

# Nuclear Geometry

## Homography, involutions, and animation

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## §1 The projective plane

### §1.1 Formal definition

The definitions presented in §1 are probably not satisfactory, so let's redo it, better this time. The idea is to “homogenize”  $\mathbb{R}^2$ , by adding a third, scalable coordinate.

#### Definition 1.1 (Real projective plane)

Formally, we define

$$\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{(0, 0, 0)\}) / \sim,$$

where  $(a, b, c) \sim (\lambda a, \lambda b, \lambda c)$  for all  $\lambda$ .

Basically what I'm saying is,  $(a, b, c)$  and  $(2a, 2b, 2c)$  are the same point; we write  $(a : b : c)$  for these homogeneous coordinates.

### §1.2 Algebraic detail

Why is this similar to  $\mathbb{R}^2$  at all? Consider the map  $\mathbb{RP}^2 \rightarrow \mathbb{R}^2$  by  $(x : y : z) \mapsto (x/z, y/z)$ . This map is injective, but only defined when  $z$  is nonzero. So  $\mathbb{RP}^2$  looks just like  $\mathbb{R}^2$ , except for some extra points of the form  $(x : y : 0)$ . Clearly these correspond to our “points at infinity,” and they lie on the line at infinity.

### §1.3 Geometric detail

So why does  $\mathbb{RP}^2$  have any notion of geometry? The key is that  $\mathbb{RP}^2$  can be thought of as the set of lines through the origin in  $\mathbb{R}^3$ . For example,  $(1 : 1 : 1)$  corresponds to all points satisfying  $x = y = z$ .

Then, we project all these lines onto the plane  $z = 1$ . This is the map  $(x : y : z) \mapsto (x/z, y/z)$ . Furthermore, rays with  $z = 0$ , i.e. parallel to the plane  $z = 1$ , intersect  $z = 1$  infinitely far away.

We can also project onto any other plane (not through the origin). Thus, the line at infinity isn't special at all! Any line can be the line at infinity by choosing a suitable plane, i.e. applying a suitable projective transformation.

## §1.4 Point-line duality

In  $\mathbb{R}^3$ , a plane through the origin has equation  $ax + by + cz = 0$ . It stands to reason this is also the equation of a line in  $\mathbb{RP}^2$ . In particular, lines may be expressed as homogeneous coordinates  $(a : b : c)$  as well!

With vectors, we might denote points as  $\mathbf{x} = (x : y : z)$  and lines as  $\ell = (a : b : c)$ . We have the following:

### Proposition 1.2 (Incidence)

The point  $\mathbf{x}$  lies on the line  $\ell$  if and only if  $\mathbf{x} \cdot \ell = 0$ .

The above basically means:  $\mathbf{x}$  lies on  $\ell$  if and only if the vectors  $\mathbf{x}$ ,  $\ell$  are orthogonal. In particular, the cross product  $u \times v$  gives the vector orthogonal to both  $u$ ,  $v$ , so

### Proposition 1.3 (Intersection)

Let  $\mathbf{x}$ ,  $\mathbf{y}$  be elements of  $\mathbb{RP}^2$ . Then

- $\mathbf{x} \times \mathbf{y}$  is the line through points  $\mathbf{x}$ ,  $\mathbf{y}$ ;
- $\mathbf{x} \times \mathbf{y}$  is the intersection of lines  $\mathbf{x}$ ,  $\mathbf{y}$ .

Note the obvious symmetry in points/lines. Hence the following thought ensues:

**If you can prove a purely projective statement about points, you can also prove it about lines, and vice versa.**

Here is an example of duality — recall this theorem:

### Theorem 1.4 (Desargues' theorem)

Let  $ABC$ ,  $XYZ$  be triangles. Then  $\overline{AX}$ ,  $\overline{BY}$ ,  $\overline{CZ}$  concur if and only if  $\overline{BC} \cap \overline{YZ}$ ,  $\overline{CA} \cap \overline{ZX}$ ,  $\overline{AB} \cap \overline{XY}$  are collinear.

I bet you can already smell the point-line duality in this one — the theorem itself is self-dual. To spell it out completely algebraically, consider the following two interpretations of the “only if” statement:

- Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{z}$  be elements of  $\mathbb{RP}^2$  referring to points  $A$ ,  $B$ ,  $C$ ,  $X$ ,  $Y$ ,  $Z$ . If

$$(\mathbf{a} \times \mathbf{x}) \cdot [(\mathbf{b} \times \mathbf{y}) \times (\mathbf{c} \times \mathbf{z})] = 0,$$

then we also have

$$((\mathbf{b} \times \mathbf{c}) \times (\mathbf{y} \times \mathbf{z})) \cdot [((\mathbf{c} \times \mathbf{a}) \times (\mathbf{z} \times \mathbf{x})) \times ((\mathbf{a} \times \mathbf{b}) \times (\mathbf{x} \times \mathbf{y}))] = 0.$$

- Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{z}$  be elements of  $\mathbb{RP}^2$  referring to lines  $BC$ ,  $CA$ ,  $AB$ ,  $YZ$ ,  $ZX$ ,  $XY$ . If

$$((\mathbf{b} \times \mathbf{c}) \times (\mathbf{y} \times \mathbf{z})) \cdot [((\mathbf{c} \times \mathbf{a}) \times (\mathbf{z} \times \mathbf{x})) \times ((\mathbf{a} \times \mathbf{b}) \times (\mathbf{x} \times \mathbf{y}))] = 0.$$

then we also have

$$(\mathbf{a} \times \mathbf{x}) \cdot [(\mathbf{b} \times \mathbf{y}) \times (\mathbf{c} \times \mathbf{z})] = 0,$$

Hence each direction of Desargues' theorem actually proves the other direction as well.

## §1.5 (Optional) Polarity in conics

**Remark.** See Vincent Huang's blog post for more details:

<https://artofproblemsolving.com/community/c2591h1740237>.

The following result from linear algebra usually helps when using projective coordinates. We'll be brief, so we won't actually use it here.

### Lemma 1.5 (Lagrange / triple product expansion)

For vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ , we have  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ .

Algebraic curves in  $\mathbb{RP}^2$  are homogeneous polynomial equations in  $x, y, z$ . For example, the unit circle is  $x^2 + y^2 = z^2$ . In particular, general conics  $\mathcal{C}$  are given by

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0.$$

Here we will consider the matrix

$$M = \begin{bmatrix} 2A & F & E \\ F & 2B & D \\ E & D & 2C \end{bmatrix}.$$

### Proposition 1.6

The conic  $\mathcal{C}$  is the set of points  $\mathbf{x}$  with  $\mathbf{x}^\top M \mathbf{x} = 0$ .

*Proof of Proposition A.6.* Expansion: it turns out

$$\mathbf{x}^\top M \mathbf{x} = 2(Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy).$$

□

### Lemma 1.7

$M$  is invertible if and only if  $\mathcal{C}$  isn't degenerate.

*Proof of Lemma A.7.* Note that for  $\mathbf{x} \in \mathcal{C}$ , the product  $M\mathbf{x}$  denotes the partial derivatives of  $\mathcal{C}$  at  $\mathbf{x}$ . If  $M$  isn't invertible, then  $M\mathbf{x} = 0$  for some  $\mathbf{x} \neq 0$ , so the partials at  $\mathbf{x}$  are all zero. If  $\mathcal{C}$  is nondegenerate, then at no point should the partials all be zero. □

### Proposition 1.8

For any point  $\mathbf{x}$ , the polar of  $\mathbf{x}$  wrt.  $\mathcal{C}$  is given by  $M\mathbf{x}$ .

First, this makes sense, since the pole of a line  $\mathbf{x}$  would then be given by  $M^{-1}\mathbf{x}$ . The pole is not well-defined when  $\mathcal{C}$  is degenerate — precisely when polarity is not well-defined either.

*Proof of Proposition A.8.* By the partials argument above,  $M\mathbf{x}$  is the tangent to  $\mathcal{C}$  at  $\mathbf{x}$  for any  $\mathbf{x} \in \mathcal{C}$ . Since multiplication by matrices represent projective transformations, this uniquely defines  $M$  as the pole-polar transformation.  $\square$

As an example of projective coordinates, we can prove La Hire's theorem:

### Theorem 1.9 (La Hire)

Let  $P, Q$  be points and  $\mathcal{C}$  a conic. Then  $P$  lies on the polar of  $Q$  if and only if  $Q$  lies on the polar of  $P$ .

*Proof.* We want to show  $\mathbf{p} \cdot M\mathbf{q} = 0$  if and only if  $\mathbf{q} \cdot M\mathbf{p} = 0$ . This is obvious by  $M = M^T$ .  $\square$

## Part I.

# Projective transformations

## §2 Introduction to homography

### §2.1 Addendum to cross ratios

We will define the cross ratio for conics:

#### Definition 2.1 (Cross ratio)

If  $A, B, X, Y, P$  lie on a conic  $\gamma$ , then

$$(AB; XY)_\gamma = \frac{\sin \angle XPA}{\sin \angle XPB} \bigg/ \frac{\sin \angle YPA}{\sin \angle YPB}.$$

In particular, one definition of “conic” is the locus of points for such the aforementioned ratio of sines is constant.

### §2.2 Purely projectiveness

A statement is **projective** if it can be expressed in terms of incidences (collinearities, concurrences, tangencies) and cross ratios.

In particular, any statement involving angles (e.g. perpendicularities, parallel lines), concyclicities, etc. is not projective. However, we can oftentimes “generalize” a problem to make it purely projective. Some common generalizations include

- replacing some circle with a general conic;
- replacing the line at infinity with a general line (and therefore parallel conditions with concurrency conditions).

## §2.3 Projective transformations

### Definition 2.2 (Homography, projective transformation)

A **homography** is a bijection on  $\mathbb{RP}^2$  preserving collinearities and cross ratio. In particular:

- four collinear points are sent to four collinear points with the same cross ratio;
- four concurrent lines are sent to four concurrent lines with the same cross ratio;
- for points on a conic  $\omega$  are sent to four points on the image of  $\omega$  with the same cross ratio.

**Remark.** It turns out that any bijective map preserving lines and conics also preserves cross ratio, i.e. is a projective transformation.

Here are some nice base conditions we can impose on projective transformations:

### Theorem 2.3 (Homographies that exist)

We can find a homography uniquely obeying each of the following:

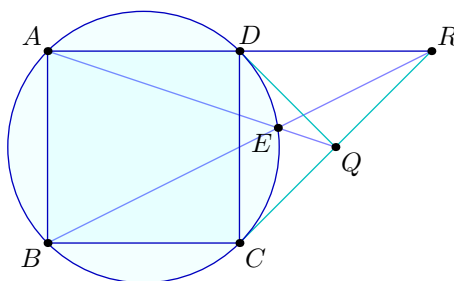
- For some four points  $A, B, C, D$  (no three collinear), we can send them to four other points  $W, X, Y, Z$  (no three collinear).
- For some circle  $\omega$  and interior points  $P, Q$ , we can send  $\omega$  to itself and send  $P$  to  $Q$ . (Note that  $Q$  is usually taken to be the center of  $\omega$ .)
- For some circle  $\omega$  and exterior line  $\ell$ , we can send  $\omega$  to itself and  $\ell$  to the line at infinity.

Here we demonstrate the power of homography:

### Example 2.4 (APMO 2013/5)

Let  $ABCD$  be a quadrilateral inscribed in a circle  $\omega$ , and let  $P$  be a point on the extension of  $\overline{AC}$  such that  $\overline{PB}$  and  $\overline{PD}$  are tangent to  $\omega$ . The tangent at  $C$  intersects  $\overline{PD}$  at  $Q$  and the line  $AD$  at  $R$ . Let  $E$  be the second point of intersection between  $\overline{AQ}$  and  $\omega$ . Prove that  $B, E, R$  are collinear.

**Walkthrough.** Hint:



## §3 Homography problems

**Problem 3.1.** Let  $ABCD$  be a quadrilateral. Define  $P = \overline{AD} \cap \overline{BC}$ ,  $Q = \overline{AB} \cap \overline{CD}$ ,  $R = \overline{AC} \cap \overline{BD}$ ,  $X_1 = \overline{PR} \cap \overline{AD}$ ,  $X_2 = \overline{PR} \cap \overline{BC}$ ,  $Y_1 = \overline{QR} \cap \overline{AB}$ ,  $Y_2 = \overline{QR} \cap \overline{CD}$ .

Prove that lines  $PQ$ ,  $X_1Y_1$ ,  $X_2Y_2$  are concurrent.

**Problem 3.2** (Butterfly theorem). Let  $M$  be the midpoint of a chord  $PQ$  of a circle, through which two other chords  $AB$  and  $CD$  are drawn. Chords  $AD$  and  $BC$  intersect chord  $PQ$  at  $X$  and  $Y$  respectively. Show that  $M$  is the midpoint of  $\overline{XY}$ .

**Problem 3.3** (APMO 2016/3). Let  $AB$  and  $AC$  be two distinct rays not lying on the same line, and let  $\omega$  be a circle with center  $O$  that is tangent to ray  $AC$  at  $E$  and ray  $AB$  at  $F$ . Let  $R$  be a point on segment  $EF$ . The line through  $O$  parallel to  $\overline{EF}$  intersects line  $AB$  at  $P$ . Let  $N$  be the intersection of lines  $PR$  and  $AC$ , and let  $M$  be the intersection of line  $AB$  and the line through  $R$  parallel to  $\overline{AC}$ . Prove that line  $MN$  is tangent to  $\omega$ .

## Part II.

# Desargue involution

## §4 Involution

### §4.1 Introduction to involution

#### Definition 4.1 (Involution)

Let  $\mathcal{P}$  be a line or conic. Then  $f : \mathcal{P} \rightarrow \mathcal{P}$  is an **involution** iff

- $f \circ f$  is the identity, and
- $f$  preserves cross ratio on  $\mathcal{P}$ .

For any point  $P \in \mathcal{P}$ , we say  $(P, f(P))$  is a **reciprocal pair**.

By the idea of “degrees of freedom,” we have the following:

#### Proposition 4.2

If  $f : \mathcal{P} \rightarrow \mathcal{P}$  preserves cross ratio and swaps some two points  $A, B$ , then  $f$  is an involution.

*Proof.* For  $X \in \mathcal{P}$ , let  $Y = f(X)$  and  $X' = f(Y)$ . Then

$$(AB; XY) \stackrel{f}{=} (BA; YX') = (AB; X'Y),$$

hence  $X = X'$ . □

### §4.2 Classifying involutions on a line

Some examples of involutions on a line:

- identity,
- reflection of a point,
- isogonal conjugation,
- inversion in general(!)

These should be easy to verify.

In fact “inversion” is the global descriptor: we have

**Proposition 4.3**

Any involution on a line  $\ell$  is an inversion of some nonzero power, possibly centered at infinity.

*Proof.* Let the involution swap  $O$  and infinity and have reciprocal pairs  $(X_1, X_2), (Y_1, Y_2)$ . Then

$$(O\infty; X_1Y_1) = (\infty O; X_2Y_2) \implies \frac{OX_1}{OY_1} = \frac{OY_2}{OX_2}.$$

Thus (in directed lengths),  $OX_1 \cdot OX_2 = OY_1 \cdot OY_2$ .  $\square$

**Remark.** If we impose a coordinate system  $x \in \mathbb{R} \cup \{\infty\}$  on  $\ell$ , then maps preserving cross ratio have the form  $x \mapsto \frac{ax+b}{cx+d}$ . Involutions are those satisfying  $a+d=0$ .

**§4.3 Classifying involutions on a conic****Proposition 4.4**

For any involution  $f$  on a conic  $\mathcal{C}$ , there is a point  $P$  such that  $f$  sends each point  $A$  to the second intersection of line  $PA$  and  $\mathcal{C}$ .

*Proof.* The statement is purely projective, so send  $\mathcal{C}$  to a circle. Consider reciprocal pairs  $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ , and take a point  $X \in \mathcal{C}$ . Invert about  $X$  sending  $\mathcal{C}$  to a line  $\ell$ . Then  $(A_1, A_2), (B_1, B_2), (C_1, C_2)$  are sent to reciprocal pairs of an involution  $f'$  on  $\ell$ .

By Proposition 4.3, for some  $O \in \ell$  we have  $OA'_1 \cdot OA'_2 = OB'_1 \cdot OB'_2 = OC'_1 \cdot OC'_2$ , so the circles  $(XA'_1A'_2), (XB'_1B'_2), (XC'_1C'_2)$  are coaxial. Then  $\overline{A_1A_2}, \overline{B_1B_2}, \overline{C_1C_2}$  concur, as needed.  $\square$

Note that we can project involutions from lines and conics. This can be helpful in both directions: to prove concurrencies or to utilize concurrencies.

**§5 Desargues involution****§5.1 Desargues' involution theorem (DIT)****Theorem 5.1 (DIT)**

Let  $ABCD$  be a quadrilateral inscribed in a conic  $\mathcal{C}$ . A line  $\ell$  intersects lines  $AB, CD, AD, BC, AC, BD$  at points  $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ ;  $\ell$  intersects  $\mathcal{C}$  at  $W_1, W_2$ . Then  $(W_1, W_2), (X_1, X_2), (Y_1, Y_2), (Z_1, Z_2)$  are pairs of some involution on  $\ell$ .

*Proof.* Take a homography  $f$  fixing  $\ell$  and sending  $W_1, W_2, X_1$  to  $W_2, W_1, X_2$ . Then  $f$  is an involution by Proposition 4.2, so  $(X_1, X_2)$  is a reciprocal pair. It suffices to show (via symmetry) that  $(Y_1, Y_2)$  is a reciprocal pair.

But

$$(X_1Y_1; W_1W_2) \stackrel{A}{=} (BD; W_1W_2) \stackrel{C}{=} (Y_2X_2; W_1W_2) = (X_2Y_2; W_2W_1),$$

so  $f(Y_1) = Y_2$  as desired.  $\square$

Sometimes the degenerate cases are useful:



**Corollary 5.2** (2 points DIT)

For points  $A, B$  on a conic  $\mathcal{C}$ , a line  $\ell$  intersects line  $AB$  at  $X$  and the tangents to  $\mathcal{C}$  at  $A, B$  at  $Y_1, Y_2$ ;  $\ell$  intersects  $\mathcal{C}$  at  $W_1, W_2$ . Then  $(W_1, W_2), (X, X), (Y_1, Y_2)$  are reciprocal pairs of some involution on  $\ell$ .

**Corollary 5.3** (3 points DIT)

Let  $ABC$  be a triangle inscribed in a conic  $\mathcal{C}$ . A line  $\ell$  intersects lines  $AB, AC, BC$  at  $X_1, X_2, Y_1$  and the tangent to  $\mathcal{C}$  at  $A$  at  $Y_2$ ;  $\ell$  intersects  $\mathcal{C}$  at  $W_1, W_2$ . Then  $(W_1, W_2), (X_1, X_2), (Y_1, Y_2)$  are reciprocal pairs of some involution on  $\ell$ .

**§5.2 Taking the dual of involutions****Definition 5.4**

Let  $P$  be a point and  $\mathcal{L}$  the set of lines containing  $P$ . Then  $f : \mathcal{L} \rightarrow \mathcal{L}$  is an involution on  $\mathcal{L}$  if

- $f \circ f$  is the identity, and
- $f$  preserves cross ratio on  $\mathcal{L}$ .

Having an involution on a pencil, we can project onto and from a line and an involution on that line. Evidently this is the main use of the following jackpot theorem:

**§5.3 Dual of Desargues' involution theorem (DDIT)**

The real magic:

**Theorem 5.5** (DDIT)

Let  $P$  be a point and  $ABCD$  a quadrilateral with inconic  $\mathcal{C}$ . Set  $E = \overline{AB} \cap \overline{CD}$  and  $F = \overline{AD} \cap \overline{BC}$ . Then if  $\overline{PX}, \overline{PY}$  are the tangents from  $P$  to  $\mathcal{C}$ , then  $(\overline{PX}, \overline{PY}), (\overline{PA}, \overline{PC}), (\overline{PB}, \overline{PD}), (\overline{PE}, \overline{PF})$  are reciprocal pairs of some involution on the pencil of lines through  $P$ .

And the degenerate cases:

**Corollary 5.6** (2 points DDIT)

Let  $A, B$  lie on a conic  $\mathcal{C}$  and let  $P$  be a point. If the tangents to  $\mathcal{C}$  at  $A, B$  intersect at  $X$  and  $\overline{PX}, \overline{PY}$  are the tangents from  $P$  to  $\mathcal{C}$ , then  $(\overline{PY}, \overline{PZ}), (\overline{PX}, \overline{PX}), (\overline{PA}, \overline{PB})$  are reciprocal pairs of some involution on the pencil of lines through  $P$ .

**Corollary 5.7** (3 points DDIT)

Let  $ABC$  be a triangle with inconic  $\mathcal{C}$  and  $P$  be a point in a plane. Let  $\mathcal{C}$  be tangent to  $\overline{BC}$  at  $D$ , and let  $\overline{PX}, \overline{PY}$  be the tangents from  $P$  to  $\mathcal{C}$ . Then  $(\overline{PX}, \overline{PY}), (\overline{PA}, \overline{PD}), (\overline{PB}, \overline{PC})$  are reciprocal pairs of some involution on the pencil of lines through  $P$ .

Demonstration:

**Example 5.8** (USAMO 2012/5)

Let  $P$  be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line passing through  $P$ . Let  $A', B', C'$  be the points where the reflections of lines  $PA, PB, PC$  with respect to  $\gamma$  intersect lines  $BC, AC, AB$  respectively. Prove that  $A', B', C'$  are collinear.

**Walkthrough.** Define  $C'_1 = \overline{AB} \cap \overline{A_1B_1}$ . Show there is an involution swapping  $(\overline{PA}, \overline{PA_1}), (\overline{PB}, \overline{PB_1}), (\overline{PC}, \overline{PC'_1})$ ; then show this involution is reflection over  $\gamma$ .

## §6 Desargue involution problems

**Problem 6.1** (China TST 2017/2/3). Let  $ABCD$  be a quadrilateral and  $\ell$  a line. Let  $\ell$  intersect lines  $AB, CD, BC, DA, AC, BD$  at points  $X, X', Y, Y', Z, Z'$ , respectively. Given that these six points on  $\ell$  are in the order  $X, Y, Z, X', Y', Z'$ , show that the circles with diameter  $\overline{XX'}, \overline{YY'}, \overline{ZZ'}$  are coaxial (i.e. share a radical axis).

**Problem 6.2** (Isogonality lemma). Let  $ABC$  be a triangle with two interior points  $P, Q$ . Suppose  $\overline{AP}, \overline{AQ}$  are isogonal wrt.  $\angle A$ . Let  $X = \overline{PB} \cap \overline{QC}$  and  $Y = \overline{PC} \cap \overline{QB}$ . Show that  $\overline{AX}, \overline{AY}$  are isogonal wrt.  $\angle A$ .

**Problem 6.3** (ISL 2007 G3). The diagonals of a trapezoid  $ABCD$  intersect at point  $P$ . Point  $Q$  lies between the parallel lines  $BC$  and  $AD$  such that  $\angle AQD = \angle CQB$ , and line  $CD$  separates points  $P$  and  $Q$ . Prove that  $\angle BQP = \angle DAQ$ .

**Problem 6.4** (CGMO 2017/7). Let  $ABCD$  be a cyclic quadrilateral with circumcircle  $\omega_1$ . Let  $E = \overline{AC} \cap \overline{BD}$  and  $F = \overline{AD} \cap \overline{BC}$ . Circle  $\omega_2$  is tangent to segments  $EB, EC$  at  $M, N$ , respectively and intersects  $\omega_1$  at points  $Q, R$ . Lines  $BC, AD$  intersect line  $MN$  at  $S, T$ , respectively. Show that  $Q, R, S, T$  are concyclic.

**Problem 6.5.** Let  $ABC$  and  $DEF$  be two triangles sharing an incircle  $\omega$  and a circumcircle  $\Omega$ . Let  $L$  be the tangency point between  $\overline{EF}$  and  $\omega$ , and let  $K$  be the tangency point between  $\overline{BC}$  and  $\omega$ . Let  $N = \overline{AL} \cap \Omega$  and  $M = \overline{DK} \cap \Omega$ . Show that lines  $AM, EF, BC, ND$  concur.

**Problem 6.6** (IMO 2019/2). In triangle  $ABC$ , point  $A_1$  lies on side  $BC$  and point  $B_1$  lies on side  $AC$ . Let  $P$  and  $Q$  be points on segments  $AA_1$  and  $BB_1$ , respectively, such that  $\overline{PQ}$  is parallel to  $\overline{AB}$ . Let  $P_1$  be a point on line  $PB_1$ , such that  $B_1$  lies strictly between  $P$  and  $P_1$ , and  $\angle PP_1C = \angle BAC$ . Similarly, let  $Q_1$  be a point on line  $QA_1$ , such that  $A_1$  lies strictly between  $Q$  and  $Q_1$ , and  $\angle CQ_1Q = \angle CBA$ .

Prove that points  $P, Q, P_1$ , and  $Q_1$  are concyclic.

**Problem 6.7** (USA TST 2018/5). Let  $ABCD$  be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at  $H$ . Denote by  $M$  and  $N$  the midpoints of  $\overline{BC}$  and  $\overline{CD}$ . Rays  $MH$  and  $NH$  meet  $\overline{AD}$  and  $\overline{AB}$  at  $S$  and  $T$ , respectively. Prove that there exists a point  $E$ , lying outside quadrilateral  $ABCD$ , such that

- ray  $EH$  bisects both angles  $\angle BES, \angle TED$ , and
- $\angle BEN = \angle MED$ .

**Problem 6.8** (ELMO SL 2018 G4). Let  $ABCDEF$  be a hexagon inscribed in a circle  $\Omega$  such that triangles  $ACE$  and  $BDF$  have the same orthocenter. Suppose that segments  $BD$  and  $DF$  intersect  $\overline{CE}$  at  $X$  and  $Y$ , respectively. Show that there is a point common to  $\Omega$ , the circumcircle of  $\triangle DXY$ , and the line through  $A$  perpendicular to  $\overline{CE}$ .

**Problem 6.9** (Serbia 2017/6). Let  $ABC$  be a triangle and let the common external tangents to the circumcircle and the  $A$ -excircle intersect line  $BC$  at  $P$  and  $Q$ . Show that  $\angle PAB = \angle CAQ$ .

**Problem 6.10** (Taiwan TST 2014/3/3). Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and let  $M$  be an arbitrary point on  $\Gamma$ . Suppose that the tangents from  $M$  to the incircle of  $ABC$  intersect  $\overline{BC}$  at two distinct points  $X_1$  and  $X_2$ . Prove that the circumcircle of triangle  $MX_1X_2$  passes through the tangency point of the  $A$ -mixtilinear incircle with  $\Gamma$ .

**Problem 6.11** (USMCA 2020/7). Let  $ABCD$  be a convex quadrilateral, and let  $\omega_A$  and  $\omega_B$  be the incircles of  $\triangle ACD$  and  $\triangle BCD$ , with centers  $I$  and  $J$ . The second common external tangent to  $\omega_A$  and  $\omega_B$  touches  $\omega_A$  at  $K$  and  $\omega_B$  at  $L$ . Prove that lines  $AK$ ,  $BL$ ,  $IJ$  are concurrent.

**Problem 6.12** (ISL 2012 G8). Let  $ABC$  be a triangle with circumcircle  $\omega$  and  $\ell$  a line without common points with  $\omega$ . Denote by  $P$  the foot of the perpendicular from the center of  $\omega$  to  $\ell$ . The lines  $BC$ ,  $CA$ ,  $AB$  intersect  $\ell$  at the points  $X$ ,  $Y$ ,  $Z$  different from  $P$ . Prove that the circumcircles of the triangles  $AXP$ ,  $BYP$ , and  $CZP$  have a common point different from  $P$  or are mutually tangent at  $P$ .

## Part III.

# Moving points

## §7 Projective maps

### §7.1 The core idea

**Problem 7.1** (ISL 2016 G5). Let  $D$  be the foot of perpendicular from  $A$  to the Euler line (the line passing through the circumcenter and the orthocenter) of an acute scalene triangle  $ABC$ . A circle  $\omega$  with center  $S$  passes through  $A$  and  $D$ , and it intersects sides  $AB$  and  $AC$  at  $X$  and  $Y$  respectively. Let  $P$  be the foot of altitude from  $A$  to  $\overline{BC}$ , and let  $M$  be the midpoint of  $\overline{BC}$ . Prove that the circumcenter of triangle  $XS Y$  is equidistant from  $P$  and  $M$ .

**Walkthrough.** Verify the problem when  $S$  is the midpoint of  $\overline{AH}$  and  $\overline{AO}$ . Then move  $S$  linearly, and apply spiral similarity at  $D$  to show  $X$ ,  $Y$  move linearly along  $\overline{AB}$ ,  $\overline{AC}$ . Since  $\triangle XLY$  has fixed shape ( $L$  being circumcenter of  $\triangle XS Y$ ),  $L$  moves along a line. This line is the perpendicular bisector of  $\overline{PM}$ .

### §7.2 Definitions and backbone

First a simple definition:

#### Definition 7.2 (Projective map)

Let  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  be conics, lines, or pencil of lines. Then a projective map  $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  preserves cross ratio.

To revisit the idea of degree-counting, we have the following:

#### Theorem 7.3

If  $f, g : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  are projective and  $f, g$  coincide at at least three values, then  $f \equiv g$ .

*Proof.* This is very much the fundamental theorem of algebra: if  $f(A) = g(A)$ ,  $f(B) = g(B)$ ,  $f(C) = g(C)$  then for any point  $D \in \mathcal{C}_1$ ,

$$(f(A)f(B); f(C)f(D)) = (AB; CD) = (g(A)g(B); g(C)g(D)) = (f(A)f(B); f(C)g(D)).$$

Therefore  $f(D) = g(D)$ . □

Thus if we can phrase problems in terms of projective maps, **it suffices to verify the problem for just three cases!**

Here are some examples of projective maps:

- For line  $\ell$  and point  $P$ , the map from  $\ell$  to  $\mathcal{C}_P$  (the pencil of lines through  $P$ ) by  $X \mapsto PX$ .
- For conic  $\gamma$  and point  $P \in \gamma$ , the map from  $\gamma$  to  $\mathcal{C}_P$  by  $X \mapsto PX$ .
- For conic  $\gamma$  and point  $P$ , the map from  $\gamma$  to  $\gamma$  by  $X \mapsto (\overline{PX} \cap \gamma)$ .
- For clines  $\gamma_1, \gamma_2$ , any inversion swapping  $\gamma_1, \gamma_2$  is projective.

Note that projection from a line to another line is the first bullet point applied twice; similar arguments hold for the second bullet point wrt. conics.

### §7.3 Example

#### Example 7.4 (IMO 2010/2)

Given a triangle  $ABC$ , with  $I$  as its incenter and  $\Gamma$  as its circumcircle, line  $AI$  intersects  $\Gamma$  again at  $D$ . Let  $E$  be a point on the arc  $BDC$ , and  $F$  a point on the segment  $BC$ , such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

If  $G$  is the midpoint of  $\overline{IF}$ , prove that lines  $EI$  and  $DG$  intersect on  $\Gamma$ .

**Walkthrough.** Animate  $E$ , let  $E'$  be reflection of  $E$  in  $\overline{AI}$ . Show the map  $E \mapsto E' \mapsto F \mapsto G$  is projective.

## §8 Tethered moving points problems

**Problem 8.1** (EGMO TST 2020/2). Let  $ABC$  be a triangle and let  $P$  be a point not lying on any of the sidelines of the triangle. Distinct points  $D, E, F$  lie on lines  $BC, CA, AB$ , respectively, such that  $\overline{DE} \parallel \overline{CP}$  and  $\overline{DF} \parallel \overline{BP}$ . Show that there exists a point  $Q$  on the circumcircle of  $\triangle AEF$  such that  $\triangle BAQ$  is similar to  $\triangle PAC$ .

**Problem 8.2** (USA TST 2019/1). Let  $ABC$  be a triangle and let  $M$  and  $N$  denote the midpoints of  $\overline{AB}$  and  $\overline{AC}$ , respectively. Let  $X$  be a point such that  $\overline{AX}$  is tangent to the circumcircle of triangle  $ABC$ . Denote by  $\omega_B$  the circle through  $M$  and  $B$  tangent to  $\overline{MX}$ , and by  $\omega_C$  the circle through  $N$  and  $C$  tangent to  $\overline{NX}$ . Show that  $\omega_B$  and  $\omega_C$  intersect on line  $BC$ .

**Problem 8.3** (Iran TST 2011/6a). Let  $BCB'C'$  be a rectangle inscribed in the circle  $\omega$  with center  $O$ . Points  $K$  and  $H$  lie on the tangents to  $\omega$  at  $B$  and  $C$ , respectively, and points  $K'$  and  $H'$  lie on the angle bisectors of  $\angle BCO$  and  $\angle CBO$ , respectively, such that  $\overline{KK'}$  and  $\overline{HH'}$  are perpendicular to  $\overline{BC}$ . Prove that  $K, H', B'$  are collinear if and only if  $H, K', C'$  are collinear.

## §9 Introduction to untethered moving points

### §9.1 General maps and degree

#### Definition 9.1 (Moving point)

A moving point is a map

$$\mathbb{R} \cup \{\infty\} \rightarrow \mathbb{RP}^2 \quad \text{by} \quad t \rightarrow (P(t) : Q(t) : R(t))$$

where  $P, Q, R$  are polynomials with no common root, and the image of  $t = \infty$  is defined by limits / continuity in  $\mathbb{RP}^2$ .

#### Definition 9.2 (Moving line)

A moving line is a map

$$\mathbb{R} \cup \{\infty\} \rightarrow \mathbb{RP}^2 \quad \text{by} \quad t \rightarrow (P(t) : Q(t) : R(t)).$$

This time, however, each  $(P(t) : Q(t) : R(t))$  refers to the line  $P(t)x + Q(t)y + R(t)z = 0$ . (See §1.4.)

#### Definition 9.3 (Degree)

The degree of a moving point or line  $(P(t) : Q(t) : R(t))$  is  $\max\{\deg P, \deg Q, \deg R\}$ .

Examples:

- Fixed points have degree 0.
- Points moving linearly have degree 1.
- Points moving projectively have degree either 1 or 2, depending on whether they move along a line or a conic.

### §9.2 Important: battle plan

Most proofs using moving points take on the following form:

- Animate some point along a line. This “parameter” has degree 1.
- Compute the degrees of all (or a good number of) points.
- Bound the degree  $d$  of a polynomial corresponding to our desired condition.
- Find  $d + 1$  special cases for our “parameter” for which the problem is true and easier to show. (Infinity is typically a valid choice.)
- Conclude via fundamental theorem of algebra that the polynomial condition is identically zero.

**Remark.** Note that computing the degrees of points can be especially tricky. If you make a single mistake here, your entire proof will fall apart and you won’t suspect a thing.

We have yet to explain how to compute the degrees of the points in a problem, and how to compute the degree of a condition. We will do so now:

## §10 Computing degrees

### §10.1 Relating degrees

#### Lemma 10.1 (Zack's lemma)

For two moving points  $A, B$  coinciding at  $k$  points (counting multiplicity), the degree of line  $AB$  is at most  $\deg A + \deg B - k$ .

#### Lemma 10.2 (Zack's lemma for lines)

For two moving lines  $\ell_1, \ell_2$  coinciding at  $k$  points (counting multiplicity), the degree of  $\ell_1 \cap \ell_2$  is at most  $\deg \ell_1 + \deg \ell_2 - k$ .

*Proof of both.* Let

$$\begin{aligned} A &= (P_1(t) : Q_1(t) : R_1(t)) \\ B &= (P_2(t) : Q_2(t) : R_2(t)), \end{aligned}$$

where  $A, B$  are either moving points or lines.

Then line  $AB$  or the intersection of lines  $A, B$  is given by

$$A \times B = (Q_1(t)R_2(t) - Q_2(t)R_1(t) : \cdots : \cdots).$$

If  $A = B$  for  $t = t_1, \dots, t_k$  then  $t - t_i$  divides each term of  $A \times B$ , so

$$A \times B = \left( \frac{Q_1(t)R_2(t) - Q_2(t)R_1(t)}{\prod(t - t_i)} : \cdots : \cdots \right).$$

□

Note the  $-k$  term is only there if our end degree is too large and we need to make a few optimizations here and there. Usually we take Zack's lemma in the following form:

- For moving points  $A, B$ , the line  $AB$  has degree at most  $\deg A + \deg B$ ;
- For moving lines  $\ell_1, \ell_2$ , the intersection  $\ell_1 \cap \ell_2$  has degree at most  $\deg \ell_1 + \deg \ell_2$ .

**Warning:** If you use some special case to decrease the degree in Zack's lemma, you cannot use it again as one of the  $d + 1$  special cases.

### §10.2 Conic doubling

#### Corollary 10.3

If a moving point  $A$  with  $\deg A = 2$  moves along a conic, it moves projectively; for any fixed  $X$  on the same conic, line  $XA$  has degree 1.

In fact we can generalize this:

**Proposition 10.4** (Conic doubling)

If  $\mathcal{C}_1, \mathcal{C}_2$  are lines or conics and  $\phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  is a projective map, then for a moving point  $P \in \mathcal{C}_1$  with degree  $d$ ,

$$\deg(\phi(P)) = \begin{cases} \deg P & \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are both lines or both conics;} \\ 2 \deg P & \mathcal{C}_1 \text{ is a line and } \mathcal{C}_2 \text{ is a conic;} \\ \frac{1}{2} \deg P & \mathcal{C}_1 \text{ is a conic and } \mathcal{C}_2 \text{ is a line.} \end{cases}$$

The proof requires work in  $\mathbb{C}$  that we will omit. (However the result still holds in  $\mathbb{R}$ .) We can generalize this even further:

**Proposition 10.5**

If  $A$  has degree  $n$ , it moves along a plane curve of degree  $k \mid n$ , and the map from  $t$  to the curve extends to a polynomial map on  $\mathbb{C}$  which is surjective and  $(n/k)$ -to-1.

**§10.3 Degree of statements**

Now we need to compute the degree of statements. To contrast with our previous use of the word “degree,” we mean the degree of the polynomial that verifies the truth of some statement; i.e. we want to show  $P \equiv 0$ .

**Theorem 10.6** (Collinearity)

If moving points  $P, Q, R$  have degrees  $a, b, c$ , then the statement “ $P, Q, R$  collinear” has degree  $a + b + c$ .

**Theorem 10.7** (Concurrency)

If moving lines  $\ell_1, \ell_2, \ell_3$  have degrees  $a, b, c$ , then the statement “ $\ell_1, \ell_2, \ell_3$  collinear” has degree  $a + b + c$ .

*Proof of both.* If  $P = (P_1(t) : P_2(t) : P_3(t))$ , etc. then the condition is equivalent to

$$\det \begin{pmatrix} P_1(t) & P_2(t) & P_3(t) \\ Q_1(t) & Q_2(t) & Q_3(t) \\ R_1(t) & R_2(t) & R_3(t) \end{pmatrix} = 0.$$

This is a polynomial of degree at most  $a + b + c$ . □

This is really all we need.

**§10.4 Examples****Example 10.8** (USA TST 2015/6)

Let  $ABC$  be a non-equilateral triangle and let  $M_a, M_b, M_c$  be the midpoints of sides  $BC, CA, AB$ , respectively. Let  $S$  be a point lying on the Euler line. Denote by  $X, Y, Z$  the second intersections of lines  $M_aS, M_bS, M_cS$  with the nine-point circle. Prove that lines  $AX, BY, CZ$  are concurrent.

**Walkthrough.** Animate  $S$ , and show the concurrence has degree 6. Then find seven values of  $S$ . (This is very easy by triangle symmetry.)

### Example 10.9

Let  $ABCD$  be a cyclic quadrilateral with circumcenter  $O$ . Let lines  $AB$  and  $CD$  meet at  $E$  and lines  $AC$  and  $BD$  meet at  $P$ . Furthermore, let lines  $EP$  and  $AD$  meet at  $K$ , and let  $M$  be the midpoint of  $\overline{AD}$ . Prove that  $BCM K$  is cyclic.

**Walkthrough.** Work in  $\mathbb{CP}^2$ . Animate  $B$ , and apply Pascal on hexagon  $IJBKMC$ .

**Remark.** In  $\mathbb{CP}^2$ , there are two “circle points”  $I, J = (1 : \pm i : 0)$  that lie on every circle.

Whenever necessary, we can work with moving points  $\mathbb{C} \cup \{\infty\} \rightarrow \mathbb{CP}^2$ . This is particularly useful to show concyclicities: to show  $A, B, C, D$  are concyclic, show  $A, B, C, D, I, J$  lie on the same conic using Pascal.

We will spend the majority of our work here in  $\mathbb{RP}^2$ ; notwithstanding, the results readily generalize.

## §11 Untethered moving points problems

**Problem 11.1** (Kariya’s theorem). Let  $ABC$  be a triangle with incenter  $I$  and intouch triangle  $DEF$ . Select  $X, Y, Z$  on rays  $ID, IE, IF$  such that  $IX = IY = IZ$ . Show that lines  $AX, BY, CZ$  concur.

**Problem 11.2** (Kiepert’s theorem). Let  $ABC$  be a triangle and select points  $X, Y, Z$  such that  $\triangle BXC, \triangle AYC, \triangle AZB$  are similar triangles with  $BX = XC, CY = YA, AZ = ZB$ . Show that lines  $AX, BY, CZ$  concur on a fixed conic through  $A, B, C$ .

**Problem 11.3** (USA TST 2020/2). Two circles  $\Gamma_1$  and  $\Gamma_2$  have common external tangents  $\ell_1$  and  $\ell_2$  meeting at  $T$ . Suppose  $\ell_1$  touches  $\Gamma_1$  at  $A$  and  $\ell_2$  touches  $\Gamma_2$  at  $B$ . A circle  $\Omega$  through  $A$  and  $B$  intersects  $\Gamma_1$  again at  $C$  and  $\Gamma_2$  again at  $D$ , such that quadrilateral  $ABCD$  is convex.

Suppose lines  $AC$  and  $BD$  meet at point  $X$ , while lines  $AD$  and  $BC$  meet at point  $Y$ . Show that  $T, X, Y$  are collinear.

**Problem 11.4** (TSTST 2019/5). Let  $ABC$  be an acute triangle with orthocenter  $H$  and circumcircle  $\Gamma$ . A line through  $H$  intersects segments  $AB$  and  $AC$  at  $E$  and  $F$ , respectively. Let  $K$  be the circumcenter of  $\triangle AEF$ , and suppose line  $AK$  intersects  $\Gamma$  at a point  $D$ . Prove that line  $HK$  and the line through  $D$  perpendicular to  $\overline{BC}$  meet on  $\Gamma$ .