 5 7 11 13 17 19 23 25 29 31 35 37 41 43 47 49

4 counterexample (1 less or 5 less, not prime)

Logic

1) On an island people either always tell the truth or always tell lies. You are approached by 2 people. Identify if they are truth-tellers or liars in the following situations.

a) The first person says "we both always tell lies"

A

If A truth $\Rightarrow$ both tell lies × contradiction

$\therefore$ A liar $\Rightarrow$ not both tell lies $\Rightarrow$ B truth

A B

b) The first person points at the second and says "he is a liar" and the second person says "neither of us are liars"

If A truth $\Rightarrow$ B is liar $\Rightarrow$ at least one liar

If A liar $\Rightarrow$ B truth $\Rightarrow$ no liars × contradiction

A B

c) The first person says "we are both telling the truth" and the second one replies "he is lying".

If A truth $\Rightarrow$ B truth $\Rightarrow$ A liar × contradiction

A liar $\Rightarrow$ at least one liar

If B liar $\Rightarrow$ A truth × contradiction If B truth $\Rightarrow$ A liar

A

d) The first person says "at least one of us is lying"

If A truth $\Rightarrow$ B liar

If A liar $\Rightarrow$ no liars × contradiction

A B

e) The first person says "exactly one of us is lying", and the second replies "actually we're both lying"

If A truth $\Rightarrow$ B liar $\Rightarrow$ at least one truth

If A liar $\Rightarrow$ either no liars or both liars $\Rightarrow$ both liars

but B liar $\Rightarrow$ at least one truth

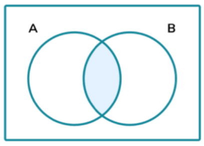
× contradiction

TMUA Proof and Logic Summary

Definitions

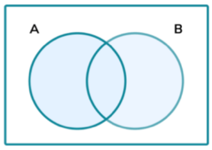
<u>**and**</u> A **and** B means A and B together ($A \cap B$)

For A **and** B to be true, **both** $\boldsymbol{A}$ **and** $\boldsymbol{B}$ must be true



<u>**or**</u> A **or** B means A or B or both ($A \cup B$)

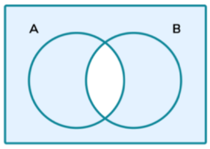
For A **or** B to be true, **either** $\boldsymbol{A}$ **or** $\boldsymbol{B}$ **or both** must be true



negation = <u>**not**</u>

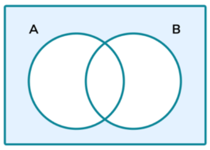
not (A **and** B**) = not** A **or not** B

not *(blue eyed* ***and*** *blonde) = not blue-eyed or not blond*
so could be one or the other but not both



not (A **or** B) = **not** A **and not** B

not (blue eyed ***or*** *blonde) = not blue-eyed and not blond*
so does not have either characteristic

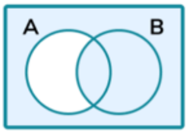


<u>**if, then**</u>

if A then B means if A is true, then B must be true (But if A is not true, then B could be true or false)

We can also write this in the following ways:

A implies B	$A \implies B$
B if A	A only if B
A is **sufficient** for B	B is **necessary** for A



The **converse** statement (swapping statements) is

'if B then A' but these are not always equivalent

The **contrapositive** statement (swapping and negating both statements) is

'if not B then not A' and this is an equivalent statement to the original.

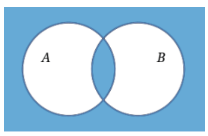
The **negation** of the statement is

'if A then not B' or 'A and not B'

<u>**if and only if**</u>

A if and only if B. We can also write this in the following ways:

A implies B and B implies A	$A \iff B$
A if B and A only if B	A iff B
A is **sufficient** and **necessary** for B	



Quantifiers

These are equivalent:

- For every *A* / for any *A* / for all *A* / for each *A* / if *A*
- For some *A* / there exists *A* / for at least one *A*

The order of a combined statement is important.

For all positive real x, **there exists** a real y such that $y^2 = x$ TRUE (pick any $x>0$)

There exists a real y, such that for all positive real x, $y^2 = x$ FALSE (value of y changes with x)

Original statement	Negation
For all A, then B	*Not every A implies B* *There exists A such that not B*
Every integer is odd	*Not every integer is odd* *There exists an integer that is not odd*

Original statement	Negation
There exists A such that B	*There is no A such that B* *For all A, not B*
There exists a prime number less than 2	*There are no prime numbers less than 2* *All prime numbers are greater than or equal to 2*

<u>Example of negation of 'nested' statements</u>

Statement	*Negated Statement*
There exists P iff for every Q there exists R	*Not P = there exists Q such that 'not R'*
For any D there exists 'E such that F'	*There exists D such that NOT '**E such that F**'* *There exists D such that for all E not F*