

Sequences

TMUA / MAT Syllabus

*N*th term and recurrence relations; arithmetic series including S_n ; geometric series including S_n and S_∞ ; binomial expansion of $(1+x)^n$; $n!$ and nC_r

Arithmetic Sequences $u_n = a + (n - 1)d$ a is the first term, d is the common difference

1. For each sequence, write down the first four terms, and find a and d .
 - a) $u_n = 5n + 3$
 - b) $u_n = 7 - 3n$
 - c) $u_n = 6 + \frac{1}{2}n$
 - d) $u_n = 2 - n$
2. For each sequence, find the n th term, and the 10th term
 - a) 5, 8, 11, 14, ...
 - b) -1, 3, 7, 11, ...
 - c) 25, 18, 11, 4, ...
 - d) -2, -7, -12, -17, ...
3.
 - a) The first term of an arithmetic sequence is 12. The fourth term is 30. find the common difference.
 - b) $u_3 = 30$ and $u_6 = 9$. Find the first negative term in the sequence.
 - c) The 7th term and twelfth term of an arithmetic sequence are 28 and 73. Find the first term and common difference.
 - d) The first three terms of an arithmetic sequence are $5p$, 20, $3p$ where p is a constant. Find the 20th term in the sequence
 - e) The first three terms of an arithmetic sequence are -8, k^2 , $17k$. Find two possible values of k .
 - f) The seventh term of an arithmetic series is 6. The sum of its fifth term and its tenth term is 16. Find the first term and the common difference of the series.

Arithmetic Series a is the first term, l is the last term, d is the common difference

$$S_n = \frac{n}{2}(2a + (n - 1)d) \qquad S_n = \frac{n}{2}(a + l)$$

4. Find the sum of the following series:

a) $2 + 6 + 10 + 14 + \dots$ (15 terms)

b) $5 + 1 + -3 + -7 + \dots$ (20 terms)

c) $5 + 7 + 9 + \dots + 75$

5. Find how many terms are needed to make the given sums:

a) $5 + 6 + 7 + 8 + \dots = 290$

b) $64 + 62 + 60 + \dots = 0$

6.

a) Find the sum of the first 50 positive even numbers.

b) Prove that the sum of the first n positive odd numbers is n^2

c) The 4th term of an arithmetic series is 15. The sum of its first three terms is 9.
Find the first term and common difference of the series.

d) The n th term of an arithmetic series is given by $u_n = 6n + 11$
Find the sum of the first twenty terms of the series

e) The sixteenth term of an arithmetic series is 6. The sum of the first sixteen terms is 456.
The sum of the first k terms of the series is zero. Determine the value of k .

Geometric Sequences

$$u_n = ar^{n-1} \quad a \text{ is the first term, } r \text{ is the common ratio}$$

7. Find the sixth and n th terms of the following geometric sequences:

- a) 2, 6, 18, 54, ...
- b) 1, -2, 4, -8, ...
- c) 100, 50, 25, 12.5, ...

8.

- a) The first and second terms of a geometric sequence are 90 and 30 respectively. Find the common ratio and the 5th term.
- b) The third and fourth terms of a geometric sequence are 36 and 27 respectively. Find the common ratio and the 5th term.
- c) The second and fifth terms of a geometric sequence are 12 and -1.5 respectively. Find the common ratio and the first term.
- d) The numbers 3, x and $(x + 6)$ are the first three terms of a geometric sequence with all positive terms. Find the possible values of x .
- e) The first three terms of a geometric sequence are given by $8-a$, $2a$, and a^2 where $a > 0$. Find the value of a , and hence the sixth term.
- f) A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, how many ants will there be after 3 years.

Geometric Series $S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$ a is the first term, r is the common ratio

$S_\infty = \frac{a}{1 - r}$ for $|r| < 1$ (series is convergent)

9. Find the sum of the following series:

- a) 1, 2, 4, 8, ... (8 terms)
- b) 30, 15, 7.5, ... (5 terms)
- c) 4, -4, 4, -4, ... (25 terms)

10. Work out whether the following series are convergent, and if so, find the sum to infinity:

- a) $1 + 0.1 + 0.01 + 0.001 + \dots$
- b) $0.4 + 0.8 + 1.6 + 3.2 + \dots$
- c) $16 + 8 + 4 + 2 + \dots$

11.

- a) A geometric series has common ratio $\frac{1}{3}$. Find the first term, given that the sum of the first four terms is 36.
- b) Find the least value of n such that the series with first term 5 and common ratio $\frac{1}{2}$ exceeds 9.
- c) The first and second term of a geometric series are 90 and 15. Find the sum to infinity.
- d) The common ratio of a geometric series is twice as large as its first term. Given that the sum to infinity is 1, find the exact value of the fifth term of the series.

Sigma Notation

$\sum_{n=1}^5 (2n - 1)$ this is the sum of $(2n - 1)$ with the values of n substituted from 1 to 5 ($=1+3+5+7+9$)

12. Calculate

a) $\sum_{n=1}^4 (3n + 1)$

e) $\sum_{r=1}^4 (r^2 + 1)$

b) $\sum_{n=4}^6 (2n + 5)$

f) $\sum_{r=1}^{20} 1$

c) $\sum_{r=1}^{20} (13r + 4)$

g) $\sum_{k=10}^{30} (4k + 11)$

d) $\sum_{n=1}^{50} \sin(90n)$

h) $\sum_{r=1}^5 (2(3^{r-1}))$

13. Show that

a) $\sum_{r=1}^n (2r) = n + n^2$

b) $\sum_{k=1}^n \left(\frac{k+5}{3}\right) = \frac{1}{6}n(n+11)$

c) $\sum_{k=1}^n (100 - 2k) = 99n - n^2$

Recurrence Relations

$u_{n+1} = f(u_n)$ for example $u_{n+1} = 2u_n + 1$ and $u_1 = 3$ give the sequence 3, 7, 15, 31, ...

14. Find the first four terms of the following sequences:

a) $u_{n+1} = u_n - 5$ and $u_1 = 16$ b) $u_{n+1} = 3u_n + 2$ and $u_1 = 1$

c) $u_{n+1} = (u_n)^2 - 1$ and $u_1 = 2$ d) $u_{n+1} = 3u_n$ and $u_1 = 1$

15.

a) Given $u_n = u_{n-1} + n$ and $u_0 = 4$, find an expression for u_n in terms of n

b) If $u_{n+1} = (3 - u_n)^2$ and $u_1 = 4$, find u_{10}

c) If $a_{n+1} = 3a_n + 4$ and $a_1 = k$ and $\sum_{r=1}^4 a_r = 32$ find k .

d) A recurrence relation is defined by $a_{n+1} = 7a_n - n^3 - 3$ and $a_1 = 1$

Find $\sum_{r=1}^5 a_r$

e) A sequence of numbers is given by the recurrence relation $u_{n+1} = \frac{1}{1 - u_n}$ $u_1 = 2$

Find u_{12} and show that $\sum_{r=1}^{12} u_r = 6$

f) A sequence of numbers is given by the recurrence relation $u_{n+1} = 5 - \frac{18}{4 + u_n}$ $u_2 = 0$

Find u_3 , u_4 , and u_1

g) A sequence of numbers is given by the recurrence relation $u_{n+1} = \frac{k - 5u_n}{u_n}$ $u_1 = 1$

If $u_3 > 6$, find the range of values of k .

h) A recurrence relation is defined for $n \geq 1$ by $a_{n+1} = k + \frac{1}{2}a_n$

If $a_1 = 520$ and $a_4 = 72$ find the value of k .

i) A recurrence relation is defined for $n \geq 1$ by $u_{n+1} = k + (-1)^n u_n$ $u_1 = 4$

Show that $u_5 = 4$ and given that $\sum_{r=1}^4 u_r = 6$ find the value of k .

j) A sequence of numbers is given by the recurrence relation $u_{n+1} = k - \frac{12}{u_n}$ $u_1 = 1$

If $4u_2 = u_3 + 1$, find the possible values of k .

k) A sequence of numbers is defined by the recurrence relation $u_{n+1} = k u_n + 6$ $u_1 = 4$

If $u_3 = 10$, find the possible values of k .

Given that the sequence converges to a limit L , determine the value of L .

l) A recurrence relation is defined for $n \geq 1$ by $a_{n+1} = \sqrt{a_n + 12}$ $a_1 = k$

Given that the sequence converges to a limit L , determine the value of L .

Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \dots + \frac{1}{r!}n(n-1)\dots(n-r+1)x^r = \sum_{r=0}^n \binom{n}{r} x^r$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n = \sum_{r=0}^n \binom{n}{r} a^{n-r}b^r$$

16.

- a) Find the coefficient of x^3 in the series expansion of $(1-5x)^4$
- b) Find the coefficient of x^6 in the series expansion of $(1+3x)^8$
- c) Find the coefficient of x^2 in the expansion of $(1+2x)^2(1-3x+3x^2+4x^3)$
- d) In the binomial expansion of $(1+kx)^6$, the coefficient of x^3 is twice as large as the coefficient of x^2 . Find the value of k .
- e) It is given that $(1-2x)(2+kx)^5 = A+Bx+240x^2+\dots$ where k , A , and B are constants. Find the possible values of k .
- f) Given that k is a non zero constant and n is a positive integer, then
$$(1+kx)^n = 1 + 40x + 120k^2x^2 + \dots$$
Find the value of k and the value of n

Trinomial Expansion

Coefficient of $a^p b^q c^r$ in $(a + b + c)^n$ where $n = p + q + r$ is $\frac{n!}{p!q!r!}$

17.

(i) Find the coefficients of the following terms of the series expansion of $(1 + x + y^2)^7$

- a) x^6
- b) y^4
- c) $x^4 y^2$
- d) xy^{10}

(ii) Find the coefficients of the following terms of the series expansion of $(1 + 2x - 3y^2)^4$

- a) x^3
- b) y^2
- c) xy^4
- d) $x^2 y^2$

(iii) Find the coefficients of the following terms of the series expansion of $(1 + 2x + 3x^2)^5$

- a) x
- b) x^2
- c) x^9
- d) x^4

(iv) Find the coefficients of the following terms of the series expansion of $(1 + 2x - x^2)^6$

- a) x^2
- b) x^3
- c) x^8
- d) x^9