Olympiad Training for Individual Study

Global Ideas and Probabilistic Method

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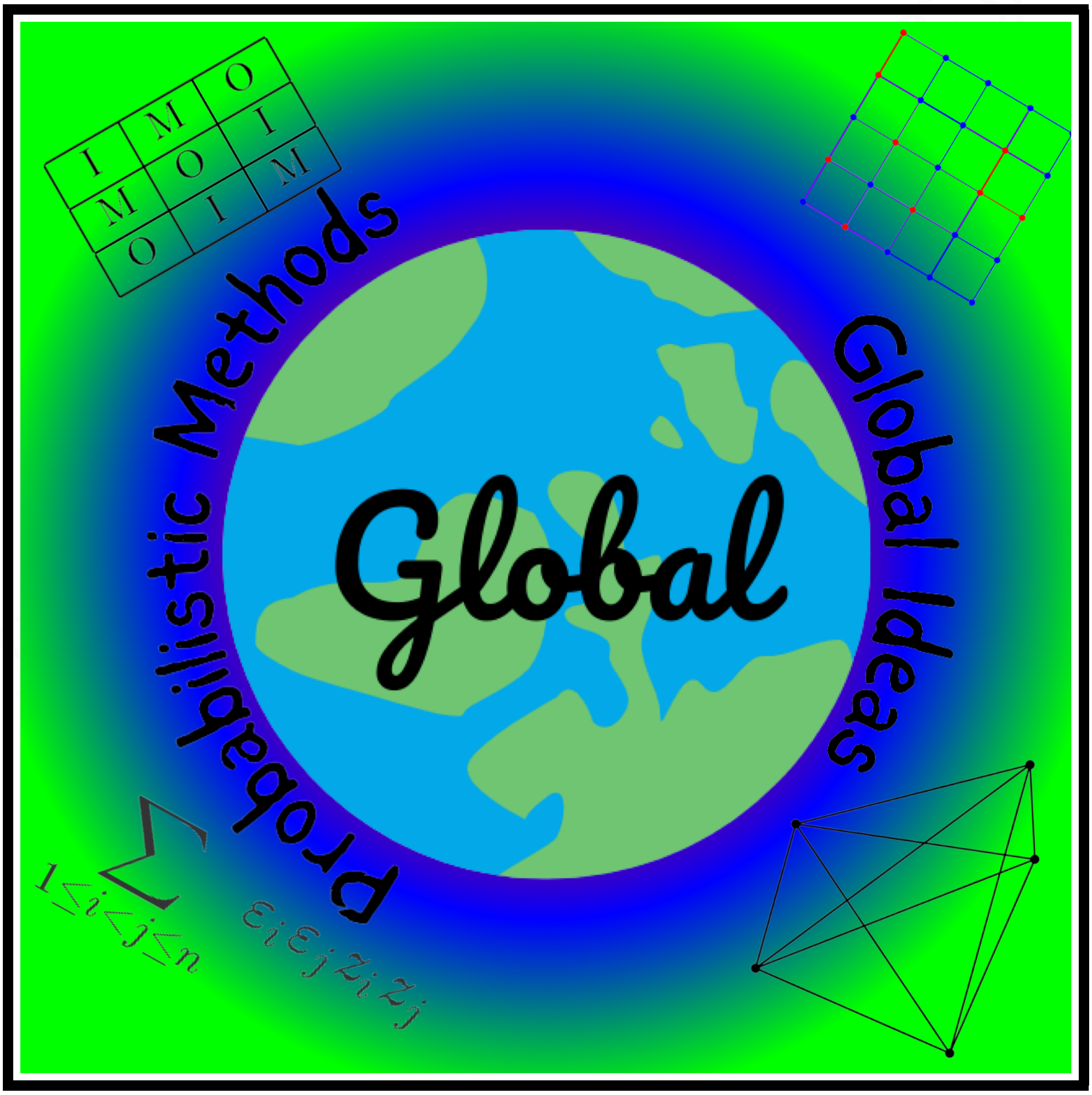

§1 Reading

- If you don't know what a graph is yet, now is a good time to learn the definitions in §6 of the *OTIS Excerpts*. Since you are just learning some terminology, you shouldn't spend more than 15 minutes on this.
- Read §7.3 of the *OTIS Excerpts*. Alternatively, you can use §3 of *Expected Uses of Probability* from my website.

Suggested exercises:

Example 1.1 (BAMO 2004/4)
Consider n real numbers with sum zero, but not all zero. Prove that it is possible to label the numbers $a_1, \dots, a_n$ in some order such that

$$a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_na_1 < 0.$$

§2 Lecture notes

This lecture is also called the following names:

- Pigeonhole
- Linearity of expectation / Probabilistic method
- Double counting
- Count in two ways
- Coloring

The similarity in all of these is that you have some sort of structure, and you can count something in two natural ways:

$$\sum_{a\in A}\sum_{b\in B} = \sum_{b\in B}\sum_{a\in A}.$$

This will give you some information.

In other words, the basic premise is that you sum over some entire space of things (often interchanging the order of summation) and use the result to conclude something.

- As a really stupid example, suppose the integers from 1 to 2022 are partitioned into 1011 pairs, and you want to show at least one pair has odd sum. But the total sum $1 + 2 + \cdots + 2022$ is odd, implying the result. The point is that you can say something about the sum of all pairs and use it to extract something that's hard to access from one individual pair.

  (Of course, this example is really stupid because you can also just count the total number of even and odd numbers.)
- Another famous example: the "handshake lemma" says that the sum of degrees in any finite simple graph is even. It follows because $\sum_v \deg v = \sum_e 2$.

- Indeed, linearity of expectation is actually proved this way. Thus, using linearity of expectation sort of hides a boilerplate "swapping the order of summation" argument.

  In particular, arguments of the form "show the expected value of X is at least r by linearity, hence some point achieves at least r" is often isomorphic to some more elaborate argument of the form "do a big sum and then swap the order of summation to compute the average, then use pigeonhole to get something at least the average".[1] An example of this connection is given in the Canada 2009 example, so make sure you understand that one and ask if you have any questions.

- In general for double-counting phrasing, one standard trope is double count ordered pairs of the form (X, Y) where Y "includes" X; the context depends on the problem. Examples of this include things like

  – (cell c, row/column containing c)

  – (element x, set S containing x).

  Similarly, if the problem has a bunch of objects and sets and asks for something about pairs of objects or pairs of sets, you might end up counting triples like

  – (set A, set B, element x in $A \cap B$)

  – (element x, element y, set S containing x and y).

  This will make more sense once you do a few of the walkthroughs.

- You can often visualize setups described in the previous bullet as an **incidence matrix**, where you make a table of 0 and 1's. For example, if you have a problem about people and committees, you can draw a matrix where the rows are people and the columns are committees, and put a 1 if a person is in a committee. Drawings like this can make it easier for you to think if you are the kind of person that benefits from pictures, e.g. you can talk about "counting the number of pairs of 1's in the same row" or similar.

I will say a few words about what kinds of problems might invite themselves to global approaches. One good signal is a problem that has a "high degree of symmetry" but for which that symmetry is broken if you try to, say, delete one thing and induct down. It's almost as if the problem "fights back" when you try to zoom in on just a small portion, by losing most of the nice symmetry properties.[2] In that case, you might be able to use global methods to get a one-hit KO instead, rather than trying to fight with individual little parts.

> **Example 2.1** (Canada 2009/2)
> Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

09CAN2

Walkthrough.

[1] Which is why I don't like "pigeonhole" classes, because they often involve problems like this, but pigeonhole is not even the main idea in that case.

[2] For example, imagine trying to do Canada 2009 by aligning one sector and then deleting it. . . , okay, this is not a great example, but you'll see some better ones later.

(a) Solve the problem by linearity of expectation.

(b) Write the proof in a "low-tech" way that doesn't quote the linearity of expectation, by considering all 200 rotations at once. This gives a "politically correct solution".

> **Example 2.2** (ELMO 2013/1)
> Let $a_1, a_2, \dots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

13ELMO1 **Walkthrough.** We say a triple $t = (a_i, a_j, a_k)$ is *large* if $a_i + a_j + a_k \ge 3m$.

(a) Show that among any three disjoint triples, at least one triple is large. Give a heuristic argument why we expect $A \ge 28$ as a result.

(b) Give a construction for $A = 28$. (Try making one element large.)

(c) We now proceed to the "global" idea of looking at every possible partition in (a) at once. Show that there are

$$C = \frac{1}{3!}\binom{9}{3,3,3} = 280$$

ways to partition the 9 elements into three disjoint triples.

(d) How many of the C partitions does each triple t appear in?

(e) Use your answer to (d) to prove $A \ge 28$, thereby solving the problem.

(f) For an alternate solution, explicitly construct a partition of the $\binom{9}{3} = 84$ triples into 28 disjoint triples. This would give another proof that $A \ge 28$.

When doing this calculation for the first time, you might be surprised that the division of seemingly random constants ends up with 28 in the end. It's important to recognize that the argument in (e) is "guaranteed" to work in a sense.

To elaborate: we constructed in (b) an example of an equality case, and every estimate we used was sharp. At the end of (e) we get some number again. The existence of the equality case means that this number *must* match the corresponding constant in (a), namely 28. This point is one of the key ideas in the Equality unit; the so-called "Sharpness Principle".

> **Example 2.3** (Canada 2006/4)
> Consider a round-robin tournament with $2k+1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a *cyclic triplet* if X beats Y, Y beats Z and Z beats X. There are no ties. Find the minimum and maximum possible number of cyclic triplets.

06CAN4 **Walkthrough.** The minimum bound is not that interesting.

(a) Give an example of a tournament with no cyclic triplet. This finds the minimum.

It's the maximum that we'll be most interested in. For a team v, let $\operatorname{outdeg} v$ denote the number of teams beaten by v. (This notation is the standard graph-theoretic one.) In order to count it, it will actually be parametrize our target in terms of degrees.

(b) Rephrase the "maximum" problem in terms of the number of *non-cyclic* triplets.

(c) By double-counting, find an expression for the number of non-cyclic triplets in terms of the outdegrees of the vertices. (Possible hint: every non-cyclic triplet can be labeled vwx with $v \to w$, $v \to x$.)

Thus we are reduced to an algebraic calculation.

(d) Use Jensen's inequality to show there are at least $(2k+1)\binom{k}{2}$ non-cyclic triplets.

(e) Give an example where equality holds; thus the maximum is $\binom{2k+1}{3} - (2k+1)\binom{k}{2}$.

> **Example 2.4** (Buffon's needle/noodle)
> You are about to drop a needle of length 1 onto a lined floor; the lines of the floor are spaced 1 unit apart. Show that the needle hits a line with probability $\frac{2}{\pi}$.

BUFFON

Walkthrough. This walkthrough is a bit unconventional, but I couldn't resist including it because it's so slick and yet not really known at all. I was first shown this argument by Qiaochu Yuan — it is more elegant, more general, and doesn't use calculus.

Rather than solving Buffon's needle, we will solve the general **Buffon's noodle** instead, where we drop a noodle of arbitrary shape.

> Given a noodle Γ of length ℓ, we let $E(\Gamma)$ denote the expected *number* of intersections of Γ with the lined floor. Determine $E(\Gamma)$.

We won't worry exactly how to define the "length" of a noodle, since arc length is a bit finnicky to define. So you have permission to handwave a bit on this walkthrough.

(a) Show that $E(\Gamma) = c \cdot \ell$ for some absolute constant c, by approximating Γ by a bunch of tiny equal straight line segments and using linearity of expectation. (You can ignore subtle technical issues with this "approximating".)

(b) Consider the special case where Γ is a circle of diameter 1. Check that $E(\Gamma) = 2$.

(c) Use the special case in (b) to extract the value of c. This solves Buffon's noodle.

(d) Use the fact that a needle of length 1 intersects the floor's ruling at most once (with probability 1) in order to extract the answer for Buffon's needle.

§2.1 Optional digression on foundations of probability

This is a long footnote about something that you may have noticed. It is not necessary for any of the problems, but might be interesting anyway.

If you're careful, some of the work done with probability may feel sketchy (especially in Buffon's needle/noodle). So I'd like to spell something out.

Probability is weird in high-school math contests because it's never defined. So even toddlers sometimes notice things ain't adding up. Examples:

- In the "pick a real number in $[0,1]$ uniformly at random", every particular real number is picked with probability 0. So how can these probabilities sum to 1?
- If we can pick a real number in $[0,1]$ uniformly at random, why can't we pick a positive integer uniformly at random? There are way more real numbers than positive integers, after all.

- Given a circle with radius 1, what's the probability a "random chord" has length at least $\sqrt{3}$? Depending on how you define "random chord", the answers differ; this is https://en.wikipedia.org/wiki/Bertrand_paradox_(probability).

Thus, probability shows up on short-answer contests with lots of waving hands; but is largely absent on proof-based contests. (Amusingly, flipping coins is considered taboo on the IMO; see IMO 2019/5 where the coins are never tossed.)

In my opinion, it's not actually "difficult" to give an airtight definition of probability — typically it's done with so-called measure spaces. If you wanted, you could spend a weekend reading the Napkin chapters on measure theory; you would find nothing surprising or tricky about the definitions, and all the "paradoxes" listed above would vanish.

But the issue is that this treatment requires a *lot* of book-keeping; it also uses topology freely, making it off-syllabus for high school. So for OTIS purposes, we won't worry about this technical hygiene.

§3 Practice problems

Instructions: Solve [36♣]. If you have time, solve [50♣]. Problems with red weights are mandatory.

> By this construction, Yahweh's work was indicated, and Yahweh's work was concealed. Thus would men know their place.
>
> Ted Chiang in *Tower of Babylon*

13BAMO4

[2♣] **Problem 1** (BAMO 2013/4). For a positive integer $n > 2$, consider the $n-1$ fractions $\frac{2}{1}, \frac{3}{2}, \dots, \frac{n}{n-1}$. The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. For which n can the product be made into 1?

13HMMTC6

[2♣] **Problem 2** (HMMT 2013). Values $a_1, \dots, a_{2013}$ are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

02IMC

[2♣] **Problem 3** (IMC 2002). An olympiad has six problems and 200 contestants. The contestants are very skilled, so each problem is solved by at least 120 of the contestants. Prove that there exist two contestants such that each problem is solved by at least one of them.

01TTSJ05

[3♣] **Problem 4** (Tournament of Towns 2001). Fifteen non-attacking rooks are placed on a 15×15 chessboard. Suppose each rook makes a move like that of a knight. Prove that after this, two of the rooks attack each other.

22PAGMO1

[3♣] **Required Problem 5** (PAGMO 2022/1). Leticia has a 9×9 toroidal board; each square thus has four orthogonal neighbors, which we call its *friends*. Leticia will paint every square one of three colors: green, blue or red. Then in each square we write a number as follows:

- If the square is green, write the number of red friends plus twice the number of blue friends.
- If the square is red, write the number of blue friends plus twice the number of green friends.
- If the square is blue, write the number of green friends plus twice the number of red friends.

Considering that Leticia can choose the coloring of the squares on the board, find the maximum possible value she can obtain when she sums the numbers in all the squares.

88CHNTST7

[3♣] **Problem 6** (China TST 1988, also Bilchfeldt's theorem). A polygon Π is given in the coordinate plane with are greater than n. Prove that there exist $n+1$ points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, ..., $P_{n+1} = (x_{n+1}, y_{n+1})$ in Π such that $x_j - x_i$ and $y_j - y_i$ are all integers for all $1 \le i < j \le n$.

85AIME14

[3♣] **Problem 7** (AIME 1985). In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten.) What was the total number of players in the tournament?

96RUS94

[3♣] **Required Problem 8** (Russia 1996)**.** In the Duma there are 1600 delegates, who have formed 16000 committees of 80 people each. Prove that one can find two committees having no fewer than four common members.

16SLC1

[2♣] **Problem 9** (Shortlist 2016 C1)**.** Titu selects two integers n and k with $n > k > 0$, and announces them to Zuming and Po-Shen. Titu then secretly tells Zuming an n-digit binary string, and then Zuming writes down all n-digit binary strings which differ from Titu's string in exactly k places (hence writing exactly $\binom{n}{k}$ strings). Po-Shen then looks at the strings written by Zuming, and tries to guess the string that Titu originally selected. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

01IBERO3

[5♣] **Problem 10** (Iberoamerican 2001/3)**.** Let X be a set with n elements. Given $k \ge 2$ subsets of X, each with at least r elements, show that we can find two of them whose intersection has at least $r - \frac{nk}{4(k-1)}$ elements.

05KOR4

[3♣] **Problem 11** (Korea 2005, added by Cordelia)**.** 11 students take a test. For any two question in the test, there are at least 6 students who solved exactly one of those two questions. Prove that there are no more than 12 questions in this test.

99AMO1

[3♣] **Problem 12** (USAMO 1999/1)**.** Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

23BAMO4

[2♣] **Problem 13** (BAMO 2023, added by Rohan Garg)**.** Zaineb makes a large necklace from beads labeled 290, 291, ..., 2023. She uses each bead exactly once, arranging the beads in the necklace any order she likes. Prove that no matter how the beads are arranged, there must be three beads in a row whose labels are the side lengths of a triangle.

19HKGTST15

[3♣] **Problem 14** (Hong Kong TST 2019/1/5)**.** Is it possible to find 24 points in $\mathbb{R}^3$, no three collinear, and 2019 planes such that each chosen plane passes through three or more of the chosen points, and every triple of chosen points lies on some chosen plane?

01IMO4

[3♣] **Required Problem 15** (IMO 2001/4)**.** Let n be an odd integer greater than 1 and let $c_1, c_2, \dots, c_n$ be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist two permutations $a \neq b$ of $\{1, 2, \dots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

17BAMO4

[3♣] **Problem 16** (BAMO 2017/4)**.** Let $\mathcal{P}$ be a convex n-gon, and let $h > 0$ be a real number. On each of the n sides of $\mathcal{P}$ we erect internally a rectangle of height h (meaning the rectangle shares a side with $\mathcal{P}$ and moreover the interiors overlap). Prove that it's possible to pick a h such that the n rectangles together cover the interior of $\mathcal{P}$, and moreover the sum of their areas is at most twice the area of $\mathcal{P}$.

22MOP3U

[5♣] **Problem 17** (MOP 2022, by Ankan Bhattacharya). The equilateral triangle Δ is partitioned into n smaller equilateral triangles with side lengths $a_1, a_2, \dots, a_n$. (The smaller equilateral triangles have sides parallel to those of Δ.) Prove that there exist $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$\sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j a_i a_j = 0.$$

04ROU

[5♣] **Problem 18** (Romania 2004). Prove that for any complex numbers $z_1, z_2, \dots, z_n$, satisfying $|z_1|^2 + |z_2|^2 + \dots + |z_n|^2 = 1$, one can select $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$\left| \sum_{k=1}^{n} \varepsilon_k z_k \right| \le 1.$$

16IMO2

[5♣] **Problem 19** (IMO 2016/2). Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- In each row and column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note that an $n \times n$ table has $4n - 2$ diagonals.

16INDTST9

[3♣] **Problem 20** (India TST 2016, added by Aayam Mathur). Let n be an odd natural number. We consider an $n \times n$ grid made up of n^2 unit squares and $2n(n+1)$ edges. We color each of these edges either red or blue. Suppose there are at most n^2 red edges. Show that there exists a unit square at least three of whose edges are blue.

04SLC1

[3♣] **Problem 21** (Shortlist 2004 C1). There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

(i) Each pair of students are in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m + 1$ students is in exactly m societies.

Find all possible values of k.

13OMOF29

[5♣] **Problem 22** (Online Math Open, Ray Li). Kevin has $2^n - 1$ cookies, each labeled with a unique nonempty subset of $\{1, 2, \dots, n\}$. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie. Determine the expected value of the number of days that Kevin eats a cookie before all cookies are gone.

[5♣] **Problem 23** (IMO 1970/6). Prove that for all sufficiently large positive integers n, within any n points on the plane in general position, at most 70% of the triangles with vertices among the points are acute.

70IMO6

[5♣] **Problem 24** (Extension of IMO 1970/6, due to Ravi Boppana)**.** Prove that for all sufficiently large positive integers n, within any n points on the plane in general position, at most 66.67% of the triangles with vertices among the points are acute.

[1♣] **Mini Survey.** Fill out feedback on the OTIS-WEB portal when submitting this problem set. Any thoughts on problems (e.g. especially nice, instructive, easy, etc.) or overall comments on the unit are welcome.

In addition, if you have any suggestions for problems to add, or want to write hints for one problem you really liked, please do so in the ARCH system!

The maximum number of [♣] for this unit is [82♣], including the mini-survey.

§4 Solutions to the walkthroughs

§4.1 Solution 1.1, BAMO 2004/4

A random labeling will work fine. Let $x_1, \dots, x_n$ be the numbers, and $a_1, \dots, a_n$ be a permutation of them. Then $\sum_i x_i = 0$, so squaring, we get

$$\sum_{i<j} x_i x_j = -\frac{1}{2}\sum_i x_i^2 < 0.$$

Now for any labeling the expected value of the sum is

$$n \cdot \mathbb{E}[a_1 a_2] < 0$$

as desired.

§4.2 Solution 2.1, Canada 2009/2

Just spin it randomly! We expect $\frac{1}{2} \cdot 200 = 100$ to line up in expectation.

(A low tech way to write the same proof: imagine 200 rotations. Over all of them, $200 \cdot 100 = 20000$ line-ups occur, so by pigeonhole, at least 100 line-ups occur for some rotation.)

§4.3 Solution 2.2, ELMO 2013/1

The answer is 28, achieved when $a_1 = \dots = a_8 = 0$ and $a_9 = 1$.

Here are three ways to see that $A \ge 28$ is necessary.

- **Linearity of expectation**: Let p be the probability a random triple is $\ge 3m$. Consider a random permutation π and the expected value of the number of triples $\{a_{\pi(3i+1)}, a_{\pi(3i+2)}, a_{\pi(3i+3)}\}$ with sum $\ge 3m$ for $i = 0, 1, 2$. On one hand it is ≥ 1, and the other hand it is $3p$, so $p \ge 1/3$ as needed.

- **Double counting**: Consider all

$$C = \frac{1}{6}\binom{9}{3,3,3} = 280$$

choices of three disjoint triples. So within each of C groups, there is at least one triple $\ge 3m$.

Consider a triple $t = (a_i, a_k, a_k)$ with sum $\ge 3m$ (there are A of these). It appears $\frac{1}{2}\binom{6}{3} = 10$ times. Therefore:

$$\frac{1}{2}\binom{6}{3} \cdot A \ge C = \frac{1}{3!}\binom{9}{3,3,3} = 280.$$

This implies $A \ge 28$.

(This is really the same as the previous solution.)

- **Explicit partition**: We explicitly break the $\binom{9}{3} = 84$ triples into 28 groups. Imagine the following array on a torus:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

  - Rows, columns, diagonals easy
  - 126, 459, 783. If three numbers occupy all different columns, then shift down cyclic.
  - Similarly, if three numbers occupy all different rows, then shift left/right cyclic.
  - 125 shapes: 476, 389 and similarly.

Thus done!

§4.4 Solution 2.3, Canada 2006/4

In what follows, we replace the word "team" with "vertex" (to match the terms from graph theory). For a team/vertex v, $\operatorname{outdeg} v$ denotes the number of teams/vertices which were beaten by v.

¶ Minimum The minimum is clearly zero — consider a tournament where there are teams of different skill levels, and no upsets.

¶ Maximum For the maximum, count the number of non-cyclic triplets. In any non-cyclic triplet there is exactly one vertex dominating the other two. So the number of non-cyclic triplets is equal to

$$\sum_v \binom{\operatorname{outdeg} v}{2}$$

which by Jensen is at least $(2k+1)\binom{k}{2}$. Hence the answer is $\binom{2k+1}{3} - (2k+1)\binom{k}{2}$.

§4.5 Solution 2.4, Buffon's needle/noodle

Rather than solving Buffon's needle, we will solve the general **Buffon's noodle** instead, where we drop a noodle of arbitrary shape.

> **Claim** — The expected *number* of intersections of a noodle of length ℓ with the rules of the floor is equal to $c \cdot \ell$, for some absolute constant c.

This is a huge surprise that the answer is in terms of length only, because you would a priori expect it to depend on the shape as well.

Idea of proof. Suppose I have a noodle of length 3 and a noodle of length 6, of any shape. You can approximate the first noodle by 300 tiny line segments of length 0.01, and the second noodle by 600 tiny line segments. Then, linearity of expectation tells you that the expected number of intersections is equal to 300ε and 600ε respectively, where ε is the probability that a single tiny line segment intersects.

In particular, the noodle of length 3 will have half as many expected intersections compared to the noodle of length 6. And of course, this works for any numbers in place of 3 and 6 (and the choice of 0.01 is arbitrary and can be replaced by any ε). This implies the claim.

To make this proof completely airtight, one would have to get into the details of what "approximate" means in the first paragraph, and to do that one would have to talk about the definition of "length" and "noodle". But this is beyond the scope of high-school math, so we won't get into these formal technicalities, and end the proof here. $\square$

Consider the special case where the noodle is a circle of diameter 1. It always intersects the floor twice, and the circle has circumference π. Hence $c = \frac{2}{\pi}$.

Finally, a needle of length 1 meets the floor at most once. So if the expected number of intersections is $2/\pi$, this is the probability of intersecting at all.