Moving point exercises

RedPig

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1 Problems

Problem 1.1 (ELMO 2017 P2). *Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.*

Problem 1.2 (ELMO SL 2018 G1). *Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.*

Problem 1.3 (China TST 2019 Test 2 P1). *AB and AC are tangents to a circle ω with center O at B, C respectively. Point P is a variable point on minor arc BC. The tangent at P to ω meets AB, AC at D, E respectively. AO meets BP, CP at U, V respectively. The line through P perpendicular to AB intersects DV at M, and the line through P perpendicular to AC intersects EU at N. Prove that as P varies, MN passes through a fixed point.*

Problem 1.4 (USA TST 2019 P1). *Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.*

Problem 1.5 (USA TST 2020 P2). *Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.*

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Problem 1.6 (USA TSTST 2019 P5). *Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.*

Problem 1.7 (IMO 2010 P2). *Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \dfrac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.*

Problem 1.8 (APMO 2016 P3). *Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC*

at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 1.9 (2019 All Russian Grade 11 P6)**.** *$\triangle ABC$ is isosceles with base BC, let D be a point on the segment AC. Let K be a point on the small arc CD of the circumcircle of $\triangle BCD$. Line CK intersects the line through A parallel to BC at T. Let M be the midpoint of segment DT. Prove that $\angle AKT = \angle CAM$.*

Problem 1.10. *Let O be the circumcircle of $\triangle ABC$, let P be an arbitrary point in the plane. Let A', B', C' be the second point of intersection of $\odot O$ with AP, BP, CP. Let X, Y, Z be the reflexion of A, B, C w.r.t line OP. Let U, V, W be the points of intersection of $A'X$, $B'Y$, $C'Z$ with BC, CA, AB, respectively. Prove that U, V, W are collinear.*

Problem 1.11 (Indonesian MO 2013 P2)**.** *Let ABC be an acute triangle and ω be its circumcircle. The bisector of $\angle BAC$ intersects ω at [another point] M. Let P be a point on AM and inside $\triangle ABC$. Lines passing P that are parallel to AB and AC intersects BC on E, F respectively. Lines ME, MF intersects ω at points K, L respectively. Prove that AM, BL, CK are concurrent.*

Problem 1.12 (IMO SL 2012 G2)**.** *Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.*

Problem 1.13 (Singapore TST 2009)**.** *Let H be the orthocentre of $\triangle ABC$ and let P be a point on the circumcircle of $\triangle ABC$, distinct from A, B, C. Let E and F be the feet of altitudes from H onto AC and AB respectively. Let $PAQB$ and $PARC$ be parallelograms. Suppose QA meets RH at X and RA meets QH at Y. Prove that XE is parallel to YF.*

Problem 1.14 (IMO 1995 P1)**.** *Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.*

Problem 1.15. *Let $\triangle ABC$ be an equilateral triangle. Let O be the circumcenter of $\triangle ABC$. Define points D, E, F on BC, AC, AB respectively such that the lines AD, BE, CF are concurrent. Let A' be the intersection of the circumcircles of AOD and ABC . Define points B' and C' similarly. Prove that AA', BB', CC' are concurrent.*

Problem 1.16. *Let ABC be a triangle and point X be an arbitrarily point. The bisector of $\angle BAC$ intersects the circumcircle AXB and AXC at M, N. The* *bisector of $\angle ABC$ and $\angle ACB$ intersect the circumcircle of BXC at P, Q. Prove that the intersection point of MQ, NP lies on the circumcircle of BXC.*

Problem 1.17. *Let ABC be a triangle with incenter I and intouch triangle DEF. Choose X, Y, Z on rays ID, IE, IF respectively with $IX = IY = IZ$. Show that lines AX, BY, CZ concur.*

Problem 1.18 (Serbia MO 2018 P1)**.** *Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC prove that $\angle PDQ = 90$.*

Problem 1.19 (EGMO 2012 P1)**.** *Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.*

Problem 1.20. *Given triangle ABC with incircle (I) which touches BC, CA, AB at D, E, F respectively. H and J are orthocenters of ABC and AEF, M is the projection of D on EF. Prove that H, J, M are colinear.*

Problem 1.21 (IMO SL 2012 G2)**.** *Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.*

Problem 1.22. *Given any point P and a triangle ABC, let X, Y, Z be the second intersection of AP, BP, CP with the circumcircle of $\triangle ABC$, we call $\triangle XYZ$ the circumcevian triangle of P with respect to a $\triangle ABC$.*

Given a triangle ABC, and three points P, Q and R on a line ℓ.

- *Let $A_P B_P C_P$ be the circumcevian triangle of P w.r.t $\triangle ABC$.*
- *Let $A_Q B_Q C_Q$ be the circumcevian triangle of Q w.r.t $\triangle A_P B_P C_P$.*
- *Let $A_R B_R C_R$ be the circumcevian triangle of R w.r.t $\triangle A_Q B_Q C_Q$.*

Prove that the lines AA_R, BB_R and CC_R concur at a point on the line ℓ. (Remark: This lemma shed insight on why so many points lie on the Euler line. Source: https://artofproblemsolving.com/community/c6h66285)

Some applications of the result: USA TST 2017 P2.