# Elementary Geometry (Version W)

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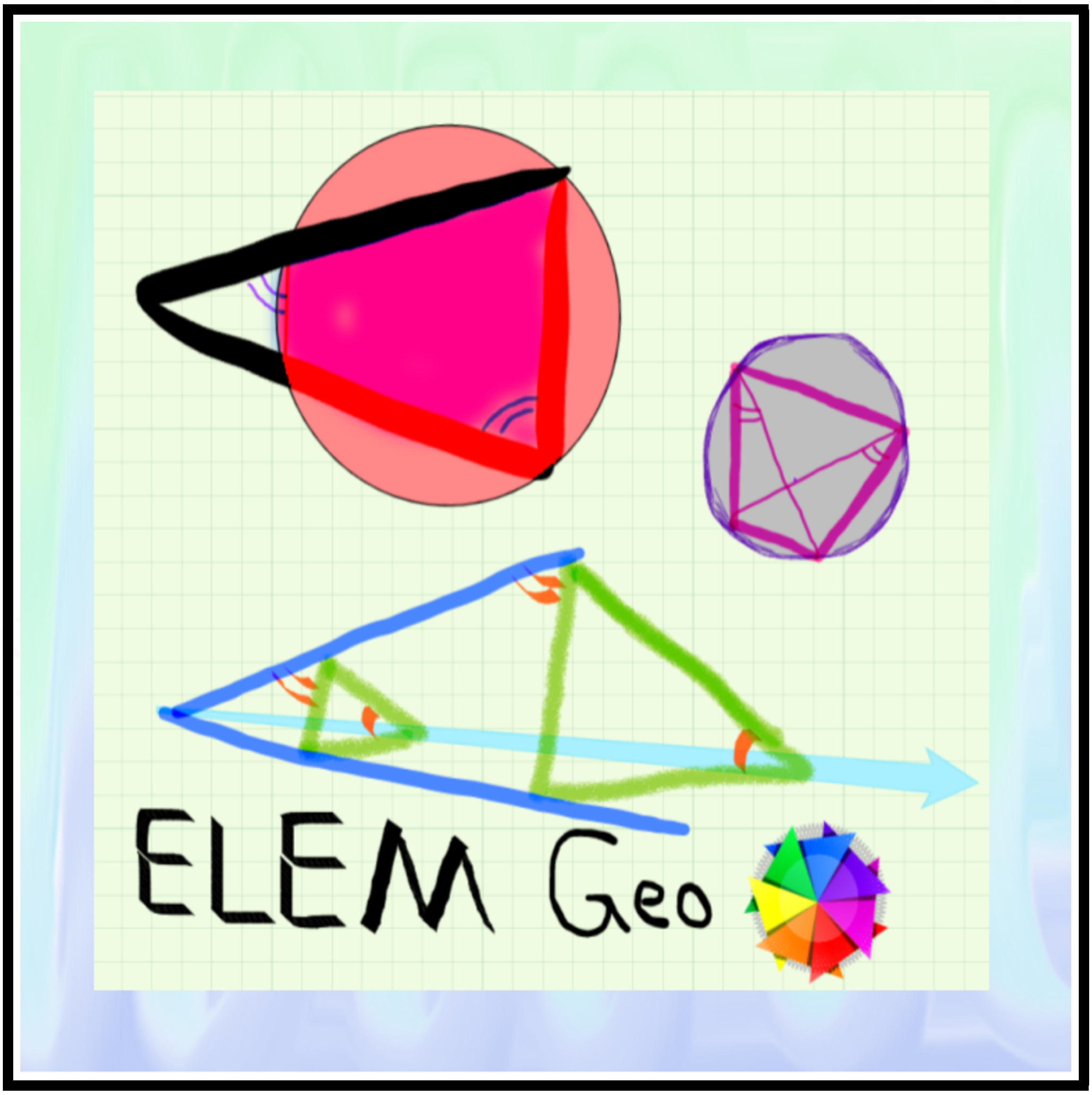
BGW-ELEMGEO



# §1 Lecture notes

This is some practice with the low-tech end of olympiad geometry, i.e. problems whose *canonical*[1] synthetic solutions essentially only use:

- Angle chasing
- Power of a point
- Homothety

> **Example 1.1** (USEMO 2021/4, Sayandeep Shee)
> Let $ABC$ be a triangle with circumcircle $\omega$, and let $X$ be the reflection of $A$ in $B$. Line $CX$ meets $\omega$ again at $D$. Lines $BD$ and $AC$ meet at $E$, and lines $AD$ and $BC$ meet at $F$. Let $M$ and $N$ denote the midpoints of $AB$ and $AC$.
> Can line $EF$ share a point with the circumcircle of triangle $AMN$?

21USEMO4 **Walkthrough.** There are lots of different approaches. We'll give the most classical one presented by the author, but there are plenty of other things that could work too!

**(a)** Let $P$ be the midpoint of $\overline{AD}$. Where else does $P$ lie?

**(b)** Show that $FB^2 = FP \cdot FA$.

**(c)** Show that $EB^2 = EN \cdot EA$.

**(d)** Using radical axis with a circle of radius zero at $B$, prove that line $EF$ is disjoint from $(AMN)$.

> **Example 1.2** (JMO 2018/3, Ray Li)
> Let $ABCD$ be a quadrilateral inscribed in circle $\omega$ with $\overline{AC} \perp \overline{BD}$. Let $E$ and $F$ be the reflections of $D$ over $\overline{BA}$ and $\overline{BC}$, respectively, and let $P$ be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of $EPD$ and $FPD$ meet $\omega$ at $Q$ and $R$ different from $D$. Show that $EQ = FR$.

18JMO3 **Walkthrough.** Most of this problem is about realizing where the points $P$, $Q$, $R$ are.

**(a)** Using what you know about the Simson line, figure out where point $P$ is.

**(b)** Determine the circumcenters of $\triangle EPD$ and $\triangle FPD$.

**(c)** Figure out where the points $Q$ and $R$ are.

**(d)** Finish the problem.

[1] I emphasize "canonical" since most official solutions can be rewritten in a way that avoids inversion, projective, etc. These problems are the ones for which even students who know this machinery would feasibly find the low-tech solution first.

**Example 1.3** (TSTST 2017/5, Ray Li)

Let $ABC$ be a triangle with incenter $I$. Let $D$ be a point on side $BC$ and let $\omega_B$ and $\omega_C$ be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that $\omega_B$ and $\omega_C$ are tangent to segment $BC$ at points $E$ and $F$, respectively. Let $P$ be the intersection of segment $AD$ with the line joining the centers of $\omega_B$ and $\omega_C$. Let $X$ be the intersection point of lines $BI$ and $CP$ and let $Y$ be the intersection point of lines $CI$ and $BP$. Prove that lines $EX$ and $FY$ meet on the incircle of $\triangle ABC$.

17TSTST5 **Walkthrough.** Let $\omega$ denote the incircle of $\triangle ABC$.

**(a)** Identify the point $Z = \overline{EX} \cap \overline{FY}$ in a good diagram. (This was worth a point! Despite this, many contestants were unable to find it.)

**(b)** Consider the positive homothety sending $\omega$ to $\omega_C$. Determine its center.

**(c)** Consider the negative homothety sending $\omega_C$ to $\omega_B$. Determine its center.

**(d)** The composition of the previous two homotheties in (b) and (c) is a negative homothety sending $\omega$ to $\omega_B$. Determine with proof the center of this homothety.

This is not as simple as the previous two parts; you will need to *use* (b) and (c) to do this part, as well as the simple observation that the center should lie on the $\angle B$ bisector.

**(e)** Conclude that $\overline{FY}$ passes through the point you claimed in (a).

Experts may notice that this walkthrough gives what is essentially a proof of Monge d'Alembert theorem.

**Example 1.4** (TSTST 2016/2, Evan Chen)

Let $ABC$ be a scalene triangle with orthocenter $H$ and circumcenter $O$ and denote by $M$, $N$ the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle $\gamma$ with diameter $\overline{AH}$ meets the circumcircle of $ABC$ at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to $\gamma$ at $G$ meets line $OM$ at $P$. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

16TSTST2 **Walkthrough.** Let $E$ and $F$ be the feet of the altitudes.

**(a)** Show that $P$ is really just the intersection of the tangents to $\gamma$ at $A$ and $G$ (and thus the line $\overline{OM}$ is just a distraction).

**(b)** Show that lines $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent, say at $R$.

**(c)** Prove that $(PAMG)$, $(MBC)$, $(MFDNE)$ are concurrent at a point $T \neq M$.

**(d)** Show that $T = \overline{PN} \cap \overline{MR}$.

**(e)** Show that $R \in \overline{HQ}$.

**(f)** Show that $R$, $G$, $T$, $Q$, $N$ are concyclic, completing the proof.

## §2 Practice problems

*Instructions*: Solve [36♣]. If you have time, solve [42♣]. Problems with red weights are mandatory.

> What kind of idiot would risk his life just to make a fake brother?
>
> Winry in *Full Metal Alchemist: Brotherhood*

REIM

[2♣] **Problem 1** (Reim's theorem). We say two lines $\ell_1$, $\ell_2$ are *antiparallel* with respect to $m_1$ and $m_2$, if the four points $\ell_i \cap m_j$ are the vertices of a cyclic quadrilateral.

Fix two lines $m_1$ and $m_2$. Prove that if $\ell_1$ is antiparallel to $\ell_2$ and $\ell_2$ is antiparallel to $\ell_3$, then lines $\ell_1$ and $\ell_3$ are either parallel or coincide.

A0AH60

[2♣] **Problem 2** (Added by Karn Chutinan). Let $ABC$ be a triangle with circumcenter $O$ and orthocenter $H$. Prove that $AO = AH$ if and only if $\angle BAC = 60^\circ$.

08IMO1

[2♣] **Problem 3** (IMO 2008/1). Let $H$ be the orthocenter of an acute-angled triangle $ABC$. The circle $\Gamma_A$ centered at the midpoint of $\overline{BC}$ and passing through $H$ intersects the sideline $BC$ at points $A_1$ and $A_2$. Similarly, define the points $B_1$, $B_2$, $C_1$, and $C_2$. Prove that six points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are concyclic.

15APMO1

[2♣] **Problem 4** (APMO 2015/1). Let $ABC$ be a triangle, and let $D$ be a point on side $BC$. A line through $D$ intersects side $AB$ at $X$ and ray $AC$ at $Y$. The circumcircle of triangle $BXD$ intersects the circumcircle $\omega$ of triangle $ABC$ again at point $Z$ distinct from point $B$. The lines $ZD$ and $ZY$ intersect $\omega$ again at $V$ and $W$ respectively. Prove that $AB = VW$.

13EGMO1

[2♣] **Problem 5** (EGMO 2013/1). The side $BC$ of the triangle $ABC$ is extended beyond $C$ to $D$ so that $CD = BC$. The side $CA$ is extended beyond $A$ to $E$ so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle $ABC$ is right-angled.

15UK

[2♣] **Problem 6** (UK 2015). Let $ABCD$ be a cyclic quadrilateral. Let $F$ be the midpoint of the arc $AB$ of its circumcircle which does not contain $C$ or $D$. Let the lines $DF$ and $AC$ meet at $P$ and the lines $CF$ and $BD$ meet at $Q$. Prove that the lines $PQ$ and $AB$ are parallel.

20BAMO4

[3♣] **Problem 7** (BAMO 2020/4). Let $ABC$ be a triangle and let $M$ be any point on segment $BC$. Denote by $O_B$ and $O_C$ the circumcenters of $\triangle ABM$ and $\triangle ACM$, respectively. Suppose $\omega$ is a circle whose center lies on $\overline{BC}$ and passes through $A$ and $M$. Lines $MO_B$ and $MO_C$ meet $\omega$ again at $K$ and $L$, respectively. Prove that lines $BK$ and $CL$ meet on $\omega$.

16HMIC2

[3♣] **Problem 8** (HMIC 2016/2). Let $ABC$ be an acute triangle with circumcenter $O$, orthocenter $H$, and circumcircle $\Omega$. Let $M$ be the midpoint of $BH$ and $N$ the midpoint of $CH$. Assume the points $M$, $N$, $O$, $H$ are distinct and lie on a circle $\omega$. Prove that the circles $\omega$ and $\Omega$ are internally tangent to each other.

15HMMTT4

[3♣] **Problem 9** (HMMT 2015 T4). Convex quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle $\Omega$; the diagonals of $ABCD$ meet at $X$. Suppose $AD < AB$, the circumcircle of triangle $BCX$ intersects segment $AB$ at a point $Y \neq B$, and ray $\overrightarrow{CY}$ meets $\Omega$ again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle $ZDB$.

04IMO1

[3♣] **Problem 10** (IMO 2004/1)**.** Let $ABC$ be an acute-angled triangle with $AB \neq AC$. The circle with diameter $BC$ intersects the sides $AB$ and $AC$ at $M$ and $N$ respectively. Denote by $O$ the midpoint of the side $BC$. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at $R$. Prove that the circumcircles of the triangles $BMR$ and $CNR$ have a common point lying on the side $BC$.

98AMO2

[2♣] **Problem 11** (USAMO 1998/2)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point $A$ on $\mathcal{C}_1$ one draws the tangent $AB$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let $C$ be the second point of intersection of ray $AB$ and $\mathcal{C}_1$, and let $D$ be the midpoint of $\overline{AB}$. A line passing through $A$ intersects $\mathcal{C}_2$ at $E$ and $F$ in such a way that the perpendicular bisectors of $\overline{DE}$ and $\overline{CF}$ intersect at a point $M$ on line $AB$. Find, with proof, the ratio $AM/MC$.

12GER3

[3♣] **Problem 12** (Germany 2012/3, added by Joel Gerlach)**.** Triangle $ABC$ has $\angle A = 120°$. Let $H$ and $O$ denote the orthocenter and circumcenter of $\triangle ABC$. Prove that

$$|AB| + |AC| = |OH|.$$

REMOROV

[3♣] **Problem 13.** Let $ABC$ be an acute triangle with circumcircle $\Gamma$ and intouch triangle $DEF$. A circle is drawn tangent to $\overline{BC}$ at $D$ and to minor arc $BC$ of $\Gamma$ at $X$. Define $Y$ and $Z$ similarly. Prove that lines $\overline{DX}$, $\overline{EY}$, $\overline{FZ}$ are concurrent.

23AMO1

[3♣] **Problem 14** (USAMO 2023/1)**.** In an acute triangle $ABC$, let $M$ be the midpoint of $\overline{BC}$. Let $P$ be the foot of the perpendicular from $C$ to $AM$. Suppose that the circumcircle of triangle $ABP$ intersects line $BC$ at two distinct points $B$ and $Q$. Let $N$ be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

06SLG2

[3♣] **Required Problem 15** (Shortlist 2006 G2)**.** Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points $K$ and $L$ lie on the line segments $AB$ and $CD$, respectively, so that $AK/KB = DL/LC$. Suppose points $P$ and $Q$ on segment $KL$ obey

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

Prove that $P$, $Q$, $B$, $C$ are concyclic.

17EGMO1

[3♣] **Problem 16** (EGMO 2017/1)**.** Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90°$ and $\angle ABC > \angle CDA$. Let $Q$ and $R$ be points on segments $BC$ and $CD$, respectively, such that line $QR$ intersects lines $AB$ and $AD$ at points $P$ and $S$, respectively. It is given that $PQ = RS$. Let the midpoint of $BD$ be $M$ and the midpoint of $QR$ be $N$. Prove that the points $M$, $N$, $A$ and $C$ lie on a circle.

20IGOB2

[3♣] **Problem 17** (Iran Geo Olympiad 2020, added by Pedro Rosalba)**.** Let $ABC$ be an isosceles triangle ($AB = AC$) with its circumcenter $O$. Point $N$ is the midpoint of the segment $BC$ and point $M$ is the reflection of the point $N$ with respect to the side $AC$. Suppose that $T$ is a point so that $ANBT$ is a rectangle. Prove that $\angle OMT = \frac{1}{2}\angle BAC$.

22USEMO4

[9♣] **Required Problem 18** (USEMO 2022/4)**.** Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points $P$, $Q$, $R$, $S$ lie in the interiors of segments $AB$, $BC$, $CD$, $DA$, respectively, such that

$$\angle PDA = \angle PCB, \quad \angle QAB = \angle QDC, \quad \angle RBC = \angle RAD, \quad \text{and} \quad \angle SCD = \angle SBA.$$

Let $\overline{AQ}$ intersect $\overline{BS}$ at $X$, and $\overline{DQ}$ intersect $\overline{CS}$ at $Y$. Prove that lines $\overline{PR}$ and $\overline{XY}$ are either parallel or coincide.

20ELMO04

[5♣] **Problem 19** (ELMO 2020/4)**.** Let $ABC$ be an acute scalene triangle $ABC$ with orthocenter $H$. Let $D$ be the foot of the $A$-altitude, and let $M$ be the midpoint of $\overline{BC}$. Let $D'$ be the reflection of $D$ over $M$. Point $P$ is chosen on line $D'H$ such that $\overline{AP} \parallel \overline{BC}$. Suppose the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at a point $G$ other than $H$. Prove that $\angle MHG = 90^\circ$.

97IMO2

[3♣] **Problem 20** (IMO 1997/2)**.** It is known that $\angle BAC$ is the smallest angle in the triangle $ABC$. The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$. The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$.

Show that $AU = TB + TC$.

15INDTST1

[5♣] **Problem 21** (India TST 2015/1)**.** Diagonals $\overline{AC}$ and $\overline{BD}$ of convex quadrilateral $ABCD$ meet at $P$. Prove that the incenters of the triangles $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PDA$ are concyclic if and only if their $P$-excenters are also concyclic.

23TSTST1

[5♣] **Required Problem 22** (TSTST 2023/1)**.** Let $ABC$ be a triangle with centroid $G$. Points $R$ and $S$ are chosen on rays $GB$ and $GC$, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

17SLG3

[9♣] **Problem 23** (Shortlist 2017 G3)**.** Let $ABC$ be an acute triangle with orthocenter $H$ and circumcenter $O$. Let $P = \overline{BH} \cap \overline{AO}$ and $Q = \overline{CH} \cap \overline{AO}$. Prove that the circumcenter of $\triangle HPQ$ lies on the $A$-median.

16EGMO4

[5♣] **Required Problem 24** (EGMO 2016/4)**.** Two circles $\omega_1$ and $\omega_2$, of equal radius intersect at different points $X_1$ and $X_2$. Consider a circle $\omega$ externally tangent to $\omega_1$ at $T_1$ and internally tangent to $\omega_2$ at point $T_2$. Prove that lines $X_1T_1$ and $X_2T_2$ intersect at a point lying on $\omega$.

99AMO6

[3♣] **Problem 25** (USAMO 1999/6)**.** Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle $\omega$ of triangle $BCD$ meets $CD$ at $E$. Let $F$ be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle $ACF$ meet line $CD$ at $C$ and $G$. Prove that the triangle $AFG$ is isosceles.

[1♣] **Mini Survey.** Fill out feedback on the OTIS-WEB portal when submitting this problem set. Any thoughts on problems (e.g. especially nice, instructive, easy, etc.) or overall comments on the unit are welcome.

In addition, if you have any suggestions for problems to add, or want to write hints for one problem you really liked, please do so in the ARCH system!

The maximum number of [♣] for this unit is [89♣], including the mini-survey.

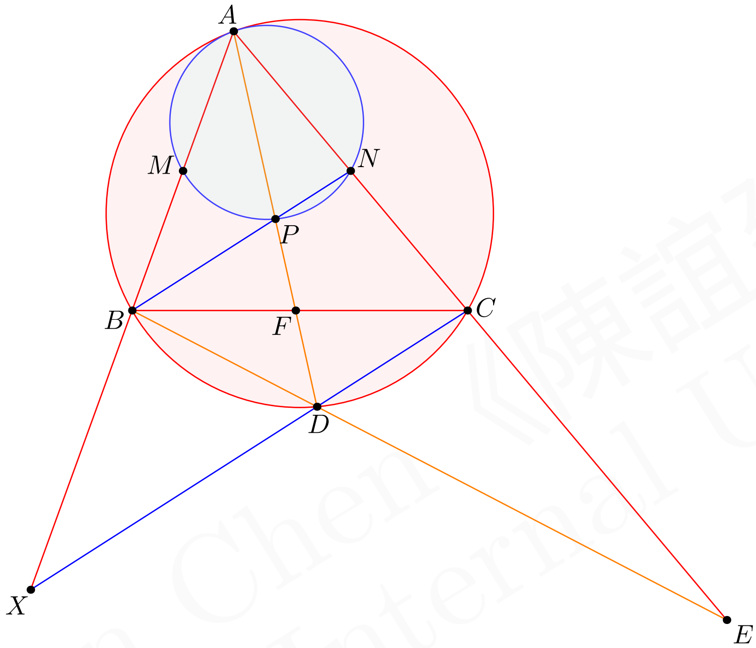
# §3 Solutions to the walkthroughs

## §3.1 Solution 1.1, USEMO 2021/4, Sayandeep Shee

The answer is no, they never intersect.

**¶ Classical solution, by author** Let $P$ denote the midpoint of $\overline{AD}$, which

- lies on $\overline{BN}$, since $\overline{BN} \parallel \overline{CX}$; and
- lies on $(AMN)$, since it's homothetic to $(ABC)$ through $A$ with factor $\frac{1}{2}$.




Now, note that

$$\begin{aligned} \measuredangle FBP = \measuredangle CBN = \measuredangle BCD = \measuredangle BAD = \measuredangle BAF &\implies FB^2 = FP \cdot FA \\ \measuredangle EBN = \measuredangle EDC = \measuredangle BDC = \measuredangle BAC = \measuredangle BAE &\implies EB^2 = EN \cdot EA. \end{aligned}$$

This means that line $EF$ is the radical axis of the circle centered at $B$ with radius zero, and the circumcircle of triangle $AMN$. Since $B$ obviously lies outside $(AMN)$, the disjointness conclusion follows.

**¶ Projective solution, by Ankit Bisain** In this approach we are still going to prove that $\overline{EF}$ is the radical axis of $(AMN)$ and the circle of radius zero at $B$, but we are not going to use the point $P$, or even points $E$ and $F$.

Instead, let $Y = \overline{EF} \cap \overline{AB}$, which by Brokard's theorem on $ABDC$ satisfies $(AB;XY) = -1$. Since $XB = XA$, it follows that $AY : YB = 2$. From here it is straightforward to verify that

$$YB^2 = \frac{1}{9}AB^2 = YM \cdot YA.$$

Thus $Y$ lies on the radical axis.

Finally, by Brokard's theorem again, if $O$ is the center of $\omega$ then $\overline{OX} \perp \overline{EF}$. Taking a homothety with scale factor 2 at $A$, it follows that the line through $B$ and the center of $(AMN)$ is perpendicular to $\overline{EF}$.

Since $\overline{EF}$ contains $Y$, it now follows that $\overline{EF}$ is the radical axis, as claimed.

**¶ Solution with inversion, projective, and Cartesian coordinates, by Ankan Bhattacharya** In what follows, let $O$ be the center of $\omega$. Note that Brokard's theorem gives that $\overline{EF}$ is the polar of $X$.

Note that since none of $E$, $F$, $X$ are points at infinity, $O$ is different from all three. We consider inversion in $\omega$ to eliminate the polar:

- The circumcircle of $\triangle AMN$, i.e. the circle with diameter $\overline{AO}$, is sent to the line $\ell$ tangent to $\omega$ at $A$.
- The line $EF$, as the polar of $X$, is sent to the circle with diameter $\overline{OX}$. (It is indeed a circle, because $O$ does not lie on line $EF$.)

Thus, if the posed question is true, then we see that $\ell$ intersects $(OX)$. We claim this is impossible.

Establish Cartesian coordinates with $A = (0,0)$ and $O = (2,0)$, so $\ell$ is the $y$-axis. Let $T$ be the center of $(OX)$: the midpoint of $\overline{OX}$. Observe:

- $B$ lies on the circle with center $(2,0)$ and radius 2.
- $X$ lies on the circle with center $(4,0)$ and radius 4.
- $T$ lies on the circle with center $(3,0)$ and radius 2.

Thus, let the coordinates of $T$ be $(x,y)$, with $(x-3)^2+y^2=4$. The intersection of $\ell$ and $(OX)$ being nonempty is equivalent to

$$\begin{aligned} & d(T,\ell)^2 \le OT^2 \\ \iff & x^2 \le (x-2)^2+y^2 \\ \iff & x^2 \le (x-2)^2+[4-(x-3)^2] \\ \iff & (x-1)^2 \le 0, \end{aligned}$$

or $x=1$ (which forces $y=0$); i.e. $T=(1,0)$. However, this forces

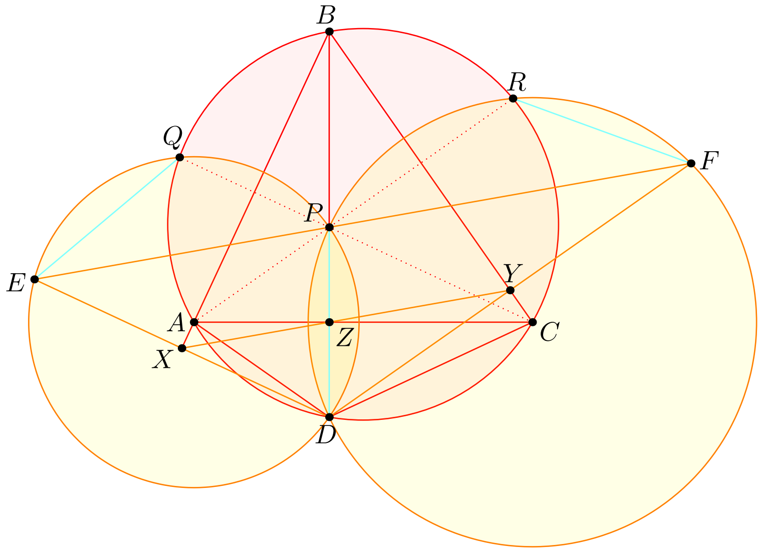
$$B = (0,0) = A,$$

which is not permitted. Thus, line $\ell$ cannot share a point with $(OX)$, and so line $EF$ cannot share a point with $(AMN)$.

## §3.2 Solution 1.2, JMO 2018/3, Ray Li

Most of this problem is about realizing where the points $P$, $Q$, $R$ are.

**¶ First solution (Evan Chen)** Let $X$, $Y$, be the feet from $D$ to $\overline{BA}$, $\overline{BC}$, and let $Z = \overline{BD} \cap \overline{AC}$. By Simson theorem, the points $X$, $Y$, $Z$ are collinear. Consequently, the point $P$ is the reflection of $D$ over $Z$, and so we conclude $P$ is the orthocenter of $\triangle ABC$.




Suppose now we extend ray $CP$ to meet $\omega$ again at $Q'$. Then $\overline{BA}$ is the perpendicular bisector of both $\overline{PQ'}$ and $\overline{DE}$; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q' = Q$. In the same way $R$ is the second intersection of ray $\overline{AP}$ with $\omega$.

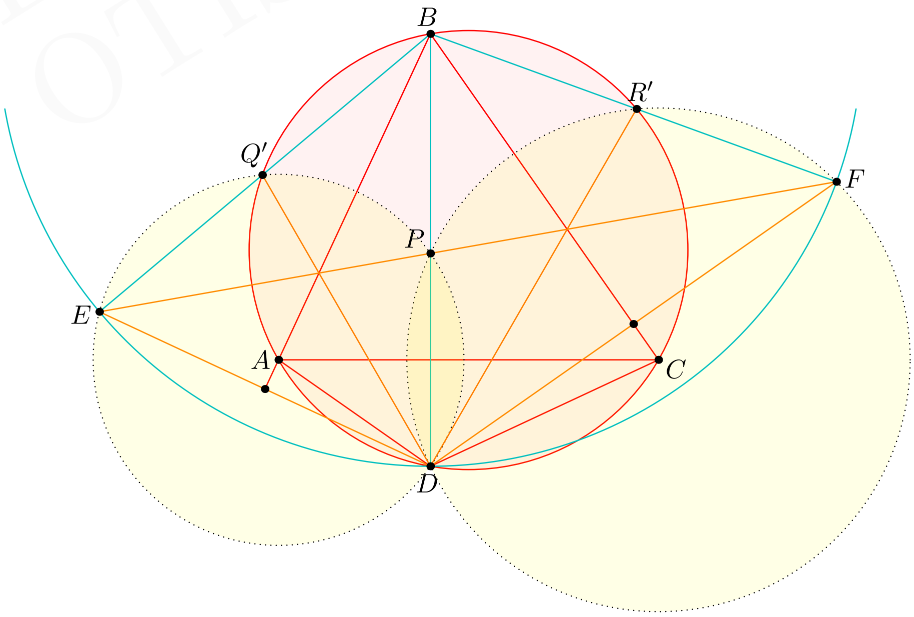
Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ = PD = FR$$

as desired.

> **Remark.** Alternatively, after identifying $P$, one can note $\overline{BQE}$ and $\overline{BRF}$ are collinear. Since $BE = BD = BF$, upon noticing $BQ = BP = BR$ we are also done.

**¶ Second solution (Danielle Wang)** Here is a solution which does not identify the point $P$ at all. We know that $BE = BD = BF$, by construction.



**Claim** — The points $B$, $Q$, $E$ are collinear. Similarly the points $B$, $R$, $F$ are collinear.

*Proof.* Work directed modulo 180°. Let $Q'$ be the intersection of $\overline{BE}$ with $(ABCD)$. Let $\alpha = \measuredangle DEB = \measuredangle BDE$ and $\beta = \measuredangle BFD = \measuredangle FDB$.

Observe that $BE = BD = BF$, so $B$ is the circumcenter of $\triangle DEF$. Thus, $\measuredangle DEP = \measuredangle DEF = 90° - \beta$. Then

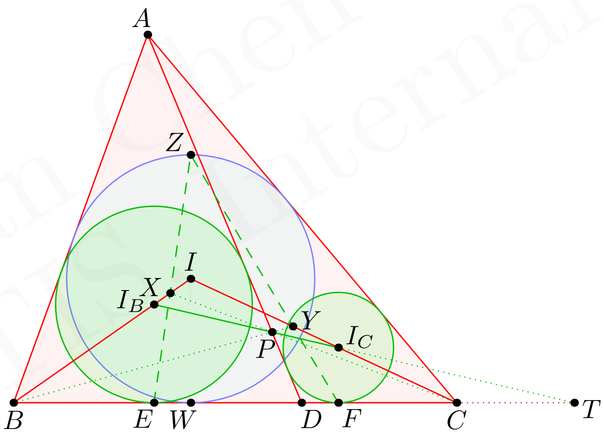
$$\begin{aligned}\measuredangle DPE &= \measuredangle DEP + \measuredangle PDE = (90° - \beta) + \alpha \\ &= \alpha - \beta + 90° \\ \measuredangle DQ'B &= \measuredangle DCB = \measuredangle DCA + \measuredangle ACB \\ &= \measuredangle DBA - (90° - \measuredangle DBC) = -(90° - \alpha) - (90° - (90° - \beta)) \\ &= \alpha - \beta + 90°.\end{aligned}$$

Thus $Q'$ lies on the desired circle, so $Q' = Q$. □

Now, by power of a point we have $BQ \cdot BE = BP \cdot BD = BR \cdot BF$, so $BQ = BP = BR$. Hence $EQ = PD = FR$.

## §3.3 Solution 1.3, TSTST 2017/5, Ray Li

**¶ First solution (homothety)** Let $Z$ be the diametrically opposite point on the incircle. We claim this is the desired intersection.




Note that:

- $P$ is the insimilicenter of $\omega_B$ and $\omega_C$
- $C$ is the exsimilicenter of $\omega$ and $\omega_C$.

Thus by Monge theorem, the insimilicenter of $\omega_B$ and $\omega$ lies on line $CP$.

This insimilicenter should also lie on the line joining the centers of $\omega$ and $\omega_B$, which is $\overline{BI}$, hence it coincides with the point $X$. So $X \in \overline{EZ}$ as desired.

**¶ Second solution (harmonic)** Let $T = \overline{I_B I_C} \cap \overline{BC}$, and $W$ the foot from $I$ to $\overline{BC}$. Define $Z = \overline{FY} \cap \overline{IW}$. Because $\angle I_B D I_C = 90°$, we have

$$-1 = (I_B I_C; PT) \stackrel{B}{=} (II_C; YC) \stackrel{F}{=} (I\infty; ZW)$$

So $I$ is the midpoint of $\overline{ZW}$ as desired.

**¶ Third solution (outline, barycentric, Andrew Gu)** Let $AD = t$, $BD = x$, $CD = y$, so $a = x + y$ and by Stewart's theorem we have

$$(x+y)(xy+t^2) = b^2x + c^2y. \tag{1}$$

We then have $D = (0 : y : x)$ and so

$$\overline{AI_B} \cap \overline{BC} = \left(0 : y + \frac{tx}{c+t} : \frac{cx}{c+t}\right)$$

hence intersection with $BI$ gives

$$I_B = (ax : cy + at : cx).$$

Similarly,

$$I_C = (ay : by : bx + at).$$

Then, we can compute

$$P = (2axy : y(at + bx + cy) : x(at + bx + cy))$$

since $P \in \overline{I_BI_C}$, and clearly $P \in \overline{AD}$. Intersection now gives

$$\begin{aligned} X &= (2ax : at + bx + cy : 2cx) \\ Y &= (2ay : 2by : at + bx + cy)\,. \end{aligned}$$

Finally, we have $BE = \frac{1}{2}(c + x - t)$, and similarly for $CF$. Now if we reflect $D = (0, \frac{s-c}{a}, \frac{s-b}{a})$ over $I = (\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s})$, we get the antipode

$$Q \coloneqq \left(4a^2 : -a^2 + 2ab - b^2 + c^2 : -a^2 + 2ac - c^2 + b^2\right).$$

We may then check $Q$ lies on each of lines $EX$ and $FY$ (by checking $\det(Q, E, X) = 0$ using the equation (1)).

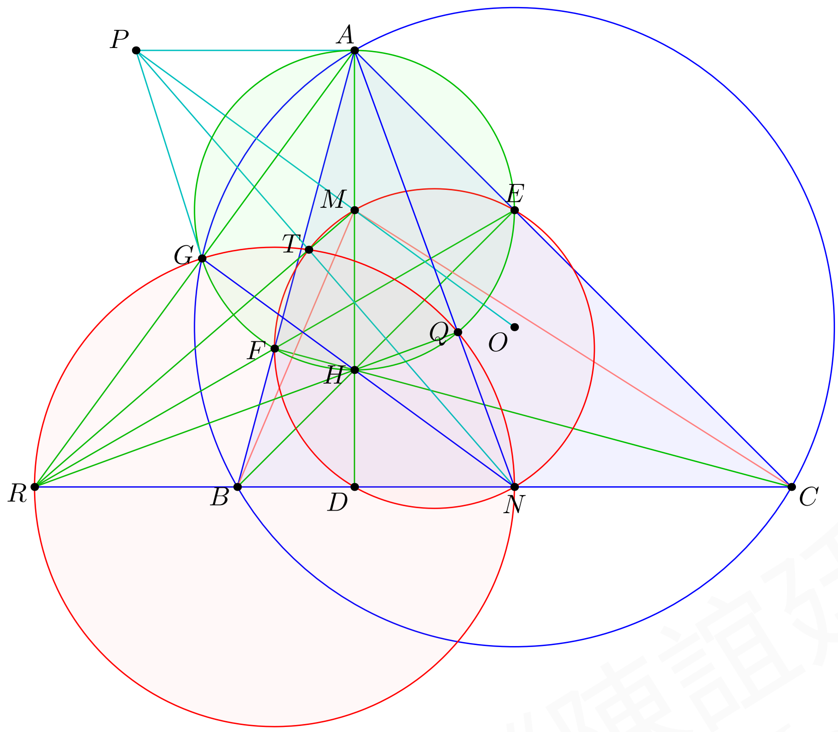
## §3.4 Solution 1.4, TSTST 2016/2, Evan Chen

We present two solutions, one using essentially only power of a point, and the other more involved.

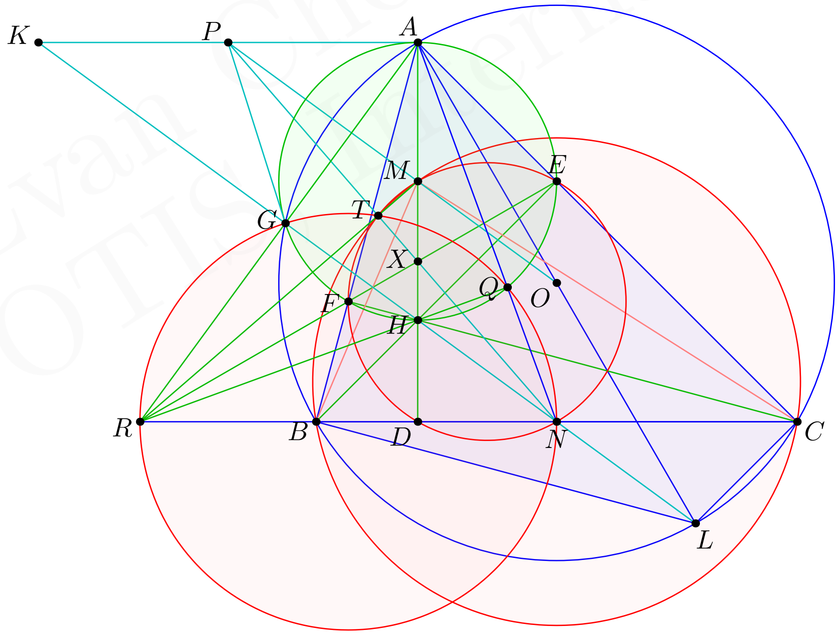
**¶ First solution (found by contestants)** Denote by $\triangle DEF$ the orthic triangle. Observe $\overline{PA}$ and $\overline{PG}$ are tangents to $\gamma$, since $\overline{OM}$ is the perpendicular bisector of $\overline{AG}$. Also note that $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent at some point $R$ by radical axis on $(ABC)$, $\gamma$, $(BC)$.

Now, consider circles $(PAGM)$, $(MFDNE)$, and $(MBC)$. They intersect at $M$ but have radical center $R$, so are coaxial; assume they meet again at $T \in \overline{RM}$, say. Then $\angle PTM$ and $\angle MTN$ are both right angles, hence $T$ lies on $\overline{PN}$.

Finally $H$ is the orthocenter of $\triangle ARN$, and thus the circle with diameter $\overline{RN}$ passes through $G$, $Q$, $N$.




**¶ Alternate solution (by proposer)** Let $L$ be diametrically opposite $A$ on the circumcircle. Denote by $\triangle DEF$ the orthic triangle. Let $X = \overline{AH} \cap \overline{EF}$. Finally, let $T$ be the second intersection of $(MFDNE)$ and $(MBC)$.




We begin with a few easy observations. First, points $H$, $G$, $N$, $L$ are collinear and $\angle AGL = 90^\circ$. Also, $Q$ is the foot from $H$ to $\overline{AN}$. Consequently, lines $AG$, $EF$, $HQ$, $BC$, $TM$ concur at a point $R$ (radical axis). Moreover, we already know $\angle MTN = 90^\circ$. This implies $T$ lies on the circle with diameter $\overline{RN}$, which is exactly the circumcircle of $\triangle GQN$.

Note by Brokard's Theorem on $AFHE$, the point $X$ is the orthocenter of $\triangle MBC$. But $\angle MTN = 90^\circ$ already, and $N$ is the midpoint of $\overline{BC}$. Consequently, points $T$, $X$,

$N$ are collinear.

Finally, we claim $P$, $X$, $N$ are collinear, which solves the problem. Note $P = \overline{GG} \cap \overline{AA}$. Set $K = \overline{HNL} \cap \overline{AP}$. Then by noting

$$-1 = (D, X; A, H) \overset{N}{=} (\infty, \overline{NX} \cap \overline{AK}; A, K)$$

we see that $\overline{NX}$ bisects segment $\overline{AK}$, as desired. (A more projective finish is to show that $\overline{PXN}$ is the polar of $R$ to $\gamma$).

**Remark.** The original problem proposal reads as follows:

> Let $ABC$ be a triangle with orthocenter $H$ and circumcenter $O$ and denote by $M$, $N$ the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose ray $OM$ meets the line parallel to $\overline{BC}$ through $A$ at $P$. Prove that the line through the circumcenter of $\triangle MBC$ and the midpoint of $\overline{OH}$ is parallel to $\overline{NP}$.

The points $G$ and $Q$ were added to the picture later to prevent the problem from being immediate by coordinates.