Entry Combinatorics

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BCX-ENTRY

§1 Reading

Read Chapter 1 of the *OTIS Excerpts* (Notes on Proofs).

Also, read the following link (if it is review for, you can skim it quickly):

https://www.math.cmu.edu/~mlavrov/arml/12-13/invariants-12-09-12.pdf

§2 Lecture notes

This is a unit meant to help familiarize you with typical arguments in olympiad combinatorics.

Unlike units like Global (in which the problems are unified by a certain type of trick), the problems in this unit aren't meant to be "one-step tricky". However they might be more demanding in terms of maturity, being diligent with details, and working out formalities. This makes them a good way to get started with combinatorics problems.

§2.1 Topics

Here are topics covered within this unit.

- **Recognizing bi-directional problems**. As mentioned in the reading, any problem about finding a minimum or maximum is automatically a two-part problem. You should separate clearly in your head the part in which you prove a bound, and when you are showing that the bound can be achieved.

  Similarly, you will see many problems of the form "find all X such that...". Remember always to separate both directions in your head.

- **Get used to specifying algorithms**. Often a problem will ask you to show that something is always possible, and the way you do so is by giving an procedure to do it, i.e. a series of steps. This is totally normal and in fact you should get used to doing this sort of thing for existence problems.

- **Invariants can show impossibility**. Conversely, if you want to show something is never possible, one typical way to do it is to define an invariant, i.e. a quantity that never changes during a step of some operation.

  This is valuable too in find-all problems, maybe especially so. For example, suppose you are confronted with a problem about whether a certain task is possible on $n \times n$ board. **One common thing to do is just try the simplest invariant and see which n (if any) are ruled out.** Sometimes it won't do much, but often you'll get a few free cases this way[1], and trying this is often so simple it would be silly not to take a quick look.

- **Induction is your friend**. This can help you formalize an argument. In particular, if you are specifying an algorithm, induction is often the way to prove that it works. Or, if you have some sort of recursion, typically you'll want to use induction to verify it.

[1] If you've done Diophantine equations before, this is similar to how you might often start by taking a few mods to get a bit of starting information. You don't expect to solve the problem instantly, just get a bit of opening information to begin a proper line of attack.

- **Counting**. Some of the problems here involve counting how many ways there are to do something; you might be used to this type of problem already from short-answer contests (e.g. AIME). There usually isn't anything unexpected here; the same solutions you saw in the AIME packet will work equally well as proofs.

§2.2 Basic recipe

In many problems, you'll be asked to find all x for which a certain task is possible. A general outline for an approach to these problems:

Step 1 Play with some examples of x to get a sense for what the answer is.

Step 2 Make a guess what you think the valid x are. Let A be your claimed set of x for which you think the task is possible.

Step 3a For $x \in A$, describe an algorithm that works, using ideas gathered from Step 1.

Step 3b For $x \notin A$ give an invariant showing the task is impossible.

Steps 3a and 3b can be performed in either order, and often one of them is obvious and the other is less obvious. On the other hand, while trying to prove one of 3a or 3b you may often find that there is some boundary case and the answer you thought you had was wrong.

§2.3 Worked examples

Example 2.1 (NIMO Winter 2014/2)

Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

14NIMOW2 **Walkthrough.** This is quite easy; it's just meant to give a concrete example of a bi-directional problem.

(a) Figure out the answer.

(b) Show that this choice of c actually works.

(c) Manually verify that none of the smaller c work.

Example 2.2 (HMMT 2016 T4)

Let $n > 1$ be an odd integer. On an $n \times n$ chessboard the center square and four corners are deleted. We wish to group the remaining $n^2 - 5$ squares into $\frac{1}{2}(n^2 - 5)$ pairs, such that the two squares in each pair intersect at exactly one point (i.e. they are diagonally adjacent, sharing a single corner).

For which odd integers $n > 1$ is this possible?

16HMMTT4 **Walkthrough.** Let's do some cases first.

(a) Can one do $n = 3$?

(b) Can one do $n = 5$?

(c) Can one do $n = 7$?

In fact, for most n the task is impossible. This is a parity argument: we seek a coloring the cells by black and white (not the usual checkerboard) so that any valid pair has different colors.

(d) Find a coloring of the squares by black and white so that diagonally adjacent squares are opposite colors. (Optionally, find all such colorings.)

(e) Use this to narrow down the set of possible n to two values.

(f) Wrap up the problem using your earlier work.

Example 2.3 (JMO 2019/1)

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

19JMO1 **Walkthrough.** First we show that if ab is even then the goal is possible. The basic idea is to use induction.

(a) If $\min(a,b) \geq 1$, and a and b are opposite parity, show that in one swap one can reduce from (a,b) to $(a-1,b-1)$.

(b) If $\min(a,b) \geq 2$, and a and b are both even, show that in two swaps one can reduce from (a,b) to $(a-2,b-2)$.

(c) Formulate a set of base cases and complete the proof via induction.

Conversely, we need to show the task is impossible if ab is odd.

(d) Let X denote the number of apples in odd-numbered bowls, and let Y denote the number of pears in odd-numbered bowls. Find a relation between X and Y that does not change under the operation.

(e) Use this to show that the task is impossible when ab is odd.

Example 2.4 (Shortlist 2012 C1)

There are n positive integers written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x>y$ and x is to the left of y, and replaces the pair (x,y) by either $(y+1,x)$ or $(x-1,x)$. Prove that she can perform at most n^n such steps.

12SLC1 **Walkthrough.** A weaker result is easier to see:

(a) Look up the term "lexicographic order" if you don't know what it means.

(b) Show the process must terminate, ignoring the bound of n^n.

The hard part is to get the bound n^n, which doesn't depend on how big the numbers are, rather only depends on n itself.

The idea is that we need to pay attention to the *relative order* of the numbers, rather than the numbers themselves. After all, the numbers on the board can be as large as Alice wants.

So for each board state B, we define a permutation π_B on $\{1, \dots, n\}$ where the number in the ith position of B is the $\pi_B(i)$th smallest number. For example,

$$B = (1337, 42, 2012, 1000, 7) \longmapsto \pi_B = 42531.$$

There is a little wrinkle: what do we do with ties?

(c) Choose a convention for breaking ties, so that each B gives an unambiguous π_B, even if some numbers of B are equal.

(d) Under your convention, is π_B "monotonic" in the lexicographic order?

(e) If you answered "yes" to (d), then prove it. If you answered "no", give a different answer to (c) and try again.

(f) Prove that the process terminates in at most $n!$ steps.

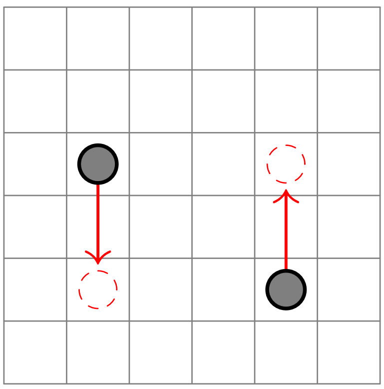
Example 2.5 (USAMO 2015/4)

Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

15AMO4 **Walkthrough.** Despite its placement, this is not an especially quick problem, which is why it is saved as the last walkthrough. It is really rather easy to mess up some details, and moreover, it also takes a while to write-up depending on how you do it.

On the other hand, the statement is pretty long-winded. Here's a tl;dr: stone moves look like the thing below, count the number of equivalence classes under stone moves.



This walkthrough will present the cleanest approach I know of, due to Ankan Bhattacharya. But you should be aware that most solutions to the problem are much worse.

(a) Let r_i denote the number of stones in the ith row, and c_j the number of stones in the jth column. Show that r_i and c_j never change.

(b) What are $\sum r_i$ and $\sum c_j$?

(c) Show that the number of $2n$-tuples $(r_1, \dots, r_n, c_1, \dots, c_n)$ satisfying (b) is $\binom{n+m-1}{m}^2$.

So the classic mistake is to assume that (c) gives the answer. In truth, this is only the beginning of the problem. To see why, let's agree that the *signature* of a piling is the tuple described in (c).

Thus we have checked that if two pilings are equivalent, then they have the same signature. But this does not mean the number of piling methods is equal to the number of signatures![2]

(d) There are two more statements we have to prove to finish the problem. What are they?

We'll actually now remodel the problem as follows. Forget about the entire grid. Instead, consider a blackboard where we write (x, y) for every stone in row x and column y. Thus there should be exactly m ordered pairs on the blackboard, one for each stone.

Thus, a stone move amounts to switching the y-coordinates of two ordered pairs.

(e) Consider two pilings which have the same signature. Describe an algorithm to reach one from the other, using the blackboard analogy instead of the grid.

(f) Moreover, show that every possible signature is achievable. (Why is this step necessary?)

(g) Put everything together to complete the solution.

§2.4 Warning for experts on invariants (and monovariants)

I deliberately choose to not super-emphasize invariants (and monovariants; in what follows, I'll just say "invariants" for brevity), which is contrary to popular wisdom.

I think you should certainly keep invariants in mind on problems that involve moving processes, and look for them when it's natural to do so. However, I have seen many intermediate to advanced students who have become trained to only search for invariants, rather than attempting to solve the problem. As you might expect, this works great for problems where an obvious invariant does exist, and works terribly on problems where no such invariant exists, or even if an invariant does exist but relies on having obtained some deeper understanding of the underlying process.

So what other methods exist besides invariants? Unfortunately, whatever the answer is, I think it has no name. But when you take the rigid or process unit, you'll start to see what I'm talking about.

[2] For example, here is another invariant: the total number of stones is m, which takes on only one value. But that certainly does not mean the answer is 1.

§3 Practice problems

Instructions: Solve [45♣]. If you have time, solve [60♣]. Problems with red weights are mandatory.

It's not denial. I'm just selective about the reality I accept.

Calvin in *Calvin and Hobbes*

Small warning: this unit is not short. Tough coach makes the beginners sweat too :P

§3.1 When is it possible?

14NIMOW3

[2♣] **Problem 1** (NIMO Winter 2014/3). The numbers 1, 2, . . . , 10 are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2+b^2+c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

18SLN1

[5♣] **Problem 2** (Shortlist 2018 N1). Determine all pairs (m, n) of positive integers for which there exists a positive integer s such that sm and sn have an equal number of divisors.

11AMO2

[9♣] **Required Problem 3** (USAMO 2011/2). An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

ZC1AD077

[3♣] **Problem 4** (Soviet Union 1978). A token is placed on the corner cell of a $n \times n$ chess-board. Each of two players moves it in turn to a neighbour (i.e. that has the common side) cell. A player loses if they move the token to a cell that was already visited.

(a) If the token starts on a corner cell, for each n, who wins?

(b) If the token starts adjacent to a corner cell, for each n, who wins?

§3.2 Mins and maxes

16JMO4

[2♣] **Problem 5** (JMO 2016/4). Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \dots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

21USEMO1

[5♣] **Problem 6** (USEMO 2021/1). Let n be a positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells c in the grid such that the entry in c is both strictly *greater* than the average of c's column and strictly *less* than the average of c's row.

11MOPR21

[5♣] **Problem 7** (MOP 2011). Seven points $A_1, \dots, A_7$ are marked on a circle γ. We draw all $\binom{7}{2} = 21$ line segments of the form $\overline{A_iA_j}$ for $1 \le i < j \le 7$. They meet at n distinct points in the interior of γ (not on the boundary). How small can n be, over all possible choices of A_i?

05TTFSA3

[3♣] **Problem 8** (Tournament of Towns 2005). Originally, every square of 8×8 chessboard contains a rook. One by one, rooks which attack an odd number of other rooks are removed. (Rooks may not jump over other rooks.) Find the maximal number of rooks that can be removed. (A rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

20INMO3

[3♣] **Problem 9** (INMO 2020/3). Consider a nonempty set $S \subseteq \{0, 1, \dots, 9\}$. It turns out that every sufficiently large positive integer can be written as the sum of two positive integers a and b, where both a and b have decimal digits only in S. How small can $|S|$ be?

§3.3 Just show it's possible or impossible

SVNTST

[3♣] **Problem 10** (Slovenia TST 2018). Let $n \ge 2$ be an integer. There are n^2 marbles, each of which is one of n colors, but not necessarily n of each color. Prove that they may be placed in n boxes with n marbles in each box, such that each box contains marbles of at most two colors.

16KAZ101

[3♣] **Problem 11** (Kazakhstan 2016, added by Tilek Askerbekov). Let $n \ge 2$ be a positive integer. Prove that all divisors of n can be written as a sequence $d_1, \dots, d_k$ such that for any $1 \le i < k$ one of $\frac{d_i}{d_{i+1}}$ and $\frac{d_{i+1}}{d_i}$ is a prime number.

21SLA1

[2♣] **Problem 12** (Shortlist 2021 A1). Let $n \ge 1$ be an integer, and let A be a subset of $\{0, 1, 2, \dots, 5^n\}$. If $|A| = 4n + 2$, prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

21PAGMO5

[9♣] **Problem 13** (PAGMO 2021/5). Celeste has an unlimited amount of each type of n types of candy, numbered type 1, type 2, ..., type n. Initially she takes $m > 0$ candy pieces and places them in a row on a table. Then, she chooses one of the following operations (if available) and executes it:

- She eats a candy of type k, and in its position in the row she places one candy type $k - 1$ followed by one candy type $k + 1$, indices modulo n.
- She chooses two consecutive candies which are the same type, and eats them.

Find all positive integers n for which Celeste can leave the table empty for any value of m and any configuration of candies on the table.

97IMO4

[3♣] **Problem 14** (IMO 1997/4). An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, 2, \dots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

§3.4 Counting

USMT5329

[5♣] **Problem 15** (USAMTS 5/3/29). David has a deck of $2n$ distinct cards, each with one of n colors so that each color appears twice. He deals the cards face-up on a table, one at a time. If there are ever two cards of the same color both face-up at once, he removes both of them.

Of the $(2n)!$ possible original orderings of the deck, determine how many have the following property: at every point, the table contains cards of at most two distinct colors.

19CAN3

[5♣] **Required Problem 16** (Canada 2019/3). Let m and n be positive integers. A $2m \times 2n$ grid of squares is colored in the usual chessboard fashion. Determine the number of ways to place mn counters on the *white* squares, at most one counter per square, so that no two counters are diagonally adjacent.

10JMO1

[2♣] **Problem 17** (JMO 2010/1). Let $P(n)$ be the number of permutations $(a_1, \dots, a_n)$ of the numbers $(1, 2, \dots, n)$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

§3.5 Extra

04PTNMA1

[2♣] **Problem 18** (Putnam 2004 A1). Basketball star Yang Liu's team statistician keeps track of the number, $S(N)$, of successful free throws he has made in his first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

21JMO4

[5♣] **Required Problem 19** (JMO 2021/4). Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

21SLC1

[5♣] **Problem 20** (Shortlist 2021 C1). Let S be an infinite set of positive integers. Assume there exist pairwise distinct $a, b, c, d \in S$ satisfying $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist pairwise distinct $x, y, z \in S$ satisfying $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

08AMO4

[5♣] **Required Problem 21** (USAMO 2008/4). For which integers $n \ge 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

(Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n-2$ triangles.)

18EGMO4

[5♣] **Problem 22** (EGMO 2018/4). Let $n \ge 3$ be an integer. Several non-overlapping dominoes are placed on an $n \times n$ board. The *value* of a row or column is the number of dominoes that cover at least one cell of that row column. A domino configuration is called *balanced* if there exists some $k \ge 1$ such that every row and column has value k.

Prove that a balanced configuration exists for every $n \ge 3$ and find the minimum number of dominoes needed in such a configuration.

23AMO5

[3♣] **Problem 23** (USAMO 2023/5). Let $n \ge 3$ be an integer. We say that an arrangement of the numbers 1, 2, ..., n^2 in an $n \times n$ table is *row-valid* if the numbers in each row can be permuted to form an arithmetic progression, and *column-valid* if the numbers in each column can be permuted to form an arithmetic progression.

For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

[1♣] **Mini Survey.** Fill out feedback on the OTIS-WEB portal when submitting this problem set. Any thoughts on problems (e.g. especially nice, instructive, easy, etc.) or overall comments on the unit are welcome.

In addition, if you have any suggestions for problems to add, or want to write hints for one problem you really liked, please do so in the ARCH system!

The maximum number of [♣] for this unit is [95♣], including the mini-survey.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, NIMO Winter 2014/2

The answer is $c = 10$. In what follows we say that a number is *good* if all its decimal digits are less than 5.

We first prove $c = 10$ is a working example for all n. When $n = 1, 2, 3$, we have 2024, 2114 and 3014, which are all good. When $n \geq 4$, we find that

$$10^n + 2014 = 1\underbrace{000\ldots000}_{n-4 \text{ zeros}}2014$$

which is good. This shows that $c = 10$ is works.

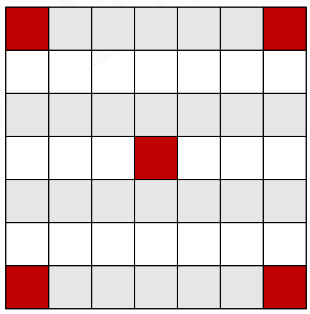
Next, we show that $c \geq 10$ is necessary.

- For $c = 1, 2, 3, 4, 5$, taking $n = 1$ gives the numbers 2015, 2016, ..., 2019, none of which are good.
- On the other hand, for $c = 6, 7, 8, 9$, taking $n = 2$ gives the numbers 2050, 2063, 2078, 2095, none of which are good.

¶ **Authorship comments** This came out by accident while I was trying to craft a different problem involving $S(n^6 + 2014)$, which became #4 on the same contest.

§4.2 Solution 2.2, HMMT 2016 T4

Constructions for $n = 3$ and $n = 5$ are easy. For $n > 5$, color the odd rows black and the even rows white. If the squares can be paired in the way desired, each pair we choose must have one black cell and one white cell, so the numbers of black cells and white cells are the same.



The number of black cells is $\frac{n+1}{2}n - 4$ or $\frac{n+1}{2}n - 5$ depending on whether the removed center cell is in an odd row. The number of white cells is $\frac{n-1}{2}n$ or $\frac{n-1}{2}n - 1$. But

$$\left(\frac{n+1}{2}n - 5\right) - \frac{n-1}{2}n = n - 5$$

so for $n > 5$ this pairing is impossible. Thus the answer is $n = 3$ and $n = 5$.

§4.3 Solution 2.3, JMO 2019/1

First we show that if ab is even then the goal is possible. We prove the result by induction on $a + b$.

- If $\min(a, b) = 0$ there is nothing to check.

- If $\min(a,b) = 1$, say $a = 1$, then b is even, and we can swap the (only) leftmost apple with the rightmost pear by working only with those fruits.
- Now assume $\min(a,b) \ge 2$ and $a+b$ is odd. Then we can swap the leftmost apple with rightmost pear by working only with those fruits, reducing to the situation of $(a-1, b-1)$ which is possible by induction (at least one of them is even).
- Finally assume $\min(a,b) \ge 2$ and $a+b$ is even (i.e. a and b are both even). Then we can swap the apple in position 1 with the pear in position $a+b-1$, and the apple in position 2 with the pear in position $a+b$. This reduces to the situation of $(a-2, b-2)$ which is also possible by induction.

Now we show that the result is impossible if ab is odd. Define

$$\begin{aligned} X &= \text{number apples in odd-numbered bowls} \\ Y &= \text{number pears in odd-numbered bowls.} \end{aligned}$$

Note that $X - Y$ does not change under this operation. However, if a and b are odd, then we initially have $X = \frac{1}{2}(a+1)$ and $Y = \frac{1}{2}(b-1)$, while the target position has $X = \frac{1}{2}(a-1)$ and $Y = \frac{1}{2}(b+1)$. So when ab is odd this is not possible.

Remark. Another proof that ab must be even is as follows.

First, note that apples only move right and pears only move left, a successful operation must take exactly ab moves. So it is enough to prove that the *number of moves* made must be even.

However, the number of fruits in odd-numbered bowls either increases by $+2$ or -2 in each move (according to whether i and j are both even or both odd), and since it ends up being the same at the end, the number of moves must be even.

Alternatively, as pointed out in the official solutions, one can consider the sums of squares of positions of fruits. The quantity changes by

$$\left[(i+1)^2 + (j-1)^2\right] - (i^2 + j^2) = 2(i-j) + 2 \equiv 2 \pmod 4$$

at each step, and eventually the sums of squares returns to zero, as needed.

§4.4 Solution 2.4, Shortlist 2012 C1

If a and b are numbers on the board, then we say $a \lessdot b$ if either

- $a < b$; or
- $a = b$ but a is to the left of b.

(In other words, $\lessdot$ is like $\lneq$ except ties are broken by position.)

For each board state B, we define a permutation π_B on $\{1, \dots, n\}$ where the number in the ith position of B is the $\pi_B(i)$th smallest number when sorting by $\lessdot$. For example,

$$B = (13,\ 9,\ 4,\ 9,\ 3,\ 7) \mapsto \pi_B = 642513$$

since 13 is the 6th smallest (i.e. largest) number on the board, the two 9's are tied for 4th smallest and 5th smallest, etc.

Claim — The permutations become lexicographically smaller each step.

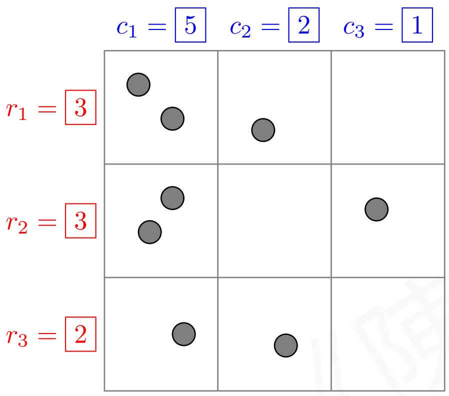
Proof. Basically immediate from construction. $\square$

Since there are $n!$ permutations, the number of moves is at most $n! < n^n$.

§4.5 Solution 2.5, USAMO 2015/4

The answer is $\binom{m+n-1}{n-1}^2$. The main observation is that the ordered sequence of column counts (i.e. the number of stones in the first, second, etc. column) is invariant under stone moves, as does the analogous sequence of row counts.

¶ Definitions Call these numbers $(c_1, c_2, \dots, c_n)$ and $(r_1, r_2, \dots, r_n)$ respectively, with $\sum c_i = \sum r_i = m$. We say that the sequence $(c_1, \dots, c_n, r_1, \dots, r_n)$ is the *signature* of the configuration. These are the $2m$ blue and red numbers shown in the example below (in this example we have $m = 8$ and $n = 3$).



Signature: $(5, 2, 1; 3, 3, 2)$

By stars-and-bars, the number of possible values $(c_1, \dots, c_n)$ is $\binom{m+n-1}{n-1}$. The same is true for $(r_1, \dots, r_m)$. So if we're just counting *signatures*, the total number of possible signatures is $\binom{m+n-1}{n-1}^2$.

¶ Outline and setup We are far from done. To show that the number of non-equivalent ways is also this number, we need to show that signatures correspond to pilings. In other words, we need to prove:

1. Check that signatures are invariant around moves (trivial; we did this already);
2. Check conversely that two configurations are equivalent if they have the same signatures (the hard part of the problem); and
3. Show that each signature is realized by at least one configuration (not immediate, but pretty easy).

Most procedures to the second step are algorithmic in nature, but Ankan Bhattacharya gives the following far cleaner approach. Rather than having a grid of stones, we simply consider the multiset of ordered pairs (x, y) corresponding to the stones. Then:

- a stone move corresponds to switching two y-coordinates in two different pairs.
- we *redefine* the signature to be the multiset (X, Y) of x and y coordinates which appear. Explicitly, X is the multiset that contains c_i copies of the number i for each i.

For example, consider the earlier example which had

- Two stones each at $(1, 1)$, $(1, 2)$.

- One stone each at $(3,1)$, $(2,1)$, $(2,3)$, $(3,2)$.

Its signature can then be reinterpreted as

$$(5,2,1;3,3,2) \longleftrightarrow \begin{cases} X = \{1,1,1,1,1,2,2,3\} \\ Y = \{1,1,1,2,2,2,3,3\}. \end{cases}$$

In that sense, the entire grid is quite misleading!

¶ Proof that two configurations with the same signature are equivalent The second part is completed just because transpositions generate any permutation. To be explicit, given two sets of stones, we can permute the labels so that the first set is (x_1, y_1), …, (x_m, y_m) and the second set of stones is (x_1, y_1'), …, (x_m, y_m'). Then we just induce the correct permutation on (y_i) to get (y_i').

¶ Proof that any signature has at least one configuration Sort the elements of X and Y arbitrarily (say, in non-decreasing order). Put a stone whose x-coordinate is the ith element of X, and whose y-coordinate is the ith element of Y, for each $i = 1, 2, \dots, m$. Then this gives a stone placement of m stones with signature (X, Y).

For example, if

$$X = \{1,1,1,1,1,2,2,3\}$$
$$Y = \{1,1,1,2,2,2,3,3\}$$

then placing stones at $(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$, $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$ gives a valid piling with this signature.