Entry Combinatorics

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18 August 2023
BCY-ENTRY

§1 Reading

Read Chapter 1 of the *OTIS Excerpts* (Notes on Proofs).

Also, read the following link (if it is review for, you can skim it quickly):

https://www.math.cmu.edu/~mlavrov/arml/12-13/invariants-12-09-12.pdf

§2 Lecture notes

This is a unit meant to help familiarize you with typical arguments in olympiad combinatorics.

Unlike units like Global (in which the problems are unified by a certain type of trick), the problems in this unit aren't meant to be "one-step tricky". However they might be more demanding in terms of maturity, being diligent with details, and working out formalities. This makes them a good way to get started with combinatorics problems.

§2.1 Topics

Here are topics covered within this unit.

- **Recognizing bi-directional problems**. As mentioned in the reading, any problem about finding a minimum or maximum is automatically a two-part problem. You should separate clearly in your head the part in which you prove a bound, and when you are showing that the bound can be achieved.

  Similarly, you will see many problems of the form "find all X such that...". Remember always to separate both directions in your head.

- **Get used to specifying algorithms**. Often a problem will ask you to show that something is always possible, and the way you do so is by giving an procedure to do it, i.e. a series of steps. This is totally normal and in fact you should get used to doing this sort of thing for existence problems.

- **Invariants can show impossibility**. Conversely, if you want to show something is never possible, one typical way to do it is to define an invariant, i.e. a quantity that never changes during a step of some operation.

  This is valuable too in find-all problems, maybe especially so. For example, suppose you are confronted with a problem about whether a certain task is possible on $n \times n$ board. **One common thing to do is just try the simplest invariant and see which n (if any) are ruled out.** Sometimes it won't do much, but often you'll get a few free cases this way[1], and trying this is often so simple it would be silly not to take a quick look.

- **Induction is your friend**. This can help you formalize an argument. In particular, if you are specifying an algorithm, induction is often the way to prove that it works. Or, if you have some sort of recursion, typically you'll want to use induction to verify it.

[1]If you've done Diophantine equations before, this is similar to how you might often start by taking a few mods to get a bit of starting information. You don't expect to solve the problem instantly, just get a bit of opening information to begin a proper line of attack.

- **Counting**. Some of the problems here involve counting how many ways there are to do something; you might be used to this type of problem already from short-answer contests (e.g. AIME). There usually isn't anything unexpected here; the same solutions you saw in the AIME packet will work equally well as proofs.

§2.2 Basic recipe

In many problems, you'll be asked to find all x for which a certain task is possible. A general outline for an approach to these problems:

Step 1 Play with some examples of x to get a sense for what the answer is.

Step 2 Make a guess what you think the valid x are. Let A be your claimed set of x for which you think the task is possible.

Step 3a For $x \in A$, describe an algorithm that works, using ideas gathered from Step 1.

Step 3b For $x \notin A$ give an invariant showing the task is impossible.

Steps 3a and 3b can be performed in either order, and often one of them is obvious and the other is less obvious. On the other hand, while trying to prove one of 3a or 3b you may often find that there is some boundary case and the answer you thought you had was wrong.

§2.3 Worked examples

Example 2.1 (NIMO Winter 2014/2)
Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

14NIMOW2 **Walkthrough.** This is quite easy; it's just meant to give a concrete example of a bi-directional problem.

(a) Figure out the answer.

(b) Show that this choice of c actually works.

(c) Manually verify that none of the smaller c work.

Example 2.2 (HMMT 2016 T4)
Let $n > 1$ be an odd integer. On an $n \times n$ chessboard the center square and four corners are deleted. We wish to group the remaining $n^2 - 5$ squares into $\frac{1}{2}(n^2 - 5)$ pairs, such that the two squares in each pair intersect at exactly one point (i.e. they are diagonally adjacent, sharing a single corner).
For which odd integers $n > 1$ is this possible?

16HMMTT4 **Walkthrough.** Let's do some cases first.

(a) Can one do $n = 3$?

(b) Can one do $n = 5$?

(c) Can one do $n = 7$?

In fact, for most n the task is impossible. This is a parity argument: we seek a coloring the cells by black and white (not the usual checkerboard) so that any valid pair has different colors.

(d) Find a coloring of the squares by black and white so that diagonally adjacent squares are opposite colors. (Optionally, find all such colorings.)

(e) Use this to narrow down the set of possible n to two values.

(f) Wrap up the problem using your earlier work.

Example 2.3 (JMO 2019/1)

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

19JMO1 **Walkthrough.** First we show that if ab is even then the goal is possible. The basic idea is to use induction.

(a) If $\min(a,b) \ge 1$, and a and b are opposite parity, show that in one swap one can reduce from (a,b) to $(a-1,b-1)$.

(b) If $\min(a,b) \ge 2$, and a and b are both even, show that in two swaps one can reduce from (a,b) to $(a-2,b-2)$.

(c) Formulate a set of base cases and complete the proof via induction.

Conversely, we need to show the task is impossible if ab is odd.

(d) Let X denote the number of apples in odd-numbered bowls, and let Y denote the number of pears in odd-numbered bowls. Find a relation between X and Y that does not change under the operation.

(e) Use this to show that the task is impossible when ab is odd.

Example 2.4 (Shortlist 2012 C1)

There are n positive integers written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x,y) by either $(y+1,x)$ or $(x-1,x)$. Prove that she can perform at most n^n such steps.

12SLC1 **Walkthrough.** A weaker result is easier to see:

(a) Look up the term "lexicographic order" if you don't know what it means.

(b) Show the process must terminate, ignoring the bound of n^n.

The hard part is to get the bound n^n, which doesn't depend on how big the numbers are, rather only depends on n itself.

The idea is that we need to pay attention to the *relative order* of the numbers, rather than the numbers themselves. After all, the numbers on the board can be as large as Alice wants.

So for each board state B, we define a permutation π_B on $\{1, \dots, n\}$ where the number in the ith position of B is the $\pi_B(i)$th smallest number. For example,

$$B = (1337, 42, 2012, 1000, 7) \longmapsto \pi_B = 42531.$$

There is a little wrinkle: what do we do with ties?

(c) Choose a convention for breaking ties, so that each B gives an unambiguous π_B, even if some numbers of B are equal.

(d) Under your convention, is π_B "monotonic" in the lexicographic order?

(e) If you answered "yes" to (d), then prove it. If you answered "no", give a different answer to (c) and try again.

(f) Prove that the process terminates in at most $n!$ steps.

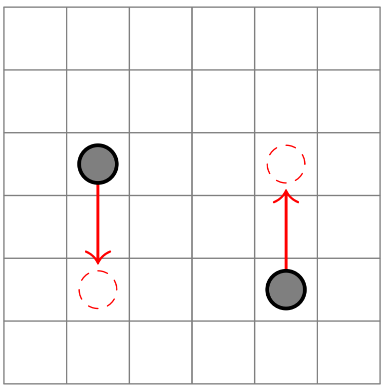
Example 2.5 (USAMO 2015/4)

Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

15AM04 **Walkthrough.** Despite its placement, this is not an especially quick problem, which is why it is saved as the last walkthrough. It is really rather easy to mess up some details, and moreover, it also takes a while to write-up depending on how you do it.

On the other hand, the statement is pretty long-winded. Here's a tl;dr: stone moves look like the thing below, count the number of equivalence classes under stone moves.



This walkthrough will present the cleanest approach I know of, due to Ankan Bhattacharya. But you should be aware that most solutions to the problem are much worse.

(a) Let r_i denote the number of stones in the ith row, and c_j the number of stones in the jth column. Show that r_i and c_j never change.

(b) What are $\sum r_i$ and $\sum c_j$?

(c) Show that the number of $2n$-tuples $(r_1, \dots, r_n, c_1, \dots, c_n)$ satisfying (b) is $\binom{n+m-1}{m}^2$.

So the classic mistake is to assume that (c) gives the answer. In truth, this is only the beginning of the problem. To see why, let's agree that the *signature* of a piling is the tuple described in (c).

Thus we have checked that if two pilings are equivalent, then they have the same signature. But this does not mean the number of piling methods is equal to the number of signatures![2]

(d) There are two more statements we have to prove to finish the problem. What are they?

We'll actually now remodel the problem as follows. Forget about the entire grid. Instead, consider a blackboard where we write (x, y) for every stone in row x and column y. Thus there should be exactly m ordered pairs on the blackboard, one for each stone.

Thus, a stone move amounts to switching the y-coordinates of two ordered pairs.

(e) Consider two pilings which have the same signature. Describe an algorithm to reach one from the other, using the blackboard analogy instead of the grid.

(f) Moreover, show that every possible signature is achievable. (Why is this step necessary?)

(g) Put everything together to complete the solution.

§2.4 Warning for experts on invariants (and monovariants)

I deliberately choose to not super-emphasize invariants (and monovariants; in what follows, I'll just say "invariants" for brevity), which is contrary to popular wisdom.

I think you should certainly keep invariants in mind on problems that involve moving processes, and look for them when it's natural to do so. However, I have seen many intermediate to advanced students who have become trained to only search for invariants, rather than attempting to solve the problem. As you might expect, this works great for problems where an obvious invariant does exist, and works terribly on problems where no such invariant exists, or even if an invariant does exist but relies on having obtained some deeper understanding of the underlying process.

So what other methods exist besides invariants? Unfortunately, whatever the answer is, I think it has no name. But when you take the rigid or process unit, you'll start to see what I'm talking about.

[2] For example, here is another invariant: the total number of stones is m, which takes on only one value. But that certainly does not mean the answer is 1.

§3 Practice problems

Instructions: Solve [45♣]. If you have time, solve [60♣]. Problems with red weights are mandatory.

Your light is expended. It is finished.

In Utter Darkness, from the *StarCraft II: Wings of Liberty* campaign

Small warning: this unit is not short. Tough coach makes the beginners sweat too :P

§3.1 When is it possible?

05AMO1

[3♣] **Problem 1** (USAMO 2005/1)**.** Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

18CAN3

[5♣] **Problem 2** (Canada 2018/3)**.** Two positive integers a and b are *prime-related* if either a/b or b/a is prime. Find all positive integers n with at least three divisors for which it's possible to arrange all the divisors of n in a circle, so that any two adjacent divisors are prime-related.

15RUS115

[2♣] **Problem 3** (Russia 2015/11.5)**.** Kelvin the Frog jumps along the coordinate line landing at integer points. It starts from point 0; its first jump has length 3, the second one has length 5, the third one has length 9, and so on (the kth jump has length $2^k + 1$). The direction of each jump is chosen by Kelvin. Is it possible that Kelvin eventually visits all points with positive integer coordinates at least once?

18CAN1

[3♣] **Required Problem 4** (Canada 2018/1, added by Michael Yang)**.** Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B. We say that an arrangement of n tokens is *collapsible* if it is possible to end up with all n tokens at the same point after a finite number of moves.

Find all integers $n \geq 1$ such that: every arrangement of n tokens is collapsible.

11ARGTST2

[2♣] **Problem 5** (Argentina TST 2011/2)**.** A wizard kidnaps 31 members from party A, 28 members from party B, 23 members from party C, and 19 members from party D, keeping them isolated in individual rooms in his castle. Every day, the kidnapped people can walk in the park and talk with each other. However, when three members of three different parties start talking with each other, the wizard reconverts them to the fourth party (there are no conversations with 4 or more people involved).

For which parties could all of the kidnapped people belong to the same party at some point in time?

21PAGMO1

[5♣] **Required Problem 6** (PAGMO 2021/1)**.** There are $n \geq 2$ coins numbered from 1 to n. These coins are placed around a circle, not necessarily in order.

In each turn, if we are on the coin numbered i, we will jump to the one i places from it, always in a clockwise order, beginning with coin number 1.

Find all values of n for which there exists an arrangement of the coins in which every coin will be visited.

22RUS93

[3♣] **Problem 7** (Russia 2022/9.3)**.** Suppose 200 positive integers are written in a row. For any two adjacent numbers in a row, the right one is either 9 times greater than or 2 times smaller than the left one. Can the sum of these 200 numbers equal 24^{2022}?

§3.2 Mins and maxes

18TMDJ6

[3♣] **Problem 8** (Tuymaada 2018/J6). The numbers $1, 2, \dots, 1024$ are written on a blackboard. The following procedure is performed ten times: partition the numbers on the board into disjoint pairs, and replace each pair with its nonnegative difference. Determine all possible values of the final number.

18JBMOSLC3

[3♣] **Problem 9** (JBMO SL 2018 C3). The cells of a 8×8 table are initially white. Alice and Bob play a game. First Alice paints n of the cells in red. Then Bob chooses 4 rows and 4 columns from the table and paints all cells in them in black. Alice wins if there is at least one red cell left. Find the least value of n such that Alice can win the game no matter how Bob plays.

21AMO4

[5♣] **Problem 10** (USAMO 2021/4). A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s, t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

22AMO5

[9♣] **Problem 11** (USAMO 2022/5). A function $f : \mathbb{R} \to \mathbb{R}$ is *essentially increasing* if $f(s) \le f(t)$ holds whenever $s \le t$ are real numbers such that $f(s) \ne 0$ and $f(t) \ne 0$.

Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \dots, x_{2022}$, there exist k essentially increasing functions $f_1, \dots, f_k$ such that

$$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \dots, 2022.$$

§3.3 Show it's possible

20CYBER5

[3♣] **Problem 12** (Cyberspace Competition 2020/5). There are 2020 positive integers written on a blackboard. Every minute, Zuming erases two of the numbers and replaces them by their sum, difference, product, or quotient. (For example, if Zuming erases the numbers 6 and 3, he may replace them with one of the numbers in the set $\{6+3, 6-3, 3-6, 6\times 3, 6\div 3, 3\div 6\} = \{9, 3, -3, 18, 2, \frac{1}{2}\}$.)

After 2019 minutes, Zuming arrives at the single number -2020 on the blackboard. Show that it is possible for Zuming to have arrived at the single number 2020 on the blackboard instead, under the same rules and using the same 2020 starting integers.

18SLC1

[5♣] **Problem 13** (Shortlist 2018 C1). Let $n \ge 3$ be an integer. Prove that there exists a set S of $2n$ distinct positive integers such that for any $m \in \{2, \dots, n\}$, the set S can be partitioned into two sets with cardinalities m and $2n - m$ with equal sums.

12CAN4

[5♣] **Problem 14** (Canada 2012, added by Carlos Rodriguez). A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.

All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

20JMO1

[2♣] **Problem 15** (JMO 2020/1). Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width.

Initially, the books are arranged in increasing order of height from left to right. In a *move*, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible.

Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

§3.4 Show it's impossible

99ELMO2

[2♣] **Problem 16** (ELMO 1999/2). Mr. Fat moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -2y)$, $(-2x, 3y)$, $(x + 1, y + 4)$ and $(x - 1, y - 4)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

19LEULER

[3♣] **Problem 17** (Added by Adilet Zauytkhan). The numbers $1, 2, \dots, 1000$ are written on the board. In a turn, one can erase any two numbers a and b and replace them with ab and $a^2 + b^2$. Prove that it is impossible to get at least 700 identical numbers by performing such operations.

§3.5 Counting

19AMO4

[3♣] **Problem 18** (USAMO 2019/4). Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \leq i \leq n$ and $0 \leq j \leq n$ (not necessarily distinct), such that

- $|S_{ij}| = i + j$, and
- $S_{ij} \subseteq S_{kl}$ if $0 \leq i \leq k \leq n$ and $0 \leq j \leq l \leq n$.

13TSTST7

[3♣] **Problem 19** (TSTST 2013/7). A country has n cities, labelled $1, 2, 3, \dots, n$. It wants to build exactly $n - 1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.

(a) For all odd n, prove that T_n is divisible by n.

(b) For all even n, prove that T_n is divisible by $n/2$.

96AMO4

[2♣] **Problem 20** (USAMO 1996/4). An n-term sequence $(x_1, x_2, \dots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

§3.6 Extra

19ESLC1

[3♣] **Problem 21** (ELMO SL 2019 C1)**.** Let $n \geq 3$ be a fixed positive integer. Elmo is playing a game with his clone. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, which player wins for each n?

97AMO1

[2♣] **Problem 22** (USAMO 1997/1)**.** Let $p_1, p_2, p_3, \dots$ be the prime numbers listed in increasing order, and let $0 < x_0 < 1$ be a real number between 0 and 1. For each positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \dots$ eventually becomes 0.

USMT4131

[5♣] **Required Problem 23** (USAMTS 4/1/31, added by Ryan Li)**.** A group of 100 friends stands in a circle. Initially, one person has 2019 mangoes, and no one else has mangoes. The friends split the mangoes according to the following rules:

- sharing: to share, a friend passes two mangoes to the left and one mango to the right.
- eating: the mangoes must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangoes. Every time a person eats a mango, they must also pass another mango to the right.

A person may only share if they have at least three mangoes, and they may only eat if they have at least two mangoes. The friends continue sharing and eating, until so many mangoes have been eaten that no one is able to share or eat anymore. Show that there are exactly eight people stuck with mangoes, which can no longer be shared or eaten.

12JMO4

[3♣] **Problem 24** (JMO 2012/4)**.** Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points $P_1, P_2, \dots, P_n$ as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \leq n$.

09SLC1

[9♣] **Required Problem 25** (Shortlist 2009 C1)**.** Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

21PAGMO4

[2♣] **Problem 26** (PAGMO 2021/4)**.** Lucía multiplies some positive one-digit numbers (not necessarily distinct) and obtains a number n greater than 10. Then, she multiplies all the digits of n and obtains an odd number. Find all possible values of the units digit of n.

[1♣] **Mini Survey.** Fill out feedback on the OTIS-WEB portal when submitting this problem set. Any thoughts on problems (e.g. especially nice, instructive, easy, etc.) or overall comments on the unit are welcome.

In addition, if you have any suggestions for problems to add, or want to write hints for one problem you really liked, please do so in the ARCH system!

The maximum number of [♣] for this unit is [96♣], including the mini-survey.

§4 Solutions to the walkthroughs

§4.1 Solution 2.1, NIMO Winter 2014/2

The answer is $c = 10$. In what follows we say that a number is *good* if all its decimal digits are less than 5.

We first prove $c = 10$ is a working example for all n. When $n = 1, 2, 3$, we have 2024, 2114 and 3014, which are all good. When $n \geq 4$, we find that

$$10^n + 2014 = 1\underbrace{000\dots000}_{n-4 \text{ zeros}}2014$$

which is good. This shows that $c = 10$ is works.

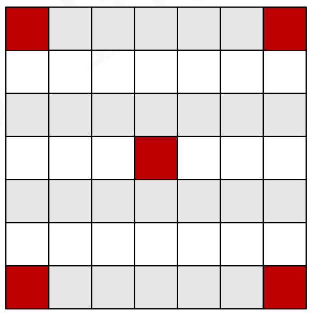
Next, we show that $c \geq 10$ is necessary.

- For $c = 1, 2, 3, 4, 5$, taking $n = 1$ gives the numbers 2015, 2016, ..., 2019, none of which are good.
- On the other hand, for $c = 6, 7, 8, 9$, taking $n = 2$ gives the numbers 2050, 2063, 2078, 2095, none of which are good.

¶ **Authorship comments** This came out by accident while I was trying to craft a different problem involving $S(n^6 + 2014)$, which became #4 on the same contest.

§4.2 Solution 2.2, HMMT 2016 T4

Constructions for $n = 3$ and $n = 5$ are easy. For $n > 5$, color the odd rows black and the even rows white. If the squares can be paired in the way desired, each pair we choose must have one black cell and one white cell, so the numbers of black cells and white cells are the same.



The number of black cells is $\frac{n+1}{2}n - 4$ or $\frac{n+1}{2}n - 5$ depending on whether the removed center cell is in an odd row. The number of white cells is $\frac{n-1}{2}n$ or $\frac{n-1}{2}n - 1$. But

$$\left(\frac{n+1}{2}n - 5\right) - \frac{n-1}{2}n = n - 5$$

so for $n > 5$ this pairing is impossible. Thus the answer is $n = 3$ and $n = 5$.

§4.3 Solution 2.3, JMO 2019/1

First we show that if ab is even then the goal is possible. We prove the result by induction on $a + b$.

- If $\min(a, b) = 0$ there is nothing to check.

- If $\min(a, b) = 1$, say $a = 1$, then b is even, and we can swap the (only) leftmost apple with the rightmost pear by working only with those fruits.
- Now assume $\min(a, b) \ge 2$ and $a + b$ is odd. Then we can swap the leftmost apple with rightmost pear by working only with those fruits, reducing to the situation of $(a - 1, b - 1)$ which is possible by induction (at least one of them is even).
- Finally assume $\min(a, b) \ge 2$ and $a + b$ is even (i.e. a and b are both even). Then we can swap the apple in position 1 with the pear in position $a + b - 1$, and the apple in position 2 with the pear in position $a + b$. This reduces to the situation of $(a - 2, b - 2)$ which is also possible by induction.

Now we show that the result is impossible if ab is odd. Define

$$\begin{aligned} X &= \text{number apples in odd-numbered bowls} \\ Y &= \text{number pears in odd-numbered bowls}. \end{aligned}$$

Note that $X - Y$ does not change under this operation. However, if a and b are odd, then we initially have $X = \frac{1}{2}(a + 1)$ and $Y = \frac{1}{2}(b - 1)$, while the target position has $X = \frac{1}{2}(a - 1)$ and $Y = \frac{1}{2}(b + 1)$. So when ab is odd this is not possible.

Remark. Another proof that ab must be even is as follows.

First, note that apples only move right and pears only move left, a successful operation must take exactly ab moves. So it is enough to prove that the *number of moves* made must be even.

However, the number of fruits in odd-numbered bowls either increases by $+2$ or -2 in each move (according to whether i and j are both even or both odd), and since it ends up being the same at the end, the number of moves must be even.

Alternatively, as pointed out in the official solutions, one can consider the sums of squares of positions of fruits. The quantity changes by

$$\left[(i + 1)^2 + (j - 1)^2\right] - (i^2 + j^2) = 2(i - j) + 2 \equiv 2 \pmod 4$$

at each step, and eventually the sums of squares returns to zero, as needed.

§4.4 Solution 2.4, Shortlist 2012 C1

If a and b are numbers on the board, then we say $a \lessdot b$ if either

- $a < b$; or
- $a = b$ but a is to the left of b.

(In other words, $\lessdot$ is like $\lneq$ except ties are broken by position.)

For each board state B, we define a permutation π_B on $\{1, \dots, n\}$ where the number in the ith position of B is the $\pi_B(i)$th smallest number when sorting by $\lessdot$. For example,

$$B = (13,\ 9,\ 4,\ 9,\ 3,\ 7) \mapsto \pi_B = 642513$$

since 13 is the 6th smallest (i.e. largest) number on the board, the two 9's are tied for 4th smallest and 5th smallest, etc.

Claim — The permutations become lexicographically smaller each step.

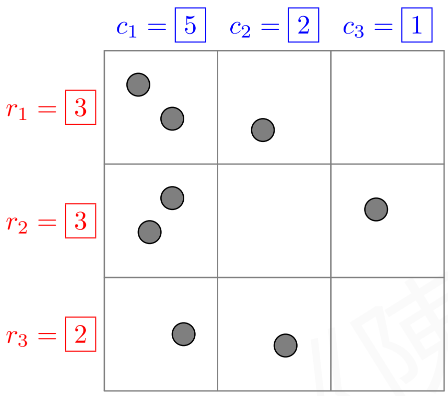
Proof. Basically immediate from construction. $\square$

Since there are $n!$ permutations, the number of moves is at most $n! < n^n$.

§4.5 Solution 2.5, USAMO 2015/4

The answer is $\binom{m+n-1}{n-1}^2$. The main observation is that the ordered sequence of column counts (i.e. the number of stones in the first, second, etc. column) is invariant under stone moves, as does the analogous sequence of row counts.

¶ Definitions Call these numbers $(c_1, c_2, \dots, c_n)$ and $(r_1, r_2, \dots, r_n)$ respectively, with $\sum c_i = \sum r_i = m$. We say that the sequence $(c_1, \dots, c_n, r_1, \dots, r_n)$ is the *signature* of the configuration. These are the $2m$ blue and red numbers shown in the example below (in this example we have $m = 8$ and $n = 3$).



Signature: $(5, 2, 1; 3, 3, 2)$

By stars-and-bars, the number of possible values $(c_1, \dots, c_n)$ is $\binom{m+n-1}{n-1}$. The same is true for $(r_1, \dots, r_m)$. So if we're just counting *signatures*, the total number of possible signatures is $\binom{m+n-1}{n-1}^2$.

¶ Outline and setup We are far from done. To show that the number of non-equivalent ways is also this number, we need to show that signatures correspond to pilings. In other words, we need to prove:

1. Check that signatures are invariant around moves (trivial; we did this already);
2. Check conversely that two configurations are equivalent if they have the same signatures (the hard part of the problem); and
3. Show that each signature is realized by at least one configuration (not immediate, but pretty easy).

Most procedures to the second step are algorithmic in nature, but Ankan Bhattacharya gives the following far cleaner approach. Rather than having a grid of stones, we simply consider the multiset of ordered pairs (x, y) corresponding to the stones. Then:

- a stone move corresponds to switching two y-coordinates in two different pairs.
- we *redefine* the signature to be the multiset (X, Y) of x and y coordinates which appear. Explicitly, X is the multiset that contains c_i copies of the number i for each i.

For example, consider the earlier example which had

- Two stones each at $(1, 1)$, $(1, 2)$.

- One stone each at $(3,1)$, $(2,1)$, $(2,3)$, $(3,2)$.

Its signature can then be reinterpreted as

$$(5,2,1;3,3,2) \longleftrightarrow \begin{cases} X = \{1,1,1,1,1,2,2,3\} \\ Y = \{1,1,1,2,2,2,3,3\}. \end{cases}$$

In that sense, the entire grid is quite misleading!

¶ Proof that two configurations with the same signature are equivalent The second part is completed just because transpositions generate any permutation. To be explicit, given two sets of stones, we can permute the labels so that the first set is (x_1, y_1), ..., (x_m, y_m) and the second set of stones is (x_1, y_1'), ..., (x_m, y_m'). Then we just induce the correct permutation on (y_i) to get (y_i').

¶ Proof that any signature has at least one configuration Sort the elements of X and Y arbitrarily (say, in non-decreasing order). Put a stone whose x-coordinate is the ith element of X, and whose y-coordinate is the ith element of Y, for each $i = 1, 2, \dots, m$. Then this gives a stone placement of m stones with signature (X, Y).

For example, if

$$X = \{1,1,1,1,1,2,2,3\}$$
$$Y = \{1,1,1,2,2,2,3,3\}$$

then placing stones at $(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$, $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$ gives a valid piling with this signature.