A walk through the Projective Plane

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May 31, 2021

Preface: what you should know

The **Method of Moving Points** is a very powerful method which can be traced back to the 90's [1]. It has been recently generalized to cover more complex cases in olympiad geometry(by yayup in the blog [link1], and some follow ups by Vladyslav Zveryk [link2], Zack Chroman [link3], . However, I still struggle a lot when reading these documents, especially when I was not familiar enough with the projective geometry. In particular, I was hardly able to tell whether I applied this method correctly on a given problem. I guess this is common issue for newcomers in algebraic geometry like me. Hence I went over several different lectures/notes in algebraic geometry to get me familiar with the topic [5, 3, 2, 4, 6]. The purpose of this note is to explain in more details from scratch what is the projective geometry and how the Method of Moving Points works.

A proper definition of the projective space requires a lot of fundamental concepts in abstract algebra, such as group [Wiki link], quotient group [Wiki link], equivalence class [Wiki link], etc. While these notions are necessary for understanding high dimensional projective space, we can get rid of most of them when focusing on planar geometry, which is the case in maths olympiad. Nevertheless, a minimum knowledge on linear algebra is still required, such as three dimensional matrix-vector product, determinant, etc. Although not necessary, for those who are encouraging, check out the full lecture (≈30h) of Linear Algebra at MIT by Gilbert Strang, available on [Youtube watchlist]. I will try to reduce at maximum the prerequisite on linear algebra in this note, hopefully this is accessible to any high school student, providing a minimum sense of what the projective space looks like.

Last but not least, applying this method can **easily** lead to errors, I suggest taking an extreme attention when considering it. Especially, apply it only if you are a hundred percent confident that you fully understand all the underlying concepts. Also, I am not sure whether this method is allowed in national/international olympiads, probably not at the moment, in which case you would need to build the theory during the writing of the solutions.

1 Why projective space?

Before even talking about geometry, let us start with a simple algebraic fact that the equation

$$x^2 + 1 = 0,$$

admits no solution in $\mathbb{R}$, i.e. there does not exist any real number $x \in \mathbb{R}$ such that $x^2 = -1$.

In contrast, amazingly, the same equation is soluble in complex space, with solution $x = \pm i$. In other words, the complex space **extends** the real space by appropriately introducing additional elements to complete the space. This extension allows us to achieve the fundamental theorem of algebra:

> *every non-constant polynomial with* **complex** *coefficients has at least one* **complex** *root.*

which is not true if we only restrict ourselves to real numbers, i.e.

> *every non-constant polynomial with* **real** *coefficients does not necessarily has one* **real** *root.*

Hence, by extending to the complex space, we achieve a **universal** way to factorize polynomials ! Such unification is extremely powerful and we can hardly go further without such fundamental object.[1]

The reason that we start with this seemingly unrelated topic is to motivate the introduction of projective plane, denoted as $\mathbb{P}^2_{\mathbb{R}}$, which can be viewed as **an extension of the ordinary Euclidean plane**[2]. One important unification that projective plane provides is that

> *in projective plane, any two distinct lines intersect in one and only one point.*

while in the ordinary Euclidean plane,

> *in Eucldiean plane, any two distinct lines are either* **parallel** *or intersect in one point.*

This unification simplifies the statement of many geometrical properties, and together with some other important characteristics, the projective space plays a fundamental role in the modern development of algebraic geometry. In the following, we start formally defining the projective plane $\mathbb{P}^2_{\mathbb{R}}$.

[1] A proper definition of completeness and vector space extension requires much deeper tools in abstract algebra and it is beyond the discussion in this note.

[2] Formally called projective completion.

2 Defining the projective plane $\mathbb{P}^2_{\mathbb{R}}$

We start with an informal description of the projective plane $\mathbb{P}^2_{\mathbb{R}}$, as the extension of the Euclidean plane with points at infinity.

Definition 2.1 (Informal Definition)**.** *The projective plane $\mathbb{P}^2_{\mathbb{R}}$ is an extension*[a] *of the Euclidean plane by adding points at infinity such that there is one and only one point at infinity along each direction in the plane.*

[a]Here extension means all the points in the Euclidean plane are preserved in the projective plane and moreover it includes some additional points

Conceptually, theses points at infinity serves as the intersection of parallel lines. In particular, all the parallel lines along the same direction intersect at an unique point at infinity. In order to make this intuition concrete, we introduce the canonical representation of the projective plane.

Definition 2.2 (Canonical representation)**.** *The points in the projective plane $\mathbb{P}^2_{\mathbb{R}}$ are represented in a three coordinate system $[x : y : z]$, with bracket and colons separating the coordinates in order to differentiate from ordinary points in the space. The canonical representation of the points in the projective plane are given by*

$$\begin{cases} [x:y:1] & \textit{represents the point } (x,y) \textit{ in the Euclidean plane} \\ [x:1:0] & \textit{represents the point at } \textbf{infinity} \textit{ along direction } (x,1), \textit{ which} \\ & \quad \textit{includes all directions not parallel to x-axis.} \\ [1:0:0] & \textit{represents the point at } \textbf{infinity} \textit{ along x-axis} \end{cases}$$

In particular, all the points at infinity has its third coordinate equals to zero.

The canonical representation provides an explicit representation to associate each point in the Euclidean plane to a point in the projective plane. Moreover, it adds in the points at infinity, which in total gives

$$\text{projective plane } \mathbb{P}^2_{\mathbb{R}} = \text{Euclidean plane} + \text{points at infinity}$$

One may wonder why do we call something like $[x : 1 : 0]$ as a point at infinity while every coordinate is finite. To answer it in an informal way, one should regard the last coordinate as the denominator in a quotient and division by zero gives infinity. There is indeed an analogy between the projective plane and the rational numbers. To make this connection clearer, we introduce a more general way to represent points in the projective plane through the homogeneous representation.

Definition 2.3 (Homogeneous representation). *For any* $(x, y, z) \in \mathbb{R}^3 \backslash (0, 0, 0)$*, i.e. at least one of* x*,* y*,* z *is non-zero, we define the relation*

$$[x : y : z] = [\lambda x : \lambda y : \lambda z] \quad \text{for any } \lambda \in \mathbb{R}^*.$$

In particular, for any $(x, y, z) \in \mathbb{R}^3 \backslash (0, 0, 0)$*, there is a one and only one point* A *in the canonical representation such that* $A = [x : y : z]$*. Therefore, every homogeneous representation* $[x : y : z]$ *is associated to a point in the projective plane, while a single point in the projective plane can be represented in multiple forms in the homogeneous representation.*

Proof. If $z \neq 0$, then $A = [\frac{x}{z} : \frac{y}{z} : 1]$; if $z = 0$ and $y \neq 0$, then $A = [\frac{x}{y} : 1 : 0]$; if $z = 0$ and $y = 0$ then $A = [1 : 0 : 0]$. Conversely, if we have A and B in the canonical representations such that $A = [x : y : z] = B$, then there is λ such that $A = \lambda B$. However, by enumerating different forms in the canonical representation, this happens if and only if $\lambda = 1$. Hence the uniqueness follows. $\square$

⚠ $[0 : 0 : 0]$ is **NOT** a point in the projective plane.

The idea that different representation can refer to the same object is similar to the representation of rational numbers. More precisely, the number $\frac{1}{2}$ can be represented by $\frac{2}{4}$ or $\frac{\pi}{2\pi}$. While $\frac{1}{2}$, $\frac{2}{4}$ or $\frac{\pi}{2\pi}$ all look different at their appearances, they indeed associate to the same number. In other words, if we use an extremely unusual notation $[x : y]$ to refer to rational number $\frac{x}{y}$, then $[1 : 2] = [2 : 4] = [\pi : 2\pi] = [\lambda : 2\lambda]$ for any $\lambda \neq 0$. In such a way, $[1 : 0] = \frac{1}{0}$ is the infinity.

In our case, projective points are given in a three-coordinates system, and by definition $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Namely, two representations refers to the same point if

"one can multiply all its coordinates by a same number λ to get the other".

One can indeed properly define a notion of quotient in such system, –quotient of group, again this notion requires sufficient prerequisites in abstract algebra, and it is beyond the discussion of this note.

To summarize, on one hand, projective plane is an extension of the Euclidean plane, where we artificially give multiple representations to the same point; on the other hand, projective plane contains a brunch of infinite points, which does not exist in the Euclidean plane. It is always useful to keep in mind the analogy of rational numbers, where, **multiple representations can refer to the same point in the projective plane** $\mathbb{P}^2_{\mathbb{R}}$**.**

Example 2.4. *Let us go through a few examples to get a better idea of the concept on homogeneous representation.*

- *Take* $A = [1 : 1 : 1]$ *in the projective plane, which is in its canonical representation. We can also use* $[2 : 2 : 2]$, $[e : e : e]$ *or* $[\pi : \pi : \pi]$ *to denote the same point* A *in the homogeneous representation system. Moreover, it refers to the point* $(1, 1)$ *in the Euclidean plane.*

- *Take* $B = [1 : 5 : 0]$ *in the homogeneous representation. Then it refers to the point* $[\frac{1}{5} : 1 : 0]$ *in the canonical representation, which corresponds to the point at infinity along the direction* $\vec{d} = (\frac{1}{5}, 1)$.

We need to reduce to the canonical form before associating a projective point back to the Euclidean plane. In particular, $[2 : 2 : 2]$ **does not** refer to the point $(2, 2)$, it refers to the point $(1, 1)$. The point $(2, 2)$ in the Euclidean plane is associated to the canonical representation $[2 : 2 : 1]$ or more generally $[2\lambda : 2\lambda : \lambda]$.

Excercise 2.5. *Transform between different representations of the points.*

(a) *Represent the projective point* $[1 : 2 : 3]$ *in canonical representation.*

(b) *What is the point in Euclidean plane corresponding to the projective point* $[0 : 4 : 1]$*? What about* $[0 : 4 : 2]$ *or* $[0 : 4 : 4]$ *?*

(c) *Represent the projective point* $[13 : 17 : 0]$ *in its canonical representation. What about* $[481 : 629 : 0]$*?*

(d) *Represent the projective point* $[e : 0 : 0]$ *in its canonical representation. What about* $[100 : 0 : 0]$ *or* $[\pi : 0 : 0]$*?*

From now on, we usually use the homogeneous representation to represent projective points since they are easier to manipulate algebraically. To see that the projective plane actually encodes the notion of "plane", we now define the projective curves.

3 Projective Curves

The curves in the Euclidean plane are usually represented as the solution of a two-variable polynomial $Q(x, y) = 0$, for instance $Q(x, y) = ax + by + c$ defining the lines and $Q(x, y) = x^2 + y^2 - 1$ defines the unit circle. Similarly, a natural way to define curves in the projective plane is via a three-variable polynomial $P(x, y, z)$, as the projective plane is defined in a three-coordinate system. However, the special rule that $[x : y : z]$ defines the same point as $[\lambda x : \lambda y : \lambda z]$ imposes a special constraint on the form of the polynomial :

$$\text{it must satisfy } P(x, y, z) = 0 \Leftrightarrow P(\lambda x, \lambda y, \lambda z) = 0.$$

To satisfy this constraint, we introduce the notion of homogeneous polynomials.

Definition 3.1. *A polynomial $P(x, y, z)$ is* **homogeneous of degree d** *if*

$$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z), \quad \textit{for any } x, y, z \in \mathbb{R}^3,\ \lambda \in \mathbb{R}^*.$$

In particular, if we take $x = y = z = 0$ then $P(0, 0, 0) = \lambda^d P(0, 0, 0)$ for any λ, implying that $P(0, 0, 0) = 0$. Hence P does not have any constant term. To illustrate some examples [3]

- $P(x, y, z) = x + y + z$ is homogeneous of degree 1.
- $P(x, y, z) = x^2 + y^2 - z^2$ is homogeneous of degree 2.
- $P(x, y, z) = x^2 - yz$ is homogeneous of degree 2.
- $P(x, y, z) = x - xyz$ is not homogeneous.

It is easy to see that if P is homogeneous then $P(x, y, z) = 0 \Leftrightarrow P(\lambda x, \lambda y, \lambda z) = 0$. Hence the solution of P is consistent with the homogeneous coordinates. This leads to the definition of projective curves:

Definition 3.2. *A* **projective curve** *is the set of projective points $[x : y : z]$ satisfying $P(x, y, z) = 0$, where $P(x, y, z)$ is a homogeneous polynomial.*

A natural question that one may come to mind would be how to extend a curve in the Euclidean plane to a curve in the projective plane. To do so, we perform the homogenization on the polynomial. More precisely, given a two variable polynomial $Q(x, y)$ with highest degree d, we define

$$P(x, y, z) = z^d Q\left(\frac{x}{z}, \frac{y}{z}\right).$$

[3] Check out the wikipedia page as well for more explanation [Wiki link]

For instance, if $Q(x,y) = x+y-1$, then $P(x,y,z) = x+y-z$; if $Q(x,y) = x^2 + y^2 - 1$ then $P(x,y,z) = x^2+y^2-z^2$; if $Q(x,y) = x^2 - y$ then $P(x,y,z) = x^2 - yz$, etc. As lines and circles are the main focus in the olympiad geometry, we consider polynomials with degree not greater than 2 in the following.

3.1 Projective Lines

A projective line is given by a degree 1 homogeneous polynomial, i.e.

Definition 3.3. *A projective line in the projective plane $\mathbb{P}^2_{\mathbb{R}}$ has the equation*

$$\ell_{[a:b:c]} \;:\; ax + by + cz = 0, \qquad \text{where } (a,b,c) \in \mathbb{R}^3 \backslash (0,0,0)$$

In other words, a point $A = [x_A : y_A : z_A]$ lies on $\ell_{[a:b:c]}$ if and only if $ax_A + by_A + cz_A = 0$.

When there is no ambiguity, we abbreiviate $[a : b : c]$ as the **line coordinates** of $\ell_{[a:b:c]}$, as opposite to the point coordinates. To understand how this definition extends the notion of lines in the ordinary sense, we illustrate with an example.

Example 3.4. *Consider the projective line $\ell_{[1:2:3]} \;:\; x + 2y + 3z = 0$, then it contains all the points $[x : y : 1]$ such that $x + 2y + 3 = 0$. In other words, all the points lie on the line $x + 2y + 3 = 0$ (this is a line in the Euclidean plane) also lies on $\ell_{[1:2:3]}$. But, $\ell_{[1:2:3]}$ also contains one and only one point at infinity $[-2 : 1 : 0]$. Hence the projective line $\ell_{[1:2:3]}$ extends the line $x + 2y + 3 = 0$ by adding the point at infinity $[-2 : 1 : 0]$.*

More generally, if a or b is non-zero, then the projective line $\ell_{[a:b:c]}$ can be viewed as an ordinary line plus a point at infinity:

$$\ell_{[a:b:c]} = \underbrace{\{[x : y : 1] \mid ax + by + c = 0\}}_{\text{an ordinary line in the Euclidean plane}} \cup \underbrace{[-b, a, 0]}_{\text{a point at infinity}}$$

This justifies the name projective line as it extends the ordinary line.

Example 3.5. *Besides all the ordinary lines, the projective plane also includes an important line:* **the line at infinity***, which has equation $z = 0$. This line includes all the point at infinity, and moreover, none of the points in the Eucldiean plane lies on this line. This line is "fictive" in the sense that we are not able to visualize it. Nevertheless, it does exist in the projective plane, characterized by its equation.*

Excercise 3.6. *With the notation of projective lines $\ell_{[a:b:c]} \;:\; ax + by + cz = 0$.*

(a) *Find a point lies both on $\ell_{[1:2:3]}$ and the line $\ell_{[3:2:1]}$? Is this point unique?*

(b) *Find a point lies both on $\ell_{[1:2:3]}$ and the line $\ell_{[-1:-2:4]}$. Is this point unique?*

(c) *Find a point lies both on* $\ell_{[1:2:3]}$ *and the line at infinity. Is this point unique?*

(d) *Find a point lies both on* $\ell_{[1:2:3]}$ *and the line* $\ell_{[-2:-4:-6]}$*? Is this point unique?*

With the notion of projective lines in hand, we are now ready to present one important property in the projective plane: two distinct lines always intersect.

Theorem 3.7. *Given two distinct projective lines* ℓ_1 *and* ℓ_2 *in the projective plane, there is one and only one point* A *lies simultaneously on* ℓ_1 *and* ℓ_2*. In other words, any two distinct projective lines intersect at one point.*

Proof. Assume that ℓ_1 is given by $a_1x + b_1y + c_1z = 0$ and ℓ_2 is given by $a_2x + b_2y + c_2z = 0$. Since ℓ_1 and ℓ_2 are distinct, the vector (a_1, b_1, c_1) and (a_2, b_2, c_2) are not aligned. Let

$$\begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \times \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1c_2 - b_2c_1 \\ c_1a_2 - a_1c_2 \\ a_1b_2 - a_2b_1 \end{pmatrix} \tag{1}$$

be the cross product of the above two vectors. Therefore x_P, y_P, z_P are not all zero, which defines a point in the projective plane $P = [x_P : y_P : z_P]$. Moreover,

$$a_1x_P + b_1y_P + c_1z_P = 0, \quad \text{and} \quad a_2x_P + b_2y_P + c_2z_P = 0$$

Hence $P \in \ell_1 \cap \ell_2$, which shows the existence of the intersection point.

The uniqueness follows from the fact that the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) forms a plane in $\mathbb{R}^3$. If a point $Q = [x_Q : y_Q : z_Q]$ lies both on ℓ_1 and ℓ_2 then the vector (x_Q, y_Q, z_Q) is orthogonal to this plane, which is necessarily parallel to the cross product, hence there is λ such that $x_Q = \lambda x_P, y_Q = \lambda y_P$ and $z_Q = \lambda z_P$, implying that $[x_Q : y_Q : z_Q] = [x_P : y_P : z_P]$.[4] $\square$

In other words, we get rid of the notion of parallelism and every two lines intersect, either in the ordinary plane, or at a point at infinity. A natural question that follows is in what situation three pairwise distinct lines ℓ_1, ℓ_2 and ℓ_3 are concurrent, i.e. passing through a common point.

[4] A more rigorous proof requires the notion of basis and dimension from linear algebra.

Theorem 3.8. *Given three pairwise distinct projective lines ℓ_1, ℓ_2 and ℓ_3 in the projective plane, assume that ℓ_i has equation $a_i x + b_i y + c_i z = 0$ for any $i \in [1,3]$. Then the lines are concurrent if and only if*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0, \tag{2}$$

where by definition

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

Proof. The lines are concurrent if and only if ℓ_3 pass through the intersection of ℓ_1 and ℓ_2. Namely, their intersection point (x_P, y_P, z_P), given by Equation 1, lies on ℓ_3 if and only if

$$a_3x_P + b_3y_P + c_3z_P = a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - a_1c_2) + c_3(a_1b_2 - a_2b_1) = 0,$$

re-organizing the terms recovers the expression of determinant. $\square$

Excercise 3.9. *Determine the point of intersection between any pair of lines, then determine whether they are concurrent. Justify the answer using the value of determinant given in Equation 2.*

- $\ell_{[1:0:0]}$, $\ell_{[0:1:0]}$, $\ell_{[0:0:1]}$.
- $\ell_{[1:2:1]}$, $\ell_{[1:2:2]}$, $\ell_{[1:2:3]}$.
- $\ell_{[1:2:1]}$, $\ell_{[0:0:1]}$, $\ell_{[3:6:4]}$.

Now we have a characterization for concurrent lines, it is natural to ask what about collinear points. Before that, we start with a seemingly evident result.

Theorem 3.10. *Given two distinct points A,B in the projective plane $\mathbb{P}^2_{\mathbb{R}}$, there is one and only one projective line pass through A and B.*

Proof. The proof is essentially the same as the proof of the Theorem 3.7. Let $A = [x_A : y_A : z_A]$ and $B = [x_B : y_B : z_B]$, let

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} \times \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} \tag{3}$$

then the line $\ell_{[a:b:c]}$ defined by the equation $ax + by + cz = 0$ passes through the two points A and B. The uniqueness follows similarly as in the proof of Theorem 3.7. $\square$

Theorem 3.11. *Given three distinct points A,B, C in the projective plane $\mathbb{P}^2_{\mathbb{R}}$, they are collinear if and only if*

$$\det \begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ z_A & z_B & z_C \end{pmatrix} = 0, \tag{4}$$

Excercise 3.12. *Determine the equation of lines AB, BC and CA, then determine whether they are collinear. Justify the answer using the value of determinant given in Equation 2.*

- $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $C = [0 : 0 : 1]$.
- $A = [1 : 2 : 1]$, $B = [1 : 2 : 2]$, $C = [1 : 2 : 3]$.
- $A = [1 : 2 : 1]$, $B = [0 : 0 : 1]$, $C = [3 : 6 : 4]$.

It is worth noticing that the properties of lines and the property of points are somehow determined by the same equation/formula. Such similarity is called duality between the points and lines in the projective plane. We will make it clearer later in Section 5. Briefly speaking, there is a one-to-one (bijective) mapping

$$\begin{aligned} \phi \colon \text{point in } \mathbb{P}^2_{\mathbb{R}} &\to \text{line in } \mathbb{P}^2_{\mathbb{R}} \\ [a : b : c] &\mapsto \ell_{[a:b:c]}. \end{aligned}$$

between the point coordinates and the line coordinates. The line pass through A and B is given by $\phi(A \times B)$,[5] and, the intersection of $\phi(X)$ and $\phi(Y)$ is $X \times Y$. This allows us to use the same three-coordinate system to represent and manipulate projective lines. All we need is to evaluate the cross product or determinant. When there is no ambiguity, we say that a line AB has coordinates $[a : b : c]$ to refer the line $\ell_{[a:b:c]}$.

Note that the notion of parallelism no longer present in $\mathbb{P}^2_{\mathbb{R}}$, as every pair of lines in the projective plan intersect. Nevertheless, we do have the notion of parallel lines in the Euclidean plane, which we could naturally extend into the projective language:

Proposition 3.13 (Parallelism and orthogonality)**.** *Let $\ell_{[a:b:c]}$ be a line in the projective space with $a \neq 0$ or $b \neq 0$.*

- *All the lines parallel to $\ell_{[a:b:c]}$ passes through $P_\infty = [b : -a : 0]$.*
- *All the lines perpendicular to $\ell_{[a:b:c]}$ passes through $Q_\infty = [a : b : 0]$.*

In particular, if $A = [x_A : y_A : z_A]$ be a (finite) point, then

- *The line through A parallel to $\ell_{[a:b:c]}$ is the line AP_∞, given by $\ell_{[az_A:bz_A:-ax_A-by_A]}$.*

[5] Here we abbreviate $A \times B$ as the cross product of their homogeneous coordinates

- *The line through A perpendicular to $\ell_{[a:b:c]}$ is the line AQ_∞, given by $\ell_{[bz_A:-az_A:bx_A-ay_A]}$.*

Proof. The restriction of the line $\ell_[a : b : c]$ in the Euclidean plane is $ax+by+c = 0$. Hence the result follows from the fact that all the parallel to it has the equation $ax+by+d = 0$ and all the perpendicular to it has the equation $bx-ay+d = 0$. $\square$

⚠ The notion of parallelism and orthogonality are properties from the Euclidean plane, it is necessary that the lines we consider are finite. It make **no sense** to talk about parallelism and orthogonality regarding the line at infinity.

While defining the distance (or metric) is possible on the projective plane, it has a quite different nature compared to the usual distance in the Euclidean plane. Hence, instead defining points based on the underlying distance, it is more convenient to define points as image of linear transformations, such as homothety or rotation. For instance, if we want to introduce the midpoint M of a segment AB, then we can state M as the image of B under the homothety of center A with ratio $1/2$.

Proposition 3.14 (Homothety)**.** *Let $A = [x_A : y_A : z_A]$ be a finite point. The homothety of center A with ratio $\lambda \neq 0$ is given by the following formula*

$$\mathcal{H} = \begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} \to \begin{pmatrix} \lambda z_A & 0 & (1-\lambda)x_A \\ 0 & \lambda z_A & (1-\lambda)y_A \\ 0 & 0 & z_A \end{pmatrix} \begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} = \begin{pmatrix} (1-\lambda)x_A z_M + \lambda z_A x_M \\ (1-\lambda)y_A z_M + \lambda z_A y_M \\ z_A z_M \end{pmatrix}$$

In particular, $\lambda = -1$ gives the formula for reflexion with respect to a given point.

Similarly, the rotation with a finite center is given by the following transformation:

Proposition 3.15 (Rotation)**.** *Let $A = [x_A : y_A : z_A]$ be a finite point. The rotation of center A with angle θ is given by the following formula*

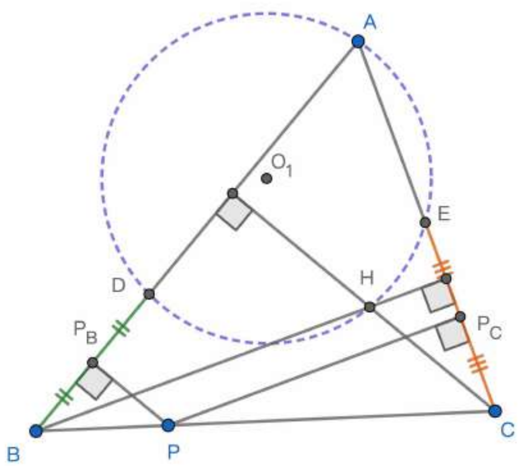
$$\mathcal{R} = \begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} \to \begin{pmatrix} z_A \cos\theta & -z_A \sin\theta & -x_A \cos\theta + y_A \sin\theta + x_A \\ z_A \sin\theta & z_A \cos\theta & -x_A \sin\theta - y_A \cos\theta + y_A \\ 0 & 0 & z_A \end{pmatrix} \begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix}$$

These basic computations provide us a straightforward method (based on linear transformation) to bash geometry problems that only involve lines, in a way similar to cartesian coordinates. Usually, computing all the coordinates in a problem is very tedious and time-consuming. Even though it is a painful task, let us perform such expensive computation once in our life, which will facilitate and motivate our later discussion.

Problem 3.16 (USA TST 2012 P1)**.** *In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.*

The first step we conduct is to translate the problem in a way that it only involves lines and linear transformations. This step is very important in the sense that it provides us the mechanism to generate the points in the problem in a projective way. In this problem, we need to a) appropriately define D and E; b) translate the statement into concurrency of lines.

- D can be obtained as follows: let P_B the orthogonal projection from P to the line AB, then D is the image of P_B under the homothety at point B with ratio 2.
- E can be defined similarly with respect to the point C.
- The concyclic property is a bit trickier. After trying some special cases, it is not hard to figure out that the fixed point is the orthocenter H of $\triangle ABC$. Hence, the statement that "A, D, E, H are concyclic" is equivalent to "the perpendicular bisectors of AD, AE and AH are concurrent". In this way, the problem can be stated without involving any circle



Proof. We provide detailed coordinates for all the points described above.

- Without loss of generality, we assume that BC is the x-axis with $B = [0 : 0 : 1]$ and $C = [1 : 0 : 1]$. The variable point P lies on BC, has coordinates $P = [t : 0 : 1]$.
- $A = [x_A : y_A : 1]$ where $y_A \neq 0$. (otherwise A lies on BC)
- The line AB and AC are determined by

$$AB : \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_A \\ -x_A \\ 0 \end{pmatrix} \quad AC : \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_A \\ 1 - x_A \\ -y_A \end{pmatrix}$$

- The lines perpendicular to AB all pass through the point $C^\perp = [y_A : -x_A : 0]$.
- The lines perpendicular to AC all pass through the point $B^\perp = [y_A : 1-x_A : 0]$
- The line through P perpendicular to the side AB and AC are respectively $PC^\perp$ and $PB^\perp$, given by

$$PC^\perp : \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} y_A \\ -x_A \\ 0 \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ -tx_A \end{pmatrix}, \quad PB^\perp : \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} y_A \\ 1-x_A \\ 0 \end{pmatrix} = \begin{pmatrix} x_A - 1 \\ y_A \\ t(1-x_A) \end{pmatrix}$$

- The orthogonal projection from P to AB and AC are respectively $P_B = AB \cap PC^\perp$ and $P_C = AC \cap PB^\perp$, given by

$$P_B = AB \cap PC^\perp : \begin{pmatrix} y_A \\ -x_A \\ 0 \end{pmatrix} \times \begin{pmatrix} x_A \\ y_A \\ -tx_A \end{pmatrix} = \begin{pmatrix} tx_A^2 \\ tx_Ay_A \\ x_A^2 + y_A^2 \end{pmatrix}$$

$$P_C = AC \cap PB^\perp : \begin{pmatrix} y_A \\ 1-x_A \\ -y_A \end{pmatrix} \times \begin{pmatrix} x_A - 1 \\ y_A \\ t(1-x_A) \end{pmatrix} = \begin{pmatrix} t(1-x_A)^2 + y_A^2 \\ y_A(1-x_A)(1-t) \\ (1-x_A)^2 + y_A^2 \end{pmatrix}$$

- The homothety at B of ratio 2 sends P_B to D, and the homothety at C of ratio 2 send P_C to E.

$$D = \begin{pmatrix} 2tx_A^2 \\ 2tx_Ay_A \\ x_A^2 + y_A^2 \end{pmatrix}, \quad E = \begin{pmatrix} (2t-1)(1-x_A)^2 + y_A^2 \\ 2y_A(1-x_A)(1-t) \\ (1-x_A)^2 + y_A^2 \end{pmatrix}$$

- The homothety at A of ratio $1/2$ sends D, E to the midpoint of AD and AE, denoted by M_D and M_E respectively

$$M_D = \begin{pmatrix} x_A(x_A^2 + y_A^2 + 2tx_A) \\ y_A(x_A^2 + y_A^2 + 2tx_A) \\ 2(x_A^2 + y_A^2) \end{pmatrix}, \quad M_E = \begin{pmatrix} (1+x_A)[(1-x_A)^2 + y_A^2] + 2(t-1)(1-x_A)^2 \\ y_A[(1-x_A)^2 + y_A^2] + 2y_A(1-x_A)(1-t)] \\ 2[(1-x_A)^2 + y_A^2] \end{pmatrix}$$

- The perpendicular bisector of AD and AE are $M_DC^\perp$ and $M_EB^\perp$, given by

$$\begin{aligned} M_DC^\perp &= \begin{pmatrix} x_A(x_A^2 + y_A^2 + 2tx_A) \\ y_A(x_A^2 + y_A^2 + 2tx_A) \\ 2(x_A^2 + y_A^2) \end{pmatrix} \times \begin{pmatrix} y_A \\ -x_A \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2x_A(x_A^2 + y_A^2) \\ 2y_A(x_A^2 + y_A^2) \\ -(x_A^2 + y_A^2 + 2tx_A)(x_A^2 + y_A^2) \end{pmatrix} = \begin{pmatrix} 2x_A \\ 2y_A \\ -(x_A^2 + y_A^2 + 2tx_A) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} M_EB^\perp &= \begin{pmatrix} (1+x_A)[(1-x_A)^2 + y_A^2] + 2(t-1)(1-x_A)^2 \\ y_A[(1-x_A)^2 + y_A^2] + 2y_A(1-x_A)(1-t)] \\ 2[(1-x_A)^2 + y_A^2] \end{pmatrix} \times \begin{pmatrix} y_A \\ 1-x_A \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2(1-x_A)((1-x_A)^2 + y_A^2) \\ 2y_A((1-x_A)^2 + y_A^2) \\ (1 - x_A^2 - y_A^2 + 2(t-1)(1-x_A))((1-x_A)^2 + y_A^2) \end{pmatrix} = \begin{pmatrix} 2(x_A - 1) \\ 2y_A \\ 1 - x_A^2 - y_A^2 + 2(t-1)(1-x_A) \end{pmatrix} \end{aligned}$$

- The altitudes $BB^\perp$ and $CC^\perp$ are given by

$$BB^\perp : \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}y_A\\1-x_A\\0\end{pmatrix} = \begin{pmatrix}x_A-1\\y_A\\0\end{pmatrix}, \quad CC^\perp : \begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}y_A\\-x_A\\0\end{pmatrix} = \begin{pmatrix}x_A\\y_A\\-x_A\end{pmatrix}$$

- The orthocenter $H = BB^\perp \cap CC^\perp$,

$$H = BB^\perp \cap CC^\perp : \begin{pmatrix}x_A-1\\y_A\\0\end{pmatrix} \times \begin{pmatrix}x_A\\y_A\\-x_A\end{pmatrix} = \begin{pmatrix}-x_Ay_A\\-x_A(1-x_A)\\-y_A\end{pmatrix} = \begin{pmatrix}x_Ay_A\\x_A(1-x_A)\\y_A\end{pmatrix}$$

- The midpoint AH is obtained the image of H under homothety at A with ratio $1/2$

$$M_H = \begin{pmatrix}2x_Ay_A\\x_A(1-x_A)+y_A^2\\2y_A\end{pmatrix}$$

- Since $AH \perp BC$, the perpendicular bisector of AH is given by

$$M_HA^\perp = \begin{pmatrix}2x_Ay_A\\x_A(1-x_A)+y_A^2\\2y_A\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-2y_A\\x_A(1-x_A)+y_A^2\end{pmatrix}$$

Finally, the perpendicular bisector of AD, AE and AH are concurrent if and only if

$$\det\begin{pmatrix}2x_A & 2(x_A-1) & 0\\ 2y_A & 2y_A & -2y_A\\ -(x_A^2+y_A^2+2tx_A) & 1-x_A^2-y_A^2+2(t-1)(1-x_A) & x_A(1-x_A)+y_A^2\end{pmatrix} = 0,$$

Expanding the expression yields a degree 2 polynomial of t, and we can check that each coefficient is zero, which finishes the proof. □

Remark 3.17. *Another way to see the determinant vanishes is to remark that* $(x_A-1)M_DC^\perp - x_AM_EB^\perp = M_HA^\perp$.

Important Discussion:

While the exact computation is quite tedious and pretty time consuming, there is one important point worth highlighting here. When **the moving point** P moves along the line BC, the other points/lines move accordingly, with **their coordinates are polynomials of the variable** $\boldsymbol{t}$. More concretely, when viewing x_A and y_A as constants, all the coordinates (points and lines) described above are degree 1 polynomial of t. And the final collinearity is to check a polynomial of degree 2, say $Q(t)$, is identically zero. Hence, if we are able to prove the statement for three different values of t, then from the fundamental theorem of algebra, the polynomial $Q(t)$ must be identically zero! This observation is quite amazing in the sense that we only need to check few special cases to prove a such complex statement. For this problem, it suffices to check that

- When P is the midpoint of BC, then D, E are the foot of B and C, hence A, D, E, H are concyclic.
- When P is B, then $D = B$, E is the reflexion point of C with respect to BH, let H_B be the foot of B on AC, then $B_H E \cdot B_H A = B_H C \cdot B_H A = B_H H \cdot BH_B$, implying that $A, E, H, B(= D)$ are concyclic.
- When P is C works similarly.

With that said, the problem becomes almost trivial under this method. The crucial step is to determine the degree of different points, also the degree of the statement. While as in this example, the polynomials all have degree one, which is the best scenario, mistakes and errors can be easily drawn when circles are involved. As a suggestion, I recommend **extreme cautious and rigorous regarding this step!**

3.2 Circles

In this section, we introduce the equation of circles:

Definition 3.18. *A circle in the projective plane* $\mathbb{P}^2_{\mathbb{R}}$ *has the equation*

$$\mathcal{C}_{(a,b),r} \ : \ (x - az)^2 + (y - bz)^2 = r^2 z^2, \quad \textit{where } (a, b) \in \mathbb{R}^2, \ r \in \mathbb{R}^+.$$

In particular, the circle $\mathcal{C}_{(a,b),r}$ does not pass any point at infinity, because if $z = 0$ then necessarily $x = y = 0$ and $[0 : 0 : 0]$ is not a point in $\mathbb{P}^2_{\mathbb{R}}$.[6] This

[6] If complex number are allowed, then any circle pass through two fix points $[1 : i : 0]$ and $[1 : -i : 0]$ at infinity.

means $\mathcal{C}_{(a,b),r}$ lies entirely in the Euclidean plane, and, it corresponds exactly to the circle with center (a,b) and radius r, i.e. $(x-a)^2+(y-b)^2=r^2$. The key proposition that allows us to deal with circles is the following result.

Proposition 3.19 (Second intersection point)**.** *Let $A=[x_A:y_A:z_A]$ be a point on the circle $\mathcal{C}_{(a,b),r}$, then for any $B=[x_B:y_B:z_B]\in\mathbb{P}^2_{\mathbb{R}}$, the second point of intersection C between the line AB and the circle is given by*

$$C=\alpha_B\begin{pmatrix}x_A\\y_A\\z_A\end{pmatrix}-2\beta_{AB}\begin{pmatrix}x_B\\y_B\\z_B\end{pmatrix}, \tag{5}$$

where

$$\begin{aligned}\alpha_B&=(x_B-az_B)^2+(y_B-bz_B)^2-r^2z_B^2,\\ \beta_{AB}&=(x_A-az_A)(x_B-az_B)+(y_A-bz_A)(y_B-bz_B)-r^2z_Az_B.\end{aligned}$$

The proof is rather straightforward by checking the constructed point C actually lies on the circle, which boils down to simple algebraic calculations. I recommend you to check it out by yourself as an exercise. Extreme rigorous is required when applying this result, as highlight below:

⚠ To apply this formula, **the point A must lie on the circle!**

⚠ In the simplest case where A is a given point on a fixed circle, the degree of C is twice the degree of B. Hence the degrees could quickly expode.

⚠ When applied in a general setting, must keep in mind that **both the center (a,b) and the radius r^2 could also be polynomial of t**.

4 Rational parametrization

In the previous section, we have defined a projective curve as the solution set of a homogeneous polynomial, i.e. $P(x, y, z) = 0$. This definition is easy to use for checking whether a point belongs to the curve: given a point $[x_0 : y_0 : z_0]$, we just need to evaluate $P(x_0, y_0, z_0)$, and check whether it vanishes. However, a down side of this definition is that it is not clear how one can draw the curve directly from the formula: imagine that you are asked to draw the curve defined by $P(x, y, z) = x^3 - xz^2 + z^3 - y^2 z$, you will be in trouble if you are not familiar with elliptic curves. This motivates the study of parametrization which provides a way to explicitly draw the curve:

Definition 4.1 (Rational parametrization). *Given (single variable) polynomials $U(t), V(t), W(t)$ with $gcd(U, V, W) = 1$, the set of points $[U(t) : V(t) : W(t)]$ for any $t \in R \cup \{\infty\}$ defines a rational parametrization of a projective curve.*

While as the notion is very natural and straightforward, there are multiple subtleties that we would like to highlight for rigorous of the presentation:

Value at ∞ The definition allows taking $t = \infty$, which deserves a special attention. A seemingly natural way to define the value at infinity is to take the limit when t goes to infinity. However, the limit of a non-constant polynomial at infinity is always infinite, which is not very helpful for defining a point. Here is the place where homogeneity comes into play, the representation refers to the same point when dividing all the coordinates by a same value: let d be the maximum degree of the polynomial among P, Q and R, then we define

$$[P(\infty) : Q(\infty) : R(\infty)] = \left[\lim_{t\to\infty} \frac{P(t)}{t^d} : \lim_{t\to\infty} \frac{Q(t)}{t^d} : \lim_{t\to\infty} \frac{R(t)}{t^d}\right].$$

Concretely, if $deg(P) < deg(S)$, then the limit is zero; else $deg(P) = deg(S)$, the limit is the value of leading coefficients.

Example 4.2. *Consider the following parametrization*

- *consider $[P(t) : Q(t) : R(t)] = [1 : t : t + 1]$, when $t \neq 0$, we can rewrite*

  $$[1 : t : t + 1] = \left[\frac{1}{t} : 1 : \frac{t+1}{t}\right],$$

  in which case the coordinates are rational function of t, justifying its name "rational parametrization". When $t \to \infty$, the limit point is $[0 : 1 : 1]$.

- *consider $[P(t) : Q(t) : R(t)] = [t^2 - 1 : 2t : t^2 + 1]$, when $t \neq 0$, we can rewrite*

  $$[t^2 - 1 : 2t : t^2 + 1] = \left[\frac{t^2-1}{t^2} : \frac{2}{t} : \frac{t^2+1}{t^2}\right].$$

  When $t \to \infty$, the limit point is $[1 : 0 : 1]$.

Irreducibility $gcd(U, V, W) = 1$ The property of irreducibility ensures that U, V, W does not vanish simultaneously at any value of t. Otherwise, assume that $U(t_0) = V(t_0) = W(t_0) = 0$, then $t - t_0$ is a common factor of U, V, W, violating the assumption. Hence the irreducibility guarantees that $[U : V : W] \neq [0 : 0 : 0]$[7], which is important since $[0 : 0 : 0]$ is not a point in the projective plane!

On the other hand, given any triple of polynomials U, V, W, we can always turn them into an irreducible form U_0, V_0, W_0. Hence we abuse the notation $[U : V : W]$ as defined by their irreducible representations $[U_0 : V_0 : W_0]$. To show a concrete example: it is clear that $[-t : t^2 : -t]$ can be reduced into $[-1 : t : -1]$. However, if we assign $t = 0$ in the expression $[-t : t^2 : -t]$, all the coordinates are zero which is not a point in the projective plane. In contrast, $t = 0$ is well defined in $[-1 : t : -1]$. Hence we should not take the value literally in a reducible representation, we need to first reduce it into the irreducible form.

Why $[U : V : W]$ defines a projective curve? By definition a projective curve is the solution set of a homogeneous polynomial. Thus it is not trivial at all that given any U, V, W, we can always find a non-zero polynomial P such that $P(U(t), V(t), W(t)) = 0$ for any t. (Try for example $U = t^2 + 1$, $V = t^2 + 2t - 2$, $W = 1$) Such finding process on the appropriate P is called **implicitization**. The procedure is constructive is based on resultant of polynomials which we will omit here. For whom are interested, please check chapter 17 in [7]. This means given any U, V, W, it is always possible to find P such that $P(U(t), V(t), W(t)) = 0$, which justifies the claim on projective curves.

Is any projective curve rational parametrizable? Unfortunately, this is not true, for example elliptic curves are not rational parametrizable (non-trivial). Hence rational curves is only a subset of projective curves:

Definition 4.3. *A projective curve $\mathcal{C}$ (defined by homogeneous polynomial) in $\mathbb{P}^2_{\mathbb{R}}$ is rational (parametrizable) if there are polynomials $U(t), V(t), W(t)$ with $gcd(U, V, W) = 1$ such that*

- *$[U(t) : V(t) : W(t)]$ lies on $\mathcal{C}$ for any $t \in \mathbb{R} \cup \{\infty\}$;*
- *For any point A on the curve $\mathcal{C}$, there is $t \in \mathbb{R} \cup \{\infty\}$ such that $A = [U(t) : V(t) : W(t)]$.*

The degree of the parametrization is the maximum degree of U, V, W.

Theorem 4.4. *A projective line is degree 1 rational parametrizable.*

[7]Indeed we also need to prove that the point at ∞ is not $[0 : 0 : 0]$, which is clear from the previous paragraph.

First proof: direct parametrization. Given the line $\ell_{[a:b:c]}$, we provide a direct parametrization case by case:

- If $a = b = 0$, then the line is the line at infinity, where $[t : 1 : 0]$ is a parametrization.
- If $a \neq 0$, then the point $[-\frac{c}{a} : 0 : 1]$ is its intersection with the x-axis. Then $[-\frac{c}{a} : 0 : 1] + t[-b : a : 0]$ gives an appropriate parametrization.
- If $b \neq 0$, then the point $[0 : -\frac{b}{c} : 1]$ is its intersection with the y-axis. Then $[0 : -\frac{b}{c} : 1] + t[-b : a : 0]$ gives an appropriate parametrization.

In all cases, we have a degree one parametrization. □

Second proof by projection. The main purpose of this second proof is to introduce a fundamental idea of projective geometry, fix a center and project from a line.

We start with the parametrization of the infinite line ℓ_∞, given by $[t : 1 : 0]$. Now consider a different line $\ell_{[a:b:c]}$ (this implies that either $a \neq 0$ or $b \neq 0$). We are going to create a bijection between the infinite line and the given line, leading to a parametrization. To do so, we take an arbitrary point A, that does not lie on either of the lines, as the projection center. Then for any point X from the infinite line, draw the line AX, intersecting $\ell_{[a:b:c]}$ at one and only one point $AX \cap \ell_{[a:b:c]}$. The function $X \to AX \cap \ell_{[a:b:c]}$, viewed as a projection of center A, gives a bijection[8] of the infinite line and the line $\ell_{[a:b:c]}$.

To provide a concrete example, let me take $A = [-ca + a : -cb + b : a^2 + b^2]$. (Exercise: justify that A lies on neither of the lines) Consider the following projection

$$\begin{aligned} f\colon \ell_\infty &\to \ell_{[a:b:c]} \\ X &\mapsto AX \cap \ell_{[a:b:c]} \end{aligned}$$

Given a moving point $X = [t : 1 : 0]$ on the infinite line, we have

$$AX = \begin{pmatrix} -ca + a \\ -cb + b \\ a^2 + b^2 \end{pmatrix} \times \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -(a^2 + b^2) \\ (a^2 + b^2)t \\ (1 - c)(a - bt) \end{pmatrix}$$

Therefore

$$AX \cap \ell_{[a:b:c]} = \begin{pmatrix} -(a^2 + b^2) \\ (a^2 + b^2)t \\ (1 - c)(a - bt) \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} (a^2c + b^2)t - (1 - c)ab \\ -(1 - c)abt + (a^2 + b^2c) \\ -(a^2 + b^2)(at + b) \end{pmatrix},$$

which gives a degree 1 parametrization. □

Example 4.5. *Given the formula in the second proof, the x-axis $y = 0$ ($\ell_{[0:1:0]}$) is parametrized by $[t : 0 : -1]$, which is clearly an appropriate (bijective) parametrization. The limit when $t \to \infty$ gives $[1 : 0 : 0]$.*

[8] Exercise: prove that this is indeed a bijection

Theorem 4.6. *A circle is degree 2 rational parametrizable.*

First proof: direct parametrization. Given a circle $\mathcal{C}_{(a,b),r}$ centered at (a,b) with radius, the parametrization $[a(t^2+1)+r(t^2-1) : b(t^2+1)+2rt : t^2+1]$ is an appropriate parametrization of the circle. $\square$

Second proof by projection. We approach in a similar way as the projections of lines, this time the projection center A needs to lie on the circle. Let us take the point $A = [a-r : b : 1]$ lying on the circle. Consider the following projection

$$\begin{aligned} f\colon \ell_\infty &\to \mathcal{C}_{(a,b),r} \\ X &\mapsto \text{second intersection of } AX \text{ and } \mathcal{C}_{(a,b),r} \end{aligned}$$

which is bijective. Now take any point $X = [t : 1 : 0]$ on the infinite line, then using the second intersection formula in (5), we have

$$\begin{aligned} \alpha_B &= (x_B - az_B)^2 + (y_B - bz_B)^2 - r^2 z_B^2 = t^2 + 1 \\ \beta_{AB} &= (x_A - az_A)(x_B - az_B) + (y_A - bz_A)(y_B - bz_B) - r^2 z_A z_B = -rt. \end{aligned}$$

This gives the parametrization

$$\alpha_B \begin{pmatrix} a-r \\ b \\ 1 \end{pmatrix} - 2\beta_{AB} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a(t^2+1)+r(t^2-1) \\ b(t^2+1)+2rt \\ t^2+1 \end{pmatrix},$$

which is indeed the same parametrization we provided in the direct parametrization. $\square$

Example 4.7. *Given the formula in the projection proof , the unit circle* $\mathcal{C}_{(0,0),1}$ *is parametrized by* $[t^2-1 : 2t : t^2+1]$, *which is clearly an appropriate (bijective) parametrization. The limit when* $t \to \infty$ *gives* $[1:0:1]$.

Definition 4.8 (Moving point). *For abbreviation, we call any rational parametrization* $[U(t) : V(t) : W(t)]$ *a moving point.*

The key idea underlying what we have discussed is to transfer geometry object into algebraic object, i.e. polynomials, and, geometric properties into algebraic identities. Hence whenever we refer to point/line, we always regard it as a moving point/line. A fixed point simply refers to constant polynomials U, V, W. The general idea is to start with a moving point on a line or a circle and construct the others accordingly:

Proposition 4.9. *A list of useful results are given in the following:*

- *A fixed point (do not depend on the moving point) has degree* 0.
- *If A has degree d_A and B has degree d_B, then the line AB has degree at most $d_A + d_B$. [Hint: apply (3)]*
- *If A has degree d_A and B has degree d_B, then the midpoint of AB has degree at most $d_A + d_B$. [Hint: apply homothety transformation as in Prop 3.14]*
- *If ℓ_1 has degree d_1 and ℓ_2 has d_2, then their intersection has degree at most $d_1 + d_2$. [Hint: apply (1)]*
- *If A has degree d_A on a* **fixed** *circle $\mathcal{C}_{(a,b),r}$, B has degree d_B, then the second intersection $C = AB \cap \mathcal{C}_{(a,b),r}$ has degree at most $d_A + 2d_B$. [Hint: apply (5)]*

⚠ The formula of the final bullet is not symmetric in A and B !

- If A is a fix point, then $C = AB \cap \mathcal{C}_{(a,b),r}$ has degree $2d_B$.
- If B is a fix point, then $C = AB \cap \mathcal{C}_{(a,b),r}$ has degree d_A.

While as the degree of the points/lines are usually increasing, there is one special case where the degree can be reduced: drawing lines between points on the circle. Let's start with an example.

Example 4.10. *Consider the point $A = [1 : 0 : 1]$ and a moving point P on the unit circle, parametrized by $[t^2 - 1 : 2t : t^2 + 1]$, then the line AP is given by*

$$AP = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} t^2 - 1 \\ 2t \\ t^2 + 1 \end{pmatrix} = \begin{pmatrix} -2t \\ -2 \\ 2t \end{pmatrix}$$

which indeed has degree one while P has degree 2!

Theorem 4.11. *Given two distinct moving points on the unit circle*[a] *$A = [P(t) : Q(t) : R(t)]$ and $B = [U(t) : V(t) : W(t)]$, then the line AB has degree at most $\frac{d_A + d_B}{2}$.*

[a]Here we mean that for any t, the point A and B always lie on the unit circle, in other words, $P^2(t) + Q^2(t) = R^2(t)$ for any t, and, $U^2(t) + V^2(t) = W^2(t)$ for any t.

Proof. We start with the following lemma:

Lemma 4.12. *Let P, Q, R by polynomials satisfying $P^2(t) + Q^2(t) = R^2(t)$ with degree at most d and $gcd(P, Q, R) = 1$. Then there exist polynomial S and T with degree at most $d/2$ such that $P = \frac{S^2-T^2}{2}$, $Q = ST$, and $R = \frac{S^2+T^2}{2}$.*

Proof of lemma. We have $Q^2 = R^2 - P^2 = (R - P)(R + P)$. Note that $gcd(R - P, R + P) = gcd(R, P) = 1$, hence both $R - P$ and $R + P$ are squares of polynomials. Thus there exists S and T such that $R - P = T^2$, $R + P = S^2$ and $Q = ST$. As a consequence $2deg(S) \leq d$ and $2deg(T) \leq d$. $\square$

According to the lemma, there are S, T such that $[P : Q : R] = [S^2 - T^2 : 2ST : S^2 + T^2]$ and X, Y such that $[U : V : W] = [X^2 - Y^2 : 2XY : X^2 + Y^2]$. Then the line AB is given by

$$AB = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \times \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} S^2 - T^2 \\ 2ST \\ S^2 + T^2 \end{pmatrix} \times \begin{pmatrix} X^2 - Y^2 \\ 2XY \\ X^2 + Y^2 \end{pmatrix} = \begin{pmatrix} 2(TX - SY)(SX - TY) \\ 2(TX - SY)(TX + SY) \\ 2(SX + TY)(SY - TX) \end{pmatrix}$$

Note that $TX - SY$ is not identically zero, otherwise the two moving points are identical for any t. Removing the common factor yields $AB = [SX - TY : SY + TX : -SX - TY]$, which has degree at most $\max(deg(S), deg(T)) + \max(deg(X), deg(Y)) \leq \frac{d_A + d_B}{2}$. $\square$

Example 4.13. *Consider the point $A = [3 : 4 : 5]$ and a moving point $P = [t^2 - 1 : 2t : t^2 + 1]$ both on the unit circle, then the line AP is given by*

$$AP = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \times \begin{pmatrix} t^2 - 1 \\ 2t \\ t^2 + 1 \end{pmatrix} = \begin{pmatrix} 4t^2 - 10t + 4 \\ 2t^2 - 8 \\ -4t^2 + 6t + 4. \end{pmatrix} = \begin{pmatrix} 2(t - 2)(2t - 1) \\ 2(t - 2)(t + 2) \\ -2(t - 2)(2t + 1). \end{pmatrix}$$

In particular, when $t = 2$, the point $P = A$, which implies $t - 2$ is a common factor of the coordinates. Factoring out $t - 2$ gives a degree one parameterization of the line AP.

> *If we take $t = 2$ directly in the equation of AP, we get $[0 : 0 : 0]$ which is not a point in $\mathbb{P}^2_{\mathbb{R}}$. This happens as the expression is reducible, where $t-2$ is a common factor.* **This shows the importance of the irreducibility.**
>
> *A question raised naturally: what does AP refer to when $P = A$? To answer it, we need to adapt the algebraic perspective where A and P are functions of t. While as A is a constant function, P moves on the circle. Thus "AP at $P = A$" is the limit of AP when P approaches A, which is also known as AA, the tangent line at A.*
>
> *In other words, AP not only relies on the absolute position of A and P, but also depend on how they move. In this simple case that A is fixed, AP depends on the trajectory of P in the neighborhood of A. Hence, even though P coincides with A when $t = 2$, the line AP is still well defined. This is a fundamental concept that will come back again and again.*

Lemma 4.14 (Coincidence Lemma)**.** *Given two non-identical moving point $A = [P(t) : Q(t) : R(t)]$ and $B = [U(t) : V(t) : W(t)]$, if $A = B$ for k different values of t, then the line AB has degree at most $deg(A) + deg(B) - k$.*

Proof. Let t_i be one of the values such that $A = B$. Then AB evaluated at t_i is $[0 : 0 : 0]$, meaning that $t - t_i$ is a common factor of the coordinates AB. This is true for any t_i such that $A = B$, therefore AB can be factorized by $\prod(t - t_i)$ with degree at most $deg(A) + deg(B) - k$. □

Remark 4.15. *In some cases, we might have a root t_0 with multiplicity larger than 1, i.e. $(t - t_0)^n$ is a common factor of the coordinates AB. However, this is in general hard to check without deriving the exact formula of AB.*

Remark 4.16. *This lemma is extremely useful for reducing the degree of points/-lines. Indeed, Example 4.13 is just a special case of applying the Coincidence Lemma, as P coincides with A at value $t = 2$ ($[t^2 - 1 : 2t : t^2 + 1]_{t=2} = [3 : 4 : 5]$).*

Degree of statement As we have seen in Problem 3.16, the final step of the proof is to check a polynomial is identically zero. We call the degree of such polynomial as the degree of the statement.

- If A, B, C has degree d_A, d_B, d_C, then the statement A, B, C are collinear is equivalent to "a degree at most $d_A + d_B + d_C$ polynomial is identically zero". [Hint: apply (4)]
- If ℓ_1, ℓ_2, ℓ_3 has degree d_1, d_2, d_3, then the statement ℓ_1, ℓ_2, ℓ_3 are concurrent is equivalent to "a degree at most $d_1 + d_2 + d_3$ polynomial is identically zero". [Hint: apply (2)]

For statement such as "A, B, C, D are concyclic", one way to reformulate it is "the perpendicular bisector of AB, AC and AD are concurrent". This turns out to be a concurrency statement after carefully evaluating the degree of each line. We will see a more advanced characterization for concyclic statement later based on angle conditions in Corollary 5.14.

Lemma 4.17 (Coincidence Lemma for statement)**.** *Given three moving points A, B and C, if $A = B$ for k different values of t, then the statement "A, B, C are collinear" has degree at most $deg(A) + deg(B) + deg(C) - k$.*

General routine to apply the method of moving point is as follows

1. Formulate the problem in the language of projective geometry. (For example, use transformations to describe the problem);
2. Determine a moving point in the problem. Animate it along a line/-circle, which define the initial parametrization (have degree one/two respectively).
3. Carefully evaluate the degree of the points/lines encountered in the problem;
4. Bound the degree d of the polynomial corresponding to the statement;
5. Inspect whether **coincidence lemma** can be applied to reduce the degree of the statement, assume that k coincidences are found.
6. Find $d - k + 1$ **distinct** values of the parameter $t \in \mathbb{R} \cup \{\infty\}$ where we can easily check the problem is true. This implies that the desired polynomial is identically zero.

Now we are ready to apply the moving point method, let us first use it to show some classic theorems.

Theorem 4.18 (**Pappus's Theorem).** *Given two distinct lines ℓ_1 and ℓ_2. Let A_1, B_1, C_1 be arbitrary points on ℓ_1, A_2, B_2, C_2 be arbitrary points on ℓ_2, then the intersections $D = A_1B_2 \cap B_1A_2$, $E = B_1C_2 \cap C_1B_2$ and $F = C_1A_2 \cap A_1C_2$ are collinear.*

Proof. We fix the points A_1, B_1, C_1, A_2, B_2 and let C_2 be a moving point on the line ℓ_2, which has degree 1. We start by evaluating the degrees of all the points we care about:

- The line A_1B_2, A_2B_1, C_1B_2 and C_1A_2 are fixed, hence $D = A_1B_2 \cap B_1A_2$ is fixed.
- The line B_1C_2 has degree $deg(B_1) + deg(C_2) = 0 + 1 = 1$, hence E as the intersection of B_1C_2 and C_1B_2 has degree $deg(B_1C_2)+deg(C_1B_2) = 1+0 = 1$.
- Similarly, A_1C_2 has degree 1 implying that F is degree 1.

Therefore, the statement D, E, F are collinear has degree $deg(D) + deg(E) + deg(F) = 0 + 1 + 1 = 2$. We start by checking whether we can reduce the statement degree via coincidence lemma. Indeed,

- When $C_2 = \ell_1 \cap \ell_2$ (which always exist in $\mathbb{P}^2_{\mathbb{R}}$), we have $E = B_1C_2 \cap C_1B_2 = \ell_1 \cap C_1B_2 = C_1$, similarly $F = C_1$. Hence by coincidence lemma for statement, the degree of the statement reduces by 1.

Therefore the degree of the statement is 1, it suffices to check two special cases:

- When $C_2 = A_2$, we have $F = C_1A_2 \cap A_1C_2 = C_1A_2 \cap A_1A_2 = A_2$. Note that $E = B_1C_2 \cap B_2C_1$ lies on $B_1C_2 = B_1A_2$ and D lies on B_1A_2 by definition, we conclude that D, E, F all lies on B_1A_2.

- When $C_2 = B_2$, we obtain similarly that D, E, F all lies on A_1B_2.

□

Theorem 4.19 (**Pascal's Theorem**). *Given a circle ω and let $ABCDEF$ be arbitrary points on ω. Then the intersections $X = AB \cap DE$, $Y = BC \cap EF$ and $Z = CD \cap FA$ are collinear.*

We leave the proof as an exercise to the readers, which is essentially the same as Pappus's theorem. The key is to apply Theorem 4.11 to reduce the degree of lines as we are working on circles. Next, we show another classic theorem, the Butterfly Theorem.

Theorem 4.20 (**Butterfly Theorem**). *Let M be the midpoint of a chord PQ of a circle Ω. Two other chords AB and CD of Ω are drawn, both passing through M. Let AD and BC intersect chord PQ at X and Y respectively. Then M is the midpoint of XY.*

Proof. We reformulate the problem as follows. Fix the circle Ω, the chord PQ and CD. Let A be a moving point on the circle, B be the second intersection of AM and Ω. Let $X = AD \cap BC$ and X' be the reflexion of X with respect to M. We show that B, X', C are collinear.

We start by evaluating the degrees of different points.

- A moves on Ω, has degree 2.

- As M is fixed, the second intersection of AM and Ω, which is B, has degree at most $deg(A) + 2deg(M) = 2$.

- As A,D both lie on Ω, AD has degree $\frac{deg(A)+deg(D)}{2} = \frac{2+0}{2} = 1$.

- The intersection of AD and PQ, which is X, has degree $deg(AD)+deg(PQ) = 1 + 0 = 1$.

- The reflexion at M, which is fixed, sends X to X', we have $deg(X') = deg(X) = 1$.

- Finally the statement B, X', C are collinear has degree $2 + 1 + 0 = 3$

Before we check special cases, we first check whether we can reduce the statement of the problem. We can easily check that

- When $A = P$, we have $X = A = P$ and $X' = Q = B$.

- Similarly $A = Q$ gives $X' = P = B$.

- When $A = D$, we have $B = C$.

Therefore, by coincidence lemma, the determinant of B, X', C (which is a polynomial of t) has degree at most $deg(B) + deg(X') + deg(C) - 3 = 2 + 1 - 3 = 0$. Finally, it suffices to check 1 special case. Indeed,

- When $A = C$, we have $B = D$ and $X = M = X'$, the collinearity clearly follows, which finishes the proof.

$\square$

Remark 4.21 (Important Discussion). *This proof involves several important aspects deserving further clarifications:*

- *The determinant of B, X', C is a polynomial of t (the moving variable). It has degree 0 means that the determinant is constant. However, the constant might not be zero so we still need to check one special case.*

- *We should be careful not confusing the coincidence lemma with the special cases, which have very different nature:*

- *When coincidence occurs, for example $B = C$, it means that the algebraic expression represented by the line BC is reducible (by a common polynomial factor), hence the degree of the algebraic objects/statement can be reduced.*

- *The special cases should be considered in their irreducible forms. In other words, even though $B = C$ (when $A = D$ in the Butterfly theorem), the collinearity of B, X', C is non-trivial, which should be interpreted as: the tangent line at C (also known as CC) passes through X'. The reason we get the tangent line is because B moves on Ω. As B approaches C, the limit of the line BC approaches the tangent line at C. Therefore, the theorem takes the following degenerated form:*

  > *Let M be the midpoint of a chord PQ. Another chord CD passes through M. The tangent line at C and D with respect to the circle Ω intersects PQ at X and Y respectively. Prove that M is the midpoint of XY.*

  Once we prove this statement (which is not hard by drawing a parallel line at C of DD), we can use $A = D$ as the special case in our final step instead of $A = C$. In such case, the point $A = D$ helps us reducing the degree twice, one for coincidence and the other as special point.

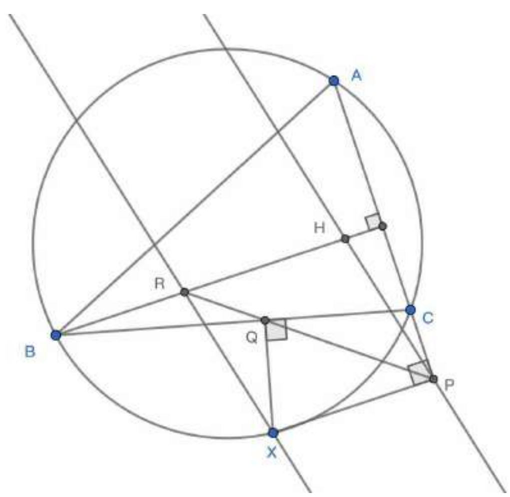
- *In the general case, whenever two points S and T coincides, the line ST refers to the tangent line to whatever the algebraic curve ST jointly defined. However, characterizing this limit line is in general difficult. We have a concrete example in our proof: when $A = P$, we have $B = Q = X'$. However characterizing the irreducible form* *of BX' is not trivial at all. As X' moves towards Q on PQ and B moves on Ω, the limit line BX' (when $B = X'$) depends on the relative speed between B and X', which is not easy to handle. Since we are unable to interpret the irreducible form of BX', we can not use $A = P$ as a special case. Nevertheless, it is used to reduce the degree via coincidence lemma.*

- **To summarize, we should be aware that coincidence lemma is intrinsically different than special cases, even though it is rare that one can use one point for both purposes (as the case $A = D$ here).**

Problem 4.22 (USA TST 2014 P1)**.** *Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point*

Proof. Let us fix $\triangle ABC$ and consider X as a moving point on its circumcircle, which is degree 2 parametrizable. We reformulate the problem as follows.

" Let H be the orthocenter of $\triangle ABC$ and X an arbitrary point of the circumcircle $\odot(ABC)$. Define $A^{\perp}$,$B^{\perp}$ be the intersection of AH,BH and the line at infinity, respectively. Let $P = XB_{\perp} \cap AC$ and $Q = XA_{\perp} \cap BC$. Denote $R = PQ \cap BH$ and $Z = RX \cap \ell_{\infty}$, show that H, P, Z are collinear."



We now evaluate the degree of the corresponding points/lines

- The lines AH, BH are fixed implying that $A^\perp$ and $B^\perp$ are fix points.
- The line $XB^\perp$ ($XA^\perp$) has degree 2.
- $P = XB^\perp \cap AC$ has degree 2, also is $Q = XA^\perp \cap BC$.
- The line PQ has degree 4.
- The point $R = PQ \cap BH$ has degree 4.
- The line RX has degree 6.
- $Z = RX \cap \ell_\infty$ has degree 6
- The statement H, P, Z are collinear is a degree $0 + 2 + 6 = 8$ problem

Hence it suffices to find 9 special cases.

- when $X = B$, then $Q = B$, $P \in BH$, hence $R \in BH$ and finally $Z \in BH = PH$. Similarly $X = A, C$ also works.
- when X be the antipole of B, then $Q = C$, $P \in AC$, hence $R = BH \cap AC$ with $XP \parallel HR$ and $XP = HR$, hence $HP \parallel RX$. Similarly X be the anti-pole of A and C also works.
- when X be $AH \cap \odot(ABC)$, then $Q = BH \cap BC$ and $XQ = QH$. Together with $RH \parallel PX$ implies $XPHR$ is a parallelogram, hence $HP \parallel RX$. Similarly X be the intersection of BH, CH with $\odot(ABC)$ works as well.

$\square$

Remark 4.23. *One may wonder that we have to check these special points are all distinct, which is not always the case. When $\triangle ABC$ is acute, the points A, B, C and their anti-poles are distinct. However, $AH \odot (ABC)$ might be the same point as the anti-pole of A, in which case $\triangle ABC$ is isosceles. Nevertheless, we have shown that for any acute non-isosceles triangle, the statement holds. To show the degenerate case, it suffices to fix X and consider A as a moving point. On one hand, this can be formulate as a finite polynomial in the coordinates of A. On the other hand, we have already shown infinite many solutions: for those A such that $\triangle ABC$ is acute non-isosceles. Hence this polynomial must be identically zero.*

As we have gone through some positive examples of moving point methods, we would like to provide a negative example which helps understanding the limitation of the moving point method.

Theorem 4.24 (Simson Line). *Let Ω be the circumcircle of triangle $\triangle ABC$. Let P be an arbitrary point on Ω and denote D, E, F the orthogonal projection from P to BC, CA, AB respectively. Prove that D,E and F are collinear.*

Tentative proof. We fix $\triangle ABC$ and Ω, let P be the moving point, which has degree 2. As BC is fixed, the orthogonal direction of BC is fixed. Hence the line ℓ through P perpendicular to BC has degree the same as P. Therefore $D = \ell \cap BC$ has degree $deg(\ell) + deg(BC) = 2 + 0 = 2$. Similarly, E, F has degree 2. Therefore, the statement D, E, F are collinear has degree 6.

Note that when $P = A$, E coincides with F. Similarly when $P = B$, D coincides with F; when $P = C$, D coincides with E. Hence by coincidence lemma, the degree of the statement is at most $6 - 3 = 3$.

Finally, we need to find 4 special cases:

- When P is the anti-pole of A, we have $E = C$, $F = B$ and D lies on BC, hence collinearity holds.
- Similarly for P being the anti-pole of B and C.

However, we still need to find another special cases. I am not able to find another trivial case (this could be wrong if you can find one). This special case should not involve angle chasing argument, otherwise we can simply apply angle chasing on the original problem, without going through the moving point process. □

The main reason we fail to apply moving point method in this example is due to the lack of special cases. It is in general not easy to determine in advance whether we have enough special points. My recommendation would be always try to apply angle chasing to see whether some properties can be easily drawn, in which case, the degree of the statement might be reduced and we get a better chance to finish up with enough special cases.

A rather surprising thing is that we can prove a generalization of Simson Line using moving point method. The generalization is as follows:

Theorem 4.25 (Generalization of Simson Line)**.** *Let Ω be the circumcircle of triangle $\triangle ABC$. Let Q be an arbitrary point, denote the second intersection of AQ, BQ, CQ with Ω by A_1, B_1, C_1. Let P be an arbitrary point on Ω and denote $D = PA_1 \cap BC$, $E = PB_1 \cap CA$, $F = PC_1 \cap AB$. Prove that Q, D, E and F are collinear.*

A quick proof would be to apply Pascal's theorem on AA_1PB_1BC, which implies that Q, D, E are collinear. If you have not recognized the Pascal's configuration, applying moving point can also easily prove that Q, D, E are collinear (basically copying the proof of Pascal's theorem).

To recover the Simson Line, it suffices to take Q to the infinite line such that the angle between OQ and OP is given by $\angle PAB + \angle PBC + \angle PCA$. (where O is the center of Ω). In this case, Q is no longer an arbitrary point, it depends on P. The surprising thing is that such coupling of P and Q makes the problem less appealing for moving point method. This example should give you a sense about the limitation of moving point method, which is less favorite when specific dependency exists. In the following, we continue our journey by introducing a way to handle angles in moving point method.

5 The space of lines

As you may have already noticed in section 3.1, there is a kind of symmetry between the collinearity in points and the concurrency between lines, governed by the same formula of determinant. We would like to clarify such symmetry in this section, which is formally called **duality**. To form the duality, we first introduce the space of lines as the set of all projective lines in $\mathbb{P}^2_{\mathbb{R}}$:

$$\mathbb{L}^2_{\mathbb{R}} = \{\ell_{[a:b:c]} \mid (a,b,c) \neq (0,0,0)\}^{9},$$

where $\ell_{[a:b:c]}$ refers to the line $ax + by + cz = 0$. A very first thing we realize is that $\mathbb{L}^2_{\mathbb{R}}$ inherits a homogeneous coordinate system from the projective plane. Indeed, each triple (a, b, c) determines a line[10], the line determined is unchanged if the triple is multiplied by a non-zero scalar, i.e. the line $\ell_{[\lambda a:\lambda b:\lambda c]}$ defined by

$$\lambda ax + \lambda by + \lambda cz = 0 \Leftrightarrow ax + by + cz = 0 \quad \forall \lambda \neq 0.$$

is the same line as $\ell_{[a:b:c]}$! Therefore $[a : b : c]$ may be taken to be homogeneous coordinates of a line in the projective plane, which is the **line coordinates as opposed to point coordinates**.

Therefore, the space of lines $\mathbb{L}^2_{\mathbb{R}}$ shares the same three-coordinate homogeneous representation as the projective plane. While $[x : y : z]$ refers to a point in projective plane, $[a : b : c]$ refers to a line $\ell_{[a:b:c]}$. The fundamental elements in $\mathbb{P}^2_{\mathbb{R}}$ are points while the fundamental elements in $\mathbb{L}^2_{\mathbb{R}}$ are lines. This leads to the concept of duality in projective geometry, the principle that the roles of points and lines can be interchanged in a theorem in projective geometry and the result will also be a theorem.

To formalize the concept of duality, whenever a homogeneous coordinates is given, we call that object a point in the associated space. In such terminology, the coordinates $[a : b : c]$, referring $\ell_{[a:b:c]}$, is a point in $\mathbb{L}^2_{\mathbb{R}}$. Hence, a point in $\mathbb{L}^2_{\mathbb{R}}$ is a line in $\mathbb{P}^2_{\mathbb{R}}$. One way to interpret such terminology is that the two spaces $\mathbb{P}^2_{\mathbb{R}}$ and $\mathbb{L}^2_{\mathbb{R}}$ operates on a different scale: the projective plane is "microscopic", operating on points, and, the space of lines is "macroscopic", operating on lines. In other words, the space of lines $\mathbb{L}^2_{\mathbb{R}}$ only care about structures and relationships between lines. Now given that we have defined the points in $\mathbb{L}^2_{\mathbb{R}}$, we can naturally define the lines in $\mathbb{L}^2_{\mathbb{R}}$ by

[9] This notation $\mathbb{L}^2_{\mathbb{R}}$ is not a standard notation, I introduce it in order to distinguish between point coordinates and line coordinates. In the language of abstract algebra, $\mathbb{L}^2_{\mathbb{R}}$ is isomorphic to the projective plane $\mathbb{P}^2_{\mathbb{R}}$, meaning that they share the same underlying structure, see [Wiki link]. Hence, the space of lines is often directly denoted by $\mathbb{P}^2_{\mathbb{R}}$. This could be confusing for someone who are not familiar with isomorphism, so I explicitly stretch the difference between point coordinates and line coordinates by introducing the notation $\mathbb{L}^2_{\mathbb{R}}$.

[10] at least one of a, b and c must be non-zero

Definition 5.1 (**Pencil of lines**)**.** *A line in $\mathbb{L}^2_{\mathbb{R}}$ is given by the equation*

$$pl_{[\alpha:\beta:\gamma]} = \{[a:b:c] \in \mathbb{L}^2_{\mathbb{R}} \text{ such that } \alpha a + \beta b + \gamma c = 0\},$$

where $(\alpha, \beta, \gamma) \neq (0,0,0)$.

This said, a line in $\mathbb{L}^2_{\mathbb{R}}$ encodes a linear structure between the set of points in $\mathbb{L}^2_{\mathbb{R}}$ (to emphasize, a point in $\mathbb{L}^2_{\mathbb{R}}$ is a line in $\mathbb{P}^2_{\mathbb{R}}$). It is then natural to ask what $pl_{[\alpha:\beta:\gamma]}$ in $\mathbb{L}^2_{\mathbb{R}}$ corresponds to in the perspective of $\mathbb{P}^2_{\mathbb{R}}$:

Example 5.2. *Let us take $\alpha = 1$, $\beta = 2$, $\gamma = 3$. From definition, $pl_{[1:2:3]}$ consists of all the lines $\ell_{[a:b:c]}$ satisfying $a+2b+3c = 0$. Note that $\ell_{[a:b:c]}$ consists of all the points (x, y, z) satisfying $ax + by + cz = 0$. Therefore $\ell_{[a:b:c]} \in pl_{[1:2:3]}$ if and only if $[1:2:3]$ lies on the line $\ell_{[a:b:c]}$. This means*

$$pl_{[1:2:3]} = \{ \text{ all the projective lines } \ell_{[a:b:c]} \text{ passing through } [1:2:3] \},$$

which is also known as the **pencil of lines** *passing through $[1:2:3]$.*

Example 5.3. *Let us take $\alpha = 1$, $\beta = 0$, $\gamma = 0$, which gives the pencil of lines passing through $[1:0:0]$. Note that $[1:0:0]$ is an infinite point, a (Euclidean) line ℓ pass through it if and only if ℓ is parallel to the vector $(1, 0)$. Therefore*

$$pl_{[1:0:0]} = \{ \text{ all the lines parallel to the } x\text{-axis} \} \cup \ell_\infty.$$

Readers already familiar with projective geometry may have noticed that in this way we immediately build the connection between cross-ratio on lines are nothing but the cross ratio (of points) in the space of line, we will come back to this later. Usually pencil of lines are defined as the family of lines passing through a given point. Here, we define the pencil of line as a line in the space of lines $\mathbb{L}^2_{\mathbb{R}}$. The two definition do match each other as we show in the example. The important thing is that the pencil of lines encodes the line structure in the space of lines, which allows us to derive the dual formulation of all the theorems in section 3.1. We recall Theorem 3.7:

Theorem 3.7. *Given two distinct projective lines ℓ_1 and ℓ_2 in the projective plane, there is one and only one point A lies simultaneously on ℓ_1 and ℓ_2. In other words, any two distinct projective lines intersect at one point.*

Now we apply it in the space of lines:

Theorem 5.4 (Theorem 3.7 in the space of lines)**.** *Given two distinct lines pl_1 and pl_2 in the space of lines $\mathbb{L}^2_{\mathbb{R}}$, there is one and only one point $P = [a_P : b_P : c_P] \in \mathbb{L}^2_{\mathbb{R}}$ lies simultaneously on pl_1 and pl_2.*

For completeness, I provide a proof below, which is really just a copy-paste of the proof on Theorem 3.7.

Proof. Assume that pl_1 is given by $\alpha_1 a + \beta_1 b + \gamma_1 c = 0$ and pl_2 is given by $\alpha_2 a + \beta_2 b + \gamma_2 c = 0$. Since pl_1 and pl_2 are distinct, the vector $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are not aligned. Let

$$\begin{pmatrix} a_P \\ b_P \\ c_P \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \times \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \tag{6}$$

be the cross product of the above two vectors. Therefore a_P, b_P, c_P are not all zero, which defines a point in the projective plane $P = [a_P : b_P : c_P]$. Moreover,

$$\alpha_1 a_P + \beta_1 b_P + \gamma_1 c_P = 0, \quad \text{and} \quad \alpha_2 a_P + \beta_2 b_P + \gamma_2 c_P = 0$$

Hence $P \in pl_1 \cap pl_2$, which shows the existence of the intersection point.

The uniqueness follows from the fact that the vectors $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ forms a plane in $\mathbb{R}^3$. If a point $Q = [a_Q : b_Q : c_Q]$ lies both on pl_1 and pl_2 then the vector (a_Q, b_Q, c_Q) is orthogonal to this plane, which is necessarily parallel to the cross product, hence there is λ such that $a_Q = \lambda a_P, b_Q = \lambda b_P$ and $c_Q = \lambda c_P$, implying that $[a_Q : b_Q : c_Q] = [a_P : b_P : c_P]$. □

Now, if we interpret the above Theorem 5.4 in the projective plane $\mathbb{P}^2_{\mathbb{R}}$, we get the dual formulation of Theorem 3.7. Note that a pencil of line is uniquely determined by its center $[\alpha : \beta : \gamma]$, and the point $[a : b : c] \in pl_{[\alpha:\beta:\gamma]}$ if and only if $\alpha a + \beta b + \gamma c = 0$, which can also be interpreted as the line $\ell_{[a:b:c]}$ pass through $[\alpha : \beta : \gamma]$. Therefore, Theorem 5.4 is indeed stating the following result:

Theorem 5.5 (Dual form of Theorem 3.7)**.** *Given two distinct points $[\alpha_1 : \beta_1 : \gamma_1]$ and $[\alpha_2 : \beta_2 : \gamma_2]$ in $\mathbb{P}^2_{\mathbb{R}}$[a], there is one and only one line $\ell_{[a_P:b_P:c_P]}$[b] pass through both points.*

[a] served as centers of pencil of lines
[b] served as the intersection point of pl_1 and pl_2 in $\mathbb{L}^2_{\mathbb{R}}$.

As a point in $\mathbb{L}^2_{\mathbb{R}}$ is a line in $\mathbb{P}^2_{\mathbb{R}}$, the role of points and lines get interchanged ! Hence the dual form of Theorem 3.7 is Theorem 3.10 ! The general routine to dualize a theorem is to restate the result in the space of lines $\mathbb{L}^2_{\mathbb{R}}$, then reinterpret it in $\mathbb{P}^2_{\mathbb{R}}$, which can also be done by simply interchanging the role between points and lines. For instance,

> *Theorem 3.7: two line intersects at a point*

interchange line and point yields

> *Theorem 3.10 two points lie (determine) on a common line.*

Similarly, it is easy to see that the dual form of Theorem 3.8 is Theorem 3.11 ! Such connection also explains why their proofs look quite similar. If we go a bit further, we can show that several well known theorems are in dual formulation:

- Desargues' theorem $\leftrightarrow$ Converse of Desargues' theorem,
- Pascal's theorem $\leftrightarrow$ Brianchon's theorem,
- Menelaus' theorem $\leftrightarrow$ Ceva's theorem.

In other words, Pascal's theorem applied on the space of lines gives Brianchon's theorem in the projective plane. The derivation is pretty straightforward, which is left to the readers.

The underlying idea of duality is to take a line as an object, in contract to points. Such abstraction allows us to manipulate and focus on the structure between lines. Before moving on, let us recap the main concepts and results:

Space of lines $\mathbb{L}^2_{\mathbb{R}}$		Projective plane $\mathbb{P}^2_{\mathbb{R}}$
a point $[a : b : c]$	$\leftrightarrow$	a line $\ell_{[a:b:c]}$
a line $pl_{[\alpha:\beta:\gamma]}$	$\leftrightarrow$	pencil of lines at center $[\alpha : \beta : \gamma]$
a point lies on a line	$\leftrightarrow$	a line pass through a point
two lines intersect	$\leftrightarrow$	two points determine a line
collinear points	$\leftrightarrow$	concurrent lines

The main idea underlying the space of lines is to manipulate lines as an object. The fundamental operation between lines is rotation, which also defines angles. More concretely, given two lines ℓ_1 and ℓ_2 intersecting in a finite point O of the Euclidean plane. By the angle $\angle(\ell_1, \ell_2)$, we mean the angle by which ℓ_1 has to be rotated counterclockwise around O until it coincides with ℓ_2. To properly operate on angles, let us first introduce the following terminology.

Definition 5.6 (Direction of line). *Given a finite line $\ell \in \mathbb{P}^2_{\mathbb{R}}$, we call the point $\ell \cap \ell_\infty$ ($\in \mathbb{P}^2_{\mathbb{R}}$) the direction of ℓ .*

First, let us justify that such definition aligns with the usual meaning of direction. Indeed, given two parallel (Euclidean) lines ℓ_1 and ℓ_2, they intersect on a unique point on the line at infinity, i.e. $\ell_1 \cap \ell_\infty = \ell_2 \cap \ell_\infty$. Therefore, according to Definition 5.6, they have the same direction, as in the common way.

Second, such definition is orientation invariant and length invariant. As opposed to the usual definition, where the direction is defined as the unit vector along the line, we need to impose the length($= 1$) and orientation appropriately.

Here, due to the property on homogeneous coordinate, we can multiply the coordinates with an arbitrary scalar λ and it still remains the same point in $\mathbb{P}^2_{\mathbb{R}}$. We illustrate it with a concrete example:

Example 5.7. *Consider the projective line $\ell_1 : x - y + z = 0$, associating to the Euclidean line $x - y + 1 = 0$. By Definition 5.6, its direction is given by*

$$\ell_1 \cap \ell_\infty = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Now consider a second line $\ell_2 : -\pi x + \pi y + 3.14z = 0$, associating to the Euclidean line $-\pi x + \pi y + 3.14 = 0$, its direction is given by

$$\ell_2 \cap \ell_\infty = \begin{pmatrix} -\pi \\ \pi \\ 3.14 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \\ 0 \end{pmatrix}.$$

As in the homogeneous coordinate system, we have $[\pi : \pi : 0] = [-1 : -1 : 0]$, the two lines share the same direction. The different sign and scale does not matter. More generally, the direction of $\ell_{[a:b:c]}$ is $[b : -a : 0]$.

With the direction of lines in hand, we are now ready to present the important rotation lemmas:

Lemma 5.8 (**Rotation lemma (fixed angle)**)**.** *Given a (Euclidean) line $\ell_{[a:b:c]}$ and a finite point $A = [x_A : y_A : z_A]$, perform the rotation $\mathcal{R}_\theta$ of angle θ (counterclockwise) at center A, then the image of $\mathcal{R}_\theta(\ell_{[a:b:c]})$ has direction*

$$D_\theta = [b \cos\theta + a \sin\theta : -a \cos\theta + b \sin\theta : 0].$$

Moreover, if A **lies on** *$\ell_{[a:b:c]}$, then the image of $\mathcal{R}_\theta(\ell_{[a:b:c]})$ is given by*

$$\mathcal{R}_\theta(\ell_{[a:b:c]}) = Ad_\theta : A \times d_\theta = \begin{pmatrix} (a \cos\theta - b \sin\theta) z_A \\ (a \sin\theta + b \cos\theta) z_A \\ (-a \cos\theta + b \sin\theta) x_A - (a \sin\theta + b \cos\theta) y_A \end{pmatrix}$$

In particular, the degree of $\mathcal{R}_\theta(\ell_{[a:b:c]})$ is at most deg(A) + deg$(\ell_{[a:b:c]})$.

Proof. To derive D_θ, it suffices to remark that rotation sends $\ell_{[a:b:c]} \cap \ell_\infty$ to $\mathcal{R}_\theta(\ell_{[a:b:c]})$, then apply the rotation formula in Proposition 3.15. $\square$

⚠ To apply this formula, the angle θ must be fixed (independent of t)!

⚠ It is possible to derive a formula when A does not lie on $\ell_{[a:b:c]}$, but we rarely use it since the degree of $\mathcal{R}_\theta(\ell_{[a:b:c]})$ get tripled, becomes $deg(A) + 3deg(\ell_{[a:b:c]})$ in such case.

Problem 5.9. *Let I be the incenter of $\triangle ABC$ and D be the foot of I on BC. Points P, Q lie on lines IB, IC such that $\measuredangle PA_1Q = 90°$. Then $\measuredangle PAQ = \frac{1}{2}\measuredangle BAC$.*

Proof. Fix $\triangle ABC$ and let P be the moving point on IB, which has degree 1. The line DP, AP has degree 1. Let $\mathcal{R}_D$ be the rotation of center D with angle $90°$, and $\mathcal{R}_A$ be the rotation of center A with angle $\frac{1}{2}\measuredangle BAC$. Then $\mathcal{R}_D(DP)$ and $\mathcal{R}_A(AP)$ both have degree 1. The statement is equivalent to CI, $\mathcal{R}_D(DP)$ and $\mathcal{R}_A(AP)$ are concurrent, which is a degree 2 statement. Hence it suffices to check 3 special cases:

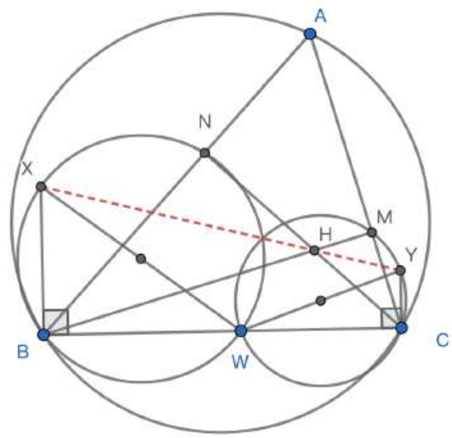
- $P = B$ then $Q = I$ and the statement holds.
- $P = I$ then $Q = C$ and the statement holds.
- $P = BI \cap \ell_\infty$. Then $DQ \perp BI$. Let $X = AQ \cap BI$, then $\measuredangle IXA = \measuredangle(IB, AQ) = \measuredangle BIC - 90° = \frac{1}{2}\measuredangle BAC$, which completes the proof.

□

Remark 5.10. *Some applications of this lemma: ISL 2006 G4, All-Russian 2009 10.7,*

Problem 5.11 (IMO 2013 Problem 4)**.** *Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.*

Proof. We fix the triangle ABC (which fix N, M, H) and consider W as a moving point on BC, which has degree 1. We first construct X and Y. As WX is the diameter of ω_1, we have $BW \perp BX$ and $NW \perp NX$. Therefore, X is the intersection of ℓ_1 (the perpendicular line to NW at N) with ℓ_2 the perpendicular line to BC at B. Note that ℓ_1 is obtained by rotating the line NW by $90°$ at center N (which is fixed), we have $deg(\ell_1)$ is at most $deg(N) + deg(NW) = 0 + 1$. As BC is fixed, ℓ_2 is a fixed line. Hence $X = \ell_1 \cap \ell_2$ has degree at most 1. Similarly, Y has degree at most 1. Finally, the collinearity of X, Y, H is a degree 2 statement, it suffices to show it for three positions of W.



- $W = B$ we have $X \in CN$ and $Y = C$, hence $XY = CN$ pass through H;
- $W = C$ similar to first case;
- $W = AH \cap BC$, then $H \in \odot BWN$, implying that $\angle XHW = 90°$. Similarly, $H \in \odot CWM$ and $\angle WHM = 90°$, hence H, X, Y are collinear.

□

The key here is to operates on the lines in order to construct the point X and Y, which are initially defined as the anti-pole of W in the corresponding circles. The rotation lemma is extremely useful when dealing with concyclic points. In this problem, we are in the sweet spot as the angle of rotation is $90°$, which is fixed. Let's try another harder problem.

Problem 5.12. *Let O be the circumcenter of $\triangle ABC$. Let Q be an arbitrary point, denote the second intersection of AQ, BQ and CQ with $\odot O$ as A_1, B_1 and C_1. Let P be an arbitrary point on the line OQ, denote the orthogonal projection from P to BC, CA, AB as A_2, B_2 and C_2. Prove that the circumcircles of $\triangle PA_1A_2$, $\triangle PB_1B_2$ and $\triangle PC_1C_2$ have two common points. (in other words coaxial)*

Proof. We first translate the coaxial property into a statement of collinearity. Denote P_A, P_B, P_C as the anti-pole of P with respect to the circumcircles of $\triangle PA_1A_2$, $\triangle PB_1B_2$ and $\triangle PC_1C_2$. Then these circles are coaxial iff their centers are collinear iff P_A, P_B and P_C are collinear. Note that $PA_2 \perp BC$, hence P_A lies on BC. Moreover $PA_1 \perp A_1P_A$, hence P_A is indeed the intersection of BC and the line through A_1 perpendicular to PA_1.

Now we are in shape to apply moving point method. We fix $\triangle ABC$ and Q, hence A_1, B_1, C_1 are fixed. Let P be a moving point on OQ which has degree 1.

The lines PA_1 has degree 1. Consider the rotation $\mathcal{R}_A$ of center A_1 and angle $90°$. Then by rotation lemma, as A_1 is fixed, $\mathcal{R}_A(PA_1)$ has the same degree as PA_1, which is degree 1. Therefore $P_A = BC \cap \mathcal{R}_A(PA_1)$ has degree 1. Similarly, P_B, P_C has degree 1. Finally, the statement P_A, P_B, P_C are collinear has degree 3. It suffices to find 4 special cases.

- Let P be one of the intersection $QO \cap \odot O$. Denote P' the anti-pole of P, then $P' = A_1P_A \cap B_1P_B \cap C_1P_C$. By Pascal's theorem to $BB_1P'C_1CA$, we have $Q = BB_1 \cap C_1C$, $P_B = B_1P' \cap CA$ and $P_C = P'C_1 \cap AB$ are collinear. Similarly, Q, P_A, P_B are collinear. Hence P_A, P_B, P_C are collinear. (2 cases)
- Let P such that $PA_1 \perp CA_1$, implying that $P_A = C$. Denote A', C' the anti-pole of A and C. By Pascal's theorem on $A_1AA'C_1CC'$, we have $Q = A_1A \cap C_1C$, $O = AA' \cap CC'$ and $A'C_1 \cap C'A_1$ are collinear. Note that $Q = OP \cap C'A_1$, we have Q is indeed $A'C_1 \cap C'A_1$. Therefore, $P_C = A$. As P_B lies on AC by definition, we have P_A, P_B, P_C are collinear. (3 cases)

□

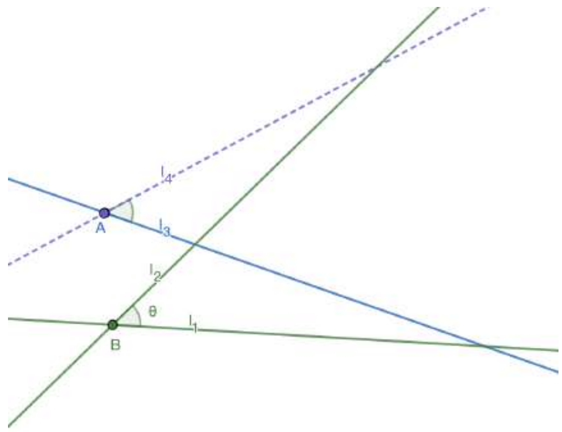
This is a rather difficult problem as the special cases are not completely trivial, the original problem is even harder to directly prove, see source. Some applications of this result includes Romania TST 6 2010, P2, AOPS problem. So far, the rotation we considered all have fixed angles (usually $90°$), but in some cases, the angle of rotation can be itself a variable. This would be useful when dealing with concyclic points. Let's see how we proceed in such case.

Lemma 5.13 (Rotation lemma (variable angle)). *Let $\ell_1 : [a_1 : b_1 : c_1], \ell_2 : [a_2 : b_2 : c_2]$ be two (Euclidean) lines. Given another finite point A and a line $\ell_3 : [a_3 : b_3 : c_3]$ passing through A. Consider the rotation $\mathcal{R}_\theta$ at center A of angle $\theta = \angle \ell_1\ell_2$, then the image of ℓ_3 has direction*

$$D_\theta = \begin{pmatrix} (a_1b_2 - a_2b_1)a_3 + (a_1a_2 + b_1b_2)b_3 \\ (a_1b_2 - a_2b_1)b_3 - (a_1a_2 + b_1b_2)a_3 \\ 0 \end{pmatrix}.$$

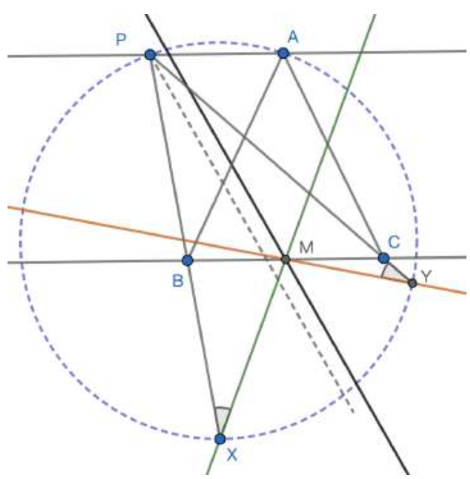
Therefore $\mathcal{R}_\theta(\ell_3) = AD_\theta$ has degree at most $deg(A) + deg(\ell_1) + deg(\ell_2) + deg(\ell_3)$.

Proof. It suffices to remark that $\cos\theta = \frac{a_1a_2+b_1b_2}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$ and $\sin\theta = \frac{a_1b_2-b_2a_1}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$. □



Corollary 5.14. *Given four lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, let $A = \ell_1 \cap \ell_2$, $B = \ell_3 \cap \ell_4$, $C = \ell_2 \cap \ell_4$, $D = \ell_1 \cap \ell_3$, then the statement A, B, C, D is concyclic has degree at most $deg(\ell_1) + deg(\ell_2) + deg(\ell_3) + deg(\ell_4)$.*

Problem 5.15 (IMO SL 2018 G2)**.** *Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.*



Proof. Fix P and $\triangle ABC$, and let X vary on PB, which has degree 1. Consider the line ℓ through M parallel to the angle bisector of $\angle BPC$, which is fixed.

Claim 1: Consider the reflexion ϕ w.r.t ℓ, then $\phi(MX) = MY$.
Proof: Let $Y' = \phi(MX) \cap AC$. Basic angle chasing shows that

$$\angle MY'P = \angle(\ell, MY) - \angle(\ell, PC) = \angle(MX, \ell) - \angle(PB, \ell) = \angle MXP.$$

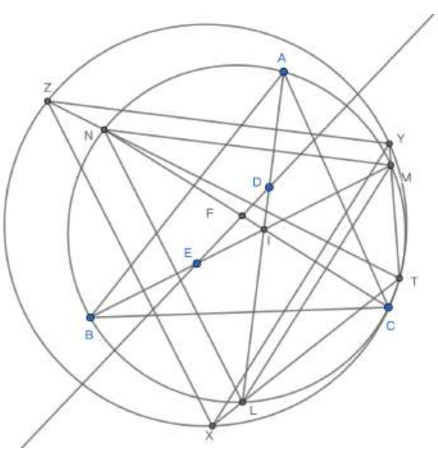
Therefore $Y' = Y$. $\square$

Hence MY has degree 1, also Y has degree 1. Consider $\ell_1 = PB$, $\ell_2 = PC$, $\ell_3 = YA$, $\ell_4 = AX$, then the concyclic of $\ell_1 \cap \ell_2 = P$, $\ell_2 \cap \ell_3 = Y$, $\ell_3 \cap \ell_4 = A$, $\ell_4 \cap \ell_1 = X$ has degree $0 + 0 + 1 + 1$ which has degree 2. It suffices to check 3 special cases.

- Let X be the foot of M on PB, then P, X, M, Y, A are concyclic.
- Let $X = AM \cap PB$, then $\angle MXC = \angle BXM = \angle MYC$, hence M, X, C, Y are concyclic, implying that $XY \perp PC$, hence P, X, Y, A are concyclic.
- Let X s.t. $Y = AM \cap PC$ works similarly as the second case.

$\square$

Problem 5.16 (IMO SL 2018 G5)**.** *Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors ℓ_A, ℓ_B, and ℓ_C of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.*



Proof. Let $X = \ell_B \cap \ell_C$, $Y = \ell_C \cap \ell_A$, $Z = \ell_A \cap \ell_B$. Let L, M, N be the second intersection of AI, BI, CI with $\odot(ABC)$. Then $LM \parallel XY$, $MN \parallel YZ$, $NL \parallel ZX$, thus LX, MY, NZ are concurrent at their homothety center T. It suffices to show that T lies on $\odot(ABC)$. We fixed A, B, C, D, let E moving on BI has degree 1, then $F = DE \cap CI$ has degree 1. ℓ_A is fixe. The midpoint of BE,CF has degree 1. As the directions of BI, CI are fixed, their orthogonal directions are fixed as well. Hence ℓ_B, ℓ_C has degree 1. And $Y = \ell_A \cap \ell_B$, $Z = \ell_C \cap \ell_A$ both have degree 1. Finally the intersection $T = MY \cap NZ$ has degree 1. Finally, the statement A, N, T, M are concyclic has degree $deg(AN) + deg(NT) + deg(TM) + deg(MA) = 0 + 1 + 1 + 0 = 2$. Therefore it suffices to check 3 special cases.

- Let $E = I$, then $F = I$, YM, ZN are the perpendicular bisector of BI and CI intersects at $L \in \odot(ABC)$.
- Let E such that $DE \parallel CI$, then $F = CI \cap \ell_\infty$. The midpoint of CF is still F. Hence $\ell_C = \ell_\infty$ and $Y = \ell_A \cap \ell_\infty$. This implies that YM is parallel to ℓ_A, which is orthogonal to AI, note that $MN \perp AI$ as well, therefore N lies on MY. This means $N = MY \cap NZ$ which lies on $\odot(ABC)$.
- Let $E = BI \cap \ell_\infty$ gives the same reasoning as the second bullet.

□

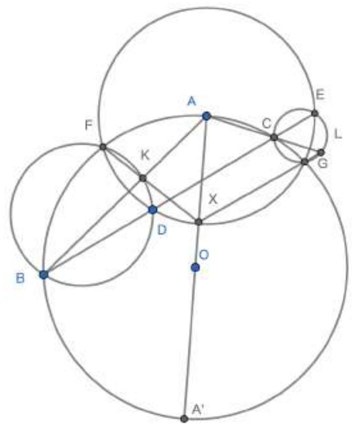
Problem 5.17 (IMO 2015 P4). *Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA. Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.*

Proof. We reformulate the problem as follows:

> Let F, G be the intersections of two circles $\odot O$, $\odot A$ with $A \in \odot O$. Let B be a point on $\odot O$ and D be a point on $\odot A$, let C and E be the second intersection of the line BD with $\odot O$ and $\odot A$ respectively. Let L be the second point of intersection of the circumcircle of triangle CGE and the line CA. Let $X = GL \cap AO$ and $K = FX \cap AB$, prove that B, D, F, K are concyclic.

We consider B as a fixed point and let D moves on $\odot A$, which has degree 2.

- First, we remark $\angle EGL = \angle ECL = \angle BCA$ is a fixed angle, hence GL is obtained by rotation of center GE and angle $\angle EGL$, we apply the rotation lemma with fixed angle, the degree of GL is at most $deg(G) + deg(GE) = 0 + 1 = 1$. ($E = BD \cap \odot A$ has degree 2 and GE has degree 1, moreover G,E belongs to the fixed circle $\odot A$, hence GE has degree $2/2 = 1$.)



- $X = EG \cap AO$ has degree 1, $K = FX \cap AB$ has degree 1.
- We remark that if D be one of the intersections of $AB \cap \odot A$, then $D = K$, hence applying the Coincidence Lemma 4.14 yields the line DK has degree at most $deg(D) + deg(K) - 2 = 2 + 1 - 2 = 1$.
- Finally, we apply corollary on KB, KD, FB, FD, the concyclic statement has degree $deg(KB) + deg(KD) + deg(FB) + deg(FD) = 0 + 1 + 0 + 1 = 2$.

So it suffices to check for 3 special cases of D.

- When $D = G$, we have $C = G$, $GL = CL = AC$, hence $X = A = K$ and $BDFK = BCFA \in \odot O$ are concyclic.
- When D is the second intersection of $BG \cap \odot A$. As $AF = AG$, we have $\angle ABF = \angle DBA$, implying that D is the reflexion of F w.r.t AB. Note that we have $C = E = G$, as E moves on $\odot A$, we have $EG = GG$ the tangent of $\odot A$ at G, which is indeed GA'. By construction, $\angle XGA' = \angle AGB$. Then as F and G are symmetric w.r.t AA', we have $\angle A'FX = \angle AGB = \angle AA'B = 90^\circ - \angle BAA' = 90^\circ - \angle BFA'$, meaning that $KF \perp FB$. By symmetry, $KD \perp BD$, hence B, D, F, K are concyclic.
- When $D = F$, we have $C = D = F$, the line FD is the tangent of $\odot A$ at F, which is indeed FA'. We need to prove that $\odot BFK$ is tangent to FA'. Note that the circle $\odot CEG = \odot FEG = \odot A$. Hence $L = FA \cap \odot A$ yields $LG \perp FG$ implying $LG \parallel AO$, implying that $X = AO \cap \ell_\infty$. Hence $FK \parallel AO$. Hence $\angle KFA' = \angle AA'F = \angle ABF$, implying that $A'F$ is tangent to $\odot BFK$.

□

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