Youth EUCLID Association

Projective Geometry

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Contents

§1 Introduction

We work in the **real projective plane** $\mathbb{RP}^2$. For now, you may treat $\mathbb{RP}^2$ as the Euclidean plane $\mathbb{R}^2$ with an added **point at infinity** along each class of parallel lines. These points at infinity lie on the **line at infinity**.

> **Remark.** For those who haven't handled points at infinity before, here's some more detail. If $\ell_1 \parallel \ell_2$, then $\ell_1 \cap \ell_2$ is the point at infinity along ℓ_1, ℓ_2.

What's nice about $\mathbb{RP}^2$ is that the following holds:

Proposition 1.1

In $\mathbb{RP}^2$,

- any two distinct points determine a unique line; and
- any two distinct lines intersect at a unique point.

§1.1 Cross ratio

Henceforth all lengths are directed.

Definition 1.2 (Cross ratio on line)

For collinear points A, B, X, Y, define

$$(AB; XY) = \frac{XA}{XB} \Big/ \frac{YA}{YB}.$$

Definition 1.3 (Cross ratio on circle)

For concyclic points A, B, X, Y, also define

$$(AB; XY) = \frac{XA}{XB} \Big/ \frac{YA}{YB}.$$

Definition 1.4 (Cross ratio on pencil of lines)

For concurrent lines a, b, x, y define

$$(ab; xy) = \frac{\sin \angle(x, a)}{\sin \angle(x, b)} \Big/ \frac{\sin \angle(y, a)}{\sin \angle(y, b)}.$$

For points P, A, B, X, Y, we use the shorthand $P(AB; XY) = (\overline{PA}, \overline{PB}; \overline{PX}, \overline{PY})$.

These three definitions above are seemingly unconnected, but the true magic begins with these two propositions:

Proposition 1.5

For points A, B, X, Y on a line ℓ and P a point not on ℓ, we have

$$(AB; XY) = P(AB; XY).$$

Proposition 1.6

For concyclic points P, A, B, X, Y, we have

$$(AB; XY) = P(AB; XY).$$

Exercise 1.7. Verify both of these. The first uses ratio lemma and the second uses extended law of sines.

We will see the power of these two propositions in the next two examples.

Example 1.8 (MOP Plank Countdown 2019)

Lines ℓ_1, ℓ_2 intersect at a single point P. Points P, A, B, C lie on ℓ_1 in that order; points P, D, E, F lie on ℓ_2 in that order. We are given $PA = 3$, $AB = 1$, $BC = 2$, $DE = 3$, $EF = 4$. If lines AD, BE, CF concur, evaluate PD.

Walkthrough. Let the concurrence point be Q.

(a) Calculate $(PA; BC)$ and $(PD; EF)$.

(b) Determine $Q(PA; BC) = Q(PD; EF)$ in two ways.

(c) Set them equal.

For points A, B, X, Y, A', B', X', Y', if $\overline{AA'}$, …concur at P, and $\overline{ABXY}$, $\overline{A'B'X'Y'}$ collinear, we have

$$(AB; XY) = P(AB; XY) = P(A'B'; X'Y') = (A'B'; X'Y'),$$

shorthand $(AB; XY) \overset{P}{=} (A'B'; X'Y')$. This also works when $ABXY$ are concyclic instead of collinear, but keep in mind: in the concyclic case, P must lie on the circle $(ABXY)$ as well.

We can also project from a circle onto itself:

Theorem 1.9

We can write $(AB; XY) \overset{P}{=} (A'B'; X'Y')$ if $P \in \overline{AA'}$, $P \in \overline{BB'}$, $P \in \overline{CC'}$, $P \in \overline{DD'}$, and one of the following holds:

- $\overline{ABXY}$, $\overline{A'B'X'Y'}$ collinear and P lies on neither line;
- $(ABXY)$ concyclic, $\overline{A'B'X'Y'}$, and P lies on $(ABXY)$ but not $\overline{A'B'X'Y'}$;
- $(ABXY)$, $(A'B'X'Y')$ concyclic and P lies on both circles;
- $(ABXYA'B'X'Y')$ concyclic and P does not lie on the circle.

Example 1.10 (AIME II 2016/10)

Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4$, $PQ = 3$, $QB = 6$, $BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 1.11. Solve the above example by perspectivity through the concurrence point.

§1.2 Uniqueness

Exercise 1.12. Show that for real $k \neq 0$ and points A, B, X, there is a unique point Y with $(AB; XY) = k$.

§2 Harmonic bundles

§2.1 Definition

Definition 2.1 (Harmonic bundles)

We say $AXBY$ is a **harmonic quadrilateral** iff $-1 = (AB; XY)$. Here, Y is the **harmonic conjugate** of X.

§2.2 Common occurrences of harmonic bundles

Harmonic bundles come up frequently. Verify the following theorems:

Theorem 2.2 (Midpoints and infinity)

If M is the midpoint of $\overline{AB}$ and ∞_{AB} the infinity point along $\overline{AB}$, then $-1 = (AB; M\infty_{AB})$.

Theorem 2.3 (Ceva-Menelaus)

In triangle ABC, concurrent cevians meet opposite sides at D, E, F, and $T = \overline{BC} \cap \overline{EF}$. Then $-1 = (BC; DT)$.

Theorem 2.4 (Complete Quadrilaterals)

Let $ABCD$ be a quadrilateral and let $K = \overline{AC} \cap \overline{BD}$, $L = \overline{AD} \cap \overline{BC}$, $M = \overline{KL} \cap \overline{AB}$, $N = \overline{KL} \cap \overline{CD}$. Then $-1 = (KL; MN)$.

Theorem 2.5 (Symmedians)

If the A-symmedian of $\triangle ABC$ hits the circumcircle at K, then $-1 = (AK; BC)$.

Theorem 2.6 (Poles and polars)

Points A, B lie on a circle ω and P lies on line AB but outside segment AB. The tangents from P touch ω at X, Y, and $Q = \overline{AB} \cap \overline{XY}$. Then $-1 = (AB; PQ)$.

Theorem 2.7 (Right angles and angle bisectors)

Let PAB be a triangle and suppose that X, Y lie on line AB. Then any two of the following imply the third:

- $-1 = (AB; XY)$;
- $\angle XAY = 90^\circ$;
- $\overline{AX}$ bisects $\angle APB$.

Exercise 2.8 (Apollonius)**.** Let k be a positive real number and let A, B be points. What is the locus of points R such that $\frac{RA}{RB} = k$?

§2.3 Two important lemmas

Lemma 2.9 (Midpoints of harmonic bundles)

Points A, B, X, Y lie on a line such that $-1 = (AB; XY)$ and M is the midpoint of $\overline{AB}$. Then

(i) $MX \cdot MY = MA^2$;

(ii) $XM \cdot XY = XB \cdot XC$.

Lemma 2.10 ("Prism lemma")

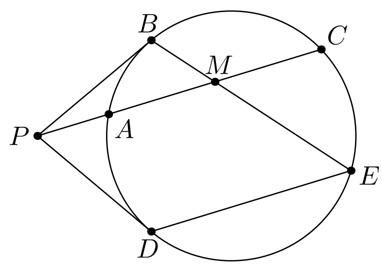
Let $\overline{PABC}$, $\overline{PDEF}$ be collinear. If $(PA; BC) = (PD; EF)$, then $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur.

§2.4 Example problems

Example 2.11 (JMO 2011/5)

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

Walkthrough. Here's the basic diagram.

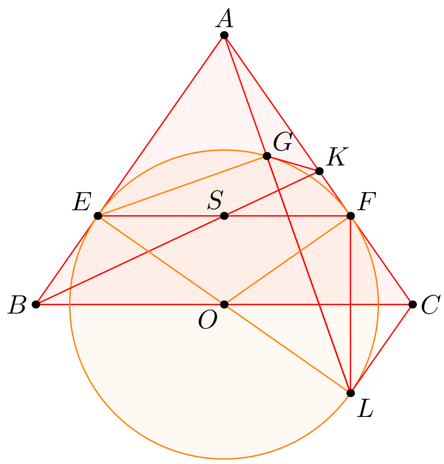


(a) Show that $-1 = (AC; BD)$.

(b) Take perspectivity through a point we haven't used yet.

Example 2.12 (China Southeast 2018/5)
Let ABC be an isosceles triangle with $AB = AC$. Suppose that the center of circle ω is the midpoint of the $\overline{BC}$, and $\overline{AB}$ and $\overline{AC}$ are tangent to ω at points E and F respectively. There is a point G that lies on ω such that $\angle AGE = 90°$. Show that if the tangent to ω at G meets $\overline{AC}$ at K, then line BK bisects $\overline{EF}$.

Walkthrough. Here's a picture:



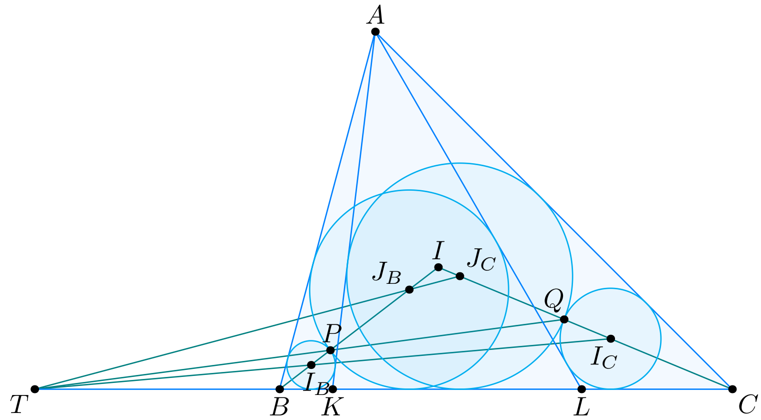
(a) Why does it suffice to prove $-1 = (AF; KC)$?

(b) Let line AG intersect ω again at L. Show that F and L are reflections across $\overline{BC}$. (Moreover, why would we construct L?)

(c) Project $(AF; KC)$ onto ω through L. Conclude.

Example 2.13 (TSTST 2019/9)
Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

Walkthrough. Let I_B, J_B, I_C, J_C denote the incenters of $\triangle ABK$, $\triangle ABL$, $\triangle ACL$, $\triangle ACK$. As a shorthand, let $r_{\triangle XYZ}$ denote the inradius of $\triangle XYZ$. The key insight can be seen in the diagram below:



First I contend that $\overline{BC}$, $\overline{I_BI_C}$, $\overline{J_BJ_C}$ always concur for any points K, L on $\overline{BC}$, regardless of whether the tangency condition holds.

(a) Show that $\angle BAI = \angle I_BJ_C = \angle J_BAI_C = \angle IAC$.

(b) Prove that $(BI_B; IJ_B) = (CI_C; IJ_C)$, and use this to conclude the concurrence by Prism lemma.

Now assume P, Q exist, and let $\overline{BC}$, $\overline{I_BI_C}$, $\overline{J_BJ_C}$ concur at T.

(c) Prove that $(BP; I_BJ_B) = (CQ; I_CJ_C)$, and use this to conclude that $T \in \overline{PQ}$ by Prism lemma again.

(d) Apply Menelaus theorem on $\triangle IBC$.

§3 Problems for perspectivity and harmonic bundles

[2♣] **Problem 3.1.** Let ABC be a triangle with incenter I and A-excenter I_A. Show that $-1 = (I, I_A; A, \overline{AI} \cap \overline{BC})$.

[2♣] **Problem 3.2** (IMO 2014/4)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that the intersection of $\overline{BM}$ and $\overline{CN}$ lies on the circumcircle of triangle ABC.

[2♣] **Problem 3.3** (ISL 1995 G3)**.** Let $ABCD$ be a tangential quadrilateral. The incircle of $\triangle ABC$ touches $\overline{AB}$ at W and $\overline{BC}$ at X. The incircle of $\triangle ACD$ touches $\overline{CD}$ at Y and $\overline{DA}$ at Z. Prove that W, X, Y, Z are concyclic.

[3♣] **Problem 3.4** (WOOT 2019/1/4)**.** Let $ABCD$ be a convex quadrilateral with $AC = BC$ and $\overline{AB} \parallel \overline{CD}$. Let H be the midpoint of $\overline{AB}$. A line passing through H intersects line AD at P and line BD at Q. Prove that $\angle ACP$ and $\angle BCQ$ are either equal or supplementary.

[2♣] **Problem 3.5.** Let ABC be a triangle and let D, E, F be the feet of the altitudes with D on $\overline{BC}$, E on $\overline{CA}$ and F on $\overline{AB}$. Let the line through D parallel to $\overline{EF}$ meet $\overline{AB}$ at X and $\overline{AC}$ at Y. Let T be the intersection of lines EF and BC and let M be the midpoint of the side $\overline{BC}$. Prove that the points T, M, X, Y are concyclic.

[5♣] **Problem 3.6** (ISL 1995 G3)**.** Let $ABCD$ be a tangential quadrilateral. The incircle of $\triangle ABC$ touches $\overline{AB}$ at W and $\overline{BC}$ at X. The incircle of $\triangle ACD$ touches $\overline{CD}$ at Y and $\overline{DA}$ at Z. Prove that W, X, Y, Z are concyclic.

[5♣] **Problem 3.7** (USMCA 2019/3)**.** Let ABC be a scalene triangle. The incircle of ABC touches $\overline{BC}$ at D. Let P be a point on $\overline{BC}$ satisfying $\angle BAP = \angle CAP$, and let M be the midpoint of $\overline{BC}$. Define Q to be on $\overline{AM}$ such that $\overline{PQ} \perp \overline{AM}$. Prove that the circumcircle of $\triangle AQD$ is tangent to $\overline{BC}$.

[3♣] **Problem 3.8** (Canada 1994/5)**.** Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively. Prove that $\angle EDH = \angle FDH$.

[3♣] **Problem 3.9.** Let ABC be a triangle with orthocenter H, and let E and F be the feet of the altitudes from B and C respectively. Let the circumcircles of $\triangle ABC$ and $\triangle AEF$ meet again at Q. Show that the angle bisectors of $\angle BQC$ and $\angle BHC$ intersect on $\overline{BC}$.

[3♣] **Problem 3.10** (Key claim from USAMO 2016/3)**.** Let ABC be a triangle. Show that the C-excenter, the foot of the B-altitude, and the reflection of the A-excenter across line AC are collinear.

[5♣] **Problem 3.11** (APMO 2013/5)**.** Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

[9♣] **Problem 3.12** (USMCA 2020/7)**.** Let $ABCD$ be a convex quadrilateral, and let ω_A and ω_B be the incircles of $\triangle ACD$ and $\triangle BCD$, with centers I and J. The second common external tangent to ω_A and ω_B touches ω_A at K and ω_B at L. Prove that lines AK, BL, IJ are concurrent.

§4 Poles and Polars

§4.1 Polarity

Definition 4.1

Let ω be a circle and P a point. Denote by P^* the inverse of P wrt. ω, and let ℓ be the line through P^* perpendicular to $\overline{OP}$. Then ℓ is the **polar** of P and P is the **pole** of ℓ.

Remark. The polar of the center of ω is the line at infinity.

Theorem 4.2 (La Hire)

Let P, Q be points and ω a circle. Then P lies on the polar of Q if and only if Q lies on the polar of P.

Corollary 4.3 (Stronger Theorem 2.6)

Let $\overline{AB}$ be a chord of ω and P, Q two points on line AB. Then $-1 = (AB; PQ)$ if and only if P lies on the polar of Q.

§4.2 Orthogonality

Definition 4.4

Two circles are **orthogonal** if they intersect at right angles. That is, $\omega_1 \perp \omega_2$ means that if P is one of their intersection points and their centers are O_1 and O_2 respectively, then $\angle O_1PO_2 = 90^\circ$.

Proposition 4.5

If $\omega_1 \perp \omega_2$, then ω_2 is mapped to itself under inversion at ω_1.

Definition 4.6 (Self-Polar Orthogonality)

Point P lies on the polar of Q wrt. ω. Then ω and the circle with diameter $\overline{PQ}$ are orthogonal.

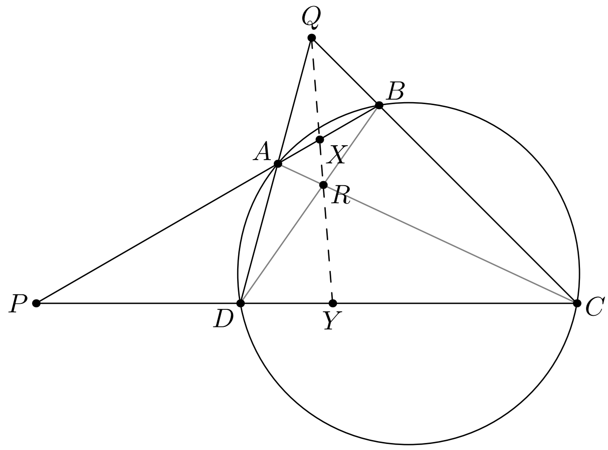
Theorem 4.7 (Brokard's theorem)

Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω and set $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{AD} \cap \overline{BC}$, and $R = \overline{AC} \cap \overline{BD}$. Then, P is the pole of $\overline{QR}$, Q is the pole of $\overline{RP}$, and R is the pole of $\overline{PQ}$. We say that $\triangle PQR$ is **self-polar** wrt. ω, and that ω is the **polar circle** of $\triangle PQR$.

In particular, the center O of ω must be the orthocenter of $\triangle PQR$.

Remark (Notes on polar circle)**.** There is a unique polar circle wrt. every triangle PQR. In particular, if ω denotes the polar circle of $\triangle PQR$, then for every point $A \in \omega$, if $B = \overline{AP} \cap \omega$, $C = \overline{AR} \cap \omega$, $D = \overline{AQ} \cap \omega$, we have $P \in \overline{CD}$, $R \in \overline{BD}$, $Q \in \overline{BC}$.

Proof of Theorem 4.7. Refer to the below diagram:



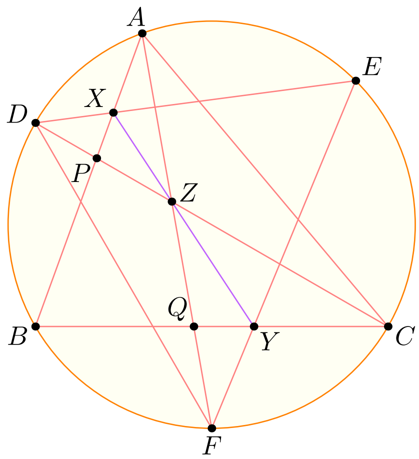
Let $X = \overline{AB} \cap \overline{QR}$, $Y = \overline{CD} \cap \overline{QR}$. By Ceva-Menelaus, $-1 = (AB; XP)$ and $-1 = (CD; YP)$, so $\overline{XY}$ is the polar of P. Applying this symmetrically gives the desired conclusion. $\square$

§4.3 Pascal's theorem

Theorem 4.8 (Pascal's theorem)

Let $ABCDEF$ be a hexagon inscribed in a conic. Then the points $\overline{AB} \cap \overline{DE}$, $\overline{BC} \cap \overline{EF}$, $\overline{CA} \cap \overline{FD}$ are collinear.

Proof of Theorem 4.8. Let $X = \overline{AB} \cap \overline{DE}$, $Y = \overline{BC} \cap \overline{EF}$, $Z = \overline{CA} \cap \overline{FD}$, $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{BC} \cap \overline{FA}$.



We have

$$(AB; PX) \overset{D}{=} (AB; CE) \overset{F}{=} (QB; CY),$$

so by the "Prism lemma," $\overline{AQ}$, $\overline{CP}$, $\overline{XY}$ concur, end proof. $\square$

The special case when the conic is the union of two lines:

Corollary 4.9 (Pappus' theorem)

Let ℓ_1, ℓ_2 be two lines, and let A, B, C be points on ℓ_1 and D, E, F be points on ℓ_2. THen the points $\overline{AE} \cap \overline{BD}$, $\overline{BF} \cap \overline{CE}$, $\overline{CD} \cap \overline{AF}$ are collinear.

The dual of Pascal's theorem:

Corollary 4.10 (Brianchon's theorem)

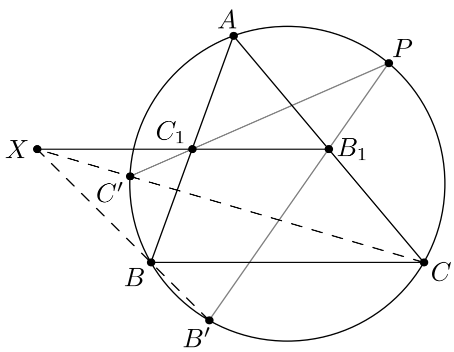
Let $ABCDEF$ be a tangential hexagon. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur.

§4.4 Example problems

Example 4.11 (ISL 2007 G5)

Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle, and let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Walkthrough. Here's part of the diagram. In what follows, $[\bullet]$ denotes area. Let XYZ be the triangle formed.



(a) In the above diagram, conjecture a collinearity involving X, and apply Pascal theorem in the form $BB'?C'C?$ to show that X lies on line B_1C_1.

(b) Add in the points Y, Z, and guess where line BY, CZ intersect.

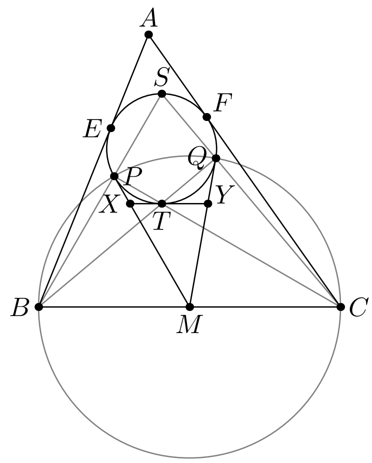
(c) Apply Pappus theorem in the form $B?CZ?Y$ to prove your conjecture in (b).

(d) Show that $[XYZ] = [XBA] + [XAC]$, and finish from here.

Example 4.12 (Iran TST 2018/1/3)

In triangle ABC, let M be the midpoint of $\overline{BC}$. Let ω be a circle inside $\triangle ABC$ tangent to $\overline{AB}$ and $\overline{AC}$ at E and F respectively. The tangents from M to ω touch ω at P and Q, so that P and B lie on the same side of $\overline{AM}$. Let $X = \overline{PM} \cap \overline{BF}$ and $Y = \overline{QM} \cap \overline{CE}$. If $2PM = BC$, prove that $\overline{XY}$ is tangent to ω.

Walkthrough. Picture:



(a) Show that B, P, Q, C are concyclic.

(b) Let $S = \overline{BP} \cap \overline{CQ}$, $T = \overline{BQ} \cap \overline{CP}$. Prove that $S, T \in \omega$ in the following two steps:

(i) show that $SPTQ$ is cyclic; and

(ii) show that $\overline{MP}$, $\overline{MQ}$ are tangent to its circumcircle.

(c) Write out some pole/polar pairs. What is the polar of X?

(d) Conclude that $\overline{XY}$ is tangent to ω at T.

Example 4.13 (IGO Advanced 2018/4)

Let $ABCD$ be a tangential quadrilateral whose diagonals $\overline{AC}$, $\overline{BD}$ are not perpendicular. The angle bisectors formed by these two diagonals intersect sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ at points K, L, M, N, respectively. Show that if quadrilateral $KLMN$ is cyclic, then quadrilateral $ABCD$ is also cyclic.

Walkthrough. Before we begin, let's recall some useful facts about bicentric quadrilaterals.

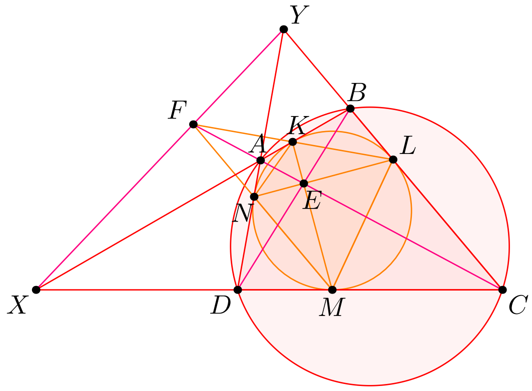
(a) This lemma is just angle chasing. Prove it.

> **Lemma** — Let $ABCD$ be a tangential quadrilateral whose incircle touches $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ at W, X, Y, Z. Then $\overline{WY} \cap \overline{XZ}$ if and only if $WXYZ$ is cyclic.

(b) Prove the following lemma by two applications of degenerate Brianchon.

> **Lemma** — Let $ABCD$ be a tangential quadrilateral whose incircle touches $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ at W, X, Y, Z. Then $\overline{AC}$, $\overline{BD}$, $\overline{WY}$, $\overline{XZ}$ concur.

Now the key observation is that $(KLMN)$ is actually the incircle of $ABCD$, and since $\overline{KM} \perp \overline{LN}$ is freely given, proving that will suffice. So we proceed by phantom points and let K', L', M', N' denote the tangency points of the incircle with $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$. Also let $E = \overline{AC} \cap \overline{BD}$, $X = \overline{AB} \cap \overline{CD}$, $Y = \overline{AD} \cap \overline{BC}$.



(c) What does the statement of the lemma in (b) imply about $\overline{K'M'} \cap \overline{L'N'}$?

(d) Use Pascal's theorem to prove that $\overline{K'L'}$, $\overline{M'N'}$, $\overline{AC}$ concur. Henceforth call the concurrence point F.

(e) Evaluate $(AC; EF)$ by setting it equal to its dual wrt. the incircle

(f) Let $K^* = \overline{AB} \cap \overline{LN}$, $L^* = \overline{BC} \cap \overline{KM}$. Show that lines AC, KL, K^*L^* concur using the Prism lemma. Call the concurrence point F'.

(g) Prove that $F = F'$ by evaluating $(AC; EF')$.

(h) What is the polar circle of $\triangle EXY$?

(i) Prove that $K = K'$, etc. and conclude.

§5 Assorted problems

[2♣] **Problem 5.1.** Let $ABCD$ be a cyclic quadrilateral. Apply Pascal's theorem on $AABCCD$, $ABBCDD$. What is the result?

[3♣] **Problem 5.2.** Let ABC be a triangle with incenter I, and let D be the foot of the angle bisector from A to $\overline{BC}$. Let γ be the circumcircle of triangle BIC, and let $\overline{PQ}$ be a chord of γ passing through D. Prove that $\overline{AD}$ bisects $\angle PAQ$.

[3♣] **Problem 5.3** (Sharygin 2013)**.** Let ABC be a triangle, and let $\overline{AD}$ denote the bisector of $\angle A$ (with D on $\overline{BC}$). Points M and N are the projections of B and C respectively to $\overline{AD}$. The circle with diameter $\overline{MN}$ intersects $\overline{BC}$ at points X and Y. Prove that $\angle BAX = \angle CAY$.

[3♣] **Problem 5.4** (ISL 2002 G7)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of $\overline{AD}$. If N is the common point of the circle Ω and $\overline{KM}$ (distinct from K), then prove Ω and the circumcircle of triangle BCN are tangent to each other.

[5♣] **Problem 5.5** (Iran TST 2017/1/5)**.** In triangle ABC, select arbitrary points P and Q on side BC such that $BP = CQ$, and P lies between B and Q. The circumcircle of $\triangle APQ$ intersects sides AB and AC at E and F, respectively. Point T is the intersection of lines EP and FQ. Two lines passing through the midpoint of $\overline{BC}$ parallel to $\overline{AB}$ and $\overline{AC}$ intersect lines EP and FQ at points X and Y, respectively.

Prove that the circumcircles of $\triangle TXY$ and $\triangle APQ$ are tangent to each other.

[5♣] **Problem 5.6** (China 2002)**.** Let $ABCD$ be a quadrilateral. Let $E = \overline{AB} \cap \overline{CD}$, $F = \overline{AD} \cap \overline{BC}$, and $P = \overline{AC} \cap \overline{BD}$. Denote by Q the foot of the perpendicular from P to $\overline{EF}$. Prove that $\angle AQD = \angle BPC$.

[3♣] **Problem 5.7.** Let Γ be a circle with center O and diameter $\overline{AB}$. Let C lie on ray AB outside of Γ. Let a line through C meet Γ at D and E, with D between C and E. Let ω be the circumcircle of $\triangle OBD$, and let $\overline{OF}$ be a diameter of ω. Let $\overline{CF}$ meet ω again at G. Prove that $OEAG$ is cyclic.

[5♣] **Problem 5.8** (MEMO 2017 T6)**.** Let ABC be an acute-angled triangle with $AB \neq AC$, circumcenter O and circumcircle Γ. Let the tangents to Γ at B and C meet each other at D, and let the line AO intersect $\overline{BC}$ at E. Denote the midpoint of $\overline{BC}$ by M and let $\overline{AM}$ meet Γ again at $N \neq A$. Finally, let $F \neq A$ be a point on Γ such that A, M, E, F are concyclic. Prove that $\overline{FN}$ bisects the segment MD.

[5♣] **Problem 5.9.** Let ABC be a triangle with incenter I and incircle ω that touches $\overline{CA}$ and $\overline{AB}$ at E and F respectively. Let G and H be reflections of E and F respectively across I, and let line GH intersect line BC at Q. Show that if M denotes the midpoint of $\overline{BC}$, then $\angle MIQ = 90°$.

[5♣] **Problem 5.10** (Iran TST 2019/3/3)**.** Let ABC be a triangle and let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Point K lies on segment NP such that $\overline{AK}$ bisects $\angle BKC$. Lines MN and BK intersect at E and lines MP and CK intersect at F. Suppose that H is the foot of the perpendicular from A to $\overline{BC}$ and L is the second intersection of the circumcircles of $\triangle AKH$ and $\triangle HEF$.

Prove that lines MK, EF, HL are concurrent.

[5♣] **Problem 5.11** (USA TST 2019/6)**.** Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

[9♣] **Problem 5.12** (Mock AIME 2019/15')**.** In triangle ABC, let D, E, and F denote the feet of the altitudes from A, B, and C, respectively, and let O denote the circumcenter of $\triangle ABC$. Points X and Y denote the projections of E and F, respectively, onto $\overline{AD}$, and $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{ZT}$ and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that line AQ bisects $\overline{BC}$.

[9♣] **Problem 5.13** (Taiwan TST 2015/3/Q6)**.** In scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of $\triangle AEF$ at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter $\overline{ST}$ is orthogonal to the nine-point circle of $\triangle BIC$.

§6 Solution to walkthroughs

§6.1 Solution 1.8 (MOP 2019)

Just use $(PA;BC) = (PD;EF)$.

§6.2 Solution 1.10 (AIME II 2016/10)

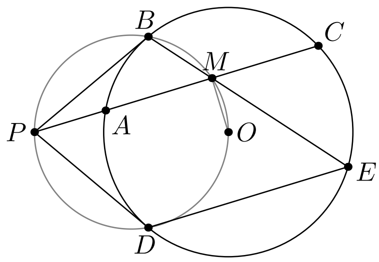
Notice that

$$ST = \frac{5\cdot 7}{13}(A,T;B,S) \stackrel{C}{=} \frac{35}{13}(A,Q;B,P) = \frac{35}{13}\cdot\frac{13}{6}\cdot\frac{3}{4} = \frac{35}{8},$$

and the requested sum is $35 + 8 = 043$.

§6.3 Solution 2.11 (JMO 2011/5)

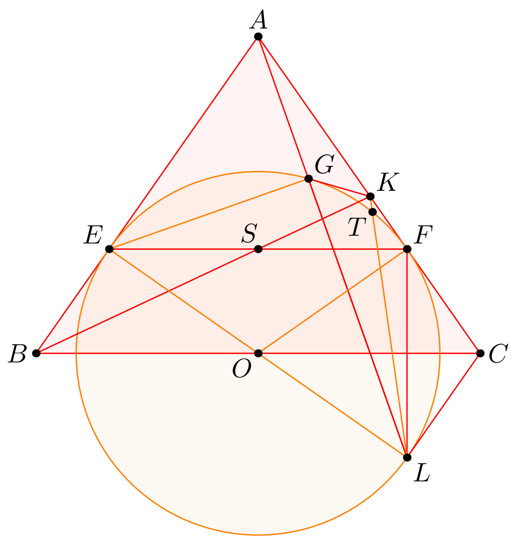
First solution, by angle chasing



Let O be the center of ω and $M = \overline{BE} \cap \overline{AC}$. Then $\measuredangle BDP = \measuredangle BED = \measuredangle BMP$, so M lies on $(OBPD)$ and $\angle OMP = 90^\circ$. It follows that M is the midpoint of $\overline{AC}$, as desired.

Second solution, by harmonic bundles We have $-1 = (AC;BD) \stackrel{E}{=} (A,C;\overline{BE}\cap\overline{AC},\infty_{AC})$, as desired.

§6.4 Solution 2.12 (China Southeast 2018/5)

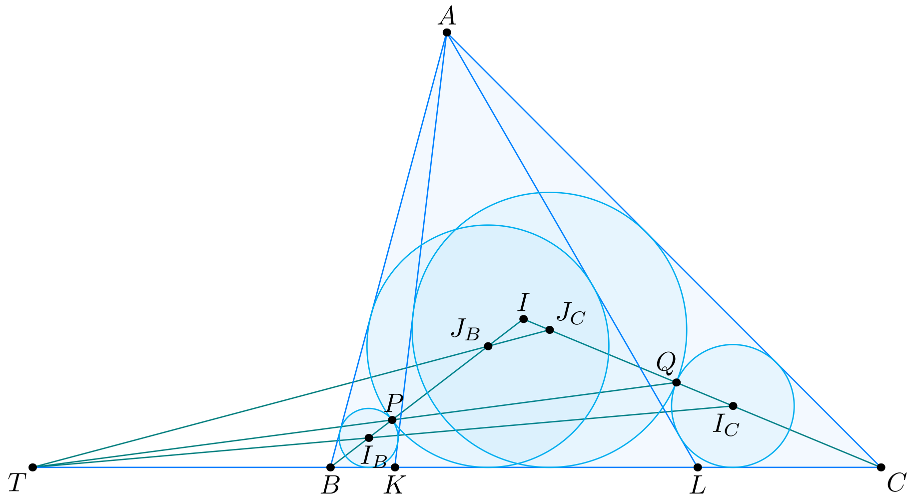


Let line AG intersect ω again at L, $\overline{LK}$ intersect ω again at T, and $\overline{BK}$ intersect $\overline{EF}$ at S. Note that since $\angle EGL = 90°$, L is the antipode of E on ω, so F and L are reflections over $\overline{BC}$. It follows that $\overline{CL}$ is tangent to ω, whence

$$-1 = (GF;TL) \stackrel{L}{=} (AF;KC) \stackrel{B}{=} (EF;S\infty_{BC}).$$

This implies that S is the midpoint of $\overline{EF}$, and we are done.

§6.5 Solution 2.13 (TSTST 2019/9)



Let I_B, J_B, I_C, J_C denote the incenters of $\triangle ABK$, $\triangle ABL$, $\triangle ACL$, $\triangle ACK$. Also let $r_{\triangle XYZ}$ denote the inradius of $\triangle XYZ$.

Lemma

For any points K, L on $\overline{BC}$ of $\triangle ABC$, if I_B, J_B, I_C, J_C denote the incenters of $\triangle ABK$, $\triangle ABL$, $\triangle ACL$, $\triangle ACK$, then $\overline{I_BI_C}$, $\overline{J_BJ_C}$, $\overline{BC}$ concur at a point T.

Proof. Rotation by $\frac{1}{2}\angle A$ about A gives

$$A(BI_B;IJ_B) = A(IJ_C;CI_C) = A(CI_C;IJ_C),$$

and thus $(BI_B;IJ_B) = (CI_C;IJ_C)$ and the result follows.[1] □

Consider the homothety centered at B sending (I_B) to (J_B); we can check that the scale factor is

$$\frac{PJ_B}{PI_B} = \frac{r_{\triangle ABL}}{r_{\triangle ABK}} = \frac{BJ_B}{BI_B} \implies -1 = (BP;I_BJ_B),$$

and similarly $-1 = (CP;I_CJ_C)$. It is immediate that $\overline{PQ}$ also passes through T. By Menelaus on $\triangle II_BI_C$,

$$-1 = \frac{IP}{PI_B} \cdot \frac{I_BT}{TI_C} \cdot \frac{I_CQ}{QI} = \frac{IP}{QI} \cdot \frac{I_BT}{TI_C} \cdot \frac{I_CQ}{PI_B} = \frac{IP}{QI} \cdot \frac{r_{\triangle ABK}}{r_{\triangle ACL}} \cdot \frac{r_{\triangle ACL}}{r_{\triangle ABK}} = \frac{IP}{QI},$$

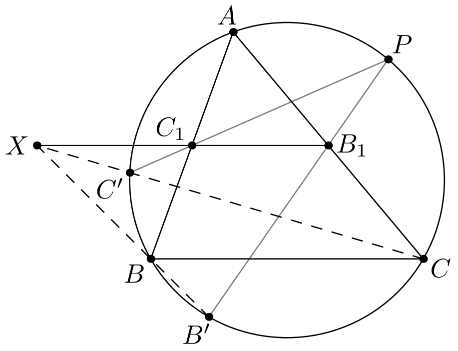
[1] Alternatively we can directly compute that

$$A(BI_B;IJ_B) = \frac{\sin\frac{1}{2}\angle BAC \cdot \sin\frac{1}{2}\angle KAL}{\sin\frac{1}{2}\angle KAC \cdot \sin\frac{1}{2}\angle BAL} = A(CI_C;IJ_C).$$

whence $IP = IQ$, as desired.

§6.6 Solution 4.11 (ISL 2007 G5)

In what follows, $[\bullet]$ denotes area. Let the triangle formed by the three lines be XYZ. I claim $[XYZ] = \frac{1}{2}[ABC]$, which will suffice. In what follows, the two claims allow the lemma to solve the problem.



Claim 1. X lies on $\overline{B_1C_1}$, and similarly $Y \in \overline{C_1A_1}$, $Z \in \overline{A_1B_1}$.

Proof. Apply Pascal theorem to $BB'PC'CA$. □

Claim 2. $\overline{AX} \parallel \overline{BY} \parallel \overline{CZ}$.

Proof. Apply Pappus theorem to $BACZA_1Y$ to show $\overline{BY} \parallel \overline{CZ}$. □

Remark. In general, for any triangle ABC with medial triangle $A_1B_1C_1$ and points X, Y, Z on $\overline{B_1C_1}$, $\overline{C_1A_1}$, $\overline{A_1B_1}$, the following two conditions are equivalent:

- $\overline{AX} \parallel \overline{BY} \parallel \overline{CZ}$;
- $A \in \overline{YZ}$, $B \in \overline{ZX}$, $C \in \overline{XY}$.

After noting the first two claims, the problem statement is equivalent to the below lemma.

Lemma (Fixed area on medial triangle)

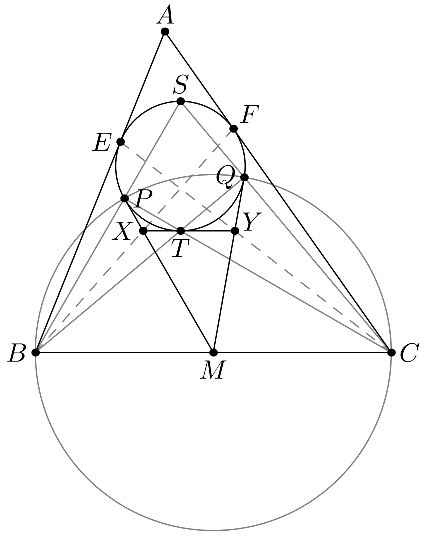
Let ABC be a triangle with medial triangle $A_1B_1C_1$. If X, Y, Z lie on $\overline{B_1C_1}$, $\overline{C_1A_1}$, $\overline{A_1B_1}$ with $\overline{AX} \parallel \overline{BY} \parallel \overline{CZ}$, then $[XYZ] = \frac{1}{2}[ABC]$.

Proof. Using directed areas, we have

$$[XYZ] = [XYA] + [XAZ] = [XBA] + [XAC] = [XBC] = \tfrac{1}{2}[ABC],$$

as needed. □

§6.7 Solution 4.12 (Iran TST 2018/1/3)



First solution, by Brokard's theorem Note that B, P, Q, C all lie on the circle Γ centered at M with diameter $\overline{BC}$. Denote $S = \overline{BP} \cap \overline{CQ}$ and $T = \overline{BQ} \cap \overline{CP}$. Since $\angle BPC = \angle BQC = 90^\circ$, we know $BCST$ is an orthocentric system, so by the Three Tangents lemma, $\overline{MP}$ and $\overline{MQ}$ are tangent to (ST) at P and Q respectively.

It follows that $\omega = (ST)$, so $\overline{ST}$ is a diameter of ω. By Brokard's theorem, B is the pole of $\overline{CE}$ and C is the pole of $\overline{BF}$ with respect to ω. It follows that X is the pole of $\overline{CP}$, so $\overline{XT}$ is tangent to ω. Similarly $\overline{YT}$ is tangent to ω, so $\overline{XY}$ is tangent to ω at T, as desired.

Second solution, by self-polar orthogonality Note that B, P, Q, C all lie on the circle Γ centered at M with diameter $\overline{BC}$. Thus ω and Γ are orthogonal, so by self-polar orthogonality, B and C lie on the polars of each other with respect to ω. This implies X is the pole of $\overline{CP}$ and Y is the pole of $\overline{BQ}$. It thus suffices to show that $T = \overline{BQ} \cap \overline{CP}$ lies on ω, but this is just angle-chasing: let N be the center of ω; then,

$$\angle PTQ = \frac{\widehat{BC} + \widehat{PQ}}{2} = 90^\circ + \frac{1}{2}\angle PMQ = 180^\circ - \frac{1}{2}\angle PNQ,$$

and the conclusion follows from inscribed angle theorem.

§6.8 Solution 4.13 (IGO Advanced 2018/4)

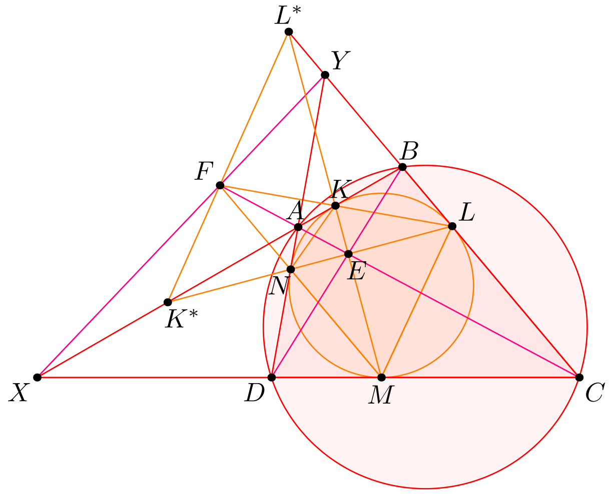
Let $E = \overline{AB} \cap \overline{CD}$, $X = \overline{AB} \cap \overline{CD}$, $Y = \overline{AD} \cap \overline{BC}$. Also let the incircle of $ABCD$ touch $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ at K', L', M', N'. I claim $K = K'$, etc.

Claim 1. $\overline{KL} \cap \overline{MN}$ is the harmonic conjugate of E wrt. $\overline{AC}$. (In particular, it lies on $\overline{AC}$.)

Proof. Let $K^* = \overline{AB} \cap \overline{LN}$ and $L^* = \overline{BC} \cap \overline{KM}$. From the angle bisectors we have $-1 = (AB; KK^*) = (CB; LL^*)$, so $\overline{AC}$, $\overline{KL}$, $\overline{K^*L^*}$ concur at a point F. But

$$-1 = (AB; KK^*) \stackrel{L^*}{=} (AC; EF).$$

The claim readily follows by symmetry. □



Claim 2. $E = \overline{K'M'} \cap \overline{L'N'}$.

Proof. Brianchon theorem on $AK'BCM'D$, $ABL'CDN'$. $\square$

Claim 3. $\overline{K'L'} \cap \overline{M'N'}$ is the harmonic conjugate of E wrt. $\overline{AC}$. (In particular, it lies on $\overline{AC}$.)

Proof. By Pascal theorem on $K'K'L'N'N'M'$, the point $F = \overline{K'L'} \cap \overline{M'N'}$ lies on $\overline{AEC}$. Also let $G = \overline{K'N'} \cap \overline{L'M'}$. Note that

$$-1 = G(K'L'; F'E') = (AC; EF),$$

where the second equality is by taking the dual wrt. $(K'L'M'N')$. $\square$

Remark. By Pascal theorem on $K'K'L'M'M'N'$ and $K'L'L'M'N'N'$, points X, Y lie on $\overline{FG}$.

Claim 4. $K = K'$, $L = L'$, $M = M'$, $N = N'$.

Proof. We have proven via Claims 2 and 3 that $(KLMN)$ and $(K'L'M'N')$ coincide as the polar circle of $\triangle EXY$. But $(K'L'M'N')$ is tangent to each side of $ABCD$, so $K = K'$ is unique, etc. $\square$

Finally, $\overline{KM} \cap \overline{LN}$ readily implies $ABCD$ is bicentric; to spell it out,

$$\angle BAD = 180^\circ - \widehat{NK} = \widehat{LM} = 180^\circ - \angle DCB,$$

as needed.

§A A digression on the projective plane

§A.1 Formal definition

The definitions presented in §1 are probably not satisfactory, so let's redo it, better this time. The idea is to "homogenize" $\mathbb{R}^2$, by adding a third, scalable coordinate.

Definition A.1 (Real projective plane)
Formally, we define
$$\mathbb{RP}^2 = \left(\mathbb{R}^3 \setminus \{(0,0,0)\}\right) / \sim,$$
where $(a,b,c) \sim (\lambda a, \lambda b, \lambda c)$ for all λ.

Basically what I'm saying is, (a,b,c) and $(2a,2b,2c)$ are the same point; we write $(a:b:c)$ for these homogeneous coordinates.

§A.2 Algebraic detail

Why is this similar to $\mathbb{R}^2$ at all? Consider the map $\mathbb{RP}^2 \to \mathbb{R}^2$ by $(x:y:z) \mapsto (x/z, y/z)$. This map is injective, but only defined when z is nonzero. So $\mathbb{RP}^2$ looks just like $\mathbb{R}^2$, except for some extra points of the form $(x:y:0)$. Clearly these correspond to our "points at infinity," and they lie on the line at infinity.

§A.3 Geometric detail

So why does $\mathbb{RP}^2$ have any notion of geometry? The key is that $\mathbb{RP}^2$ can be thought of as the set of lines through the origin in $\mathbb{R}^3$. For example, $(1:1:1)$ corresponds to all points satisfying $x=y=z$.

Then, we project all these lines onto the plane $z=1$. This is the map $(x:y:z) \mapsto (x/z, y/z)$. Furthermore, rays with $z=0$, i.e. parallel to the plane $z=1$, intersect $z=1$ infinitely far away.

We can also project onto any other plane (not through the origin). Thus, the line at infinity isn't special at all! Any line can be the line at infinity by choosing a suitable plane, i.e. applying a suitable projective transformation.

§A.4 Point-line duality

In $\mathbb{R}^3$, a plane through the origin has equation $ax+by+cz=0$. It stands to reason this is also the equation of a line in $\mathbb{RP}^2$. In particular, lines may be expressed as homogeneous coordinates $(a:b:c)$ as well!

With vectors, we might denote points as $\mathbf{x} = (x:y:z)$ and lines as $\ell = (a:b:c)$. We have the following:

Proposition A.2 (Incidence)
The point $\mathbf{x}$ lies on the line ℓ if and only if $\mathbf{x} \cdot \ell = 0$.

The above basically means: $\mathbf{x}$ lies on ℓ if and only if the vectors $\mathbf{x}$, ℓ are orthogonal. In particular, the cross product $u \times v$ gives the vector orthogonal to both u, v, so

Proposition A.3 (Intersection)
Let $\mathbf{x}$, $\mathbf{y}$ be elements of $\mathbb{RP}^2$. Then

- $\mathbf{x} \times \mathbf{y}$ is the line through points $\mathbf{x}$, $\mathbf{y}$;
- $\mathbf{x} \times \mathbf{y}$ is the intersection of lines $\mathbf{x}$, $\mathbf{y}$.

Note the obvious symmetry in points/lines. Hence the following thought ensues:

If you can prove a purely projective statement about points, you can also prove it about lines, and vice versa.

Here is an example of duality — recall this theorem:

Theorem A.4 (Desargues' theorem)
Let ABC, XYZ be triangles. Then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ concur if and only if $\overline{BC} \cap \overline{YZ}$, $\overline{CA} \cap \overline{ZX}$, $\overline{AB} \cap \overline{XY}$ are collinear.

I bet you can already smell the point-line duality in this one — the theorem itself is self-dual. To spell it out completely algebraically, consider the following two interpretations of the "only if" statement:

- Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be elements of $\mathbb{RP}^2$ referring to points A, B, C, X, Y, Z. If

$$(\mathbf{a} \times \mathbf{x}) \cdot [(\mathbf{b} \times \mathbf{y}) \times (\mathbf{c} \times \mathbf{z})] = 0,$$

  then we also have

$$((\mathbf{b} \times \mathbf{c}) \times (\mathbf{y} \times \mathbf{z})) \cdot [((\mathbf{c} \times \mathbf{a}) \times (\mathbf{z} \times \mathbf{x})) \times ((\mathbf{a} \times \mathbf{b}) \times (\mathbf{x} \times \mathbf{y}))] = 0.$$

- Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be elements of $\mathbb{RP}^2$ referring to lines BC, CA, AB, YZ, ZX, XY. If

$$((\mathbf{b} \times \mathbf{c}) \times (\mathbf{y} \times \mathbf{z})) \cdot [((\mathbf{c} \times \mathbf{a}) \times (\mathbf{z} \times \mathbf{x})) \times ((\mathbf{a} \times \mathbf{b}) \times (\mathbf{x} \times \mathbf{y}))] = 0.$$

  then we also have

$$(\mathbf{a} \times \mathbf{x}) \cdot [(\mathbf{b} \times \mathbf{y}) \times (\mathbf{c} \times \mathbf{z})] = 0,$$

Hence each direction of Desargues' theorem actually proves the other direction as well.

§A.5 Polarity in conics

Remark. See Vincent Huang's blog post for more details:
https://artofproblemsolving.com/community/c2591h1740237.

The following result from linear algebra usually helps when using projective coordinates. We'll be brief, so we won't actually use it here.

Lemma A.5 (Lagrange / triple product expansion)

For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, we have $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Algebraic curves in $\mathbb{RP}^2$ are homogeneous polynomial equations in x, y, z. For example, the unit circle is $x^2 + y^2 = z^2$. In particular, general conics $\mathscr{C}$ are given by

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0.$$

Here we will consider the matrix

$$M = \begin{bmatrix} 2A & F & E \\ F & 2B & D \\ E & D & 2C \end{bmatrix}.$$

Proposition A.6

The conic $\mathscr{C}$ is the set of points $\mathbf{x}$ with $\mathbf{x}^\mathsf{T} M \mathbf{x} = 0$.

Proof of Proposition A.6. Expansion: it turns out

$$\mathbf{x}^\mathsf{T} M \mathbf{x} = 2\left(Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy\right).$$

□

Lemma A.7

M is invertible if and only if $\mathscr{C}$ isn't degenerate.

Proof of Lemma A.7. Note that for $\mathbf{x} \in \mathscr{C}$, the product $M\mathbf{x}$ denotes the partial derivatives of $\mathscr{C}$ at $\mathbf{x}$. If M isn't invertible, then $M\mathbf{x} = 0$ for some $\mathbf{x} \neq 0$, so the partials at $\mathbf{x}$ are all zero. If $\mathscr{C}$ is nondegenerate, then at no point should the partials all be zero. □

Proposition A.8

For any point $\mathbf{x}$, the polar of $\mathbf{x}$ wrt. $\mathscr{C}$ is given by $M\mathbf{x}$.

First, this makes sense, since the pole of a line $\mathbf{x}$ would then be given by $M^{-1}\mathbf{x}$. The pole is not well-defined when $\mathscr{C}$ is degenerate — precisely when polarity is not well-defined either.

Proof of Proposition A.8. By the partials argument above, $M\mathbf{x}$ is the tangent to $\mathscr{C}$ at $\mathbf{x}$ for any $\mathbf{x} \in \mathscr{C}$. Since multiplication by matrices represent projective transformations, this uniquely defines M as the pole-polar transformation. □

As an example of projective coordinates, we can prove La Hire's theorem:

Theorem A.9 (La Hire)

Let P, Q be points and $\mathscr{C}$ a conic. Then P lies on the polar of Q if and only if Q lies on the polar of P.

Proof. We want to show $\mathbf{p} \cdot M\mathbf{q} = 0$ if and only if $\mathbf{q} \cdot M\mathbf{p} = 0$. This is obvious by $M = M^\mathsf{T}$. □