Linearity of Power of a Point

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In this article we present a powerful technique to solve olympiad geometry problems - linearity of power of a point. After introducing theory, we present applications to several problems which have appeared on various contests.

1 Introduction

Many olympiad geometry problems can be reduced to computing the power of a point with respect to some circle. For example, showing that a point lies on a circle is equivalent to showing it has power zero. *Linearity of Power of a Point* gives us an efficient method of computing this power.

We first introduce the theory of linearity of power of a point which is very minimal. There are not many formal prerequisites; however, the reader should be familiar with power of a point.

To show the full power of the technique, we will employ barycentric coordinates. It should be noted that the reader does not need extensive prior knowledge of barycentric coordinates. The reader may also choose to simply ignore any discussion mentioning barycentric coordinates.

2 Theory

We define the power of a point P w.r.t. a circle ω as $\mathbb{P}(P,\omega) = PO^2 - r^2$, where O and r denote the center and radius of ω. Here is the primary result we are concerned with.

Theorem 1 (Linearity of Power of a Point)
Let $\mathbb{P}(P,\omega_1,\omega_2) = \mathbb{P}(P,\omega_1) - \mathbb{P}(P,\omega_2)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(P) := \mathbb{P}(P,\omega_1,\omega_2)$. Then F is linear.

Proof. It suffices to show that for points A and B, that

$$\mathbb{P}(C,\omega_1,\omega_2) = k\mathbb{P}(A,\omega_1,\omega_2) + (1-k)\mathbb{P}(B,\omega_1,\omega_2)$$

where $C = kA + (1-k)B$. Let O_1 and O_2 be the centers of ω_1 and ω_2 respectively. Without loss of generality, let $AB = 1$. Then C is the point on line AB such that $AC = 1-k$ and $BC = k$. Applying Stewart's theorem to $\triangle O_1AB$, we see that

$$kAO_1^2 + (1-k)BO_1^2 = k(1-k) + CO_1^2.$$

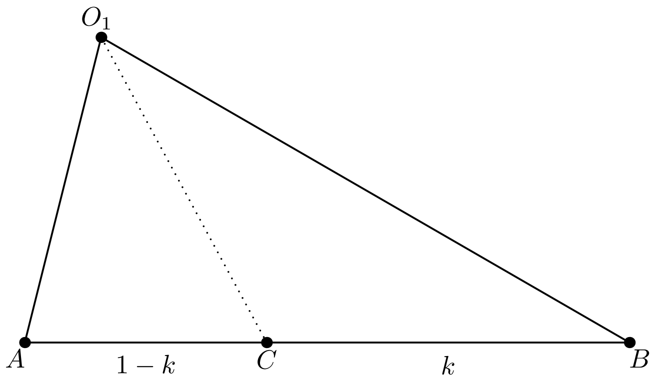

Figure 1: Using Stewart's Theorem

Similarly applying Stewart's to $\triangle O_2AB$, we have

$$kAO_2^2 + (1-k)BO_2^2 = k(1-k) + CO_2^2.$$

Subtracting these two equalities yields

$$k(AO_1^2 - AO_2^2) + (1-k)(BO_1^2 + BO_2^2) = CO_1^2 - CO_2^2.$$

Adding $-r_1^2 + r_2^2$ to each sides results in

$$k(AO_1^2 - r_1^2 - AO_2^2 + r_2^2) + (1-k)(BO_1^2 - r_1^2 + BO_2^2 + r_2^2) = CO_1^2 - r_1^2 - CO_2^2 + r_2^2$$

as desired. □

3 Standard Examples

We present several examples.

3.1 ELMO Shortlist 2013/G3

Example 2 (ELMO Shortlist 2013/G3)
In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

We let M denote the midpoint of BC. We wish to show that $\odot(AEF)$ always passes through some fixed point P on AM. The idea is to show that M has a fixed power w.r.t. $\odot(AEF)$. Linearity of power of a point gives us a simple way to compute this value. The fact that $\vec{M} = \frac{1}{2}\vec{B} + \frac{1}{2}\vec{C}$ further motivates linearity.

Proof. Let ω and ω_1 denote the circumcircles of $\triangle ABC$ and $\triangle AEF$. We compute

$$\begin{aligned}\mathbb{P}(M, \omega_1, \omega) &= \frac{1}{2}(\mathbb{P}(B, \omega_1, \omega) + \mathbb{P}(C, \omega_1, \omega)) \\ &= \frac{1}{2}(BE \cdot BA - 0 + CF \cdot CA - 0) \\ &= \frac{1}{2}(BD \cdot BC + CD \cdot BC) \\ &= \frac{BC^2}{2}.\end{aligned}$$

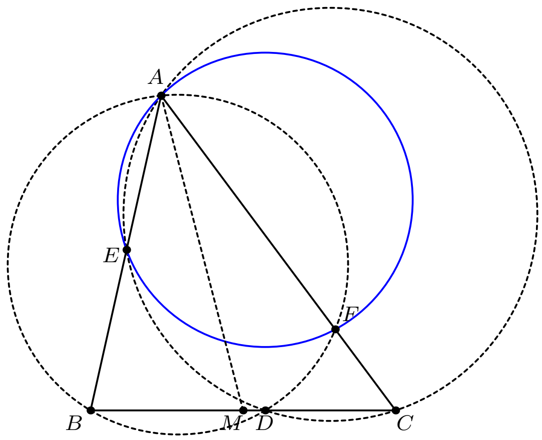

Figure 2: ELMO Shortlist 2013/G3

So

$$\frac{BC^2}{4} - \mathbb{P}(M, \omega_1) = \mathbb{P}(M, \omega_1, \omega) = \frac{BC^2}{2}.$$

Therefore, $\mathbb{P}(M, \omega_1)$ is constant, so ω_1 passes through a fixed point on $\triangle AEF$. $\square$

Generally, we like to choose the circumcircle of a triangle, since it evokes friendly powers. Here, M, B, C all had nice powers. In particular, $\mathbb{P}(B, \omega) = \mathbb{P}(C, \omega) = 0$!

3.2 USAMO 2013/1

Example 3 (USAMO 2013/1)

In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

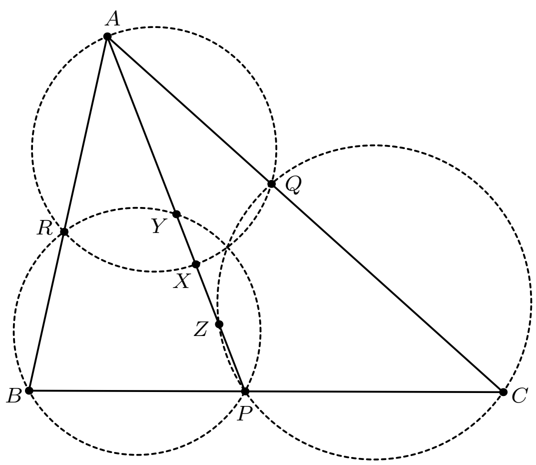

Figure 3: USAMO 2013/1

Proof. Let $BP/PC = \lambda$. Then we let A denote the circle of radius 0.

$$\begin{aligned} PA \cdot AX = \mathbb{P}(P, A, \omega_A) &= \lambda\mathbb{P}(C, A, \omega_A) + (1-\lambda)\mathbb{P}(B, A, \omega_A) \\ &= \lambda(CQ \cdot CQ - CA^2) + \lambda(BR \cdot BQ - BQ^2) \\ &= \lambda(CA \cdot AQ) + (1-\lambda)(BA \cdot AR) \\ &= \lambda(CA \cdot AQ - BA \cdot AR) + BA \cdot AR \end{aligned}$$

So therefore

$$\begin{aligned} \lambda &= \frac{PA \cdot AX - BA \cdot AR}{CA \cdot AQ - BA \cdot AR} \\ &= \frac{PA \cdot AX - AY \cdot AP}{AX \cdot AP - AY \cdot AP} \\ &= \frac{AX - AY}{AX - AY} = \frac{XY}{YZ} \end{aligned}$$

as desired. □

3.3 USAMO 2015/2, USAJMO 2015/3

The following problem managed to nearly sweep the USAJMO group. However, after guessing the circle (by drawing a good diagram), the problem is not hard with linearity of power of a point.

Example 4 (USAMO 2015/2, USAJMO 2015/3)

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

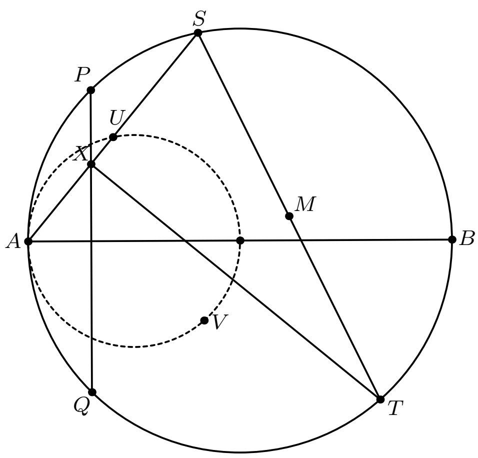

Figure 4: USAMO 2015/2, USAJMO 2015/3

Proof. (Due to Kapil Chandran) Let U, V be the midpoints of AS, AT, respectively, and let γ be the circle of diameter AO passing through A, O, U, V.

Then since $M = \frac{1}{2} \cdot S + \frac{1}{2} \cdot T$, we obtain

$$\mathbb{P}(M, \gamma) - MS \cdot MT = \mathbb{P}(M, \gamma, \omega) = \left(\frac{1}{2}\right) \mathbb{P}(S, \gamma, \omega) + \left(\frac{1}{2}\right) \mathbb{P}(T, \gamma, \omega).$$

It follows that $\mathbb{P}(M, \gamma) = \frac{1}{4}\left(AS^2 + AT^2 - ST^2\right)$. But from the Law of Cosines, we have

$$AS^2 + AT^2 - ST^2 = 2 \cdot AS \cdot AT \cos \angle SAT = 2 \cdot AS \cdot AX,$$

where the last step follows from examining right triangle $\triangle AXT$. Now notice that $\triangle APX \sim \triangle ASP \implies AS \cdot AX = AP$. It follows that $\mathbb{P}(M, \gamma)$ is fixed. Thus if O_1 is the center of γ (the midpoint of $\overline{AO}$), then from the Power of a Point Formula, we see that MO_1^2 is fixed. Thus, M lies on a fixed circle centered at O_1. □

4 Barycentric Coordinates

4.1 USA December TST 2012/1

This is a good example of the use of barycentric coordinates in conjunction with linearity of power of a point.

Example 5 (USA December TST 2012/1)
In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

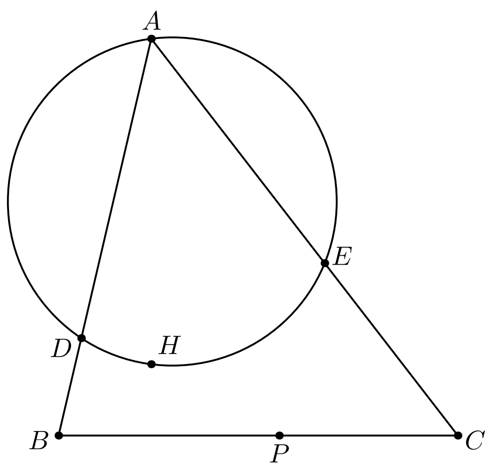

Figure 5: USA December TST 2012/1

Proof. (Due to Kapil Chandran) Let M, N be the projections of P onto AB, AC, respectively, and denote Γ and ω as the circumcircles of $\triangle ABC$ and $\triangle ADE$. We will prove that as P varies, ω always passes through H, the orthocenter of $\triangle ABC$.

Recall that $H = \frac{1}{\tan A \tan B \tan C} (\tan A : \tan B : \tan C)$ in barycentric coordinates. So we write

$$\mathbb{P}(H,\omega,\Gamma) = \frac{\tan A \cdot \mathbb{P}(A,\omega,\Gamma) + \tan B \cdot \mathbb{P}(B,\omega,\Gamma) + \tan C \cdot \mathbb{P}(C,\omega,\Gamma)}{\tan A \tan B \tan C}$$

This allows us to compute

$$\begin{aligned}\tan A \tan B \tan C \cdot \mathbb{P}(H,\omega,\Gamma) &= \tan B(BD \cdot BA) + \tan C(CE \cdot CA) \\ &= \frac{PM}{BM} \cdot BD \cdot BA + \frac{PN}{CN} \cdot CE \cdot CA \\ &= 2 \cdot PM \cdot BA + 2 \cdot PN \cdot CA \\ &= 4[ABP] + 4[ACP] = 4[ABC],\end{aligned}$$

which is fixed. It follows that $\mathbb{P}(H,\omega)$ is fixed as well. Thus, it suffices to show that $\mathbb{P}(H,\omega) = 0$ for one choice of P.

When P is the midpoint of $\overline{BC}$, we see that D, E are the projections of B, C onto AB, AC, respectively. Thus, ω is in fact the circle of diameter $\overline{AH}$, implying that $\mathbb{P}(H,\omega) = 0$. It follows that $\mathbb{P}(H,\omega) = 0$ for any P, i.e. $H \in \omega$. □

4.2 IMO Shortlist 2012/G6

Example 6 (IMO Shortlist 2012/G6)

Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

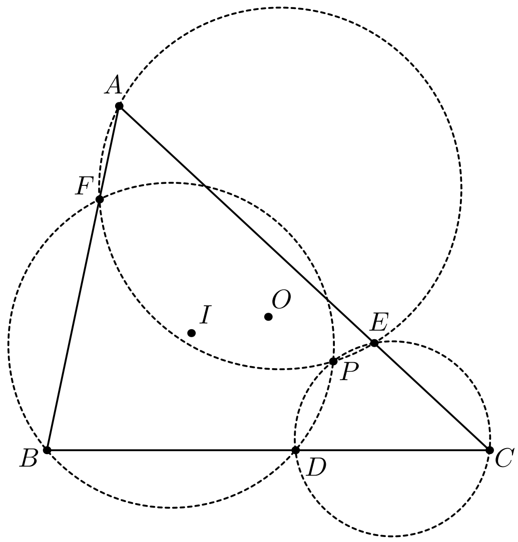

Figure 6: IMO Shortlist 2012/G6

Proof. (Due to AoPS user math154) Let $\omega, \omega_A, \omega_B, \omega_C$ denote the circumcircles of $\triangle ABC, \triangle AEF, \triangle BFD, \triangle CDE$ respectively. We wish to show that $\mathbb{P}(P,\omega) = \mathbb{P}(I,\omega)$. First notice that $AE + AF = BC$ and that P lies on $\omega_a, \omega_b, \omega_c$ by Miquel's theorem. We invoke barycentric coordinates and linearity of power of a point.

Let $P = (x, y, z)$. Then

$$\begin{aligned} 0 - \mathbb{P}(P, \omega) = \mathbb{P}(P, \omega_A, \omega) &= x\mathbb{P}(A, \omega_A, \omega) + y\mathbb{P}(B, \omega_A, \omega) + z\mathbb{P}(C, \omega_A, \omega) \\ &= y(BF \cdot BA) + z(CE \cdot CA) \end{aligned}$$

Taking cyclic sums, we have that

$$\begin{aligned} -\mathbb{P}(P, \omega) \sum_{\text{cyc}} a &= a(yc \cdot BF + zb \cdot CE) + b(za \cdot CD + xc \cdot AF) + c(xb \cdot AE + ya \cdot BD) \\ &= xbc(AE + AF) + yca(BF + BD) + zab(CD + CE) \\ &= abc(x + y + z) \end{aligned}$$

Therefore, $\mathbb{P}(P, \omega) = \frac{abc}{a+b+c} = -2Rr = OI^2 - R^2 = \mathbb{P}(I, \omega)$ as desired. $\square$

5 Problems

Problems are sorted in roughly increasing difficulty. Each solution uses linearity of power of a point in a key step. The problems below do not require barycentric coordinates.

1. In triangle ABC points E and F lie on sides AC and BC such that segments AE and BF have equal length. Let $\odot(ACF)$ and $\odot(BCE)$ intersect for a second time at D. Prove that CD bisects $\angle ACB$.

2. (USAJMO 2012/1) Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

3. (2016 Taiwan TST) Let O be the circumcenter of triangle ABC, and ω be the circumcircle of triangle BOC. Line AO intersects with circle ω again at the point G. Let M be the midpoint of side BC, and the perpendicular bisector of BC meets circle ω at the points O and N. Prove that the midpoint of the segment AN lies on the radical axis of the circumcircle of triangle OMG, and the circle whose diameter is AO.

4. Given triangle ABC inscribed in $\odot(O)$. Let M be the midpoint of BC, H be the projection of A onto BC. OH meets AM at P. Prove that P lies on the radical axis of $\odot(BOC)$ and the nine-point circle of triangle ABC.

5. (USA TSTST 2015/2) Let $\triangle ABC$ be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of $\triangle ABC$.

6. Let Γ and ω be two circles tangent at a point A. Let B, C be points on Γ and draw tangents BY, CZ to ω. Prove that $\frac{AB}{AC} = \frac{BY}{CZ}$.

5.1 Barycentric Coordinates

These problems make use of linearity of power of a point in conjunction with barycentric coordinates.

1. (AoPS user ABCDE) Let ABC be a triangle and let D be the foot of the altitude from A to BC. Let X and Y be points on AB and AC different from A with $DX = DY = DA$. Let the circumcircle of $BCXY$ be ω_A, and define ω_B and ω_C similarly. Prove that the radical center of ω_A, ω_B, and ω_C is the symmedian point of ABC.

2. (2010 Romania JBMO TST) Let ABC be a triangle with incenter I and circumcenter O. Prove that the radical axis of the reflection of the circle with diameter IB over IC and the reflection of the circle with diameter IC over IB passes through O.

3. (IMO Shortlist 2004/G7) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.