

The Method of Animation

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1 A Short Refresher on Projective Coordinates

In this section, we'll be defining projective coordinates, also called homogeneous coordinates. Recall that the projective plane \mathbb{RP}^2 is defined geometrically by taking the usual plane \mathbb{R}^2 and adding points at infinity for each direction, and a line at infinity that passes through all these infinity points. This space has the important property that every pair of lines intersects exactly once. Our goal is to describe an algebraic coordinate system for this space.

The idea is to “homogenize” \mathbb{R}^2 . We do this by adding a third coordinate, to get a space that looks like \mathbb{R}^3 , and saying points are preserved under scaling by a constant. Formally, we define

$$\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{(0, 0, 0)\}) / \sim,$$

where $(a, b, c) \sim (\lambda a, \lambda b, \lambda c)$ for each λ . This notation means that we take one point for each “direction” in \mathbb{R}^3 —one point for each line through the origin, and notate them in the same way we do \mathbb{R}^3 , but where any two points which differ by a constant are the same. For example, $(a, b, c) = (2a, 2b, 2c)$. We write $(a : b : c)$ for these homogenous coordinates.

First, why does this space look like \mathbb{R}^2 at all? Well, for any point $(x, y) \in \mathbb{R}^2$ we associate the point $(x : y : 1)$ —no two of these points are the same, so we have an **injection** from \mathbb{R}^2 to \mathbb{RP}^2 . But, in general we can go the other direction too, sending $(x : y : z) \rightarrow (x/z, y/z)$ —this map is well defined since it sends $(x : y : z)$ to the same point as $(\lambda x : \lambda y : \lambda z)$. But it's only defined when z is nonzero! So \mathbb{RP}^2 looks like \mathbb{R}^2 , except plus all the points of the form $(x : y : 0)$, which is the so-called **line at infinity**.¹

Now, let's take a moment to defend the idea that this space has a notion of geometry. For example, what is a line here? It turns out out lines in general have the equation

$$ax + by + cz = 0.$$

Indeed, when $z \neq 0$ this restricts to the usual $ax + by + c = 0$ line in euclidean geometry. Moreover, these lines are also invariant up to scaling; the lines defined by

$$ax + by + cz = 0 \text{ and } (\lambda a)x + (\lambda b)y + (\lambda c)z = 0$$

¹For readers more familiar with barycentric coordinates, it's fine to consider those instead—barycentric coordinates use the same construction as classical projective coordinates, but take the line at infinity to be $x + y + z = 0$ instead of $z = 0$.

are the same line. So we can identify each line with the triple $(a : b : c)$ in homogenous coordinates—this is the so-called **point-line duality**.

Finally, it's worth mentioning that general algebraic curves are just homogenous polynomial equations in x, y, z —for example, the unit circle is given by $x^2 + y^2 = z^2$.

Now that we have the basics down, let's try animating!

2 Animation Theory

Let's start by defining a general **moving point**:

Definition 1

We define a moving point as a map

$$\begin{aligned} \mathbb{R} \cup \{\infty\} &\rightarrow \mathbb{RP}^2 \\ t &\rightarrow (P(t) : Q(t) : R(t)) \end{aligned}$$

where P, Q, R are polynomials with no common root, and the image of $t = \infty$ is defined by continuity in \mathbb{RP}^2 .

When necessary, it can be convenient to extend this definition to take $t \in \mathbb{C} \cup \{\infty\}$ and the point in \mathbb{CP}^2 . Though we won't go into it here, this is very possible and all the properties of moving points carry over.

Exercise 2. What sort of curve does the moving point $t \rightarrow (1 : t : t^2)$ draw? How about $(t : t + 1 : 2t - 1)$?

Note that we can also define a moving **line** in the same way, using the definition of the projective coordinates of lines; the line $ax + by + cz = 0$ corresponds to the triple $(a : b : c)$.

Now, we say a moving point has **degree** given by the maximum degree of the three polynomial that define it. Note here that **constant points have degree 0**.

Our general technique for solving problems will be the following:

1. Animate some point in the problem which moves along a line. This will define our parameter, and definitionally have degree one.
2. Compute the degrees of all the points and lines in the problem.
3. Bound the degree d of the polynomial corresponding to the result we want to show.
4. Find $d + 1$ special cases—values our initial moving point can take—where the problem is true.
5. Conclude by the Fundamental Theorem of Algebra; a polynomial of degree d with $d + 1$ zeros must be identically zero.

The hardest part of the process is usually going the second step. However, we have one important tool that, when applied correctly, will allow us to bound the degrees of most points we'll come across in practice:

Lemma 3 (Moving Points Lemma)

If two moving points A and B coincide for k values of t (counting multiplicity), the degree of line AB is at most

$$\deg(A) + \deg(B) - k.$$

Similarly, if two moving lines ℓ_1, ℓ_2 coincide k times, their intersection is a moving point of degree at most

$$\deg(\ell_1) + \deg(\ell_2) - k.$$

Proof: Let

$$\begin{aligned} A &= (P_1(t) : Q_1(t) : R_1(t)) \\ B &= (P_2(t) : Q_2(t) : R_2(t)). \end{aligned}$$

Then the line AB has projective coordinates

$$A \times B = (Q_1(t)R_2(t) - Q_2(t)R_1(t) : - : -).$$

Then if $A = B$ for $t = t_0$, $(t - t_0)$ divides each term of $A \times B$, since they're all 0 when $t = t_0$. Then we can factor it out, getting

$$A \times B = \left(\frac{Q_1(t)R_2(t) - Q_2(t)R_1(t)}{t - t_0} : - : - \right),$$

which has one smaller degree. We can do this for each of the k values of t_0 , eventually getting a point with degree at most $\deg(A) + \deg(B) - k$. ■

At this point we should provide an important warning: If you use some special case (particular value of t) to decrease the degree of some point using Lemma 3, you **cannot use it again in counting $d + 1$ special cases where the problem is true to finish (step 4)**. This is very important, since double counting special cases can invalidate an entirely correct solution.

Corollary 4

If a moving point A moves along a conic and has degree 2, it moves projectively; for any fixed X on the same conic, XA has degree 1.

In fact, we can extend this to the following general result about projective maps:

Corollary 5 (Conic Doubling)

If Γ_1, Γ_2 are lines or conics and $\phi : \Gamma_1 \rightarrow \Gamma_2$ is a projective map (think a projection or composition of projections), then for a moving point $P \in \Gamma_1$ of degree d ,

$$\deg(\phi(P)) = \begin{cases} \deg(P) & \Gamma_1 \text{ and } \Gamma_2 \text{ are both lines or both conics} \\ 2 \deg(P) & \Gamma_1 \text{ is a line and } \Gamma_2 \text{ is a conic} \\ \deg(P)/2 & \Gamma_1 \text{ is a conic and } \Gamma_2 \text{ is a line} \end{cases}$$

We omit the proof of these corollaries since it requires us to extend our definitions to work over \mathbb{C} —however, the results still hold for a real variable t . This result, which we call **conic doubling**, extends to a general phenomenon on higher degree curves:

Theorem 6

If A has degree n , it moves along a plane curve of degree $k \mid n$, and the map from t to the curve extends to a polynomial map on \mathbb{C} which is surjective and m -to-1, where $mk = n$.

2.1 Degrees of Statements

Now, we have all the basic tools we'll need to compute the degrees of points. However, in order to finish problems, we'll need to be able to compute the degrees of *conditions*. This is a slightly different notion of degree; here, we use degree to mean the degree of the polynomial corresponding to the truth of a statement—that is, a polynomial P such that a given condition holds for all t with $P(t) = 0$.

Lemma 7

If moving points P, Q, R have degrees a, b, c , the condition “ P, Q, R collinear” has degree at most $a + b + c$.

Proof: If $P = (P_1(t), P_2(t), P_3(t))$, etc., P, Q, R collinear is equivalent to

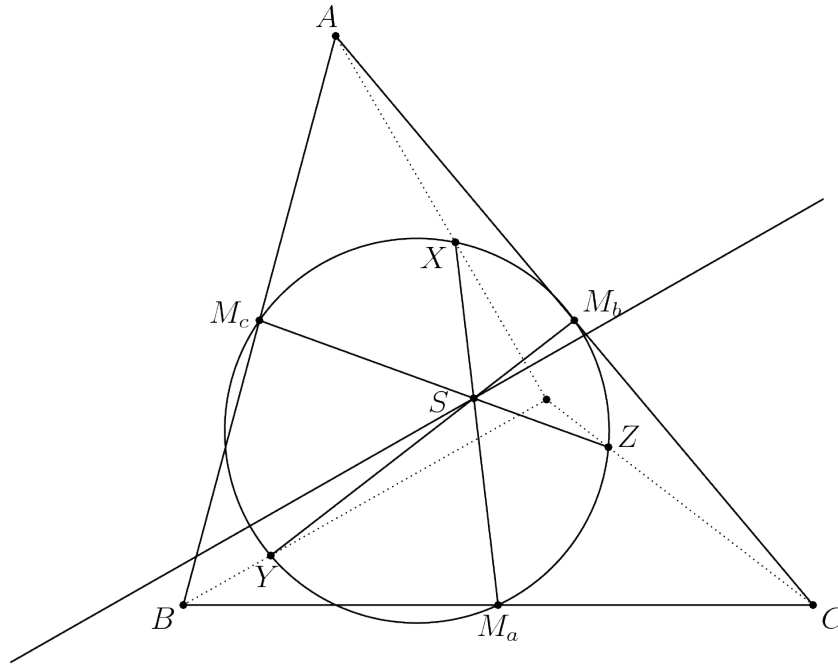
$$\det \begin{pmatrix} P_1(t) & P_2(t) & P_3(t) \\ Q_1(t) & Q_2(t) & Q_3(t) \\ R_1(t) & R_2(t) & R_3(t) \end{pmatrix} = 0,$$

which is a polynomial in t of degree at most $a + b + c$. Similarly, if we have three moving lines of degrees a, b, c , the three lines being concurrent is a condition of degree at most $a + b + c$ as well. ■

In general, this is going to be all we need; our goal will always be to reduce the problem to a statement about collinearity or concurrency. Now we have some powerful tools, so let's solve a problem:

Example 8 (USA TST 2015/6)

Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB , respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.



Proof: We have a free degree of freedom, so let's do the obvious thing; animate S along the Euler line. So we first note that X, Y, Z each have degree 2; clearly the map from $S \rightarrow X$ given by projection through M_a is projective, so X, Y, Z move projectively on the nine-point circle. Then lines AX, BY, CZ each have degree two as well, by the Moving Points Lemma. Then their concurrency is a polynomial of degree at most 6. So it suffices to find seven values of S for which the problem is true. This seems like a lot, but it's really not bad at all, as it turns out there are a ton of values for S that make the problem nice:

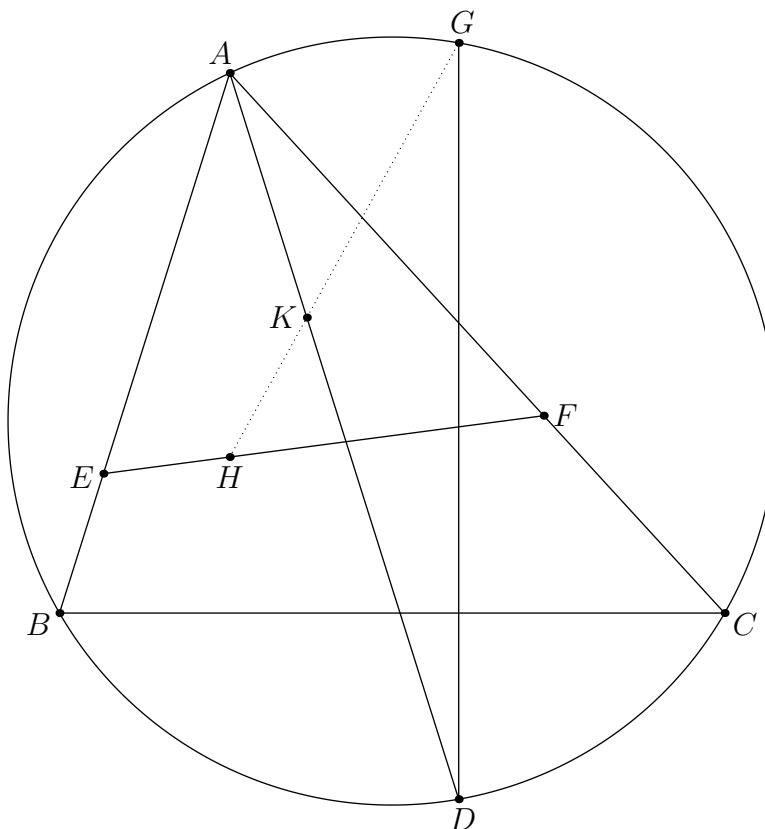
- If S lies on the nine-point circle, then $X = Y = Z = S$, so the lines are concurrent. This gives two values.
- If S lies on M_bM_c , say, then $Y = M_c$ and $Z = M_b$. So the lines concur at A . This gives three values at S ; one for each side of the medial triangle.
- If $S = G$ is the centroid, then all the lines are medians and concur at G .
- If $S = N$ is the nine-point center, then X is the midpoint of AH , so the three lines concur at the orthocenter.

This gives the desired seven points. We could have found more; though these are the simplest examples, it's also very possible to solve the problem for, say, $S = H$ or $S = O$. ■

This problem demonstrates the power of this technique: though we needed to verify a lot of cases, the problem is much easier when we get to choose the value of S , so we have the ability to do so. Before we move on to the exercises, let's go through one more example:

Example 9 (TSTST 2019/5)

Let ABC be an acute triangle with orthocenter H and circumcircle Γ . A line through H intersects segments AB and AC at E and F , respectively. Let K be the circumcenter of AEF , and suppose line AK intersects Γ again at a point D . Prove that line HK and the line through D perpendicular to BC meet on Γ .



Proof: Let G be the point on Γ such that $DG \perp BC$. We'll show that H, K, G collinear.

Let $\ell = EF$, which is some line through the fixed point H . Animate ℓ with degree 1 on the pencil of lines through H . Note that $E = \ell \cap AB$ and $F = \ell \cap AC$ are projective (degree 1), so the

perpendicular bisectors of AE and AF have degree 1, which means that their intersection K has degree ≤ 2 by the Moving Points Lemma.

We see that the map

$$\ell \mapsto \ell^\perp \mapsto AD$$

projective, where ℓ^\perp is the line through A perpendicular to ℓ . This is because ℓ^\perp and AD are isogonal in $\angle A$, so $\ell^\perp \mapsto AD$ is a reflection in the angle bisector of $\angle A$, and thus a projective map. Thus, AD is projective, so $\ell \mapsto D$ is projective, since A lies on Γ .

Finally, $D \mapsto G$ is projective since the map is given by projecting through the point at ∞ in the direction perpendicular to BC . Thus, $\ell \mapsto G$ is projective on the conic Γ , so G has degree 2.

The problem is asking us to show H, K, G collinear. By Lemma 7, this is a degree $0 + 2 + 2 = 4$ statement, so it suffices to check the problem for 5 values of ℓ .

1. If ℓ is the line through H anti-parallel to BC , (so $BCEF$ cyclic), then A, K, H are collinear. Thus, $D = AH \cap \Gamma$, and $G = A$. Thus, H, K, G are all collinear on the A -altitude.
2. If ℓ is the B -altitude, then $E = B$ and F is the foot from B to AC . This means that (AEF) is the circle with diameter AB , so K is the midpoint of AB , so $D = B$. This means that G is the antipode of C . It's clear that the antipode of C , the midpoint of AB , and H are collinear (orthocenter reflections), so this case works.
3. The case where ℓ is the C -altitude works similarly.
4. Suppose ℓ is the line through H parallel to AC . Then, K is the point at infinity in the direction perpendicular to AB , so D is the antipode of B . Thus, $G = C$, and G, K, H are collinear on the C -altitude.
5. The case where ℓ is the line through H parallel to AB works similarly.

■

3 When to Animate

In general, it can be hard to tell when animation is going to work. The most foolproof method to tell is: **Are there a lot of easy special cases?** Shoot for at least 4 or 5, and remember that degenerate cases generally do count². This usually only takes a couple minutes to check, and it's definitely worth it. For example, in TST 2015/6, there are a bazillion values of S for which the problem is really easy, but in general it's very difficult. By contrast, if you can only think of one or two, it's likely going to be very hard to animate. For this reason, symmetry is often very helpful in turning one example into two or three.

However, if this isn't enough to tell, it's usually worth giving a shot, especially if:

²Certainly count them for your preliminary assessment; when writing up you should be careful, as degenerate cases that result in points not being defined need to be dealt with through the Moving Points Lemma

- There aren't a lot of circles in the problem—intersecting lines is incredibly easy, but when circles get involved degrees can get really large really fast.
- The problem isn't conditional—problems with a condition attached are going to be very hard to animate, since you have to animate in a way that preserves the condition. However, if you can rephrase the problem to be unconditional, e.g. by drawing the diagram in a different order, then it could be possible.
- Conversely, if the problem has an extra degree of freedom (for example, a variable $D \in BC$ for a triangle problem), it's almost always worth attempting animation.

Finally, it's worth noting that our perspective in this handout has solely been to be able to compute the degrees, without much geometric insight. Choice of set-ups can often reduce your work, and so it is useful to try many different set-ups and choices of point to animate.

4 Problems

1. Try animating your favorite geometry problem!
2. (Kariya's Theorem): Let ABC be a triangle with incenter I and intouch triangle DEF . Choose X, Y, Z on rays ID, IE, IF respectively with $IX = IY = IZ$. Show that lines AX, BY, CZ concur.
3. (IMO 2010/2) Given a triangle ABC , with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D . Let E be a point on the arc BDC , and F a point on the segment BC , such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF , prove that the meeting point of the lines EI and DG lies on Γ .
4. (USA TST 2019/1) Let ABC be a triangle and let M and N denote the midpoints of \overline{AB} and \overline{AC} , respectively. Let X be a point such that \overline{AX} is tangent to the circumcircle of triangle ABC . Denote by ω_B the circle through M and B tangent to \overline{MX} , and by ω_C the circle through N and C tangent to \overline{NX} . Show that ω_B and ω_C intersect on line BC .
5. (2016 ISL G5) Let D be the foot of perpendicular from A to the Euler line of an acute scalene triangle ABC . A circle ω with centre S passes through A and D , and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC , and let M be the midpoint of BC . Prove that the circumcenter of triangle $XS Y$ is equidistant from P and M .
6. (IMO 2019/2) In triangle ABC , point A_1 lies on side BC and point B_1 lies on side AC . Let P and Q be points on segments AA_1 and BB_1 , respectively, such that PQ is parallel to AB . Let P_1 be a point on line PB_1 , such that B_1 lies strictly between P and P_1 , and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1 , such that A_1 lies strictly between Q and Q_1 , and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1 , and Q_1 are concyclic.

7. (RMM 2016/1) Let ABC be a triangle and let D be a point on the segment BC , $D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E . The circle ACD meets the segment AB again at an interior point F . Let A' be the reflection of A in the line BC . The lines $A'C$ and DE meet at P , and the lines $A'B$ and DF meet at Q . Prove that the lines AD , BP and CQ are concurrent (or all parallel)
8. Let ABC be a triangle. We let ℓ_a denote the second common external tangent of the B and C excircles (other than line BC), and define ℓ_b and ℓ_c similarly. Let D be a variable point on ℓ_a . The second tangent from D to the C -excircle (other than ℓ_a) meets ℓ_b at E . The second tangent from D to the B -excircle (other than ℓ_a) meets ℓ_c at F . Prove that line EF is tangent to the A -excircle.
9. Let ABC be a scalene triangle with circumcircle Γ . Let M be the midpoint of BC . A variable point P is selected in the line segment AM . The circumcircles of triangles BPM and CPM intersect Γ again at points D and E , respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y , respectively. Prove that as P varies, the circumcircle of AXY passes through a fixed point T distinct from A .
10. Let (Ω, ω) be a pair of circles such that there exists a triangle with circumcircle Ω and incircle ω , and let P on Ω . Show that the Simson lines of P with regard to each of the triangles with circumcircle Ω and incircle ω concur.
11. (ISL 2016 G6) Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P . Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD . Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF . Prove that $PQ \perp AC$.
12. (Taiwan TST 2014/3) Let M be any point on the circumcircle of triangle ABC . Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2 . Prove that the circumcircle of triangle MX_1X_2 intersects the circumcircle of ABC again at the tangency point of the A -mixtilinear incircle.
13. (110 Geometry Problems # 77) Let ABC be a triangle with circumcircle Γ and incircle γ . Let A' be the mixtilinear excircle touch point, and let $A'B'$ and $A'C'$ be tangent to γ with $B', C' \in \Gamma$. Let X be the tangency point of $B'C'$ with γ which exists by Poncelet's Porism. Show that (XBC) is tangent to γ .