# TMUA Proof

***Syllabus***

Logic of arguments (true/false, and/or, if, then, not) converse, contrapositive, necessary sufficient; quantifiers; mathematical proof (deduction, proof by cases, proof by contradiction, disproof by counterexample); identifying errors in proofs.

## Logic of Arguments

1a. Consider the following statement: "If it is my birthday, I will eat some cake"

What conclusion can I draw from each of the following statements:

i) It is my birthday

ii) It is not my birthday

iii) I eat some cake

iv) I do not eat some cake

1b. Consider the following statement: "If it rains the ground will get wet"

What conclusion can I draw from each of the following statements:

i) The ground is wet

ii) The ground is not wet

iii) It is raining

iv) It is not raining

1c. Consider the following statement: "If I am in Paris, then I am in France"

What conclusion can I draw from each of the following statements:

i) I am in Paris

ii) I am in France

iii) I am in London

iv) I am at the Eiffel Tower

1d. Consider the following statement: "If a shape is a square, then it is a quadrilateral"

What conclusion can I draw from each of the following statements:

i) The shape is a square

ii) The shape is a quadrilateral

iii) The shape is not a quadrilateral

iv) The shape is a rhombus

An if/then statement is called a **conditional statement**.

'**if**' is the hypothesis, and '**then**' is the conclusion.

It tells us what happens if the hypothesis is true, but doesn't tell us anything about what happens if the hypothesis is false.

Example: **If** it is raining **then** I will wear a coat.

This tells me that if it rains, I will wear a coat, but if it is not raining I may or may not wear a coat.

The **converse** of a statement swaps the hypothesis and conclusion and is *not always true*.

If I wear a coat, then it is raining (eg I might wear a coat when it is cold but not raining).

However the original statement does tell me something when I am not wearing a coat. In this case it can't be raining, because if it was I would wear a coat which is a contradiction. Therefore I can say:

If I am not wearing a coat, then it is not raining.

This is called the ***contrapositive*** and is always logically equivalent to the original statement. It is obtained by negating both hypothesis and conclusion and swapping them.

2. Write the contrapositive of the following statements:

a) If I have enough money, I will go on holiday

b) If I pass my driving test, I will get my driving licence

c) Ben will not go to school only if he is sick

d) If you do not study, you will not do well in your exams

e) I wear a hat if it is sunny

3. Write the contrapositive of the following mathematical statements:

a) If an integer is not equal to 2, then it is not an even prime

b) If a shape has 4 sides, it is a quadrilateral

c) A number is even only if the square of the number is even

d) $f(a) > 0$ **if** $a > 0$

e) $a^2 < a$ is sufficient for $a < 1$

The following statements are all equivalent:

**If** an animal is a zebra, **then** it has stripes

**All** zebras have stripes

**Any** zebra has stripes

Being a zebra **implies** having stripes

An animal is a zebra **only if** it has stripes

An animal has stripes **if** it is a zebra

An animal with **no** stripes is **not** a zebra

Being a zebra is a **sufficient** condition for an animal to have stripes

Having stripes is **necessary** for an animal to be a zebra

**Sufficient** If P is sufficient for Q, then Q must happen if P happens, so we have '**if P then Q**'

**Necessary** If P is necessary for Q, then Q can't happen without P, so we have '**if Q then P**'

4. Rewrite the following true statements in the form **If… Then …**

a) The ground gets wet when it rains

b) All mammals have hair

c) I always go to bed when I am sick

d) A fruit is yellow if it is a banana

e) I am in Paris only if I am in France

5. Rewrite the following true mathematical statements in the form **If… Then …**

a) Any rectangle is a quadrilateral

b) Any triangle has 3 sides

c) The number 2 is the only even prime number

d) $x > 10$ if $x > 100$

e) $k < 1$ when $k^2 < 1$

f) $p^2 < p$ only if $p < 1$

The following statements are all equivalent

| | |
|---|---|
| **If** $a < b$ **then** $f(a) < f(b)$ | If P then Q |
| $a < b$ **implies that** $f(a) < f(b)$ | P implies Q |
| $a < b$ **only if** $f(a) < f(b)$ | P only if Q |
| $a < b$ **is sufficient for** $f(a) < f(b)$ | P is sufficient for Q |
| $f(a) < f(b)$ **if** $a < b$ | Q if P |
| $f(a) < f(b)$ **is necessary for** $a < b$ | Q is necessary for P |
| **If** $f(a) \geq f(b)$ **then** $a \geq b$ | If 'not Q' then 'not P' (*contrapositive*) |
| $f(a) \geq f(b)$ **implies** $a \geq b$ | 'Not Q' implies 'not P' |
| $f(a) \geq f(b)$ **only if** $a \geq b$ | 'Not Q' only if 'not P' |
| $f(a) \geq f(b)$ **is sufficient for** $a \geq b$ | 'Not Q' is sufficient for 'not P' |
| $a \geq b$ **if** $f(a) \geq f(b)$ | 'Not P' if 'not Q' |
| $a \geq b$ **is necessary for** $f(a) \geq f(b)$ | 'Not P' is necessary for 'not Q' |

The **contrapositive** is always logically equivalent to the original statement. It is obtained by negating both 'if' and 'then' parts of the statement and swapping them.

The **converse** of a statement is found by swapping the 'if' and 'then' parts of the statement, but does not always result in a true statement.

| | |
|---|---|
| If I am in London, then I am in England | This is TRUE |
| If I am in England, then I am in London | Not necessarily true - I could be in Bristol |

The **converse is true** when the original statement is an 'if and only if' statement

| | |
|---|---|
| If I am in London, then I am in the capital of England | This is TRUE |
| If I am in the capital of England, then I am in London | This is TRUE |

Therefore I am in London **if and only if** I am in the capital of England

How to answer: Is P sufficient for Q?

Look for a counter example - an instance where P is true but Q is not true

Or look for proof that P implies Q (if P then Q is true)

How to answer: Is P necessary for Q?

Look for a counter example - an instance where Q is true but not P

Or look for proof that Q implies P

6. Complete the statements with one of the following:

'necessary' 'sufficient' 'necessary and sufficient' 'not necessary and not sufficient'

a) being a bird is a ___________ condition for having feathers

b) being a robin is a __________ condition for being a bird

c) having 2 legs is a ________________ condition for being a bird

d) having feathers is a ______________ condition for being able to fly

e) being an odd number is a _________ condition for being a prime number greater than 10

f) being greater then 20 is a ________________ condition for being a greater than 10

g) being a rectangle is a _____________ condition for being a square

h) $x^2 = 1$ is a _______________ condition for $x = 1$

i) $x^2 < 5$ is a _______________ condition for $x^2 < 10$

j) $x^2 < 1$ is a _______________ condition for $-1 < x < 1$

k) $ab < ac$ is a _____________ condition for $b < c$

7. Are the following statements true or false?

| | | | |
|---|---|---|---|
| a) | $x > 5$ | if | $x > 10$ |
| b) | $x < 8$ | only if | $x < 3$ |
| c) | x is even | if and only if | $(x + 1)$ is odd |
| d) | $ab = ac$ | if and only if | $b = c$ |
| e) | $a^2 < a$ | if | $a < 1$ |
| f) | $a^2 < a$ | only if | $a < 1$ |
| g) | $a^2 < a$ | if and only if | $-1 < a < 1$ |
| h) | an even number is prime | if and only if | it is 2 |
| i) | an odd number is prime | if and only if | it is 3 |
| j) | a triangle is equilateral | if and only if | all it angles are 60° |
| k) | a triangle is isosceles | if | it is equilateral |

1. $a$ and $b$ are real numbers and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $a < b$ **then** $f(a) < f(b)$
B $a < b$ **only if** $f(a) < f(b)$
C $f(a) < f(b)$ is **sufficient** for $a < b$
D $f(a) < f(b)$ is **necessary** for $a < b$

2. $a$ is a real number and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $a > 0$ **then** $f(a) > 0$
B $a > 0$ **only if** $f(a) > 0$
C $a > 0$ is **sufficient** for $f(a) > 0$
D $a > 0$ is **necessary** for $f(a) > 0$

3. $a$ is a real number and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $f(a) > 0$ **then** $|a| < 1$
B $f(a) > 0$ **if** $|a| < 1$
C $|a| < 1$ **only if** $f(a) > 0$
D $|a| < 1$ is **sufficient** for $f(a) > 0$

4. Consider the four options below about a particular statement:

A The statement is true if $x^2 < 1$
B The statement is true if and only if $x^2 < 1$
C The statement is true if $x^2 < 4$
D The statement is true if and only if $x^2 < 4$

Given that exactly one of these options is correct, which one is it?

5. Given that exactly one of the following statements is true, which one is it?

A $x$ is not an even prime **only if** $x = 2$

B **if** $x$ is an even prime, **then** $x \neq 2$

C $x \neq 2$ is **sufficient** for $x$ to be an even prime

D $x \neq 2$ is **necessary** for $x$ to be an even prime

E $x = 2$ **if and only if** $x$ is not an even prime

F $x$ is not an even prime **only if** $x \neq 2$

6. f is a function and *a is* a real number.
Given that exactly one of the following statements is true, which one is it?

A $a \leq 0$ **only if** $f(a) \leq 0$

B $f(a) > 0$ **if** $a > 0$

C $f(a) > 0$ is **sufficient** for $a > 0$

D $f(a) \leq 0$ is **necessary** for $a \leq 0$

E **If** $f(a) > 0$ **then** $a > 0$

F $a > 0$ **if** $f(a) > 0$

7. f is a function and *a, b* are real numbers.
Given that exactly one of the following statements is true, which one is it?

A $f(a) \geq f(b)$ if and only if $a \geq b$

B $f(a) \geq f(b)$ only if $a < b$

C $f(a) < f(b)$ if $a \geq b$

D $a \geq b$ if $f(a) \geq f(b)$

E $a < b$ only if $f(a) \geq f(b)$

F $a < b$ only if $f(a) < f(b)$

## Quantifiers

The words 'all', 'some, 'none' are examples of quantifiers. These tell us how many instances satisfy the statement, and a statement containing one or more of these words is called a **quantified statement**.

In English there are many ways to write these statements:

All: All / Every / Each / Any / If

All even numbers are divisible by 2

For all even numbers $x$, $x$ is divisible by 2

For each/every even number $x$, $x$ is divisible by 2

If $x$ is an even number, then $x$ is divisible by 2

Some: Some / At least one / There exists

Some even numbers are prime

For some even number $x$, $x$ is prime

There exists an even number $x$, such that $x$ is prime

There is at least one even number $x$, for which $x$ is prime

None: No / Not any / There does not exist / There are no

No real square numbers are negative

There are not any real square numbers that are negative

There does not exist a real square number $x$ for which $x$ is negative

There is no real square number $x$ such that $x$ is negative

Consider the order of quantifiers:

When the same type of quantifier is used, the order does not matter:

**For all** odd numbers x and **all** even numbers $y$ the sum of $x$ and $y$ is odd

**For all** even numbers $y$ and **all** odd numbers $x$ the sum of $x$ and $y$ is odd

However, with different quantifiers, the order changes the meaning of the statement:

**For all** positive real $x$, **there exists** a real $y$ such that $y^2 = x$

This is TRUE as we can choose any positive value for x and find a value of y that makes the equation true by calculating $y = \sqrt{x}$

**There exists** a real $y$, such that for all positive real $x$, $y^2 = x$

However this is FALSE as there is not a single value of y that makes the equation true; the value of $y$ that we need changes with our choice of $x$

**Negation (denial not opposite)**

The negation of a statement is achieved by placing 'not' in front of the statement. In reality there are often multiple ways of phrasing this in English. Be careful not to infer too much from a negation, for example 'not hot' does not mean cold - it just means not hot (eg it could be warm).

| **Statement** | **Negation** |
| --- | --- |
| He is a doctor | Not 'He is a doctor' = He is not a doctor |
| She is tall | Not 'She is tall' = She is not tall (She is short would be incorrect) |

1. I am hungry
2. They do their homework
3. It is not raining
4. The melon is not ripe

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To negate *(A and B)* we use *not (A and B)* which is the same as *not A or not B*

| | |
| --- | --- |
| I have blue eyes **and** blond hair | Either I do not have blue eyes **or** I do not have blond hair (**or** I do not have either) |

5. My socks are blue and stripy
6. I play hockey and basketball
7. I had lunch with Bill and Ben
8. Neither my brother nor sister will help me

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To negate *(A or B)* we use *not (A or B)* which is the same as *not A and not B / neither A nor B*

| | |
| --- | --- |
| I study English **or** German | I do not study English **and** I do not study German |

9. Jan drinks tea or coffee
10. The man is called Jim or John
11. The children eat apples or bananas
12. It is not hot or sunny

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*Note how the words '**and**' & '**or**' swap when we negate a statement*

To negate *(for all A, then B)* we use *not (for all A, then B)* which is the same as *not every A implies B / there exists A such that not B*

| **Statement** | **Negation** |
|---|---|
| Everyone like pizza | **Not** everyone likes pizza / At least one person doesn't like pizza / Some people don't like pizza / There exists someone who doesn't like pizza |

13. All vegetarians eat carrots

14. My teacher is always right

15. All dogs bark

16. There are no prime numbers that are even

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To negate *(there exists A such that B)* we use *not (there exists A such that B)* which is the same as *there is no A such that B / for all A, not B*

| | |
|---|---|
| There is a prime number less than 2 | There are no prime numbers less than 2<br>All prime numbers are greater than or equal to 2 |

17. Some boys like football

18. At least one square number is less than 3

19. There exist some birds who can not fly

20. Not every integer is odd

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To negate *(if A, then B)* we use *not (if A, then B)* which is the same as *if A, then not B / A and not B*

| | |
|---|---|
| If the sun shines, I will wear a hat | If the sun shines, I will not wear a hat |

21. If it is raining I will take an umbrella

22. I will receive a gold medal if I win

23. If $a < b$ then $f(a) < f(b)$

24. $f(a) > 0$ if $a > 0$

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*Note how the words '**for all**' & '**there exists**' swap when we negate a statement*

**Negation of ‘Nested’ statements**

A Consider the statement: The class can complete their homework online, if for **every** student in the class, the student has online access

The negation is: A class can **not** complete their homework online, if there is **at least** one student who does **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has online access

Then the statement becomes: P is true if **for every** Q, **there exists** R

The negation of this is: P is **not** true if **there exists** Q such that **not** R

B Consider the statement: The class can complete their homework online, if **for every** student in the class, the student **has** a friend who has online access

The negation is: A class can **not** complete their homework online if **there exists** a student in the class, **all** of whose friends do **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has a friend

S = friend has online access

Then the statement becomes: P is true if **for every** Q, **there exists** ‘R such that S’

The negation of this is: P is **not** true if **there exists** Q such that **not** ‘**there exists** R such that S’

or: P is **not** true if **there exists** Q such that ‘**for all** R **not** S’

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Write the negation of the following nested statements:

1) A set of integers P is the set of even numbers iff for any integer $n$ in P, $\frac{n}{2}$ is also an integer.

2) A set of integers P is the set of square numbers iff for any integer $n$ in P, there exists an integer k such that $k^2 = n$

**Counter Examples**

A counter example is one example which disproves a statement. It proves a statement is not true.

1) Find a counter example to the following statements:

a) All quadrilaterals with equal side length are squares

b) The square root of a number is always less than the number

c) If a three-dimensional solid has a circular base, then it is a cylinder

d) If n is an integer and $n^2$ is divisible by 4, then n is divisible by 4

e) If $p$ is an odd prime then $p+2$ is also an odd prime

f) The sum of 2 numbers is always greater than both numbers

g) $10k^2 + 1$ is prime if k is an odd prime

h) For all real x, 5x > 4x

i) For all real x, $\sqrt{1 - sin^2x} = cosx$

2) A set of five signs has a letter printed on the left and a number printed on the right

A 8 B 4 C 1 D 7 E 3

Which sign(s) provide a counterexample to the following statements:

a) Every card that has a vowel on the left has an even number on the right

b) Every card that has an even number on the right has a vowel on the left.

c) Every card that has a consonant on the left has a prime number on the right

d) Every card that has a prime number on the right has a vowel on the left

3) How many counter examples are there to the following statements:

a) All odd numbers between 2 and 20 are prime.

b) If n is a prime integer less than or equal to 10, then $n^2 + 2$ is also prime

c) A whole number n less than or equal to 50, is prime if it is 1 less or 5 less than a multiple of 6

4) Prove the following statements, or find a counter example to disprove them:

a) For all real $x$, $3x < 3^x$

b) For all real $x$, if $ax = bx$ then $a = b$

c) If a positive integer $p$ has remainder 1 when divided by 3, then $p^3$ also has remainder 1 when divided by 3

d) For all integers $p$ and $q$, if $p < q$, then $p^2 < q^2$

e) For all integers $n$, $9n^2 + 24n$ is not prime

f) If a positive integer $p$ is prime, then $2p + 1$ is also prime

g) For consecutive even integers $p$ and $q$, $p^3 - q^3$ is a multiple of 8

h) For all integers $n$, if $n$ is prime, then $(-1)^n = -1$

i) $4^n + 3^{n-2} + 3$ is divisible by 5 for all integers $n \geq 2$

j) If $p$ and $q$ are irrational, such that $p \neq q$, then $p + q$ is irrational

k) If $p$ is rational and $q$ is irrational, then $log_p q$ is irrational

**Logic**

1) On an island people either always tell the truth or always tell lies. You are approached by 2 people. Identify if they are truth-tellers or liars in the following situations.

a) The first person says “we both always tell lies”

b) The first person points at the second and says “he is a liar” and the second person says “neither of us are liars”

c) The first person says “we are both telling the truth” and the second one replies “he is lying”.

d) The first person says “at least one of us is lying”

e) The first person says “exactly one of us is lying”, and the second replies “actually we’re both lying”