On the Desargues' Involution Theorem

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As the title suggests, this article will deal with powerful theorems in projective geometry: Desargues' Involution Theorem and its variants. In addition, we will present some Olympiad problems which can be solved with these theorems. Readers are expected to be familiar with projective geometry, inversion, and some basics of conics.

§1 What is involution?

As you may have guessed, this article will deal with involutions. In general, an involution is any function $f : A \to A$ satisfying $f(f(x)) = x$ for every $x \in A$, but let us restrict a bit more by adding the following condition.

Definition 1. Let $\mathcal{P}$ be the set of all points on a fixed line or a fixed conic. Then a function $f : \mathcal{P} \to \mathcal{P}$ is called *an involution* if and only if it satisfies two conditions.

(i) f preserves cross-ratio, i.e. for any points $A, B, C, D \in \mathcal{P}$,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)).$$

(ii) $f(f(A)) = A$ for every point $A \in \mathcal{P}$.

Furthermore, we call a pair in form $(A, f(A))$ *reciprocal pair*.

We give the following observation.

Theorem 2

Let $\mathcal{P}$ be the set of all points on a fixed line or a fixed conic. If a function $f : \mathcal{P} \to \mathcal{P}$ preserves cross ratio and has two points $A, A' \in \mathcal{P}$ satisfying both $f(A) = A'$ and $f(A') = A$, then f is an involution.

Proof. Let $P \in \mathcal{P}, Q = f(P)$ and $P' = f(Q)$. Then, we have

$$(A, A'; P, Q) = (f(A), f(A'); f(P), f(Q)) = (A', A; Q, P') = (A, A'; P', Q),$$

which implies that $P' = P$ as desired. □

Now we will classify involution on a line and involution on a conic. Both classifications are very different, so we will split the discussion into two subsections.

§1.1 Involution on a line

Some readers may want to see some examples of involutions on a line ℓ. The most obvious ones are identity function (trivial) and reflection across a fixed point on ℓ. Furthermore, isogonal-conjugating is an involution as well (it preserves cross-ratio because it is a composition of reflection and projection on to ℓ). But in fact, even the inversion around a point on ℓ is an involution.

Theorem 3

Let ℓ be a line. Then, the inversion (possibly a negative inversion) around a fixed point on ℓ is an involution on ℓ.

Proof. Let A, B, C, D be any four points on line ℓ, and let A', B', C', D' be their inverted image. It suffices to show that

$$(A, B; C, D) = (A', B'; C', D')$$

Let O and p be the center and the power of this inversion, respectively (thus $p = \overline{OA} \cdot \overline{OA'} = \ldots$). Using directed lengths, we find that

$$\begin{aligned}(A', B' : C', D') &= \frac{(\overline{OA'} - \overline{OC'})(\overline{OB'} - \overline{OD'})}{(\overline{OB'} - \overline{OC'})(\overline{OA'} - \overline{OD'})} \\ &= \frac{\left(\frac{p}{\overline{OA}} - \frac{p}{\overline{OC}}\right)\left(\frac{p}{\overline{OB}} - \frac{p}{\overline{OD}}\right)}{\left(\frac{p}{\overline{OB}} - \frac{p}{\overline{OC}}\right)\left(\frac{p}{\overline{OA}} - \frac{p}{\overline{OD}}\right)} \\ &= \frac{(\overline{OC} - \overline{OA})(\overline{OD} - \overline{OB})}{(\overline{OC} - \overline{OB})(\overline{OD} - \overline{OA})} \\ &= (A, B; C, D),\end{aligned}$$

as desired. □

Furthermore, the converse of Theorem 3 holds as well.

Theorem 4

Any involution on a line ℓ is either an inversion of a nonzero (possibly negative) power, or a reflection across a fixed point.

Proof. Let the involution swap point P and point ∞ (i.e. the point at infinity on ℓ), swap points X_1 and X_2, and swap points Y_1 and Y_2.

Case 1. $P \neq \infty$

Then, we have

$$(P, \infty; X_1, Y_1) = (\infty, P; X_2, Y_2) \implies \frac{PX_1}{PY_1} = \frac{PY_2}{PX_2}$$

Hence, $\overline{PX_1} \cdot \overline{PX_2} = \overline{PY_1} \cdot \overline{PY_2}$ (lengths are directed). Therefore, this involution must be an inversion centered at P.

Case 2. $P = \infty$

Then, we have

$$(\infty, X_1; Y_1, Y_2) = (\infty, X_2; Y_2, Y_1) \implies \frac{X_1Y_1}{X_1Y_2} = \frac{X_2Y_2}{X_2Y_1}.$$

With directed length, this means that $X_1Y_1 = X_2Y_2$, or $\overline{X_1Y_1}$ and $\overline{X_2Y_2}$ share the same midpoint M. The involution is the reflection across M. □

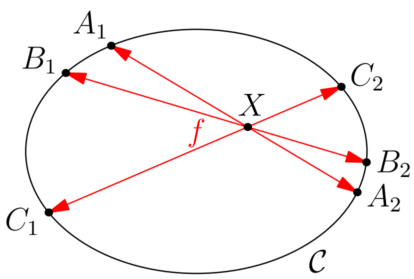
§1.2 Involution on a conic

Surprisingly, the description of an involution on a conic is very simple.

Theorem 5

Let $\mathcal{C}$ be a conic. Then for any involution f on $\mathcal{C}$, there exists a fixed point X such that f takes point A to the second intersection of XA and $\mathcal{C}$.

Note. The converse of this obviously holds: the projection from X is known to preserve cross ratio (say by inversion around X when $\mathcal{C}$ is a circle), hence it must be an involution.



Proof. Since the statement is purely projective, we can take any projective transformation which sends $\mathcal{C}$ to a circle. Therefore, without loss of generality, we may assume that $\mathcal{C}$ be a circle.

Let $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ be any three reciprocal pairs of f. Perform an inversion around any point $P \in \mathcal{C}$; it takes $\mathcal{C}$ to a line ℓ, and $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ become reciprocal pairs of an involution f' on a line ℓ (due to the projection from P). We must show that the circles $\odot(PA_1A_2)$, $\odot(PB_1B_2)$, $\odot(PC_1C_2)$ are coaxial.

By Theorem 4, there exists a point $K \in \ell$ such that $KA_1 \cdot KA_2 = KB_1 \cdot KB_2 = KC_1 \cdot KC_2$ (lengths are directed). Therefore, K lies on radical axis of these three circles. However, they have a common point P, hence they have the pairwise radical axis PK. □

We would like to remark that we can project an involution from a line to a conic. This can be a possible way to prove that three chords are concurrent.

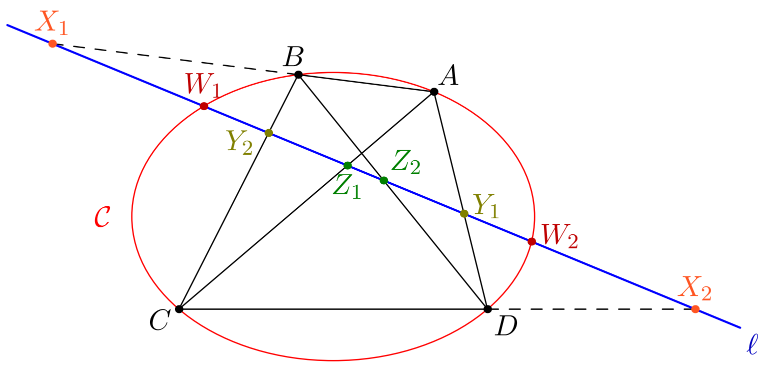
§2 Desargues' Involution Theorem (DIT)

Now it may seem that involutions are useless because it is just an inversion. However, our definition gives a more projective viewpoint of an inversion. Using the definitions we have built so far, we are ready to present the main theorem.

§2.1 The main theorem

Theorem 6 (Desargues' Involution Theorem / DIT)

Let $ABCD$ be a quadrilateral. A line ℓ intersects the lines AB, CD, AD, BC, AC, BD at points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$, respectively. A conic $\mathcal{C}$ passing through points A, B, C, D intersects ℓ at points W_1, W_2. Then, the pairs (W_1, W_2), (X_1, X_2), (Y_1, Y_2), (Z_1, Z_2) are reciprocal pairs of some involution on ℓ.



The proof of this theorem requires cross-ratio on a conic. Make sure that you know (or convince yourself) that projecting from a line to a conics from a point at that conic preserves cross-ratio.

Proof. [2] Recall that there exists a projective transformation f which fixes a line ℓ and sends points W_1, W_2, X_1 to W_2, W_1, X_2, respectively. By Theorem 2, f is an involution, hence (X_1, X_2) is a reciprocal pair too. Now, it suffices to prove that $(Y_1, Y_2), (Z_1, Z_2)$ are reciprocal pairs too. To that end, observe that

$$(X_1, Y_1; W_1, W_2) \stackrel{A}{=} (B, D; W_1, W_2)_{\mathcal{C}} \stackrel{C}{=} (Y_2, X_2; W_1, W_2) = (X_2, Y_2; W_2, W_1)$$

But we already know that $f(X_1) = X_2, f(W_1) = W_2, f(W_2) = W_1$. Therefore, $f(Y_1) = Y_2$, and hence (Y_1, Y_2) is a reciprocal pair of f. Similarly, (Z_1, Z_2) is a reciprocal pair of f too, and we are done. $\square$

Note. You will rarely require all four reciprocal pairs at once.

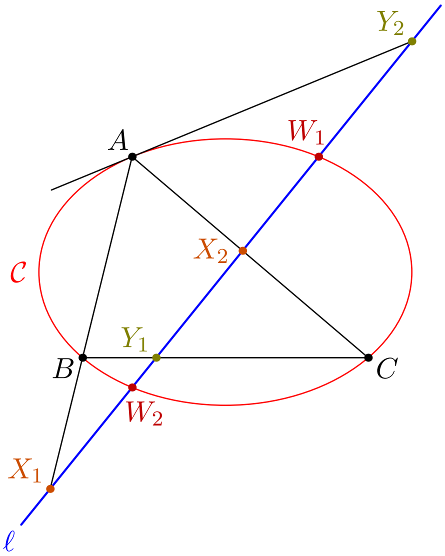
The degenerate cases of this theorem hold. Here, we present the three points and two points version of this theorem. Convince yourself that the following statements are true.

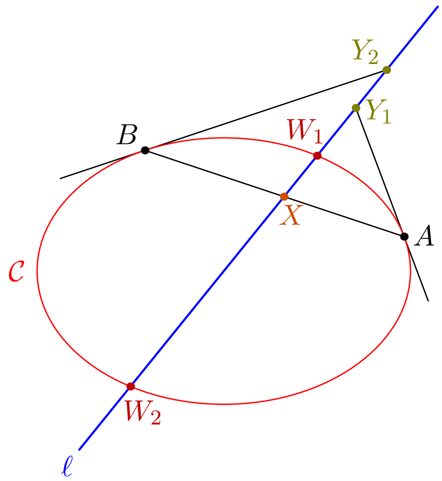
Theorem 7 (3 Points DIT)

Let ABC be a triangle inscribed in a conic $\mathcal{C}$. A line ℓ intersects lines AB, AC, BC at X_1, X_2, Y_1 and intersects the line tangent to $\mathcal{C}$ at A at Y_2. Line ℓ also intersects $\mathcal{C}$ at W_1, W_2. Then, the pairs (W_1, W_2), (X_1, X_2), (Y_1, Y_2) are reciprocal pairs of some involution on ℓ.

Theorem 8 (2 Points DIT)

Let A, B be points on a conic $\mathcal{C}$. A line ℓ intersects lines AB at X and intersects lines tangent to $\mathcal{C}$ at A, B at Y_1, Y_2. Line ℓ also intersects $\mathcal{C}$ at W_1, W_2. Then, the pairs (W_1, W_2), (X, X), (Y_1, Y_2) are reciprocal pairs of some involution on ℓ.





§2.2 Dual of Desargues' Involuion Theorem (DDIT)

In projective geometry, one of the key transformations is the pole-polar transformation, which turn colinearity into concurrency. This transformation makes a purely projective theorem come in pairs (e.g. Pascal's and Brianchon's, Newton's and Brokard's). Dersargues' Involution Theorem also have it's pair (or dual).

Definition 9. Let P be a point on the plane. Let $\mathcal{L}$ be the set of all line that passes through P. Then, $f : \mathcal{L} \to \mathcal{L}$ is *an involution on a pencil of lines* if and only if.

(i) For every $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD} \in \mathcal{L}$, we have

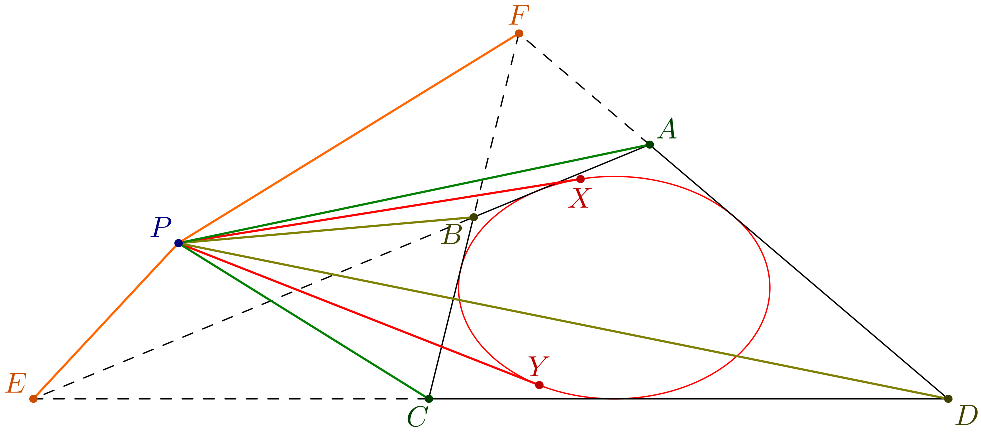
$$(\overline{PA}, \overline{PB}; \overline{PC}, \overline{PD}) = (f(\overline{PA}), f(\overline{PB}); f(\overline{PC}), f(\overline{PD})).$$

(ii) $f(f(\ell)) = \ell$ for every $\ell \in \mathcal{L}$. Furthermore, we call a pair in form $(\ell, f(\ell))$ *reciprocal pair.*

Of course, if we have an involution on a pencil, we can project onto a line to get an involution on a line.

Theorem 10 (Dual of Desargues' Involution Theorem , DDIT)

Let P, A, B, C, D be fives points on a plane, and let $\overline{AB} \cap \overline{CD} = E, \overline{AD} \cap \overline{BC} = F$. Let $\mathcal{C}$ be a conic tangent to the lines AB, CD, AD, BC, and let $\overline{PX}, \overline{PY}$ be the tangents to $\mathcal{C}$ passing through P. Then, the pairs $(\overline{PX}, \overline{PY}), (\overline{PA}, \overline{PC}), (\overline{PB}, \overline{PD}), (\overline{PE}, \overline{PF})$ are reciprocal pairs of some involution on the pencil of lines passing through P.



Proof. Take the pole-polar transformation with respect to $\mathcal{C}$. Then, the above theorem is equivalent to the original Desargues' Involution Theorem. $\square$

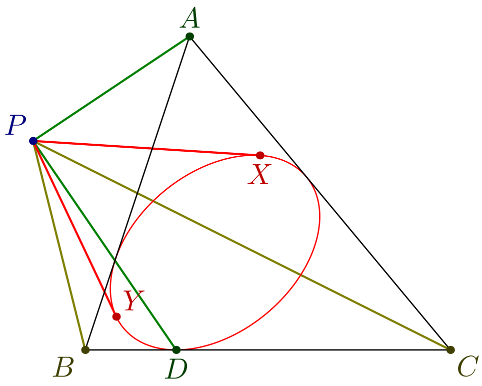
Usually, this dual statement is more useful than the original one. The three points and two points versions are also true.

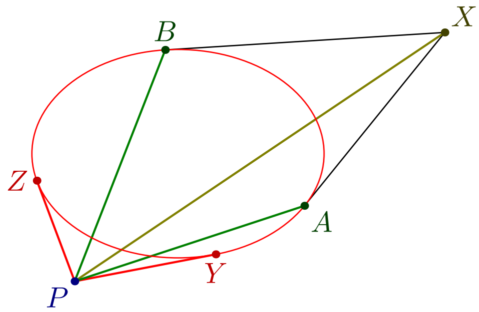
Theorem 11 (3 Points DDIT)

Let ABC be a triangle and P be a point on a plane. A conic $\mathcal{C}$ which is tangent to lines BC, AC, AB meets BC at D. Let PX, PY be the two tangents to $\mathcal{C}$ that passes through P. Then, the pairs $(\overline{PX}, \overline{PY})$, $(\overline{PA}, \overline{PD})$, $(\overline{PB}, \overline{PC})$ are reciprocal pairs of some involution on the pencil of lines passing through P.

Theorem 12 (2 Points DDIT)

Let A, B be points on a conic $\mathcal{C}$ and let P be a point on a plane. The tangents to $\mathcal{C}$ at A, B intersect at X. Let PY, PZ be the two tangents to $\mathcal{C}$ that passes through P. Then, the pairs $(\overline{PY}, \overline{PZ})$, $(\overline{PX}, \overline{PX})$, $(\overline{PA}, \overline{PB})$ are reciprocal pairs of some involution on the pencil of lines passing through P.



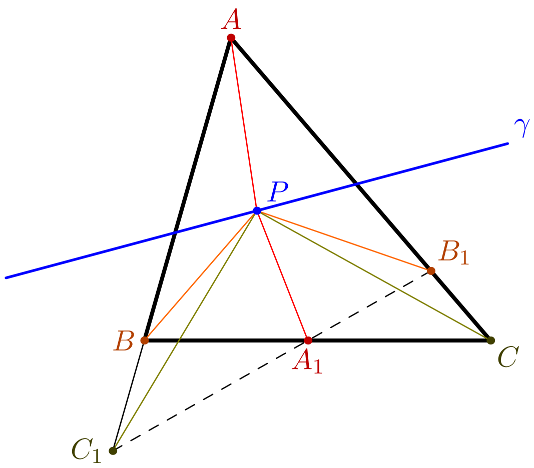


Note : Usually, we only use the special case where $\mathcal{C}$ is a circle.

§3 Examples

Let us present the first problem, which is known to be very direct with DDIT.

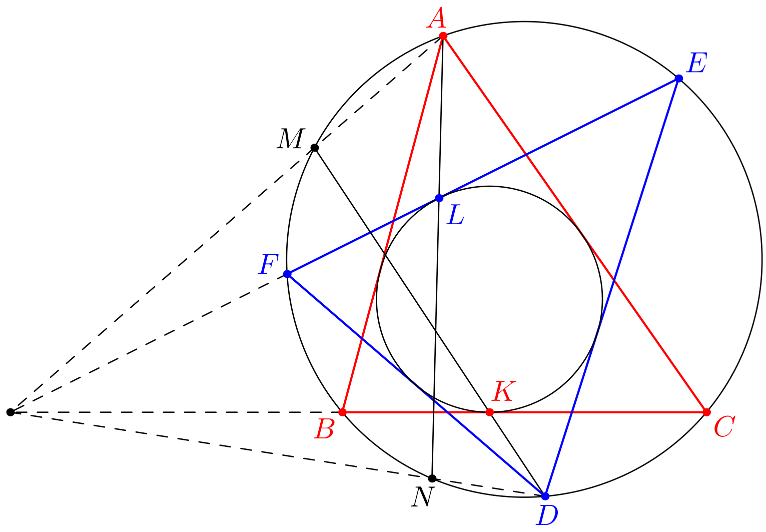
Example 1 (USAMO 2012 P5)**.** Let P be a point in the plane of $\triangle ABC$, and γ be a line passing through P. Let A_1, B_1, C_1 be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A_1, B_1, C_1 are collinear.



Solution. By reflection, there is an involution swapping $(\overline{PA}, \overline{PA_1}), (\overline{PB}, \overline{PB_1}), (\overline{PC}, \overline{PC_1})$. If $C_1' = \overline{A_1B_1} \cap \overline{AB}$, then by DDIT, this involution must swap $(\overline{PC}, \overline{PC_1'})$. Hence, $C_1 = C_1'$, and we are done. □

Let we present a use of involution on a conic. This problem is actually very straightforward with involution but is very difficult to solve in other ways.

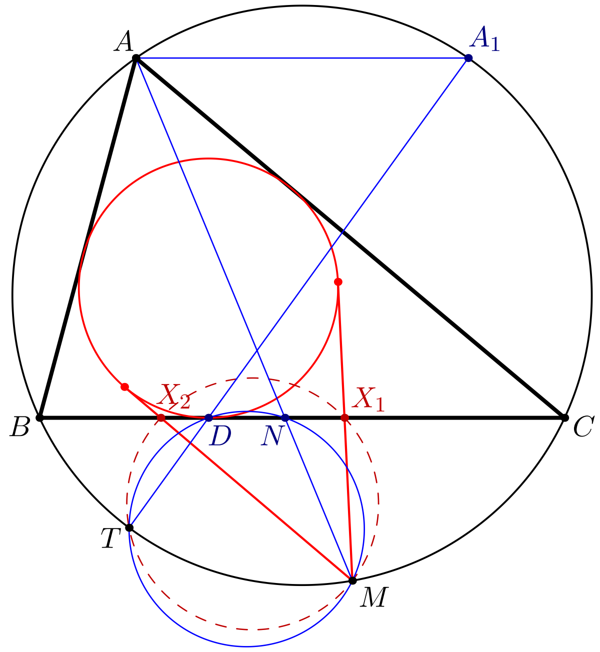
Example 2 (EulerMacaroni on AoPS)**.** Let ABC and DEF be two triangles sharing the same incircle ω and circumcircle γ. Let L be the tangency point of EF to ω and let K be the tangency point of BC to ω. Let the lines AL and DK meet γ again at N and M, respectively. Show that the lines AM, EF, BC, ND are concurrent.



Solution. By DDIT (on D and $ABKC$), there exists an involution swapping $(\overline{DA}, \overline{DK})$, $(\overline{DB}, \overline{DC})$, $(\overline{DE}, \overline{DF})$. Projecting this involution to γ, we obtain an involution swapping (A, M), (B, C), (E, F). By Theorem 5, the lines AM, EF, BC are concurrent. Similarly, the lines EF, BC, ND are concurrent, and we are done. □

We would like to close this article with the following extremely difficult problem from Taiwan TST. No one solved this during exam, but it has a short solution with DDIT.

Example 3 (Taiwan TST3 2014 P3)**.** Let M be any point on the circumcircle of $\triangle ABC$. Suppose that the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that the circumcircle of $\triangle MX_1X_2$ intersects the circumcircle of $\triangle ABC$ again at the tangency point of the A-mixtilinear incircle.



Solution. Let $\odot(I)$ denote the incircle of $\triangle ABC$, which touches BC at D, and let T be the tangency point of the A-mixtilinear incircle.

By DDIT, there exists an involution swapping $(\overline{MX_1}, \overline{MX_2})$, $(\overline{MB}, \overline{MC})$, $(\overline{MD}, \overline{MA})$. Projecting onto the line BC and defining $\overline{AM} \cap \overline{BC} = N$, we obtain an involution swapping $(D, N), (B, C), (X_1, X_2)$.

By Theorem 4, let K be the center of inversion and consider the circles $\odot(ABMC)$, $\odot(MDN)$ $\odot(MX_1X_2)$. We have $KB \cdot KC = KD \cdot KN = KX_1 \cdot KX_2$. Thus, MK is the pairwise radical axis of these three circles, therefore they must meet at another point.

Now, it suffices to show that the point M, D, N, T are concyclic. To that end, extend $\overline{TD}$ to meet $\odot(ABC)$ at A_1. It is a well known mixtilinear incircle property (see [1]) that $AA_1 \parallel BC$. Hence, we are done because $\angle ANB = \angle A_1AM = \angle A_1TM = \angle DTM$. □

§4 Practice Problems

Here are a few practice problems I have assembled. On the easier end are the problems which is nearly immediate when using (D)DIT, but on the harder end are the problems which applying (D)DIT is a tiny step in it. The problems are sorted roughly by difficulty. Have fun!

§4.1 Beginner

Problem 1. Use involution to prove Pappus' theorem.

Problem 2 (China TST 2017)**.** Let $ABCD$ be a quadrilateral and let ℓ be a line. Let ℓ intersect the lines AB, CD, BC, DA, AC, BD at points X, X', Y, Y', Z, Z', respectively. Given that these six points on ℓ are in the order X, Y, Z, X', Y', Z', show that the circles with diameter XX', YY', ZZ' are coaxial.

Problem 3 (Generalization of Serbia MO 2018)**.** A quadrilateral $ABCD$ has an incircle centered at I. Point X is placed outside this quadrilateral so that $\angle AXC$ and $\angle BXD$ has a common angle bisector. Show that this bisector passes through I.

Problem 4 (Parallelogram Isogonality Lemma)**.** Let P be a point inside $\triangle ABC$ such that $\angle ABP = \angle ACP$. If Q is the point which $BPCQ$ is a parallelogram, prove that $\angle BAP = \angle CAQ$.

Problem 5 (Existence of Orthotransversal)**.** Let P be a point inside $\triangle ABC$. Place points X, Y, Z on the lines BC, CA, AB, respectively so that $\angle APX = \angle BPY = \angle CPZ = 90°$. Prove that the points X, Y, Z are colinear.

§4.2 Intermediate

Problem 6 (Serbia MO 2017)**.** Let ω be the circumcircle of $\triangle ABC$, and let ω_A be the A-excircle. Let the two common tangents of ω, ω_A cut BC at points P, Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

Problem 7 (CGMO 2017)**.** Let $ABCD$ be a cyclic quadrilateral with circumcircle ω_1. Lines AC and BD meet at E, while lines AD and BC meet at F. Circle ω_2 is tangent to segments EB and EC at M and N, respectively, and ω_2 also intersects with the circle ω_1 at Q and R. The lines BC, AD intersect the line MN at S and T. Show that Q, R, S, T are concyclic.

Problem 8 (USMCA 2020)**.** Let $ABCD$ be a convex quadrilateral, and let ω_A and ω_B be the incircles of $\triangle ACD$ and $\triangle BCD$, with centers I and J. The second common external tangent to ω_A and ω_B touches ω_A at K and ω_B at L. Prove that lines AK, BL, IJ are concurrent.

Problem 9 (ISL 2005)**.** Let ABC be a triangle, and M be the midpoint of the side BC. Let γ be the incircle of $\triangle ABC$. The median AM of $\triangle ABC$ intersects γ at X and Y. The lines through X and Y, parallel to BC, intersect the incircle γ again at X_1 and Y_1. The lines AX_1 and AY_1 intersect BC at P and Q. Prove that $BP = CQ$.

Problem 10 (No source)**.** Let $ABCD$ be a convex quadrilateral. The incenter and B-excenter of $\triangle BCA$ are I and J respectively; the incenter and B-excenter of $\triangle BCD$ are I' and J' respectively. Lines IJ' and $I'J$ intersect at P. Prove that CP bisects $\angle ACD$.

Problem 11 (titanian on AoPS)**.** Let ABC be a triangle with orthocenter H, circumcenter O, and circumcircle Ω. Let M_A, M_B, M_C be the midpoints of sides BC, CA, AB. Lines AM_A, BM_B, CM_C intersect Ω again at P_A, P_B, P_C, respectively. Rays M_AH, M_BH, M_CH intersect Ω at Q_A, Q_B, Q_C. Prove that lines P_AQ_A, P_BQ_B, P_CQ_C and OH are concurrent.

Problem 12 (EGMO 2018)**.** Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

§4.3 Advanced

Problem 13 (Serbia 2013)**.** Let O be the circumcenter of an acute triangle ABC, and let M_a, M_b and M_c be the midpoints of sides BC, AC and AB, respectively. The circumcircles of triangles BOC and $M_aM_bM_c$ intersect at two different points X and Y inside $\triangle ABC$. Prove (without inversion) that $\angle BAX = \angle CAY$.

Problem 14 (IMO 2019)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 15 (ISL 2012 G8)**.** Let ABC be a triangle with circumcenter O and ℓ be a line. Denote by P the foot from O to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circles $\odot(AXP)$, $\odot(BYP)$ and $\odot(CZP)$ are coaxial.

Problem 16 (China TST 2017)**.** In a non-isosceles triangle ABC, M_A, M_B, M_C are the midpoints of sides BC, CA, AB, respectively. The line (different from line BC) through M_A that is tangent to the incircle of triangle $\triangle ABC$ intersects the line $M_B M_C$ at X. Define Y, Z similarly. Prove that X, Y, Z are collinear.

Problem 17 (ISL 2015)**.** Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA. Let the segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or all parallel.

Problem 18 (Summer MO 2020)**.** Let $\triangle ABC$ be an acute scalene triangle with incenter I and incircle ω. Two points X and Y are chosen on minor arcs AB and AC, respectively, of the circumcircle of triangle $\triangle ABC$ such that XY is tangent to ω at P and $\overline{XY} \perp \overline{AI}$. Let ω be tangent to sides AC and AB at E and F, respectively. Denote the intersection of lines XF and YE as T.

Prove that if the circumcircles of triangles $\triangle TEF$ and $\triangle ABC$ are tangent at some point Q, then lines PQ, XE, and YF are concurrent.

§4.4 Some fun with conics

Problem 19. Let H be the orthocenter of $\triangle ABC$, and let D, E, F be the feet of altitude from A, B, C to the sides BC, CA, AB, respectively. Let $\mathcal{H}$ be a hyperbola that passes through A, B, C, H, and let P, Q be isogonal conjugates with respect to $\triangle DEF$. Prove that P lies on the polar of Q with respect to $\mathcal{H}$.

Problem 20. Let A, B, C, D, E, F be six points lying on a conic $\mathcal{C}$. Suppose that $\triangle ABC$ and $\triangle DEF$ share the common orthocenter H. Prove that H lies on the radical axis of $\odot(ABC)$ and $\odot(DEF)$.

References

[1] Evan Chen. A guessing game: Mixtilinear incircles, 2015.

[2] Michael Woltermann. 63. Desargues' Involution Theorem, 2010.