Analysis

Xiaolong Pan

December 14, 2023

Contents

Chapter 1

Vector Fields and Flows

1.1 Vector fields

Vector fields are familiar objects of study in multivariable calculus. In that setting, a vector field on an open subset $U \subseteq \mathbb{R}^n$ is simply a continuous map from U to $\mathbb{R}^n$, which can be visualized as attaching an arrow to each point of U. In this section we show how to extend this idea to smooth manifolds.

1.1.1 Vector fields on manifolds

If M is a smooth manifold with or without boundary, a vector field on M is a section of the map $\pi : TM \to M$. More concretely, a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \mathrm{id}_M$$

or equivalently, $X_p \in T_pM$ for each $p \in M$.

We are primarily interested in **smooth vector fields**, the ones that are smooth as maps from M to TM, when TM is given the smoothmanifold structure described in Proposition **??**. In addition, for some purposes it is useful to consider maps from M to TM that would be vector fields except that they might not be continuous. A **rough vector field** on M is a (not necessarily continuous) map $X : M \to TM$ satisfying $X_p \in T_pM$. Just as for functions, if X is a vector field on M, the **support** of X is defined to be the closure of the set $\{p \in M : X_p \neq 0\}$. A vector field is said to be **compactly supported** if its support is a compact set.

Suppose M is a smooth n-manifold (with or without boundary). If $X : M \to TM$ is a rough vector field and $(U, (x^i))$ is any smooth coordinate chart for M, we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

This defines n functions $X^i : U \to \mathbb{R}$, called the **component functions** of X in the given chart.

Proposition 1.1.1.1 (Smoothness Criterion for Vector Fields). *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.*

Proof. Let (x^i, v^i) be the natural coordinates on $\pi^{-1}(U) \subseteq TM$ associated with the chart $(U, (x^i))$. By definition of natural coordinates, the coordinate representation of $X : M \to TM$ on U is

$$\widehat{X} = (x^1, \dots, x^n, X^1(x), \dots, X^n(x))$$

where X^i is the i-th component function of X in x^i-coordinates. It follows immediately that smoothness of X in U is equivalent to smoothness of its component functions. □

Example 1.1.1.2 (Coordinate Vector Fields). If $(U, (x^i))$ is any smooth chart on M, the assignment

$$p \mapsto \frac{\partial}{\partial x^i}\Big|_p$$

determines a vector field on U, called the i-th coordinate vector field and denoted by $\partial/\partial x^i$. It is smooth because its component functions are constants.

Example 1.1.1.3 (The Euler Vector Field). The vector field V on $\mathbb{R}^n$ whose value at $x \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x^1}\Big|_x + \cdots + x^n \frac{\partial}{\partial x^n}\Big|_x$$

is smooth because its coordinate functions are linear. It vanishes at the origin, and points radially outward everywhere else. It is called the Euler vector field because of its appearance in Euler's homogeneous function theorem

Example 1.1.1.4. Let θ be any angle coordinate on a proper open subset $U \subseteq S^1$, and let $d/d\theta$ denote the corresponding coordinate vector field. Because any other angle coordinate $\widetilde{\theta}$ differs from θ by an additive constant in a neighborhood of each point, the transformation law for coordinate vector fields (**??**) shows that $d/d\theta = d/d\widetilde{\theta}$ on their common domain. For this reason, there is a globally defined vector field on S^1 whose coordinate representation is $d/d\theta$ with respect to any angle coordinate. It is a smooth vector field because its component function is constant in any such chart. We denote this global vector field by $d/d\theta$, even though, strictly speaking, it cannot be considered as a coordinate vector field on the entire circle at once.

Example 1.1.1.5 (Angle Coordinate Vector Fields on Tori). On the n-dimensional torus T^n, choosing an angle function θ^i for the i-th circle factor, yields local coordinates $(\theta^1, \dots, \theta^i)$ for T^n. An analysis similar to that of the previous example shows that the coordinate vector fields $d/d\theta^1, \dots, d/d\theta^n$ are smooth and globally defined on T^n.

If $U \subseteq M$ is open, the fact that T_pU is naturally identified with T_pM for each $p \in U$ (Proposition **??**) allows us to identify TU with the open subset $\pi^{-1}(U) \subseteq TM$. Therefore, a vector field on U can be thought of either as a map from U to TU or as a map from U to TM, whichever is more convenient. If X is a vector field on M, its restriction $X|_U$ is a vector field on U, which is smooth if X is.

The next lemma is a generalization of Lemma **??** to vector fields, and is proved in much the same way. If M is a smooth manifold with or without boundary and $A \subseteq M$ is an arbitrary subset, a **vector field along A** is a continuous map $X : A \to TM$ satisfying $\pi \circ X = \mathrm{id}_A$ (or in other words $X_p \in T_pM$ for each $p \in A$). We call it a **smooth vector field along A** if for each $p \in A$, there is a neighborhood V of p in M and a smooth vector field $\widetilde{X}$ on V that agrees with X on $V \cap A$.

Lemma 1.1.1.6 (Extension Lemma for Vector Fields). *Let M be a smooth manifold with or without boundary.*

(a) Let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\operatorname{supp}(\widetilde{X}) \subseteq U$.

(b) Let $S \subseteq M$ is an embedded submanifold with or without boundary. Given $\mathfrak{X} \in \mathfrak{X}(S)$, then there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Every such vector field extends to all of M if and only if S is properly embedded.

Proof. Note the if $\psi \in C^\infty(M)$ and X is a vector field of M, then fX is also a vector field of M, since $f(p)X_p \in T_pM$ for all $p \in M$. This suggests that we can apply the proof of Lemma **??**.

The proof for the second part is similar to Lemma **??**. □

As an important special case, any vector at a point can be extended to a smooth vector field on the entire manifold.

Proposition 1.1.1.7. *Let M be a smooth manifold with or without boundary. Given $p \in M$ and $v \in T_pM$, there is a smooth global vector field X on M such that $X_p = v$.*

Proof. The assignment $p \mapsto v$ is an example of a vector field along the set $\{p\}$ as defined above. It is smooth because it can be extended, say, to a constant-coefficient vector field in a coordinate neighborhood of p. Thus, the proposition follows from the extension lemma with $A = \{p\}$ and $U = M$. □

If M is a smooth manifold with or without boundary, it is standard to use the notation $\mathfrak{X}(M)$ to denote the set of all smooth vector fields on M. It is a vector space under pointwise addition and scalar multiplication:

$$(aX + bY)_p = aX_p + bY_p$$

The zero element of this vector space is the zero vector field, whose value at each $p \in M$ is $0 \in T_pM$. In addition, smooth vector fields can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX : M \to TM$ by

$$(fX)_p = f(p)X_p$$

Proposition 1.1.1.8. *Let M be a smooth manifold with or without boundary.*

(a) If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.

(b) $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

For example, the basis expression for a vector field X can also be written as an equation between vector fields instead of an equation between vectors at a point:

$$X = X^i \frac{\partial}{\partial x^i}$$

where X^i is the i-th component function of X in the given coordinates.

1.1.1.1 Local and Global Frames

Suppose M is a smooth n-manifold with or without boundary. An ordered k-tuple $(X_1, \ldots, X_k)$ of vector fields defined on some subset $A \subseteq M$ is said to be **linearly independent** if $(X_1|_p, \ldots, X_k|_p)$ is a linearly independent k-tuple in T_pM for each $p \in A$, and is said to **span the tangent bundle** if the k-tuple spans T_pM at each $p \in A$. A local frame for M is an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ defined on an open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle; thus the vectors $(E_1|_p, \ldots, E_n|_p)$ form a basis for T_pM at each $p \in U$. It is called a **global frame** if $U = M$, and a smooth frame if each of the vector fields E_i is smooth. We often use the shorthand notation (E_i) to denote a frame $(E_1, \ldots, E_n)$. If M has dimension n, then to check that an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ is a local frame, it suffices to check either that it is linearly independent or that it spans the tangent bundle.

Example 1.1.1.9 (Local and Global Frames).

(a) The standard coordinate vector fields form a smooth global frame for $\mathbb{R}^n$.

(b) If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M (possibly with boundary), then the coordinate vector fields form a smooth local frame $(\partial/\partial x^i)$ on U, called a **coordinate frame**. Every point of M is in the domain of such a local frame.

(c) The vector field $d/d\theta$ defined in Example 1.1.1.4 constitutes a smooth global frame for the circle.

The next proposition shows that local frames are easy to come by.

Proposition 1.1.1.10 (Completion of Local Frames). *Let M be a smooth n-manifold with or without boundary.*

(a) If $(v_1, \ldots, v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p \in M$ with $1 \le k \le n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i|_p = v_i$ for all i.

(b) If $(X_1, \ldots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an open subset $U \subseteq M$ with $1 \le k < n$, then for each $p \in U$ there exist smooth vector fields $X_{k+1}, \ldots, X_n$ in a neighborhood V of p such that $X_1, \ldots, X_n$ is a smooth local frame for M on $U \cap V$.

(c) If $(X_1, \ldots, X_n)$ is a linearly independent n-tuple of smooth vector fields along a closed subset $A \subseteq M$ then there exists a smooth local frame $(\widetilde{X}_1, \ldots, \widetilde{X}_n)$ on some neighborhood of A such that $\widetilde{X}_i|_A = X_i$ for all i.

Proof. Part (a) is immediate, for example, we can choose a chart (U, φ) for p and define X_i to be constant: $X_i|_p = v^i \partial/\partial x^i$.

For (b), let (U, φ) be a chart at p, define $X_{k+1}, \ldots, X_n$ to be constant on U such that $X_1|_p, \ldots, X_n|_p$ is linearly independent. Since the matrix $[X_i^j(p)]$ is nonsingular at p, there is a neighborhood U of p in which $[X_i^j]$ is invertible, thus (X_i) is a smooth local frame on $U \cap V$.

Finally, if $(X_1, \ldots, X_n)$ is a linearly independent n-tuple, then at each point of A we can extend it into a local frame. Now the claim follows by using a partition of unity. □

For subsets of $\mathbb{R}^n$, there is a special type of frame that is often more useful for geometric problems than arbitrary frames. A k-tuple of vector fields $(E_1, \ldots, E_k)$ defined on some subset $A \subseteq \mathbb{R}^n$ is said to be orthonormal if for each $p \in A$, the vectors $(E_1|_p, \ldots, E_k|_p)$ are orthonormal with respect to the Euclidean dot product (where we identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ in the usual way). A (local or global) frame consisting of orthonormal vector fields is called an **orthonormal frame**.

Example 1.1.1.11. The standard coordinate frame is a global orthonormal frame on $\mathbb{R}^n$. For a less obvious example, consider the smooth vector fields defined on $\mathbb{R}^2 - \{0\}$ by

$$E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \quad E_2 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}. \tag{1.1.1.1}$$

where $r = \sqrt{x^2 + y^2}$. A straightforward computation shows that (E_1, E_2) is an orthonormal frame for $\mathbb{R}^2$ over the open subset $\mathbb{R}^2 - \{0\}$ Geometrically, E_1 and E_2 are unit vector fields tangent to radial lines and circles centered at the origin, respectively.

The next lemma describes a useful method for creating orthonormal frames.

Lemma 1.1.1.12 (Gram-Schmidt Algorithm for Frames). *Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over an open subset $U \subseteq \mathbb{R}^n$. Then there is a smooth orthonormal frame (E_j) over U such that* $\operatorname{span}(X_1|_p, \ldots, X_j|_p) = \operatorname{span}(E_1|_p, \ldots, E_j|_p)$ *for each j and each $p \in U$.*

Proof. Applying the GramSchmidt algorithm to the vectors $(X_j|_p)$ at each $p \in U$, we obtain an n-tuple of rough vector fields $(E_1, \ldots, E_n)$ given inductively by

$$E_j = \frac{X_j - \sum_{i=1}^{j-1}(X_j, E_i)E_i}{|X_j - \sum_{i=1}^{j-1}(X_j, E_i)E_i|}$$

For each j and each $p \in U$ we have $X_{j+1}|_p \notin \operatorname{span}(E_1|_p, \ldots, E_j|_p)$ (which is equal to $\operatorname{span}(X_1|_p, \ldots, X_j|_p)$), so the denominator above is a nowhere-vanishing smooth function on U. Therefore, this formula defines (E_j) as a smooth orthonormal frame on U that satisfies the conclusion of the lemma. □

Although smooth local frames are plentiful, global ones are not. A smooth manifold with or without boundary is said to be **parallelizable** if it admits a smooth global frame. The manifolds $\mathbb{R}^n$, S^1, T^n are all parallelizable, and all Lie groups are parallelizable. We will see later that parallelizability of M is intimately connected to the question of whether its tangent bundle is diffeomorphic to the product $M \times \mathbb{R}^n$.

The simplest example of a nonparallelizable manifold is S^2, but the proof of this fact will have to wait until we have developed more machinery. In fact, using more advanced methods from algebraic topology, it was shown that S^1, S^3, and S^7 are the only spheres that are parallelizable. Thus these are the only positive-dimensional spheres that can possibly admit Lie group structures. The first two do: $S^1 \approx \mathrm{U}(2)$ and $S^3 \approx \mathrm{SU}(2)$. But it turns out that S^7 has no Lie group structure.

1.1.1.2 Vector Fields as Derivations of $C^\infty(M)$

An essential property of vector fields is that they define operators on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf : U \to \mathbb{R}$, defined by

$$(Xf)(p) = X_p f$$

(Be careful not to confuse the notations fX and Xf: the former is the smooth vector field on U obtained by multiplying X by f, while the latter is the real-valued function on U obtained by applying the vector field X to the smooth function f.) Because the action of a tangent vector on a function is determined by the values of the function in an arbitrarily small neighborhood, it follows that Xf is locally determined. In particular, for any open subset $V \subseteq U$,

$$(Xf)|_V = X(f|_V) \tag{1.1.1.2}$$

This construction yields another useful smoothness criterion for vector fields.

Proposition 1.1.1.13. *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. The following are equivalent:*

(a) X is smooth.

(b) For every $f \in C^\infty(M)$, the function Xf is smooth on M.

(c) For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function Xf is smooth on U.

Proof. We will prove that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

To prove $(a) \Rightarrow (b)$, assume X is smooth, and let $f \in C^\infty(M)$. For any $p \in M$, we can choose smooth coordinates (x^i) on a neighborhood U of p. Then for $x \in U$, we can write

$$Xf(x) = \left(X^i(x) \frac{\partial}{\partial x^i}\bigg|_x \right) f = X^i(x) \frac{\partial f}{\partial x^i}(x)$$

Since the component functions X^i are smooth on U by Proposition 1.1.1.1, it follows that Xf is smooth in U. Since the same is true in a neighborhood of each point, Xf is smooth on M.

To prove $(b) \Rightarrow (c)$, suppose $U \subseteq M$ is open and $f \in C^\infty(M)$. For any $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighborhood of p and supported in U, and define $\widetilde{f} = \psi f$, extended to be zero on $M \setminus \operatorname{supp}(\psi)$. Then $X\widetilde{f}$ is smooth by assumption, and is equal to Xf in a neighborhood of p by (1.1.1.2). This shows that Xf is smooth in a neighborhood of each point of U.

Finally, to prove $(c) \Rightarrow (a)$, suppose Xf is smooth whenever f is smooth on an open subset of M. If (x^i) are any smooth local coordinates on $U \subseteq M$, we can think of each coordinate x^i as a smooth function on U. Applying X to one of these functions, we obtain

$$Xx^i = X^j \frac{\partial}{\partial x^j}(x^i) = X^i$$

Because Xx^i is smooth by assumption, it follows that the component functions of X are smooth, so X is smooth. □

One consequence of the preceding proposition is that a smooth vector field $X \in \mathfrak{X}(M)$ defines a map from $C^\infty(M)$ to itself by $f \mapsto Xf$. This map is clearly linear over $\mathbb{R}$. Moreover, the product rule for tangent vectors translates into the following product rule for vector fields:

$$X(fg) = fXg + gXf \tag{1.1.1.3}$$

In general, a map $X : C^\infty(M) \to C^\infty(M)$ is called a **derivation** if it is linear over $\mathbb{R}$ and satisfies (1.1.1.3) for all $f, g \in C^\infty(M)$.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields.

Proposition 1.1.1.14. *Let M be a smooth manifold with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathfrak{X}(M)$.*

Proof. We just showed that every smooth vector field induces a derivation. Conversely, suppose $D : C^\infty(M) \to C^\infty(M)$ is a derivation. We need to concoct a vector field X such that $Df = Xf$ for all f. From the discussion above, it is clear that if there is such a vector field, its value at $p \in M$ must be the derivation at p whose action on any smooth real-valued function f is given by

$$X_p f = (Df)(p)$$

The linearity of D guarantees that this expression depends linearly on f, and the fact that D is a derivation yields the product rule for tangent vectors. Thus, the map $X_p : C^\infty(M) \to \mathbb{R}$ so defined is indeed a tangent vector, that is, a derivation of $C^\infty(M)$ at p. This defines X as a rough vector field. Because $Xf = Df$ is smooth whenever $f^\infty(M)$, this vector field is smooth by Proposition 1.1.1.13. □

Because of this result, we sometimes identify smooth vector fields on M with derivations of $C^\infty(M)$, using the same letter for both the vector field and the derivation.

1.1.2 Vector fields and smooth maps

Suppose $F : M \to N$ is smooth and X is a vector field on M, and suppose there happens to be a vector field Y on N with the property that for each $p \in M$, $dF_p(X_p) = Y_{F(p)}$. In this case, we say the vector fields X and Y are ***F*-related**. The next proposition shows how F-related vector fields act on smooth functions.

Proposition 1.1.2.1. *Suppose $F : M \to N$ is a smooth map between manifolds with or without boundary, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N,*

$$X(F^* f) = F^*(Yf). \tag{1.1.2.1}$$

where $F^* f$ is defined to be $f \circ F$.

Proof. For any $p \in M$ and any smooth real-valued f defined in a neighborhood of $F(p)$,

$$X(F^* f)(p) = X_p(f \circ F) = dF_p(X_p)(f),$$

while

$$(F^*(Yf))(p) = (Yf)(F(p)) = Y_{F(p)} f.$$

Thus, (1.1.2.1) is true for all f if and only if $dF_p X_p = Y_{F(p)}$ for all p, i.e., if and only if X and Y are F-related. □

Example 1.1.2.2. Let $F : \mathbb{R} \to \mathbb{R}^2$ be the smooth map $F(t) = (\cos t, \sin t)$. Then

$$\partial F(t) = (-\cos t, \sin t)$$

Therefore, $d/dt \in \mathfrak{X}(\mathbb{R})$ is F-related to the vector field $Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

It is important to remember that for a given smooth map $F : M \to N$ and vector field $X \in \mathfrak{X}(M)$, there may not be any vector field on N that is F-related to X. There is one special case, however, in which there is always such a vector field, as the next proposition shows.

Proposition 1.1.2.3. *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Proof. For $Y \in \mathfrak{X}(N)$ to be F-related to X means that $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$. If F is a diffeomorphism, therefore, we define Y by

$$Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$$

It is clear that Y, so defined, is the unique (rough) vector field that is F-related to X. Note that $Y : N \to TN$ is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

It follows that Y is smooth. □

In the situation of the preceding proposition we denote the unique vector field that is F-related to X by $F_* X$, and call it the pushforward of X by F. Remember, it is only when F is a diffeomorphism that $F_* X$ is defined. The proof of Proposition 1.1.2.3 shows that $F_* X$ is defined explicitly by the formula

$$(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}) \tag{1.1.2.2}$$

As long as the inverse map F^{-1} can be computed explicitly, the pushforward of a vector field can be computed directly from this formula.

Example 1.1.2.4 (Computing the Pushforward of a Vector Field). Let M and N be the following open submanifolds of $\mathbb{R}^2$:

$$M = \{(x,y) : y > 0 \text{ and } x + y > 0\}$$

$$N = \{(u,v) : u > 0 \text{ and } v > 0\}$$

and define $F : M \to N$ by $F(x,y) = (x + y, x/y + 1)$. Then F is a diffeomorphism because its inverse is easily computed: just solve $(u,v) = (x + y, x/y + 1)$ for x and y to obtain the formula $(x,y) = F^{-1}(u,v) = (u - u/v, u/v)$. Let us compute the pushforward $F_* X$, where X is the following smooth vector field on M:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}$$

The differential of F at a point $(x,y) \in M$ is represented by its Jacobian matrix

$$\partial F(x,y) = \begin{pmatrix} 1 & 1 \\ \dfrac{1}{y} & -\dfrac{x}{y^2} \end{pmatrix}$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$\partial F(u - u/v, u/v) = \begin{pmatrix} 1 & 1 \\ \dfrac{v}{u} & \dfrac{v(1-v)}{u} \end{pmatrix}$$

For any $(u,v) \in N$,

$$X_{F^{-1}(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial x}\bigg|_{F^{-1}(u,v)}$$

Therefore, applying (1.1.2.3) with $p = (u,v)$ yields the formula for $F_* X$:

$$F_* X_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u}\bigg|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v}\bigg|_{(u,v)}$$

The next corollary follows directly from Proposition 1.1.2.3.

Corollary 1.1.2.5. *Suppose $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. For any $f \in C^\infty(N)$,*

$$F^*\big((F_* X) f\big) = X(F^* f).$$

1.1.2.1 Vector Fields and Submanifolds

If $S \subseteq M$ is an immersed or embedded submanifold (with or without boundary), a vector field X on M does not necessarily restrict to a vector field on S, because X_p may not lie in the subspace $T_p S \subseteq T_p M$ at a point $p \in S$. Given a point $p \in S$, a vector field X on M is said to be **tangent to S at p** if $X_p \in T_p S \subseteq T_p M$. It is **tangent to S** if it is tangent to S at every point of S.

Proposition 1.1.2.6. *Let M be a smooth manifold, $S \subseteq M$ be an embedded submanifold with or without boundary, and X be a smooth vector field on M. Then X is tangent to S if and only if $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ such that $f|_S = 0$.*

Proof. This is an immediate consequence of Proposition **??**. □

Suppose $S \subseteq M$ is an immersed submanifold with or without boundary, and Y is a smooth vector field on M. If there is a vector field $X \in \mathfrak{X}(S)$ that is ι-related to Y, where $\iota : S \hookrightarrow M$ is the inclusion map, then clearly Y is tangent to S, because $Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$. The next proposition shows that the converse is true.

Proposition 1.1.2.7 (Restricting Vector Fields to Submanifolds). *Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold with or without boundary, and let $\iota : S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, that is ι-related to Y.*

Proof. The fact that Y is tangent to S means by definition that Y_p is in the image of $d\iota_p$ for each p. Thus, for each p there is a vector $X_p \in T_pS$ such that $Y_p = d\iota_p(X_p)$. Since $d\iota_p$ is injective, X_p is unique, so this defines X as a rough vector field on S. If we can show that X is smooth, it is the unique vector field that is ι-related to Y. It suffices to show that it is smooth in a neighborhood of each point.

Let p be any point in S. Since an immersed submanifold (with or without boundary) is locally embedded, there is a neighborhood V of p in S that is embedded in M. Let $(U, (x^i))$ be a slice chart (or boundary slice chart) for V in M centered at p, so that $V \cap U$ is the subset where $x^{k+1} = \cdots = x^n = 0$ (and $x^k \geq 0$ if $\partial S \neq \varnothing$), and $(x^1, \ldots, x^k)$ form local coordinates for S in $V \cap U$. If $Y = Y^i \partial/\partial x^i$ in these coordinates, it follows from our construction that X has the coordinate representation $Y^1\partial/\partial x^1 + \cdots + Y^k \partial/\partial x^k$, which is clearly smooth on $V \cap U$. □

1.1.3 Lie brackets

In this section we introduce an important way of combining two smooth vector fields to obtain another vector field.

Let X and Y be smooth vector fields on a smooth manifold M. Given a smooth function $f : M \to \mathbb{R}$, we can apply X to f and obtain another smooth function Xf. In turn, we can apply Y to this function, and obtain yet another smooth function $YXf = Y(Xf)$. The operation $f \mapsto YXf$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following example shows.

Example 1.1.3.1. Define vector fields $X = \partial/\partial x$ and $Y = x\partial/\partial y$ on $\mathbb{R}^2$, and let $f(x, y) = x$, $g(x, y) = y$. Then direct computation shows that

$$XY(fg) = 2x, \quad fXYg + gXYf = x$$

so XY is not a derivation of $C^\infty(M)$.

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function XYf. Applying both of these operators to f and subtracting, we obtain an operator $[X, Y] : C^\infty(M) \to C^\infty(M)$, called the **Lie bracket of $\boldsymbol{X}$ and $\boldsymbol{Y}$**, defined by

$$[X, Y]f = XYf - YXf$$

The key fact is that this operator is a vector field.

Lemma 1.1.3.2. *The Lie bracket of any pair of smooth vector fields is a smooth vector field.*

Proof. By Proposition 1.1.1.14, it suffices to show that $[X, Y]$ is a derivation of $C^\infty(M)$. For arbitrary $f, g \in C^\infty(M)$, we compute

$$\begin{aligned}
[X, Y](fg) &= X\big(Y(fg)\big) - Y\big(X(fg)\big) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= X(fYg) + X(gYf) - Y(fXg) - Y(gXf) \\
&= fXYg + YgXf + gXYf + YfXg - fYXg - XgYf - gYXf - XfYg \\
&= fXYg - fYXg + gXYf - gYXf = f[X, Y]g + g[X, Y]f
\end{aligned}$$

□

Remark 1.1.3.3. From the proof above, we can see why we substract the two operators XY and YX:

$$XY(fg) = X(fYg + gYf) = fXYg + gXYf + YgXf + XgYf$$

the terms $fXYg$ and $gXYf$ are order-two operators, which are expected in our product rule of XY. If we can cancle the last two terms, this will turn out to be a derivative. Now the observation that $YgXf + XgYf$ is symmeteric on X and Y lead our definition.

The value of the vector field $[X, Y]f$ at a point $p \in M$ is the derivation at p given by the formula

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf)$$

However, this formula is of limited usefulness for computations, because it requires one to compute terms involving second derivatives of f that will always cancel each other out. The next proposition gives an extremely useful coordinate formula for the Lie bracket, in which the cancellations have already been accounted for.

Proposition 1.1.3.4 (Coordinate Formula for the Lie Bracket). *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \partial/\partial x^i$ and $Y = Y^i \partial/\partial x^i$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{1.1.3.1}$$

or more concisely,

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j} \tag{1.1.3.2}$$

Proof. Because we know already that $[X, Y]$ is a smooth vector field, its action on a function is determined locally: $([X, Y]f)|_U = [X, Y](f|_U)$. Thus it suffices to compute in a single smooth chart, where we have

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j} \right) - Y^j \frac{\partial}{\partial x^j}\left(X^i \frac{\partial f}{\partial x^i} \right) \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \end{aligned}$$

where in the last step we have used the fact that mixed partial derivatives of a smooth function can be taken in any order. Interchanging the roles of the dummy indices i and j in the second term, we obtain (1.1.3.1). □

One trivial application of (1.1.3.1) is to compute the Lie brackets of the coordinate vector fields $(\partial/\partial x^i)$ in any smooth chart: because the component functions of the coordinate vector fields are all constants, it follows that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \quad \text{for all } i, j.$$

(This also follows from the definition of the Lie bracket, and is essentially a restatement of the fact that mixed partial derivatives of smooth functions commute.) Here is a slightly less trivial computation.

Example 1.1.3.5. Define smooth vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ by

$$\begin{aligned} X &= x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}. \\ Y &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}. \end{aligned}$$

Then (1.1.3.2) yields

$$\begin{aligned} [X, Y] &= X(1)\frac{\partial}{\partial x} + X(y)\frac{\partial}{\partial z} - Y(x)\frac{\partial}{\partial x} - Y(1)\frac{\partial}{\partial y} - Y(x(y+1))\frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial z} - \frac{\partial}{\partial x} - (y+1)\frac{\partial}{\partial z} = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \end{aligned}$$

Proposition 1.1.3.6 (Properties of the Lie Bracket). *The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:*

(a) Bilinearity: For $a, b \in \mathbb{R}$,

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z]. \\ [X, aY + bZ] &= a[X, Y] + b[X, Z]. \end{aligned}$$

(b) Antisymmetry:

$$[X, Y] = -[Y, X]$$

(c) Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(*d*) *For* $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X. \tag{1.1.3.3}$$

Proof. Bilinearity and antisymmetry are obvious consequences of the definition. The proof of the Jacobi identity is just a computation

$$\begin{aligned}
&[X,[Y,Z]]f + [Y,[Z,X]]f + [Z,[X,Y]]f \\
&= X[Y,Z]f - [Y,Z]Xf + Y[Z,X]f - [Z,X]Yf + Z[X,Y]f - [X,Y]Zf \\
&= X(YZ - ZY)f - (YZ - ZY)Xf + Y(ZX - XZ)f - (ZX - XZ)Yf \\
&\quad + Z(XY - YX)f - (XY - YX)Zf \\
&= XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf - ZXYf + XZYf \\
&\quad + ZXYf - ZYXf - XYZf + YXZf \\
&= 0
\end{aligned}$$

So is the last equality:

$$\begin{aligned}
[fX, gY] &= fX(gY) - gY(fX) = fgXY + (fXg)Y - gfYX - (gYf)X \\
&= fg[X,Y] + (fXg)Y - (gYf)X
\end{aligned}$$

as needed. □

The significance of part (*d*) of this proposition might not be evident at this point, but it will become clearer in the next section, where we will see that it expresses the fact that the Lie bracket satisfies product rules with respect to both of its arguments.

Proposition 1.1.3.7 (Naturality of the Lie Bracket). *Let* $F : M \to N$ *be a smooth map between manifolds with or without boundary, and let* $X_1, X_2 \in \mathfrak{X}(M)$ *and* $Y_1, Y_2 \in \mathfrak{X}(N)$ *be vector fields such that* X_i *is F-related to* Y_i *for* $i = 1, 2$. *Then* $[X_1, X_2]$ *is F-related to* $[Y_1, Y_2]$.

Proof. Using Proposition 1.1.2.1 and the fact that X_i and Y_i are F-related,

$$X_1X_2(f\circ F) = X_1\big(X_2(f\circ F)\big) = X_1\big((Y_2f)\circ F\big) = (Y_1Y_2f)\circ F$$

Similarly

$$X_2X_1(f\circ F) = (Y_2Y_1f)\circ F$$

Thus

$$[X_1,X_2](f\circ F) = X_1X_2(f\circ F) - X_2X_1(f\circ F) = (Y_1Y_2f)\circ F - (Y_2Y_1f)\circ F = ([Y_1,Y_2]f)\circ F$$

□

When applied in special cases, this result has the following important corollaries. First we consider the case in which the map is a diffeomorphism.

Corollary 1.1.3.8 (Pushforwards of Lie Brackets). *Suppose* $F : M \to N$ *is a diffeomorphism and* $X_1, X_2 \in \mathfrak{X}(M)$. *Then* $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

Proof. This is just the special case of Proposition 1.1.3.7 in which F is a diffeomorphism and $Y_i = F_*X_i$. □

Remark 1.1.3.9. Let $\mathsf{Vec}_\mathbb{R}$ denote the category of real vector spaces and linear maps, and Diff_1 the category of smooth manifolds and diffeomorphisms. Then we have a covariant functor

$$\mathfrak{X} : \mathsf{Diff}_1 \to \mathsf{Vec}_\mathbb{R}, \quad M \to \mathfrak{X}(M), F \mapsto F_*$$

and its product $\mathfrak{X}\times\mathfrak{X}$. Then Corollary 1.1.3.8 shows that the Lie bracket is a natural transformation from $\mathfrak{X}\times\mathfrak{X}$ to $\mathfrak{X}$.

$$\begin{tikzcd}
\mathfrak{X}(M)\times\mathfrak{X}(M) \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "F_*\times F_*"'] & \mathfrak{X}(M) \arrow[d, "F_*"] \\
\mathfrak{X}(N)\times\mathfrak{X}(N) \arrow[r, "{[\cdot,\cdot]}"] & \mathfrak{X}(N)
\end{tikzcd}$$

This is called the **naturality**.

The second special case is that of the inclusion of a submanifold.

Corollary 1.1.3.10 (Brackets of Vector Fields Tangent to Submanifolds). *Let M be a smooth manifold and let S be an immersed submanifold with or without boundary in M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then $[Y_1, Y_2]$ is also tangent to S.*

Proof. By Proposition 1.1.2.7, there exist smooth vector fields X1 and X2 on S such that X_i is ι-related to Y_i for $i = 1, 2$, where $\iota : S \hookrightarrow M$ is the inclusion. By Proposition 1.1.3.7, $[X_1, X_2]$ is ι-related to $[Y_1, Y_2]$, which is therefore tangent to S. □

1.1.4 Exercise

Exercise 1.1.1. Euler's homogeneous function theorem: Let μ be a real number, a smooth function $f : \mathbb{R}^n - \{0\} \to \mathbb{R}$ is said to be **positively homogeneous of degree c** if

$$f(\lambda x) = \lambda^\mu f(x).$$

Prove that f is positively homogeneous of degree c if and only if $Vf = \mu f$, where V is the Euler vector field defined in Example 1.1.1.3.

Proof. Assume that the equality $f(\lambda x) = \lambda^\mu f(x)$, then

$$\frac{d}{d\lambda} f(\lambda x) = \partial f(\lambda x) \frac{d}{d\lambda}(\lambda x) = \left(\frac{\partial f}{\partial x^1}(\lambda x), \dots, \frac{\partial f}{\partial x^n}(\lambda x)\right) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = x^i \frac{\partial f}{\partial x^i}(\lambda x) = \mu \lambda^{\mu-1} f(x)$$

We choose $\lambda = 1$ at both sides, then $Vf = x^i \partial f / \partial x^i = \mu f(x)$.

Conversely, assume $Vf = \mu f$ holds, then we substitute x by λx to get

$$\lambda x^i \frac{\partial f}{\partial x^i}(\lambda x) = \mu f(\lambda x)$$

One note that

$$\frac{df(\lambda x)}{d\lambda} = x_i \frac{\partial f}{\partial x_i}(\lambda x),$$

therefore we have

$$\frac{df(\lambda x)}{d\lambda} = \frac{\mu}{\lambda} f(\lambda x).$$

One can solve this equation for λ to get $f(\lambda x) = \lambda^\mu f(x)$. Thus f is homogeneous of degree μ. □

Exercise 1.1.2. Let M be a smooth manifold with boundary. Show that there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an atlas of M, and let (x_α^i) be the coordinate corresponding to $(U_\alpha, \varphi_\alpha)$. For brevity, let $V = \partial/\partial x_\alpha^n|_{U_\alpha}$. Then V_α is an inward-pointing vector field in $\mathfrak{X}(U_\alpha)$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$. For each $\alpha \in A$, the product $\psi_\alpha V_\alpha$ is a smooth vector field on U_α which has a smooth extension to all of M, if we define it to be zero outside $\operatorname{supp}(\psi_\alpha)$. Then we define $V : M \to TM$ by

$$V = \sum_{\alpha \in A} \psi_\alpha V_\alpha.$$

Observe that if $v_1, \dots, v_k \in T_pM$ are inward, $\lambda_1, \dots, \lambda_k$ are positive, then $\sum_{i=1}^k \lambda_i v_i$ is also inward. Thus V_p is inward-pointing at each point $p \in \partial M$. □

Exercise 1.1.3. Let $\mathbb{H}$ be the algebra of quaternions and let $\mathcal{S} \subseteq \mathbb{H}$ be the group of unit quaternions.

(a) Show that if $p \in \mathbb{H}$ is imaginary, then $X_q := q \cdot p$ is tangent to $\mathcal{S}$ at each $q \in \mathcal{S}$.

(b) Define vector fields X_1, X_2, X_3 on $\mathbb{H}$ by

$$X_1|_q = q\mathbf{i}, \quad X_2|_q = q\mathbf{j} \quad X_3|_q = q\mathbf{k}$$

Show that these vector fields restrict to a smooth left-invariant global frame on $\mathcal{S}$.

(c) Under the isomorphism $\mathbb{R}^4 \cong \mathbb{H}$, show that these vector fields have the following coordinate representations:

$$X_1 = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4}$$
$$X_2 = -x^3\frac{\partial}{\partial x^1} - x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial x^4}$$
$$X_3 = -x^4\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial x^4}$$

Proof. The tangent space of $\mathcal{S}$ can be identified with

$$T_q\mathcal{S} = \{p \in \mathbb{H} : (p,q) = 0\}$$

Also, from the condition on p, we have

$$(pq,q) = pqq^* + q(pq)^* = pqq^* + qq^*p^* = |q|(p+p^*) = 0$$

Thus qp is tangent to $\mathcal{S}$.

Now part (b) is a consequence of (a), since $\mathbf{i},\mathbf{j},\mathbf{k}$ are imaginary, and they are left-invariant on $\mathbb{H}$: the map L_p is linear, so $d(L_p) = p$, and

$$d(L_p)_e X_e = p \cdot X_e$$

Finally, for the coordinates we compute:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\mathbf{i} = a\mathbf{i} - b - c\mathbf{k} + d\mathbf{j} = -b + a\mathbf{i} + d\mathbf{j} - c\mathbf{k}$$
$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\mathbf{j} = a\mathbf{j} + b\mathbf{k} - c - d\mathbf{i} = -c - d\mathbf{i} + a\mathbf{j} + b\mathbf{k}$$
$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\mathbf{k} = a\mathbf{k} - b\mathbf{j} + c\mathbf{i} - d = -d + c\mathbf{i} - b\mathbf{j} + a\mathbf{k}$$

□

Exercise 1.1.4. Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F : M \to M$ be the map $F(x,y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}$$

Proof. The map F is a diffeomorphism because it has an inverse $(x,y) = F^{-1}(u,v) = (\sqrt{u/v}, \sqrt{uv})$. The differential of F at (x,y) is given by

$$\partial F(x,y) = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}$$

and thus $dF_{F^{-1}(u,v)}$ is given by

$$\partial F(\sqrt{u/v}, \sqrt{uv}) = \begin{pmatrix} \sqrt{uv} & \sqrt{\frac{u}{v}} \\ -\frac{v\sqrt{v}}{\sqrt{u}} & \sqrt{\frac{v}{u}} \end{pmatrix}$$

Thus

$$F_*X_{(u,v)} = dF_{F^{-1}(u,v)}X_{F^{-1}(u,v)} = 2u\frac{\partial}{\partial u}, \quad F_*Y_{(u,v)} = dF_{F^{-1}(u,v)}Y_{F^{-1}(u,v)} = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}$$

We can also apply the formula

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$$

□

Exercise 1.1.5. For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x,y) : x > 0\}$

(a) $X = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$

(b) $Y = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$

(c) $Z = (x^2+y^2)\dfrac{\partial}{\partial x}$

Proof. We define a map $F(x,y) = (\sqrt{x^2+y^2}, \arctan y/x)$. The differential of F is given by

$$\partial F(x,y) = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{x}{x^2+y^2} & -\dfrac{x}{x^2+y^2} \end{pmatrix}$$

Since the inverse of F is $(x,y) = F^{-1}(r,\theta) = (r\cos\theta, r\sin\theta)$, we have

$$\partial F(r\cos\theta, r\sin\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{pmatrix}$$

Hence the pushforward is given by

$$F_*X = dF_{F^{-1}(r,\theta)}X_{F^{-1}(r,\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = r\frac{\partial}{\partial r}.$$

$$F_*Y = dF_{F^{-1}(r,\theta)}Y_{F^{-1}(r,\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} r\cos\theta \\ -r\sin\theta \end{pmatrix} = r(\cos^2\theta - \sin^2\theta)\frac{\partial}{\partial r} + \sin 2\theta\frac{\partial}{\partial\theta}.$$

$$F_*Z = dF_{F^{-1}(r,\theta)}Z_{F^{-1}(r,\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} r^2 \\ 0 \end{pmatrix} = r^2\cos\theta\frac{\partial}{\partial r} + r\sin\theta\frac{\partial}{\partial\theta}$$

□

Exercise 1.1.6. Show that there is a smooth vector field on S^2 that vanishes at exactly one point.

Proof. Let $\varphi : S^2 - \{N\} \to \mathbb{R}^2$ be the stereographic projection on the north pole:

$$(u,v) = \varphi(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z}), \quad (x,y,z) = \varphi^{-1}(u,v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big)$$

Consider the vector field $\partial/\partial u$ on $\mathbb{R}^2$: we can pull it back on $S^2 - \{N\}$ to define a vector field. Now change to $\psi : S^2 - \{S\} \to \mathbb{R}^2$, we find

$$(\bar{u},\bar{v}) = \psi\circ\varphi^{-1}(u,v) = \psi\Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big) = (\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2})$$

So the vector field $\partial/\partial u$ has the form

$$\frac{\partial}{\partial u} = \frac{\partial\bar{u}}{\partial u}\frac{\partial}{\partial\bar{u}} + \frac{\partial\bar{v}}{\partial u}\frac{\partial}{\partial\bar{v}} = \frac{v^2-u^2}{(v^2+u^2)^2}\frac{\partial}{\partial\bar{u}} + \frac{2uv}{u^2+v^2}\frac{\partial}{\partial\bar{v}} = 2(\bar{v}^2-\bar{u}^2)\frac{\partial}{\partial\bar{v}} + 2\bar{u}\bar{v}\frac{\partial}{\partial\bar{v}}$$

Thus we can extend this vector field to the north pole. The resulting vector field only vanishes at the north pole. □

Exercise 1.1.7. For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

(a) $X = y\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}, \quad Y = \dfrac{\partial}{\partial y};$

(b) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}, \quad Y = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y};$

(c) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}, \quad Y = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}.$

Proof. We compute

$$[X,Y] = X(1)\frac{\partial}{\partial y} - Y(y)\frac{\partial}{\partial z} - Y(-2xy^2)\frac{\partial}{\partial y} = -\frac{\partial}{\partial z} + 4xy\frac{\partial}{\partial y}$$

$$[X,Y] = X(y)\frac{\partial}{\partial z} + X(-z)\frac{\partial}{\partial y} - Y(x)\frac{\partial}{\partial y} - Y(-y)\frac{\partial}{\partial x} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

$$[X,Y] = X(x)\frac{\partial}{\partial y} + X(y)\frac{\partial}{\partial x} - Y(x)\frac{\partial}{\partial y} - Y(-y)\frac{\partial}{\partial x} = -y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

□

Exercise 1.1.8. Let M and N be smooth manifolds. Given vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, we can define a vector field $X \oplus Y$ on $M \times N$ by

$$(X \oplus Y)_{(p,q)} = (X_p, Y_q)$$

Prove that $X \oplus Y$ is smooth if X and Y are smooth, and $[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, Y_1] \oplus [X_2, Y_2]$.

Proof. We have

$$\begin{aligned}
[X_1 \oplus Y_1, X_2 \oplus Y_2] &= (X_1X_2^i\frac{\partial}{\partial x^i}, Y_1Y_2^j\frac{\partial}{\partial y^j}) - (X_2X_1^i\frac{\partial}{\partial x^i}, Y_2Y_1^j\frac{\partial}{\partial y^j}) \\
&= (X_1X_2^i\frac{\partial}{\partial x^i} - X_2X_1^i\frac{\partial}{\partial x^i}, Y_1Y_2^j\frac{\partial}{\partial y^j} - Y_2Y_1^j\frac{\partial}{\partial y^j}) \\
&= ([X_1, X_2], [Y_1, Y_2]) = [X_1, Y_1] \oplus [X_2, Y_2]
\end{aligned}$$

□

Exercise 1.1.9. Show that $\mathbb{R}^3$ with the cross product is a Lie algebra.

Exercise 1.1.10. Let $A \subseteq \mathfrak{X}(\mathbb{R}^3)$ be the subspace spanned by $\{X, Y, Z\}$, where

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Show that A is a Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$, which is isomorphic to $\mathbb{R}^3$ with the cross product.

Proof. We compute that

$$[X,Y] = X(z)\frac{\partial}{\partial x} + X(-x)\frac{\partial}{\partial z} - Y(y)\frac{\partial}{\partial z} - Y(-z)\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -Z$$

$$[Y,Z] = Y(x)\frac{\partial}{\partial y} + Y(-y)\frac{\partial}{\partial x} - Z(z)\frac{\partial}{\partial x} - Z(-x)\frac{\partial}{\partial z} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = -X$$

$$[Z,X] = Z(y)\frac{\partial}{\partial z} + Z(-z)\frac{\partial}{\partial y} - X(x)\frac{\partial}{\partial y} - X(-y)\frac{\partial}{\partial x} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} = -Y$$

□

Exercise 1.1.11.

(a) Given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, show that the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra with the bracket defined by

$$[(X,Y),(X',Y')] = ([X,X'],[Y,Y'])$$

(b) Suppose G and H are Lie groups. Prove that $\mathfrak{Lie}(G \times H)$ is isomorphic to $\mathfrak{Lie}(G) \oplus \mathfrak{Lie}(H)$.

Proof. We check the Jacobi identity:

$$\begin{aligned}
&[(X_1,Y_1),[(X_2,Y_2),(X_3,Y_3)]] + [(X_2,Y_2),[(X_3,Y_3),(X_1,Y_1)]] + [(X_3,Y_3),[(X_1,Y_1),(X_2,Y_2)]] \\
&= [(X_1,Y_1),([X_2,X_3],[Y_2,Y_3])] + [(X_2,Y_2),([X_3,X_1],[Y_3,Y_1])] + [(X_3,Y_3),([X_1,X_2],[Y_1,Y_2])] \\
&= ([X_1,[X_2,X_3]],[Y_1,[Y_2,Y_3]]) + ([X_2,[X_3,X_1]],[Y_2,[Y_3,Y_1]]) + ([X_3,[X_1,X_2]],[Y_3,[Y_1,Y_2]]) \\
&= ([X_1,[X_2,X_3]] + [X_2,[X_3,X_1]] + [X_3,[X_1,X_2]]\,,\ [Y_1,[Y_2,Y_3]] + [Y_2,[Y_3,Y_1]] + [Y_3,[Y_1,Y_2]]) \\
&= 0
\end{aligned}$$

Thus $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra.

□

Exercise 1.1.12. Prove that if G is an abelian Lie group, then $\mathfrak{Lie}(G)$ is abelian.

Proof. If G is abelian, then the inverse map $i : G \to G$ is a homomorphism. By Proposition 2.2.2.1 this induces a homomorphism $i_* : \mathfrak{Lie}(G) \to \mathfrak{Lie}(G)$. By Exercise 2.1.1 we have $di_e(X) = -X$, so we conclude $i_*(X) = -X$. Thus

$$i_*[X,Y] = YX - XY = [i_*X, i_*Y] = [-X,-Y] = [X,Y]$$

This implies $[X,Y] = 0$ for all $X, Y \in \mathfrak{Lie}(G)$. □

Exercise 1.1.13. Suppose $F : G \to H$ is a Lie group homomorphism. Show that the kernel of $F_* : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ is the Lie algebra of $\ker F$.

Proof. By the isomorphism $\mathfrak{Lie}(G) \cong T_eG$, the kernel of F_* can be identified with $\ker dF_e$. Let $\iota : \ker F \hookrightarrow G$, then by Proposition **??**, since $\ker F$ is the level set $F^{-1}(e)$, we have $T_e \ker F = \ker dF_e$. Then by the canonical isomorphism $\mathfrak{Lie}(\ker F) \cong T_e \ker F$, we get the claim. □

Exercise 1.1.14. Let G and H be Lie groups, and suppose $F : G \to H$ is a Lie group homomorphism that is also a local diffeomorphism. Show that the induced homomorphism $F_* : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ is an isomorphism of Lie algebras.

Proof. Note that F is a local diffeomorphism if and only if it is a local diffeomorphism at e_G. But if this holds, then dF_e is an isomorphism from $T_{e_G}G$ to $T_{e_H}H$. □

Exercise 1.1.15. Theorem 2.2.3.1 implies that the Lie algebra of any Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ is canonically isomorphic to a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$, with a similar statement for Lie subgroups of $\mathrm{GL}_n(\mathbb{C})$. Under this isomorphism, show that

$$\begin{aligned}
\mathfrak{Lie}(\mathrm{SL}_n(\mathbb{R})) &\cong \mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr} A = 0\} \\
\mathfrak{Lie}(\mathrm{SL}_n(\mathbb{C})) &\cong \mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr} A = 0\} \\
\mathfrak{Lie}(\mathrm{SO}(n)) &\cong \mathfrak{o}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) : A^T + A = 0\} \\
\mathfrak{Lie}(\mathrm{U}(n)) &\cong \mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A^* + A = 0\} \\
\mathfrak{Lie}(\mathrm{SU}(n)) &\cong \mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})
\end{aligned}$$

Proof. Recall Exercise 2.1.2, we have

$$d(\det)_{I_n}(X) = \mathrm{tr}(X)$$

so $\mathfrak{sl}(n,\mathbb{R}) \cong T_{I_n}\mathrm{SL}_n(\mathbb{R}) = \ker d(\det)_{I_n} = \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr} A = 0\}$. □

1.2 Integral curves and flows

1.2.1 Integral curves

Suppose M is a smooth manifold with or without boundary. If $\gamma : J \to M$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}M$. Now we describe a way to work backwards: given a tangent vector at each point, we seek a curve whose velocity at each point is equal to the given vector there.

If V is a vector field on M, an **integral curve** of V is a differentiable curve $\gamma : J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \quad \text{for} \quad t \in j$$

If $0 \in J$, the point $\gamma(0)$ is called the **starting point of γ**.

Example 1.2.1.1 (Integral Curves).

(a) Let (x, y) be standard coordinates on $\mathbb{R}^2$, and let $V = \partial/\partial$ be the first coordinate vector field. It is easy to check that the integral curves of V are precisely the straight lines parallel to the x-axis, with parametrizations of the form $(a + t, b)$ for constants a and b. Thus, there is a unique integral curve starting at each point of the plane, and the images of different integral curves are either identical or disjoint.

(b) Let $W = x\partial/\partial y - y\partial/\partial x$ on $\mathbb{R}^2$. If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (x(t), y(t))$, then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve translates to

$$x'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} = x(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} - y(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)}$$

Comparing the components of these vectors, we see that this is equivalent to the system of ordinary differential equations

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

These equations have the solutions

$$x(t) = a\cos t - b\sin t, \quad y(t) = a\sin t + b\cos t.$$

for arbitrary constants a and b, and thus each curve of the form above is an integral curve of W. When $(a, b) = (0, 0)$, this is the constant curve $\gamma(t) \equiv (0, 0)$; otherwise, it is a circle traversed counterclockwise. Since $\gamma(0) = (a, b)$, we see once again that there is a unique integral curve starting at each point $(a, b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint.

As the second example above illustrates, finding integral curves boils down to solving a system of ordinary differential equations in a smooth chart. Suppose V is a smooth vector field on M and $\gamma : J \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $(\gamma^1(t), \ldots, \gamma^n(t))$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written

$$\dot{\gamma}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = V^i(\gamma(t))\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

which reduces to the following autonomous system of ordinary differential equations

$$\begin{cases} \dot{\gamma}^1(t) = V^1(\gamma^1(t), \ldots, \gamma^n(t)), \\ \dot{\gamma}^2(t) = V^2(\gamma^1(t), \ldots, \gamma^n(t)), \\ \qquad\vdots \\ \dot{\gamma}^i(t) = V^n(\gamma^1(t), \ldots, \gamma^n(t)). \end{cases} \tag{1.2.1.1}$$

The fundamental fact about such systems is the existence, uniqueness, and smoothness theorem. (This is the reason for the terminology *integral curves*, because solving a system of ODEs is often referred to as integrating the system.) We will derive detailed consequences of that theorem later; for now, we just note the following simple result.

Proposition 1.2.1.2. *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon.\varepsilon) \to M$ that is an integral curve of V starting at p.*

Proof. This is just the existence statement of ODEs, applied to the coordinate representation of V. □

The next two lemmas show how affine reparametrizations affect integral curves.

Lemma 1.2.1.3 (Rescaling Lemma). *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma} : \widetilde{J} \to M$ defined by $\widetilde{J}(t) = J(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t : at \in J\}$.*

Proof. One way to see this is as a straightforward application of the chain rule in local coordinates. Somewhat more invariantly, we can examine the action of $\widetilde{\gamma}'(t)$ on a smooth real-valued function f defined in a neighborhood of a point $\widetilde{\gamma}(t_0)$. By the chain rule and the fact that γ is an integral curve of V,

$$\widetilde{\gamma}'(t_0)f = \frac{d}{dt}\Big|_{t=t_0}(f \circ \widetilde{\gamma})(t) = \frac{d}{dt}\Big|_{t=t_0}(f \circ \gamma)(at) = a(f \circ \gamma)'(at_0) = a\gamma'(at_0)f = aV_{\widetilde{\gamma}(t_0)}f.$$

as needed. □

Lemma 1.2.1.4 (Translation Lemma). *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma} : \widehat{J} \to M$ defined by $\widehat{J}(t) = J(t+b)$ is an integral curve of the vector field V, where $\widehat{J} = \{t : t+b \in J\}$.*

Proof. A direct computation shows

$$\widehat{\gamma}'(t_0) = \frac{d}{dt}|_{t=t_0}\gamma(t+b) = \gamma'(t_0+b) = X_{\gamma(t_0+b)} = X_{\widehat{\gamma}(t_0)}.$$

as needed. □

Proposition 1.2.1.5 (Naturality of Integral Curves). *Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

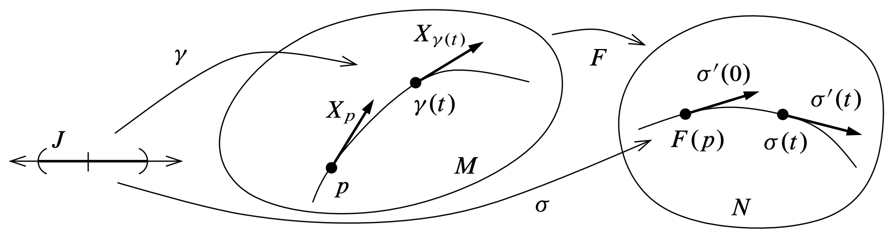


Figure 1.1: Flows of F-related vector fields.

Proof. Suppose first that X and Y are F-related, and $\gamma : J \to M$ is an integral curve of X. If we define $\sigma : J \to N$ by $\sigma = F \circ \gamma$, then

$$\sigma'(t) = (F \circ \gamma)'(t) = dF_{\gamma(t)}\big(\gamma'(t)\big) = dF_{\gamma(t)}(X_{\gamma(t)}) = Y_{\gamma(t)},$$

so σ is an integral curve of Y.

Conversely, suppose F takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Since $F \circ \gamma$ is an integral curve of Y starting at $F(p)$, we have

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p\big(\gamma'(0)\big) = dF_p(X_p)$$

which shows that X and Y are F-related. □

1.2.2 Flow

Here is another way to visualize the family of integral curves associated with a vector field. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$, and suppose that for each point $p \in M$, V has a unique integral curve starting at p and defined for all $t \in \mathbb{R}$, which we denote by $\theta^{(p)} : \mathbb{R} \to M$. (It may not always be the case that every integral curve is defined for all t, but for purposes of illustration let us assume so for the time being.) For each $t \in \mathbb{R}$, we can define a map $\theta_t : M \to M$ by sending each $p \in M$ to the point obtained by following for time t the integral curve starting at p:

$$\theta_t(p) = \theta^{(p)}(t)$$

Each map θ_t *slides* the manifold along the integral curves for time t. The translation lemma implies that $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at $q := \theta^{(p)}(s)$; since we are assuming uniqueness of integral curves, $\theta^{(q)}(t) = \theta^{(p)}(t+s)$. When we translate this into a statement about the maps θ_t, it becomes

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$$

Together with the equation $\theta_0(p) = p$, which holds by definition, this implies that the map $\theta : \mathbb{R} \times M \to M$ is an action of the additive group $\mathbb{R}$ on M. Motivated by these observations, we define a **global flow** on M (also called a **one-parameter group action**) to be a continuous left $\mathbb{R}$-action on M, that is, a continuous map $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

$$\theta(t, \theta(s, p)) = \theta(t+s, p), \quad \theta(0, p) = p.$$

Given a global flow θ on M, we define two collections of maps as follows:

- For each $t \in \mathbb{R}$, define a continuous map $\theta_t : M \to M$:

  $$\theta_t(p) = \theta(t, p)$$

  The defining properties of θ are equivalent to the **group laws**

  $$\theta_t \circ \theta_s(p) = \theta_{t+s}(p), \quad \theta_0 = \mathrm{id}_M.$$

  As is the case for any continuous group action, each map $\theta_t : M \to M$ is a homeomorphism, and if the flow is smooth, θ_t is a diffeomorphism.

- For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by

  $$\theta^{(p)}(t) = \theta(t, p)$$

  The image of this curve is the orbit of p under the group action.

The next proposition shows that every smooth global flow is derived from the integral curves of some smooth vector field in precisely the way we described above. If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = \dot{\theta}^{(p)}(0)$$

The assignment $p \mapsto V_p$ is a (rough) vector field on M which is called the **infinitesimal generator of θ**, for reasons we will explain below.

Proposition 1.2.2.1. *Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.*

Proof. To show that V is smooth, it suffices by Proposition 1.1.1.13 to show that Vf is smooth for every smooth real-valued function f defined on an open subset $U \subseteq M$. For any such f and any $p \in U$, just note that

$$Vf(p) = V_p f = \dot{\theta}^{(p)}(0) f = \left.\frac{d}{dt}\right|_{t=0} f\left(\theta^{(p)}(t)\right) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} f\left(\theta(0, p)\right)$$

Because $f(\theta(t, p))$ is a smooth function of (t, p) by composition, so is its partial derivative with respect to t. Thus, $Vf(p)$ depends smoothly on p, so V is smooth.

Next we need to show that $\theta^{(p)}$ is an integral curve of V, which means that $\dot{\theta}^{(p)}(t) = V_{\theta^{(p)}(t)}$ for all $p \in M$ and all $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$ be arbitrary, and set $q = \theta^{(p)}(t_0) = \theta_{t_0}(p)$, so what we have to show is $\dot{\theta}^{(p)}(t_0) = V_q$. By the group law, for all t,

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t \circ \theta_{t_0}(p) = \theta_{t+t_0}(p) = \theta^{(p)}(t + t_0)$$

Therefore, for any smooth real-valued function f defined in a neighborhood of q,

$$V_q f = \dot{\theta}^{(q)}(0) f = \frac{d}{dt}\Big|_{t=0} (f \circ \theta^{(q)})(t) = \frac{d}{dt}\Big|_{t=0} (f \circ \theta^{(p)})(t + t_0) = \dot{\theta}^{(p)}(t_0) f$$

which was to be shown. □

Example 1.2.2.2. The two vector fields on the plane described in Example 1.2.1.1 both had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. Using the results of that example, we can write down the flows explicitly.

(a) The flow of $V = \partial/\partial x$ in $\mathbb{R}^2$ is the map $\tau : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\tau_t(x, y) = (x + t, y)$$

For each nonzero $t \in \mathbb{R}$, τ_t 0translates the plane to the right $(t > 0)$ or left $(t < 0)$ by a distance $|t|$.

(b) The flow of $W = x\partial/\partial y - y\partial/\partial x$ is the map $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\theta_t(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For each $t \in \mathbb{R}$, θ_t rotates the plane through an angle t about the origin.

1.2.3 The fundamental theorem on flows

We have seen that every smooth global flow gives rise to a smooth vector field whose integral curves are precisely the curves defined by the flow. Conversely, we would like to be able to say that every smooth vector field is the infinitesimal generator of a smooth global flow. However, it is easy to see that this cannot be the case, because there are smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$. Here are two examples.

Example 1.2.3.1. Let $M = \mathbb{R}^2 - \{0\}$ with standard coordinates (x, y), and let V be the vector field $\partial/\partial x$ on M. The unique integral curve of V starting at $(-1, 0) \in M$ is $\gamma(t) = (t - 1, 0)$. However, in this case, γ cannot be extended continuously past $t = 1$. This is intuitively evident because of the hole in M at the origin; to prove it rigorously, suppose $\widetilde{\gamma}$ is any continuous extension of γ past $t = 1$. Then $\gamma(t) \to \widetilde{\gamma}(1) \in \mathbb{R}^2 - \{0\}$ as $t \to 1^-$. But we can also consider γ as a map into $\mathbb{R}^2$ by composing with the inclusion $M \hookrightarrow \mathbb{R}^2$, and it is obvious from the formula that $\gamma(t) \to (0, 0)$ as $t \mapsto 1^-$. Since limits in $\mathbb{R}^2$ are unique, this is a contradiction.

Example 1.2.3.2. For a more subtle example, let M be all of $\mathbb{R}^2$ and let $W = x^2\partial/\partial x$. You can check easily that the unique integral curve of W starting at $(1, 0)$ is

$$\gamma(t) = (\frac{1}{1-t}, 0)$$

This curve also cannot be extended past $t = 1$, because its x-coordinate is unbounded as $t \to 1^-$.

For this reason, we make the following definitions. If M is a manifold, a **flow domain** for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the slice $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0. A **flow** on M is a continuous map $\theta : \mathcal{D} \to M$ where $\mathcal{D}$ is a flow domain, that satisfies the following group laws: for all $p \in M$,

$$\theta(0, p) = p \tag{1.2.3.1}$$

and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,

$$\theta\big(t, \theta(s, p)\big) = \theta(t + s, p) \tag{1.2.3.2}$$

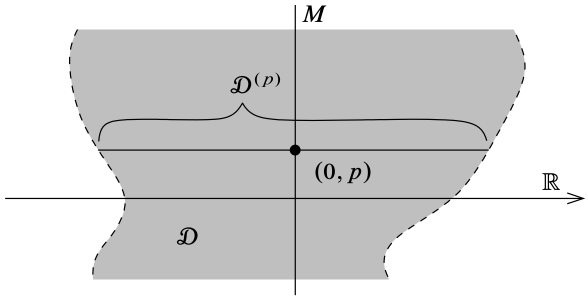


Figure 1.1: A flow domain.

We sometimes call θ a **local flow** to distinguish it from a global flow as defined earlier. The unwieldy term **local one-parameter group action** is also used.

If θ is a flow, we define $\theta_t(p) = \theta(t, p)$ whenever $(t, p) \in \mathcal{D}$, just as for a global flow. For each $t \in \mathbb{R}$, we also define

$$M_t = \{p \in M : (t, p) \in \mathcal{D}\}$$

so that

$$p \in M_t \iff t \in \mathcal{D}^{(p)} \iff (t, p) \in \mathcal{D}$$

If θ is smooth, the infinitesimal generator of θ is defined by $V_p = \dot{\theta}^{(p)}(0)$.

Proposition 1.2.3.3. *If $\theta : \mathcal{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

Proof. The proof is essentially identical to the analogous proof for global flows, Proposition 1.2.2.1. In the proof that V is smooth, we need only note that for any $p_0 \in M$, $\theta(t, p)$ is defined and smooth for all (t, p) sufficiently close to $(0, p_0)$ because $\mathcal{D}$ is open. In the proof that $\theta^{(p)}$ is an integral curve, we need to verify that all of the expressions make sense. Suppose $t_0 \in \mathcal{D}^{(p)}$. Because both $\mathcal{D}^{(p)}$ and $\mathcal{D}^{(\theta_{t_0}(p))}$ are open intervals containing 0, there is a positive number ε such that $t + t_0 \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta_{t_0}(p))}$ whenever $|t| < \varepsilon$, and then $\theta_t(\theta_{t_0}(p)) = \theta_{t+t_0}(p)$ by definition of a flow. The rest of the proof goes through just as before. $\square$

The next theorem is the main result of flows. A **maximal integral curve** is one that cannot be extended to an integral curve on any larger open interval, and a **maximal flow** is a flow that admits no extension to a flow on a larger flow domain.

Theorem 1.2.3.4 (Fundamental Theorem on Flows). *Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*

(a) For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t - s : t \in \mathcal{D}^{(p)}\}$.

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Proof. Proposition 1.2.1.2 shows that there exists an integral curve starting at each point $p \in M$. Suppose $\gamma, \widetilde{\gamma} : J \to M$ are two integral curves of V defined on the same open interval J such that $\gamma(t_0) = \widetilde{\gamma}(t_0)$ for some $t_0 \in J$. Let $\mathcal{S}$ be the set of $t \in J$ such that $\gamma(t) = \widetilde{\gamma}(t)$. Clearly, $\mathcal{S} \neq \varnothing$, because $t_0 \in \mathcal{S}$ by hypothesis, and $\mathcal{S}$ is closed in J by continuity. On the other hand, suppose $t_1 \in \mathcal{S}$. Then in a smooth coordinate neighborhood around the point $p = \gamma(t_1)$, γ and $\widetilde{\gamma}$ are both solutions to same ODE with the same initial condition $\gamma(t_1) = \widetilde{\gamma}(t_1) = p$. By the uniqueness theorem of ODEs, $\gamma \equiv \widetilde{\gamma}$ on an interval containing t_1, which implies that $\mathcal{S}$ is open in J. Since J is connected, $\mathcal{S} = J$, which implies that $\gamma \equiv \widetilde{\gamma}$ on all of J. Thus, any two integral curves that agree at one point agree on their common domain.

For each $p \in M$, let $\mathcal{D}^{(p)}$ be the union of all open intervals $J \subseteq \mathbb{R}$ containing 0 on which an integral curve starting at p is defined. Define $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ by letting $\theta^{(p)}(t) = \gamma(t)$, where γ is any integral curve starting at p and defined on an open interval containing 0 and t. Since all such integral curves agree at t by the argument above, $\theta^{(p)}(t)$ is well defined, and is obviously the unique maximal integral curve starting at p.

Now let $\mathcal{D} = \{(t,p) \in \mathbb{R} \times M : t \in \mathcal{D}^{(p)}\}$ and define $\theta(t,p) = \theta^{(p)}(t)$. As usual, we also write $\theta_t(p) = \theta(t,p)$. By definition, θ satisfies property (a) in the statement of the fundamental theorem: for each $p \in M$, $\theta^{(p)}(t)$ is the unique maximal integral curve of V starting at p. To verify the group laws, fix any $p \in M$ and $s \in \mathcal{D}^{(p)}$, and write $q = \theta(s,p) = \theta^{(p)}(s)$. The curve $\gamma : \mathcal{D}^{(p)} - s \to M$ defined by $\gamma(t) = \theta^{(p)}(t+s)$ starts at q, and the translation lemma shows that γ is an integral curve of V. By uniqueness of ODE solutions, γ agrees with $\theta^{(q)}$ on their common domain, which is equivalent to the second group law (1.2.3.2), and the first group law (1.2.3.1) is immediate from the definition. By maximality of $\theta^{(q)}$, the domain of γ cannot be larger than $\mathcal{D}^{(q)}$, which means that $\mathcal{D}^{(p)} - s \subseteq \mathcal{D}^{(q)}$. Since $0 \in \mathcal{D}^{(p)}$, this implies that $-s \in \mathcal{D}^{(q)}$, and the group law implies that $\theta^{(q)}(-s) = p$. Applying the same argument with $(-s,q)$ in place of (s,p), we find that $\mathcal{D}^{(q)} + s \subseteq \mathcal{D}^{(p)}$, which is the same as $\mathcal{D}^{(q)} \subseteq \mathcal{D}^{(p)} - s$. This proves (b).

Next we show that $\mathcal{D}$ is open in $\mathbb{R} \times M$ (so it is a flow domain), and that $\theta : \mathcal{D} \to M$ is smooth. Define a subset $W \subseteq \mathcal{D}$ as the set of all $(t,p) \in \mathcal{D}$ such that θ is defined and smooth on a product neighborhood of (t,p) of the form $J \times U$ where $J \subseteq \mathbb{R}$ is an open interval containing 0 and t, $U \subseteq M$ is a neighborhood of p. Then W is open in $\mathbb{R} \times M$, and the restriction of θ to W is smooth, so it suffices to show that $W = \mathcal{D}$. Suppose this is not the case. Then there exists some point $(\tau, p_0) \in \mathcal{D} - W$. For simplicity, assume $\tau > 0$, the argument for $\tau < 0$ is similar.

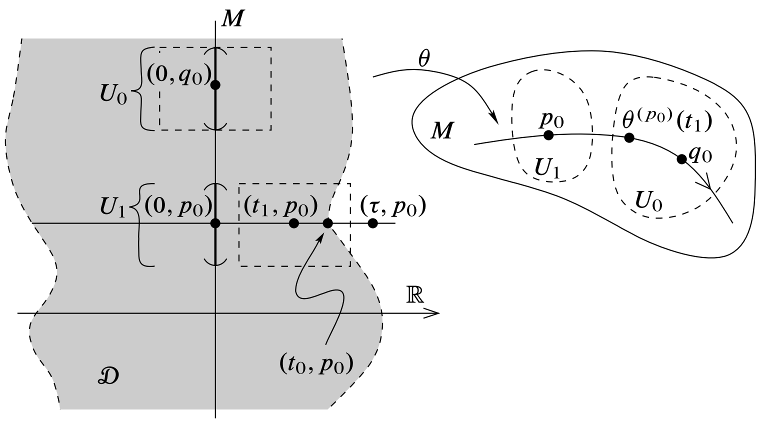


Figure 1.2: Proof that $\mathcal{D}$ is open.

Let $t_0 = \sup\{t \in \mathbb{R} : (t,p_0) \in W\}$. By the existence and uniquess theorem of ODEs applied in smooth coordinates around p_0, we know that θ is defined and smooth in some product neighborhood of $(0,p_0)$, so $t_0 > 0$. Since $t_0 \le \tau$ and $\mathcal{D}^{(p_0)}$ is an open interval containing 0 and τ, it follows that $t_0 \in \mathcal{D}^{(p_0)}$. Let $q_0 = \theta^{(p_0)}(t_0)$. By the ODE theorem again, there exist $\varepsilon > 0$ and a neighborhood U_0 of q_0 such that $(-\varepsilon, \varepsilon) \times U_0 \subseteq W$. We will use the group law to show that θ extends smoothly to a neighborhood of (t_0, p_0), which is a contradiction.

Choose some $t_1 < t_0$ such that $t_1 + \varepsilon > t_0$ and $\theta^{(p_0)}(t_1) \in U_0$. Since $t_1 < t_0$, we have $(t_1, p_0) \in W$, and so there is a product neighborhood $(t_1 - \delta, t_1 + \delta) \times U_1 \subseteq W$. Because $\theta(t_1, p_0) \in U_0$, we can choose U_1 small enough that θ maps $\{t_1\} \times U_1$ into U_0. Define $\widetilde{\theta} : [0, t_1 + \varepsilon) \times U_1 \to M$ by

$$\widetilde{\theta}(t,p) = \begin{cases} \theta_t(p) & p \in U_1, 0 \le t \le t_1, \\ \theta_{t-t_1} \circ \theta_{t_1}(p) & p \in U_1, t_1 - \varepsilon < t < t_1 + \varepsilon \end{cases}$$

The group law for θ guarantees that these definitions agree where they overlap, and our choices of U_1, t_1, and θ ensure that this defines a smooth map. By the translation lemma, each map $t \mapsto \theta_t(p)$ is an integral curve of V, so $\widetilde{\theta}$ is a smooth extension of θ to a neighborhood of (t_0, p_0), contradicting our choice of t_0. This completes the proof that $W = \mathcal{D}$.

Finally, we prove (c). The fact that M_t is open is an immediate consequence of the fact that $\mathcal{D}$ is open.

From part (b) we deduce

$$p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow \mathcal{D}^{(\theta_t(p))} = \mathcal{D}^{(p)} - t \Rightarrow -t \in \mathcal{D}^{(\theta_t(p))} \Rightarrow \theta_t(p) \in M_{-t}$$

which shows that θ_t maps M_t to M_{-t}. Moreover, the group laws then show that $\theta_{-t} \circ \theta_t$ is equal to the identity on M_t. Reversing the roles of t and $-t$ shows that $\theta_t \circ \theta_{-t}$ is the identity on M_{-t}, which completes the proof. □

The flow whose existence and uniqueness are asserted in the fundamental theorem is called the **flow generated by V**, or just the **flow of V**. Now we give an example of this.

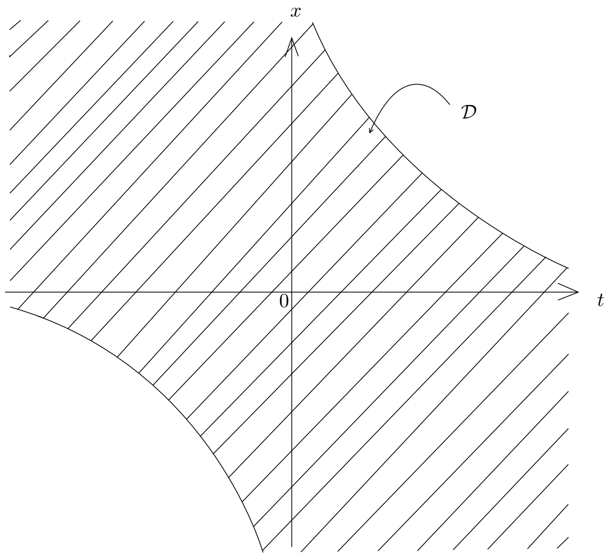


Figure 1.3: The flow domain in Example 1.2.3.5.

Example 1.2.3.5. Let $V = x^2\partial/\partial x$ be a vector field on $\mathbb{R}$, we can check that the unique integral curve of V starting at x is

$$\theta(t,x) = \gamma_x(t) = \begin{cases} \dfrac{1}{x^{-1} - t} & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Thus the flow domain of V is

$$\mathcal{D} = \{(t,x) : t > 0, x < 1/t\} \cup \{(t,x) : t < 0, x > 1/t\}.$$

Now for $x > 0$ and $t \in (-\infty, 1/x)$, we check that

$$\theta(t,x) = \frac{1}{x^{-1} - t}, \quad \mathcal{D}^{(\theta(t,x))} = (-\infty, \frac{1}{x} - t) = \mathcal{D}^{(x)} - t.$$

Theorem 1.2.3.6 (Naturality of Flows). *Suppose M and N are smooth manifolds, $F : M \to N$ is a smooth map, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ \downarrow{\scriptstyle \theta_t} & & \downarrow{\scriptstyle \eta_t} \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

Proof. By Proposition 1.2.1.5, for any $p \in M$, the curve $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F \circ \theta^{(p)}(0) = F(p)$. By uniqueness of integral curves, therefore, the maximal integral curve $\eta^{(F(p))}$ must be defined at least on the interval $\mathcal{D}^{(p)}$, and $F \circ \theta^{(p)} = \eta^{(F(p))}$ on that interval. This means that

$$p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow t \in \mathcal{D}^{(F(p))} \Rightarrow F(p) \in N_t,$$

which is equivalent to $F(M_t) \subseteq N_t$, and

$$F\big(\theta^{(p)}(t)\big) = \eta^{(F(p))}(t) \quad \text{for} \quad t \in \mathcal{D}^{(p)}$$

which is equivalent $\eta_t \circ F = F \circ \theta_t$ for all $p \in M_t$. ☐

The next corollary is immediate.

Corollary 1.2.3.7 (Diffeomorphism Invariance of Flows). *Let $F : M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow of F^*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.*

1.2.3.1 Complete vector fields

As we observed earlier in this chapter, not every smooth vector field generates a global flow. The ones that do are important enough to deserve a name. We say that a smooth vector field is **complete** if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

We will show below that all compactly supported smooth vector fields, and therefore all smooth vector fields on a compact manifold, are complete. The proof will be based on the following lemma.

Lemma 1.2.3.8 (Uniform Time Lemma). *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.*

Proof. Suppose for the sake of contradiction that for some $p \in M$, the domain $\mathcal{D}^{(p)}$ of $\theta^{(p)}$ is bounded above. (A similar proof works if it is bounded below.) Let $b = \sup \mathcal{D}^{(p)}$, let t_0 be a positive number such that $b - \varepsilon < t_1 < b$, and let $q = \theta^{(p)}(t_0)$. The hypothesis implies that $\theta^{(q)}(t)$ is defined at least for $t \in (-\varepsilon, \varepsilon)$. Define a curve $\gamma : (-\varepsilon, t_0 + \varepsilon) \to M$ by

$$\gamma(t) = \begin{cases} \theta^{(p)}(t), & -\varepsilon < t < b \\ \theta^{(q)}(t - t_0), & t_0 - \varepsilon < t < t_0 + \varepsilon \end{cases}$$

These two definitions agree where they overlap, because

$$\theta^{(q)}(t - t_0) = \theta_{t-t_0}(q) = \theta_{t-t_0}\theta_{t_0}(p) = \theta_t(p)$$

by the group law for θ. By the translation lemma, γ is an integral curve starting at p. Since $t_0 + \varepsilon > b$, this is a contradiction. ☐

Theorem 1.2.3.9. *Every compactly supported smooth vector field on a smooth manifold is complete.*

Proof. Suppose V is a compactly supported vector field on a smooth manifold M, and let $K = \operatorname{supp}(V)$. For each $p \in K$, there is a neighborhood U_p of p and a positive number ε_p such that the flow of V is defined at least on $(-\varepsilon_p, \varepsilon_p) \times U_p$. By compactness, finitely many such sets $U_{p_1}, \ldots, U_{p_k}$ cover K. With $\varepsilon := \min\{\varepsilon_{p_i}\}$, it follows that every maximal integral curve starting in K is defined at least on $(-\varepsilon, \varepsilon)$. Since $V \equiv 0$ outside of K, every integral curve starting in $M - K$ is constant and thus can be defined on all of $\mathbb{R}$. Thus the hypotheses of the uniform time lemma are satisfied, so V is complete. ☐

Corollary 1.2.3.10. *On a compact smooth manifold, every smooth vector field is complete.*

Left-invariant vector fields on Lie groups form another class of vector fields that are always complete.

Theorem 1.2.3.11. *Every left-invariant vector field on a Lie group is complete.*

Proof. Let G be a Lie group, let $X \in \mathfrak{Lie}(G)$, and let $\theta : \mathcal{D} \to G$ denote the flow of X. There is some $\varepsilon > 0$ such that $\theta^{(e)}$ is defined on $(-\varepsilon, \varepsilon)$.

Let $g \in G$ be arbitrary. Because X is L_g-related to itself, it follows from Proposition 1.2.1.5 that the curve $L_g \circ \theta^{(e)}$ is an integral curve of X starting at g and therefore is equal to $\theta^{(g)}$. This shows that for each $g \in G$, the integral curve $\theta^{(p)}$ is defined at least on $(-\varepsilon, \varepsilon)$, so the uniform time lemma guarantees that X is complete. ☐

Here is another useful property of integral curves.

Lemma 1.2.3.12 (Escape Lemma). *Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, the set $\gamma\big([t_0, b)\big)$ is not contained in any compact subset of M.*

Proof. Let $p = \gamma(0)$ and θ denote the flows of V so that $\gamma(t) = \theta^{(p)}(t)$. Assume that there is a $t_0 \in J$ such that $\gamma\big([t_0, b)\big)$ lies in a compact set $K \subseteq M$, we claim that γ can be extended past b.

- We use sequentially compactness: let (x_n) be any sequence tends to b from below, since $\{\gamma(x_n)\}$ is contained in a compact set, it has a subsequence converging to a point $q \in M$.
- By theorem 1.2.3.4, there is $\varepsilon > 0$ and a neighborhood of q such that θ is defined on $(-\varepsilon, \varepsilon) \times U$.
- By discrading finitely many terms, we may assume that $\gamma(x_i) \in U$ and $t_i > b - \varepsilon$. Then define $\sigma : [t_0, b + \varepsilon) \to M$ by
$$\sigma(t) = \begin{cases} \gamma(t) & t_0 \le t < b, \\ \theta_{t-t_i} \circ \theta_{t_i}(p), & t_i - \varepsilon < t < t_i + \varepsilon. \end{cases}$$
This map is defined for all $t \in [b, b + \varepsilon)$ since $t_i \to b$, and the two definitions agree where they overlap, as
$$\theta_{t-t_i} \circ \theta_{t_i}(p) = \theta_t(p) = \gamma(t)$$

This contradicts the maximality, so our claim follows. □

1.2.4 Flowouts

Flows provide the technical apparatus for many geometric constructions on manifolds. Most of those constructions are based on the following general theorem, which describes how flows behave in the vicinity of certain submanifolds.

Theorem 1.2.4.1 (Flowout Theorem). *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded k-dimensional submanifold, and $V \subseteq \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta : \mathcal{D} \to M$ be the flow of V, let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$, and let $\Phi = \theta|_{\mathcal{O}}$.*

(a) $\Phi : \mathcal{O} \to M$ is an immersion.

(b) $\partial/\partial t \in \mathfrak{X}(\mathcal{O})$ is Φ-related to V.

(c) There exists a smooth positive function $\delta : S \to \mathbb{R}$ such that the restriction of Φ to $\mathcal{O}_\delta$ is injective, where $\mathcal{O}_\delta \subseteq \mathcal{O}$ is the flow domain
$$\mathcal{O}_\delta = \{(t, p) \in \mathcal{O} : |t| < \delta(p)\} \tag{1.2.4.1}$$
Thus, $\Phi(\mathcal{O}_\delta)$ is an immersed submanifold of M containing S, and V is tangent to this submanifold.

(d) If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of M.

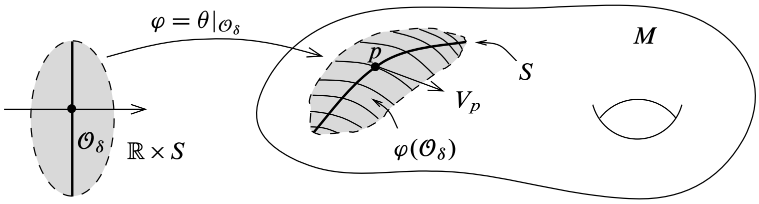


Figure 1.1: A flowout.

Remark 1.2.4.2. The submanifold $\Phi(\mathcal{O}_\delta) \subseteq M$ is called a **flowout from S along V**.

Proof. First we prove (b). Fix some $p \in S$, and let $\sigma : \mathcal{D}^{(p)} \to \mathbb{R} \times S$ be the curve $\sigma(t) = (t, p)$. Then $\Phi \circ \sigma(t) = \theta(t, p)$ is an integral curve of V, so for any $t_0 \in \mathcal{D}^{(p)}$ it follows that

$$d\Phi_{(t_0,p)}\Big(\frac{\partial}{\partial t}\Big|_{(t_0,p)}\Big) = (\Phi \circ \sigma)'(t_0) = V_{\Phi(t_0,p)}.$$

Next we prove (a). The restriction of Φ to $\{0\} \times S$ is the composition of the diffeomorphism $\{0\} \times S \approx S$ with the embedding $S \hookrightarrow M$, so it is an embedding. Thus, the restriction of $d\Phi_{(0,p)}$ to T_pS (viewed as a subspace of $T_{(0,p)}\mathcal{O} \cong T_0\mathbb{R} \oplus T_pS$) is the inclusion $T_pS \hookrightarrow T_pM$. If $(E_1, \ldots, E_k)$ is any basis for T_pS, it follows that $d\Phi_{(0,p)}$ maps the basis $(\partial/\partial t|_{(0,p)}, E_1, \ldots, E_k)$ for $T_{(0,p)}\mathcal{O}$ to $(V_p, E_1, \ldots, E_k)$. Since V_p is not tangent to S, this $(k+1)$-tuple is linearly independent and thus $d\Phi_{(0,p)}$ is injective.

To show $d\Phi$ is injective at other points, we argue as in the proof of the equivariant rank theorem. Given $(t_0, p_0) \in \mathcal{O}$, let $\tau_{t_0} : \mathcal{O} \to \mathbb{R} \times S$ be the translation $\tau_{t_0}(t, p) = (t + t_0, p)$. By the group law for θ, the following diagram commutes (where the horizontal maps might be defined only in open subsets containing $(0, p_0)$ and p_0, respectively):

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\tau_{t_0}"] \arrow[d, "\Phi"'] & \mathcal{O} \arrow[d, "\Phi"] \\ M \arrow[r, "\theta_{t_0}"] & M \end{tikzcd}$$

Both horizontal maps in the diagram above are local diffeomorphisms. Taking differentials, we obtain

$$\begin{tikzcd} T_{(0,p_0)}\mathcal{O} \arrow[r, "d(\tau_{t_0})_{(0,p_0)}"] \arrow[d, "d\Phi_{(0,p_0)}"'] & T_{(t_0,p_0)}\mathcal{O} \arrow[d, "d\Phi_{(t_0,p_0)}"] \\ T_{p_0}M \arrow[r, "d(\theta_{t_0})_{p_0}"] & T_{\Phi(t_0,p_0)}M \end{tikzcd}$$

Because the horizontal maps are isomorphisms, the two vertical maps have the same rank. Since we have already shown that $d\Phi_{(0,p_0)}$ has full rank, so does $d\Phi_{(t_0,p_0)}$. This completes the proof that Φ is an immersion.

Next we prove (c). Given a point $p_0 \in S$, choose a slice chart $(U, (x^i))$ for S in M centered at p_0, so that $U \cap S$ is the set where $x^{k+1} = \cdots = x^n = 0$ (where $n = \dim M$). Because V is not tangent to S, one of the last $n - k$ components of V_{p_0}, say $V^j_{p_0}$, must be nonzero. Shrinking U if necessary, we may assume that there is a constant $c > 0$ such that

$$|V^j_p| \geq c \quad \text{for} \quad p \in U. \tag{1.2.4.2}$$

Since $\Phi^{-1}(U)$ is open in $\mathbb{R} \times S$, we may choose a number $\varepsilon_{p_0} > 0$ and a neighborhood W_{p_0} of p_0 in S such that $(-\varepsilon_{p_0}, \varepsilon_{p_0}) \times W_{p_0} \subseteq \mathcal{O}$ and $\Phi\big((-\varepsilon_{p_0}, \varepsilon_{p_0}) \times W_{p_0}\big) \subseteq U$. Write the component functions of Φ in these local coordinates as

$$\Phi(t, p) = \big(\Phi^1(t, p), \ldots, \Phi^n(t, p)\big)$$

Because Φ is the restriction of the flow, the component function Φ^j satisfies

$$\frac{\partial \Phi^j}{\partial t}(t, p) = V^j\big(\Phi(t, p)\big), \quad \Phi^j(0, p) = p$$

By (1.2.4.2) and the fundamental theorem of calculus, $|\Phi^j(t, p)| \geq c|t|$, and thus for $(t, p) \in (-\varepsilon_{p_0}, \varepsilon_{p_0}) \times W_{p_0}$ we conclude that $\Phi(t, p) \in S$ if and only if $t = 0$.

Choose a smooth partition of unity $\{\psi_p : p \in S\}$ subordinate to the open cover $\{W_p : p \in S\}$ of S, and define $f : S \to \mathbb{R}$ by

$$f(q) = \sum_{p \in S} \varepsilon_p \psi_p(q)$$

Then f is smooth and positive. For each $q \in S$, there are finitely many $p \in S$ such that $\psi_p(q) > 0$; if p_1 is one of these points such that ε_{p_1} is maximum among all such ε_p, then

$$f(q) \leq \varepsilon_{p_1} \sum_{p \in S} \psi_p(q) = \varepsilon_{p_1}$$

It follows that if $(t,q) \in \mathcal{O}$ such that $|t| < f(q)$, then $(t,q) \in (-\varepsilon_{p_1}, \varepsilon_{p_1}) \times W_{p_1}$, so $\Phi(t,q) \in S$ if and only if $t = 0$.

Let $\delta = f/2$. We will show that $\Phi|_{\mathcal{O}_\delta}$ is injective, where $\mathcal{O}_\delta$ is defined by (1.2.4.1). Suppose $\Phi(t,q) = \Phi(t',q')$ for some $(t,q), (t',q') \in \mathcal{O}_\delta$. By renaming the points if necessary, we may arrange that $f(q') \leq f(q)$. Our assumption means that $\theta_t(q) = \theta_{t'}(q')$, and the group law for θ then implies that $\theta_{t-t'}(q) = q' \in S$. The fact that (t,q) and (t',q') are in $\mathcal{O}_\delta$ implies that

$$|t - t'| \leq |t| + |t'| \leq \frac{1}{2} f(q) + \frac{1}{2} f(q') \leq f(q)$$

which forces $|t - t'| = 0$ by our previous argument, and thus $q = q'$.

Only (d) remains. If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta}$ is an injective smooth immersion between manifolds of the same dimension, so it is an embedding (Proposition **??**(d)) and a diffeomorphism onto an open submanifold (Proposition **??**). □

1.2.4.1 Regular points and singular points

If V is a vector field on M, a point $p \in M$ is said to be a singular point of V if $V_p = 0$, and a regular point otherwise. The next proposition shows that the integral curves starting at regular and singular points behave very differently from each other.

Proposition 1.2.4.3. *Let V be a smooth vector field on a smooth manifold M, and let $\theta : \mathcal{D} \to M$ be the flow generated by V. If $p \in M$ is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) = p$. If p is a regular point, then $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is a smooth immersion.*

Proof. If $V_p = 0$, then the constant curve $\gamma : \mathbb{R} \to M$ given by $\gamma(t) = p$ is clearly an integral curve of V, so by uniqueness and maximality it must be equal to $\theta^{(p)}$.

To verify the second statement, we prove its contrapositive: if $\theta^{(p)}$ is not an immersion, then p is a singular point. The assumption that $\theta^{(p)}$ is not an immersion means that $\det \theta^{(p)}(s) = 0$ for some $s \in \mathcal{D}^{(p)}$. Write $q = \theta^{(p)}(s)$. Then the argument in the preceding paragraph implies that $\mathcal{D}^{(q)} = \mathbb{R}$ and $\theta^{(q)}(t) = q$ for all $t \in \mathbb{R}$. It follows from Theorem 1.2.3.4(b) that $\mathcal{D}^{(p)} = \mathbb{R}$ as well, and for all $t \in \mathbb{R}$ the group law gives

$$\theta^{(p)}(t) = \theta_t(p) = \theta_{t-s} \circ \theta_s(p) = \theta_{t-s}(q) = q.$$

Setting $t = 0$ yields $p = q$, and thus $\theta^{(p)}(t) \equiv p$ and $V_p = 0$. □

If $\theta : \mathcal{D} \to M$ is a flow, a point $p \in M$ is called an **equilibrium point** of θ if $\theta(t,p) = p$ for all $t \in \mathcal{D}^{(p)}$. Proposition 1.2.4.3 shows that the equilibrium points of a smooth flow are precisely the singular points of its infinitesimal generator. The next theorem completely describes, up to diffeomorphism, exactly what a vector field looks like in a neighborhood of a regular point.

Theorem 1.2.4.4 (Canonical Form Near a Regular Point). *Let V be a smooth vector field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\partial/\partial s^1$. If $S \subseteq M$ is any embedded hypersurface with $p \in S$ and $V_p \notin T_pS$, then the coordinates can also be chosen so that s^1 is a local defining function for S.*

Proof. If no hypersurface S is given, choose any smooth coordinates $(U, (x^i))$ centered at p, and let $S \subseteq U$ be the hypersurface defined by $x^j = 0$ where j is chosen so that $V^j(p) \neq 0$. (Recall that p is a regular point of V.)

Regardless of whether S was given or was constructed as above, since $V_p \notin T_pS$, we can shrink S if necessary so that V is nowhere tangent to S. The flowout theorem then says that there is a flow domain $\mathcal{O}_\delta \subseteq \mathbb{R} \times S$ such that the flow of V restricts to a diffeomorphism Φ from $\mathcal{O}_\delta$ onto an open subset $W \subseteq M$ containing S. There is a product neighborhood $(-\varepsilon, \varepsilon) \times W_0$ of $(0,p)$ in $\mathcal{O}_\delta$. Choose a smooth local parametrization $X : \Omega \to S$ whose image is contained in W_0, where Ω is an open subset of $\mathbb{R}^{n-1}$ with coordinates denoted by $(s^2, \ldots, s^n)$. It follows that the map $\Psi : (-\varepsilon, \varepsilon) \times \Omega \to M$ given by

$$\Psi(t, s^2, \ldots, s^n) := \Phi(t, X(s^2, \ldots, s^n))$$

is a diffeomorphism onto a neighborhood of p in M. Because the diffeomorphism $(t, s^2, \ldots, s^n) \mapsto (t, X(s^2, \ldots, s^n))$ push $\partial/\partial t$ forward to itself and $\Phi_*(\partial/\partial t) = V$, it follows that $\Psi_*(\partial/\partial t) = V$. Thus Ψ^{-1} is a smooth coordinate chart in which V has the coordinate representation $\partial/\partial t$. Renaming t to s^1 completes the proof. □

The proof of the canonical form theorem actually provides a technique for finding coordinates that put a given vector field V in canonical form, at least when the corresponding system of ODEs can be explicitly solved: begin with a hypersurface S to which V is not tangent and a local parametrization $X : \Omega \to S$, and form the composite map $\Psi(t,s) = \theta_t(X(s))$, where θ is the flow of V. The desired coordinate map is then the inverse of Ψ. The procedure is best illustrated by an example.

Example 1.2.4.5. Let $W = x\partial/\partial y - y\partial/\partial x$ on $\mathbb{R}^2$. We computed the flow of W in Example 1.2.2.2. The point $(1,0) \in \mathbb{R}^2$ is a regular point of W, because $W_{(1,0)} = \partial/\partial y|_{(1,0)}$. Because W has nonzero y-coordinate there, we can take S to be the x-axis, parametrized by $X(s) = (s,0)$. We define $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(t,s) = \theta_t(s,0) = (s\cos t, s\sin t)$$

and then solve locally for (t,s) in terms of (x,y) to obtain the following coordinate map in a neighborhood of $(1,0)$:

$$(t,s) = \Psi^{-1}(x,y) = (\arctan(y/s), \sqrt{x^2+y^2})$$

It is easy to check that $W = \partial/\partial t$ in these coordinates.

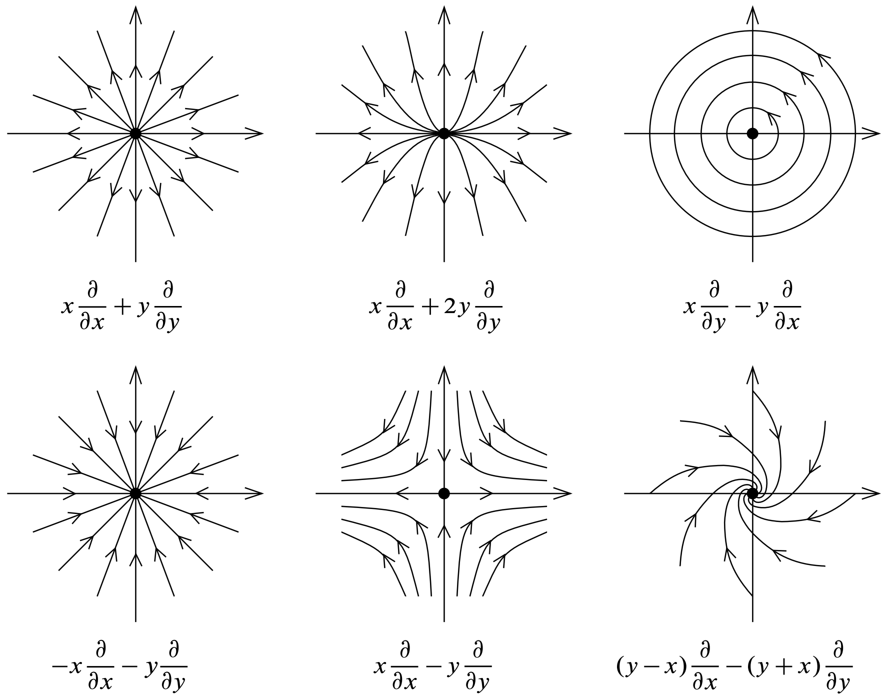


Figure 1.2: Examples of flows near equilibrium points.

The canonical form theorem shows that a flow in a neighborhood of a regular point behaves, up to diffeomorphism, just like translation along parallel coordinate lines in $\mathbb{R}^n$. Thus all of the interesting local behavior of the flow is concentrated near its equilibrium points. The flow around equilibrium points can exhibit a wide variety of behaviors, such as closed orbits surrounding the equilibrium point, orbits converging to the equilibrium point as $t \to +\infty$ or $-\infty$, and many more complicated phenomena. Some typical 2-dimensional examples are illustrated in Figure 1.2.

1.2.5 Flows and flowouts on manifolds with boundary

On a manifold with boundary, the definitions of flow domain, flow, and infinitesimal generator of a flow are exactly the same as on a manifold without boundary. In general, a smooth vector field on a manifold with boundary need not generate a flow, because, for example, the integral curves starting at some boundary points might be defined only on half-open intervals. But there is a variant of the flowout theorem for manifolds with boundary, which has many important applications.

Suppose M is a smooth manifold with nonempty boundary. The next theorem describes a sort of *one-sided flowout* from ∂M determined by a vector field that is inward pointing everywhere on ∂M.

Theorem 1.2.5.1 (Boundary Flowout Theorem). *Let M be a smooth manifold with nonempty boundary, and let N be a smooth vector field on M that is inward-pointing at each point of ∂M. There exist a smooth function $\delta : \partial M \to \mathbb{R}^+$ and a smooth embedding $\Psi : \mathcal{P}_\delta \to M$ where*

$$\mathcal{P}_\delta = \{(t,p) : p \in \partial M, 0 \le t < \delta(p)\} \subseteq \mathbb{R} \times \partial M$$

such that $\Phi(\mathcal{P}_\delta)$ is a neighborhood of ∂M, and for each $p \in \partial M$ the map $t \mapsto \Psi(t,p)$ is an integral curve of N starting at p.

Proof. We can define a flow in the same way as the flow theorem, but the domain is now not open. For every $p \in \partial M$, $\theta^{(p)}$ is a curve starting at p point into M. Let $f(p)$ be the first time that $\theta^{(p)}$ hits the boundary again, and define $\delta(p) = f(p)/2$. Then we can define $\mathcal{P}_\delta$, and it is easy to see Φ is an immersion. Assume $\Phi(t,p) = \Phi(t',p')$ with $(t,p),(t',p') \in \mathcal{P}_\delta$. Let's assume $t \le t'$, then from $\theta_t(p) = \theta_{t'}(p')$ and the group law, we obtain

$$\theta_{t'-t}(p') = p \in \partial M$$

But $t' - t \le t' < \delta(p') < f(p')$, this contradicts our definition. Thus Φ is an injective immersion. Since ∂M has dimension $(n-1)$, it is in fact an embedding. $\square$

Let M be a smooth manifold with boundary. A neighborhood of ∂M is called a **collar neighborhood** if it is the image of a smooth embedding $[0,1) \times \partial M \to M$ that restricts to the obvious identification $\{0\} \times \partial M \to \partial M$.

Theorem 1.2.5.2 (Collar Neighborhood Theorem). *If M is a smooth manifold with nonempty boundary, then ∂M has a collar neighborhood.*

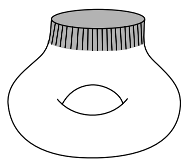


Figure 1.1: A collar neighborhood of the boundary.

Proof. By the result of Exercise 1.1.2, there exists a smooth vector field $N \in \mathfrak{X}(M)$ whose restriction to ∂M is everywhere inward-pointing. Let $\delta : M \to \mathbb{R}^+$ and $\Phi : \mathcal{P}_\delta \to M$ be as in Theorem 1.2.5.1, and define a map $\psi : [0,1) \times \partial M \to \mathcal{P}_\delta$ by $\psi(t,p) = (t\delta(p), p)$. Then is a diffeomorphism that restricts to the identity on $\{0\} \times \partial M$ and therefore the map $\Phi \circ \psi : [0,1) \times \partial M \to M$ is a smooth embedding with open image that restricts to the usual usual identification $\{0\} \times \partial M \to \partial M$. The image of $\Phi \circ \psi$ is a collar neighborhood of ∂M. $\square$

Our first application of the collar neighborhood theorem shows (among other things) that every smooth manifold with boundary is homotopy equivalent to its interior.

Theorem 1.2.5.3. *Let M be a smooth manifold with nonempty boundary, and let $\iota : \operatorname{Int} M \hookrightarrow M$ denote inclusion. There exists a proper smooth embedding $R : M \to \operatorname{Int} M$ such that both $\iota \circ R : M \to M$ and $R \circ \iota : \operatorname{Int} M \to \operatorname{Int} M$ are smoothly homotopic to identity maps. Therefore, ι is a homotopy equivalence.*

Proof. Theorem 1.2.5.2 shows that ∂M has a collar neighborhood C_0 in M, which is the image of a smooth embedding $E_0 : [0,1) \times \partial M \to M$ satisfying $E_0(0,x) = x$ for all $x \in \partial M$. Let $f : M \to \mathbb{R}^+$ be a smooth positive exhaustion function. Note that

$$W = \{(t,x) : f(E_0(t,x)) > f(x) - 1\} \subseteq [0,1) \times \partial M$$

is an open subset containing $\{0\} \times \partial M$. Using a partition of unity as in the proof of Theorem 1.2.4.1, we may construct a smooth positive function $\delta : \partial M \to \mathbb{R}^+$ such that $(t,x) \in W$ whenever $0 \le t < \delta(x)$. Define

$$E : [0,1) \times \partial M \to M, \quad E(t,x) = E_0(t\delta(x), x).$$

Then E is a diffeomorphism onto a collar neighborhood C of ∂M, and by construction

$$f(E(t,x)) > f(x) - 1 \quad \text{for all } (t,x) \in [0,1) \times \partial M.$$

We claim that for each $a \in (0,1)$, the set $E([0,a] \times \partial M)$ is closed in M: let p be a limit point of $E([0,a] \times \partial M)$ in M, then there is a sequence $\{(t_i,x_i)\}$ in $[0,a] \times \partial M$ such that $E(t_i,x_i) \to p$. Then $f(E(t_i,x_i))$ remains bounded, and thus $f(x_i) < f(E(t_i,x_i)) + 1$ also remains bounded. Since ∂M is closed in M, $f|_{\partial M}$ is also an exhaustion function, and therefore the sequence $\{x_i\}$ lies in some compact subset K of ∂M. But then (t_i,x_i) is contained in the compact subset $[0,a] \times K$, so by passing to a subsequence, we may assume $(t_i,x_i) \to (t_0,x_0)$. Therefore, $p = E(t_0,x_0) \in E([0,a] \times \partial M)$, implying $E([0,a] \times \partial M)$ is closed.

To simplify notation, we will use this embedding to identify C with $[0,1) \times \partial M$ and denote a point in C as an ordered pair (s,x), with $s \in [0,1)$ and $x \in \partial M$; thus $(s,x) \in \partial M$ if and only if $s = 0$. For any $a \in (0,1]$, let

$$C(a) = \{(s,x) \in C : 0 \leq s < a\} = E([0,a) \times \partial M) \quad \text{and} \quad M(a) = M - C(a)$$

To see that $M(a)$ is a regular domain, note first that it is closed in M because it is the complement of the open set $C(a)$. Let $p \in M(a)$ be arbitrary. If $p \notin E([0,a] \times \partial M)$, then p has a neighborhood in $\operatorname{Int} M$ contained in $M(a)$ by the argument above. If $p \in E([0,a] \times \partial M)$, then $p = E(a,x)$ for some $x \in \partial M$. Let U be a smooth chart of x, then $E([a,1) \times U)$ is a slice boundary chart for p. This proves $M(a)$ is a properly embedded submanifold.

Let $\psi : [0,1) \to [1/3,1)$ be an increasing diffeomorphism that satisfies $\psi(s) = s$ for $2/3 \leq s < 1$, and define $R : M \to \operatorname{Int} M$ by

$$R(p) = \begin{cases} p, & p \in \operatorname{Int}(M(2/3)); \\ (\psi(s),x), & p = (s,x) \in C. \end{cases}$$

These definitions both give the identity map on the set $C - C(2/3)$ where they overlap, so R is smooth by the gluing lemma. It is a diffeomorphism onto the closed subset $M(1/3)$, so it is a proper smooth embedding of M into $\operatorname{Int} M$.

Define $H : M \times I \to M$ by

$$H(p,t) = \begin{cases} p, & p \in \operatorname{Int}(M(2/3)); \\ (ts + (1-t)\psi(s),x), & p = (s,x) \in C \end{cases}$$

As before, H is smooth, and a straightforward verification shows that it is a homotopy from $\iota \circ R$ to id_M. If $p \in \operatorname{Int} M$, then $H(p,t) = p$ for all $t \in I$, so the restriction of H to $(\operatorname{Int} M)\times$ is a smooth homotopy from $R \circ \iota$ to $\mathrm{id}_{\operatorname{Int} M}$. □

Theorem 1.2.5.3 is the main ingredient in the following generalization of the Whitney approximation theorem.

Theorem 1.2.5.4 (Whitney Approximation for Manifolds with Boundary). *If M and N are smooth manifolds with boundary, then every continuous map from M to N is homotopic to a smooth map.*

Proof. Theorem **??** takes care of the case in which $\partial N = \varnothing$, so we may assume that $\partial N \neq \varnothing$. Let $F : M \to N$ be a continuous map, let $\iota : \operatorname{Int} N \to N$ be inclusion, and let $R : M \to \operatorname{Int} M$ be the map constructed in Theorem 1.2.5.3, so that $\iota \circ R : N \to N$ is smoothly homotopic to id_N. Theorem 1.2.5.3 shows that $R \circ F : M \to \operatorname{Int} N$ is homotopic to a smooth map G. It follows that

$$\iota \circ G \simeq \iota \circ R \circ F \simeq F$$

so $\iota \circ G : M \to N$ is a smooth map homotopic to F. □

The next theorem generalizes the main result of Theorem **??** to the case of maps into a manifold with boundary.

Theorem 1.2.5.5. *Suppose M and N are smooth manifolds with or without boundary. If $F : M \to N$ are homotopic smooth maps, then they are smoothly homotopic.*

Proof. Theorem **??** takes care of the case $\partial N = \varnothing$, so we may assume that N has nonempty boundary. Let $\iota : \operatorname{Int} N \to N$ and $R : N \to \operatorname{Int} N$ be as in Theorem 1.2.5.3. Then $R \circ F$ and $R \circ G$ are homotopic smooth maps from M to $\operatorname{Int} N$, so Theorem **??** shows that they are smoothly homotopic to each other. Thus we have smooth homotopies

$$F \simeq \iota \circ R \circ F \simeq \iota \circ R \circ G \simeq G$$

By transitivity of smooth homotopy, it follows that F is smoothly homotopic to G. □

The following theorem is probably the most important application of the collar neighborhood theorem.

Theorem 1.2.5.6 (Attaching Smooth Manifolds Along Boundaries). *Let M and N be smooth n-manifolds with nonempty boundaries, and suppose $h : \partial N \to \partial M$ is a diffeomorphism. Let $M \cup_h N$ be the adjunction space formed by identifying each $x \in \partial N$ with $h(x) \in \partial M$. Then $M \cup_h N$ is a topological manifold without boundary, and has a smooth structure such that there are regular domains $M', N' \subseteq M \cup_h N$ diffeomorphic to M and N, respectively, and satisfying*

$$M' \cup N' = M \cup_h N, \quad M' \cap N' = \partial M = \partial N \tag{1.2.5.1}$$

If M and N are both compact, then $M \cup_h N$ is compact, and if they are both connected, then $M \cup_h N$ is connected.

Proof. For simplicity, let $X = M \cup_h N$ denote the quotient space and $\pi : M \amalg N$ the quotient map. Let $V \subseteq M$ and $W \subseteq N$ be collar neighborhoods of ∂M and ∂N, respectively, and denote the corresponding diffeomorphisms by

$$\alpha : [0, 1) \times \partial M \to V \quad \text{and} \quad \beta : [0, 1) \times \partial N \to W.$$

Define a continuous map $\Phi : V \amalg W \to (-1, 1) \times \partial M$ by

$$\Phi(x) = \begin{cases} (-t, p), & x = \alpha(t, p) \in V; \\ (t, h(q)), & x = \beta(t, q) \in W. \end{cases}$$

Then the restriction of Φ to V or W is a topological embedding with closed image, from which it follows easily that Φ is a closed map. Because Φ is constant on the fibers of π, it descends to a continuous map $\widetilde{\Phi} : \pi(V \amalg W) \to (-1, 1) \times \partial M$. This map is bijective, and it is a homeomorphism because it too is a closed map: if $K \subseteq \pi(V \amalg W)$ is closed, then $\pi^{-1}(K)$ is closed in $V \amalg W$, and therefore $\widetilde{\Phi}(K) = \Phi(\pi^{-1}(K))$ is closed. Thus, $\pi(V \amalg W)$ is a topological n-manifold. On the other hand, the restriction of π to the saturated open subset $\operatorname{Int} M \amalg \operatorname{Int} N$ is an injective quotient map and thus a homeomorphism onto its image; this shows that X is locally Euclidean of dimension n. Since X is the union of the second-countable open subsets $\pi(\operatorname{Int} M \amalg \operatorname{Int} N)$ and $\pi(V \amalg W)$, it is second-countable. Any two fibers in $M \amalg N$ can be separated by saturated open subsets, so X is Hausdorff. Thus it is a topological n-manifold.

We define a collection of charts on X as follows:

$$\begin{aligned} &(\pi(U), \varphi \circ \pi^{-1}|_{\pi(U)}), && \text{for each smooth chart } (U, \varphi) \text{ for } \operatorname{Int} M \text{ or } \operatorname{Int} N; \\ &(\widetilde{\Phi}^{-1}(U), \varphi \circ \widetilde{\Phi}|_{\widetilde{\Phi}^{-1}(U)}), && \text{for each smooth chart } (U, \varphi) \text{ for } (-1, 1) \times \partial M. \end{aligned}$$

These maps are compositions of homeomorphisms, so they define coordinate charts on X, and it is straightforward to check that they are all smoothly compatible and thus define a smooth structure on X. The restriction of π to M is continuous, closed, and injective, and thus it is a proper embedding. In terms of any of the smooth charts constructed above and corresponding charts on M, π has a coordinate representation that is either an identity map or an inclusion map, so it is a smooth embedding, and its image M' is therefore a regular domain in X. Similar considerations apply to N, and the relations (1.2.5.1) follow immediately from the definitions.

If M and N are compact, then X is the union of the compact sets M' and N', so it is compact; and if they are connected, then X is the union of the connected sets M' and N' with points of $\partial M' = \partial N'$in common, so it is connected. □

Corollary 1.2.5.7. *Suppose M and N are smooth n-manifolds with boundary, $A \subseteq \partial M$ and $B \subseteq \partial N$ are nonempty subsets that are unions of components of the respective boundaries, and $h : B \to A$ is a diffeomorphism. Then $M \cup_h N$ is a topological manifold with boundary, and can be given a smooth structure such that M and N are diffeomorphic to regular domains in $M \cup_h N$.*

Example 1.2.5.8 (Connected Sums). Let M_1, M_2 be connected smooth manifolds of dimension n. For $i = 1, 2$, let U_i be a regular coordinate ball centered at some point $p_i \in M_i$, and let $M_i' = M_i - U_i$. Exercise **??** shows that each M_i' is a smooth manifold with boundary whose boundary is diffeomorphic to S^{n-1}. A smooth connected sum of M_1 and M_2, denoted by $M_1 \# M_2$, is a smooth manifold formed by choosing a diffeomorphism from ∂M_1 to ∂M_2 and attaching M_1' and M_2' along their boundaries.

If M is any smooth manifold with boundary, Theorem 1.2.5.3 shows that M can be properly embedded into a smooth manifold without boundary (namely, a copy of $\operatorname{Int} M$). The next example shows a different way that M can be so embedded; this construction has the advantage of embedding M into a compact manifold when M itself is compact.

Example 1.2.5.9 (The Double of a Smooth Manifold with Boundary). Let M be a smooth manifold with boundary. The **double of M** is the manifold $D(M) = M \cup_{\mathrm{id}} M$, where $\mathrm{id} : \partial M \to \partial M$ is the identity map of ∂M, it is obtained from $M \amalg M$ by identifying each boundary point in one copy of M with the same boundary point in the other. It is a smooth manifold without boundary, and contains two regular domains diffeomorphic to M. It is easy to check that $D(M)$ compact if and only if M is compact, and connected if and only if M is connected. (It is useful to extend the definition to manifolds without boundary by defining $D(M) = M \amalg M$ when $\partial M = \varnothing$.)

Although vector fields on manifolds with boundary do not always generate flows, there is one circumstance in which they do: when the vector field is everywhere tangent to the boundary. To prove this, we begin with the following special case.

Lemma 1.2.5.10. *Suppose M is a smooth manifold and $D \subseteq M$ is a regular domain. If V is a smooth vector field on M that is tangent to ∂D, then every integral curve of V that starts in D remains in D as long as it is defined.*

Proof. Suppose $\gamma : J \to M$ is an integral curve of V with $\gamma(0) \in D$. Define $\mathcal{T} = \{t \in J : \gamma(t) \in D\}$. We will show that $\mathcal{T}$ is both open and closed in J; since J is an interval, this implies $\mathcal{T} = J$ and proves the lemma.

Since D is closed in M (by definition of a regular domain), $\mathcal{D}$ is closed in J by continuity. To prove it is open, suppose $t_0 \in \mathcal{T}$. If $\gamma(t_0) \in \operatorname{Int} D$, then a neighborhood of t_0 is contained in $\mathcal{T}$ by continuity, so we can assume $\gamma(t_0) \in \partial D$. Because V is tangent to ∂D, Proposition 1.1.2.7 shows that there is a smooth vector field $W = V|_{\partial D}$ that is ι-related to V, where $\iota : \partial D \hookrightarrow M$ is inclusion. Let $\widetilde{\gamma} : (t_0 - \varepsilon, t_0 + \varepsilon) \to \partial D$ be an integral curve of W with $\widetilde{\gamma}(t_0) = \gamma(t_0)$. By naturality of integral curves (Proposition 1.2.1.5), $\iota \circ \widetilde{\gamma}$ is an integral curve of V with the same initial condition, so by uniqueness it must be equal to γ where both are defined. This shows that $\gamma(t) \subseteq \partial D \subseteq D$ for t in some neighborhood of t_0, so $\mathcal{T}$ is open in J as claimed. $\square$

Theorem 1.2.5.11 (Flows on Manifolds with Boundary). *The conclusions of Theorem 1.2.3.4 remain true if M is a smooth manifold with boundary and V is a smooth vector field on M that is tangent to ∂M.*

Proof. We can consider M as a regular domain in its double $D(M)$. By the extension lemma for vector fields, we can extend V to a smooth vector field $\widetilde{V}$ on $D(M)$. Let $\widetilde{\theta} : \mathcal{D} \to D(M)$ be the flow of $\widetilde{V}$, and let $\mathcal{D} = (\mathbb{R} \times M) \cap \widetilde{\mathcal{D}}$ and $\theta = \widetilde{\theta}|_M$. Then Lemma 1.2.5.10 guarantees that θ maps $\mathcal{D}$ into M, and the rest of the conclusions follow from Theorem 1.2.3.4 applied to $\widetilde{V}$. $\square$

For manifolds with boundary, the canonical form theorem has the following variant.

Theorem 1.2.5.12 (Canonical Form Near a Regular Point on the Boundary). *Let M be a smooth manifold with boundary and let V be a smooth vector field on M that is tangent to ∂M. If $p \in \partial M$ is a regular point of V, there exist smooth boundary coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\partial/\partial s^1$.*

1.2.6 Lie derivatives

Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is the flow of V. For any smooth vector field W on M, define a rough vector field on M, denoted by $\mathfrak{L}_V W$ and called the Lie derivative of W with respect to V, by

$$(\mathfrak{L}_V W)_p = \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) - W_p}{t}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighborhood of p, and θ_{-t} is the inverse of θ_t, so both $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$ and W_p are elements of T_pM.

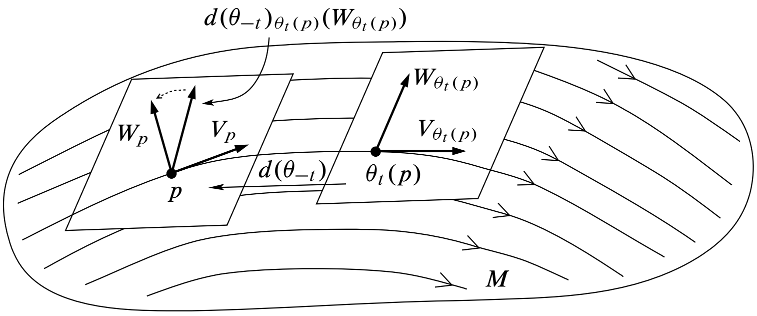


Figure 1.1: The Lie derivative of a vector field.

If M has nonempty boundary, this definition of $\mathfrak{L}_V W$ makes sense as long as V is tangent to ∂M so that its flow exists by Theorem 1.2.5.11.

Lemma 1.2.6.1. *Suppose M is a smooth manifold with or without boundary, and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume in addition that V is tangent to ∂M. Then $(\mathfrak{L}_V W)_p$ exists for every $p \in M$, and $\mathfrak{L}_V W$ is a smooth vector field.*

Proof. Let θ be the flow of V. For arbitrary $p \in M$, let $(U, (x^i))$ be a smooth chart containing p. Choose an open interval J_0 containing 0 and an open subset $U_0 \subseteq U$ containing p such thatθ maps $J_0 \times U_0$ into U. For $(t, x) \in J_0 \times U_0$, write the component functions of θ as $(\theta^1(t,x), \ldots, \theta^n(t,x))$. Then for any $(t,x) \in J_0 \times U_0$, the matrix of $d(\theta_{-t})_{\theta_t(p)} : T_{\theta_t(x)}M \to T_xM$ is

$$\left(\frac{\partial \theta^i}{\partial x^j}(-t, \theta_t(x))\right)$$

Therefore,

$$d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta_t(x)) W^j(\theta_t(x)) \frac{\partial}{\partial x^i}\bigg|_x$$

Because θ^i and W^j are smooth functions, the coefficient of $\partial/\partial x^i|_x$ depends smoothly on (t,x). It follows that $(\mathfrak{L}_V W)_x$, which is obtained by taking the derivative of this expression with respect to t and setting $t = 0$, exists for each $x \in U_0$ and depends smoothly on x. □

The definition of $\mathfrak{L}_V W$ is not very useful for computations, because typically the flow is difficult or impossible to write down explicitly. Fortunately, there is a simple formula for computing the Lie derivative without explicitly finding the flow.

Theorem 1.2.6.2. *If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathfrak{L}_V W = [V, W]$.*

Proof. Suppose $V, W \in \mathfrak{X}(M)$, and let $\mathcal{R}(V) \subseteq M$ be the set of regular points of V (the set of points $p \in M$ such that $V_p \neq 0$). Note that $\mathcal{R}(V)$ is open in M by continuity, and its closure is the support of V. We will show that $(\mathfrak{L}_V W)_p = [V, W]_p$ for all $p \in M$, by considering three cases.

First let $p \in \mathcal{R}(V)$. In this case, we can choose smooth coordinates (u^i) on a neighborhood of p in which V has the coordinate representation $V = \partial/\partial u^i$ (Theorem 1.2.4.4). In these coordinates, the flow of V is $\theta_t(u) = (u^1 + t, u^2, \ldots, u^n)$. For each fixed t, the matrix of $d(\theta_{-t})_{\theta_t(x)}$ in these coordinates is the identity at every point. Consequently, for any $u \in U$,

$$\begin{aligned} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) &= d(\theta_{-t})_{\theta_t(p)}\left(W^j(u^1 + t, u^2, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_{\theta_t(u)}\right) \\ &= W^j(u^1 + t, u^2, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_u \end{aligned}$$

Using the definition of the Lie derivative, we obtain

$$(\mathfrak{L}_V W)_u = \frac{d}{dt}\bigg|_{t=0} W^j(u^1 + t, u^2, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_u = \frac{\partial W^j}{\partial u^1}(u^1, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_u$$

On the other hand, by virtue of formula (1.1.3.1) for the Lie bracket in coordinates, $[V, W]_u$ is easily seen to be equal to the same expression.

Now let $p \in \mathrm{supp}(V)$. Because $\mathrm{supp}(V)$ is the closure of $\mathcal{R}(V)$, it follows by continuity that $(\mathfrak{L}_V W)_p = [V, W]_p$ for $p \in \mathrm{supp}(V)$.

Finally, let $p \in M \setminus \mathrm{supp}(V)$. In this case, $V \equiv 0$ on a neighborhood of p. On the one hand, this implies that θ_t is equal to the identity map in a neighborhood of p for all t, so $d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = W_p$, which implies $(\mathfrak{L}_V W)_p = 0$. On the other hand, $[V, W] = 0$ by formula (1.1.3.1). □

This theorem allows us to extend the definition of the Lie derivative to arbitrary smooth vector fields on a smooth manifold M with boundary. Given $V, W \in \mathfrak{X}(M)$, we define $(\mathfrak{L}_V W)_p$ for $p \in \partial M$ by embedding M in a smooth manifold $\widetilde{M}$ without boundary (such as the double of M), extending V and W to smooth vector fields on $\widetilde{M}$, and computing the Lie derivative there. By virtue of the preceding theorem, $(\mathfrak{L}_V W)_p = [V, W]_p$ is independent of the choice of extension.

Theorem 1.2.6.2 also gives us a geometric interpretation of the Lie bracket of two vector fields: it is the directional derivative of the second vector field along the flow of the first. A number of nonobvious properties of the Lie derivative follow immediately from things we already know about Lie brackets.

Corollary 1.2.6.3. *Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.*

- *(a)* $\mathfrak{L}_V W = -\mathfrak{L}_W V$.
- *(b)* $\mathfrak{L}_V [W, X] = [\mathfrak{L}_V W, X] + [W, \mathfrak{L}_V X]$.
- *(c)* $\mathfrak{L}_{[V,W]} X = \mathfrak{L}_V \mathfrak{L}_W X - \mathfrak{L}_W \mathfrak{L}_V X$.
- (d) *If $g \in C^\infty(M)$, then $\mathfrak{L}_V(gW) = (Vg)W + g\mathfrak{L}_V W$.*
- (e) *If $F : M \to N$ is a diffeomorphism, then $F_*(\mathfrak{L}_V X) = \mathfrak{L}_{F_* V} F_* X$.*

Proof. Recall the Jacobi identity

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

Thus

$$\begin{aligned} \mathfrak{L}_V [W, X] &= [V, [W, X]] = -[W, [X, V]] - [X, [V, W]] \\ &= [[V, W], X] + [W, [V, X]] \\ &= [\mathfrak{L}_V W, X] + [W, \mathfrak{L}_V X] \end{aligned}$$

Similarly,

$$\begin{aligned} \mathfrak{L}_{[V,W]} X &= [[V, W], X] = [V, [W, X]] + [W, [X, V]] \\ &= [V, [W, X]] - [W, [V, X]] \\ &= \mathfrak{L}_V [W, X] - \mathfrak{L}_W [V, X] \\ &= \mathfrak{L}_V \mathfrak{L}_W X - \mathfrak{L}_W \mathfrak{L}_V X \end{aligned}$$

Part (d) is a direct computation:

$$\begin{aligned} \mathfrak{L}_V(gW) &= [V, gW] = V(gW) - gWV = (Vg)W + gVW - gWV \\ &= (Vg)W + g[V, W] = (Vg)W + g\mathfrak{L}_V W \end{aligned}$$

and the last statment is a result of natruality of Lie bracket. □

Part (d) of this corollary gives a meaning to the mysterious formula (1.1.3.3) for Lie brackets of vector fields multiplied by functions: because the Lie bracket $[fV, gW]$ can be thought of as the Lie derivative $\mathfrak{L}_{fV}(gW)$, it satisfies a product rule in g and W, and because it can also be thought of as $-\mathfrak{L}_{gW}(fV)$, it satisfies a product rule in f and V as well. Expanding out these two product rules yields (1.1.3.3):

$$\begin{aligned} \mathfrak{L}_{fV}(gW) &= (fVg)W + g\mathfrak{L}_{fV} W = (fVg)W - g\mathfrak{L}_W(fV) \\ &= (fVg)W - g(Wf + f\mathfrak{L}_W V) \end{aligned}$$

$$= (fVg)W - (gWf) + fg\mathfrak{L}_V W$$

If V and W are vector fields on M and θ is the flow of V, the Lie derivative $(\mathfrak{L}_V W)_p$, by definition, expresses the t-derivative of the time-dependent vector $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$ at $t = 0$. The next proposition shows how it can also be used to compute the derivative of this expression at other times.

Proposition 1.2.6.4. *Suppose M is a smooth manifold with or without boundary and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any (t_0, p) in the domain of θ,*

$$\frac{d}{dt}\bigg|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = d(\theta_{-t_0})\big((\mathfrak{L}_V W)_{\theta_{t_0}(p)}\big)$$

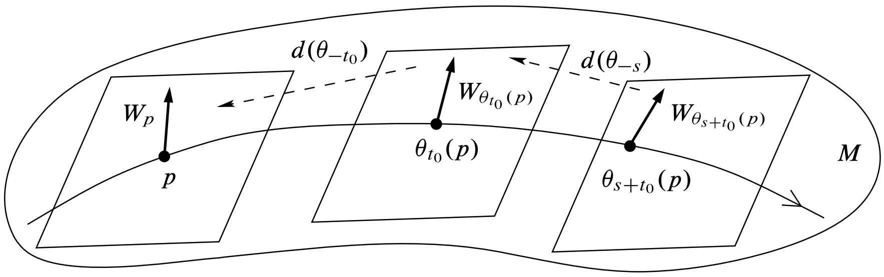


Figure 1.2: Proof of Proposition 1.2.6.4.

Proof. Let $p \in M$ be arbitrary, let $\mathcal{D}^{(p)} \subseteq \mathbb{R}$ denote the domain of the integral curve $\theta^{(p)}$, and consider the map $X : \mathcal{D}^{(p)} \to T_pM$ given by $X(t) = d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$. The argument in the proof of Lemma 1.2.6.1 shows that X is a smooth curve in the vector space T_pM. Making the change of variables $t = t_0 + s$ we obtain

$$\begin{aligned}
X'(t_0) = \frac{d}{dt}\bigg|_{s=0} X(t_0+s) &= \frac{d}{dt}\bigg|_{s=0} d(\theta_{-t_0-s})(W_{\theta_{s+t_0}(p)}) \\
&= \frac{d}{dt}\bigg|_{s=0} d(\theta_{-t_0}) \circ d(\theta_{-s})(W_{\theta_s \circ \theta_{t_0}(p)}) \\
&= d(\theta_{-t_0}) \frac{d}{dt}\bigg|_{s=0} d(\theta_{-s})(W_{\theta_s \circ \theta_{t_0}(p)}) \\
&= d(\theta_{-t_0})\big((\mathfrak{L}_V W)_{\theta_{t_0}(p)}\big)
\end{aligned}$$

as needed. □

1.2.7 Commuting vector fields

Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. We say that V and W **commute** if $VWf = WVf$ for every smooth function f, or equivalently if $[V, W] \equiv 0$. If θ is a smooth flow, a vector field W is said to be invariant under θ if W is θ_t-related to itself for each t; more precisely, this means that $W|_{M_t}$ is θ_t-related for each t, or equivalently that $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$ for all (t, p) in the domain of θ. The next proposition shows that these two concepts are intimately related.

Theorem 1.2.7.1. *For smooth vector fields V and W on a smooth manifold M, the following are equivalent:*

(a) V and W commute.

(b) W is invariant under the flow of V.

(c) V is invariant under the flow of W.

Proof. Suppose $V, W \in \mathfrak{X}(M)$, and let θ denote the flow of V. If (b) holds, then $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$ whenever (t, p) is in the domain of θ. Applying $d(\theta_{-t})_{\theta_t(p)}$ to both sides, we then conclude that $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$, which obviously implies $\mathfrak{L}_V W = [V, W] = 0$ directly from the definition of the Lie derivative. The same argument shows that (c) implies (a).

To prove that (a) implies (b), assume that $[V, W] = \mathfrak{L}_V W = 0$. Let $p \in M$ be arbitrary, and let $X(t) = d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$ for $t \in \mathcal{D}^{(p)}$. Proposition 1.2.6.4 shows that $X'(t) \equiv 0$. Since $X(0) = W_p$, this implies that $X(t) \equiv W_p$ for all $t \in \mathcal{D}^{(p)}$, and applying $d(\theta_t)_p$ to both sides yields the identity that says W is invariant under θ. The same proof also shows that (a) implies (c). □

Corollary 1.2.7.2. *Every smooth vector field is invariant under its own flow.*

Proof. Use the preceding proposition together with the fact that $[V, V] \equiv 0$. □

The deepest characterization of commuting vector fields is in terms of the relationship between their respective flows. The next theorem says that two vector fields commute if and only if their flows commute. But before we state the theorem formally, we need to examine exactly what this means. Suppose V and W are smooth vector fields on M, and let θ and denote their respective flows. If V and W are complete, it is clear what we should mean by saying their flows commute: simply that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all $s, t \in \mathbb{R}$. However, if either V or W is not complete, the most we can hope for is that this equation holds for all s and t such that both sides are defined. Unfortunately, even when the vector fields commute, their flows might not commute in this naive sense, because there are examples of commuting vector fields V and W and particular choices of t, s, and p for which both $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are defined, but they are not equal. Here is the problem: if $\theta_t \circ \psi_s(p)$ is defined for $t = t_0$ and $s = s_0$, then by the properties of flow domains, it must be defined for all t in some open interval containing 0 and t_0, but the analogous statement need not be true of s—there might be values of s between 0 and s_0 for which the integral curve of V starting at $\psi_s(p)$ does not extend all the way to $t = t_0$.

Thus we make the following definition. If θ and ψ are flows on M, we say that θ and ψ commute if the following condition holds for every $p \in M$, whenever J and K are open intervals containing 0 such that one of the expressions $\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined for all $(t, s) \in J \times K$, both are defined and they are equal. For global flows, this is the same as saying that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all s and t.

Theorem 1.2.7.3. *Smooth vector fields commute if and only if their flows commute.*

Proof. Let V and W be smooth vector fields on a smooth manifold M, and let θ and denote ψ their respective flows. Assume first that V and W commute. Suppose that $p \in M$, and J and K are open intervals containing 0 such that $\psi_s \circ \theta_t(p)$ is defined for all $(s, t) \in J \times K$. (The same proof with V and W reversed works under the assumption that the other expression is defined on such a rectangle.) By Theorem 1.2.7.1, the hypothesis implies that V is invariant under ψ. Fix any $s \in J$, and consider the curve $\gamma : K \to M$ defined by $\gamma(t) = \psi_s \circ \theta_t(p) = \psi_s(\theta^{(p)}(t))$. This curve satisfies $\gamma(0) = \psi_s(p)$, and its velocity at $t \in K$ is

$$\gamma'(t) = \frac{d}{dt}\big(\psi_s(\theta^{(p)}(t))\big) = d(\psi_s)(\dot\theta^{(p)}(t)) = d(\psi_s)(V_{\theta^{(p)}(t)}) = V_{\gamma(t)}$$

Thus, γ is an integral curve of V starting at $\psi_s(p)$. By uniqueness, therefore,

$$\gamma(t) == \theta^{(\psi_s(p))}(t) = \theta_t\big(\psi_s(p)\big)$$

This proves that θ and ψ commute.

Conversely, assume that the flows commute, and let $p \in M$. If $\varepsilon > 0$ is chosen small enough that $\psi_s \circ \theta_t(p)$ is defined whenever $|s| < \varepsilon$ and $|t| < \varepsilon$, then the hypothesis guarantees that $\psi_s \circ \theta_t(p) = \theta_t \circ \psi_s(p)$ for all such s and t. This can be rewritten in the form

$$\psi^{(\theta_t(p))}(s) = \theta_t(\psi^{(p)}(s))$$

Differentiating this relation with respect to s, we get

$$W_{\theta_t(p)} = \frac{d}{ds}\bigg|_{s=0}\psi^{(\theta_t(p))}(s) = \frac{d}{ds}\bigg|_{s=0}\theta_t(\psi^{(p)}(s)) = d(\theta_t)_p(W_p)$$

This means W_p is invariant under θ_t, thus by Theorem 1.2.7.1 V commutes with W. □

1.2.7.1 Commuting frames

Suppose M is a smooth n-manifold. Recall that a local frame for M is an n-tuple (E_i) of vector fields defined on an open subset $U \subseteq M$ such that $(E_i|_p)$ forms a basis for T_pM at each $p \in U$. A smooth local frame (E_i) for M is called a commuting frame if $[E_i, E_j] = 0$ for all i and j.

Example 1.2.7.4 (Commuting and Noncommuting Frames).

(a) The simplest examples of commuting frames are the coordinate frames. Given any smooth coordinate chart $(U, (x^i))$ for a smooth manifold M, (1.1.3.1) shows that the coordinate frame $(\partial/\partial x^i)$ is a commuting frame.

(b) The frame (E_1, E_2) for $\mathbb{R}^2$ over $\mathbb{R}^2 - \{0\}$ defined by (1.1.1.1) is not a commuting frame, because a straightforward computation shows that

$$[E_1, E_2] = \frac{y}{r^2}\frac{\partial}{\partial x} - \frac{x}{r^2}\frac{\partial}{\partial y} \neq 0$$

Because every coordinate frame is a commuting frame, and because Lie brackets are invariantly defined, it follows that a necessary condition for a smooth frame to be expressible as a coordinate frame in some smooth chart is that it be a commuting frame. Thus, the computation above shows that (E_1, E_2) cannot be expressed as a coordinate frame for $\mathbb{R}^2$ with respect to any choice of smooth local coordinates.

The next theorem shows that commuting is also a sufficient condition for a smooth frame to be locally expressible as a coordinate frame.

Theorem 1.2.7.5 (Canonical Form for Commuting Vector Fields). *Let M be a smooth n-manifold, and let $(V_1, \dots, V_k)$ be a linearly independent k-tuple of smooth commuting vector fields on an open subset $W \subseteq M$. For each $p \in W$, there exists a smooth coordinate chart $(U, (s^i))$ centered at p such that $V_i = \partial/\partial s^i$ for $1 \le i \le k$. If $S \subseteq W$ is an embedded codimension-k submanifold and p is a point of S such that T_pS is complementary to the span of $(V_1|_p, \dots, V_k|_p)$, then the coordinates can also be chosen such that $S \cap U$ is the slice defined by $s^1 = \cdots = s^k = 0$.*

Proof. Let $p \in W$ be arbitrary. If no submanifold S is given, just let S be any smooth embedded codimension-k submanifold S whose tangent space at p is complementary to the span of $(V_1|_p, \dots, V_k|_p)$ (e.g., an appropriate coordinate slice) Let $(U, (s^i))$ be a slice chart for S centered at p, with $U \subseteq W$, and with $S \cap U$ equal to the slice $\{x \in U : x^1 = \cdots = x^k = 0\}$. Our assumptions ensure that the vectors $\{V_1|_p, \dots, V_k|_p, \partial/\partial x^{k+1}|_p, \dots, \partial/\partial x^n|_p\}$ span T_pM. Since the theorem is purely local, we may as well consider $V_1, \dots, V_k$ as vector fields on $U \subseteq \mathbb{R}^n$, and consider S to be the subset of U where the first k coordinates vanish. The basic idea of this proof is similar to that of the flowout theorem, except that we have to do a bit of extra work to make use of the hypothesis that the vector fields commute.

Let θ_i denote the flow of V_i for $i = 1, \dots, k$. There exist $\varepsilon > 0$ and a neighborhood Y of p in U such that the composition $(\theta_1)_{t_1} \circ \cdots \circ (\theta_k)_{t_k}$ is defined on Y and maps Y into U whenever $|t_1|, \dots, |t_k|$ are all less than ε. (To see this, just choose $\varepsilon_k > 0$ and $U_k \subseteq U$ such that θ_k maps $(-\varepsilon_k, \varepsilon_k) \times U_k$ into U, and then inductively choose ε_i and U_i such that θ_i maps $(-\varepsilon_i, \varepsilon_i) \times U_i$ into U_{i+1}. Taking $\varepsilon = \min_i\{\varepsilon_i\}$ and $Y = U_1$ does the trick.)

Define $\Omega \subseteq \mathbb{R}^{n-k}$ by

$$\Omega = \{(s^{k+1}, \dots, s^n) \in \mathbb{R}^{n-k} : (0, \dots, 0, s^{k+1}, \dots, s^n) \in Y\}$$

and define $\Phi : (-\varepsilon, \varepsilon)^k \times \Omega \to U$ by

$$\Phi(s^1, \dots, s^k, s^{k+1}, \dots, s^n) = (\theta_1)_{s_1} \circ \cdots \circ (\theta_k)_{s_k}(0, \dots, 0, s^{k+1}, \dots, s^n)$$

By construction, $\Phi(\{0\} \times \Omega) = S \cap Y$.

We show next that $\partial/\partial s^i$ is Φ-related to V_i for $1 \le i \le k$. Because the flows θ_i commute, for any $i \in \{1, \dots, k\}$ and any $s_0 \in (-\varepsilon, \varepsilon)^k \times \Omega$ we have

$$\begin{aligned} d\Phi_{s_0}\left(\frac{\partial}{\partial s^i}\Big|_{s_0}\right) f &= \frac{\partial}{\partial s^i}\Big|_{s_0} f\big(\Phi(s^1, \dots, s^n)\big) \\ &= \frac{\partial}{\partial s^i}\Big|_{s_0} f\big((\theta_1)_{s_1} \circ \cdots \circ (\theta_k)_{s^k}(0, \dots, 0, s^{k+1}, \dots, s^n)\big) \end{aligned}$$

$$= \frac{\partial}{\partial s^i}\bigg|_{s_0} f\big((\theta_i)_{s^i} \circ (\theta_1)_{s_1} \circ \cdots \circ (\theta_{i-1})_{s^{i-1}} \circ (\theta_{i+1})_{s^{i+1}} \\ \circ \cdots \circ (\theta_k)_{s^k}(0, \ldots, 0, s^{k+1}, \ldots, s^n)\big)$$

For any $q \in M$, $t \mapsto (\theta_i)_t(q)$ is an integral curve of V_i, so this last expression is equal to $V_i|_{\Phi(s_0)} f$, which proves the claim.

Next we show that $d\Phi_0$ is invertible. The computation above shows that

$$d\Phi_0\Big(\frac{\partial}{\partial s^i}\Big|_0\Big) = V_i|_p, \quad 1 \le i \le k.$$

On the other hand, since $\Phi(0, \ldots, 0, s^{k+1}, \ldots, s^n) = (0, \ldots, 0, s^{k+1}, \ldots, s^n)$, it follows immediately that

$$d\Phi_0\Big(\frac{\partial}{\partial s^i}\Big|_0\Big) = \frac{\partial}{\partial x^i}\Big|_0, \quad k+1 \le i \le n.$$

It follows that $d\Phi_0$ takes $(\partial/\partial s^1|_0, \ldots, \partial/\partial s^n|_0)$ to $(V_1|_p, \ldots, V_k|_p, \partial/\partial x^{k+1}|_p, \ldots, \partial/\partial x^n|_p)$. By the inverse function theorem, Φ is a diffeomorphism in a neighborhood of 0, and $\varphi = \Phi^{-1}$ is a smooth coordinate map that takes V_i to $\partial/\partial s^i$ for $1 \le i \le k$, and takes S to the slice $s^1 = \cdots = s^k = 0$. □

Just as in the case of a single vector field, the proof of Theorem 1.2.7.5 suggests a technique for finding explicit coordinates that put a set of commuting vector fields into canonical form, as long as their flows can be found explicitly. The method can be summarized as follows: Begin with an $(n-k)$-dimensional submanifold S whose tangent space at p is complementary to the span of $(V_1|_p, \ldots, V_k|_p)$. Then define Φ by starting at an arbitrary point in S and following the k flows successively for k arbitrary times. Because the flows commute, it does not matter in which order they are applied. An example will help to clarify the procedure.

Example 1.2.7.6. Consider the following two vector fields on $\mathbb{R}^2$:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

A computation shows that $[V, W] = 0$. The flow of V is

$$\theta_t(x, y) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

and an easy verification shows that the flow of W is

$$\eta_t(x, y) = (e^t x, e^t y)$$

At $p = (1, 0)$, V_p and W_p are linearly independent. Because $k = n = 2$ in this case, we can take the subset S to be the single point $\{(1, 0)\}$, and define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(s, t) = \eta_t \circ \theta_s(1, 0) = e^t \sin\cos s, e^t \sin s$$

In this case, we can solve for $(s, t) = \Phi^{-1}(x, y)$ explicitly in a neighborhood of $(1, 0)$ to obtain the coordinate map

$$(s, t) = (\arctan y/x, \log\sqrt{x^2 + y^2})$$

1.2.8 Time-dependent vector fields

All of the systems of differential equations we have encountered so far have been autonomous ones, meaning that when they are written in the form (1.2.1.1), the functions V_i on the right-hand sides do not depend explicitly on the independent variable t. However, nonautonomous ODEs do arise in manifold theory, so it is worth exploring how the results of this chapter can be extended to cover this case.

Let M be a smooth manifold. A **time-dependent vector field** on M is a continuous map $V : J \times M \to TM$, where $J \subseteq \mathbb{R}$ is an interval, such that $V(t, p) \in T_pM$ for each $(t, p) \in J \times M$. This means that for each $t \in J$, the map $V_t : M \to TM$ defined by $V_t(p) = V(t, p)$ is a vector field on M. If V is a time-dependent vector field on M, an **integral curve of V** is a differentiable curve $\gamma : J_0 \to M$, where J_0 is an interval contained in J, such that

$$\gamma'(t) = V(t, \gamma(t)) \quad \text{for all } t \in J_0.$$

Every ordinary vector field $X \in \mathfrak{X}(M)$ determines a time-dependent vector field defined on $\mathbb{R} \times M$, just by setting $V(t,p) = X_p$.

A time-dependent vector field might not generate a flow, because two integral curves starting at the same point but at different times might follow different paths, whereas all integral curves of a flow through a given point have the same image. As a substitute for the fundamental theorem on flows, we have the following theorem.

Theorem 1.2.8.1 (Fundamental Theorem on Time-Dependent Flows). *Let M be a smooth manifold, let $J \subseteq \mathbb{R}$ be an open interval, and let $V : J \times M \to TM$ be a smooth time-dependent vector field on M. There exist an open subset $\mathcal{E} \subseteq J \times J \times M$ and a smooth map $\psi : \mathcal{E} \to M$ called the **time-dependent flow** of V, with the following properties:*

(a) For each $t_0 \in J$ and $p \in M$, the set $\mathcal{E}^{(t_0,p)} = \{t \in J : (t, t_0, p) \in \mathcal{E}\}$ is an open interval containing t_0, and the smooth curve $\psi^{(t_0,p)} : \mathcal{E}^{(t_0,p)} \to M$ defined by $\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$ is the unique maximal integral curve of V with initial condition $\psi^{(t_0,p)}(t_0) = p$.

(b) If $t_1 \in \mathcal{E}^{(t_0,p)}$ and $q = \psi^{(t_0,p)}(t_1)$, then $\mathcal{E}^{(t_1,q)} = \mathcal{E}^{(t_0,p)}$ and $\psi^{(t_1,q)} = \psi^{(t_0,p)}$.

(c) For each $(t_1, t_0) \in J \times J$, the set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \mathcal{E}\}$ is open in M, and the map $\psi_{t_1,t_0} : M_{t_1,t_0} \to M$ defined by $\psi_{t_1,t_0}(p) = \psi(t_1, t_0, p)$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse ψ_{t_0,t_1}.

(d) If $p \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and

$$\psi_{t_2,t_1} \circ \psi_{t_1,t_0}(p) = \psi_{t_2,t_0}(p). \tag{1.2.8.1}$$

Proof. This can be proved by following the outline of the proof of Theorem 1.2.3.4, using the uniqueness theorem for nonautonomous differential equations. However, it is much quicker to use the following trick to reduce it to the time-independent case.

Consider the smooth vector field $\widetilde{V}$ on $J \times M$ defined by

$$\widetilde{V}_{(s,p)} = \left(\frac{\partial}{\partial s}\bigg|_s, V(s,p) \right).$$

where s is the standard coordinate on $J \subseteq \mathbb{R}$, and we identify $T_{(s,p)}(J \times M)$ with $T_s J \oplus T_p M$ as usual. Let $\widetilde{\theta} : \widetilde{\mathcal{D}} \to J \times M$ denote the flow of $\widetilde{V}$. If we write the component functions of $\widetilde{\theta}$ as

$$\widetilde{\theta}(t, (s,p)) = \big(\alpha(t,(s,p)), \beta(t,(s,p))\big),$$

then $\alpha : \widetilde{\mathcal{D}} \to J$ and $\psi : \widetilde{\mathcal{D}} \to M$ satisfy

$$\begin{aligned} &\frac{\partial \alpha}{\partial t}(t,(s,p)) = 1, && \alpha(0,(s,p)) = s; \\ &\frac{\partial \beta}{\partial t}(t,(s,p)) = V\big(\alpha(t,(s,p)), \psi(t,(s,p))\big), && \beta(0,(s,p)) = p. \end{aligned}$$

It follows immediately that $\alpha(t,(s,p)) = t + s$, and therefore β satisfies

$$\frac{\partial \beta}{\partial t}(t,(s,p)) = V(t + s, \beta(t,(s,p))). \tag{1.2.8.2}$$

Let $\mathcal{E}$ be the subset of $\mathbb{R} \times J \times M$ defined by

$$\mathcal{E} = \{(t, t_0, p) : (t - t_0, (t_0, p)) \in \widetilde{\mathcal{D}}\}.$$

Clearly, $\mathcal{E}$ is open in $\mathbb{R} \times J \times M$ because $\widetilde{\mathcal{D}}$ is. Moreover, since α maps $\widetilde{\mathcal{D}}$ into J, if $(t, t_0, p) \in \mathcal{E}$, then $t = \alpha(t - t_0, (t_0, p)) \in J$, which implies that $\mathcal{E} \subseteq J \times J \times M$. The fact that each set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \widetilde{\mathcal{E}}\}$ is open in M follows immediately from the fact that $\mathcal{E}$ is open.

Now define $\psi : \mathcal{E} \to M$ by

$$\psi(t, t_0, p) = \beta(t - t_0, (t_0, p)).$$

Then is smooth because β is, and it follows from (1.2.8.2) that $\psi^{(t,t_0,p)}$ is an integral curve of V with initial condition $\psi^{(t_0,p)}(t_0) = p$.

To prove uniqueness, suppose $t_0 \in J$ and $\gamma : J_0 \to M$ is any integral curve of V defined on some open interval $J_0 \subseteq J$ containing t_0 and satisfying $\gamma(t_0) = p$. Define a smooth curve $\widetilde{\gamma} : J_0 \to J \times M$ by $\gamma(t) = (t, \gamma(t))$. Then $\widetilde{\gamma}$ is easily seen to be an integral curve of $\widetilde{V}$ with initial condition $\widetilde{\gamma}(t_0) = (t_0, p)$. By uniqueness and maximality of integral curves of $\widetilde{V}$, we must have $\widetilde{\gamma}(t) = \widetilde{\theta}(t - t_0, (t_0, p))$ on its whole domain, which implies that the domain of γ is contained in that of $\psi^{(t_0,p)}$, and $\gamma = \psi^{(t_0,p)}$ on that domain. It follows that $\psi^{(t_0,p)}$ is the unique maximal integral curve of V passing through p at $t = t_0$. This completes the proof of (a).

To prove (b), suppose $t_1 \in \mathcal{E}^{(t_0,p)}$ and $q = \psi^{(t_0,p)}(t_1)$. Then both $\psi^{(t_1,q)}$ and $\psi^{(t_0,p)}$ are integral curves of V that pass through q when $t = t_1$, so by uniqueness and maximality they must have the same domain and be equal on that domain.

Next, we prove (d). Suppose $p \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, and set $q = \psi_{t_1,t_0}(p) = \psi^{(t_0,p)}(t_1)$. Then (b) implies that $\psi^{(t_1,q)}(t_2) = \psi^{(t_0,p)}(t_2)$. Unwinding the definitions yields (1.2.8.1).

Finally, we prove (c). Suppose $(t_0, t_1) \in J \times J$. We have already noted that M_{t_1,t_0} is open in M. To show that $\psi_{t_1,t_0}(M_{t_1,t_0}) \subseteq M_{t_0,t_1}$, let p be a point of M_{t_1,t_0}, and set $q = \psi_{t_1,t_0}(p)$. Part (b) implies that $\mathcal{E}^{(t_0,p)} = \mathcal{E}^{(t_1,q)}$, and thus $t_0 \in \mathcal{E}^{(t_0,p)} = \mathcal{E}^{(t_1,q)}$. This is equivalent to $(t_0, t_1, q) \in \mathcal{E}$, which in turn means $q \in M_{t_0,t_1}$ as claimed. To see that $\psi_{t_1,t_0} : M_{t_1,t_0} \to M_{t_0,t_1}$ is a diffeomorphism, just note that the same argument as above implies that $\psi_{t_0,t_1} : M_{t_0,t_1} \to M_{t_1,t_0}$ and then (d) implies that $\psi_{t_1,t_0} \circ \psi_{t_0,t_1}(p) = p$ for all $p \in M_{t_0,t_1}$, and similarly that $\psi_{t_0,t_1} \circ \psi_{t_1,t_0}(p) = p$ for all $p \in M_{t_1,t_0}$. □

Example 1.2.8.2. Define a time-dependent vector field V on $\mathbb{R}^n$ by

$$V(t, x) = \frac{1}{t} x^i \frac{\partial}{\partial x^i}\Big|_x, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n.$$

Suppose $t_0 \in (0, +\infty)$ and $x_0 \in \mathbb{R}^n$ are arbitrary, and let $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ denote the integral curve of V with initial condition $\gamma(t_0) = x_0$. Then the components of γ satisfy the following nonautonomous system of differential equations:

$$\begin{aligned} \dot{\gamma}^i(t) &= \frac{1}{t} \gamma^i(t), \\ \gamma^i(t_0) &= x_0^i. \end{aligned}$$

The maximal solution to this system is $\gamma^i(t) = t x_0^i / t_0$, defined for all $t > 0$. Therefore, the time-dependent flow of V is given by

$$\psi(t, t_0, x) = tx/t_0 \quad \text{for} \quad (t, t_0, x) \in (0, +\infty) \times (0, +\infty) \times \mathbb{R}^n.$$

1.2.9 First-order partial differential equations

One of the most powerful applications of the theory of flows is to partial differential equations. In its most general form, a **partial differential equation** (PDE) is any equation that relates an unknown function of two or more variables with its partial derivatives up to some order and with the independent variables. The **order** of the PDE is the highest-order derivative of the unknown function that appears.

The number of specialized techniques that have been developed to solve partial differential equations is staggering. However, it is a remarkable fact that real-valued first-order PDEs can be reduced to ordinary differential equations by means of the theory of flows, and thus can be solved using only ODEs and a little differential-geometric insight but no specialized PDE theory. In this section, we describe how this is done for two special classes of first-order equations: first, linear equations; and then, somewhat more generally, quasilinear equations (which we define below). A PDE that is not quasilinear is said to be fully nonlinear.

In coordinates, any first-order PDE for a single unknown function can be written

$$F\left(x^1, \dots, x^n, \frac{\partial u}{\partial x^1}(x), \dots, \frac{\partial u}{\partial x^n}(x)\right) = 0. \tag{1.2.9.1}$$

where u is an unknown function of n variables and F is a given smooth function of $2n + 1$ variables. (Smoothness is not strictly necessary, but we assume it throughout for simplicity.) The theory we will describe applies only when F and u are realvalued, so we assume that as well.

Without further restrictions, most PDEs have a multitude of solutions—for example, the PDE $\partial u/\partial x = 0$ in the plane is solved by any smooth function u that depends on y alone—so in order to get a unique solution one generally stipulates that the solution should satisfy some extra conditions. For first-order equations, the appropriate condition is to specify "initial values" on a hypersurface: given a smooth hypersurface $S \subseteq \mathbb{R}^n$ and a smooth function $\varphi : S \to \mathbb{R}$, we seek a smooth function u that solves the PDE and also satisfies the initial condition

$$u|_S = \varphi. \tag{1.2.9.2}$$

The problem of finding a solution to (1.2.9.1) in a neighborhood of S subject to the initial condition (1.2.9.2) is called a **Cauchy problem**.

Not every Cauchy problem has a solution: for example, in $\mathbb{R}^2$, the equation $\partial u/\partial x = 1$ has no solution with $u = 0$ on the x-axis, because the equation and the initial condition contradict each other. To avoid such difficulties, one usually assumes that the Cauchy problem (1.2.9.1)–(1.2.9.2) is **noncharacteristic**, meaning that there is a certain geometric relationship between the equation and the initial data, which is sufficient to guarantee the existence of a solution near S. As we study the Cauchy problem in increasing generality, we will describe the oncharacteristic condition separately for each type of equation we treat.

1.2.9.1 Linear equations

The first type of equation we will treat is a first-order linear PDE, which is one that depends linearly or affinely on the unknown function and its derivatives. In coordinate form, the most general such equation can be written

$$a^1(x)\frac{\partial u}{\partial x^1}(x) + \cdots + a^n(x)\frac{\partial u}{\partial x^n}(x) + b(x)u(x) = f(x), \tag{1.2.9.3}$$

where $a_1, \ldots, a_n, b$ and f are smooth, real-valued functions defined on some open subset $\Omega \subseteq \mathbb{R}^n$, and u is an unknown smooth function on Ω.

It should come as no surprise that flows of vector fields play a role in the solution of (1.2.9.3), because the first n terms on the left-hand side represent the action on u of a smooth vector field $A \in \mathfrak{X}(\Omega)$:

$$A_x = a^1(x)\frac{\partial}{\partial x^1}\Big|_x + \cdots + a^n(x)\frac{\partial}{\partial x^n}\Big|_x. \tag{1.2.9.4}$$

In terms of A, we can rewrite (1.2.9.3) in the simple form $Au + bu = f$. In this form, it makes sense on any smooth manifold, and is no more difficult to solve in that generality, so we state our first theorem in that context. The Cauchy problem for $Au + bu = f$ with initial hypersurface S is said to be **noncharacteristic** if A is nowhere tangent to S.

Theorem 1.2.9.1 (The Linear First-Order Cauchy Problem). *Let M be a smooth manifold. Suppose we are given an embedded hypersurface $S \subseteq M$, a smooth vector field $A \in \mathfrak{X}(M)$ that is nowhere tangent to S, and functions $b, f \in C^\infty(M)$ and $\varphi \in C^\infty(S)$. Then for some neighborhood U of S in M, there exists a unique solution $u \in C^\infty(U)$ to the noncharacteristic Cauchy problem*

$$Au + bu = f, \quad u|_S = \varphi. \tag{1.2.9.5}$$

Proof. The flowout theorem gives us a neighborhood $\mathcal{O}_\delta$ of $\{0\} \times S$ in $\mathbb{R} \times S$, a neighborhood U of S in M, and a diffeomorphism $\Phi : \mathcal{O}_\delta \to U$ that satisfies $\Phi(0,p) = p$ for $p \in S$ and $\Phi_*(\partial\partial t) = A$. Let us write $\widehat{u} = u \circ \Phi$, $\widehat{f} = f \circ \Phi$ and $\widehat{b} = b \circ \Phi$. Proposition 1.1.2.1 shows that $\partial\widehat{u}/\partial t = (Au) \circ \Phi$. Thus, $u \in C^\infty(U)$ satisfies (1.2.9.5) if and only if $\widehat{u}$ satisfies

$$\begin{aligned} &\frac{\partial \widehat{u}}{\partial t}(t,p) = \widehat{f}(t,p) - \widehat{b}(t,p)\widehat{u}(t,p), \quad (t,p) \in \mathcal{O}_\delta, \\ &\widehat{u}(0,p) = \varphi(p), \quad p \in S. \end{aligned} \tag{1.2.9.6}$$

For each fixed $p \in S$, this is a linear first-order ODE initial value problem for $\widehat{u}$ on the interval $-\delta(p) < t < \delta(p)$. As is shown in ODE texts, such a problem always has a unique solution on the whole interval, which can be written explicitly as

$$\widehat{u}(t,p) = e^{-\int_0^t \widehat{b}(s,p)ds}\Big(\varphi(p) + \int_0^t \widehat{f}(\tau,p)e^{\int_0^\tau \widehat{b}(s,p)ds}d\tau\Big).$$

This is a smooth function of (t,p) (as can be seen by choosing local coordinates for S and differentiating under the integral signs). Therefore, $u = \widehat{u} \circ \Phi^{-1}$ is the unique solution on U to (1.2.9.5). □

This proof shows how to write down an explicit solution to the Cauchy problem, provided the flow of the vector field A can be found explicitly. The computations are usually easiest if we first choose a (local or global) parametrization $X : \Omega \to S$, and substitute $X(s)$ for p in (1.2.9.6). This amounts to using the canonical coordinates of Theorem 1.2.4.4 to transform the Cauchy problem to an ODE.

Example 1.2.9.2. Suppose we wish to solve the following Cauchy problem for a smooth function $u(x,y)$ in the plane:

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = x, \quad u(x,0) = x \text{ when } x > 0. \tag{1.2.9.7}$$

The vector field acting on u on the left-hand side is the vector field W of Example 1.2.4.5. The initial hypersurface S is the positive x-axis, and this problem is noncharacteristic because W is nowhere tangent to S. (Notice that this would not be the case if we took S to be the entire x-axis.) Using the computations of Example 1.2.4.5, we find that the transformation $(x,y) = \Psi(t,s) = (s\cos t, s\sin t)$ pushes $\partial/\partial t$ forward to W, and thus transforms (1.2.9.7) to the ODE initial value problem

$$\frac{\partial \widehat{u}}{\partial t}(t,s) = s\cos t, \quad \widehat{u}(0,s) = s.$$

This is solved by $\widehat{u} = s\sin t + s$. Substituting for (t,s) in terms of (x,y), we obtain the solution $u(x,y) = y + \sqrt{x^2+y^2}$ to the original problem.

1.2.9.2 Quasilinear equations

The preceding results extend easily to certain nonlinear partial differential equations. A PDE is called quasilinear if it can be written as an affine equation in the highest-order derivatives of the unknown function, with coefficients that may depend on the function itself and its derivatives of lower order. Thus, in coordinates, a quasilinear first-order PDE is a differential equation of the form

$$a^1(x,u(x))\frac{\partial u}{\partial x^1} + \cdots + a^n(x,u(x))\frac{\partial u}{\partial x^n} = f(x,u(x)) \tag{1.2.9.8}$$

for an unknown real-valued function $u(x^1,\dots,x^n)$, where $a^1,\dots,a^n$ and f are smooth real-valued functions defined on some open subset $W \subseteq \mathbb{R}^{n+1}$. (For simplicity, this time we concentrate only on the local problem, and restrict our attention to open subsets of Euclidean space.)

We wish to solve a Cauchy problem for this equation with initial condition

$$u|_S = \varphi, \tag{1.2.9.9}$$

where $S \subseteq \mathbb{R}^n$ is a smooth, embedded hypersurface, and $\varphi : S \to \mathbb{R}$ is a smooth function whose graph is contained in W. A quasilinear Cauchy problem is said to be noncharacteristic if the vector field A^φ along S defined by

$$A^\varphi|_x = a^1(x,\varphi(x))\frac{\partial}{\partial x^1}\Big|_x + \cdots + a^n(x,\varphi(x))\frac{\partial}{\partial x^n}\Big|_x$$

is nowhere tangent to S. (Notice that in this case the noncharacteristic condition depends on the initial value φ, not just on the initial hypersurface.) We will show that a noncharacteristic Cauchy problem always has local solutions.

Theorem 1.2.9.3 (The Quasilinear Cauchy Problem). *If the Cauchy problem (1.2.9.8)–(1.2.9.9) is noncharacteristic, then for each $p \in S$ there exists a neighborhood U of p in M on which there exists a unique solution u to (1.2.9.8)–(1.2.9.9).*

Proof. The key is to convert the dependent variable u to an additional independent variable. (This is a trick that is useful in many different contexts.) Define the **characteristic vector field** for (1.2.9.8) to be the vector field ξ on $W \subseteq \mathbb{R}^{n+1}$ given by

$$\xi_{(x,z)} = a^1(x,z)\frac{\partial}{\partial x^1}\Big|_{(x,z)} + \cdots + a^n(x,z)\frac{\partial}{\partial x^n}\Big|_{(x,z)} + f(x,z)\frac{\partial}{\partial z}\Big|_{(x,z)} \tag{1.2.9.10}$$

where we write $(x,z) = (x^1,\ldots,x^n,z)$. Suppose u is a smooth function defined on an open subset $V \subseteq \mathbb{R}^n$ whose graph $\Gamma(u) = \{(x,u(x)) : x \in V\}$ is contained in W. Then (1.2.9.8) holds if and only if $\xi(z - u(x)) = 0$ at all points of $\Gamma(u)$. Since $z - u(x)$ is a defining function for $\Gamma(u)$, it follows from Corollary **??** that u solves (1.2.9.8) if and only if ξ is tangent to $\Gamma(u)$. The idea is to construct the graph of u as the flowout by ξ from a suitable initial submanifold.

Let $\Gamma(\varphi) = \{(x,\varphi(x)) : x \in S\}$ denote the graph of φ; it is an $(n-1)$-dimensional embedded submanifold of W. The projection $\pi : W \to \mathbb{R}^n$ onto the first n variables maps $\Gamma(\varphi)$ diffeomorphically onto S, so if ξ were tangent to $\Gamma(\varphi)$ at some point $(x,\varphi(x))$, then $d\pi(\xi_{x,\varphi(x)})$ would be tangent to S at x. However, a direct computation using (1.2.9.10) shows that

$$d\pi(\xi_{(x,\varphi(x))}) = A^{\varphi}|_x.$$

so the noncharacteristic assumption guarantees that ξ is nowhere tangent to $\Gamma(\varphi)$.

We can apply the flowout theorem to the vector field ξ starting on $\Gamma(\varphi) \subseteq W$ to obtain an immersed n-dimensional submanifold $\mathcal{S} \subseteq W$ containing $\Gamma(\varphi)$, such that ξ is everywhere tangent to $\mathcal{S}$. If we can show that $\mathcal{S}$ is the graph of a smooth function u, at least locally near $\Gamma(\varphi)$, then u will be a solution to our problem.

Let $p \in S$ be arbitrary. At $(p,\varphi(p)) \in \Gamma(\varphi) \subseteq \mathcal{S}$, the tangent space to $\mathcal{S}$ is spanned by the vector $\xi_{(p,\varphi(p))}$ together with $T_{(p,\varphi(p))}\Gamma(\varphi)$. The restriction of π to $\Gamma(\varphi)$ is a diffeomorphism onto S, so $d\pi$ maps $T_{(p,\varphi(p))}\Gamma(\varphi)$ isomorphically onto T_pS. On the other hand, as we noted above, $d\pi$ takes $\xi_{(p,\varphi(p))}$ to $A^{\varphi}|_p$. By the noncharacteristic assumption, $A^{\varphi}|_p \notin T_pS$, so $d\pi$ is injective on $T_{(p,\varphi(p))}\mathcal{S}$, and thus for dimensional reasons $T_{(p,\varphi(p))}\mathbb{R}^{n+1} = T_{(p,\varphi(p))}\mathcal{S} \oplus \ker d\pi_{(p,\varphi(p))}$. Because $\ker d\pi_{(p,\varphi(p))}$ is the tangent space to $\{p\} \times \mathbb{R}$, it follows that $\mathcal{S}$ intersects $\{p\} \times \mathbb{R}$ transversely at $(p,\varphi(p))$. By Corollary **??**, there exist a neighborhood V of $(p,\varphi(p))$ in $\mathcal{S}$ and a neighborhood U of p in $\mathbb{R}^n$ such that V is the graph of a smooth function $u : U \to \mathbb{R}$. This function solves the Cauchy problem in U.

To prove uniqueness, we might need to shrink U. Because $\mathcal{S}$ is a flowout, it is the image of some open subset $\mathcal{O}_\delta \subseteq \mathbb{R} \times \Gamma(\varphi)$ under the flow of ξ. Choose V small enough that it is the image under the flow of a set of the form $(-\varepsilon,\varepsilon) \times Y \subseteq \mathcal{O}_\delta$, for some $\varepsilon > 0$ and some neighborhood Y of $(p,\varphi(p))$ in $\Gamma(\varphi)$. With this assumption, $\Gamma(u)$ is exactly the union of the images of the integral curves of ξ starting at points of Y and flowing for time $|t| < \varepsilon$. Suppose $\widetilde{u}$ is any other solution to the same Cauchy problem on the same open subset U. As we noted above, this means that ξ is tangent to the graph of $\widetilde{u}$, and the initial condition ensures that $Y = \Gamma(\varphi) \cap U \subseteq \Gamma(\widetilde{u})$. Since the graph of $\widetilde{u}$ is a properly embedded submanifold of $U \times \mathbb{R}$, Exercise 1.2.2 shows that each integral curve of ξ in $U \times \mathbb{R}$ starting at a point of Y must lie entirely in the graph of $\widetilde{u}$. Thus $\Gamma(u) \subseteq \Gamma(\widetilde{u})$. Since u and $\widetilde{u}$ are defiend both on U, this then implies $\widetilde{u} = u$ on U. □

To find an explicit solution to a quasilinear Cauchy problem, we begin by choosing a smooth local parametrization of S, written as $s \mapsto X(s)$ for $s = (s^2,\ldots,s^n) \in \Omega \subseteq \mathbb{R}^{n-1}$. Then the map $\widetilde{X} : \Omega \to \mathbb{R}^{n+1}$ given by $\widetilde{X}(s) = \big(X(s), \varphi(X(s))\big)$ is a local parametrization of $\Gamma(\varphi)$, and a local parametrization of $\mathcal{S}$ is given by

$$\Psi(s,t) = \theta_t(\widetilde{X}(s)).$$

where θ is the flow of ξ. To rewrite $\mathcal{S}$ as a graph, we just invert the map $\pi \circ \Psi : \Omega \to \mathbb{R}^n$ locally to get $(t,s^2,\ldots,s^n)$ in terms of $(x^1,\ldots,x^n)$; then the z-component of Ψ, written as a function of $(x^1,\ldots,x^n)$, is a solution to the Cauchy problem.

Example 1.2.9.4 (A Quasilinear Cauchy Problem). Suppose we wish to solve the following quasilinear Cauchy problem in the plane:

$$(u+1)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad u(x,0) = x.$$

The initial hypersurface S is the x-axis, and the initial value is $\varphi(x,0) = x$. The vector field A^{φ} is $(x+1)\partial/\partial x + \partial/\partial y$, which is nowhere tangent to the x-axis, so this problem is noncharacteristic.

The characteristic vector field is the following vector field on $\mathbb{R}^3$:

$$\xi = (z+1)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

We wish to find the flowout of ξ starting from $\Gamma(\varphi)$. We can parametrize S by $X(s) = (s, 0)$ for $s \in \mathbb{R}$, and then $\Gamma(\varphi)$ is parametrized by $\widetilde{X}(s) = (s, 0, s)$. Solving the system of ODEs associated to ξ with initial conditions $(x, y, z) = (s, 0, s)$, we find that the flowout of ξ starting from $\widetilde{X}(s)$ is parametrized by

$$\Psi(t, s) = (s + t + st, t, s).$$

The image of this map is the graph of our solution. To reparametrize the graph in terms of x and y, we invert the map $\pi \circ \Psi$; that is, we solve $(x, y) = (s + t + st, t)$ (locally) for t and s, yielding

$$(t, s) = (y, \frac{x - y}{1 + y}).$$

and therefore on the flowout manifold we have $z = s = (x - y)/(1 + y)$. The solution to our Cauchy problem is $u(x, y) = (x - y)/(1 + y)$. Note that it is defined only in a neighborhood of S (the set where $y > -1$), not on the whole plane.

1.2.10 Exercise

Exercise 1.2.1. Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.

(a) We say γ is periodic if there is a number $T > 0$ such that $\gamma(t) = \gamma(t + T)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and nonconstant.

(b) Show that if γ is periodic and nonconstant, then there exists a unique positive number T (called the period of γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}$, S^1 or $\mathbb{R}^0$.

Proof. Assume that γ is neither constant nor injective. Then there is t_1, t_2 such that $\gamma(t_1) = \gamma(t_2)$, and so

$$\gamma'(t) = V_{\gamma(t_1)} = V_{\gamma(t_2)} = \gamma(t).$$

Now $\widetilde{\gamma}(t) := \gamma(t + t_2 - t_1)$ is an integral curve such that $\widetilde{\gamma}(t_1) = \gamma(t_1)$ and $\widetilde{\gamma}'(t_1) = \gamma(t_1)$. Thus by uniqueness they must equal. That is, in a neighborhood of t_1, we have

$$\gamma(t + t_2 - t_1) = \gamma(t)$$

then by the connectivity, we can show γ is periodic.

Note that the set of periods of γ forms a closed subgroup of $(\mathbb{R}, +)$, and therefore by the lemma below is neither $\mathbb{R}$ or $\{nT : n \in \mathbb{Z}\}$ for some T.

Finally, assume that γ is not constant. Then by Proposition 1.2.4.3, $\gamma'(t)$ naver vanishes, so it is an immersion. If γ is injective, then γ is a diffeomorphism from $\mathbb{R}$ to $\operatorname{im}\gamma$. If γ is periodic, then $\operatorname{im}\gamma = \gamma([0, T])$, thus is compact. $\square$

Lemma 1.2.10.1. *Let H be a nontrivial subgroup of $(\mathbb{R}, +)$, then exactly one of the following cases holds:*

- *$H = \{nT : n \in \mathbb{Z}\}$ for some $T > 0$.*
- *H is dense.*

Proof. In fact, define

$$T := \inf\{x \in H : x > 0\}.$$

We have tow cases:

- If $T > 0$, then every element in H is isolated since 0 is isolated in H. It is a general result that an additive subgroup of $\mathbb{R}^n$ is discrete if and only if it is generated by linearly independent vectors $a_1, \ldots, a_m \in \mathbb{R}^n$, $m \le n$.

- If $T = 0$, then we can find a null sequence $\{a_n\}$ in H. For each n, the set $\{ka_n : k \in \mathbb{Z}\}$ gives a partition of $\mathbb{R}$ such that every number of $\mathbb{R}$ is with diatance $|a_n|$ to some element ka_n. Since $a_n \to 0$, this implies that H is dense.

$\square$

Exercise 1.2.2. Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold, and V is a smooth vector field on M that is tangent to S.

(a) Show that for any integral curve γ of V such that $\gamma(t_0) \in S$, there exists $\varepsilon > 0$ such that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq S$.

(b) Now assume S is properly embedded. Show that every integral curve that intersects S is contained in S.

Proof. Lemma 1.2.5.10. Let V be the vector field on $\mathbb{R}^2$:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

and S be the upper semi-sphere, which is not properly embedded. $\square$

Exercise 1.2.3. For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $S^{2n-1} \subseteq \mathbb{C}^n$ by $\theta_t(z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nonvanishing vector field on S^{2n-1}.

Proof. The infinitesimal generator is

$$V_z = \dot{\theta}^{(z)}(t) = ie^{it}z$$

Since $|V_z| = |z| = 1$, V_z is non-vanishing. $\square$

Exercise 1.2.4. Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.

Proof. Let V be a nowhere vanishing smooth vector field on M. Since M is copmact, V is complete. Let $\theta : \mathbb{R} \times M \to M$ be the flow of V. For each point $p \in M$, since p is a regular point of V, there is a chart $(U_p, (x^i))$ of p and $\varepsilon_p > 0$ such that

$$\theta(t, x) = (x^1 + t, x^2, \ldots, x^n) \quad \text{for} \quad (t, x) \in (-\varepsilon, \varepsilon) \times U_p.$$

Thus the restriction of θ_t on U_p has no fixed point for $t \in (-\varepsilon, \varepsilon)$. Since M is compact, it is covered by finitely many charts $\{U_{p_i}\}$, and we can choose $\varepsilon_i := \min\{\varepsilon_{p_i}\}$. Let $|t| < \varepsilon$, then θ_t has no fixed point.

The homotopy is defined by

$$H(t, p) = \theta(\frac{t\varepsilon}{2}, p)$$

$\square$

Exercise 1.2.5. Let M be a smooth manifold and let $S \subseteq M$ be a compact embedded submanifold. Suppose $V \subseteq \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Show that there exists $\varepsilon > 0$ such that the flow of V restricts to a smooth embedding $(-\varepsilon, \varepsilon) \times S \to M$.

Proof. From the flowout theorem, the restriction $\Phi : \mathcal{O}_\delta \to M$ is a immersion, hence a local embedding. Now use the compactness of S. $\square$

Exercise 1.2.6. Suppose M is a smooth manifold and $S \subseteq M$ is an embedded hypersurface. Suppose further that there is a smooth vector field V defined on a neighborhood of S and nowhere tangent to S. Show that S has a neighborhood in M diffeomorphic to $(-1, 1) \times S$, under a diffeomorphism that restricts to the obvious identification $\{0\} \times S \approx S$.

Proof. By the flowout theorem, $\Phi : \mathcal{O}_\delta \to M$ is a diffeomorphism. Now let $\sigma : (-1, 1) \times S \to \mathcal{O}_\delta$ be defined as

$$\sigma(t, p) = (t\delta(p), p)$$

Since δ is a smooth function, σ is a diffeomorphism. Thus $\Phi \circ \sigma$ satisfies the condition. $\square$

Exercise 1.2.7. Suppose M_1 and M_2 are connected smooth n-manifolds and $M_1 \# M_2$ is their smooth connected sum. Show that the smooth structure on $M_1 \# M_2$ can be chosen in such a way that there are open subsets $\widetilde{M}_1, \widetilde{M}_2 \subseteq M_1 \# M_2$ that are diffeomorphic to $M_1 - \{p_1\}$ and $M_2 - \{p_2\}$, respectively, such that $\widetilde{M}_1 \cap \widetilde{M}_2 = M_1 \# M_2$ and $\widetilde{M}_1, \widetilde{M}_2$ is diffeomorphic to $(-1, 1) \times S^{n-1}$.

1.3 Distributions and foliations

1.3.1 Distributions and involutivity

Let M be a smooth manifold. A **distribution** on M of rank k is a rank-k subbundle of TM. It is called a **smooth distribution** if it is a smooth subbundle. Distributions are also sometimes called **tangent distributions**, **k-plane fields**, or **tangent subbundles**.

Often a rank-k distribution is described by specifying for each $p \in M$ a linear subspace $D_p \subseteq T_pM$ of dimension k, and letting $D = \bigcup_{p \in M} D_p$. It then follows from the local frame criterion for subbundles that D is a smooth distribution if and only if each point of M has a neighborhood U on which there are smooth vector fields $X_1, \dots, X_k : U \to TM$ such that $X_1|_q, \dots, X_k|_q$ form a basis for D_q at each $q \in U$. In this case, we say that D is the distribution **(locally) spanned by the vector fields $X_1, \dots, X_k$**.

1.3.1.1 Integral manifolds and involutivity

Suppose $D \subseteq TM$ is a smooth distribution. A nonempty immersed submanifold $N \subseteq M$ is called an **integral manifold** of D if $T_pN = D_p$ at each point $p \in N$. The main question we want to address in this section is that of the existence of integral manifolds.

Before we proceed with the general theory, let us describe some examples of distributions and integral manifolds.

Example 1.3.1.1 (Distributions and Integral Manifolds).

(a) If V is a nowhere-vanishing smooth vector field on a manifold M, then V spans a smooth rank-1 distribution on M. The image of any integral curve of V is an integral manifold of D.

(b) In $\mathbb{R}^n$, the vector fields $\partial/\partial x^1, \dots, \partial/\partial x^k$ span a smooth distribution of rank k. The k-dimensional affine subspaces parallel to $\mathbb{R}^k$ are integral manifolds.

(c) Let R be the distribution on $\mathbb{R}^n - \{0\}$ spanned by the radial vector field $x^i \partial/\partial x^i$, and let $R^\perp$ be its orthogonal complement bundle. Then $R^\perp$ is a smooth rank-$(n-1)$ distribution on $\mathbb{R}^n - \{0\}$. Through each point $x \in \mathbb{R}^n - \{0\}$, the sphere of radius $|x|$ around 0 is an integral manifold of $R^\perp$.

(d) Let D be the smooth distribution on $\mathbb{R}^3$ spanned by the following vector fields:

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

It turns out that D has no integral manifolds. To get an idea why, suppose N is an integral manifold through the origin. Because X and Y are tangent to N, any integral curve of X or Y that starts in N has to stay in N, at least for a short time. Thus, N contains an open subset of the x-axis (which is an integral curve of X). It also contains, for each sufficiently small x, an open subset of the line parallel to the y-axis and passing through $(x, 0, 0)$ (which is an integral curve of Y). Therefore, N contains an open subset of the (x, y)-plane. However, the tangent plane to the (x, y)-plane at any point p off of the x-axis is not equal to D_p. Therefore, no such integral manifold exists.

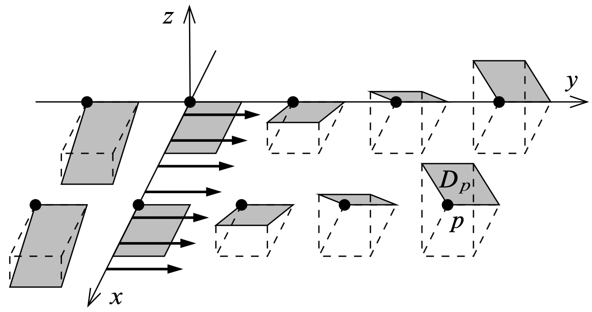


Figure 1.1: A smooth distribution with no integral manifolds.

The last example shows that in general, integral manifolds may fail to exist. Suppose D is a smooth distribution on M. We say that D is **involutive** if given any pair of smooth local sections of D, their Lie bracket is also a local section of D. The next proposition shows that the involutivity condition can be expressed concisely in terms of Lie algebras.

Proposition 1.3.1.2. *Let $D \subseteq TM$ be a smooth distribution, and let $\Gamma(D) \subseteq TM$ denote the space of smooth global sections of D. Then D is involutive if and only if $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. If D is involutive, the definition implies that $\Gamma(D)$ is closed under Lie brackets. Because it is also a linear subspace of $\mathfrak{X}(M)$, it is a Lie subalgebra.

Conversely, suppose $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$, and let X, Y be smooth local sections of D over an open subset $U \subseteq M$. Given $p \in U$, let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on a neighborhood of p and supported in U. Then ψX and ψY are smooth global sections of D, so their Lie bracket is also a section of D by hypothesis. This Lie bracket is $[\psi X, \psi Y] = \psi^2[X,Y] + (\psi X\psi)Y - (\psi Y\psi)X$, which is equal to $[X,Y]$ in a neighborhood of p. Thus, $[X,Y] \in D_p$ for each $p \in U$, so D is involutive. □

A smooth distribution D on M is said to be **integrable** if each point of M is contained in an integral manifold of D.

Proposition 1.3.1.3. *Every integrable distribution is involutive.*

Proof. Let $D \subseteq TM$ be an integrable distribution. Suppose X and Y are smooth local sections of D defined on some open subset $U \subseteq M$. Let p be any point in U, and let N be an integral manifold of D containing p. The fact that X and Y are sections of D means that X and Y are tangent to N. By Corollary 1.1.3.10, $[X,Y]$ is also tangent to N, and therefore $[X,Y]_p \in D_p$. Since this is true at each $p \in U$, it follows that D is involutive. □

Note, for example, that the distribution D of Example 1.3.1.1(d) is not involutive, because $[X,Y] = -\partial/\partial z$, which is not a section of D.

The next lemma shows that the involutivity condition does not have to be checked for every pair of smooth vector fields, just those of a smooth local frame in a neighborhood of each point.

Lemma 1.3.1.4 (Local Frame Criterion for Involutivity). *Let $D \subseteq TM$ be a smooth distribution. If in a neighborhood of every point of M there exists a smooth local frame $V_1, \dots, V_k$ for D such that $[V_i, V_j]$ is a section of D for each i, j, then D is involutive.*

Proof. Suppose the hypothesis holds, and suppose X and Y are smooth local sections of D over some open subset $U \subseteq M$. Given $p \in U$, choose a smooth local frame $V_1, \dots, V_k$ satisfying the hypothesis in a neighborhood of p, and write $X = X^i V_i$ and $Y = Y^i V_i$ in that neighborhood. Then, using (1.1.3.3),

$$[X,Y] = [X^i V_i, Y^j V_j] = X^i Y^j [V_i, V_j] + X^i (V_i Y^j) V_j - Y^j (V_j X^i) V_i.$$

It follows from the hypothesis that this last expression is a section of D. □

1.3.1.2 Involutivity and differential forms

Differential forms yield an alternative way to describe distributions and involutivity.

Lemma 1.3.1.5 (1-Form Criterion for Smooth Distributions). *Suppose M is a smooth n-manifold and $D \subseteq TM$ is a distribution of rank k. Then D is smooth if and only if each point $p \in M$ has a neighborhood U on which there are smooth 1-forms $\omega^1, \dots, \omega^{n-k}$ such that for each $q \in U$,*

$$D_q = \bigcap_{i=1}^{n-k} \ker \omega^i|_q. \tag{1.3.1.1}$$

Proof. First suppose that there exist such forms $\omega^1, \dots, \omega^{n-k}$ in a neighborhood of each point. The assumption (1.3.1.1) together with the fact that D has rank k implies that the forms $\omega^1, \dots, \omega^{n-k}$ are independent on U for dimensional reasons. By Proposition **??**, we can complete them to a smooth coframe $(\omega^1, \dots, \omega^n)$ on a (possibly smaller) neighborhood of each point. If $(E_1, \dots, E_n)$ is the dual frame, it is easy to check that D is locally spanned by $E_{n-k+1}, \dots, E_n$, so it is smooth by the local frame criterion.

Conversely, suppose D is smooth. In a neighborhood of any $p \in M$, there are smooth vector fields $Y_1, \dots, Y_k$ spanning D. By Proposition **??** again, we can complete these vector fields to a smooth local frame $Y_1, \dots, Y_n$ for M in a neighborhood of p. With the dual coframe denoted by $(\varepsilon^1, \dots, \varepsilon^n)$, it follows easily that D is characterized locally by

$$D_q = \bigcap_{i=1}^{n-k} \ker \varepsilon^i|_q.$$

This completes the proof. □

If D is a rank-k distribution on a smooth n-manifold M, any $n-k$ linearly independent 1-forms $\omega^1, \dots, \omega^{n-k}$ defined on an open subset $U \subseteq M$ and satisfying (1.3.1.1) for each $q \in U$ are called **local defining forms** for D. More generally, if $0 \le p \le n$, we say that a p-form $\omega \in \Omega^p(M)$ **annihilates** $\boldsymbol{D}$ if $\omega(X_1, \dots, X_p) = 0$ whenever $X_1, \dots, X_p$ are local sections of D. (In the case $p = 0$, only the zero function annihilates D.)

Lemma 1.3.1.6. *Suppose M is a smooth n-manifold and D is a smooth rank-k distribution on M. Let $\omega^1, \dots, \omega^{n-k}$ be smooth local defining forms for D over an open subset $U \subseteq M$. A smooth p-form η defined on U annihilates D if and only if it can be expressed in the form*

$$\eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i \tag{1.3.1.2}$$

for some smooth $(p-1)$-forms $\beta^1, \dots, \beta^{n-k}$ on U.

Proof. It is easy to check that any form η that satisfies (1.3.1.2) in a neighborhood of each point annihilates D. Conversely, suppose that η annihilates D on U. In a neighborhood of each point we can complete the $(n-k)$-tuple $\omega^1, \dots, \omega^{n-k}$ to a smooth local coframe $\omega^1, \dots, \omega^n$ for M. If $(E_1, \dots, E_n)$ is the dual frame, then D is locally spanned by $E_{n-k+1}, \dots, E_n$. In terms of this coframe, any $\eta \in \Omega^p(M)$ can be written locally in a unique way as

$$\eta = {\sum_I}' \eta_I \, \omega^{i_1} \wedge \cdots \wedge \omega^{i_p},$$

where the coefficients η_I are determined by $\eta_I = \eta(E_{i_1}, \dots, E_{i_p})$. Thus, η annihilates D in U if and only if $\eta_I = 0$ whenever $n-k+1 \le i_1 < \cdots < i_p \le n$, in which case η can be written locally as

$$\eta = {\sum_{I : i_1 \le n-k}}' \eta_I \, \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} = \sum_{i_1 = 1}^{k} \omega^{i_1} \wedge \left({\sum_{I'}}' \eta_{i_1 I'} \, \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} \right),$$

where we have written $I' = (i_2, \dots, i_p)$. This holds in a neighborhood of each point of U; patching together with a partition of unity, we obtain a similar expression on all of U. □

When expressed in terms of differential forms, the involutivity condition translates into a statement about exterior derivatives.

Theorem 1.3.1.7 (1-Form Criterion for Involutivity). *Suppose $D \subseteq TM$ is a smooth distribution. Then D is involutive if and only if the following condition is satisfied:*

If η is any smooth 1-form that annihilates D on an open subset $U \subseteq M$, then $d\eta$ also annihilates D on U. (1.3.1.3)

Proof. First, assume that D is involutive, and suppose η is a smooth 1-form that annihilates D on $U \subseteq M$. Then for any smooth local sections X, Y of D, formula (3.3.3.5) for $d\eta$ gives

$$d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]).$$

The hypothesis implies that each of the terms on the right-hand side is zero on U.

Conversely, suppose D satisfies (1.3.1.3), and suppose X and Y are smooth local sections of D. If $\omega^1, \dots, \omega^{n-k}$ are smooth local defining forms for D, then (3.3.3.5) shows that for each $1 \le i \le n-k$,

$$\omega^i([X, Y]) = X(\omega^i(Y)) - Y(\omega^i(X)) - d\omega^i(X, Y) = 0,$$

which implies that $[X, Y]$ takes its values in D. Thus D is involutive. □

Just like the Lie bracket condition for involutivity, the exterior derivative condition need only be checked for a particular set of smooth defining forms in a neighborhood of each point, as the next proposition shows.

Proposition 1.3.1.8 (Local Coframe Criterion for Involutivity). *Let D be a smooth distribution of rank k on a smooth n-manifold M, and let $\omega^1, \dots, \omega^{n-k}$ be smooth defining forms for D on an open subset $U \subseteq M$. The following are equivalent:*

(a) D is involutive on U.

(b) $d\omega^1, \dots, d\omega^{n-k}$ annihilate D.

(c) There exist smooth 1-forms $\{\alpha^i_j : 1 \le i, j \le n-k\}$ such that

$$d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j \quad \text{for} \quad 1 \le i \le n-k.$$

(d) For each $i = 1, \dots, n-k$ we have

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0.$$

Proof. By Theorem 1.3.1.7, if D is involutive on U then $d\omega^1, \dots, \omega^{n-k}$ annihilate D, so (a) implies (b). Now assume (b). If η is a smooth 1-form that annihilates D on U, then by Proposition 1.3.1.6 there are smooth functions β^i on U such that

$$\eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i.$$

Therefore

$$d\eta = \sum_{i=1}^{n-k} d\omega^i \wedge \beta^i - \sum_{i=1}^{n-k} \omega^i \wedge d\beta^i.$$

Since all $d\omega^i$'s annihilate D, $d\eta$ also annihilates D on U. Thus D is involutive on U by Theorem 1.3.1.7. This proves $(a) \Leftrightarrow (b)$. By Lemma 1.3.1.6 we see (c) is equivalent to (b).

For (d), it is clear that (c) implies (d). Now assume (d) holds. In a neighborhood of each point we can complete the $(n-k)$-tuple $\omega^1, \dots, \omega^{n-k}$ to a smooth local coframe $\omega^1, \dots, \omega^n$ for M. If $(E_1, \dots, E_n)$ is the dual frame, then D is locally spanned by $E_{n-k+1}, \dots, E_n$. In terms of this coframe, $d\omega^i$ can be written as

$$d\omega^i = \sum_I{}' \alpha_I \omega^{i_1} \wedge \omega^{i_2}.$$

The fact that $d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0$ means $\alpha_I = 0$ when $n-k+1 \le i_1 < i_2 \le n$. Therefore, similar to the proof of Lemma 1.3.1.6, we can write

$$d\omega^i = \sum_{i_1 \le n-k} \alpha_I \omega^{i_1} \wedge \omega^{i_2} = \sum_{i=1}^{n-k} \omega^i \wedge \Big(\sum_{i_1 < i_2} \omega_{i_2} \Big).$$

By Lemma 1.3.1.6 this implies $d\omega^i$ annihilates D. Since this holds for all i, we get (b). □

With a bit more algebraic terminology, there is an elegant way to express the involutivity condition in terms of differential forms. Recall that we have defined the graded algebra of smooth differential forms on a smooth n-manifold M as $\Omega^*(M) = \bigoplus_{p=0}^n \Omega^p(M)$. An **ideal** in $\Omega^*(M)$ is a linear subspace $\mathcal{I} \subseteq \Omega^*(M)$ that is closed under wedge products with arbitrary elements of $\Omega^*(M)$. Now suppose D is a smooth distribution on a smooth n-manifold M. Let $\mathcal{I}^p(D)$ denote the space of smooth p-forms that annihilate D, and let $\mathcal{I}(D) = \bigoplus_{p=0}^n \mathcal{I}^p(D)$.

Any ideal of the form $\mathcal{I}(D)$ for some smooth distribution D is sometimes called a **Pfaffian system**. An ideal $\mathcal{I} \subseteq \Omega^*(M)$ is said to be a **differential ideal** if $d(\mathcal{I}) \subseteq \mathcal{I}$, that is, if $\eta \in \mathcal{I}$ implies $d\eta \in \mathcal{I}$.

Proposition 1.3.1.9 (Differential Ideal Criterion for Involutivity). *Let M be a smooth manifold. A smooth distribution $D \subseteq TM$ is involutive if and only if $\mathcal{I}(D)$ is a differential ideal in $\Omega^*(M)$.*

Proof. One direction is given by Theorem 1.3.1.7, the other is proved by Lemma 1.3.1.6. □

1.3.2 The Frobenius theorem

Given a rank-k distribution $D \subseteq TM$, let us say that a smooth coordinate chart (U, φ) on M is **flat** for D if $\varphi(U)$ is a cube in $\mathbb{R}^n$, and at points of U, D is spanned by the first k coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^k$. In any such chart, each slice of the form $\{x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$ for constants $c^{k+1}, \dots, c^n$ is an integral manifold of D. This is the nicest possible local situation for integral manifolds. We say that a distribution $D \subseteq TM$ is **completely integrable** if there exists a flat chart for D in a neighborhood of each point of M. Obviously, every completely integrable distribution is integrable and therefore involutive. In summary,

$$\text{completely integrable} \implies \text{integrable} \implies \text{involutive.}$$

The next theorem is the main result of this section, and indeed one of the central theorems in smooth manifold theory. It says that the implications above are actually equivalences:

$$\text{completely integrable} \iff \text{integrable} \iff \text{involutive.}$$

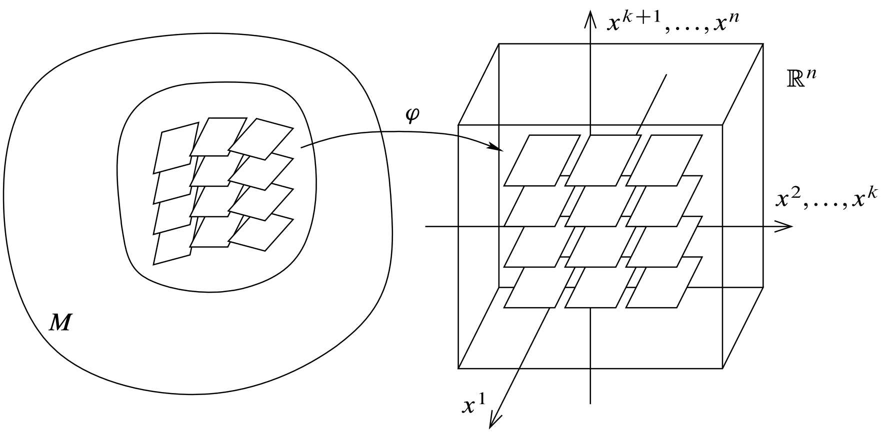


Figure 1.1: A flat chart for a distribution.

Theorem 1.3.2.1 (Frobenius). *Every involutive distribution is completely integrable.*

Proof. The canonical form theorem for commuting vector fields (Theorem 1.2.7.5) implies that any distribution locally spanned by independent smooth commuting vector fields is completely integrable, because the coordinate chart whose existence is guaranteed by that theorem is flat (after shrinking the domain if necessary so the image is a cube). Thus, it suffices to show that every involutive distribution is locally spanned by independent smooth commuting vector fields.

Let D be an involutive distribution of rank k on an n-dimensional manifold M, and let $p \in M$. Since complete integrability is a local question, by passing to a smooth coordinate neighborhood of p, we may replace M by an open subset $U \subseteq \mathbb{R}^n$, and choose a smooth local frame $X_1, \dots, X_k$ for D. By reordering the coordinates if necessary, we may assume that D_p is complementary to the subspace of $T_p\mathbb{R}^n$ spanned by $\partial/\partial x^{k+1}|_p, \dots, \partial/\partial x^n|_p$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k coordinates. This induces a smooth bundle homomorphism $d\pi : T\mathbb{R}^n \to T\mathbb{R}^k$, which can be written

$$d\pi\Big(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_q \Big) = \sum_{i=1}^k v^i \frac{\partial}{\partial x^i}\Big|_{\pi(q)}.$$

Because $d\pi|_D$ is the composition of the inclusion $D \hookrightarrow U$ followed by $d\pi$, it is a smooth bundle homomorphism. Thus, the matrix entries of $d\pi|_{D_q}$ with respect to the frames $(X_i|_q)$ and $(\partial/\partial x^j|_{\pi(q)})$ are smooth functions of q.

By our choice of coordinates, $D_p \subseteq T_p\mathbb{R}^n$ is complementary to the kernel of $d\pi_p$, so the restriction of $d\pi_p$ to D_p is bijective. By continuity, therefore, the same is true of $d\pi_{D_q}$ for q in a neighborhood of

p, and the matrix entries of $(d\pi|_{D_q})^{-1} : T_{\pi(q)}\mathbb{R}^k \to D_q$ are also smooth functions of q. Define a new smooth local frame $V_1, \ldots, V_k$ for D in a neighborhood of p by

$$V_i|_q = (d\pi|_{D_q})^{-1} \frac{\partial}{\partial x^i}\bigg|_{\pi(q)}.$$

The theorem will be proved if we can show that $[V_i, V_j] = 0$ for all i, j.

First observe that V_i and $\partial/\partial x^i$ are π-related, because

$$\frac{\partial}{\partial x^i}\bigg|_{\pi(q)} = (d\pi|_{D_q})V_i|_q = d\pi_q(V_i|_q).$$

Therefore, by the naturality of Lie brackets,

$$d\pi_q([V_i, V_j]_q) = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_{\pi(q)} = 0.$$

Since involutivity of D implies that $[V_i, V_j]$ takes its values in D, and $d\pi$ is injective on each fiber of D, this implies that $[V_i, V_j] = 0$ for each q, thus completing the proof. □

For later use in our treatment of overdetermined partial differential equations, we note the following easy corollary to the proof.

Corollary 1.3.2.2. *Suppose M is a smooth manifold, D is an involutive rank-k distribution on M, and $S \subseteq M$ is a codimension-k embedded submanifold. If $p \in S$ is a point such that T_pS is complementary to D_p, then there is a flat chart $(U, (s^i))$ for D centered at p in which $S \cap U$ is the slice $s^1 = \cdots = s^k = 0$.*

Proof. The proof of the theorem showed that locally D is spanned by k commuting vector fields, and then the corollary follows from Theorem 1.2.7.5. □

As is often the case, embedded in the proof of the Frobenius theorem is a technique for finding integral manifolds. The idea is to use a coordinate projection to find commuting vector fields spanning the same distribution, and then use the technique of Example 1.2.7.6 to find a flat chart. Here is an example.

Example 1.3.2.3. Let $D \subseteq T\mathbb{R}^3$ be the distribution spanned by the vector fields

$$\begin{aligned} X &= x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \end{aligned}$$

We have

$$[X, Y] = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} = -Y,$$

so D is involutive. Let us try to find a flat chart in a neighborhood of the origin. Since D is complementary to the span of $\partial/\partial z$, the coordinate projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi(x, y, z) = (x, y)$ induces an isomorphism $d\pi|_{D_{(x,y,z)}} : D_{(x,y,z)} \to T_{(x,y)}\mathbb{R}^2$ for each (x, y, z). The proof of the Frobenius theorem shows that if we can find smooth local sections V, W of D that are π-related to $\partial/\partial x$ and $\partial/\partial y$, respectively, they will be commuting vector fields spanning D. It is easy to check that V, W have this property if and only if they take their values in D and are of the form

$$\begin{aligned} V &= \frac{\partial}{\partial x} + u(x, y, z)\frac{\partial}{\partial z}, \\ W &= \frac{\partial}{\partial y} + v(x, y, z)\frac{\partial}{\partial z}, \end{aligned}$$

for some smooth real-valued functions u, v. A bit of linear algebra shows that the vector fields

$$\begin{aligned} V &= Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \\ W &= X - xY = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \end{aligned}$$

do the trick. Sparing the details, we find that the flow of V is

$$\alpha_t(x, y, z) = (x + t, y, z + ty),$$

and that of W is

$$\beta_t(x, y, z) = (x, y + t, z + tx).$$

Thus, by the procedure of Example 1.2.7.6, we can define the inverse Φ of our coordinate map by starting on the z-axis and flowing out along these two flows in succession:

$$\Phi(u, v, w) = \alpha_u \circ \beta_v(0, 0, w) = \alpha_u(0, 0, w) = (u, v, w + uv).$$

The flat coordinates we seek are given by inverting the map $(x, y, z) = \Phi(u, v, w) = (u, v, w + uv)$, to yield

$$(u, v, w) = \Phi^{-1}(x, y, z) = (x, y, z - xy).$$

It follows that the integral manifolds of D are the level sets of $w(x, y, z) = z - xy$.

The next proposition is one of the most important consequences of the Frobenius theorem.

Proposition 1.3.2.4 (Local Structure of Integral Manifolds). *Let D be an involutive distribution of rank k on a smooth manifold M, and let $(U, (x^i))$ be a flat chart for D. If H is any integral manifold of D, then $H \cap U$ is a union of countably many disjoint open subsets of parallel k-dimensional slices of U, each of which is open in H and embedded in M.*

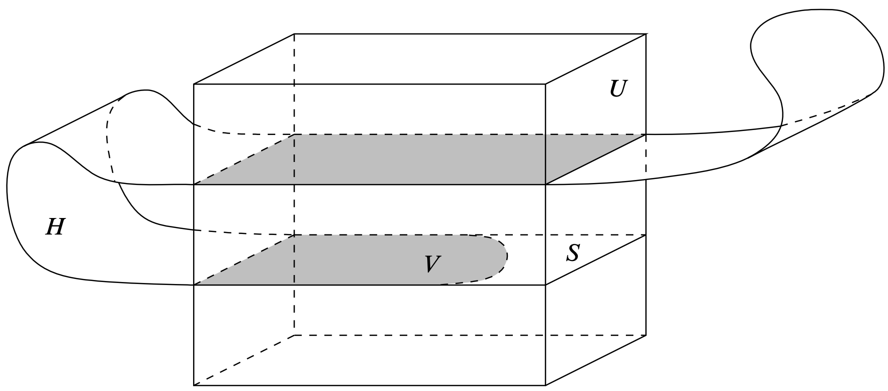


Figure 1.2: The local structure of an integral manifold.

Proof. Let H be an integral manifold of D. Because the inclusion map $\iota : H \hookrightarrow M$ is continuous, $H \cap U = \iota^{-1}(U)$ is open in H, and thus consists of a countable disjoint union of connected components, each of which is open in H.

Let V be any component of $H \cap U$. We show first that V is contained in a single slice. Since $dx^{k+1}, \ldots, dx^n$ are local defining forms for D, it follows that the pullbacks of these 1-forms to V are identically zero. Because V is connected, this implies that $x^{k+1}, \ldots, x^n$ are all constant on V, so V lies in a single slice S.

Because S is embedded in M, the inclusion map $V \hookrightarrow M$ is also smooth as a map into S by Corollary **??**. The inclusion $V \hookrightarrow S$ is thus an injective smooth immersion between manifolds of the same dimension, and therefore a local diffeomorphism, an open map, and a homeomorphism onto an open subset of S. The inclusion map $V \hookrightarrow M$ is a composition of the smooth embeddings $V \hookrightarrow S \hookrightarrow M$, so it is a smooth embedding. $\square$

The preceding proposition implies the following important result about integral manifolds, which we will use in our study of Lie subgroups at the end of this section. Recall that a smooth submanifold $H \subseteq M$ is said to be weakly embedded in M if every smooth map $F : N \to M$ whose image lies in H is smooth as a map from N to H.

Theorem 1.3.2.5. *Every integral manifold of an involutive distribution is weakly embedded.*

Proof. Let M be a smooth n-manifold, let $H \subseteq M$ be an integral manifold of an involutive rank-k distribution D on M, and suppose $F : N \to M$ is a smooth map such that $F(N) \subseteq H$. Let $p \in N$ be arbitrary, and set $q = F(p) \in H$. Let $(y^1, \ldots, y^n)$ be flat coordinates for D on a neighborhood U of q, and let (x^i) be smooth coordinates for N on a connected neighborhood B of p such that $F(B) \subseteq U$. With the coordinate representation of F written as

$$(y^1, \ldots, y^n) = (F^1(x), \ldots, F^n(x)).$$

The fact that $F(N) \subseteq H \cap U$ means that the coordinate functions $F^{k+1}(x), \ldots, F^n(x)$ take on only countably many values. Because B is connected, the intermediate value theorem implies that these coordinate functions are constant, and thus $F(B)$ lies in a single slice $S \subseteq U$. Because $S \cap H$ is an open subset of H that is embedded in M, it follows that $F|_B$ is smooth from B into $S \cap H$, and thus by composition, $F|_B : D \hookrightarrow (S \cap H) \hookrightarrow H$ is smooth into H. $\square$

1.3.3 Foliations

When we put together all of the maximal integral manifolds of an involutive rank-k distribution on a smooth manifold M, we obtain a partition of M into k-dimensional submanifolds that fit together locally like the slices in a flat chart.

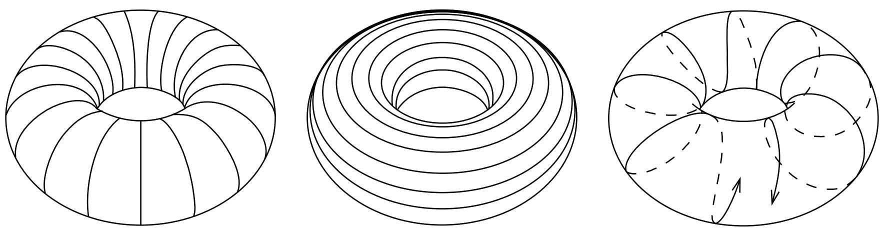
To express more precisely what we mean by fitting together, we need to extend our notion of a flat chart slightly. Let M be a smooth n-manifold, and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M. A smooth chart (U, φ) for M is said to be **flat for** $\mathcal{F}$ if $\varphi(U)$ is a cube in $\mathbb{R}^n$, and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $\{x^{k+1} = c^{k+1}, \ldots, x^n = c^n\}$. We define a **foliation of dimension k on M** to be a collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M (called the **leaves of the foliation**), whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

Example 1.3.3.1 (Foliations).

(a) The collection of all k-dimensional affine subspaces of $\mathbb{R}^n$ parallel to $\mathbb{R}^k \times \{0\}$ is a k-dimensional foliation of $\mathbb{R}^n$.

(b) The collection of open rays of the form $\{\lambda x : \lambda > 0\}$ as x ranges over $\mathbb{R}^n - \{0\}$ is a 1-dimensional foliation of $\mathbb{R}^n - \{0\}$.

(c) The collection of all spheres centered at 0 is an $(n-1)$-dimensional foliation of $\mathbb{R}^n - \{0\}$.



(a) (b) (c)

Figure 1.1: Foliations of the torus.

(d) If M and N are connected smooth manifolds, the collection of subsets of the form $M \times \{q\}$ as q ranges over points in N forms a foliation of $M \times N$, each of whose leaves is diffeomorphic to M. For example, the collection of all circles of the form $S^1 \times \{q\} \subseteq T^2$ for $q \in S^1$ yields a foliation of the torus T^2. A different foliation of T^2 is given by the collection of circles of the form $\{p\} \times S^1$.

(e) If α is a fixed real number, the images of all curves of the form

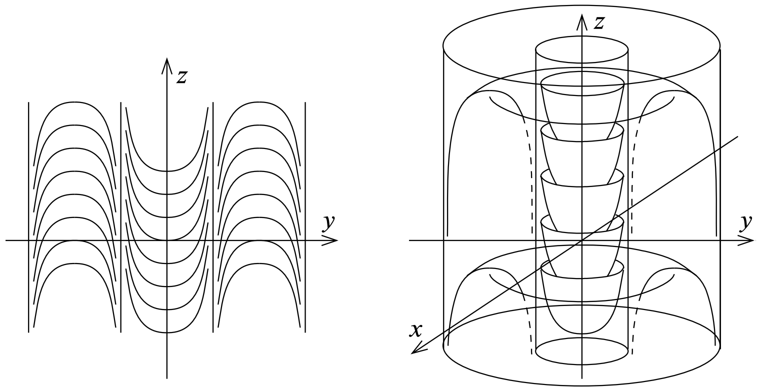
$$\gamma_\theta(t) = (e^{it}, e^{i(\alpha t + \theta)})$$

as θ ranges over $\mathbb{R}$ form a 1-dimensional foliation of the torus. If α is rational, each leaf is an embedded circle; whereas if α is irrational, each leaf is dense.

(f) The collection of connected components of the curves in the (y,z)-plane defined by the following equations is a foliation of $\mathbb{R}^2$:

$$z = \sec y + c,\ c \in \mathbb{R};$$
$$y = (k + \frac{1}{2})\pi,\ k \in \mathbb{Z}.$$

(g) If we revolve the curves of the previous example around the z-axis, we obtain a 2-dimensional foliation of $\mathbb{R}^3$ in which some of the leaves are diffeomorphic to disks and some are diffeomorphic to cylinders.



(a) (b)

Figure 1.2: Foliations of $\mathbb{R}^2$ and $\mathbb{R}^3$.

The main fact about foliations is that they are in one-to-one correspondence with involutive distributions. One direction, expressed in the next proposition, is an easy consequence of the definition.

Proposition 1.3.3.2. *Let $\mathcal{F}$ be a foliation on a smooth manifold M. The collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on M.*

The Frobenius theorem allows us to conclude the following converse, which is much more profound. By the way, it is worth noting that this result is one of the two primary reasons why the notion of immersed submanifold has been defined.

Theorem 1.3.3.3 (Global Frobenius Theorem). *Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

The theorem will be an easy consequence of the following lemma.

Lemma 1.3.3.4. *Suppose $D \subseteq TM$ is an involutive distribution, and let $\{N_\alpha\}_{\alpha\in A}$ be any collection of connected integral manifolds of D with a point in common. Then $N = \bigcup_{\alpha\in A} N_\alpha$ has a unique smooth manifold structure making it into a connected integral manifold of D.*

Proof. If we can construct a topology and smooth manifold structure making N into an integral manifold of D, then Theorem **??** shows that the topology and smooth structure are uniquely determined, because integral manifolds are weakly embedded.

To construct the topology, first we need to show that $N_\alpha \cap N_\beta$ is open in N_α and in N_β for each $\alpha, \beta \in A$. Let $q \in N_\alpha \cap N_\beta$ be arbitrary, and choose a flat chart for D on a neighborhood W of q. Let V_α, V_β denote the components of $N_\alpha \cap W$ and $N_\beta \cap W$, respectively, containing q. By Proposition 1.3.2.4, V_α and V_β are open subsets of single slices with the subspace topology, and since both contain q, they both must lie in the same slice S. Thus $V_\alpha \cap V_\beta$ is open in S and also in both N_α and N_β, so q has a neighborhood in N_α and a neighborhood in N_β contained in $N_\alpha \cap N_\beta$.

Define a topology on N by declaring a subset $U \subseteq N$ to be open if and only if $U \cap N_\alpha$ is open in N_α for each N_α. Using the result of the previous paragraph, it is easy to check that this is a topology and that each N_α is an open subspace of N. With this topology, N is locally Euclidean of dimension k, because each $q \in N$ has a coordinate neighborhood V in some N_α, and V is an open subset of N because

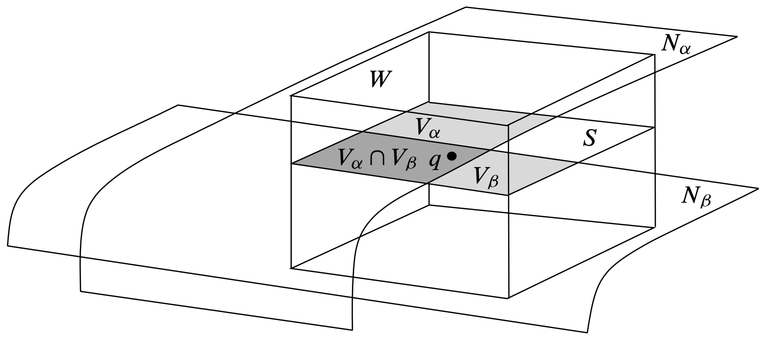


Figure 1.3: A union of integral manifolds.

N_α is open in N. Moreover, the inclusion map $N \hookrightarrow M$ is continuous: for any open subset $U \subseteq M$, $U \cap N$ is open in N because $U \cap N_\alpha$ is open in N_α for each α.

To see that N is Hausdorff, let q_1, q_2 be distinct points of N. There are disjoint open subsets $U_1, U_2 \subseteq M$ containing q_1 and q_2, respectively, and because inclusion $N \hookrightarrow M$ is continuous, $N \cap U_1$ and $N \cap U_2$ are disjoint open subsets of N containing q_1 and q_2.

Next we show that N is second-countable. We can cover M with countably many flat charts for D, say $\{W_i\}$. It suffices to show that $N \cap W_i$ is contained in a countable union of slices for each i, because any open subset of a single slice is second countable, and thus N can be expressed as a union of countably many subsets, each of which is second-countable and open in N.

Let p_0 be a point contained in N_α for every α. Let us say that a slice S of some W_k is **accessible from p_0** if there is a finite sequence of indices $i_1, \dots, i_m$ and for each i_j a slice $S_{i_j} \subseteq W_{i_j}$ with the properties that $p \in S_{i_1}$, $S_{i_m} = S$, and $S_{i_j} \cap S_{i_j+1} \neq \varnothing$ for each $j = 1, \dots, m-1$.

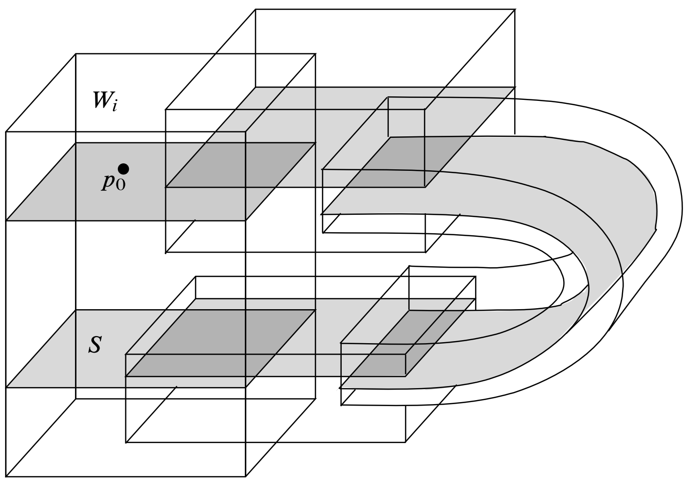


Figure 1.4: A slice S accessible from p_0

Let W_k be one of our countable collection of flat charts, and suppose $S \subseteq W_k$ is a slice that contains a point $q \in N$. Then q is contained in one of the integral manifolds N_α. Because p_0 is also in N_α, there is a continuous path $\gamma : [0,1] \to N_\alpha$ connecting p_0 and q. Since $\gamma([0,1])$ is compact, there exist finitely many numbers $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for each $j = 1, \dots, m$, the set $\gamma([t_{j-1}, t_j])$ is contained in one of the flat charts W_{i_j}. Since $\gamma([t_{j-1}, t_j])$ is connected, it is contained in a single component of $W_{i_j} \cap N_\alpha$ and therefore in a single slice $S_{i_j} \subseteq W_{i_j}$. For each $j = 1, \dots, m-1$, the slices S_{i_j} and $S_{i_{j+1}}$ have the point $\gamma(t_j)$ in common, so it follows that the slice S is accessible from p_0.

This shows that every slice of some W_k containing a point of N is accessible from p_0, and therefore every slice intersecting N must be accessible from p_0. To complete the proof of second-countability, we just note that each S_{i_j} is itself an integral manifold, and therefore it meets at most countably many slices of $W_{i_{j+1}}$ by Proposition 1.3.2.4; thus, there are only countably many slices accessible from p_0. Therefore, N is a topological manifold of dimension k. It is connected because it is a union of connected subspaces with a point in common.

To construct a smooth structure on N, we define an atlas consisting of all charts of the form $(S \cap N, \psi)$, where S is a single slice of some flat chart, and $\psi : S \to \mathbb{R}^k$ is the map whose coordinate representation in the flat chart is projection onto the first k coordinates: $(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \mapsto (x^1, \ldots, x^k)$. Because any slice is an embedded submanifold, its smooth structure is uniquely determined, and thus whenever two such slices S_1 and S_2 overlap the transition map $\psi_2 \circ \psi_1^{-1}$ is smooth. With respect to this smooth structure, the inclusion map $N \hookrightarrow M$ is a smooth immersion (because it is a smooth embedding on each slice), and the tangent space to N at each point $q \in N$ is equal to D_q (because this is true for slices). □

Proof of the global Frobenius theorem. For each $p \in M$, let L_p be the union of all connected integral manifolds of D containing p. By the preceding lemma, L_p is a connected integral manifold of D containing p, and it is clearly maximal. If any two such maximal integral manifolds L_p and $L_{p'}$ intersect, their union $L_p \cup L_{p'}$ is an integral manifold containing both p and p', so by maximality $L_p = L_{p'}$. Thus, the various maximal connected integral manifolds are either disjoint or identical.

If (U, φ) is any flat chart for D, then $L_p \cap U$ is a countable union of open subsets of slices by Proposition 1.3.2.4. For any such slice S, if $L_p \cap S$ is neither empty nor all of S, then $L_p \cup S$ is a connected integral manifold properly containing L_p, which contradicts the maximality of L_p. Therefore, $L_p \cap U$ is precisely a countable union of slices, so the collection $\{L_p : p \in M\}$ is the desired foliation. □

Suppose M is a smooth manifold and $\Phi : M \to M$ is a diffeomorphism. A distribution D on M is said to be **Φ-invariant** if $d\Phi(D) = D$ or more precisely if for each $x \in M$, $d\Phi_x(D_x) = D_{\Phi(x)}$. Similarly, a foliation $\mathcal{F}$ on M is said to be **Φ-invariant** if for each leaf L of $\mathcal{F}$, the submanifold $\Phi(L)$ is also a leaf of $\mathcal{F}$.

Proposition 1.3.3.5. *Let M be a smooth manifold and $\Phi : M \to M$ be a diffeomorphism. Suppose D is an involutive distribution on M and $\mathcal{F}$ is the foliation it determines. Then D is Φ-invariant if and only if $\mathcal{F}$ is Φ-invariant.*

Proof. Since the tangent space of a point on the leaf of the foliation is the fiber of the distribution, if $\mathcal{F}$ is Φ-invariant, then D is also Φ-invariant. Conversely, if D is Φ-invariant, then for any point $p \in M$, we have $d\Phi_p(D) = D_{\Phi(p)}$. Let L be a leaf of $\mathcal{F}$, then $\Phi(L)$ is also an integral manifold. By definition, this implies $\mathcal{F}$ is Φ-invariant. □

1.3.4 Overdetermined systems of partial differential equations

The partial differential equations we considered in Section 1.2 were all single equations for one unknown function. In some applications, it is necessary to consider systems of PDEs that are **overdetermined**, which means that there are more equations than unknown functions. In general, overdetermined systems have solutions only if they satisfy certain compatibility conditions. For some first-order systems, the compatibility condition can be interpreted as a statement about involutivity of a distribution, and the Frobenius theorem can be used to prove local existence and uniqueness of solutions.

First, we consider certain linear systems. Suppose W is an open subset of $\mathbb{R}^n$ and m is a positive integer less than or equal to n. Consider the following system of

$$\begin{cases} a_1^1(x)\dfrac{\partial u}{\partial x^1}(x) + \cdots + a_1^n(x)\dfrac{\partial u}{\partial x^n}(x) = f_1(x), \\ \qquad\qquad\qquad\qquad \vdots \\ a_m^1(x)\dfrac{\partial u}{\partial x^1}(x) + \cdots + a_m^n(x)\dfrac{\partial u}{\partial x^n}(x) = f_m(x) \end{cases} \tag{1.3.4.1}$$

where $(a_i^j(x))$ is an $n \times m$ matrix of smooth real-valued functions and $f_1, \ldots, f_m$ are smooth real-valued functions on W. The case $m = 1$ is covered by Theorem 1.2.9.1, so this discussion is useful primarily when $m > 1$.

If we let A_i denote the vector field $a_i^j \partial/\partial x^j$, the system (1.3.4.1) can be written more succinctly as $A_i u = f_i$, $i = 1, \ldots, m$. To avoid redundant or degenerate systems of equations, we assume that the matrix $(a_i^j(x))$ has rank m at each point of W, or equivalently that the vector fields $A_1, \ldots, A_m$ are linearly independent. The following theorem is an analogue of Theorem 1.2.9.1 for the overdetermined case.

Theorem 1.3.4.1. *Let $W \subseteq \mathbb{R}^n$ be an open subset and let m be an integer such that $1 \leq m \leq n$. Suppose we are given an embedded codimension-m submanifold $S \subseteq W$, a linearly independent m-tuple of smooth vector fields $(A_1, \ldots, A_m)$ on W whose span is complementary to T_pS at each $p \in S$, and functions $f_1, \ldots, f_m \in C^\infty(W)$. Suppose also that there are smooth functions $c_{ij}^k \in C^\infty(W)$ for $i, j, k = 1, \ldots, m$ such that the following compatibility conditions are satisfied:*

$$[A_i, A_j] = c_{ij}^k A_k, \tag{1.3.4.2}$$

$$A_i f_j - A_j f_i = c_{ij}^k f_k. \tag{1.3.4.3}$$

Then for each $p \in S$, there is a neighborhood U of p such that for every $\varphi \in C^\infty(S \cap U)$, there exists a unique solution $u \in C^\infty(U)$ to the following overdetermined Cauchy problem:

$$A_i u = f_i \quad \text{for } i = 1, \ldots, m \tag{1.3.4.4}$$

$$u|_{S\cap U} = \varphi. \tag{1.3.4.5}$$

Proof. Let D be the distribution on W spanned by $A_1, \ldots, A_m$, and let $p \in S$ be arbitrary. It follows from (1.3.4.2) that D is involutive, so by Corollary 1.3.2.2, on some neighborhood U of p there is a flat chart for D centered at p that is also a slice chart for S. Label the coordinates in this chart as $(v, w) = (v^1, \ldots, v^m, w^1, \ldots, w^{n-m})$ so that $S \cap U$ is the slice where $v^1 = \cdots = v^m = 0$, and each $w =$ constant slice is an integral manifold of D in U, which we denote by H_w. Because (1.3.4.4) is a coordinate-independent statement, we can replace A_i and f_i by their coordinate representations in U, solve the equation there, and then use the inverse coordinate transformation to convert the solution back to the original coordinates.

By the definition of a flat chart, the vectors $\partial/\partial v^1|_q, \ldots, \partial/\partial v^m|_q$ span D_q for each $q \in U$. Therefore the n-tuple $(A_1, \ldots, A_m, \partial/\partial w^1, \ldots, \partial/\partial w^{n-m})$ is a smooth local frame for U. Let $(\alpha^1, \ldots, \alpha^m, \beta^1, \ldots, \beta^{n-m})$ denote the dual coframe, and define a smooth 1-form $\omega \in \Omega^1(U)$ by $\omega = f_k \alpha^k$ (with the implied summation from 1 to m). The system of equations $A_i u = f_i$ is satisfied if and only if $du(A_i) = \omega(A_i)$ for $i = 1, \ldots, m$, which is equivalent to saying that the pullback of $du - \omega$ to each H_w is equal to zero.

Using formula (3.3.3.5) for the exterior derivative together with (1.3.4.2), we obtain

$$d\alpha^k(A_i, A_j) = A_i(\alpha^k(A_j)) - A_j(\alpha^k(A_i)) - \alpha^k([A_i, A_j]) = -c_{ij}^k$$

for each $i, j, k = 1, \ldots, m$. It then follows from (1.3.4.3) that

$$\begin{aligned} d\omega(A_i, A_j) &= (df_k \wedge \alpha^k + f_k \alpha^k)(A_i, A_j) = df^k(A_i)\alpha^k(A_j) - df_k(A_j)\alpha^k(A_j) - f_k c_{ij}^k \\ &= (A_i f_k)\delta_j^k - (A_j f_k)\delta_i^k - f_k c_{ij}^k = A_i f_j - A_j f_i - f_k c_{ij}^k = 0. \end{aligned}$$

Since $(A_1, \ldots, A_m)$ restricts to a frame on each integral manifold H_w, this shows that the pullback of ω to each H_w is closed.

Given $\varphi \in S \cap U$, let $u = u_0 + u_1$, where $u_0, u_1 \in C^\infty(U)$ are defined by

$$u_0(v, w) = \varphi(0, w), \quad u_1(v, w) = \int_0^1 \omega_k(tv, w) v^k dt$$

and $\omega = \omega_k \, dv^k$ is the coordinate expression for ω.

Recall that a flat chart is cubical by definition, and thus star-shaped, so the integral is well defined for all $(v, w) \in U$, and differentiation under the integral sign shows that u_1 is a smooth function of (v, w). Because $u_0|_{S\cap U} = \varphi$ and $u_1|_{S\cap U} = 0$, it follows that u satisfies the initial condition (1.3.4.5).

The function u_0 satisfies $A_1 u_0 = \cdots = A_m u_0 = 0$ because it is independent of the v-coordinates. On the other hand, for each fixed w, u_1 is the potential function on H_w for $\iota^*\omega$ given by formula (**??**), where $\iota : H_w \to U$ is the inclusion. The proof of Theorem **??** shows that $\iota^* du = \iota^*\omega$ for each w. It follows that $A_k u = A_k(u_1) = f_k$ for each $k = 1, \ldots, m$, so u is a solution to (1.3.4.4) as well.

To prove uniqueness, suppose $\widetilde{u}$ is any other solution to (1.3.4.4)–(1.3.4.5) on U, and let $\psi = u - \widetilde{u}$. Then $A_k \psi = 0$ for each k, so is independent of v. It follows that $\psi(v, w) = \psi(0, w)$, which is zero because u and $\widetilde{u}$ satisfy (1.3.4.5). □

Next we apply the Frobenius theorem to a class of nonlinear overdetermined linear PDEs. These are equations for a vector-valued function $u = (u^1, \ldots, u^m)$ that express all first partial derivatives of

u in terms of the independent variables and the values of u. We explain it first in the case of a single real-valued function u of two independent variables (x,y), in which case the notation is considerably simpler.

Suppose we seek a solution u to the system

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = \alpha(x,y,u(x,y)), \\ \dfrac{\partial u}{\partial y}(x,y) = \beta(x,y,u(x,y)). \end{cases} \tag{1.3.4.6}$$

where α and β are smooth real-valued functions defined on some open subset $W \subseteq \mathbb{R}^3$. This is an overdetermined system of (possibly nonlinear) first-order PDEs. To determine the compatibility conditions that α and β must satisfy for solvability of (1.3.4.6), assume u is a smooth solution on some open subset of $\mathbb{R}^2$. Because $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$, (1.3.4.6) implies

$$\frac{\partial}{\partial y}(\alpha(x,y,u(x,y))) = \frac{\partial}{\partial x}(\beta(x,y,u(x,y))).$$

and therefore by the chain rule

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}. \tag{1.3.4.7}$$

This is true at a point $(x,y,z) \in W$ provided there is a smooth solution u with $u(x,y) = z$. In particular, (1.3.4.7) is a necessary condition for (1.3.4.6) to have a solution in a neighborhood of each point (x_0, y_0) with freely specified initial value $u(x_0, y_0) = z_0$. Using the Frobenius theorem, we can show that this condition is sufficient.

Proposition 1.3.4.2. *Suppose α and β are smooth real-valued functions defined on some open subset $W \subseteq \mathbb{R}^3$ and satisfying (1.3.4.7) there. For each $(x_0, y_0, z_0) \in W$, there exist a neighborhood U of (x_0, y_0) in $\mathbb{R}^2$ and a unique smooth function $u : U \to \mathbb{R}$ satisfying (1.3.4.6) and $u(x_0, y_0) = z_0$.*

Proof. The idea of the proof is that the system (1.3.4.6) determines the partial derivatives of u in terms of its values, and therefore determines the tangent plane to the graph of u at each point in terms of the coordinates of the point on the graph. This collection of tangent planes defines a smooth rank-2 distribution on W, and (1.3.4.7) is equivalent to the involutivity condition for this distribution.

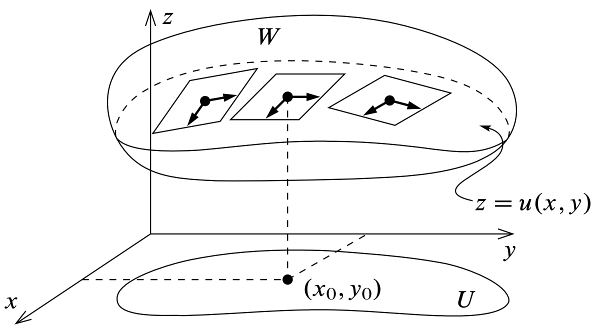


Figure 1.1: Solving for u by finding its graph.

If there is a solution u on an open subset $U \subseteq \mathbb{R}^2$, the map $F : U \to W$ given by

$$F(x,y) = (x,y,u(x,y))$$

is a smooth global parametrization of the graph $\Gamma(u) \subseteq U \times \mathbb{R}$. At any point $p = F(x,y)$, the tangent space $T_p\Gamma(u)$ is spanned by the vectors

$$dF\left(\left.\frac{\partial}{\partial x}\right|_{(x,y)}\right) = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}(x,y)\left.\frac{\partial}{\partial z}\right|_p,$$

$$dF\Big(\frac{\partial}{\partial y}\Big|_{(x,y)}\Big) = \frac{\partial}{\partial x} + \frac{\partial u}{\partial y}(x,y)\frac{\partial}{\partial z}\Big|_p.$$

The system (1.3.4.6) is satisfied if and only if

$$\begin{aligned} dF\Big(\frac{\partial}{\partial x}\Big|_{(x,y)}\Big) &= \frac{\partial}{\partial x} + \alpha(x,y,u(x,y))\frac{\partial}{\partial z}\Big|_p, \\ dF\Big(\frac{\partial}{\partial y}\Big|_{(x,y)}\Big) &= \frac{\partial}{\partial x} + \beta(x,y,u(x,y))\frac{\partial}{\partial z}\Big|_p. \end{aligned} \tag{1.3.4.8}$$

Let X and Y be the vector fields

$$\begin{aligned} X &= \frac{\partial}{\partial x} + \alpha(x,y,z)\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} + \beta(x,y,z)\frac{\partial}{\partial z}. \end{aligned} \tag{1.3.4.9}$$

on W, and let D be the distribution on W spanned by X and Y. Because (1.3.4.8) says that $T_p\Gamma(u)$ is spanned by X_p and Y_p, a necessary condition for the system (1.3.4.6) to be satisfied is that $\Gamma(u)$ be an integral manifold of D. On the other hand, this condition is also sufficient: if $\Gamma(u)$ is an integral manifold, then $dF(\partial/\partial x)$ and $dF(\partial/\partial y)$ must both be linear combinations of X and Y, and comparing $\partial/\partial x$ and $\partial/\partial y$ components shows that this can happen only if (1.3.4.8) holds.

A straightforward computation using (1.3.4.7) shows that $[X,Y] \equiv 0$, so given any point $p = (x_0, y_0, z_0) \in W$, there is an integral manifold N of D containing p. Let $\Phi : V \to \mathbb{R}$ be a defining function for N on some neighborhood V of p; for example, we could take Φ to be the third coordinate function in a flat chart. The tangent space to N at each point $p \in N$ (namely D_p) is equal to the kernel of $d\Phi_p$. Since $\partial/\partial z|_p \notin D_p$ at any point p, this implies that $\partial\Phi/\partial z \neq 0$ at p, so by the implicit function theorem N is the graph of a smooth function $z = u(x,y)$ in some neighborhood of p. It is easily verified that u is a solution to the problem. Uniqueness follows immediately from Proposition 1.3.2.4. □

There is a straightforward generalization of this result to higher dimensions. The general statement of the theorem is a bit complicated, but verifying the necessary conditions in specific examples usually just amounts to computing mixed partial derivatives and applying the chain rule.

Proposition 1.3.4.3. *Suppose W is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and $\alpha = (\alpha^i_j) : W \to \mathcal{M}_{m\times n}(\mathbb{R})$ is a smooth matrix-valued function satisfying*

$$\frac{\partial \alpha^i_j}{\partial x^k} + \alpha^l_k \frac{\partial \alpha^i_j}{\partial z^l} = \frac{\partial \alpha^i_k}{\partial x^j} + \alpha^l_j \frac{\partial \alpha^i_k}{\partial z^l} \quad \text{for all } i,j,k,$$

where we denote a point in $\mathbb{R}^n \times \mathbb{R}^m$ by $(x,z) = (x^1,\dots,x^n,z^1,\dots,z^m)$. For any $(x_0,z_0) \in W$, there is a neighborhood U of x_0 in $\mathbb{R}^n$ and a unique smooth function $u : U \to \mathbb{R}^m$ such that $u(x_0) = z_0$ and the Jacobian of u satisfies

$$\frac{\partial u^i}{\partial x^j}(x^1,\dots,x^n) = \alpha^i_j(x^1,\dots,x^n,u^1(x),\dots,u^m(x)).$$

Chapter 2

Lie Groups and Lie Algebras

2.1 Lie groups

2.1.1 Lie groups and homomorphisms

A **Lie group** is a smooth manifold G (without boundary) that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g,h) = gh, \quad i(g) = g^{-1}$$

are both smooth. A Lie group is, in particular, a topological group.

The group operation in an arbitrary Lie group is denoted by juxtaposition, except in certain abelian groups such as Rn in which the operation is usually written additively. It is traditional to denote the identity element of an arbitrary Lie group by the symbol e.

The following alternative characterization of the smoothness condition is sometimes useful.

Proposition 2.1.1.1. *If G is a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g,h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Proof. Let we denote this map by σ. We have

$$i(g) = \sigma(e,g), \quad m(g,h) = \sigma(g, i(h)).$$

Thus if σ is smooth, so is i, and hence m. □

If G is a Lie group, any element $g \in G$ defines maps $L_g, R_g : G \to G$, called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

Because L_g can be expressed as the composition of smooth maps

$$G \xrightarrow{\iota_g} G \times G \xrightarrow{m} G$$

where $\iota_g(h) = (g,h)$ and m is multiplication, it follows that L_g is smooth. It is actually a diffeomorphism of G, because $L_{g^{-1}}$ is a smooth inverse for it. Similarly, $R_g : G \to G$ is a diffeomorphism.

Example 2.1.1.2 (Lie Groups). Each of the following manifolds is a Lie group with the indicated group operation.

(a) The general linear group $\mathrm{GL}_n(\mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open submanifold of the vector space $\mathcal{M}_n(\mathbb{R})$. Multiplication is smooth because the matrix entries of a product matrix AB are polynomials in the entries of A and B. Inversion is smooth by Cramer's rule.

(b) Let $\mathrm{GL}_n^+(\mathbb{R})$ denote the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with positive determinant. Because $\det(AB) = (\det A)(\det B)$ and $\det A^{-1} = (\det A)^{-1}$, it is a subgroup of $\mathrm{GL}_n(\mathbb{R})$ and because it is the preimage of $(0,\infty)$ under the continuous determinant function, it is an open subset of $\mathrm{GL}_n(\mathbb{R})$ and therefore an n^2-dimensional manifold. The group operations are the restrictions of those of $\mathrm{GL}_n(\mathbb{R})$ so they are smooth. Thus $\mathrm{GL}_n^+(\mathbb{R})$ is a Lie group.

(c) Suppose G is an arbitrary Lie group and $H \subseteq G$ is an open subgroup (a subgroup that is also an open subset). By the same argument as in part (b), H is a Lie group with the inherited group structure and smooth manifold structure.

(d) The **complex general linear group** $\mathrm{GL}_n(\mathbb{C})$ is the group of invertible complex $n \times n$ matrices under matrix multiplication. It is an open submanifold of $\mathcal{M}_n(\mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold, and it is a Lie group because matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries.

(e) If V is any real or complex vector space, $\mathrm{GL}(V)$ denotes the set of invertible linear maps from V to itself. It is a group under composition. If V has finite dimension n, any basis for V determines an isomorphism of $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, so $\mathrm{GL}(V)$ is a Lie group. The transition map between any two such isomorphisms is given by a map of the form $A \mapsto B^{-1}AB$ (where B is the transition matrix between the two bases), which is smooth. Thus, the smooth manifold structure on $\mathrm{GL}(V)$ is independent of the choice of basis.

(f) The real number field $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition, because the coordinates of $x - y$ are smooth. Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ are Lie groups under addition.

(g) The set $\mathbb{R}^*$ of nonzero real numbers is a 1-dimensional Lie group under multiplication. In fact, it is exactly $\mathrm{GL}_1(\mathbb{R})$ if we identify a 1×1 matrix with the corresponding real number.) The subset $\mathbb{R}^+$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group. The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication, which can be identified with $\mathrm{GL}_1(\mathbb{C})$.

(h) The circle $S^1 \subseteq \mathbb{C}^*$ is a smooth manifold and a group under complex multiplication. With appropriate angle functions as local coordinates on open subsets of S^1, multiplication and inversion have the smooth coordinate expressions $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ and $\theta_1 \mapsto -\theta$, and therefore S^1 is a Lie group, called the **circle group**.

(i) Given Lie groups $G_1, \ldots, G_k$, their direct product is the product manifold $G_1 \times \cdots \times G_k$ with the group structure given by componentwise multiplication:

$$(g_1, \ldots, g_k)(g_1', \ldots, g_k') = (g_1 g_1', \ldots, g_k g_k')$$

It is a Lie group. For example, the n-torus $T^n = S^1 \times \cdots \times S^1$ is an n-dimensional abelian Lie group.

(j) Any group with the discrete topology is a topological group, called a **discrete group**. If in addition the group is finite or countably infinite, then it is a zero-dimensional Lie group, called a **discrete Lie group**.

Now we consider maps between Lie groups. If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism. In this case we say that G and H are **isomorphic Lie groups**.

Example 2.1.1.3 (Lie Group Homomorphisms).

(a) The inclusion map $S^1 \hookrightarrow \mathbb{C}^*$ is a Lie group homomorphism.

(b) Considering $\mathbb{R}$ as a Lie group under addition, and $\mathbb{R}^*$ as a Lie group under multiplication, the map $\exp : \mathbb{R} \to \mathbb{R}^*$ is smooth, and is a Lie group homomorphism. The image of exp is the open subgroup $\mathbb{R}^+$ consisting of positive real numbers, and $\exp : \mathbb{R} \to \mathbb{R}^+$ is a Lie group isomorphism with inverse $\log : \mathbb{R}^+ \to \mathbb{R}$. Similarly, $\exp : \mathbb{C} \to \mathbb{C}^*$ is a Lie group homomorphism. It is surjective but not injective, because its kernel is $2\pi i\mathbb{Z}$.

(c) The map $\varepsilon : \mathbb{R} \to S^1$ defined by $\varepsilon(x) = e^{2\pi i x}$ is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers. Similarly, the map $\varepsilon^n : \mathbb{R}^n \to T^n$ defined by $\varepsilon^n(x_1, \ldots, x^n) = (e^{2\pi i x^1}, \ldots, e^{2\pi i x^n})$ is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.

(d) The determinant function $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is smooth because $\det A$ is a polynomial in the matrix entries of A. It is a Lie group homomorphism because $\det(AB) = (\det A)(\det B)$. Similarly, $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^*$ is a Lie group homomorphism.

(e) If G is a Lie group and $g \in G$, conjugation by g is the map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$. Because group multiplication and inversion are smooth, C_g is smooth, and a simple computation shows that it is a group homomorphism. In fact, it is an isomorphism, because it has $C_{g^{-1}}$ as an inverse. A subgroup $H \subseteq G$ is said to be normal if $C_g(H) = H$ for every $g \in G$.

The next theorem is important for understanding many of the properties of Lie group homomorphisms.

Theorem 2.1.1.4. *Every Lie group homomorphism has constant rank.*

Proof. Let $F : G \to H$ be a Lie group homomorphism, and let e_G and e_H denote the identity elements of G and H, respectively. Suppose g_0 is an arbitrary element of G. We will show that dF_{g_0} has the same rank as dF_{e_G}. The fact that F is a homomorphism means that for all $g \in G$,

$$F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}F(g)$$

or in other words, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[d, "L_{g_0}"'] & H \arrow[d, "L_{F(g_0)}"] \\ G \arrow[r, "F"] & H \end{tikzcd}$$

Taking differentials of both sides at the identity and using Proposition **??**(b), we find that

$$dF_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{F(g_0)})_{e_H} \circ dF_{e_H}$$

Left multiplication by any element of a Lie group is a diffeomorphism, so both $d(L_{g_0})_{e_G}$ and $d(L_{F(g_0)})_{e_H}$ are isomorphisms. Because composing with an isomorphism does not change the rank of a linear map, it follows that dF_{g_0} and dF_{e_G} have the same rank. □

Corollary 2.1.1.5. *A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.*

Proof. The global rank theorem shows that a bijective Lie group homomorphism is a diffeomorphism. □

2.1.2 The universal covering group

Covering space theory yields the following important result about Lie groups.

Theorem 2.1.2.1 (Universal Covering Group). *Let G be a connected Lie group. Then there exists a simply connected Lie group $\widetilde{G}$, called the* ***universal covering group*** *of G, that admits a smooth covering map $\pi : \widetilde{G} \to G$ that is also a Lie group homomorphism with kernel isomorphic to $\pi_1(G)$ and is a discrete central group in $\widetilde{G}$.*

Proof. Let $\widetilde{G}$ be the universal covering manifold of G and $\pi : \widetilde{G} \to G$ be the corresponding smooth covering map. Then $\pi \times \pi : \widetilde{G} \times \widetilde{G} \to G \times G$ is also a smooth covering map.

Let $m : G \times G \to G$ and $i : G \to G$ denote the multiplication and inversion maps of G, respectively, and let $\widetilde{e}$ be an arbitrary element of the fiber $\pi^{-1}(e)$. Since $\widetilde{G}$ is simply connected, the lifting criterion for covering maps guarantees that the map $m \circ (\pi \times \pi) : \widetilde{G} \times \widetilde{G} \to G$ has a unique continuous lift $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ satisfying $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$ and $\pi \circ \widetilde{m} = m \circ (\pi \times \pi)$:

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m}"] \arrow[d, "\pi\circ\pi"'] & \widetilde{G} \arrow[d, "\pi"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Because π is a surjective local diffeomorphism and $\pi \circ \widetilde{m}$ is smooth, it follows from Proposition **??**(b) that $\widetilde{m}$ is smooth. By the same reasoning, $i \circ \pi : \widetilde{G} \to G$ has a smooth lift $\widetilde{i}$ such that $\widetilde{i}(\widetilde{e}) = \widetilde{e}$ and

$\pi \circ \widetilde{i} = i \circ \pi$:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{i}"] \arrow[d, "\pi"'] & \widetilde{G} \arrow[d, "\pi"] \\ G \arrow[r, "i"] & G \end{tikzcd}$$

We define multiplication and inversion in $\widetilde{G}$ to be $\widetilde{m}$ and $\widetilde{i}$. Then from the commutative diagrams we have

$$\pi(x, y) = \pi(x)\pi(y), \quad \pi(x^{-1})\pi(x)^{-1}. \tag{2.1.2.1}$$

It remains only to show that $\widetilde{G}$ is a group with these operations, for then it is a Lie group because $\widetilde{m}$ and $\widetilde{i}$ are smooth, and (2.1.2.1) shows that π is a homomorphism.

First we show that $\widetilde{e}$ is an identity for multiplication in $\widetilde{e}$. Consider the map $f : \widetilde{G} \to \widetilde{G}$ defined by $f(x) = \widetilde{e}x$. Then (2.1.2.1) implies that

$$\pi \circ f(x) = \pi(\widetilde{e}x) = \pi(\widetilde{e})\pi(x) = e\pi(x) = \pi(x)$$

so f is a lift of $\pi : \widetilde{G} \to G$. Since the identity map $\mathrm{id}_{\widetilde{G}}$ is already a lift of π, and it agrees with f at a point because $f(\widetilde{e}) = \widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$, the unique lifting property of covering maps implies that $f = \mathrm{id}_{\widetilde{G}}$, or equivalently, $\widetilde{e}x = x$ for all x in $\widetilde{G}$. The same argument shows that $x\widetilde{e} = x$.

Next, to show that multiplication in $\widetilde{G}$ is associative, consider the two maps $\alpha_L, \alpha_R : \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ defined by

$$\alpha_L(x, y, z) = (xy)z, \quad \alpha_R(x, y, z) = x(yz)$$

Then (2.1.2.1) applied repeatedly implies that

$$\pi \circ \alpha_L(x, y, z) = (\pi(x)\pi(y))\pi(z) = \pi(x)(\pi(y)\pi(z)) = \pi \circ \alpha_R(x, y, z)$$

so α_L and α_R are both lifts of the same map $\alpha(x, y, z) = \pi(x)\pi(y)\pi(z)$. Because α_L and α_R agree at $(\widetilde{e}, \widetilde{e}, \widetilde{e})$, they are equal.

Finally, let $x_0 \in \widetilde{G}$. To show $x_0x_0^{-1} = x_0^{-1}x_0 = \widetilde{e}$ we define a map $g : \widetilde{G} \to \widetilde{G}$ by $g(x) = x_0^{-1}x_0x$. With the same argument we can show that g is a lift of π, thus equal to $\mathrm{id}_{\widetilde{G}}$. Hence we claim $x_0^{-1}x_0 = \widetilde{e}$, and similarly $x_0x_0^{-1} = \widetilde{e}$, so $\widetilde{G}$ is a group.

Let $\varphi \in \mathrm{Aut}_\pi(\widetilde{G})$, we consider the map $\widetilde{\varphi} : g \mapsto \varphi(\widetilde{e})g$. We have

$$\pi \circ \widetilde{\varphi}(g) = \pi(\varphi(\widetilde{e})g) = e \cdot \pi(g) = \pi(g),$$

so $\widetilde{\varphi}$ is a lifting of $\pi : \widetilde{G} \to G$. Since φ is also a lifting of $\pi : \widetilde{G} \to G$ and $\varphi(\widetilde{e}) = \widetilde{\varphi}(\widetilde{e})$, we conclude that $\varphi = \widetilde{\varphi}$. With this observation, we define a map

$$\mathrm{ev} : \mathrm{Aut}_\pi(\widetilde{G}) \to \ker \pi, \quad \varphi \mapsto \varphi(\widetilde{e}).$$

Then ev is a homomorphism because

$$\mathrm{ev}(\varphi \circ \psi) = (\varphi \circ \psi(\widetilde{e})) = \varphi(\psi(\widetilde{e})) = \varphi(\widetilde{e}) \cdot \psi(\widetilde{e}) = \mathrm{ev}(\varphi) \cdot \mathrm{ev}(\psi).$$

Since $\ker \pi$ has the same cardinality as $\mathrm{Aut}_\pi(\widetilde{G})$, we conclude that $\mathrm{Aut}_\pi(\widetilde{G})$ is isomorphic to $\mathrm{Aut}_\pi(\widetilde{G})$, and hence to $\pi_1(G)$, as groups. The group $\ker \pi$ is central because it is a discrete normal subgroup of a connected group. □

Theorem 2.1.2.2. *Let G be a connected Lie group. Then the universal covering group of G is unique in the following sense: if $\widetilde{G}$ and $\widetilde{G}'$ are simply connected Lie groups that admit smooth covering maps $\pi : \widetilde{G} \to G$ and $\pi' : \widetilde{G}' \to G$ that are also Lie group homomorphisms, then there exists a Lie group isomorphism $\Phi : \widetilde{G} \to \widetilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. Assume that $\widetilde{G}$ and $\widetilde{G}'$ are universal covering groups of G. Since the universal covering manifold is unique, there exists a diffeomorphism $\Phi : \widetilde{G} \to \widetilde{G}'$ such that $\pi' \circ \Phi = \pi$ and $\Phi(e_{\widetilde{G}}) = e_{\widetilde{G}'}$. Now we only need to show Φ is a homomorphism, which means

$$\Phi \circ \widetilde{m} = \widetilde{m}' \circ (\Phi \times \Phi),$$

where $\widetilde{m}$ and $\widetilde{m}'$ are the multiplication maps in $\widetilde{G}$ and $\widetilde{G}'$, respectively. It is clear that both $\Phi \circ \widetilde{m}$ and $\widetilde{m}' \circ (\Phi \times \Phi)$ are lifts of the same map $m \circ (\pi \times \pi)$, where m is the multiplication map of G. Moreover, these two maps agree on $(\widetilde{e}, \widetilde{e})$. Therefore by the uniqueness of liftings we get the require result. □

2.1.3 Lie subgroups

Suppose G is a Lie group. A **Lie subgroup** of G is a subgroup of G endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of G. The following proposition shows that embedded subgroups are automatically Lie subgroups.

Proposition 2.1.3.1. *Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold. Then H is a Lie subgroup.*

Proof. We need only check that multiplication $H \times H \to H$ and inversion $H \to H$ are smooth maps. Because multiplication is a smooth map from $G \times G$ into G, its restriction is clearly smooth from $H \times H$ into G (this is true even if H is merely immersed). Because H is a subgroup, multiplication takes $H \times H$ into H, and since H is embedded, this is a smooth map into H by Corollary **??**. A similar argument applies to inversion. This proves that H is a Lie subgroup. □

The simplest examples of embedded Lie subgroups are the open subgroups. The following lemma shows that the possibilities for open subgroups are limited.

Lemma 2.1.3.2. *Suppose G is a Lie group and $H \subseteq G$ is an open subgroup. Then H is an embedded Lie subgroup. In addition, H is closed, so it is a union of connected components of G.*

Proof. If H is open in G, then it is embedded by Proposition **??**. By Proposition **??** H is both open and closed, so it is a union of components. □

If G is a Lie group, the connected component of G containing the identity is called the **identity component** of G and denoted by G_0.

Proposition 2.1.3.3. *Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

Proof. Since G_0 is connected, it gengerates a connected open subgroup H of G. But G_0 is a connected component contained in H, so it follows that $H = G_0$.

Let $g \in G$ be an arbitrary element, and consider the subgroup $gG_0g^{-1} = L_g \circ R_{g^{-1}}(G_0)$. Since L_g and $R_{g^{-1}}$ are both diffeomorphisms, it follows that gG_0g^{-1} is an open connected subgroup containing the identity. Since G_0 is a connected component, it follows that $gG_0g^{-1} \subseteq G_0$, thus they must equal. This shows G_0 is normal.

Let H be a connected open subgroup. Then H contains the identity, hence is contained in G_0. Moreover, by Lemma 2.1.3.2 H is also closed, therefore by the connectness of G_0 we get $H = G_0$, which means G_0 is the only open connected subgroup.

Now let C be a connected component of G, and choose a element $g \in C$. Then $L_{g^{-1}}(C)$is a connected open subset containing the identity, thus is contained in G_0. Since the same argument gives $L_g(G_0) \subseteq C$, we conclude $L_g(G_0) = C$. □

Now we move beyond the open subgroups to more general Lie subgroups. The following proposition shows how to produce many more examples of embedded Lie subgroups.

Proposition 2.1.3.4. *Let $F : G \to H$ be a Lie group homomorphism. The kernel of F is a properly embedded Lie subgroup of G, whose codimension is equal to the rank of F.*

Proof. Because F has constant rank, its kernel $F^{-1}(e)$ is a properly embedded submanifold of codimension equal to rank$F$. It is thus a Lie subgroup by Proposition 2.1.3.1. □

Complementary to the preceding result about kernels is the following result about images.

Proposition 2.1.3.5. *If $F : G \to H$ is an injective Lie group homomorphism, the image of F has a unique smooth manifold structure such that $F(G)$ is a Lie subgroup of H and $F : G \to F(G)$ is a Lie group isomorphism.*

Proof. Since a Lie group homomorphism has constant rank, it follows from the global rank theorem that F is a smooth immersion. Proposition **??** shows that $F(G)$ has a unique smooth manifold structure such that it is an immersed submanifold of H and F is a diffeomorphism onto its image. It is a Lie group (because G is), and it is a subgroup for algebraic reasons, so it is a Lie subgroup. Because $F : G \to F(G)$ is a group isomorphism and a diffeomorphism, it is a Lie group isomorphism. □

Example 2.1.3.6 (Embedded Lie Subgroups).

(a) The subgroup $\mathrm{GL}_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is an open subgroup and thus an embedded Lie subgroup. It is indeed the identity component of $\mathrm{GL}_n(\mathbb{R})$.

(b) The circle S^1 is an embedded Lie subgroup of $\mathbb{C}^*$ because it is a subgroup and an embedded submanifold.

(c) The set $\mathrm{SL}_n(\mathbb{R})$ of $n \times n$ real matrices with determinant equal to 1 is called the **special linear group** of degree n. Because $\mathrm{SL}_n(\mathbb{R})$ is the kernel of the Lie group homomorphism $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$, it is a properly embedded Lie subgroup. Because the determinant function is surjective, it is a smooth submersion by the global rank theorem, so $\mathrm{SL}_n(\mathbb{R})$ has dimension $n^2 - 1$.

(d) Let n be a positive integer, and define a map $\beta : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{R})$ by replacing each complex matrix entry $a + ib$ with the 2×2 block $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$:

$$\beta \begin{pmatrix} a_1^1 + ib_1^1 & \cdots & a_n^1 + ib_n^1 \\ \vdots & & \vdots \\ a_1^n + ib_1^n & \cdots & a_n^n + ib_n^n \end{pmatrix} = \begin{pmatrix} a_1^1 & -b_1^1 & \cdots & a_n^1 & -b_n^1 \\ b_1^1 & a_1^1 & & b_n^1 & a_n^1 \\ \vdots & & & \vdots & \\ a_1^n & -b_1^n & \cdots & a_n^n & -b_n^n \\ b_1^n & a_1^n & & b_n^n & a_n^n \end{pmatrix}$$

It is straightforward to verify that β is an injective Lie group homomorphism whose image is a properly embedded Lie subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$. Thus, $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to this Lie subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$.

(e) The subgroup $\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ consisting of complex matrices of determinant 1 is called the **complex special linear group** of degree n. It is the kernel of the Lie group homomorphism $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^*$. This homomorphism is surjective, so it is a smooth submersion by the global rank theorem. Therefore, $\mathrm{SL}_n(\mathbb{C}) = \ker \det$ is a properly embedded Lie subgroup whose dimension is $2n^2 - 2$.

Finally, here is an example of a Lie subgroup that is not embedded.

Example 2.1.3.7 (A Dense Lie Subgroup of the Torus). Let $H \subseteq T^2$ be the dense submanifold of the torus that is the image of the immersion $\gamma : \mathbb{R} \to T^2$ defined in Example **??**. It is easy to check that β is an injective Lie group homomorphism, and thus H is an immersed Lie subgroup of T^2 by Proposition 2.1.3.5.

In general, smooth submanifolds can be closed without being embedded (as is, for example, the figure-eight curve) or embedded without being closed (as is the open unit ball in $\mathbb{R}^n$). However, as the next theorem shows, Lie subgroups have the remarkable property that closedness and embeddedness are not independent. This means that every embedded Lie subgroup is properly embedded.

Theorem 2.1.3.8. *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then H is closed in G if and only if it is embedded.*

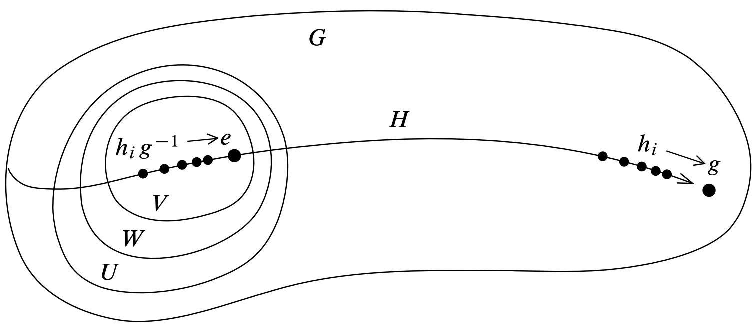


Figure 2.1: An embedded Lie subgroup is closed.

Proof. Assume first that H is embedded in G. To prove that H is closed, let g be an arbitrary point of $\overline{H}$. Then there is a sequence (h_i) of points in H converging to g. Let U be the domain of a slice chart for H containing the identity, and let W be a smaller neighborhood of e such that $\overline{W} \subseteq U$. By Lemma **??**, there is a neighborhood V of e with the property that $g_1 g_2^{-1} \in W$ whenever $g_1, g_2 \in V$.

Because $h_i g^{-1} \to e$, by discarding finitely many terms of the sequence we may assume that $h_i g^{-1} \in V$ for all i. This implies that

$$h_i h_j^{-1} = (h_i g^{-1})(h_j g^{-1})^{-1} \in W$$

for all i and j. Fixing j and letting $i \to \infty$, we find that $h_i h_j^{-1} \to g h_j^{-1} \in \overline{W} \subseteq U$. Since $H \cap U$ is a slice, it is closed in U, and therefore $g h_j^{-1} \in H$, which implies $g \in H$. Thus H is closed.

Conversely, assume H is a closed Lie subgroup, and let $m = \dim H$ and $n = \dim G$. We need to show that H is an embedded submanifold of G. If $m = n$, then H is embedded by Proposition **??**, so we may assume henceforth that $m < n$.

It suffices to show that for some $h_1 \in H$, there is a neighborhood U_1 of h_1 in G such that $H \cap U_1$ is an embedded submanifold of U_1; for then if h is any other point of H, right translation $R_{h_1^{-1}h}$ is a diffeomorphism of G that takes H to H, and takes U_1 to a neighborhood U_1' of h such that $H \cap U_1'$ is embedded in U_1', so it follows from the local slice criterion that H is an embedded submanifold of G.

Because every immersed submanifold is locally embedded (Proposition **??**), there exist a neighborhood V of e in H and a slice chart (U, φ) for V in G centered at e. By shrinking U if necessary, we may assume that it is a coordinate cube, and $U \cap V$ is the set of points whose coordinates are of the form $\{x^{m+1} = \cdots = x^n = 0\}$. Let $S \subseteq U$ be the set of points with coordinates of the form $\{x^1 = \cdots = x^m = 0\}$; it is the slice perpendicular to $U \cap V$ in these coordinates. Then S is an embedded submanifold of U and hence of G. Note that in these coordinates, T_eV is spanned by the first m coordinate vectors and T_eS by the last $n - m$, so $T_eG = T_eV \oplus T_eS$.

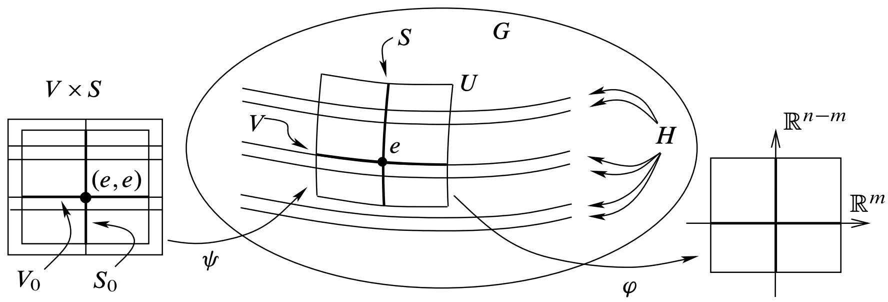


Figure 2.2: Finding a slice chart.

Now consider the map $\psi : V \times S \to G$ obtained by restricting group multiplication: $\psi(v, s) = vs$. Since $\psi(v, e) = v$ for $v \in V$ and $\psi(e, s) = s$ for $s \in S$, it follows easily that the differential of ψ at (e, e) satisfies $d\psi(X, 0) = X$ and $d\psi(0, Y) = Y$ for $X \in T_eV$ and $Y \in T_eS$, and therefore $d\psi_{(e,e)}$ is bijective. By the inverse function theorem, there are connected neighborhoods W_0 of (e, e) in $V \times S$ and U_0 of e in G such that $\psi : W_0 \to U_0$ is a diffeomorphism. Shrinking the neighborhoods if necessary, we may assume that $W_0 = V_0 \times S_0$, where V_0 and S_0 are neighborhoods of e in V and S, respectively.

Let $K = S_0 \cap H$. There are two things we need to show about the set K:

(a) $\psi(V_0 \times K) = H \cap U_0$.

(b) K is a discrete set in the topology of H.

To prove (a), let $(v, s) \in V_0 \times S_0$ be arbitrary. Since H is a subgroup and $V_0 \subseteq H$, it follows that $vs \in H$ if and only if $s \in H$, which is to say that $\psi(v, s) \in H \cap U_0$ if and only if $(v, s) \in V_0 \times K$. To prove (b), suppose $h \in K$. Right translation $R_h : H \to H$ is a diffeomorphism of H taking e to h and taking V_0 to a neighborhood V_h of h in H. Note that $V_h = R_h(V_0) = \psi(V_0 \times \{h\})$, while $K = \psi(\{e\} \times K)$. Since ψ is injective on $V_0 \times S_0$, it follows that

$$V_h \cap K = \psi(\{e\} \times \{h\}) = \{h\}$$

Thus each point $h \in K$ is isolated in H, which implies that K is discrete.

Since K is a discrete subset of the manifold H, it is countable, and since H is closed in G, it follows that $K = S_0 \cap H$ is closed in S_0. Thus, by Corollary **??**, there is a point $h_1 \in K$ that is isolated in S_0. (This step fails if H is not closed-for example, if H were a dense subgroup of the torus, then K would be dense in S_0.) This means there is a neighborhood S_1 of h_1 in S_0 such that $S_1 \cap H = \{h_1\}$. Then $U_1 = \psi(V_0 \times S_1)$ is a neighborhood of h_1 in G with the property that $U_1 \cap H$ is the slice $V_0 \times \{h_1\}$ in U_1. As explained at the beginning of the proof, the existence of such a neighborhood for one point of H implies that H is embedded. □

2.1.4 Group actions and equivariant maps

If M is a topological space and G is a topological group, an action of G on M is said to be a **continuous action** if the defining map $G \times M \to M$ or $M \times G \to M$ is continuous. In this case, M is said to be a (left or right) ***G*-space**. If in addition M is a smooth manifold with or without boundary, G is a Lie group, and the defining map is smooth, then the action is said to be a **smooth action**. We are primarily interested in smooth actions of Lie groups on smooth manifolds.

Lie group actions typically arise in situations involving some kind of *symmetry*. For example, if M is a vector space or smooth manifold endowed with a metric or other geometric structure, the set of diffeomorphisms of M that preserve the structure (called the **symmetry group of the structure**) frequently turns out to be a Lie group acting smoothly on M.

In the following, suppose $\theta : G \times M \to M$ is a left action of a group G on a set M.

- For each $p \in M$, the **orbit** of p, denoted by $G \cdot p$, is the set of all images of p under the action by elements of G:
$$G \cdot p = \{g \cdot p : g \in G\}$$
- For each $p \in M$, the **isotropy group** or **stabilizer** of p, denoted by G_p, is the set of elements of G that fix p:
$$G_p = \{g \in G : g \cdot p = p\}$$
The definition of a group action guarantees that G_p is a subgroup of G.
- The action is said to be **transitive** if for every pair of points $p, q \in M$, there exists $g \in G$ such that $g \cdot p = q$, or equivalently if the only orbit is all of M.
- The action is said to be **free** if the only element of G that fixes any element of M is the identity: $g \cdot p = p$ for some $p \in M$ implies $g = e$, or equivalently if every isotropy group is trivial.

Example 2.1.4.1 (Lie Group Actions).

(a) If G is any Lie group and M is any smooth manifold, the trivial action of G on M is defined by $g \cdot p = p$ for all $g \in G$ and $p \in M$. It is a smooth action, for which each orbit is a single point and each isotropy group is all of G.

(b) The natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ is the left action given by matrix multiplication: $(A, x) \mapsto Ax$, considering $x \in \mathbb{R}^n$ as a column matrix. This is an action because $I_n x = x$ and matrix multiplication is associative: $ABx = A(Bx)$. It is smooth because the components of Ax depend polynomially on the matrix entries of A and the components of x. Because any nonzero vector can be taken to any other by some invertible linear transformation, there are exactly two orbits: $\{0\}$ and $\mathbb{R}^n - \{0\}$.

(c) Every Lie group G acts smoothly on itself by left translation. Given any two points $g_1, g_2 \in G$, there is a unique left translation of G taking g_1 to g_2, namely left translation by $g_2 g_1^{-1}$; thus the action is both free and transitive. More generally, if H is a Lie subgroup of G, then the restriction of the multiplication map to $H \times G \to G$ defines a smooth and free (but generally not transitive) left action of H on G. Similar observations apply to right translations.

(d) Every Lie group acts smoothly on itself by conjugation: $g \cdot h = ghg^{-1}$.

(e) An action of a discrete group Γ on a manifold M is smooth if and only if for each $g \in \Gamma$, the map $p \mapsto g \cdot p$ is a smooth map from M to itself. Thus, for example, $\mathbb{Z}^n$ acts smoothly and freely on $\mathbb{R}^n$ by left translation:

Another important class of Lie group actions arises from covering maps. Suppose E and M are topological spaces, and $\pi : E \to M$ is a (topological) covering map. An **automorphism of π** (also called a **deck transformation** or **covering transformation**) is a homeomorphism $\varphi : E \to E$ such that

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

The set $\mathrm{Aut}_\pi(E)$ of all automorphisms of π, called the **automorphism group** of π, is a group under composition, acting on E on the left. It can be shown that $\mathrm{Aut}_\pi(E)$ acts transitively on each fiber of π if and only if π is a **normal covering map**, which means that $\pi_*(\pi_1(E, q))$ is a normal subgroup of $\pi_1(M, \pi(q))$ for every $q \in E$.

Proposition 2.1.4.2. *Suppose E and M are smooth manifolds with or without boundary, and $\pi : E \to M$ is a smooth covering map. With the discrete topology, the automorphism group $\mathrm{Aut}_\pi(E)$ is a zero-dimensional Lie group acting smoothly and freely on E.*

Proof. Suppose $\varphi \in \mathrm{Aut}_\pi(E)$ is an automorphism that fixes a point $p \in E$. We can consider φ as a lift of π:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ E \arrow[ur, "\varphi"] \arrow[r, "\pi"'] & M \end{tikzcd}$$

Since the identity map of E is another such lift that agrees with φ at p, the unique lifting property of covering maps guarantees that $\varphi = \mathrm{id}_E$. Thus, the action of $\mathrm{Aut}_\pi(E)$ is free.

To show that $\mathrm{Aut}_\pi(E)$ is a Lie group, we need only verify that it is countable. Let $q \in E$ be arbitrary, let $p = \pi(q) \in M$ and let $U \subseteq M$ be an evenly covered neighborhood of p. Because E is second-countable, $\pi^{-1}(U)$ has countably many components, and because each component contains exactly one point of $\pi^{-1}(p)$, it follows that $\pi^{-1}(p)$ is countable. Let $\theta^{(q)} : \mathrm{Aut}_\pi(E) \to E$ be the map $\theta^{(q)}(\varphi) = \varphi(q)$. Then $\theta^{(q)}$ maps $\mathrm{Aut}_\pi(E)$ into $\pi^{-1}(p)$, and the fact that the action is free implies that it is injective; thus $\mathrm{Aut}_\pi(E)$ is countable.

Smoothness of the action follows from Theorem **??**. □

2.1.4.1 Equivariant Maps

Suppose G is a Lie group, and M and N are both smooth manifolds endowed with smooth left G-actions θ and φ, respectively. A map $F : M \to N$ is said to be **equivariant** with respect to the given G-actions if the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g"'] & N \arrow[d, "\varphi_g"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

Example 2.1.4.3. Let $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ be any fixed nonzero vector. Define smooth left actions of $\mathbb{R}$ on $\mathbb{R}^n$ and T^n by

$$t \cdot (x^1, \ldots, x^n) = (x^1 + tv^1, \ldots, x^n + tv^n), \quad t \cdot (z^1, \ldots, z^n) = (e^{2\pi i t v^1} z^1, \ldots, e^{2\pi i t v^n} z^n)$$

for $t \in \mathbb{R}$, $(x^1, \ldots, x^n) \in \mathbb{R}^n$ and $(z^1, \ldots, z^n) \in T^n$. The smooth map $\varepsilon^n : \mathbb{R}^n \to T^n$ given by

$$\varepsilon^n(x^1, \ldots, x^n) = (e^{2\pi i x^1}, \ldots, e^{2\pi i x^n})$$

is equivariant with respect to these actions.

The following generalization of Theorem 2.1.1.4 is an extremely useful tool for proving that certain maps have constant rank.

Theorem 2.1.4.4 (Equivariant Rank Theorem). *Let M and N be smooth manifolds and let G be a Lie group. Suppose $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G-action on M and any smooth G-action on N. Then F has constant rank.*

Therefore, if F is surjective, it is a smooth submersion; if it is injective, it is a smooth immersion; and if it is bijective, it is a diffeomorphism.

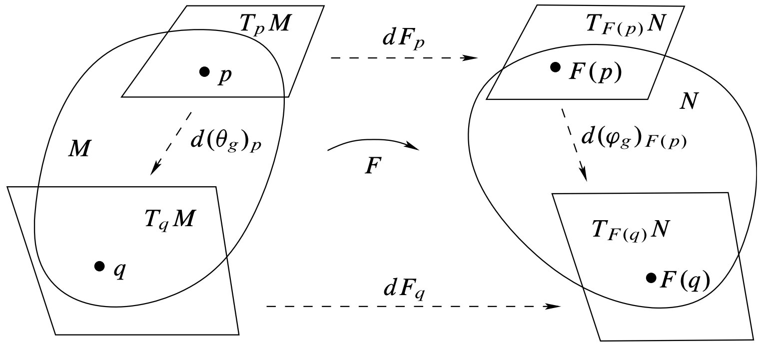


Figure 2.1: The equivariant rank theorem.

Proof. Let θ and φ denote the G-actions on M and N, respectively, and let p and q be arbitrary points in M. Choose $g \in G$ such that $\theta_g(p) = q$. (Such a g exists because we are assuming that G acts transitively on M.) Because $\varphi_g \circ F = \theta_g \circ F$, the following diagram commutes

$$\begin{tikzcd} T_pM \arrow[r, "dF_p"] \arrow[d, "d(\theta_g)_p"'] & T_pN \arrow[d, "d(\varphi_g)_{F(p)}"] \\ T_pM \arrow[r, "dF_q"] & T_qN \end{tikzcd}$$

Because the vertical linear maps in this diagram are isomorphisms, the horizontal ones have the same rank. In other words, the rank of F is the same at any two arbitrary points $p, q \in M$, so F has constant rank. The final statement follows from the global rank theorem. □

Here is an important application of the equivariant rank theorem. Suppose G is a Lie group, M is a smooth manifold, and $\theta : G \times M \to M$ is a smooth left action. For each $p \in M$, define a map $\theta^{(p)} : G \to M$ by

$$\theta^{(p)}(g) = g \cdot p$$

This is often called the orbit map, because its image is the orbit $G \cdot p$. In addition, the preimage $(\theta^{(p)})^{-1}(p)$ is the isotropy group G_p.

Proposition 2.1.4.5 (Properties of the Orbit Map). *Suppose θ is a smooth left action of a Lie group G on a smooth manifold M. For each $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank, so the isotropy group $G_p = (\theta^{(p)})^{-1}(p)$ is a properly embedded Lie subgroup of G. The induced map $F_p : G/G_p \to M$ on G/G_p is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M.*

Proof. The orbit map is smooth because it is equal to the composition

$$G \approx G \times \{p\} \hookrightarrow G \times M \xrightarrow{\theta} M$$

It follows from the definition of a group action that $\theta^{(p)}$ is equivariant with respect to the action of G on itself by left translation and the given action on M:

$$\theta^{(p)}(g'g) = (gg') \cdot p = g' \cdot (g \cdot p) = g' \cdot \theta^{(p)}(g)$$

Since G acts transitively on itself, the equivariant rank theorem shows that $\theta^{(p)}$ has constant rank. Thus, G_p is a properly embedded submanifold by Theorem **??**, and a Lie subgroup by Proposition 2.1.3.1.

Now by quotient manifold theorem, the group G/G_p has a unique smooth structure such that $\pi : G \to G_p$ is a smooth submersion. Therefore the induced map $F_p : G/G_p \to M$ defined by $F_p(gG_p) = \theta^{(p)}(g)$ is an injective smooth equivalent map with image $G \cdot p$. By the equivariant rank theorem, F_p is then a smooth immersion, and thus the orbit (endowed with a suitable topology and smooth structure) is an immersed submanifold by Proposition **??**. □

Example 2.1.4.6 (The Orthogonal Group). A real $n \times n$ matrix A is said to be orthogonal if as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ it preserves the Euclidean dot product:

$$(Ax) \cdot (Ay) = x \cdot y \quad \text{for} \quad x, y \in \mathbb{R}^n$$

The set $\mathrm{O}(n)$ of all orthogonal $n \times n$ matrices is a subgroup of $\mathrm{GL}_n(\mathbb{R})$, called the **orthogonal group** of degree n. It is easy to check that a matrix A is orthogonal if and only if it takes the standard basis of $\mathbb{R}^n$ to an orthonormal basis, which is equivalent to the columns of A being orthonormal. Since the (i, j)-entry of the matrix $A^T A$ is the dot product of the i-th and j-th columns of A, this condition is also equivalent to the requirement that $A^T A = I_n$.

Define a smooth map $\Phi : \mathrm{GL}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ by $\Phi(A) = A^T A$. Then $\mathrm{O}(n)$ is equal to the level set $\Phi^{-1}(I_n)$. To show that Φ has constant rank and therefore that $\mathrm{O}(n)$ is an embedded Lie subgroup, we show that Φ is equivariant with respect to suitable right actions of $\mathrm{GL}_n(\mathbb{R})$. Let $\mathrm{GL}_n(\mathbb{R})$ act on itself by right multiplication, and define a right action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathcal{M}_n(\mathbb{R})$ by

$$X \cdot B = B^T X B \quad \text{for} \quad X \in \mathrm{GL}_n(\mathbb{R}), B \in \mathcal{M}_n(\mathbb{R})$$

It is easy to check that this is a smooth action, and Φ is equivariant because

$$\Phi(AB) = (AB)^T AB = B^T A^T AB = B^T \Phi(A) B = \Phi(A) \cdot B$$

Thus, $\mathrm{O}(n)$ is a properly embedded Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. It is compact because it is closed and bounded in $\mathcal{M}_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$: closed because it is a level set of Φ, and bounded because every $A \in \mathrm{O}(n)$ has columns of norm 1, and therefore satisfies $|A| = \sqrt{n}$.

To determine the dimension of $\mathrm{O}(n)$, we need to compute the rank of Φ. Because the rank is constant, it suffices to compute it at the identity $I_n \in \mathrm{GL}_n(\mathbb{R})$. Thus for any $B \in T_{I_n}\mathrm{GL}_n(\mathbb{R}) = \mathcal{M}_n(\mathbb{R})$, let $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{GL}_n(\mathbb{R})$ be the curve $\gamma(t) = I_n + tB$, and compute

$$d\Phi_{I_n} B = \frac{d}{dt}\Big|_{t=0} \Phi \circ \gamma(B) = \frac{d}{dt}\Big|_{t=0} (I_n + tB)^T (I_n + B) = B^T + B$$

From this formula, it is evident that the image of $d\Phi_{I_n}$ is contained in the vector space of symmetric matrices. Conversely, if $B \in \mathcal{M}_n(\mathbb{R})$ is an arbitrary symmetric $n \times n$ matrix, then $d\Phi_{I_n}(1/2B) = B$. It follows that the image of $d\Phi_{I_n}$ is exactly the space of symmetric matrices. This is a linear subspace of $\mathcal{M}_n(\mathbb{R})$ of dimension $n(n+1)/2$, because each symmetric matrix is uniquely determined by its values on and above the main diagonal. It follows that $\mathrm{O}(n)$ is an embedded Lie subgroup of dimension

$$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

Example 2.1.4.7 (The Special Orthogonal Group). The **special orthogonal group** of degree n is defined as $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R})$. Because every matrix $A \in \mathrm{O}(n)$ satisfies

$$1 = \det I_n = \det(A^T A) = (\det A^T)(\det A) = (\det A)^2$$

it follows that $\det A = \pm 1$ for all $A \in \mathrm{O}(n)$. Therefore, $\mathrm{SO}(n)$ is the open subgroup of $\mathrm{O}(n)$ consisting of matrices of positive determinant, and is therefore also an embedded Lie subgroup of dimension $n(n-1)/2$ in $\mathrm{GL}_n(\mathbb{R})$. It is a compact group because it is a closed subset of $\mathrm{O}(n)$.

Example 2.1.4.8 (The Unitary Group). For any complex matrix A, the adjoint of A is the matrix A^* formed by conjugating the entries of A and taking the transpose: $A = \overline{A^T}$. For any positive integer n, the **unitary group** of degree n is the subgroup $\mathrm{U}(n) \subseteq \mathrm{GL}_n(\mathbb{C})$ consisting of complex $n \times n$ matrices whose columns form an orthonormal basis for $\mathbb{C}^n$ with respect to the Hermitian dot product $z \cdot w = \sum_i z^i \overline{w^i}$. It is straightforward to check that $\mathrm{U}(n)$ consists of those matrices A such that $A^* A = I_n$.

Similar to Example 2.1.4.6, we define a map $\Psi : \mathrm{GL}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ by $\Psi(A) = A^*A$. Then $\mathrm{U}(n) = \Psi^{-1}(I_n)$. We check that

$$\Psi(AB) = B^*A^*AB = B^*\Psi(A)B$$

so if we define an action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{M}_n(\mathbb{C})$ by

$$X \cdot B = B^*XB \quad \text{for} \quad X \in \mathrm{GL}_n(\mathbb{R}), B \in \mathcal{M}_n(\mathbb{C})$$

Then Ψ is an equivariant map, thus has constant rank. Now, to compute $d\Psi_{I_n}$, we define a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{GL}_n(\mathbb{C})$ by $\gamma(t) = I_n + tB$ where $B \in T_{I_n}\mathrm{GL}_n(\mathbb{C}) = \mathcal{M}_n(\mathbb{C})$. We observe that

$$d\Psi_{I_n}B = \frac{d}{dt}\bigg|_{t=0} \Psi \circ \gamma(B) = \frac{d}{dt}\bigg|_{t=0} (I_n + tB)^*(I_n + B) = B^* + B$$

Hence the image of $d\Psi_{I_n}$ is the **Hermitan matrix**, that is, matrix B such that $B^* = B$. The dimension of the set of all Hermitan matrix is

$$n + 2 \cdot \frac{n(n-1)}{2} = n^2$$

Thus $\mathrm{U}(n)$ is an properly embedded Lie subgroup of dimension $2n^2 - n^2 = n^2$.

Example 2.1.4.9 (The Special Unitary Group). The group $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$ is called the complex special unitary group of degree n. For $A \in \mathrm{U}(n)$ we have

$$1 = \det I_n = \det(A^*A) = (\det A^*)(\det A) = |\det A|^2$$

Thus there is an induced map $\det : \mathrm{U}(n) \to S^1$. Since $\dim S^1 = 1$ and $\mathrm{SU}(n) = \det^{-1}(1)$, it follows that $\mathrm{SU}(n)$ is a properly embedded (n^2-1)-dimensional Lie subgroup of $\mathrm{U}(n)$. Since the composition of smooth embeddings $\mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is again a smooth embedding, this implies that $\mathrm{SU}(n)$ is also embedded in $\mathrm{GL}_n(\mathbb{C})$.

2.1.5 Semidirect products

Suppose N and H are Lie groups, and $\theta : H \times N \to N$ is a smooth left action of H on N. We define a new Lie group $N \rtimes_\theta H$, called a **semidirect product** of H and N, as follows. As a smooth manifold, $N \rtimes_\theta H$ is just the Cartesian product $N \times H$ but the group multiplication is defined by

$$(n_1, h_1)(n_2, h_2) = (n_1\theta_{h_1}(n_2), h_1h_2)$$

Example 2.1.5.1 (The Euclidean Group). If we consider $\mathbb{R}^n$ as a Lie group under addition, then the natural action of $\mathrm{O}(n)$ on $\mathbb{R}^n$ is an action by automorphisms. The resulting semidirect product $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ is called the **Euclidean group**; its multiplication is given by $(b, A)(b', A') = (b + Ab', AA')$. It acts on $\mathbb{R}^n$ via

$$(b, A) \cdot x = b + Ax$$

It can be checked that

$$\big((b, A)(b', A')\big) \cdot x = (b + Ab', AA')x = b + Ab' + AA'x = (b, A) \cdot (b' + A'x) = (b, A) \cdot ((b', A') \cdot x)$$

so this action is well defined. This action preserves lines, distances, and angle measures, and thus all of the relationships of Euclidean geometry.

A faithful representation of $\mathrm{E}(n)$ is given by the map $\rho : \mathrm{E}(n) \to \mathrm{GL}_{n+1}(\mathbb{R})$ defined in block form by

$$\rho(b, A) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

The following result gives a extremely useful method to characterize semidirect products. We will use it to give some examples of Lie groups.

Proposition 2.1.5.2. *Suppose G, N, and H are Lie groups. Prove that G is isomorphic to a semidirect product $N \rtimes H$ if and only if there are Lie group homomorphisms $\varphi : G \to H$ and $\psi : H \to G$ such that $\varphi \circ \psi = \mathrm{id}_H$ and $\ker \varphi \cong N$.*

Proof. If $G \approx N \rtimes H$, then the projection on H and the embedding ot H in G satisfy the conditions.
Conversely, assume the existence of φ and ψ:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, hook] & G \arrow[r, "\varphi"] & H \arrow[l, bend left, "\psi"] \arrow[r] & 1 \end{tikzcd}$$

Since this sequence is split exact, we can regard N, H as subgroups of G such that $N \cap H = \{e\}$ and $NH = G$. Since N is normal in G, H act on N by conjugation, so the map $(n, h) \mapsto nh$ is an isomorphsim from G to $N \rtimes H$. □

Proposition 2.1.5.3. *The following Lie groups are isomorphic to semidirect products as shown.*

(a) $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes \mathrm{O}(1)$.

(b) $\mathrm{U}(n) \cong \mathrm{SU}(n) \rtimes \mathrm{U}(1)$.

(c) $\mathrm{GL}_n(\mathbb{R}) \cong \mathrm{SL}_n(\mathbb{R}) \rtimes \mathbb{R}^*$.

(d) $\mathrm{GL}_n(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{C}) \rtimes \mathbb{C}^*$.

Proof. In order to apply Proposition 2.1.5.2, we consider the maps

$$\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*, \quad \psi : \mathbb{R}^* \to \mathrm{GL}_n(\mathbb{R}), \quad x \mapsto xI_n$$

and

$$\det : \mathrm{O}(n) \to \{\pm 1\}, \quad \psi : \{\pm 1\} \to \mathrm{O}(n), \quad x \mapsto xI_n.$$

we have $\ker \det = \det^{-1}(1) = \mathrm{SL}_n(\mathbb{R})$, or $\ker \det = \mathrm{SO}(n)$. The claim therefore follows. □

There is a close connection between representations and group actions. Let G be a Lie group and V be a finite-dimensional vector space. An action of G on V is said to be a linear action if for each $g \in G$, the map from V to itself given by $x \mapsto g \cdot x$ is linear. For example, if $\rho : G \to \mathrm{GL}(V)$ is a representation of G, there is an associated smooth linear action of G on V given by $g \cdot x := \rho(g)x$. The next proposition shows that every linear action is of this type.

Proposition 2.1.5.4. *Let G be a Lie group and V be a finite-dimensional vector space. A smooth left action of G on V is linear if and only if it is of the form $g \cdot x = \rho(g)x$ for some representation ρ of G.*

Proof. Every action induced by a representation is evidently linear. To prove the converse, assume that we are given a linear action of G on V. The hypothesis implies that for each $g \in G$ there is a linear map $\rho(g) \in \mathrm{GL}(V)$ such that $g \cdot x = \rho(g)x$ for all $x \in V$. The group action condition implies $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, so $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. Thus, to show that it is a Lie group representation, we need only show that it is smooth. Choose a basis $\{E_i\}$ for V, and for each i let $\pi^i : V \to \mathbb{R}$ be the projection onto the i th coordinate with respect to this basis. If we let $(\rho^i_j(g))$ denote the matrix entries of $\rho(g)$ with respect to this basis, it follows that $\rho^i_j(g) = \pi^i(g \cdot E_j)$, so each function ρ^i_j is a composition of smooth functions. Because the matrix entries form global smooth coordinates for $\mathrm{GL}(V)$, this implies that ρ is smooth. □

2.1.6 Exercise

Exercise 2.1.1. Let G be a Lie group.

(a) Let m be the multiplication, show that the differential $dm_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$dm_{(e,e)}(X, Y) = X + Y \quad \text{for} \quad X, Y \in T_eG$$

(b) Let i be the inverse map, show that the differential $di_e : T_eG \to T_eG$ is given by

$$di_e(X) = -X \quad \text{for} \quad X \in T_eG$$

Proof. For part (a), fix $g, h \in G$. Then since $T_{(g,h)}(G \times G) \cong T_g G \oplus T_h G$, we have

$$dm_{(g,h)}(X, Y) = d(m \circ j^h)_g(X) + d(m \circ \iota_g)_h Y$$

for $X \in T_g G, Y \in T_h G$, where $j^h(x) = (x, h)$ and $\iota_g(x) = (g, x)$. Thus we have

$$dm_{(g,h)}(X, Y) = d(R_h)_g X + d(L_g)_h Y$$

For part (b), let $X \in T_e G$ and define a map

$$\alpha : G \to G \times G, \quad g \mapsto (g, i(g))$$

Then we have

$$d\alpha_e(X) = (X, di_e X)$$

Since $m \circ \alpha = e$, we conclude by the chain rule

$$dm_{e,e} \circ d\alpha_e(X) = dm_{(e,e)}(X, di_e(X)) = X + di_e(X) = 0$$

This implies $di_e(X) = -X$.

Now note that for any $g \in G$ we have $i = R_{g^{-1}} \circ i \circ L_{g^{-1}}$ since

$$R_{g^{-1}} \circ i \circ L_{g^{-1}}(x) = R_{g^{-1}} \circ i(g^{-1}x) = x^{-1}gg^{-1} = x^{-1}$$

So using this and the chain rule we have

$$di_g(X) = d(R_{g^{-1}})_e \circ di_e \circ d(L_{g^{-1}})_g(X) = -d(R_{g^{-1}})_e \circ d(L_{g^{-1}})_g(X) \quad \text{for} \quad X \in T_g G$$

which is the desired claim. □

Exercise 2.1.2. Let $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ denote the determinant function. For any $A \in \mathcal{M}_n(\mathbb{R})$, show that

$$\left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) = \mathrm{tr} A.$$

From this, conclude that, for $X \in \mathrm{GL}_n(\mathbb{R})$ and $B \in T_X\mathrm{GL}_n(\mathbb{R}) \approx \mathcal{M}_n(\mathbb{R})$,

$$d(\det)_X(B) = (\det X)\mathrm{tr}(X^{-1}B).$$

Proof. Let $B(t) = (b_{ij}(t)) : I \to \mathcal{M}_n(\mathbb{R})$ be a smooth curve such that $B(0) = I_n$. Then expanding the determinant along the first column, we have

$$\det(B(t)) = \sum_{i=1}^{n} (-1)^{i+1} b_{1i}(t) \cdot \det(B_{1i}(t)),$$

where B_{ij} is the minor of B at position (i, j). Now differentiate both sides, we get

$$\left.\frac{d}{dt}\right|_{t=0} \det(B(t)) = \sum_{i=1}^{n} (-1)^{i+1} \left[b'_{1i}(0) \cdot \det(B_{1i}(0)) + b_{1i}(0) \cdot \left.\frac{d}{dt}\right|_{t=0} \det(B_{11}(t)) \right].$$

By $B(0) = I_n$, the RHS is simplified into

$$\left.\frac{d}{dt}\right|_{t=0} \det(B(t)) = b'_{11}(0) + \left.\frac{d}{dt}\right|_{t=0} \det(B_{1i}(t)).$$

Now by a simple induction we get

$$\left.\frac{d}{dt}\right|_{t=0} \det(B(t)) = b'_{11}(0) + \cdots + b'_{nn}(0) = \mathrm{tr}(B'(0)).$$

Now we define the curve $\gamma(t) = X + tB$ for $B \in T_X\mathrm{GL}_n(\mathbb{R})$, and observe that

$$\det(X + tB) = \det(X) \det(I_n + tX^{-1}B)$$

Thus by Corollary **??**,

$$d(\det)_X(B) = \left.\frac{d}{dt}\right|_{t=0} (\det \circ \gamma) = \det(X) \left.\frac{d}{dt}\right|_{t=0} \det(I_n + tX^{-1}B) = \det(X)\mathrm{tr}(X^{-1}B)$$

which is the desired result. □

Exercise 2.1.3. Suppose a connected topological group G acts continuously on a discrete space K. Show that the action is trivial.

Proof. Let $\rho : G \times K \to K$ be the action. Note that the connected component of $G \times K$ is $G \times \{x\}$ for $x \in K$. For any $x_0 \in K$, the set $U := \rho^{-1}(x_0)$ is closed and open in $G \times K$, thus is of the form $G \times \{y\}$. Since $(e, x_0) \subseteq U$, we conclude $U = G \times \{x_0\}$. Thus ρ is trivial on x_0. Since x_0 is arbitrary, this implies ρ is trivial on K. □

Exercise 2.1.4. Considering S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, define an action of S^1 on S^{2n+1}, called the **Hopf action**, by

$$z(w^1, \dots, w^{n+1}) = (zw^1, \dots, zw^{n+1})$$

Show that this action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$ whose union is S^{2n+1}.

Proof. Note that this action is free on S^1, so by Proposition 2.1.4.5 each orbit is an immersed submanifold. Since S^1 is a compact set, it is in fact an embedded submanifold. So each orbit is a unite circle in $\mathbb{C}^{n+1}$. □

Exercise 2.1.5. Show that $\mathrm{SO}(2)$, $U(1)$, and S^1 are all isomorphic as Lie groups.

Proof. The subgroup $\mathrm{SO}(2)$ takes the following from

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

Thus $\mathrm{SO}(2) \approx S^1$. The subgroup $\mathrm{U}(1)$ consists of 1×1 matrix of modulus 1, which may be regarded as S^1. □

Exercise 2.1.6. Prove that $\mathrm{SU}(2)$ is diffeomorphic to S^3.

Proof. Let $U \in \mathrm{GL}_n(\mathbb{C})$,

$$U = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

From the conditions $\det U = 1$ and $U^* U = I_n$, we get

$$\begin{cases} z_1 z_4 - z_2 z_3 = 1 \\ |z_1|^2 + |z_2|^2 = 1 \\ |z_3|^2 + |z_4|^2 = 1 \\ z_1 \bar{z}_3 + z_2 \bar{z}_4 = 0 \end{cases}$$

Let's assume $z_1 = z\bar{z}_4$ where $z \in \mathbb{C}$, then $z_2 = -z\bar{z}_3$. Plugging this into the equations we get

$$|z|^2(|z_3|^2 + |z_4|^2) = 1, \quad z(|z_3|^2 + |z_4|^2) = 1$$

Thus $z = 1$, and U has the following form

$$\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

where $|z_1|^2 + |z_2|^2 = 1$. Since S^3 can be viewed as the set $\{|z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$, this implies $\mathrm{SU}(2) \approx S^3$. □

Exercise 2.1.7. Let $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ be the quaternions. The multiplication of $\mathbb{H}$ is given by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$$

By considering $\mathbb{H}$ as a two dimensional complex vector space, find a representation of $\mathbb{H}$ into $\mathrm{GL}_2(\mathbb{C})$.

Proof. Thus a representation $\rho : \mathbb{H} \to \mathrm{GL}_2(\mathbb{C})$ given by

$$\rho(z, w) = \begin{pmatrix} z & w \\ -\overline{w} & \bar{z} \end{pmatrix}.$$

Since the multiplication in $\mathbb{H}$ can be written as

$$(a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c})$$

when the base ring is commutative, this can be realized into marix multiplications.

Note that

$$\rho(\mathbf{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\mathbf{i}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(\mathbf{j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(\mathbf{k}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

Due to this, $\mathbb{H}$ has another definition: consider $\mathcal{M}_2(\mathbb{C})$, the set

$$\mathbb{H} \cong \{M \in \mathcal{M}_2(\mathbb{C}) : M^*M = aI_2, a \in \mathbb{R}, \det(M) \geq 0\}$$

Note that $\mathbb{C}$ can also be defined in the same way:

$$\mathbb{C} \cong \{M \in \mathcal{M}_2(\mathbb{R}) : M^*M = aI_2, a \in \mathbb{R}, \det(M) \geq 0\}$$

□

2.2 Lie algebras

2.2.1 The Lie algebra of a Lie group

One of the most important applications of Lie brackets occurs in the context of Lie groups. Suppose G is a Lie group. Recall that G acts smoothly and transitively on itself by left translation: $L_g(h) = gh$. A vector field X on G is said to be **left-invariant** if it is invariant under all left translations, in the sense that it is L_g-related to itself for every $g \in G$. More explicitly, this means

$$d(L_g)_{g'}(X_{g'}) = X_{gg'} \quad \text{for all } g, g' \in G.$$

Since L_g is a diffeomorphism, this can be abbreviated by writing $(L_g)_*X = X$ for every $g \in G$.

Because $(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$, the set of all smooth left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$. But it is much more than that. The central fact is that it is closed under Lie brackets.

Proposition 2.2.1.1. *Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G. Then $[X,Y]$ is also left-invariant.*

Proof. Let $g \in G$ be arbitrary. Since $(L_g)_*X = X$ and $(L_g)_*Y = Y$ by definition of left-invariance, it follows from Corollary 1.1.3.8 that

$$(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$$

Thus, $[X,Y]$ is L_g-related to itself for each g, which is to say it is left-invariant. □

A **Lie algebra** (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the bracket from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, usually denoted by $(X,Y) \mapsto [X,Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

- Bilinearity: For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X,Z] + b[Y,Z], \quad [X, aY + bZ] = a[X,Y] + b[X,Z].$$

- Antisymmetry:

$$[X,Y] = -[Y,X]$$

- Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Notice that the Jacobi identity is a substitute for associativity, which does not hold in general. If $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a Lie subalgebra of $\mathfrak{g}$ if it is closed under brackets. In this case $\mathfrak{h}$ is itself a Lie algebra with the restriction of the same bracket.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $A : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if it preserves brackets: $A[X,Y] = [AX, AY]$. An invertible Lie algebra homomorphism is called a **Lie algebra isomorphism**. If there exists a Lie algebra isomorphism from $\mathfrak{g}$ to $\mathfrak{h}$, we say that they are **isomorphic as Lie algebras**.

Example 2.2.1.2 (Lie Algebras).

(a) The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket by Proposition 1.1.3.6.

(b) If G is a Lie group, the set of all smooth left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$ and is therefore a Lie algebra.

(c) The vector space $\mathcal{M}_n(\mathbb{R})$ $n \times n$ real matrices becomes an n^2-dimensional Lie algebra under the **commutator bracket**:

$$[A, B] = AB - BA$$

Bilinearity and antisymmetry are obvious from the definition, and the Jacobi identity follows from a straightforward calculation. When we are regarding $\mathcal{M}_n(\mathbb{R})$ as a Lie algebra with this bracket, we denote it by $\mathfrak{gl}(n, \mathbb{R})$.

(d) Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$-dimensional (real) Lie algebra obtained by endowing $\mathcal{M}_n(\mathbb{C})$ with the commutator bracket.

(e) If V is a vector space, the vector space of all linear maps from V to itself becomes a Lie algebra, which we denote by $\mathfrak{gl}(V)$, with the commutator bracket:

$$[A, B] = A \circ B - B \circ A$$

Under our usual identification of $n \times n$ matrices with linear maps from $\mathbb{R}^n$ to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

(f) Any vector space V becomes a Lie algebra if we define all brackets to be zero. Such a Lie algebra is said to be **abelian**. (The name reflects the fact that brackets in most Lie algebras, as in the preceding examples, are defined as commutators in terms of underlying associative products, so all brackets are zero precisely when the underlying product is commutative; it also reflects the connection between abelian Lie algebras and abelian Lie groups.)

Example (b) is the most important one. The Lie algebra of all smooth left-invariant vector fields on a Lie group G is called the Lie algebra of G, and is denoted by $\mathfrak{Lie}(G)$. The fundamental fact is that $\mathfrak{Lie}(G)$ is finite-dimensional, and in fact has the same dimension as G itself, as the following theorem shows.

Theorem 2.2.1.3. *Let G be a Lie group. The evaluation map $\varepsilon : \mathfrak{Lie}(G) \to T_eG$, given by $\varepsilon(X) = X_e$, is a vector space isomorphism. Thus, $\mathfrak{Lie}(G)$ is finite-dimensional, with dimension equal to* $\dim G$.

Proof. It is clear from the definition that ε is linear over $\mathbb{R}$. It is easy to prove that it is injective: if $\varepsilon(X) = X_e = 0$ for some $X \in \mathfrak{Lie}(G)$, then left-invariance of X implies that $X_g = d(L_g)_e X_e = 0$ for every $g \in G$, so $X = 0$.

To show that ε is surjective, let $v \in T_eG$ be arbitrary, and define a (rough) vector field v^L on G by

$$v^L_g = d(L_g)_e(v) \tag{2.2.1.1}$$

If there is a left-invariant vector field on G whose value at the identity is v, clearly it has to be given by this formula.

First we need to check that v^L is smooth. By Proposition 1.1.1.13, it suffices to show that $v^L f$ is smooth whenever $f \in C^\infty(G)$. Choose a smooth curve $\gamma : (-\delta, \delta) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then for all $g \in G$,

$$(v^L f)(g) = v^L_g f = d(L_g)_e(v)(f) = \gamma'(0)(f \circ L_g) = \frac{d}{dt}\Big|_{t=0} (f \circ L_g \circ \gamma)(t)$$

If we define $\varphi : (-\delta, \delta) \times G \to G$ by $\varphi(t, g) = f \circ L_g \circ \gamma(t) = f(g\gamma(t))$, the computation above shows that $(v^L f)(g) = \partial\varphi/\partial t(0, g)$. Because φ is a composition of group multiplication, f and γ, it is smooth. It follows that $\partial\varphi/\partial t(0, g)$ depends smoothly on g, so $v^L f$ is smooth.

Next we show that v^L is left-invariant, which is to say that $d(L_h)_g(v^L|_g) = v^L|_{hg}$ for all $g, h \in G$. This follows from the definition of v^L and the fact that $L_h \circ L_g = L_{hg}$:

$$d(L_h)_g(v^L|_g) = d(L_h)_g \circ d(L_g)_e(v) = d(L_{hg})_e(v) = v^L|_{hg}$$

Thus $v^L \in \mathfrak{Lie}(G)$. Since L_e (left translation by the identity) is the identity map of G, it follows that $\varepsilon(v^L) = v$, so ε is surjective. $\square$

Given any vector $v \in T_eG$, we continue to use the notation v^L to denote the smooth left-invariant vector field defined by (2.2.1.1).

It is worth observing that the preceding proof also shows that the assumption of smoothness in the definition of $\mathfrak{Lie}(G)$ is unnecessary.

Corollary 2.2.1.4. *Every left-invariant rough vector field on a Lie group is smooth.*

Proof. Let X be a left-invariant rough vector field on a Lie group G, and let $v = X_e$. The fact that X is left-invariant implies that $X = v^L$, which is smooth. □

The existence of global left-invariant vector fields also yields the following important property of Lie groups. Recall that a smooth manifold is said to be *parallelizable* if it admits a smooth global frame. If G is a Lie group, a local or global frame consisting of left-invariant vector fields is called a **left-invariant frame**.

Corollary 2.2.1.5. *Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.*

Proof. If G is a Lie group, every basis for $\mathfrak{Lie}(G)$ is a left-invariant smooth global frame for G. □

Example 2.2.1.6. Let us determine the Lie algebras of some familiar Lie groups.

(a) Euclidean space $\mathbb{R}^n$: If we consider $\mathbb{R}^n$ as a Lie group under addition, left translation by an element $b \in \mathbb{R}^n$ is given by the affine map $L_b(x) = b + x$, whose differential dL_b is represented by the identity matrix in standard coordinates. Thus a vector field $X_i\partial/\partial x^i$ is left-invariant if and only if its coefficients X_i are constants. Because the Lie bracket of two constant-coefficient vector fields is zero, the Lie algebra of $\mathbb{R}^n$ is abelian, and is isomorphic to $\mathbb{R}^n$ itself with the trivial bracket. In brief, $\mathfrak{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.

(b) The circle group S^1: In terms of appropriate angle coordinates, each left translation has a local coordinate representation of the form $\theta \mapsto \theta + c$. Since the differential of this map is the identity matrix, it follows that the vector field $d/d\theta$ is left-invariant, and is therefore a basis for the Lie algebra of S^1. This Lie algebra is 1-dimensional and abelian, and therefore $\mathfrak{Lie}(S^1) \cong \mathbb{R}$. Similarly, $\mathfrak{Lie}(T^n) \cong \mathbb{R}^n$.

The Lie groups $\mathbb{R}^n$, S^1, and T^n are abelian, and as the discussion above shows, their Lie algebras turn out also to be abelian. This is no accident: every abelian Lie group has an abelian Lie algebra. Later, we will see that the converse is true provided that the group is connected.

We conclude this section by analyzing the Lie algebra of the most important nonabelian Lie group of all, the general linear group. Theorem 2.2.1.3 gives a vector space isomorphism between $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R}))$ and the tangent space to $\mathrm{GL}_n(\mathbb{R})$ at the identity matrix I_n. Because $\mathrm{GL}_n(\mathbb{R})$ is an open subset of the vector space $\mathfrak{gl}_n(\mathbb{R})$, its tangent space is naturally isomorphic to $\mathfrak{gl}_n(\mathbb{R})$ itself. The composition of these two isomorphisms gives a vector space isomorphism $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R})) \cong \mathfrak{gl}(n, \mathbb{R})$.

The vector spaces $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R}))$ and $\mathfrak{gl}(n, \mathbb{R})$ have independently defined Lie algebra structures—the first coming from Lie brackets of vector fields, and the second from commutator brackets of matrices. The next proposition shows that the natural vector space isomorphism between these spaces is in fact a Lie algebra isomorphism.

Proposition 2.2.1.7 (Lie Algebra of the General Linear Group). *The composition of the natural maps*

$$\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R})) \to T_{I_n}\mathrm{GL}_n(\mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}) \tag{2.2.1.2}$$

gives a Lie algebra isomorphism between $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R}))$ and the matrix algebra $\mathfrak{gl}(n, \mathbb{R})$.

Proof. Using the matrix entries (X^i_j) as global coordinates on $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathfrak{gl}_n(\mathbb{R})$, the natural isomorphism $T_{I_n}\mathrm{GL}_n(\mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R})$ takes the form

$$A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n} \Longleftrightarrow A^i_j$$

Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. Any matrix $A = (A^i_j) \in \mathfrak{gl}(n, \mathbb{R})$ determines a left-invariant vector field $A^L \in \mathfrak{g}$ defined by (2.2.1.1), which in this case becomes

$$A^L|_Y = d(L_Y)_{I_n}(A) = d(L_Y)_{I_n}\left(A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n}\right)$$

Since L_Y is the restriction to $\mathrm{GL}_n(\mathbb{R})$ of the linear map $A \mapsto YA$ on $\mathfrak{gl}(n,\mathbb{R})$, its differential is represented in coordinates by exactly the same linear map. In other words, the left-invariant vector field A^L determined by A is the one whose value at $Y \in \mathrm{GL}_n(\mathbb{R})$ is

$$A^L|_Y = Y^i_j A^j_k \frac{\partial}{\partial X^i_k}\Big|_Y \tag{2.2.1.3}$$

Thus by useing the coordinates function (X^i_j), we can write A^L as

$$A^L = X^i_j A^j_k \frac{\partial}{\partial X^i_k}$$

Given two matrices $A, B \in \mathfrak{gl}(n,\mathbb{R})$, the Lie bracket of the corresponding left-invariant vector fields is given by

$$\begin{aligned}
[A^L, B^L] &= \left[X^i_j A^j_k \frac{\partial}{\partial X^i_k}, X^p_q B^q_r \frac{\partial}{\partial X^p_r}\right] \\
&= X^i_j A^j_k \frac{\partial(X^p_q B^q_r)}{\partial X^i_k} \frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial(X^i_j A^j_k)}{\partial X^p_r} \frac{\partial}{\partial X^i_k} \\
&= X^i_j A^j_k B^k_r \frac{\partial}{\partial X^i_r} - X^p_q B^q_r A^r_k \frac{\partial}{\partial X^p_k} \\
&= X^i_j A^j_k B^k_r \frac{\partial}{\partial X^i_r} - X^i_j B^j_k A^k_r \frac{\partial}{\partial X^i_k} \\
&= X^i_j (A^j_k B^k_r - B^j_k A^k_r) \frac{\partial}{\partial X^i_k}
\end{aligned}$$

where we have used the fact that $\partial X^p_q / \partial X^i_k$ is equal to 1 if $p = i$ and $q = k$, and 0 otherwise, and A^i_j and B^i_j are constants. Evaluating this last expression when X is equal to the identity matrix, we get

$$[A^L, B^L]_{I_n} = (A^i_k B^k_r - B^i_k A^k_r) \frac{\partial}{\partial X^i_k}\Big|_{I_n}$$

This is the vector corresponding to the matrix commutator bracket $[A, B]$. Since the left-invariant vector field $[A^L, B^L]$ is determined by its value at the identity, this implies that

$$[A^L, B^L] = [A, B]^L$$

which is precisely the statement that the composite map is a Lie algebra isomorphism. □

There is an analogue of this result for abstract vector spaces. If V is any finitedimensional real vector space, recall that we have defined $\mathrm{GL}(V)$ as the Lie group of invertible linear transformations of V, and $\mathfrak{gl}(V)$ as the Lie algebra of all linear transformations. Just as in the case of $\mathrm{GL}_n(\mathbb{R})$, we can regard $\mathrm{GL}(V)$ as an open submanifold of $\mathfrak{gl}(V)$, and thus there are canonical vector space isomorphisms

$$\mathfrak{Lie}(\mathrm{GL}(V)) \to T_{\mathrm{id}}\mathrm{GL}(V) \to \mathfrak{gl}(V) \tag{2.2.1.4}$$

Corollary 2.2.1.8. *If V is any finite-dimensional real vector space, the composition of the canonical isomorphisms in (2.2.1.4) yields a Lie algebra isomorphism between $\mathfrak{Lie}(\mathrm{GL}(V))$ and $\mathfrak{gl}(V)$.*

2.2.2 Induced Lie algebra homomorphisms

The importance of the Lie algebra of a Lie group stems, in large part, from the fact that each Lie group homomorphism induces a Lie algebra homomorphism, as the next theorem shows.

Theorem 2.2.2.1 (Induced Lie Algebra Homomorphisms). *Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Suppose $F : G \to H$ is a Lie group homomorphism. For every $X \in g$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X. With this vector field denoted by F_*X, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.*

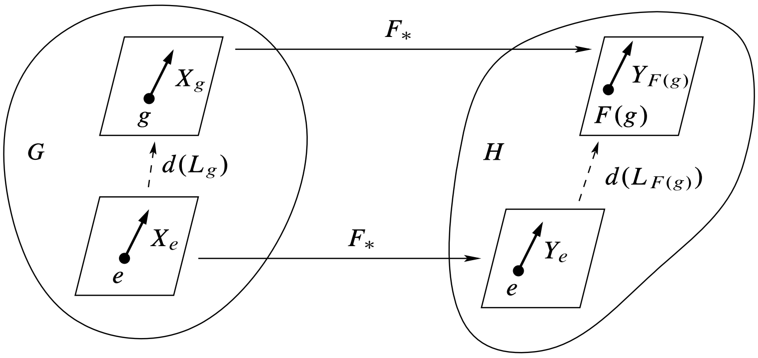


Figure 2.1: The induced Lie algebra homomorphism.

Proof. If there is any vector field $Y \in \mathfrak{h}$ that is F-related to X, it must satisfy $Y_e = dF_e(X_e)$, and thus it must be uniquely determined by

$$Y = (Y_e)^L = (dF_e(X_e))^L$$

To show that this Y is F-related to X, we note that the fact that F is a homomorphism implies

$$\begin{aligned} F(gg') = F(g)F(g') &\Rightarrow F(L_g g') = L_{F(g)}F(g') \\ &\Rightarrow F \circ L_g = L_{F(g)} \circ F \\ &\Rightarrow dF \circ d(L_g) = d(L_{F(g)}) \circ dF \end{aligned}$$

Thus,

$$dF(X_g) = dF \circ d(L_g)(X_e) = d(L_{F(g)}) \circ dF(X_e) = d(L_{F(g)})(Y_e) = Y_{F(g)}$$

This says precisely that X and Y are F-related.

For each $X \in g$, let F_*X denote the unique vector field in $\mathfrak{h}$ that is F-related to X. It then follows immediately from the naturality of Lie brackets that $F_*[X, Y] = [F_*X, F_*Y]$, so F_* is a Lie algebra homomorphism. □

The map $F_* : \mathfrak{g} \to \mathfrak{h}$ whose existence is asserted in this theorem is called the **induced Lie algebra homomorphism**. Note that the theorem implies that for any left-invariant vector field $X \in g$, F_*X is a well-defined smooth vector field on H, *even though F may not be a diffeomorphism*.

Proposition 2.2.2.2 (Properties of Induced Homomorphisms).

(a) The homomorphism $(\mathrm{id})_* : \mathfrak{Lie}(G) \to \mathfrak{Lie}(G)$ *induced by the identity map of G is the identity of* $\mathfrak{Lie}(G)$.

(b) If $F_1 : G \to H$ and $F_2 : H \to K$ are Lie group homomorphisms, then

$$(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_*$$

(c) Isomorphic Lie groups have isomorphic Lie algebras.

Proof. These all follow from the property of the differential F_*. □

Therefore, we obtain a functor from the category of Lie groups to the category of Lie algebras.

2.2.3 Lie subalgebras and Lie subgroups

If G is a Lie group and $H \subseteq G$ is a Lie subgroup, we might hope that the Lie algebra of H would be a Lie subalgebra of that of G. However, elements of $\mathfrak{Lie}(H)$ are vector fields on H, not G, and so, strictly speaking, are not elements of $\mathfrak{Lie}(G)$. Nonetheless, the next proposition gives us a way to view $\mathfrak{Lie}(H)$ as a subalgebra of $\mathfrak{Lie}(G)$.

Proposition 2.2.3.1. *Suppose $H \subseteq G$ is a Lie subgroup, and $\iota : H \hookrightarrow G$ is the inclusion map. There is a Lie subalgebra $\mathfrak{h} \in \mathfrak{Lie}(G)$ that is canonically isomorphic to $\mathfrak{Lie}(H)$, characterized by either of the following descriptions:*

$$\mathfrak{h} = \iota_*(\mathfrak{Lie}(H)) = \{X \in \mathfrak{Lie}(G) : X_e \in T_eH\}$$

Proof. Because the inclusion map $\iota : H \hookrightarrow G$ is a Lie group homomorphism, $\iota_*(\mathfrak{Lie}(H))$ is a Lie subalgebra of $\mathfrak{Lie}(G)$. By the way we defined the induced Lie algebra homomorphism, this subalgebra is precisely the set of left-invariant vector fields on G whose values at the identity are of the form $d\iota_e(v)$ for some $v \in T_eH$. Since the differential $d\iota_e : T_eH \to T_eG$ is the inclusion of T_eH as a subspace in T_eG, the two characterizations of $\mathfrak{h}$ are equal. Since $d\iota_e$ is injective on T_eH, it follows that ι_* is injective on $\mathfrak{Lie}(H)$; since it is surjective by definition of $\mathfrak{h}$, it is an isomorphism between $\mathfrak{Lie}(H)$ and $\mathfrak{h}$. □

Using this proposition, whenever H is a Lie subgroup of G, we often identify $\mathfrak{Lie}(H)$ as a subalgebra of $\mathfrak{Lie}(G)$. As we mentioned above, elements of $\mathfrak{Lie}(H)$ are not themselves left-invariant vector fields on G. But the preceding proposition shows that every element of $\mathfrak{Lie}(H)$ corresponds to a unique element of $\mathfrak{Lie}(G)$, determined by its value at the identity, and the injection of $\mathfrak{Lie}(H)$ into $\mathfrak{Lie}(G)$ thus determined respects Lie brackets; so by thinking of $\mathfrak{Lie}(H)$ as a subalgebra of $\mathfrak{Lie}(G)$ we are not committing a grave error.

This identification is especially illuminating in the case of Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$. We showed above that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is naturally isomorphic to the matrix algebra $\mathfrak{gl}(n,\mathbb{R})$. We can now prove a similar result for $\mathrm{GL}_n(\mathbb{C})$. Just as in the real case, our usual identification of $\mathrm{GL}_n(\mathbb{C})$ as an open subset of $\mathfrak{gl}(n,\mathbb{C})$ yields a sequence of vector space isomorphisms

$$\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C})) \xrightarrow{\varepsilon} T_{I_n}\mathrm{GL}_n(\mathbb{C}) \xrightarrow{\varphi} \mathfrak{gl}(n,\mathbb{C}) \tag{2.2.3.1}$$

where ε is the evaluation map and φ is the usual identification between the tangent space to an open subset of a vector space and the vector space itself.

Proposition 2.2.3.2. *The composition of the maps in* (2.2.3.1) *yields a Lie algebra isomorphism between* $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C}))$ *and the matrix algebra* $\mathfrak{gl}(n,\mathbb{C})$.

Proof. The Lie group homomorphism $\beta : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{R})$ that we constructed in Example 2.1.3.6(d) induces a Lie algebra homomorphism

$$\beta_* : \mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C})) \to \mathfrak{Lie}(\mathrm{GL}_{2n}(\mathbb{R}))$$

Composing β_* with our canonical isomorphisms yields a commutative diagram

$$\begin{tikzcd}
\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C})) \arrow[r, "\varepsilon"] \arrow[d, "\beta_*"] & T_{I_n}\mathrm{GL}_n(\mathbb{C}) \arrow[r, "\varphi"] \arrow[d, "d\beta_{I_n}"] & \mathfrak{gl}(n,\mathbb{C}) \arrow[d, "\alpha"] \\
\mathfrak{Lie}(\mathrm{GL}_{2n}(\mathbb{R})) \arrow[r, "\varepsilon"] & T_{I_{2n}}\mathrm{GL}_{2n}(\mathbb{R}) \arrow[r, "\varphi"] & \mathfrak{gl}(2n,\mathbb{R})
\end{tikzcd}$$

in which $\alpha = \varphi \circ d\beta_{I_n} \circ \varphi^{-1}$. Proposition 2.2.1.8 showed that the composition of the isomorphisms in the bottom row is a Lie algebra isomorphism; we need to show the same thing for the top row.

It is easy to see from the formula in Example 2.1.3.6(d) that β is (the restriction of) a linear map. It follows that $d\beta_{I_n}$ is given by exactly the same formula as β, as is $\alpha : \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(2n,\mathbb{R})$. Because $\beta(AB) = \beta(A)\beta(B)$, it follows that α preserves matrix commutators:

$$\alpha[A,B] = \alpha(AB - BA) = \alpha(A)\alpha(B) - \alpha(B)\alpha(A) = [\alpha(A), \alpha(B)]$$

Thus α is an injective Lie algebra homomorphism from $\mathfrak{gl}(n,\mathbb{C})$ to $\mathfrak{gl}(2n,\mathbb{R})$ (considering both as matrix algebras). Replacing the bottom row by the images of the vertical maps, we obtain a commutative diagram of vector space isomorphisms

$$\begin{tikzcd}
\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C})) \arrow[r, "\cong"] \arrow[d, "\beta_*"] & \mathfrak{gl}_n(\mathbb{C}) \arrow[d, "\alpha"] \\
\beta_*(\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{C}))) \arrow[r, "\cong"] & \alpha(\mathfrak{gl}(2n,\mathbb{R}))
\end{tikzcd}$$

in which the bottom map and the two vertical maps are Lie algebra isomorphisms; it follows that the top map is also a Lie algebra isomorphism. □

A distribution D on a Lie group G is said to be **left-invariant** if it is invariant under every left translation.

Lemma 2.2.3.3. *Let G be a Lie group. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{Lie}(G)$, then the subset $D = \bigcup_{g \in G} D_g$, where*

$$D_g = \{X_g : X \in \mathfrak{h}\} \subseteq T_g G,$$

is a smooth left-invariant involutive distribution on G.

Proof. Each $X \in \mathfrak{h}$ is a left-invariant vector field on G. Thus, for any $g, g' \in G$, the differential $d(L_{g'g^{-1}})$ restricts to an isomorphism from D_g to $D_{g'}$. It follows that D_g has the same dimension for each g, and D is left-invariant. Any basis $(X_1, \dots, X_k)$ for $\mathfrak{h}$ is a global smooth frame for D, so D is smooth. Moreover, because $[X_i, X_j] \in \mathfrak{h}$ for all $1 \le i, j \le k$, it follows from Lemma 1.3.1.4 that D is involutive. □

Theorem 2.2.3.4 (Lie Subgroups Are Weakly Embedded). *Every Lie subgroup is an integral manifold of an involutive distribution, and therefore is a weakly embedded submanifold.*

Proof. Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Theorem 2.2.3.1 shows that the Lie algebra of H is canonically isomorphic to the Lie subalgebra $\mathfrak{h} = \iota_*(\mathfrak{Lie}(H)) \subseteq \mathfrak{Lie}(G)$, where $\iota : H \to G$ is inclusion. Let $D \subseteq TG$ be the involutive distribution determined by $\mathfrak{h}$ as in Lemma 2.2.3.3. It follows from the definitions that at each point $\mathfrak{h} \in H$, the tangent space $T_h H$ is equal to D_h, so H is an integral manifold of D. It then follows from Theorem 1.3.2.5 that H is weakly embedded in G. □

Theorem 2.2.3.5 (The Lie Subgroup Associated with a Lie Subalgebra). *Suppose G is a Lie group and $\mathfrak{g}$ is its Lie algebra. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.*

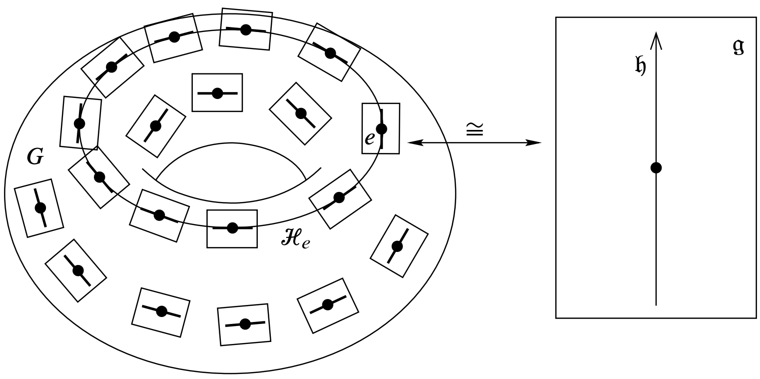


Figure 2.1: Finding a subgroup whose Lie algebra is $\mathfrak{h}$.

Proof. Suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let $D \subseteq TG$ be the involutive distribution of $\mathfrak{h}$. Let $\mathcal{H}$ denote the foliation determined by D, and for any $g \in G$, let $\mathcal{H}_g$ denote the leaf of $\mathcal{H}$ containing g. Because D is left-invariant, it follows from Proposition 1.3.3.5 that each left translation takes leaves to leaves: for any $g, g' \in G$, we have $L_g(\mathcal{H}_{g'}) = \mathcal{H}_{gg'}$. Define $H = \mathcal{H}_e$, the leaf containing the identity. We will show that H is the desired Lie subgroup.

First, to see that H is a subgroup, observe that for any $h, h' \in H$,

$$hh' = L_h(h') \in L_h(H) = L_h(\mathcal{H}_e) = \mathcal{H}_h = H.$$

Similarly,

$$h^{-1} = h^{-1}e = L_{h^{-1}}(\mathcal{H}_e) = L_{h^{-1}}(\mathcal{H}_h) = \mathcal{H}_e = H.$$

To show that H is a Lie group, we need to show that the map $\mu : (h, h') \mapsto h^{-1}h'$ is smooth as a map from $H \times H$ to H. Because $H \times H$ is a submanifold of $G \times G$, it is immediate that $\mu : H \times H \to G$ is smooth. Since H is an integral manifold of an involutive distribution, Theorem 1.3.2.5 shows that it is weakly embedded, so μ is also smooth as a map into H.

The fact that H is a leaf of $\mathcal{H}$ implies that the Lie algebra of H is $\mathfrak{h}$, because the tangent space to H at the identity is $D_e = \{X_e : X \in \mathfrak{h}\}$. To see that H is the unique connected subgroup with Lie algebra $\mathfrak{h}$, suppose $\widetilde{H}$ is any other connected subgroup with the same Lie algebra. Any such Lie subgroup is easily seen to be an integral manifold of D, so by maximality of $H = \mathcal{H}_e$, we must have $\widetilde{H} \subseteq H$. On the other hand, if U is the domain of a flat chart for D containing the identity, then by Proposition 1.3.2.4, $\widetilde{H} \cap U$ is a union of open subsets of slices. Since the slice containing e is an open subset of H, this implies that $\widetilde{H}$ contains a neighborhood of the identity in H. Since any neighborhood of the identity generates H (Proposition **??**), this implies that $\widetilde{H} = H$. □

Since nonclosed subgroups can be pathological, Theorem 2.2.3.5 is most useful in cases where the connected Lie subgroup H is actually closed. The following result gives one condition under which this is guaranteed to be the case.

Proposition 2.2.3.6. *Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and that $\mathfrak{h}$ is a maximal commutative subalgebra of $\mathfrak{g}$, meaning that $\mathfrak{h}$ is commutative and h is not contained in any larger commutative subalgebra of $\mathfrak{g}$. Then the connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$ is closed.*

Proof. Since $\mathfrak{h}$ is commutative, H is also commutative, since every element of H is a product of exponentials of elements of $\mathfrak{h}$ (Proposition 2.2.4.10). It easily follows that the closure $\overline{H}$ of H in G is also commutative. Since H is connected, $\overline{H}$ is also connected. Now, since $\overline{H}$ is commutative, its Lie algebra $\mathfrak{h}'$ is also commutative. But since $\mathfrak{h}$ was maximal commutative, we must have $\mathfrak{h}' = \mathfrak{h}$. Since, also, $\overline{H}$ is connected, we conclude that $H = \overline{H}$, showing that H is closed. □

2.2.4 One-parameter subgroups and the exponential map

Suppose G is a Lie group. Since left-invariant vector fields are naturally defined in terms of the group structure of G, one might reasonably expect to find some relationship between the group law for the flow of a left-invariant vector field and group multiplication in G. We begin by exploring this relationship.

2.2.4.1 One-Parameter Subgroups

A **one-parameter subgroup** of G is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$, with $\mathbb{R}$ considered as a Lie group under addition. By this definition, a one-parameter subgroup is not a Lie subgroup of G, but rather a homomorphism into G.

Theorem 2.2.4.1 (Characterization of One-Parameter Subgroups). *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.*

Proof. First suppose γ is the maximal integral curve of some left-invariant vector field $X \in \mathfrak{Lie}(G)$ starting at the identity. Because left-invariant vector fields are complete (Theorem 1.2.3.11), γ is defined on all of $\mathbb{R}$. Left-invariance means that X is L_g-related to itself for every $g \in G$, so L_g takes integral curves of X to integral curves of X. Applying this with $g = \gamma(s)$ for some $s \in \mathbb{R}$, we conclude that the curve $t \mapsto L_{\gamma(s)}(\gamma(t))$ is an integral curve starting at $\gamma(s)$. But the curve $t \mapsto \gamma(t+s)$ is also an integral curve with the same initial point, so they are equal:

$$\gamma(s)\gamma(t) = \gamma(s+t).$$

This says precisely that $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup.

Conversely, suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup, and let $X = \gamma_*(d/dt) \in \mathfrak{Lie}(G)$, treating d/dt as a left-invariant vector field on $\mathbb{R}$. Since $\gamma(0) = e$, we just have to show that γ is an integral curve of X. Recall that $\gamma_*(d/dt)$ is defined as the unique left-invariant vector field on G that is γ_*-related to d/dt (see Theorem 2.2.2.1). Therefore, for any $t_0 \in \mathbb{R}$,

$$\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) = X_{\gamma(t_0)}.$$

so γ is an integral curve of X. □

Given $X \in \mathfrak{Lie}(G)$, the one-parameter subgroup determined by X in this way is called the **one-parameter subgroup generated by X**. Because left-invariant vector fields are uniquely determined by their values at the identity, it follows that each one-parameter subgroup is uniquely determined by its initial velocity in T_eG, and thus there are one-to-one correspondences

$$\text{one-parameter subgroups of } G \leftrightarrow \mathfrak{Lie}(G) \leftrightarrow T_eG.$$

The one-parameter subgroups of $\mathrm{GL}_n(\mathbb{R})$ are not hard to compute explicitly.

Proposition 2.2.4.2. *For any $A \in \mathfrak{gl}(n,\mathbb{R})$, let*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{2.2.4.1}$$

This series converges to an invertible matrix $e^A \in \mathrm{GL}_n(\mathbb{R})$, and the one-parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$ generated by $A \in \mathfrak{gl}(n,\mathbb{R})$ is $\gamma(t) = e^{tA}$.

Proof. First, we verify convergence. The matrix multiplication satisfies $|AB| \leq |A||B|$, where the norm is the Frobenius norm on $\mathfrak{gl}(n,\mathbb{R})$. It follows by induction that $|A^k| \leq |A|^k$. The Weierstrass M-test then shows that (2.2.4.1) converges uniformly on any bounded subset of $\mathfrak{gl}(n,\mathbb{R})$, by comparison with the series $\sum_{k=0}^{\infty} c^k/k!$.

Fix $A \in \mathfrak{gl}(n,\mathbb{R})$. Under our identification of $\mathfrak{gl}(n,\mathbb{R})$ with $\mathfrak{Lie}(\mathrm{GL}_n(\mathbb{R}))$, the matrix A corresponds to the left-invariant vector field A^L given by (2.2.1.1). Thus, the one-parameter subgroup generated by A is an integral curve of A^L on $\mathrm{GL}_n(\mathbb{R})$, and therefore satisfies the ODE initial value problem

$$\gamma'(t) = A^L|_{\gamma(t)}, \quad \gamma(0) = I_n.$$

Using (2.2.1.3), the condition for γ to be an integral curve can be rewritten as

$$\dot{\gamma}^i_k(t) = \gamma^i_j(t) A^j_k,$$

or in matrix notation

$$\gamma'(t) = \gamma(t)A.$$

We will show that $\gamma(t) = e^{tA}$ satisfies this equation. Since $\gamma(0) = I_n$, this implies that γ is the unique integral curve of A^L starting at the identity and is therefore the desired one-parameter subgroup.

To see that γ is differentiable, we note that differentiating the series (2.2.4.1) formally term by term yields the result

$$\gamma'(t) = \sum_{k=1}^{\infty} \frac{A^k}{(k-1)!} = \gamma(t)A = A\gamma(t).$$

Since the differentiated series converges uniformly on bounded sets, the term-by-term differentiation is justified. By smoothness of solutions to ODEs, γ is a smooth curve.

It remains only to show that $\gamma(t)$ is invertible for all t, so that γ actually takes its values in $\mathrm{GL}_n(\mathbb{R})$. If we let $\sigma(t) = \gamma(t)\gamma(-t) = e^{tA}e^{-tA}$, then σ is a smooth curve in $\mathfrak{gl}(n,\mathbb{R})$, and by the previous computation and the product rule it satisfies

$$\sigma'(t) = \big(\gamma(t)A\big)\gamma(-t) - \gamma(t)\big(A\gamma(-t)\big) = 0.$$

It follows that σ is the constant curve $\sigma(t) = I_n$, which is to say that $\gamma(t) = \gamma(-t)^{-1}$. □

Proposition 2.2.4.3. *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in T_eH.*

Proof. Let $\gamma : \mathbb{R} \to H$ be a one-parameter subgroup. Then the composite map

$$\mathbb{R} \xrightarrow{\gamma} H \hookrightarrow G$$

is a Lie group homomorphism and thus a one-parameter subgroup of G, which clearly satisfies $\gamma'(0) \in T_eH$.

Conversely, suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup whose initial velocity lies in T_eH. Let $\widetilde{\gamma} : \mathbb{R} \to H$ be the one-parameter subgroup of H with the same initial velocity $\widetilde{\gamma}'(0) = \gamma'(0) \in T_eH \subseteq T_eG$. As in the preceding paragraph, by composing with the inclusion map, we can also consider $\widetilde{\gamma}$ as a one-parameter subgroup of G. Since γ and $\widetilde{\gamma}$ are both one-parameter subgroups of G with the same initial velocity, they must be equal. □

Example 2.2.4.4. If H is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$, the preceding proposition shows that the one-parameter subgroups of H are precisely the maps of the form $\gamma(t) = e^{tA}$ for $A \in \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is the subalgebra corresponding to $\mathfrak{Lie}(H)$ as in Theorem 2.2.3.1. For example, taking $H = \mathrm{O}(n)$, this shows that the one-parameter subgroups of $\mathrm{O}(n)$ are the maps of the form $\gamma(t) = e^{tA}$ for an arbitrary skew-symmetric matrix A. In particular, this shows that the exponential of any skew-symmetric matrix is orthogonal.

2.2.4.2 The Exponential Map

In the preceding part we saw that the matrix exponential maps $\mathfrak{gl}(n, \mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})$ and takes each line through the origin to a one-parameter subgroup. This has a powerful generalization to arbitrary Lie groups.

Given a Lie group G with Lie algebra $\mathfrak{g}$, we define a map $\exp : \mathfrak{g} \to G$, called the **exponential map of G**, as follows: for any $X \in \mathfrak{g}$, we set

$$\exp(X) = \gamma(1),$$

where γ is the one-parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity. The following proposition shows that, like the matrix exponential, this map sends the line through X to the one-parameter subgroup generated by X.

Proposition 2.2.4.5 ((One-Parameter Subgroups). *Let G be a Lie group. For any $X \in \mathfrak{Lie}(G)$, $\gamma(t) = \exp tX$ is the one-parameter subgroup of G generated by X.*

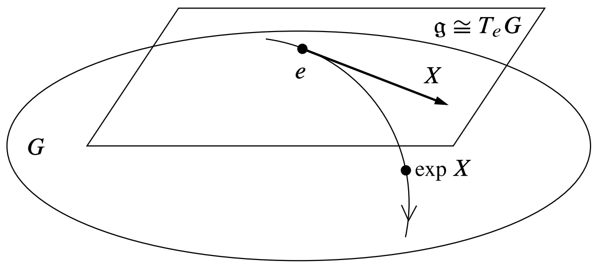


Figure 2.1: The exponential map.

Proof. Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup generated by X, which is the integral curve of X starting at e. For any fixed $t \in \mathbb{R}$, it follows from the rescaling lemma that $\widetilde{\gamma}(s) = \gamma(ts)$ is the integral curve of tX starting at e, so

$$\exp tX = \widetilde{\gamma}(1) = \gamma(t)$$

as needed. $\square$

Here are two simple but important examples.

Example 2.2.4.6. The results of the preceding part show that the exponential map of $\mathrm{GL}_n(\mathbb{R})$ (or any Lie subgroup of it) is given by $\exp A = e^A$. This, obviously, is the reason for the term exponential map.

Example 2.2.4.7. If V is a finite-dimensional real vector space, a choice of basis for V yields isomorphisms $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{R})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$. The analysis of the $\mathrm{GL}_n(\mathbb{R})$ case then shows that the exponential map of $\mathrm{GL}(V)$ can be written in the form

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where we consider $A \in \mathfrak{gl}(V)$ as a linear map from V to itself, and $A^k = A \circ \cdots \circ A$ is the k-fold composition of A with itself.

Proposition 2.2.4.8 (Properties of the Exponential Map). *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

(a) The exponential map is a smooth map from $\mathfrak{g}$ to G.

(b) *For any* $X \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.

(c) *For any* $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.

(d) *For any* $X \in \mathfrak{g}$ *and* $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.

(e) *The differential* $(d\exp)_0 : T_0\mathfrak{g} \to T_eG$ *is the identity map, under the canonical identifications of both* $T_0\mathfrak{g}$ *and* T_eG *with* $\mathfrak{g}$ *itself.*

(f) *The exponential map restricts to a diffeomorphism from some neighborhood of* 0 *in* $\mathfrak{g}$ *to a neighborhood of* e *in* G.

(g) *If* H *is another Lie group,* $\mathfrak{h}$ *is its Lie algebra, and* $\Phi : G \to H$ *is a Lie group homomorphism, the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"] & H \end{tikzcd}$$

(h) *The flow* θ *of a left-invariant vector field* X *is given by* $\theta_t = R_{\exp tX}$ *(right multiplication by* $\exp tX$*).*

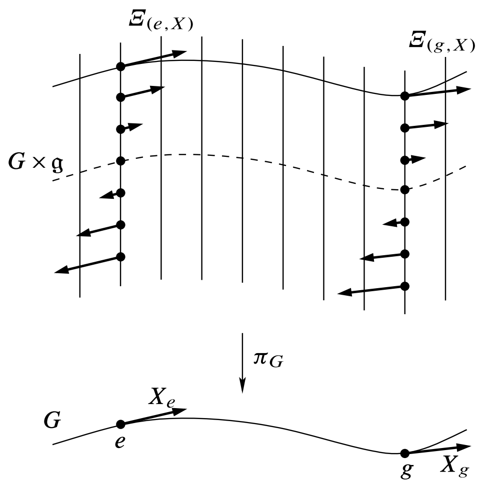


Figure 2.2: Proof that the exponential map is smooth.

Proof. In this proof, for any $X \in \mathfrak{g}$ we let θ_X denote the flow of X. To prove (a), we need to show that the expression $\theta_X^{(e)}(1)$ depends smoothly on X, which amounts to showing that the flow varies smoothly as the vector field varies. This is a situation not covered by the fundamental theorem on flows, but we can reduce it to that theorem by the following simple trick. Define a vector field Ξ on the product manifold $G \times \mathfrak{g}$ by

$$\Xi_{(g,X)} = (X_g, 0) \in T_gG \oplus T_X\mathfrak{g} \cong T_{(g,X)}(G \times \mathfrak{g}).$$

To see that Ξ is a smooth vector field, choose any basis $(X_1, \ldots, X_k)$ for $\mathfrak{g}$, and let (x^i) be the corresponding global coordinates for $\mathfrak{g}$, defined by $(x^i) \leftrightarrow x^iX_i$. Let (w^i) be any smooth local coordinates for G. If $f \in C^\infty(G \times \mathfrak{g})$ is arbitrary, then locally we can write

$$\Xi f(w^i, x^i) = x^jX_jf(w^i, x^i),$$

where each vector field X_j differentiates f only in the w^i-directions. Since this depends smoothly on (w^i, x^i), it follows from Proposition 1.1.1.13 that Ξ is smooth. It is easy to verify that the flow Θ of Ξ is given by

$$\Theta_t(g, X) = (\theta_X(t, g), X).$$

By the fundamental theorem on flows, Θ is smooth. Since $\exp X = \pi_G(\Theta_1(e, X))$, where $\pi_G : G \times \mathfrak{g} \to G$ is the projection, it follows that exp is smooth.

Next, (b) and (c) follow immediately from Proposition 2.2.4.5, because $t \mapsto \exp tX$ is a group homomorphism from $\mathbb{R}$ to G. Then (d) for nonnegative n follows from (b) by induction, and for negative n it follows from (c).

To prove (e), let $X \in \mathfrak{g}$ be arbitrary, and let $\sigma : \mathbb{R} \to \mathfrak{g}$ be the curve $\sigma(t) = tX$. Then $\sigma'(0) = X$, and Proposition 2.2.4.5 implies

$$(d\exp)_0(X) = (d\exp)_0(\sigma'(0)) = (\exp\circ\sigma)'(0) = \frac{d}{dt}\Big|_{t=0} \exp tX = X.$$

Part (f) then follows immediately from (e) and the inverse function theorem.

Next, to prove (g) we need to show that $\exp(\Phi_* X) = \Phi(\exp X)$ for every $X \in g$. In fact, we will show that for all $t \in \mathbb{R}$,

$$\exp(t\Phi_* X) = \Phi(\exp tX).$$

The left-hand side is, by Proposition 2.2.4.5, the one-parameter subgroup generated by $\Phi_* X$. Thus, if we put $\sigma(t) = \Phi(\exp tX)$, it suffices to show that $\sigma : \mathbb{R} \to H$ is a Lie group homomorphism satisfying $\sigma'(0) = (\Phi_* X)_e$. It is a Lie group homomorphism because it is the composition of the homomorphisms Φ and $t \mapsto \exp tX$. The initial velocity is computed as follows:

$$\sigma'(0) = \frac{d}{dt}\Big|_{t=0} \Phi(\exp tX) = d\Phi_e\left(\frac{d}{dt}\Big|_{t=0} \exp tX\right) = d\Phi_e(X_e) = (\Phi_* X)_e.$$

Finally, to show that $(\theta_X)_t = R_{\exp tX}$, we use the fact that for any $g \in G$, the left multiplication map L_g takes integral curves of X to integral curves of X. Thus, the map $t \mapsto L_g(\exp tX)$ is the integral curve starting at g, which means it is equal to $\theta_X^{(g)}(t)$. It follows that

$$R_{\exp tX}(g) = g \exp tX = L_g(\exp tX) = \theta_X^{(g)}(t) = (\theta_X)_t(g).$$

This completes the proof. □

The exponential map yields the following alternative characterization of the Lie subalgebra of a subgroup.We will use this later when we study normal subgroups.

Proposition 2.2.4.9. *Let G be a Lie group, and let $H \subseteq G$ be a Lie subgroup. With $\mathfrak{Lie}(H)$ considered as a subalgebra of $\mathfrak{Lie}(G)$ in the usual way, the exponential map of H is the restriction to $\mathfrak{Lie}(H)$ of the exponential map of G, and*

$$\mathfrak{Lie}(H) = \{X \in \mathfrak{Lie}(G) : \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

Proof. The fact that the exponential map of H is the restriction of that of G is an immediate consequence of Proposition 2.2.4.3. To prove the second assertion, by the way we have identified $\mathfrak{Lie}(H)$ as a subalgebra of $\mathfrak{Lie}(G)$, we need to establish the following equivalence for every $X \in \mathfrak{Lie}(G)$:

$$\exp tX \in H \text{ for all } t \in \mathbb{R} \iff X_e \in T_e H.$$

Assume first that $\exp tX \in H$ for all t. Since H is weakly embedded in G by Theorem 2.2.3.4, it follows that the curve $t \mapsto \exp tX$ is smooth as a map into H, and thus $X_e = \gamma'(0) \in T_e H$. Conversely, if $X_e \in T_e H$, then Proposition 2.2.4.3 implies that $\exp tX \in H$ for all t. □

Notice that Proposition 2.2.4.8(d) does not imply $\exp(X + Y) = (\exp X)(\exp Y)$ for arbitrary X, Y in the Lie algebra. In fact, for connected groups, this is true only when the group is abelian.

Proposition 2.2.4.10. *If G is a connected Lie group, then G is generated by the image of the exponential map.*

Proof. This comes from the fact that if G is a connected topological group, then it is generated by any neighborhood of the identity (Proposition **??**). □

Example 2.2.4.11. Let $G = \mathrm{SO}(3)$. Then $\mathfrak{g} = \mathfrak{so}(3)$ consists of skew-symmetric 3×3 matrices. One possible choice of a basis in $\mathfrak{so}(3)$ is

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can explicitly describe the corresponding subgroups in G. Namely,

$$\exp(tJ_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

is rotation around x-axis by angle t; similarly, J_y, J_z generate rotations around y, z axes. The easiest way to show this is to note that such rotations do form a one-parameter subgroup; thus, they must be of the form $\exp(tJ)$ for some $J \in \mathfrak{so}(3)$, and then compute the derivative to find J.

By Theorem 2.2.4.10, elements of the form $\exp(tJ_x), \exp(tJ_y), \exp(tJ_z)$ generate $SO(3)$. For this reason, it is common to refer to J_x, J_y, J_z as "infinitesimal generators" of $SO(3)$. Thus, in a certain sense $SO(3)$ is generated by three elements.

Corollary 2.2.4.12. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and assume G is connected. Then any Lie group homomorphism $\Phi : G \to H$ is uniquely determined by $\Phi_* : \mathfrak{g} \to \mathfrak{h}$.*

Proof. Let g be any element of G. Since G is connected, Corollary 2.2.4.10 tells us that every element of G can be written as $\exp(x_1) \cdots \exp(X_m)$, with $X_i \in \mathfrak{g}$. Let $\varphi = \Phi_*$, then by Proposition 2.2.4.8,

$$\Phi(g) = \Phi_1(\exp(x_1) \cdots \exp(X_m)) = \exp(\phi(X_1)) \cdots \exp(\phi(X_m)).$$

Therefore Φ is determined by φ. □

Corollary 2.2.4.13. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. If G is abelian, then $\mathfrak{g}$ is commutative. The converse holds if G is connected.*

Proof. If G is abelian, then the inverse map $i : G \to G$ is a Lie group homomorphism, therefore by By Exercise 2.1.1,

$$-[X, Y] = i_*[X, Y] = [i_*X, i_*Y] = [di_e(X), di_e(Y)] = [-X, -Y] = [X, Y]$$

for any $X, Y \in \mathfrak{g}$. This implies that $\mathfrak{g}$ is commutative.

Conversely, if G is connected then any element of G can be written as in Corollary 2.2.4.10. Thus by Proposition 2.2.5.1, G abelian. □

2.2.5 The closed subgroup theorem

Recall that in Theorem 2.1.3.8 we showed that a Lie subgroup is embedded if and only if it is closed. In this section, we use the exponential map to prove a much stronger form of that theorem, showing that if a subgroup of a Lie group is topologically a closed subset, then it is actually an embedded Lie subgroup.

We begin with a simple result that shows how group multiplication in G is reflected to first order in the vector space structure of its Lie algebra.

Proposition 2.2.5.1. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $X, Y \in \mathfrak{g}$ be such that $[X, Y] = 0$. Then*

$$(\exp X)(\exp Y) = \exp(X + Y) = (\exp Y)(\exp X).$$

Proof. Let θ_X and θ_Y be the flows determined by X and Y, respectively. Since $[X, Y] = 0$, these two flows commute. Therefore

$$\theta_X(t)\theta_Y(s)\theta_X(-t) = \theta_Y(s).$$

Applying this at $e \in G$ and using Proposition 2.2.4.8(h), we then get

$$(\exp tX)(\exp sY)(\exp(-tX)) = \exp sY.$$

So $\exp tX, \exp sY$ commute for all values of s, t. In particular, this implies $(\exp tX)(\exp tY)$ is a one-parameter subgroup; computing the tangent vector at $t = 0$, we see that

$$(\exp tX)(\exp tY) = \exp(t(X + Y)).$$

Evaluating at $t = 1$ gives the claim. □

Proposition 2.2.5.2. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For any $X, Y \in \mathfrak{g}$, there is a smooth function $Z : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ for some $\varepsilon > 0$ such that the following identity holds for all $t \in (-\varepsilon, \varepsilon)$:*

$$(\exp tX)(\exp tY) = \exp\big(t(X+Y) + t^2 Z(t)\big). \tag{2.2.5.1}$$

Proof. Since the exponential map is a diffeomorphism on some neighborhood of the origin in $\mathfrak{g}$, there is some $\varepsilon > 0$ such that the map $\varphi : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ defined by

$$\varphi(t) = \exp^{-1}(\exp tX \exp tY)$$

is smooth. It obviously satisfies $\varphi(0) = 0$ and

$$\exp tX \exp tY = \exp \varphi(t).$$

Observe that we can write φ as the composition

$$\mathbb{R} \xrightarrow{e_X \times e_Y} G \times G \xrightarrow{m} G \xrightarrow{\exp^{-1}} \mathfrak{g}$$

where $e_X(t) = \exp tX$ and $e_Y(t) = \exp tY$. The result of Exercise 2.1.1 shows that $dm_{(e,e)}(X, Y) = X + Y$ for $X, Y \in T_e G$, which implies

$$\varphi'(0) = \big((d\exp)_0\big)^{-1}(e_X'(0) + e_Y'(0)) = X + Y.$$

Therefore, Taylor's theorem yields

$$\varphi(t) = t(X+Y) + t^2 Z(t)$$

for some smooth function Z. □

Corollary 2.2.5.3. *Under the hypotheses of the preceding proposition,*

$$\lim_{n\to\infty} \left(\left(\exp \frac{t}{n} X\right)\left(\exp \frac{t}{n} Y\right)\right)^n = \exp t(X+Y). \tag{2.2.5.2}$$

Proof. Formula (2.2.5.1) implies that for any $t \in \mathbb{R}$ and any sufficiently large $n \in \mathbb{Z}$,

$$\left(\exp \frac{t}{n} X\right)\left(\exp \frac{t}{n} Y\right) = \exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2} Z\left(\frac{t}{n}\right)\right),$$

and then Proposition 2.2.4.8(d) yields

$$\left(\left(\exp \frac{t}{n} X\right)\left(\exp \frac{t}{n} Y\right)\right)^n = \exp\left(t(X+Y) + \frac{t^2}{n} Z\left(\frac{t}{n}\right)\right).$$

Fixing t and taking the limit as $n \to \infty$, we obtain (2.2.5.2). □

Theorem 2.2.5.4 (Closed Subgroup Theorem). *Suppose G is a Lie group and $H \subseteq G$ is a closed subgroup of G. Then H is an embedded Lie subgroup.*

Proof. By Proposition 2.1.3.1, it suffices to show that H is an embedded submanifold of G. We begin by identifying a subspace of $\mathfrak{Lie}(G)$ that will turn out to be the Lie algebra of H.

Let $\mathfrak{g} = \mathfrak{Lie}(G)$, and define a subset $\mathfrak{h} \subseteq \mathfrak{g}$ by

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

We need to show that $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$. It is obvious from the definition that $\mathfrak{h}$ is closed under scalar multiplication: if $X \in \mathfrak{h}$, then $tX \in \mathfrak{h}$ for all $t \in R$. Suppose $X, Y \in \mathfrak{h}$, and let $t \in \mathbb{R}$ be arbitrary. Then $\exp\big((t/n)X\big)$ and $\exp\big((t/n)Y\big)$ are in H for each positive integer n, and because H is a closed subgroup of G, (2.2.5.2) implies that $\exp t(X+Y) \in H$. Thus $X + Y \in \mathfrak{h}$, so $\mathfrak{h}$ is a subspace.

Next we show that there is a neighborhood U of the origin in $\mathfrak{g}$ on which the exponential map of G is a diffeomorphism, and which has the property that

$$\exp(U \cap \mathfrak{h}) = (\exp U) \cap H. \tag{2.2.5.3}$$

Note that, by the definition of $\mathfrak{h}$ we already have $\exp(U \cap \mathfrak{h}) \subseteq (\exp U) \cap H$, so we only need to show that U can be chosen small enough that $(\exp U) \cap H \subseteq \exp(U \cap \mathfrak{h})$. Assume this is not possible.

Take a vector subspace $\mathfrak{h}'$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Let $\Phi : \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \to G$ be the map

$$\Phi(X, Y) = \exp X \exp Y.$$

By the result of Exercise 2.1.1, $d\Phi_0(X, Y) = X + Y$, so Φ is a diffeomorphism in some neighborhood of $(0, 0)$. Choose neighborhoods U_0 of 0 in $\mathfrak{g}$ and $\widetilde{U}_0$ of $(0, 0)$ in $\mathfrak{h} \oplus \mathfrak{h}'$ such that both $\exp|_{U_0}$ and $\Phi|_{\widetilde{U}_0}$ are diffeomorphisms onto their images. Let $\{U_i\}$ be a countable neighborhood basis for $\mathfrak{g}$ at 0. If we set $V_i = \exp(U_i)$ and $\widetilde{U}_i = \Phi^{-1}(V_i)$, then and are neighborhood bases for G at e and $\mathfrak{h} \oplus \mathfrak{h}'$ at $(0, 0)$, respectively. We may assume that $U_i \subseteq U_0$ and $\widetilde{U}_i \subseteq \widetilde{U}_0$ for each i.

Our assumption implies that for each i, there exists $h_i \in (\exp U_i) \cap H$ such that $h_i \notin \exp(U_i \cap \mathfrak{h})$. This means $h_i = \exp Z_i$ for $Z_i \in U_i$. Because $\exp(U_i) = \Phi(\widetilde{U}_i)$, we can also write

$$h_i = \exp X_i \exp Y_i$$

for $(X_i, Y_i) \in \widetilde{U}_i$. If $Y_i = 0$, then $\exp X_i = \exp Z_i$; but exp is bijective on U_0, this implies $X_i = Z_i \in U_i \cap \mathfrak{h}$, which contradicts our assumption that $h_i \notin \exp(U_i \cap \mathfrak{h})$. Observe that $Y_i \to 0$ and $\exp Y_i = (\exp X_i)^{-1} h_i \in H$.

Choose an inner product on $\mathfrak{h}'$ and let $|\cdot|$ denote the norm associated with this inner product. If we define $c_i = |Y_i|$, then $c_i \to 0$. Consider the sequence $Y_i/|Y_i|$, which lies on the unit sphere in $\mathfrak{h}'$. By replacing it by a subsequence, we may assumeso that $\lim_{n\to\infty} Y_i/|Y_i| = Y$, with $|Y| = 1$ by continuity. We will show that $\exp tY \in H$ for all $t \in \mathbb{R}$, which implies $Y \in \mathfrak{h}$. Since $\mathfrak{h} \cap \mathfrak{h}' = 0$, this is a contradiction.

Let $t \in \mathbb{R}$ and set $n_i = [\frac{t}{c_i}]$, so that

$$\left| n_i - \frac{t}{c_i} \right| \leq 1,$$

Then $|n_i v c_i - t| \leq c_i \to 0$, implies $n_i |Y_i| \to t$. Thus,

$$\lim_{n\to\infty} \exp(n_i Y_i) = \lim_{n\to\infty} \exp(n_i c_i \frac{Y_i}{c_i}) = \exp(tY).$$

But $\exp(n_i Y_i) = (\exp Y_i)^{n_i} \in H$, so the fact that H is closed implies $\exp tY \in H$. This completes the proof of the existence of U satisfying (2.2.5.3).

Choose any linear isomorphism $E : \mathfrak{g} \to \mathbb{R}^m$ that sends $\mathfrak{h}$ to $\mathbb{R}^k$. The composite map $\varphi = E \circ \exp^{-1} : \exp(U) \to \mathbb{R}^m$ is then a smooth chart for G, and $\varphi\big((\exp U) \cap H\big) = E(U \cap \mathfrak{h})$ is the slice obtained by setting the last $m - k$ coordinates equal to zero. Moreover, if $h \in H$ is arbitrary, the left translation map L_h is a diffeomorphism from $\exp U$ to a neighborhood of h. Since H is a subgroup, $L_h(H) = H$, and so

$$L_h\big((\exp U) \cap H\big) = L_h(\exp U) \cap H.$$

and $\varphi \circ L_h^{-1}$ is easily seen to be a slice chart for H in a neighborhood of h. Thus, H is an embedded submanifold of G, hence a Lie subgroup. □

Corollary 2.2.5.5. *If G is a Lie group and H is any subgroup of G, the following are equivalent:*

(i) *H is closed in G.*

(ii) *H is an embedded submanifold of G.*

(iii) *H is an embedded Lie subgroup of G.*

Proposition 2.2.5.6. *Every continuous homomorphism of Lie groups is smooth.*

Proof. Let $\varphi : G \to H$ be a continuous homomorphism, then

$$\Gamma_\varphi = \{(g, \varphi(g)) : g \in G\}$$

is a closed subgroup, and hence a Lie subgroup of $G \times H$. Consider the projection $p : \Gamma_\varphi \to G$ given by the compositoin:

$$\Gamma_\varphi \hookrightarrow G \times H \xrightarrow{\pi_G} G$$

Then p is a bijective, smooth Lie group homomorphism, and a diffeomorphism by Corollary 2.1.1.5. Thus $\varphi = \pi_2 \circ p^{-1}$ is smooth. □

Corollary 2.2.5.7. *Let G be a Lie group, then with the given topology, there is only one smooth structure that makes G into a Lie group.*

Proof. Let $\widetilde{G}$ denote the same set G with the same topology, but a different smooth structure such that G is a Lie group. Then the identity map $\mathrm{id} : G \to \widetilde{G}$ is a continuous homomorphism, hence smooth by Proposition 2.2.5.6. Similarly, the map $\mathrm{id}^{-1} : \widetilde{G} \to G$ is also smooth, thus id is a diffeomorphism. This implies the smooth structure on G and $\widetilde{G}$ coincide. ☐

Now suppose S is a Lie subgroup of G, which is only immersed by our definition. Then by Proposition **??** the closure $\bar{S}$ is also a subgroup, and Theorem 2.2.5.4 says it is embedded. Thus we have the following result.

Proposition 2.2.5.8. *Suppose G is a Lie group, then every Lie subgroup of G is either a properly embedded submanifold of G, or a dense subset of a properly embedded submanifold.*

2.2.6 Infinitesimal generators of group actions

We have showed that a complete vector field on a manifold generates an action of $\mathbb{R}$ on the manifold. In this part, using the Frobenius theorem and properties of the exponential map, we show how to generalize this notion to actions of higher-dimensional groups.

To begin, we need to specify what we mean by an "infinitesimal generator" of a Lie group action. For reasons that will become apparent, in this section we work primarily with right actions. Because $\mathbb{R}$ is abelian, global flows can be considered either as left actions or as right actions, so everything in this part applies to global flows without modification.

Suppose we are given a smooth right action of a Lie group G on a smooth manifold M; which we denote either by $\theta : M \times G \to M$ or $(p, g) \mapsto p \cdot g$, depending on context. Each element $X \in \mathfrak{Lie}(G)$ determines a smooth global flow on M:

$$(t, p) \mapsto p \cdot \exp tX.$$

Let $\widehat{X} \in \mathfrak{X}(M)$ be the infinitesimal generator of this flow, so for each $p \in M$,

$$\widehat{X}_p = \frac{d}{dt}\Big|_{t=0} p \cdot \exp tX. \tag{2.2.6.1}$$

Thus we obtain a map $\widehat{\theta} : \mathfrak{Lie}(G) \to \mathfrak{X}(M)$, defined by $\widehat{\theta}(X) = \widehat{X}$.

There is a useful alternative characterization of $\widehat{X}$ in terms of the orbit map $\theta^{(p)} : G \to M$ defined by $\theta^{(p)}(g) = p \cdot g$. Since $\gamma(t) = \exp tX$ is a smooth curve in G whose initial velocity is $\gamma'(0) = X_e$, it follows from **??** that for each $p \in M$ we have

$$d(\theta^{(p)})_e(X_e) = (\theta^{(p)} \circ \gamma)' = \frac{d}{dt}\Big|_{t=0} p \cdot \exp tX = \widehat{X}_p. \tag{2.2.6.2}$$

The following result is thus immediate.

Proposition 2.2.6.1. *Suppose G is a Lie group and θ is a smooth right action of G on a smooth manifold M. For any $X \in \mathfrak{Lie}(G)$ and $p \in M$, the vector fields X and $\widehat{X}$ are $\theta^{(p)}$-related.*

Proof. Let $X \in \mathfrak{Lie}(G)$ and $p \in M$ be arbitrary, and write $\widehat{X} = \widehat{\theta}(X)$. Note that the group law $p \cdot gg' = (p \cdot p) \cdot p'$ translates to

$$\theta^{(p)} \circ L_g(g') = \theta^{(p \cdot g)}(g') \tag{2.2.6.3}$$

Let $g \in G$ be arbitrary, and write $q = \theta^{(p)}(g)$. Then (2.2.6.1) yields $\theta^{(p)} \circ L_g = \theta^{(q)}$. Using this together with (2.2.6.1) and the fact that X is left-invariant, we obtain

$$X_q = d(\theta^{(q)})_e(X_e) = d(\theta^{(p)} \circ L_g)_e(X_e) = d(\theta^{(p)})_g \circ d(L_g)_e(X_e) = d(\theta^{(p)})_g(X_g).$$

which proves the claim. ☐

Proposition 2.2.6.2. *Suppose G is a Lie group and θ is a smooth right action of G on a smooth manifold M. Then the map $\widehat{\theta} : \mathfrak{Lie}(G) \to \mathfrak{X}(M)$ defined above is a Lie algebra homomorphism.*

Proof. For each $p \in M$, it follows from (2.2.6.2) that $\widehat{X}_p$ depends linearly on X, so $\widehat{\theta}$ is a linear map. Given $p \in M$, Proposition 2.2.6.1 together with the naturality of Lie brackets implies that $[X, Y]$ is $\theta^{(p)}$-related to $[\widehat{X}, \widehat{Y}]$. This means, in particular, that

$$[\widehat{X}, \widehat{Y}]_p = d(\theta^{(p)})_e([X, Y]_e) = \widehat{[X, Y]}_e.$$

Since every point of M is in the image of some orbit map, we conclude that $[\widehat{\theta}(X), \widehat{\theta}(Y)] = \widehat{\theta}([X, Y])$, as claimed. □

The Lie algebra homomorphism $\widehat{\theta} : \mathfrak{Lie}(G) \to \mathfrak{X}(M)$ defined above is known as the **infinitesimal generator** of θ. More generally, if $\mathfrak{g}$ is an arbitrary finitedimensional Lie algebra, any Lie algebra homomorphism $\widehat{\theta} : \mathfrak{g} \to \mathfrak{X}(M)$ is called a **(right) $\mathfrak{g}$-action** on M. A $\mathfrak{g}$-action $\widehat{\theta}$ is said to be **complete** if for every $X \in \mathfrak{g}$, the vector field $\widehat{\theta}(X)$ is complete.

Just as every complete vector field generates an $\mathbb{R}$-action, the next theorem shows that, at least for simply connected groups, every complete Lie algebra action generates a Lie group action.

Theorem 2.2.6.3 (**Fundamental Theorem on Lie Algebra Actions**). *Let M be a smooth manifold, let G be a simply connected Lie group, and let $\mathfrak{g} = \mathfrak{Lie}(G)$. Suppose $\widehat{\theta} : \mathfrak{g} \to \mathfrak{X}(M)$ is a complete $\mathfrak{g}$-action on M. Then there is a unique smooth right G-action on M whose infinitesimal generator is $\widehat{\theta}$.*

Proof. We begin by defining a distribution D on $G \times M$; we will show that D is involutive, and then each leaf will turn out to be the graph of an orbit map $\theta^{(p)} : G \to M$. For brevity, given $X \in g$, we use the notation $\widehat{X}$ for $\widehat{\theta}(X) \in \mathfrak{X}(M)$.

Define D as follows: for each $X \in \mathfrak{g}$, define a smooth vector field $\widetilde{X}$ on $G \times M$ by

$$\widetilde{X}_{(g,p)} = (X_g, \widehat{X}_p) \in T_g G \oplus T_p M \cong T_{(g,p)}(G \times M).$$

In the notation of Exercise 1.1.8, this is $X \oplus \widehat{X}$. Then for each $(g, p) \in G \times M$, let $D_{(g,p)}$ be the set of all vectors of the form $\widetilde{X}$ as X ranges over $\mathfrak{g}$. If $X_1, \ldots, X_k$ is a basis for $\mathfrak{g}$, then the smooth vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_k$ are independent and span D, so D is a smooth distribution whose rank is equal to the dimension of G. To see that it is involutive, note that Exercise 1.1.8 and the fact that $\widetilde{\theta}$ is a Lie algebra homomorphism imply

$$[\widetilde{X}_i, \widetilde{X}_j] = [X_i \oplus \widehat{X}_i, X_j \oplus \widehat{X}_j] = [X_i, X_j] \oplus [\widehat{X}_i, \widehat{X}_j] = [X_i, X_j] \oplus \widehat{[X_i, X_j]} = \widetilde{[X_i, X_j]}.$$

Let $\mathcal{S}$ denote the foliation determined by D, and for each $(g, p) \in G \times M$ let $\mathcal{S}_{(g,p)}$ denote the leaf of $\mathcal{S}$ containing (g, p).

Next we show that D is invariant under a certain G-action on $G \times M$. Combining the natural action of G on itself by left translation with the trivial action of G on M, we get a left action of G on $G \times M$ given by

$$\psi_g(g', p) = (gg', p).$$

A straightforward computation shows

$$d(\psi_g)_{(g',p)}(\widetilde{X}_{(g',p)}) = d(\psi_g)_{(g',p)}(X_{g'}, \widehat{X}_p) = (d(L_g)_{g'}(X_{g'}), \widehat{X}_p) = (X_{gg'}, \widehat{X}_p) = \widetilde{X}_{(gg',p)}$$

so D is invariant under g for each $g \in G$. It follows that g takes leaves of $\mathcal{S}$ to leaves of $\mathcal{S}$.

Let $\pi_G : G \times M \to G$ and $\pi_M : G \times M \to M$ denote the projections. Let $p \in M$ be arbitrary, let $\mathcal{S}_p = \mathcal{S}_{(e,p)} \subseteq G \times M$ denote the leaf containing (e, p), and let $\Pi_p = \pi_G|_{\mathcal{S}_p} : \mathcal{S}_p \to G$. We will show that Π_p is a smooth covering map. To begin with, at each point $(g, q) \in \mathcal{S}_p$, $d(\Pi_p)_{(g,q)}(\widetilde{X}_{(g,q)}) = X_g$ for all $X \in \mathfrak{g}$, so Π_p is a smooth submersion, and for dimensional reasons it is a local diffeomorphism.

To show that Π_p is a covering map, choose a connected neighborhood U of e in G small enough that the exponential map of G is a diffeomorphism from some neighborhood V of 0 in $\mathfrak{g}$ onto U, and for any $g \in G$, consider the neighborhood gU of g. We will show that gU is evenly covered by constructing local sections. For each $q \in M$ such that (g, q) is in the fiber $\Pi_p^{-1}(g)$, define a map $\sigma_q : gU \to G \times M$ by

$$\sigma_q(g \exp X) = (g \exp X, \eta_{\widehat{X}}(1, q)),$$

where $X \in V$ and $\eta_{\widehat{X}}$ denotes the flow of $\widehat{X}$. It follows immediately from the definition that σ_q is smooth and satisfies $\pi_G \circ \sigma_q = \mathrm{id}_{gU}$, so to show that σ_q is a local section of Π_p, it suffices to show that it takes its values in $\mathcal{S}_p$. A straightforward computation shows that $\gamma(t) = (g \exp tX, \eta_{\widehat{X}}(t,q))$ is an integral curve of $\widehat{X}$ starting at (g,q), from which it follows easily that $\sigma_q(g \exp X) = \gamma(1) \in \mathcal{S}_p$. It is smooth because it is a local section of the local diffeomorphism.

For each $(g,q) \in \Pi_p^{-1}(g)$, the set $\sigma_q(gU)$ is a connected open subset of $\mathcal{S}_p$, which is mapped diffeomorphically onto gU by Π_p. To complete the proof that Π_p is a covering map, we need only prove that every point in $\Pi_p^{-1}(g)$ is in exactly one such set. First suppose $(g',q') \in \Pi_p^{-1}(gU)$. Then $\Pi_p(g',q') \in gU$ means that $g' = g \exp X$ for some $X \in V$. If we let $q = \eta_{\widehat{X}}(-1,q')$, then the group law for $\eta_{\widehat{X}}$ implies that $q' = \eta_{\widehat{X}}(1,q)$ and therefore $(g',q') = \sigma_q(g \exp X)$. On the other hand, suppose two such sets $\sigma_q(gU)$ and $\sigma_{q'}(gU)$ intersect nontrivially. Then for some $X, X' \in V$, we have

$$(g \exp X, \eta_{\widehat{X}}(1,q)) = (g \exp X', \eta_{\widehat{X'}}(1,q'))$$

which implies that $X = X'$ and therefore $\eta_{\widehat{X}}(1,q) = \eta_{\widehat{X}}(1,q')$; then flowing back along the integral curve of $\widehat{X}$ for time -1 shows that $q = q'$. This completes the proof that Π_p is a smooth covering map. Because we are assuming G is simply connected, Π_p is actually a diffeomorphism.

Now for each $p \in M$, define $\theta^{(p)} : G \to M$ by $\theta^{(p)}(g) = \pi_M \circ \Pi_p^{-1}$ (so $\mathcal{S}_p$ is the graph of $\theta^{(p)}$), and define an action of G on M by $p \cdot g = \theta^{(p)}(g)$. This is equivalent to declaring that $p \cdot g = q$ if and only if $\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)}$. To show that this is an action, assuming $p \cdot g = q$ and $q \cdot g' = r$, we need to show that $p \cdot gg' = r$.

Equivalently, assuming that $\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)}$ and $\mathcal{S}_{(e,q)} = \mathcal{S}_{(g',r)}$, we need to show that $\mathcal{S}_{(e,p)} = \mathcal{S}_{(gg',r)}$. This follows from ψ-invariance:

$$\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)} = \psi_g(\mathcal{S}_{e,q}) = \psi_g(\mathcal{S}_{g',r}) = \mathcal{S}_{gg',r}.$$

It remains to show that the action is smooth, that $\widehat{\theta}$ is its infinitesimal generator, and that it is the unique such action. For $g = \exp X$ near the identity, the discussion above shows that the action can be expressed as

$$p \cdot g = \theta^{(p)}(\exp X) = \pi_M \circ \sigma_p(\exp X) = \eta_{\widehat{X}}(1,p). \tag{2.2.6.4}$$

An argument analogous to the one we used to prove smoothness of the exponential map (with $\Xi_{(p,x)} = (\widehat{X}, 0)$ on $M \times \mathfrak{g}$) shows that this depends smoothly on X and p and thus on g and p. But since any neighborhood of the identity generates G (Proposition 2.2.4.10), every element of G can be expressed as a finite product of elements of the form $\exp X$ for $X \in V$, so it follows that $(p,g) \mapsto p \cdot g$ can be written as a finite composition of smooth maps. The fact that the infinitesimal generator of the action is $\widehat{\theta}$ is an immediate consequence of (2.2.6.4). Uniqueness is clear from our construction, so the proof is completed. □

2.2.7 The Lie correspondence

Many of our results about Lie groups show how essential properties of a Lie group are reflected in its Lie algebra, and vice versa. This raises a natural question: To what extent is the correspondence between Lie groups and Lie algebras (or at least between their isomorphism classes) one-to-one? We have already seen that the assignment that sends a Lie group to its Lie algebra and a Lie group homomorphism to its induced Lie algebra homomorphism is a functor from the category of Lie groups to the category of finite-dimensional Lie algebras. Because functors take isomorphisms to isomorphisms, it follows that isomorphic Lie groups have isomorphic Lie algebras. The converse is easily seen to be false: both $\mathbb{R}^n$ and T^n have n-dimensional abelian Lie algebras, which are obviously isomorphic to each other, but $\mathbb{R}^n$ and T^n are certainly not isomorphic Lie groups. However, as we will see in this part, if we restrict our attention to simply connected Lie groups, then we do obtain a one-toone correspondence.

In order to prove this correspondence, we need a way to construct an isomorphism between simply connected Lie groups when we are given an isomorphism between their algebras. Theorem 2.2.2.1 showed that every Lie group homomorphism gives rise to a Lie algebra homomorphism. Using the fundamental theorem on Lie algebra actions, we can prove the following partial converse.

Theorem 2.2.7.1. *Suppose G and H are Lie groups with G simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$, there is a unique Lie group homomorphism $\Phi : G \to H$ such that $\varphi = \Phi_*$.*

Proof. The Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ is, in particular, a complete $\mathfrak{g}$-action on H (since every left-invariant vector field is complete). Thus, by Theorem 2.2.6.3, there is a unique smooth right G-action $\theta : H \times G \to H$ for which φ is the infinitesimal generator. Let us use the notation $\widehat{X} = \varphi(X)$ for $X \in \mathfrak{g}$. Define a smooth map $\Phi : G \to H$ by $\Phi(g) = \theta(e, g)$ (where e is the identity in H). We will show that Φ is the desired homomorphism.

Proposition 2.2.6.1 shows that for each $h \in H$ and each $X \in g$, the vector fields X and $\widehat{X}$ are $\theta^{(h)}$-related. By Proposition 1.2.1.5, $\theta^{(h)}$ takes integral curves of X to integral curves of $\widehat{X}$. Therefore, $\gamma(t) = \theta(h, \exp tX)$ is the integral curve of $\widehat{X}$ starting at h. On the other hand, because $\widehat{X}$ is a left-invariant vector field on H, left translation in H takes integral curves of $\widehat{X}$ to integral curves of $\widehat{X}$. For any $h \in H$ and $X \in \mathfrak{g}$, therefore,

$$h\theta(e, \exp tX) = \theta(h, \exp tX). \tag{2.2.7.1}$$

Applying this with $h = \theta(e, g)$ for some $g \in G$, we get

$$\theta(e, g)\theta(e, \exp tX) = \theta(\theta(e, g), \exp tX) = \theta(e, g \exp tX)$$

(The last equality follows from the fact that θ is an action.) Rewritten in terms of Φ, this says

$$\Phi(g)\Phi(\exp tX) = \Phi(g \exp tX).$$

Since G is connected, it is generated by the image of the exponential map, so this implies that Φ is a homomorphism.

To see that $\varphi = \Phi_*$, let $X \in g$ be arbitrary. The fact that φ is the infinitesimal generator of θ means

$$\varphi(X)_e = \frac{d}{dt}\Big|_0 (e \cdot \exp tX) = \frac{d}{dt}\Big|_{t=0} \Phi(\exp tX) = d\Phi_e(X_e).$$

Since Φ_* is determined by the action of $d\Phi_e$, this implies $\Phi_* = \varphi$.

The proof is completed by invoking Proposition 2.2.4.12, which shows that Φ is the unique homomorphism with this property. □

Corollary 2.2.7.2. *If G and H are simply connected Lie groups with isomorphic Lie algebras, then G and H are isomorphic.*

Proof. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H, respectively, and let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra isomorphism between them. By the preceding theorem, there are Lie group homomorphisms $\Phi : G \to H$ and $\Psi : H \to G$ satisfying $\Phi_* = \varphi$ and $\Psi_* = \varphi^{-1}$. Both the identity map of G and the composition $\Psi \circ \Phi$ are Lie group homomorphisms from G to itself whose induced Lie algebra homomorphisms are equal to the identity, so the uniqueness part of Theorem 2.2.7.1 implies that $\Psi \circ \Phi = \mathrm{id}_G$. Similarly, $\Phi \circ \Psi = \mathrm{id}_H$, so Φ is a Lie group isomorphism. □

Now we are ready for our main theorem.

Theorem 2.2.7.3 (The Lie Correspondence). *There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups, given by associating each simply connected Lie group with its Lie algebra.*

Proof. We need to show that the functor that sends a simply connected Lie group to its Lie algebra is both surjective and injective up to isomorphism. Injectivity is precisely the content of Corollary 2.2.7.2.

To prove surjectivity, suppose $\mathfrak{g}$ is any finite-dimensional Lie algebra. By Ado's theorem, we may replace $\mathfrak{g}$ by an isomorphic Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{gl}_n(\mathbb{R})$. By Theorem 2.2.3.5, there is a connected Lie subgroup $G_0 \subseteq \mathrm{GL}_n(\mathbb{R})$ that has $\mathfrak{g}_0$ as its Lie algebra. If G is the universal covering group of G_0, then Proposition 2.1.2.1 shows that $\mathfrak{Lie}(G) \cong \mathfrak{Lie}(G_0) \cong \mathfrak{g}_0$. □

Corollary 2.2.7.4. *The categories of finite-dimensional Lie algebras and connected, simply-connected Lie groups are equivalent.*

2.2.8 The adjoint representation

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For any $g \in G$, the conjugation map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$ is a Lie group homomorphism. We let $\mathrm{Ad}_g = (C_g)_* : \mathfrak{g} \to \mathfrak{g}$ denote its induced Lie algebra homomorphism.

Proposition 2.2.8.1 (The Adjoint Representation). *If G is a Lie group with Lie algebra $\mathfrak{g}$, the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a Lie group representation, called the **adjoint representation** of G.*

Proof. Because $C_{g_1 g_2} = C_{g_1} \circ C_{g_2}$ for any $g_1, g_2 \in G$, it follows immediately that $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$, and Ad_g is invertible with inverse $\mathrm{Ad}_{g^{-1}}$.

To see that Ad is smooth, let $C : G \times G \to G$ be the smooth map defined by $C(g, h) = ghg^{-1}$. Let $X \in \mathfrak{g}$ and $g \in G$ be arbitrary. Then $\mathrm{Ad}_g X$ is the left-invariant vector field whose value at $e \in G$ is

$$((C_g)_* X)_e = d(C_g)_e(X_e) = \frac{d}{dt}\Big|_{t=0} C_g(\exp tX) = \frac{d}{dt}\Big|_{t=0} C(g, \exp tX) = dC_{(g,e)}(0, X_e).$$

Because $dC : T(G \times G) \to TG$ is a smooth bundle homomorphism, this expression depends smoothly on g and X. Smooth coordinates on $\mathrm{GL}(\mathfrak{g})$ are obtained by choosing a basis (E_i) for $\mathfrak{g}$ and using matrix entries with respect to this basis as coordinates. If (ε^j) is the dual basis, the matrix entries of $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ are given by $(\mathrm{Ad}_g)_i^j = \varepsilon^j(\mathrm{Ad}_g E_i)$. The computation above with $X = E_i$ shows that these are smooth functions of g. ☐

There is also an adjoint representation for Lie algebras. Given a finite-dimensional Lie algebra $\mathfrak{g}$, for each $X \in \mathfrak{g}$, define a map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_X(Y) = [X, Y]$.

Proposition 2.2.8.2. *The map ad_X is a derivation of the bracket:*

$$\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)].$$

Proof. The Jacobi identity can be written into the following form:

$$\mathrm{ad}_X([Y, Z]) = [X, [Y, Z]] = -[[Y, Z], X] = [[Z, X], Y] + [[X, Y], Z] = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$$

thus the claim follows. ☐

Proposition 2.2.8.3. *For any Lie algebra $\mathfrak{g}$, the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation, called the **adjoint representation** of $\mathfrak{g}$.*

Proof. Let $X, Y, Z \in \mathfrak{g}$, we check that

$$\begin{aligned} \mathrm{ad}_{[X,Y]}(Z) &= [[X, Y], Z] = -[[Y, Z], X] - [[Z, X], Y] = [X, [Y, Z]] - [Y, [X, Z]] \\ &= \mathrm{ad}_X \mathrm{ad}_Y(Z) - \mathrm{ad}_Y \mathrm{ad}_X(Z). \end{aligned}$$

Therefore $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y]$. That is, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism. ☐

Using the exponential map, we can show that these two representations are intimately related.

Theorem 2.2.8.4. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra, and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G. The induced Lie algebra representation $\mathrm{Ad}_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by* ad.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \end{tikzcd}$$

Proof. Let $X \in \mathfrak{g}$ be arbitrary. Then $\mathrm{Ad}_* X$ is determined by its value at the identity, which we can interpret as an element of $\mathfrak{gl}(\mathfrak{g})$. Because $t \mapsto \exp tX$ is a smooth curve in G whose velocity vector at $t = 0$ is X_e, we can compute the action of $\mathrm{Ad}_* X$ on an element $Y \in \mathfrak{g}$ by

$$(\mathrm{Ad}_* X)_e Y = \big(d(\mathrm{Ad})_e(X_e)\big) Y = \Big(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp tX}\Big) Y = \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}_{\exp tX} Y\big).$$

As an element of $\mathfrak{g}$, $\mathrm{Ad}_{\exp tX}Y$ is a left-invariant vector field on G, and thus is itself determined by its value at the identity. Using the fact that $\mathrm{Ad}_g = (C_g)_* = (R_{g^{-1}})_* \circ (L_g)_*$, its value at $e \in G$ can be computed as

$$\begin{aligned}\left(\mathrm{Ad}_{\exp tX}Y\right)_e &= \left((R_{\exp(-tX)})_* \circ (L_{\exp tX})_*(Y)\right)_e\\ &= d(R_{\exp(-tX)}) \circ d(L_{\exp tX})(Y_e)\\ &= d(R_{\exp(-tX)})(Y_{\exp tX}).\end{aligned}$$

where we use the left-invariance of Y. Recall that the flow of X is given by $\theta_t(g) = R_{\exp tX}(g)$. Therefore,

$$\left(\mathrm{Ad}_{\exp tX}Y\right)_e = d(\theta_{-t})(Y_{\theta_t(e)}).$$

Taking the derivative with respect to t and setting $t = 0$, we obtain

$$\left((\mathrm{Ad}_* X)_e Y\right)_e = \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})(Y_{\theta_t(e)}) = (\mathfrak{L}_X Y)_e = [X, Y]_e.$$

Since $(\mathrm{Ad}_* X)_e Y$ is determined by its value at e, this completes the proof. ☐

2.2.9 Ideals and normal subgroups

Let G be a connected Lie group. Then the image of exp generates G by Proposition **??**. Therefore, for Lie groups, the normality of a Lie subgroup can be detected by its Lie subalgebra: we have the following criterion.

Lemma 2.2.9.1. *Let G be a connected Lie group, and let $H \subseteq G$ be a connected Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Then H is normal in G if and only if*

$$(\exp X)(\exp Y)(\exp(-X)) \in H \quad \textit{for all } X \in \mathfrak{g} \textit{ and } Y \in \mathfrak{h}. \tag{2.2.9.1}$$

Proof. Note that $\exp(-X) = (\exp X)^{-1}$. Thus if H is normal, then (2.2.9.1) holds by definition. Conversely, suppose (2.2.9.1) holds, and choose open subsets $V \subseteq \mathfrak{g}$ containing 0 and $U \subseteq G$ containing the identity such that $\exp : V \to U$ is a diffeomorphism. Since the exponential map of H is the restriction of that of G, after shrinking V if necessary, we may assume that the restriction of exp to $V \cap \mathfrak{h}$ is a diffeomorphism from $V \cap \mathfrak{h}$ to a neighborhood U_0 of the identity in H. Shrinking V still further, we may assume also that $X \in V$ if and only if $-X \in V$. Then (2.2.9.1) implies that $ghg^{-1} \in H$ whenever $g \in U$ and $h \in U_0$.

Since H is generated by U_0, it follows that for any $g \in U$ and $h \in H$ we have

$$ghg^{-1} = gh_1 \cdots h_m g^{-1} = (gh_1g^{-1}) \cdots (gh_mg^{-1}) \in H.$$

Similarly, any $g \in G$ can be written $g_1 \cdots g_k$ with $g_1, \ldots, g_k \in U$, so it follows by induction on k that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. ☐

The following theorem is an immediate consequence of the lemma above.

Theorem 2.2.9.2 (Ideals and Normal Subgroups). *Let G be a connected Lie group, and suppose $H \subseteq G$ is a connected Lie subgroup. Then H is a normal subgroup of G if and only if $\mathfrak{Lie}(H)$ is an ideal in $\mathfrak{Lie}(G)$.*

Proof. Write $g \in \mathfrak{Lie}(G)$ and $h = \mathfrak{Lie}(H)$, considering $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$. For any $g \in G$, the commutative diagram in Proposition 2.2.4.8(g) applied to the Lie group homomorphism $C_g(h) = ghg^{-1}$ yields

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "C_g"] & G \end{tikzcd} \tag{2.2.9.2}$$

Suppose that $\mathfrak{h}$ is an ideal. Applying this to $Y \in \mathfrak{h}$ with $g = \exp X$, we obtain

$$\exp(\mathrm{Ad}_{\exp X} Y) = C_{\exp X}(\exp Y) = (\exp X)(\exp Y)(\exp(-X)).$$

We prove that H is noraml by showing the left side is in H. In fact, by Proposition 2.2.4.8(g) agian we have

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\
G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g})
\end{tikzcd}$$

Formula (2.2.4.1) for the exponential map of the group $\mathrm{GL}(\mathfrak{g})$ reads

$$\mathrm{Ad}_{\exp X} Y = (\exp(\mathrm{ad}_X))Y = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}_X)^k}{k!} Y.$$

Whenever $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, we have $(\mathrm{ad}_X)Y = [X, Y] \in \mathfrak{h}$, and by induction $(\mathrm{ad}_X)^k Y \in \mathfrak{h}$ for all k. Therefore, $\mathrm{Ad}_{\exp X} Y \in \mathfrak{h}$, and so $\exp(\mathrm{Ad}_{\exp X} Y) \in H$.

Conversely, suppose H is normal. Given $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, note that (2.2.9.2) applied to sY with $g = \exp tX$ implies

$$\exp(\mathrm{Ad}_{\exp tX}(sY)) = (\exp tX)(\exp sY)(\exp(-tX)) \in H.$$

Since $\mathrm{Ad}_{\exp tX}$ is linear over $\mathbb{R}$, it follows that for all $s \in \mathbb{R}$,

$$\exp(s\mathrm{Ad}_{\exp tX} Y) = \exp(\mathrm{Ad}_{\exp sX}(sY)) \in H,$$

so $\mathrm{Ad}_{\exp tX} Y \in \mathfrak{h}$ by Proposition 2.2.4.9. From the proof of Theorem 2.2.8.4, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp tX} Y = [X, Y],$$

and therefore $[X, Y] \in \mathfrak{h}$, so $\mathfrak{h}$ is an ideal. □

2.3 Quotient manifolds

In this section we consider groups arcting on manifolds and the orbit space of it. We will find a sufficient condition for a group action under which the orbit space is again a smooth manifolds. Finally we will apply this to covering manifolds.

2.3.1 Proper actions

Lemma 2.3.1.1. *Let G be a group acting by homeomorphisms on a topological space X, and let $\mathcal{O} \subseteq X \times X$ be the subset defined by*

$$\mathcal{O} = \{(x_1, x_2) \in X \times X : x_1 = g \cdot x_2 \text{ for some } g \in G\}$$

*It is called the **orbit relation** because $(x_1, x_2) \in \mathcal{O}$ if and only if x_1 and x_2 are in the same orbit.*

(a) The quotient map $X \to X/G$ is an open map.

(b) X/G is Hausdorff if and only if $\mathcal{O}$ is closed in $X \times X$.

Proof. (a) Let π be the quotient map, and let U be a non-empty open set in X, then $\pi(U)$ is open in X/G iff $\pi^{-1}(\pi(U))$ is open in X. But

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

is a union of open sets, since $x \mapsto g \cdot x$ is a homeomorphism. Now (b) follows from Proposition **??**. □

It is easy to construct smooth actions by Lie groups on smooth manifolds whose orbit spaces are themselves manifolds, and others whose orbit spaces are not. Here are a few examples.

Example 2.3.1.2 (Orbit Spaces of Smooth Lie Group Actions).

(a) Let G be any group and let M be any smooth manifold. The trivial action has one-point sets as orbits and $M/G = M$, so the orbit space is a smooth manifold for silly reasons.

(b) The simplest nontrivial example to keep in mind is the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times \mathbb{R}^n$ by translation in the $\mathbb{R}^k$ factor: $v \cdot (x+y) = (x+v, y)$. The orbits are the affine subspaces parallel to $\mathbb{R}^k$, and the orbit space $(\mathbb{R}^k \times \mathbb{R}^n)/\mathbb{R}^k$ is homeomorphic to $\mathbb{R}^n$. The quotient map $\pi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth submersion.

(c) The circle group S^1 acts on the plane $\mathbb{C}$ by complex multiplication: $z \cdot w = zw$. The orbits are circles centered at the origin and the singleton $\{0\}$. The orbit space is homeomorphic to $[0, \infty)$. Thus the orbit space is not a manifold.

(d) An even more dramatic example of how an orbit space can fail to be a manifold is given by the natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ by matrix multiplication. In this case, there are two orbits, $\{0\}$ and $\mathbb{R}^n - \{0\}$, and the only open subsets in the quotient topology are the empty set, the whole set, and the singleton $\{[\mathbb{R}^n - \{0\}]\}$. This orbit space is not even Hausdorff, let alone a manifold.

(e) The restriction of the natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ to $\mathrm{O}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ defines a smooth left action of $\mathrm{O}(n)$ on $\mathbb{R}^n$. In this case, the orbits are the origin and the spheres centered at the origin. As in (c), the orbit space is homeomorphic to $[0, \infty)$.

(f) If we delete the origin from each of the three preceding examples, we obtain orbit spaces that are manifolds: the quotient of $\mathbb{C} - \{0\}$ by S^1 is homeomorphic to $\mathbb{R}^+$, as is the quotient of $\mathbb{R}^n - \{0\}$ by $\mathrm{O}(n)$; and the quotient of $\mathbb{R}^n - \{0\}$ by $\mathrm{GL}_n(\mathbb{R})$ is a single point.

(g) Further restricting the natural action to $\mathrm{O}(n) \times S^{n-1} \to S^{n-1}$, we obtain an action of $\mathrm{O}(n)$ on S^{n-1}. It is smooth by Corollary **??**, because S^{n-2} is an embedded submanifold of $\mathbb{R}^n$. This action is transitive, so the quotient space is a singleton.

In (c), (d), and (e) above, the problematic point is the origin. In each case, the origin is the only point that is fixed by every element of the group. Recall that we defined a free action to be one for which every isotropy group is trivial. In the examples above, the action of $\mathbb{R}^k$ on $\mathbb{R}^n$ in part (b) is free, as is the action of S^1 on $\mathbb{C}^\times$ described in (f); the other examples are not. Of course, freeness is not necessary for an action to have a smooth manifold quotient, as (a) shows. Nor is it sufficient by itself, as the next example shows.

Example 2.3.1.3. Let α be an irrational number, and let $\mathbb{R}$ act on T^2 by

$$t \cdot (w, z) = (e^{2\pi i t} w, e^{2\pi i \alpha t} w).$$

This is a smooth action, and it is free and has dense orbits. This means that the only saturated open subsets of T^2 are $\varnothing$ and T^2, so the orbit space $T^2/\mathbb{R}$ has the trivial topology. In particular, it is not Hausdorff and therefore not a manifold.

To avoid pathological cases such as these, we need to introduce one more restriction on our group actions. A continuous action of a topological group G on a topological space E is said to be a **proper action** if the continuous map $\Theta : G \times E \to E \times E$ defined by

$$\Theta(g, e) = (g \cdot e, e) \tag{2.3.1.1}$$

is a proper map. It should be noted that this is a weaker condition than requiring the map $G \times E \to E$ defining the group action to be a proper map.

The following result explains our definition.

Proposition 2.3.1.4. *If a Lie group acts continuously and properly on a manifold, then the orbit space is Hausdorff.*

Proof. Let $\mathcal{O} \subseteq M \times M$ be the orbit relation. By Lemma 2.3.1.1 the orbit space is Hausdorff if and only if $\mathcal{O}$ is closed in $M \times M$. But $\mathcal{O}$ is just the image of the map $\Theta : G \times M \to M \times M$ defined by (2.3.1.1). Since M is a locally compact Hausdorff space, the same is true of $M \times M$, so it is compactly generated, and it follows that Θ is a closed map. Thus the orbit relation is closed and M/G is Hausdorff. □

It is not always easy to tell whether a given action is proper. The next proposition gives two alternative characterizations of proper actions that are often useful.

Proposition 2.3.1.5 (Characterizations of Proper Actions). *Let M be a manifold, and let G be a Lie group acting continuously on M. The following are equivalent.*

(i) *The action is proper.*

(ii) *If (p_i) is a sequence in M and (g_i) is a sequence in G such that both (p_i) and $(g_i \cdot p_i)$ converge, then a subsequence of (g_i) converges.*

(iii) *For every compact subset $K \subseteq M$, the set $G_K = \{g \in G : (g \cdot K) \cap K \neq \varnothing\}$ is compact.*

Proof. Let $\Theta : G \times M \to M \times M$ be the map defined by (2.3.1.1). We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).
Suppose first that Θ is proper, and let (p_i), (g_i) be sequences satisfying the hypotheses of (ii). Let U and V be precompact neighborhoods of the points $p = \lim_i p_i$ and $q = \lim_i(g_i \cdot p_i)$, respectively. The assumption means that the points $\Theta(g_i, p_i)$ all lie in the compact set $\overline{V} \times \overline{U}$ when i is large enough. Since $\Theta^{-1}(\overline{V} \times \overline{U})$ is also compact, there is a converging subsequence of (g_i, p_i) in $G \times M$. In particular, this means that a subsequence of (g_i) converges in G, and therefore (ii) holds.

Assume next that (ii) holds, and let K be a compact subset of M. To show that G_K is compact, suppose (g_i) is any sequence of points in G_K. This means that for each i, there exists $p_i \in (g_i \cdot K) \cap K$, which is to say that $p_i \in K$ and $g_i^{-1} \cdot p_i \in K$. After passing to a subsequence, we may assume that (p_i) converges, and then passing to a subsequence of that, we may assume also that $(g_i^{-1} \cdot p_i)$ converges. By (ii), there is a subsequence (g_{i_k}) such that $(g_{i_k}^{-1})$ converges, which implies that (g_{i_k}) also converges. Since each subsequence of G_K has a convergent subsequence, G_K is compact.

Suppose G_K is compact for every compact set $K \subseteq M$. Given a compact subset $L \subseteq M \times M$, let $K = \pi_1(L) \cup \pi_2(L) \subseteq M$, where $\pi_1, \pi_2 : M \times M \to M$ are the projections onto the first and second factors, respectively. Then

$$\Theta^{-1}(L) \subseteq \Theta^{-1}(K \times K) = \{(g, p) : g \cdot p \in K \text{ and } g \in K\} \subseteq G_k \times K$$

Since $M \times M$ is Hausdorff, L is closed in $M \times M$, and so $\Theta^{-1}(L)$ is closed in $G \times M$ by continuity. Thus $\Theta^{-1}(L)$ is a closed subset of the compact set $G_K \times K$ and is therefore compact. □

Corollary 2.3.1.6. *Every continuous action by a compact Lie group on a manifold is proper.*

Proof. If (p_i) and (g_i) are sequences satisfying the hypotheses of Proposition 2.3.1.5(b), then a subsequence of (g_i) converges, for the simple reason that every sequence in G has a convergent subsequence. □

Now we consider the properties of a proper action. These will help us to tell whether an action is not proper.

Proposition 2.3.1.7 (Orbits of Proper Actions). *Suppose θ is a proper smooth action of a Lie group G on a smooth manifold M. For any point $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is a proper map, and thus the orbit $G \cdot p = \theta^{(p)}(G)$ is closed in M. Moreover, the induced map $F : G/G_p \to M$ given by $F(gG_p) = \theta^{(p)}(g)$ is a smooth embedding, so the orbit of p is a properly embedded submanifold.*

Proof. If $K \subseteq M$ is compact, then $(\theta^{(p)})^{-1}(K)$ is closed in G by continuity, and since it is contained in $G_{K \cup \{p\}}$, it is compact by Proposition 2.3.1.5. Therefore, $\theta^{(p)}$ is a proper map, which implies that $G \cdot p = \theta^{(p)}(G)$ is closed. The final statement of the theorem then follows from Proposition 2.1.4.5 and **??**(b). □

The preceding results yield some simple necessary conditions for an action to be proper.

Proposition 2.3.1.8. *If a Lie group G acts properly on a manifold M, then each orbit is a embedded subamnifold of M, and each isotropy group is compact.*

Proof. The first statement follows immediately from Proposition 2.3.1.7, and the second from Proposition 2.3.1.5, using the fact that the isotropy group of a point $p \in M$ is the set G_K for $K = \{p\}$. □

Example 2.3.1.9. We can see in two ways that the action of $\mathbb{R}^+$ on $\mathbb{R}^n$ given by

$$t \cdot (x^1, \ldots, x^n) = (tx^1, \ldots, tx^n)$$

is not proper: the isotropy group of the origin is all of $\mathbb{R}^+$, which is not compact; and the orbits of other points are open rays, which are not closed in $\mathbb{R}^n$.

2.3.2 The quotient manifold theorem

Now we prove that smooth, free, and proper group actions always yield smooth manifolds as orbit spaces. The basic idea of the proof is that if G acts smoothly, freely, and properly on M, the set of orbits forms a foliation of M whose leaves are embedded submanifolds diffeomorphic to G.

Theorem 2.3.2.1 (Quotient Manifold Theorem). *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a fiber bundle with fiber G.*

Proof. Before we get started, let us establish some notation. Throughout the proof, we assume without loss of generality that G acts on the left. Let $\mathfrak{g}$ denote the Lie algebra of G, and write $k = \dim G$, $m = \dim M$, and $n = m - k$. Let $\theta : G \times M \to M$ denote the action and $\Theta : G \times M \to M \times M$ the proper map $\Theta(g, p) = (g \cdot p, p)$.

First, we take care of the easy part: the uniqueness of the smooth structure. Suppose M/G has two different smooth structures such that $\pi : M \to M/G$ is a smooth submersion. Let $(M/G)_1$ and $(M/G)_2$ denote M/G with the first and second smooth structures, respectively. By Theorem **??**, the identity map is smooth from $(M/G)_1$ to $(M/G)_2$:

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "\pi"] & \\ (M/G)_1 \arrow[r, "\mathrm{id}"'] & (M/G)_2 \end{tikzcd}$$

The same argument shows that it is also smooth in the opposite direction, so the two smooth structures are identical; this proves uniqueness.

Next, we construct bundle atlas for M. Let $p \in M$. First we note that, by Proposition 2.3.1.7 the orbit of p is a proper embeddde submanifold, and therefore there exists a slice chart (W, φ) centered at p for $G \cdot p$. Write the coordinate of φ as

$$(x, y) = (x^1, \ldots, x^k, y^1, \ldots, y^n)$$

so that $(G \cdot p) \cap W$ is the slice $\{y^1 = \cdots = y^n = 0\}$, and let S be the the submanifold of W defined by $\{x^1 = \cdots = x^k = 0\}$, then T_pM decompose as the following direct sum:

$$T_pM = T_p(G \cdot p) \oplus T_pS. \tag{2.3.2.1}$$

First let's show that $(d\theta)_{(e,p)}$ is surjective and hence an isomorphism. In fact, let $i : G \to G \times S$ and $j : S \to G \times S$ be the inclusions defined by

$$i(g) = (g, p), \quad j(q) = (e, q).$$

Then $\theta \circ i = \theta^{(p)}$ and $\theta \circ j = \iota_S : S \hookrightarrow M$. Therefore the image of $(d\theta)_{(e,p)}$ contains the subspace $T_p(G \cdot p)$ and T_pS, and by (2.3.2.1) is the whole space T_pM.

With this, by the inverse function theorem there is a neighborhood $X \times Y \subseteq G \times S$ of (e, p) such that $\theta|_{X \times Y}$ is a diffeomorphism into M. We now claim that Y can be chosen small enough that each G-orbit intersects Y in at most a single point.

Assum that this is not the case, then if $\{Y_i\}$ is a countable neighborhood basis for Y at p, for each i there exists distinct points $p_i, p_i' \in Y_i$ that are in the same orbit, namely

$$g_i \cdot p_i = p_i' \quad \text{for some } g_i \in G.$$

By our choice of $\{Y_i\}$, both sequences (p_i) and (p_i') converge to p. Because G acts properly, Proposition 2.3.1.5(b) shows that we may pass to a subsequence and assume that $g_i \to g \in G$. By continuity, therefore,

$$g \cdot p = \lim_i g_i \cdot p_i = \lim p_i' = p.$$

Since G acts freely, this implies $g = e$. This means $g_i \in X$ for i large enough, and then contradicts the fact that $\theta|_{X \times Y}$ is injective.

Now with this shrinked Y, it is clear that the map $\tau = \theta|_{G\times Y} : G \times Y \to M$ is injective. Moreover, from the equality

$$d\theta|_{(e,q)} = d\big(\theta \circ (L_{g^{-1}} \times \mathrm{id})\big)|_{(g,q)} = d\theta|_{e,q} \circ d(L_{g^{-1}} \times \mathrm{id})_{(g,q)}$$

we see that θ is a diffeomorphism from $G \times Y$ to a neighborhood U of M. Endow M/G with its quotient topology, it is clear that $\pi|_Y : Y \to \pi(U)$ is bijective by our choice. Moreover, if $W \subseteq Y$ is an open subset, then

$$\pi(W) = (\pi \circ \tau)(G \times W)$$

which is open in M/G, and thus $\pi : Y \to \pi(U)$ is a homeomorphism. This also shows that M/G is locally Euclidean. Now since π is an open map and the action is proper, it is easy to show that M/G is a topological manifold.

Finally, we need to show that M/G has a smooth structure such that π is a fiber bundle. To do this, we define Φ to be the composition of the maps

$$U \xrightarrow{(\theta|_{G\times Y})^{-1}} G \times Y \xrightarrow{\mathrm{id}_G \times \pi|_Y} G \times \pi(U)$$

then by our previous arguments, $\Phi : U \to G \times Y$ is a homeomorphism such that the action of G is turned into the left multiplication of G on $G \times Y$. This clearly gives a bundle atlas for M, and if (Φ, U, Y) and $(\Psi, \widetilde{U}, \widetilde{Y})$ are different bundle charts with expressions

$$\Phi(p) = (\chi(p), \pi(p)), \quad \Psi(p) = (\chi'(p), \pi(p)),$$

then we have

$$\Phi \circ \Psi^{-1}(g, \pi(p)) = \Phi\big(g \cdot (\pi|_{\widetilde{Y}})^{-1}(\pi(p))\big) = \big(g \cdot \chi\big((\pi|_{\widetilde{Y}})^{-1}(\pi(p))\big), \pi(p)\big).$$

This shows the transition map of two induced charts on M/G are smoothly compactible, and therefore M/G is a smooth manifold, and $\pi : M \to M/G$ is a G-bundle. □

2.3.3 Covering manifolds

Before proceeding, it is useful to have an alternative characterization of properness for free actions of discrete Lie groups.

Lemma 2.3.3.1. *Suppose a discrete Lie group Γ acts continuously and freely on a manifold E. The action is proper if and only if the following conditions both hold:*

(i) *Every point $p \in E$ has a neighborhood U such that for each $g \in \Gamma$, $(g \cdot U) \cap U = \varnothing$ unless $g = e$.*

(ii) *If $p, p' \in E$ are not in the same Γ-orbit, there exist neighborhoods V of p and V' of p' such that $(g \cdot V) \cap V' = \varnothing$ for all $g \in \Gamma$.*

Proof. First, suppose that the action is free and proper, and let $\pi : E \to E/\Gamma$ denote the quotient map. By Proposition 2.3.1.4, E/Γ is Hausdorff. If $p, p' \in E$ are not in the same orbit, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$, then $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the conclusion of condition (ii): any point of $(g \cdot W) \cap W'$ is mapped both in W and W' (by the definition of the quotient map), but $W \cap W' = \varnothing$.

To prove (i), let $p \in E$, then p has a neighborhood V contained in a compact set K. By Proposition 2.3.1.5, the set Γ_K is a compact subset of Γ, and hence finite because Γ is discrete. Write $\Gamma_K = \{e, g_1, \dots, g_m\}$. Since the action is free and E is Hausdorff, for each g_i there are disjoint neighborhoods W_i of p and W_i' of $g_i \cdot p$. Let

$$U = V \cap W_1 \cap (g_1^{-1} \cdot W_1) \cap \cdots \cap W_m \cap (g_m^{-1} \cdot W_m).$$

We will show that U satisfies (i). In fact, if $g = g_i$ for some i, then $q \in U \subseteq g_i^{-1} \cdot W_i'$ implies $g_i \cdot q \in W_i'$, which is disjoint from W_i and therefore from U. On the other hand, if $g \in \Gamma$ is not the identity and not one of the g_i's, then for any $q \in U \subseteq K$, we have $g \cdot p \in g \cdot K$, which is disjoint from K and therefore also from U.

Conversely, assume that (i) and (ii) hold. Suppose (g_i) is a sequence in Γ and (p_i) is a sequence in E such that $p_i \to p$ and $g_i \cdot p_i \to p'$. If p and p' are in different orbits, there exist neighborhoods V of p

and V' of p' as in (ii); but for large enough i, we have $p_i \in V$ and $g_i \cdot p_i \in V'$, which contradicts the fact that $(g \cdot U) \cap U' = \varnothing$ for all $g \in \Gamma$. Thus, p and p' are in the same orbit, so there exists $g \in \gamma$ such that $g \cdot p = p'$. This implies $g^{-1} g_i \cdot p_i \to p$. Choose a neighborhood U of p as in (i), and let i be large enough that p_i and $g^{-1} g_i \cdot p_i$ are both in U. Because $(g^{-1} g_i \cdot U) \cap U = \varnothing$, it follows that $g^{-1} g_i = e$. So $g_i = g$ when i is large enough, which certainly converges. By Proposition 2.3.1.5(b), the action is proper. □

In the literature, we call a continuous action satisfying (i) a **covering space action**, and simply refer to actions satisfying (i) and (ii) as **free and proper actions**. It can be shown that any covering space action on a topological space yields a covering map, though the quotient space need not be Hausdorff.

Proposition 2.3.3.2. *Let M be a smooth manifold, and let $\pi : E \to M$ be a smooth covering map. With the discrete topology, the automorphism group $\mathrm{Aut}_\pi(E)$ acts smoothly, freely, and properly on E.*

Proof. We already showed in Proposition 2.1.4.2 that the action is smooth and free. To show it is proper, we will show that it satisfies conditions (i) and (ii) of Lemma 2.3.3.1.

First, if $p \in E$ is arbitrary, choose $W \subseteq M$ to be an evenly covered neighborhood of $\pi(p)$. If U is the component of $\pi^{-1}(W)$ containing p, then it is easy to check that U satisfies (i).

Second, if $p, p' \in E$ are in different orbits and $\pi(p) \neq \pi(p')$, then just as in the proof Lemma 2.3.3.1, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$, and it follows that $V = \pi^{-1}(V)$ and $V' = \pi^{-1}(W')$ satisfy (ii). If p and p' are in different orbits and $\pi(p) = \pi(p')$, let W be an evenly covered neighborhood of $\pi(p)$, and let V, V' be the components of $\pi^{-1}(W)$ containing p and p', respectively. For any $g \in \mathrm{Aut}_\pi(E)$, a simple connectedness argument shows that $g \cdot V$ is a component of $\pi^{-1}(W)$; if it had nontrivial intersection with V it would have to be equal to V, which would imply $g \cdot p = p'$, a contradiction. □

The quotient manifold theorem yields an important partial converse to the preceding proposition.

Proposition 2.3.3.3. *Suppose E is a connected smooth manifold and Γ is a discrete Lie group acting smoothly, freely, and properly on E. Then the orbit space E/Γ is a topological manifold and has a unique smooth structure such that $\pi : E \to E/\Gamma$ is a smooth normal covering map.*

Proof. It follows from the quotient manifold theorem that E/Γ has a unique smooth manifold structure such that π is a smooth submersion. Because a smooth covering map is in particular a smooth submersion, any other smooth manifold structure on E making Γ into a smooth covering map must be equal to this one. Because $\dim E = \dim E - \dim \Gamma = \dim E$, π is a local diffeomorphism. Thus, to prove the theorem, it suffices to show that π is a normal covering map.

Let $p \in E$. By Lemma 2.3.3.1, p has a neighborhood U in E satisfying

$$(g \cdot U) \cap U = \varnothing \quad \text{for all } g \in \Gamma \text{ except } g = e. \tag{2.3.3.1}$$

Shrinking U if necessary, we may assume it is connected. Let $V = \pi(U)$, which is open in E/Γ by Lemma 2.3.1.1. Because $\pi^{-1}(V)$ is the union of the disjoint connected open subsets $g \cdot U$ for $g \in \Gamma$, to show that π is a covering map we need only show that π is a homeomorphism from each such set onto V. For each $g \in \Gamma$, the following diagram commutes:

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & g \cdot U \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Since $g : U \to g \cdot U$ is a homeomorphism (in fact, a diffeomorphism), it suffices to show that $\pi : U \to V$ is a homeomorphism. We already know that it is surjective, continuous, and open. To see that it is injective, suppose $\pi(q) = \pi(q')$ for $q, q' \in U$, which means that $q' = g \cdot q$ for some $g \in \Gamma$. By (2.3.3.1), this can happen only if $g = e$, which is to say that $q = q'$. This completes the proof that π is a smooth covering map. Because elements of Γ act as automorphisms of π, and Γ acts transitively on fibers by definition, the covering is normal. □

Example 2.3.3.4 (Proper Discrete Group Actions).

(a) The discrete Lie group $\mathbb{Z}^n$ acts smoothly and freely on $\mathbb{R}^n$ by translation. If (x_i) and (m_i) are sequences in $\mathbb{R}^n$ and $\mathbb{Z}^n$, respectively, such that $x_i \to x$ and $m_i + x_i \to y$, then $m_i \to y - x$, so the action is proper by Proposition 2.3.1.5(b). The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to the n-torus T^n, and Theorem 2.3.3.3 says that there is a unique smooth structure on T^n making the quotient map into a smooth covering map. To verify that this smooth structure on T^n is the same as the one we defined previously, we just check that the covering map $\mathbb{R}^n \to T^n$ given by $(x^1, \dots, x^n) \mapsto (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$ is a local diffeomorphism with respect to the product smooth structure on T^n, and makes the same identifications as the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$; thus Theorem **??** implies that $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to T^n.

(b) The two-element group $\{\pm 1\}$ acts on S^n by multiplication. This action is smooth and free, and it is proper because the group is compact. This defines a smooth structure on $S^n/\{\pm 1\}$. In fact, this orbit space is diffeomorphic to $\mathbb{RP}^n$, which can be seen as follows. Let $q : S^n \to \mathbb{RP}^n$ be the smooth covering map. This map makes the same identifications as the quotient map $S^n \to S^n/\{\pm 1\}$. By Theorem **??**, therefore, $S^n/\{\pm 1\}$ is diffeomorphic to $\mathbb{RP}^n$.

2.3.4 Homogeneous spaces

Some of the most interesting group action are transitive ones. A smooth manifold endowed with a transitive smooth action by a Lie group G is called a **homogeneous *G*-space**.

Here are some important examples of homogeneous spaces.

Example 2.3.4.1 (Homogeneous Spaces).

(a) The natural action of $\mathrm{O}(n)$ on S^{n-1} is transitive. Thus, S^{n-1} is a homogeneous space of $\mathrm{O}(n)$.

(b) The natural action of $\mathrm{O}(n)$ restricts to a smooth action of $\mathrm{SO}(n)$ on S^{n-1}. When $n = 1$, this action is trivial because $\mathrm{SO}(1)$ is the trivial group. But if $n > 1$, then $\mathrm{SO}(n)$ acts transitively on S^{n-1}. To see this, it suffices to show that for any $v \in S^{n-1}$, there is a matrix $A \in \mathrm{SO}(n)$ taking the first standard basis vector e_1 to v. Since $O(n)$ acts transitively, there is a matrix $A \in \mathrm{O}(n)$ taking e_1 to v. Either $\det A = 1$, in which case $A \in \mathrm{SO}(n)$, or $\det A = -1$, in which case the matrix obtained by multiplying the second column of A by -1 is in $\mathrm{SO}(n)$ and takes e_1 to v. Thus for $n \geq 2$, S^{n-1} is also a homogeneous space of $\mathrm{SO}(n)$.

(c) The Euclidean group $\mathrm{E}(n)$ defined in Example 2.1.5.1 acts on $\mathbb{R}^n$ by rigid motions. Because any point in $\mathbb{R}^n$ can be taken to any other by a translation, $\mathrm{E}(n)$ acts transitively on $\mathbb{R}^n$, so $\mathbb{R}^n$ is a homogeneous $\mathrm{E}(n)$-space.

(d) The group $\mathrm{SL}_2(\mathbb{R})$ acts smoothly and transitively on the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The resulting complex-analytic diffeomorphisms from $\mathbb{U}$ to itself are called **Möbius transformations**.

(e) For $n \geq 1$, the natural action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ restricts to natural smooth actions of both $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ on S^{2n-1}, identified with the set of unit vectors in $\mathbb{C}^n$. The natural action of $\mathrm{U}(n)$ on S^{2n-1} is transitive for all $n \geq 1$, and that of $\mathrm{SU}(n)$ is transitive for all $n \geq 2$.

Next we describe a construction that can be used to generate a great number of homogeneous spaces, as quotients of Lie groups by closed subgroups. Let G be a Lie group and $H \subseteq G$ be a Lie subgroup. A subset of G of the form

$$gH = \{gh : h \in H\}$$

for some $g \in G$ is called a **left coset** of H. The left cosets form a partition of G, and the quotient space determined by this partition (i.e., the set of left cosets with the quotient topology) is called the **left coset space of *G* modulo *H***, and is denoted by G/H. Two elements $g_1, g_2 \in G$ are in the same left coset of H if and only if $g_1^{-1} g_2 \in H$; in this case we write $g_1 \equiv g_2 \bmod H$ and say g_1 and g_2 are congruent modulo H.

Theorem 2.3.4.2 (Homogeneous Space Construction Theorem). *Let G be a Lie group and let H be a closed subgroup of G. The left coset space G/H is a topological manifold of dimension equal to* $\dim G - \dim H$*, and has a unique smooth structure such that the quotient map $\pi : G \to G/H$ is an H-bundle. The left action of G on G/H given by*

$$g_1 \cdot (g_2 H) = (g_1 g_2) H$$

turns G/H into a homogeneous G-space.

Proof. If we let H act on G by right translation, then $g_1, g_2 \in G$ are in the same H-orbit if and only if $g_1 h = g_2$ for some $h \in H$, which is the same as saying that g_1 and g_2 are in the same coset of H. In other words, the orbit space determined by the right action of H on G is precisely the left coset space G/H.

The subgroup H is a properly embedded Lie subgroup of G by the closed subgroup theorem, and the H-action on G is smooth because it is simply the restriction of the multiplication map of G. It is a free action because $gh = g$ implies $h = e$. To see that it is proper, we use Proposition 2.3.1.5(b). Suppose (g_i) is a convergent sequence in G and (h_i) is a sequence in H such that $(g_i h_i)$ converges in G. By the continuity of multiplication, $h_i = g_i^{-1}(g_i h_i)$ converges to a point in G, and since H is closed in G and has the subspace topology, it follows that (h_i) converges in H.

The quotient manifold theorem now implies that G/H has a unique smooth manifold structure such that the quotient map $\pi : G \to G/H$ is a principal H-bundle. Since a product of smooth submersions is a smooth submersion, it follows that $\mathrm{id}_G \times \pi : G \times G \to G \times G/H$ is also a smooth submersion. Consider the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "\mathrm{id}_G \times \pi"'] & G \arrow[d, "\pi"] \\ G \times G/H \arrow[r, "\theta"] & G/H \end{tikzcd}$$

where m is group multiplication and θ is the action of G on G/H. It is straightforward to check that $\pi \circ m$ is constant on the fibers of $\mathrm{id}_G \times \pi$, and therefore θ is well defined and smooth by Theorem **??**. It is immediate from the definition that θ satisfies the conditions for a group action. Finally, it is easy to verify that the action is transitive. □

The homogeneous spaces constructed in this theorem turn out to be of central importance because, as the next theorem shows, every homogeneous space is equivalent to one of this type.

Theorem 2.3.4.3 (Homogeneous Space Characterization Theorem). *Let G be a Lie group, let M be a homogeneous G-space, and let p be any point of M. Then the isotropy group G_p is a closed subgroup of G, and the map $F : G/G_p \to M$ defined by $F(gG_p) = g \cdot p$ is an equivariant diffeomorphism.*

Proof. For simplicity, let us write $H = G_p$. Note that H is closed by continuity, because $H = (\theta^{(p)})^{-1}(p)$, where $\theta^{(p)} : G \to M$ is the orbit map.

To see that F is well defined, assume that $g_1 H = g_2 H$, which means that $g_1 = g_2 h$ for some $h \in H$. Then

$$F(g_1 H) = g_1 \cdot p = (g_2 h) \cdot p = g_2 \cdot p = F(g_2 H).$$

Also, F is equivariant, because

$$F(g' g H) = (g' g) \cdot p = g' \cdot (g \cdot p) = g' \cdot F(gH).$$

It is smooth because it is obtained from the orbit map $\theta^{(p)} : G \to M$ by passing to the quotient.

Next, we show that F is bijective. Given any point $q \in M$, by transitivity there is a group element $g \in G$ such that $g \cdot p = q$, and thus $F(gH) = q$. On the other hand, if $F(g_1 H) = F(g_2 H)$, then $g_1 \cdot p = g_2 \cdot p$ implies $g_1 H = g_2 H$. Thus, F is an equivariant smooth bijection, so it is a diffeomorphism by the equivariant rank theorem. □

Applying the characterization theorem to the examples of transitive group actions we discussed earlier, we see that some familiar spaces are diffeomorphic to coset space of Lie groups.

Example 2.3.4.4 (Examples of fiber bundles of Lie groups).

(a) Consider again the natural action of $O(n)$ on S^{n-1}. If we choose our base point in S^{n-1} to be the north pole $N = (0, \ldots, 0, 1)$, then it is easy to check that the isotropy group is $O(n)$, thought of as orthogonal transformations of $\mathbb{R}^n$ that fix the last variable. Thus we get a fiber bundle

$$\begin{tikzcd} O(n-1) \arrow[r] & O(n) \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

where $n \geq 1$.

(b) For the action of $SO(n)$ on S^{n-1}, the isotropy group is $SO(n-1)$, so there is also a fiber bundle

$$\begin{tikzcd} SO(n-1) \arrow[r] & SO(n) \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

where we require $n \geq 2$.

(c) Because the Euclidean group $E(n)$ acts transitively on $\mathbb{R}^n$, and the isotropy group of the origin is the subgroup $O(n)$, we have a fiber bundle

$$\begin{tikzcd} O(n) \arrow[r] & E(n) \arrow[d] \\ & \mathbb{R}^n \end{tikzcd}$$

(d) Next, consider the transitive action of $SL_2(\mathbb{R})$ on the upper half-plane by Möbius transformations. Direct computation shows that the isotropy group of the point i consists of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $a^2 + b^2 = 1$. This subgroup is exactly $SO(2) \subseteq SL_2(\mathbb{R})$, so the characterization theorem gives rise to a fiber bundle

$$\begin{tikzcd} SO(2) \arrow[r] & SL_2(\mathbb{R}) \arrow[d] \\ & \mathbb{U} \end{tikzcd}$$

(e) By the same argument as (c) and (d), we have the following fiber bundles

$$\begin{tikzcd} U(n-1) \arrow[r] & U(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

for $n \geq 1$ and

$$\begin{tikzcd} SU(n-1) \arrow[r] & SU(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

for $n \geq 2$. In particular, since $U(0) = SU(1) = \{1\}$, this gives the identifications

$$S^1 \approx U(1), \quad S^3 \approx SU(2).$$

A highly useful application of the homogeneous space characterization theorem is to put smooth manifold structures on sets that admit transitive Lie group actions.

Theorem 2.3.4.5. *Suppose X is a set, and we are given a transitive action of a Lie group G on X such that for some point $p \in X$, the isotropy group G_p is closed in G. Then X has a unique smooth manifold structure with respect to which the given action is smooth. With this structure, $\dim X = \dim G - \dim G_p$.*

Proof. Theorem 2.3.4.2 shows that G/G_p is a smooth manifold of dimension equal to $\dim G - \dim G_p$. The map $F : G/G_p \to X$ defined by $F(gG_p) = g \cdot p$ is an equivariant bijection by exactly the same argument as we used in the proof of the characterization theorem, Theorem 2.3.4.3. If we define a topology and smooth structure on X by declaring F to be a diffeomorphism, then the given action of G on X is smooth because it can be written $(g, x) \mapsto F(g \cdot F^{-1}(x))$.

If $\widetilde{X}$ denotes the set X with any smooth manifold structure such that the given action is smooth, then the homogeneous space characterization theorem shows that the map F is also an equivariant diffeomorphism from G/G_p to $\widetilde{X}$, so the topology and smooth structure of $\widetilde{X}$ are equal to those constructed above. □

Example 2.3.4.6 (Grassmannians). Let $\mathrm{Gr}_k(\mathbb{R}^n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. The general linear group $\mathrm{GL}_n(\mathbb{R})$ acts transitively on $\mathrm{Gr}_k(\mathbb{R}^n)$: given two subspaces A and A', choose bases for both subspaces and extend them to bases for $\mathbb{R}^n$, and then the linear transformation taking the first basis to the second also takes A to A'. The isotropy group of the subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$ is

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in \mathrm{GL}_k(\mathbb{R}), B \in \mathcal{M}_{k\times(n-k)}(\mathbb{R}), D \in \mathrm{GL}_{n-k}(\mathbb{R}) \right\},$$

which is easily seen to be closed in $\mathrm{GL}_n(\mathbb{R})$. Therefore, $\mathrm{Gr}_k(\mathbb{R}^n)$ has a unique smooth manifold structure making the natural $\mathrm{GL}_n(\mathbb{R})$ action smooth.

Example 2.3.4.7 (Flag Manifolds). Let V be a real vector space of dimension $n > 1$, and let $K = (k_1, \ldots, k_m)$ be a finite sequence of integers satisfying $0 < k_1 < \cdots < k_m < n$. A **flag in V of type K** is a nested sequence of linear subspaces $S_1 \subsetneq \cdots \subsetneq S_m$, with $\dim S_i = k_i$ for each i. The set of all flags of type K in V is denoted by $F_K(V)$. (For example, if $K = (k)$, then $F_K(V)$ is the Grassmannian $\mathrm{Gr}_k(V)$) For a given type $(k_1, \ldots, k_m)$, let $\{v_1, \ldots, v_n\}$ be a basis of V, we define the standard flag of type K to be

$$\mathrm{span}(v_1, \ldots, v_{k_1}) \subsetneq \cdots \subsetneq \mathrm{span}(v_1, \ldots, v_{k_m}).$$

Then for any flag $V = (V_1, \ldots, V_m)$ of type K, we can choose a matrix $T \in \mathrm{GL}_n(V)$ such that $T(v_1, \ldots, v_{k_i}) = V_i$. This means $\mathrm{GL}_n(V)$ acts transitively on $F_K(V)$. Moreover, similar to Example 2.3.4.6, the isotropy group of the standard flag consists of blocked upper-triangular matrices and thus is closed. Therefore $F_K(V)$ has a unique smooth manifold structure making it into a homogeneous $\mathrm{GL}(V)$-space. With this structure, $F_K(V)$ is called a **flag manifold**.

2.3.5 Lie quotient groups

Theorem 2.3.5.1 (Quotient Theorem for Lie Groups). *Suppose G is a Lie group and $H \subseteq G$ is a closed normal subgroup. Then G/H is a Lie group, and the quotient map $\pi : G \to G/K$ is a surjective Lie group homomorphism whose kernel is H.*

Proof. By the homogeneous space construction theorem, G/K is a smooth manifold and π is a smooth submersion; and by the first isomorphism theorem, G/K is a group and π is a surjective group homomorphism with kernel K. Thus, the only thing that needs to be verified is that multiplication and inversion in G/K are smooth, both of which follow easily from Theorem **??**. □

Theorem 2.3.5.2 (First Isomorphism Theorem for Lie Groups). *If $F : G \to H$ is a Lie group homomorphism, then the kernel of F is a closed normal Lie subgroup of G, the image of F has a unique smooth manifold structure making it into a Lie subgroup of H, and F descends to a Lie group isomorphism $\widetilde{F} : G/\ker F \to \mathrm{im}\, F$. If F is surjective, then $G/\ker F$ is smoothly isomorphic to H.*

Proof. By group theorems, $\ker F$ is a normal subgroup and $\mathrm{im}\, F$ is a subgroup, and F descends to a group isomorphism $\widetilde{F} : G/\ker F \to \mathrm{im}\, F$. By continuity, $\ker F$ is closed in G, so it follows from Theorem 2.3.5.1 that $G/\ker F$ is a Lie group and the projection $\pi : G \to G/\ker F$ is a surjective Lie group homomorphism. Because π is surjective and has constant rank, it is a smooth submersion by the global rank theorem, so the characteristic property of surjective smooth submersions (Theorem **??**) guarantees that $\widetilde{F}$ is smooth. Since $\operatorname{im} F$ is the image of the injective Lie group homomorphism $\widetilde{F}$, Proposition 2.1.3.5 shows that it has a smooth manifold structure with respect to which it is a Lie subgroup of H and $\widetilde{F} : G/\ker F \to \operatorname{im} F$ is a Lie group isomorphism. The uniqueness of the smooth structure on ImF follows from Corollary 2.2.5.7. The last statement follows immediately just by substituting $H = \operatorname{im} F$. $\square$

As an application, we can show that the center of a Lie group G is a closed Lie subgroup.

Definition 2.3.5.3. Let $\mathfrak{g}$ be a Lie algebra. The center of $\mathfrak{g}$ is defined by

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

It is clear that $\mathfrak{z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Proposition 2.3.5.4. *Let G be a connected Lie group. Then its center $Z(G)$ is a closed Lie subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$.*

Proof. Let $g \in G$, $X \in \mathfrak{g}$. It follows from the identity

$$\exp(\operatorname{Ad}_g(tX)) = g(\exp tX)g^{-1}$$

that g commutes with all elements of one-parameter subgroup $\exp(tx)$ if and only if $\operatorname{Ad}_g(x) = x$. Since for a connected Lie group, elements of the form $\exp tX$ generate G, we see that $g \in Z(G)$ if and only if $g \in \ker \operatorname{Ad}$. Now the result follows from Proposition 2.3.5.2 and Proposition 2.2.3.1. $\square$

The quotient group $G/Z(G)$ is usually called the **adjoint group** associated with G and denoted AdG. The corresponding Lie algebra is ad$\mathfrak{g}$.

The results of Theorem 2.3.5.1 and Theorem 2.3.5.2 are particularly significant when we apply them to a **discrete subgroup**, that is, a subgroup that is a discrete space in the subspace topology.

Proposition 2.3.5.5. *Every discrete subgroup of a Lie group is a closed Lie subgroup of dimension zero.*

Proof. Let G be a Lie group and $\Gamma \subseteq G$ be a discrete subgroup. With the subspace topology, Γ is a countable discrete space and thus a zero-dimensional Lie group. By the closed subgroup theorem, Γ is a closed Lie subgroup of G if and only if it is a closed subset. A discrete subset is closed if and only if it has no limit points in G, so assume for the sake of contradiction that g is a limit point of Γ in G. By discreteness, there is a neighborhood U of e in G such that $U \cap \gamma = \{e\}$, and then by Lemma **??** there is a smaller neighborhood V of e such that $g_1 g_2^{-1} \in U$ whenever $g_1, g_2 \in V$. Then $Vg = \{hg : h \in V\}$ is a neighborhood of g, and because G is Hausdorff, Vg contains infinitely many points of Γ. Let γ_1, γ_2 be two distinct points in $\Gamma \in Vg$. Then $\gamma_1 g, \gamma_2 g \in V$, which implies

$$\gamma_1\gamma_2^{-1} = (\gamma_1 g)(\gamma_2 g)^{-1} \in U.$$

Since $U \cap \Gamma = \{e\}$, this implies $\gamma_1 = \gamma_2$, contradicting our assumption that γ_1 and γ_2 are distinct. $\square$

Theorem 2.3.5.6 (Quotients of Lie Groups by Discrete Subgroups). *Let G be a connected Lie group and Γ be a discrete subgroup of G. Then G/Γ is a smooth manifold and the quotient map $\pi : G \to G/\Gamma$ is a smooth normal covering map.*

Proof. The proof of Theorem 2.3.4.2 showed that Γ acts smoothly, freely, and properly on G on the right, and its quotient G/Γ is a smooth manifold. The theorem is then an immediate consequence of Theorem 2.3.3.3. $\square$

Example 2.3.5.7. Let C be a cube centered at the origin in $\mathbb{R}^3$. The set of orientation-preserving orthogonal transformations of $\mathbb{R}^3$ that take C to itself is a finite subgroup $\Gamma \subseteq \mathrm{SO}(3)$, and $\mathrm{SO}(3)/\Gamma$ is a connected smooth 3-manifold with finite fundamental group and with S^3 as its universal covering space. Similar examples are obtained from the symmetry groups of other regular polyhedra, such as a regular tetrahedron, octahedron, dodecahedron, or icosahedron.

Combining the results of Theorems 2.3.5.2 and 2.3.5.6, we obtain the following important characterization of homomorphisms with discrete kernels.

Theorem 2.3.5.8 (Covering homomorphisms). *Let G and H be connected Lie groups and $F : G \to H$ be a Lie group homomorphism. Then the following are equivalent:*

(i) *F is surjective and has discrete kernel.*

(ii) *F is a smooth covering map.*

(iii) *F is a local diffeomorphism.*

(iv) *The induced homomorphism $F_* : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ is an isomorphism.*

Proof. We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (iv). First, assume that F is surjective with discrete kernel $\Gamma \subseteq G$. Then Theorem 2.3.5.6 implies that the quotient map $\pi : G \to G/\Gamma$ is a smooth covering map, and Theorem 2.3.5.2 shows that F descends to a Lie group isomorphism $\widetilde{F} : G/\Gamma \to H$. This means that $F = \widetilde{F} \circ \pi$, which is a composition of a smooth covering map followed by a diffeomorphism and therefore is itself a smooth covering map. This proves that (i) $\Rightarrow$ (ii). The next implication, (ii) $\Rightarrow$ (iii), follows from Proposition **??**.

Under the assumption that F is a local diffeomorphism, each level set is an embedded 0-dimensional submanifold by the submersion level set theorem, so $\ker F$ is discrete. Since a local diffeomorphism is an open map, $F(G)$ is an open subgroup of H, and thus by Proposition 2.1.3.2, it is all of H. This shows that (iii) $\Rightarrow$ (i).

The implication (iii) $\Rightarrow$ (iv) is clear. Conversely, if F_* is an isomorphism, the inverse function theorem implies F is a local diffeomorphism in a neighborhood of $e \in G$. Because Lie group homomorphisms have constant rank, this means that $\operatorname{rank} F = \dim G = \dim H$, which implies that F is a local diffeomorphism everywhere, and thus (iv) $\Rightarrow$ (iii). □

This theory allows us to give a precise description of all the connected Lie groups with a given Lie algebra.

Proposition 2.3.5.9. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The connected Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}$ are (up to isomorphism) precisely those of the form G/Γ, where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, and Γ is a discrete normal subgroup of G.*

Proof. Given $\mathfrak{g}$, by Theorem 2.2.7.3 there exists a simply connected Lie group G with Lie algebra isomorphic to $\mathfrak{g}$. Suppose H is any other connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$, and let $\varphi : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ be a Lie algebra isomorphism. Theorem 2.2.7.1 guarantees that there is a Lie group homomorphism $\Phi : G \to H$ such that $\Phi_* = \varphi$. Because φ is an isomorphism, Theorem 2.3.5.8 implies that Φ is surjective and its kernel is a discrete normal subgroup of G. It follows that H is isomorphic to $G/\ker\Phi$ by Theorem 2.3.5.2. □

2.3.6 Connectivity and fundamental groups of Lie groups

Another application of homogeneous space theory is to identify the connected components of many familiar Lie groups. The key result is the following proposition.

Proposition 2.3.6.1. *Suppose a topological group G acts continuously, freely, and properly on a topological space M. If G and M/G are connected, then M is connected.*

Proof. Assume for the sake of contradiction that G and M/G are connected but M is not. Then there are disjoint nonempty open subsets $U, V \subseteq M$ whose union is M. Each G-orbit in M is the image of G under an orbit map $\theta^{(p)} : G \to M$; since G is connected, each orbit must lie entirely in one set U or V.

By Lemma 2.3.1.1, $\pi(U)$ and $\pi(V)$ are both open in M/G. If $\pi(U) \cap \pi(V)$ were not empty, some G-orbit in M would contain points of both U and V, which we have just shown is impossible. Thus, $\pi(U)$ and $\pi(V)$ are disjoint nonempty open subsets of M/G whose union is M/G, contradicting the assumption that M/G is connected. □

Proposition 2.3.6.2. *For each $n \geq 1$, the Lie groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are connected. The group $\mathrm{O}(n)$ has exactly two components, one of which is $\mathrm{SO}(n)$.*

Proof. First, we prove by induction on n that $\mathrm{SO}(n)$ is connected. For $n = 1$ this is obvious, because $\mathrm{SO}(1)$ is the trivial group. Now suppose we have shown that $\mathrm{SO}(n-1)$ is connected for some $n \geq 2$. Because the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is diffeomorphic to S^{n-1} and therefore is connected, Proposition 2.3.6.1 and the induction hypothesis imply that $\mathrm{SO}(n)$ is connected. A similar argument applies to $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, using the facts that $\mathrm{U}(n)/\mathrm{U}(n-1) \approx \mathrm{SU}(n)/\mathrm{SU}(n-1) \approx S^{2n-1}$.

As we noted that $\mathrm{O}(n)$ is equal to the union of the two open subsets $\mathrm{O}^+(n)$ and $\mathrm{O}^-(n)$ consisting of orthogonal matrices with determinant $+1$ and -1, respectively. By the argument in the preceding paragraph, $\mathrm{O}^+(n) = \mathrm{SO}(n)$ is connected. On the other hand, if A is any orthogonal matrix whose determinant is -1, then left translation L_A is a diffeomorphism from $\mathrm{O}^+(n)$ to $\mathrm{O}^-(n)$, so $\mathrm{O}^-(n)$ is connected as well. Therefore, the components of $\mathrm{O}(n)$ are $\mathrm{O}^+(n)$ and $\mathrm{O}^-(n)$. □

Determining the components of the general linear groups is a bit more involved. Let $\mathrm{GL}_n^+(\mathbb{R})$ and $\mathrm{GL}_n^-(\mathbb{R})$ denote the open subsets of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with positive determinant and negative determinant, respectively.

Proposition 2.3.6.3. *The connected components of* $\mathrm{GL}_n(\mathbb{R})$ *are* $\mathrm{GL}_n^+(\mathbb{R})$ *and* $\mathrm{GL}_n^-(\mathbb{R})$.

Proof. By continuity of the determinant function, $\mathrm{GL}_n^+(\mathbb{R})$ and $\mathrm{GL}_n^-(\mathbb{R})$ are nonempty, disjoint, open subsets of $\mathrm{GL}_n(\mathbb{R})$ whose union is $\mathrm{GL}_n(\mathbb{R})$, so all we need to prove is that both subsets are connected. We begin by showing that $\mathrm{GL}_n^+(\mathbb{R})$ is connected. It suffices to show that it is path-connected, which will follow once we show that for any $A \in \mathrm{GL}_n^+(\mathbb{R})$, there is a continuous path in $\mathrm{GL}_n^+(\mathbb{R})$ from A to the identity matrix I_n.

Let $A \in \mathrm{GL}_n^+(\mathbb{R})$ be arbitrary, and let $(A_1, \dots, A_n)$ denote the columns of A, considered as vectors in $\mathbb{R}^n$. The Gram-Schmidt algorithm shows that there is an orthonormal basis $(Q_1, \dots, Q_n)$ for $\mathbb{R}^n$ with the property that $\mathrm{span}(Q_1, \dots, Q_k) = \mathrm{span}(A_1, \dots, A_k)$ for each k. Thus, we can write

$$\begin{aligned} A_1 &= R_1^1 Q_1, \\ A_2 &= R_2^1 Q_1 + R_2^2 Q_2, \\ &\vdots \\ A_n &= R_n^1 Q_1 + \cdots + R_n^n Q_n, \end{aligned}$$

for some constants R_i^j. Replacing each Q_i by $-Q_i$ if necessary, we may assume that each number R_i^i (no summation) is positive. In matrix notation, this is equivalent to $A = QR$, where Q is orthogonal and R is upper triangular with positive entries on the diagonal. Since the determinant of R is the product of its diagonal entries and $(\det Q)(\det R) = \det A > 0$, it follows that $Q \in \mathrm{SO}(n)$.

Let $R_t = tI_n + (1-t)R$ for $t \in [0,1]$. It is immediate that each matrix R_t is upper triangular with positive diagonal entries, so $R_t \in \mathrm{GL}_n^+(\mathbb{R})$. Therefore, the path $\gamma : [0,1] \to \mathrm{GL}_n^+(\mathbb{R})$ given by $\gamma(t) = QR_t$ satisfies $\gamma(0) = A$ and $\gamma(1) = Q \in \mathrm{SO}(n)$. Because $\mathrm{SO}(n)$ is connected, there is a path in $\mathrm{SO}(n)$ from Q to the identity matrix. This shows that $\mathrm{GL}_n^+(\mathbb{R})$ is path-connected. Any matrix B with $\det B < 0$ yields a diffeomorphism $L_B : \mathrm{GL}_n^+(\mathbb{R}) \to \mathrm{GL}_n^-(\mathbb{R})$, so $\mathrm{GL}_n^-(\mathbb{R})$ is connected as well. This completes the proof. □

In fact, our argument goes further. Recall that with a fiber bundle we can get a exact sequence of homotopy groups. From this we can compute the fundamental groups of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. We establish some results here.

Proposition 2.3.6.4. *The fundamental groups of* $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *and given by*

$$\pi_1(\mathrm{O}(n)) = \pi_1(\mathrm{SO}(n)) = \begin{cases} \{1\} & n = 1, 2, \\ \mathbb{Z}/2\mathbb{Z} & n \geq 3. \end{cases}$$

Proof. Since $\mathrm{SO}(n)$ is a connected component of $\mathrm{O}(n)$, we only need to consider the fundamental group of $\mathrm{SO}(n)$. The case $n = 1, 2$ are trivial. Also, we know that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm 1\} \cong \mathbb{RP}^3$, therefore

$$\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}.$$

Recall that we have a fiber bundle

$$\begin{tikzcd} \mathrm{SO}(n-1) \arrow[r] & \mathrm{SO}(n) \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

from which we get a exact sequence

$$\pi_2(S^{n-1}) \longrightarrow \pi_1(\mathrm{SO}(n-1)) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(S^{n-1}) \longrightarrow \{1\}$$

For $n \geq 3$ we have $\pi_2(S^{n-1}) = \{1\}$, so we conclude that

$$\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(n-1)) \text{ for } n \geq 3.$$

This establishes the result. □

Proposition 2.3.6.5. *The fundamental groups of* $\mathrm{U}(n)$ *and* $\mathrm{SU}(n)$ *and given by*

$$\pi_1(\mathrm{U}(n)) = \mathbb{Z}, \quad \pi(\mathrm{SU}(n)) = \{1\}.$$

Proof. We have also seen that $\mathrm{U}(1) \cong S^1$ and $\mathrm{SU}(2) \cong S^3$ in Example 2.3.4.4, and we have the following fiber bundles:

$$\begin{tikzcd} \mathrm{U}(n-1) \arrow[r] & \mathrm{U}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{SU}(n-1) \arrow[r] & \mathrm{SU}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

These give the claim. □

Proposition 2.3.6.6. *The fundamental groups of* $\mathrm{GL}_n(\mathbb{R})$*,* $\mathrm{SL}_n(\mathbb{R})$*,* $\mathrm{GL}_n(\mathbb{C})$*, and* $\mathrm{SL}_n(\mathbb{C})$ *are given by*

$$\pi_1(\mathrm{GL}_n(\mathbb{R})) = \pi_1(\mathrm{SL}_n(\mathbb{R})) = \begin{cases} \{1\} & n = 1, 2, \\ \mathbb{Z}/2\mathbb{Z} & n \geq 3. \end{cases}$$

and

$$\pi_1(\mathrm{GL}_n(\mathbb{C})) = \mathbb{Z}, \quad \pi_1(\mathrm{SL}_n(\mathbb{C})) = \{1\}.$$

Proof. This follows from the fact that $\mathrm{SL}_n(\mathbb{R})$ is homotopic equivalent to $\mathrm{SO}(n)$ and $\mathrm{SL}_n(\mathbb{C})$ is homotopic equivalent to $\mathrm{SU}(n)$. □

Chapter 3

Tensors and forms on manifolds

3.1 Tensors

3.1.1 Multilinear algebra

Suppose $V_1, \dots, V_k$, and W are vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is said to be **multilinear** if it is linear as a function of each variable. Let us write $\mathcal{L}(V_1, \dots, V_k; W)$ for the set of all multilinear maps from $V_1 \times \cdots \times V_k$ to W. It is a vector space under the usual operations of pointwise addition and scalar multiplication.

Example 3.1.1.1 (Some Familiar Multilinear Functions).

(a) The inner product in $\mathbb{R}^n$ is a scalar-valued bilinear function of two vectors, used to compute lengths of vectors and angles between them.

(b) The cross product in $\mathbb{R}^3$ is a vector-valued bilinear function of two vectors, used to compute areas of parallelograms and to find a third vector orthogonal to two given ones.

(c) The determinant is a real-valued multilinear function of n vectors in $\mathbb{R}^n$, used to detect linear independence and to compute the volume of the parallelepiped spanned by the vectors.

(d) The bracket in a Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-valued bilinear function of two elements of $\mathfrak{g}$.

The next example is probably not as familiar, but it is extremely important

Example 3.1.1.2 (Tensor Products of Covectors). Suppose V is a vector space, and $\omega, \eta \in V^*$. Define a function $\omega \otimes \eta : V \times V \to \mathbb{R}$ by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$$

where the product on the right is just ordinary multiplication of real numbers. The linearity of ω and η guarantees that $\omega \otimes \eta$ is a bilinear function of v_1 and v_2, so it is an element of $\mathcal{L}(V_1, V_2; \mathbb{R})$. For example, if (e^1, e^2) denotes the standard dual basis for $(\mathbb{R}^2)^*$, then $e^1 \otimes e^2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the bilinear function

$$e^1 \otimes e^2\big((w, x), (y, z)\big) = wz$$

The last example can be generalized to arbitrary multilinear functions as follows: let $V_1, \dots, V_k$ and $W_1, \dots, W_l$ be real vector spaces, and suppose $F \in \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$ and $G \in \mathcal{L}(W_1, \dots, W_l; \mathbb{R})$. Define a function

$$F \otimes G : V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l \to \mathbb{R}$$

by

$$F \otimes G(v_1, \dots, v_k, w_1, \dots, w_l) = F(v_1, \dots, v_k)G(w_1, \dots, w_k)$$

It follows from the multilinearity of F and G that $F \otimes G$ depends linearly on each argument v_i or w_j separately, so $F \otimes G$ is an element of $\mathcal{L}(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$, called the **tensor product of F and G**.

Proposition 3.1.1.3. *The tensor product operation is bilinear and associative: $F \otimes G$ depends bilinearly on F and G, and $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.*

Because of the result of the preceding proposition, we can write tensor products of multilinear functions unambiguously without parentheses. For example, if $\omega^j \in V_j^*$ for $1 \leq j \leq k$, then $\omega^1 \otimes \cdots \otimes \omega^k$ is the multilinear function given by

$$\omega^1 \otimes \cdots \otimes \omega^k(v_1, \ldots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$$

The tensor product operation is important in part because of its role in the following proposition.

Proposition 3.1.1.4 (A Basis for the Space of Multilinear Functions). *Let $V_1, \ldots, V_k$ be real vector spaces of dimensions $n_1, \ldots, n_k$, respectively. For each $j \in \{1, \ldots, k\}$, let $(E_1^{(j)}, \ldots, E_{n_j}^{(j)})$ be a basis for V_j, and let $(\varepsilon_{(j)}^1, \ldots, \varepsilon_{(j)}^{n_j})$ be the corresponding dual basis for V_j^*. Then the set*

$$\mathcal{B} = \{\varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} : 1 \leq i_j \leq n_\ell, 1 \leq j \leq k\}$$

is a basis for $\mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$, which therefore has dimension equal to $n_1 \cdots n_k$.

Proof. Suppose $F \in \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$ is arbitrary. For each ordered k-tuple $i = (i_1, \ldots, i_k)$ of integers with $1 \leq i_j \leq n_j$, define a number $F_{i_1 \ldots i_k}$ by

$$F_{i_1 \ldots i_k} = F(E_{i_1}^{(1)}, \ldots, E_{i_k}^{(k)}).$$

Then

$$F = F_{i_1 \ldots i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}.$$

Also, $\mathcal{B}$ is linearly independent by the observation

$$\varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}(E_{i'_1}^{(1)}, \ldots, E_{i'_k}^{(k)}) = \delta_{i'_1 \ldots i'_k}^{i_1 \ldots i_k}.$$

□

3.1.1.1 Abstract tensor products of vector spaces

For any set S, a **formal linear combination** of elements of S is a function $f : S \to \mathbb{R}$ such that $f(S) = 0$ for all but finitely many $s \in S$. The **free vector space** on S, denoted by $F(S)$, is the set of all formal linear combinations of elements of S. Under pointwise addition and scalar multiplication, $F(S)$ becomes a vector space over $\mathbb{R}$.

For each element $x \in S$, there is a function δ_x that takes the value 1 on x and zero on all other elements of S; typically we identify this function with x itself, and thus think of S as a subset of $F(S)$. Then we see S is a basis of $F(S)$.

Proposition 3.1.1.5 (Characteristic Property of the Free Vector Space). *For any set S and any vector space W, every map $A : S \to W$ has a unique extension to a linear map $\widetilde{A} : F(S) \to W$:*

$$\begin{tikzcd} F(S) \arrow[r, dashed, "\widetilde{A}"] & W \\ S \arrow[u] \arrow[ur, "A"'] & \end{tikzcd}$$

Now let $V_1, \ldots, V_k$ be real vector spaces. Consider the free vector space $F(V_1 \times \cdots \times V_k)$, let R is the subspace of $F(V_1 \times \cdots \times V_k)$ spanned by all elements of the following forms:

$$(v_1, \ldots, av_i, \ldots, v_k) - a(v_1, \ldots, v_k)$$
$$(v_1, \ldots, v_i + v'_i, \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v'_i, \ldots, v_k)$$

with $v_j, v'_j \in V_j$, $i \in \{1, \ldots, k\}$, and $a \in \mathbb{R}$.

We define the **tensor product** of $V_1, \ldots, V_k$ to be

$$V_1 \otimes \cdots \otimes V_k = F(V_1 \times \cdots \times V_k)/R$$

and let $\pi : F(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k$ be the natural projection. The equivalence class of an element $(v_1, \ldots, v_k)$ in $V_1 \otimes \cdots \otimes V_k$ is denoted by

$$v_1 \otimes \cdots \otimes v_k = \pi(v_1, \ldots, v_k)$$

and is called the **tensor product of $v_1, \ldots, v_k$**.

Proposition 3.1.1.6 (Characteristic Property of the Tensor Product Space). *Let $V_1, \ldots, V_k$ be finite-dimensional real vector spaces. If $A : V_1 \times \cdots \times V_k \to X$ any multilinear map into a vector space X, then there is a unique linear map $\widetilde{A} : V_1 \otimes \cdots \otimes V_k \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "\pi"] & X \\ V_1 \otimes \cdots \otimes V_k \arrow[ur, dashed, "\widetilde{A}"'] & \end{tikzcd}$$

where π is the map $\pi(v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_k$.

Proposition 3.1.1.7 (A Basis for the Tensor Product Space). *Suppose $V_1, \ldots, V_k$ are real vector spaces of dimensions $n_1, \ldots, n_k$, respectively. For each $1 \leq j \leq k$, suppose $(E_1^{(j)}, \ldots, E_{n_j}^{(j)})$ is a basis for V_j. Then the set*

$$\mathcal{C} = \{E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$$

Proposition 3.1.1.8 (Associativity of Tensor Product Spaces). *Let V_1, V_2, V_3 be finite-dimensional real vector spaces. There are unique isomorphisms*

$$V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$$

under which elements of the forms $v_1 \otimes (v_2 \otimes v_3)$and $(v_1 \otimes v_2) \otimes v_3$ all correspond.

The connection between tensor products in this abstract setting and the more concrete tensor products of multilinear functionals that we defined earlier is based on the following proposition.

Proposition 3.1.1.9. *If $V_1, \ldots, V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$$

Proof. First, define a map $\Phi : V_1^* \times \cdots \times V_k^* \to \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$ by

$$\Phi(\omega^1, \ldots, \omega^k)(v_1, \ldots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$$

The expression on the right depends linearly on each v_i, so $\Phi(\omega^1, \ldots, \omega^k)$ is indeed an element of the space $\mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$. It is easy to check that Φ is multilinear as a function of $\omega^1, \ldots, \omega^k$, so by the characteristic property it descends uniquely to a linear map $\widetilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$, which satisfies

$$\widetilde{\Phi}(\omega^1 \otimes \cdots \otimes \omega^k)(v_1, \ldots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k).$$

It follows immediately from the definition that $\widetilde{\Phi}$ takes abstract tensor products to tensor products of covectors. It also takes the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis for $\mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$, so it is an isomorphism. □

Using this canonical isomorphism, we henceforth use the notation $V_1^* \otimes \cdots \otimes V_k^*$ to denote either the abstract tensor product space or the space $\mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$, focusing on whichever interpretation is more convenient for the problem at hand. Since we are assuming the vector spaces are all finite-dimensional, we can also identify each V_j with its second dual space V_j^{**}, and thereby obtain another canonical identification

$$V_1 \otimes \cdots \otimes V_k \cong \mathcal{L}(V_1^*, \ldots, V_k^*; \mathbb{R})$$

3.1.1.2 Covariant and contravariant tensors on a vector space

Let V be a finite-dimensional real vector space. If k is a positive integer, a **covariant k-tensor on** V is an element of the k-fold tensor product $V^* \otimes \cdots \otimes V^*$, which we typically think of as a real-valued multilinear function of k elements of V:

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ folds}} \to \mathbb{R}$$

The number k is called the **rank of α**. A 0-tensor is, by convention, just a real number. We denote the vector space of all covariant k-tensors on V by the shorthand notation

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ folds}}$$

Let us look at some examples.

Example 3.1.1.10 (Covariant Tensors). Let V be a finite-dimensional vector space.

(a) Every linear functional $\omega : V \to \mathbb{R}$ is multilinear, so a covariant 1-tensor is just a covector. Thus, $T^1(V^*)$ is equal to V^*.

(b) A covariant 2-tensor on V is a real-valued bilinear function of two vectors, also called a **bilinear form**. One example is the inner product on $\mathbb{R}^n$. More generally, every inner product is a covariant 2-tensor.

(c) The determinant, thought of as a function of n vectors, is a covariant n-tensor on $\mathbb{R}^n$.

For some purposes, it is important to generalize the notion of covariant tensors as follows. For any finite-dimensional real vector space V, we define the space of contravariant tensors on V of rank k to be the vector space

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ folds}}$$

In particular, $T^1(V) = V$, and by convention $T^0(V) = \mathbb{R}$. Because we are assuming that V is finite-dimensional, it is possible to identify this space with the set of multilinear functionals of k covectors:

$$T^k(V) \cong \{\text{multilinear functions } \alpha : V^* \times \cdots \times V^* \to \mathbb{R}\}$$

Even more generally, for any nonnegative integers k, l, we define the space of **mixed tensors on V of type (k, l)** as

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ folds}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ folds}}$$

Some of these spaces are identical:

$$\begin{aligned}
&T^{(0,0)}(V) = T^0(V) = T^0(V^*) = \mathbb{R} \\
&T^{(1,0)} = T^1(V) = V,\ T^{(0,1)}(V) = T^1(V^*) = V^* \\
&T^{(k,0)} = T^k(V),\ T^{(0,k)}(V) = T^k(V^*)
\end{aligned}$$

When V is finite-dimensional, any choice of basis for V automatically yields bases for all of the tensor spaces over V. The following corollary follows immediately from Proposition 3.1.1.7.

Corollary 3.1.1.11. *Let V be an n-dimensional real vector space. Suppose (E_i) is any basis for V and (ε^j) is the dual basis for V^*. Then the basis for the tensor spaces $T^{(k,l)}(V)$ is given by*

$$\mathcal{B} = \{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \le i_1, \ldots, i_k, j_1, \ldots, j_l \le n\}$$

Therefore, $\dim T^{(k,l)}(V) = n^{k+l}$.

In particular, once a basis is chosen for V, every covariant k-tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

For example, $T^2(V^*)$ is the space of bilinear forms on V, and every bilinear form can be written as $\beta = \beta_{ij} \varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}).

A less obvious, but extremely important, identification is the following:

$$T^{(1,1)}(V) \cong \mathrm{End}(V)$$

where $\mathrm{End}(V)$ denotes the space of linear maps from V to itself (also called the endomorphisms of V). This is a special case of the following proposition.

Proposition 3.1.1.12. *Let V be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between $T^{(k+1,l)}(V)$ and the space of multilinear maps*

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ folds}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ folds}} \to V$$

Proof. Consider the map $\Phi : \mathcal{L}(V^* \times \cdots \times V^* \times V \times \cdots \times V; V)$ defined by letting ΦA be the $(k+1, l+1)$ tensor defined by

$$\Phi A(\alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{l+1}) = \alpha_{k+1} A(\alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_l)$$

where $\alpha_i \in V^*, \beta_j \in V$. It is clear that this is an isomorphism. $\square$

We can use the result of Proposition 3.1.1.12 to define a natural operation called **trace** or **contraction**, which lowers the rank of a tensor by 2. In one special case, it is easy to describe: the operator $\mathrm{tr} : T^{(1,1)}(V) \to \mathbb{R}$ is just the trace of F when it is regarded as an endomorphism of V, or in other words the sum of the diagonal entries of any matrix representation of F. Since the trace of a linear endomorphism is basis-independent, this is well defined. More generally, we define $\mathrm{tr} : T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$ by letting $(\mathrm{tr}F)(\omega^1, \dots, \omega^k, v_1, \dots, v_l)$ be the trace of the $(1,1)$-tensor

$$F(\omega^1, \dots, \omega^k, \cdot, v_1, \dots, v_l, \cdot) \in T^{(1,1)}(V).$$

In terms of a basis, the components of $\mathrm{tr}F$ are

$$(\mathrm{tr}F)^{i_1 \dots i_k}_{j_1 \dots j_l} = F^{i_1 \dots i_k m}_{j_1 \dots j_l m}$$

In other words, just set the last upper and lower indices equal and sum.

3.1.2 Symmetric tensors

Let V be a finite-dimensional vector space. A covariant k-tensor α on V is said to be symmetric if its value is unchanged by interchanging any pair of arguments:

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

whenever $1 \leq i < j \leq k$.

The set of symmetric covariant k-tensors is a linear subspace of the space $T^k(V^*)$ of all covariant k-tensors on V; we denote this subspace by $\Sigma^k(V^*)$. There is a natural projection from $T^k(V^*)$ to $\Sigma^k(V^*)$ defined as follows. First, let $\mathfrak{S}_k$ denote the symmetric group on k elements. Let $\mathfrak{S}_k$ act on $T^k(V^*)$ by

$$\sigma \cdot \alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

We define a projection $\mathrm{Sym} : T^k(V^*) \to \Sigma^k(V^*)$ called symmetrization by

$$\mathrm{Sym}\alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \cdot \alpha$$

More explicitly, this means that

$$\mathrm{Sym}\alpha(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Proposition 3.1.2.1 (Properties of Symmetrization). *Let α be a covariant tensor on a finite-dimensional vector space.*

(a) $\mathrm{Sym}\alpha$ *is symmetric.*

(b) $\mathrm{Sym}\alpha = \alpha$ *if and only if* α *is symmetric.*

Proof. Suppose $\alpha \in T^k(V^*)$. If $\tau \in \mathfrak{S}_k$ is any permutation, then

$$\tau \cdot \mathrm{Sym}\alpha = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \tau\cdot(\sigma\cdot\alpha) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}(\tau\sigma)\cdot\alpha = \frac{1}{k!}\sum_{\eta\in\mathfrak{S}_k}\eta\cdot\alpha = \mathrm{Sym}\alpha.$$

This shows that $\mathrm{Sym}\alpha$ is symmetric.

If α is symmetric, then it follows that $\sigma\cdot\alpha = \alpha$ for every $\sigma\in\mathfrak{S}_k$, so it follows immediately that $\mathrm{Sym}\alpha = \alpha$. On the other hand, if $\mathrm{Sym}\alpha = \alpha$, then α is symmetric because part (a) shows that $\mathrm{Sym}\alpha$ is. □

If α and β are symmetric tensors on V, then $\alpha\otimes\beta$ is not symmetric in general. However, using the symmetrization operator, it is possible to define a new product that takes a pair of symmetric tensors and yields another symmetric tensor. If $\alpha\in\Sigma^k(V^*)$ and $\beta\in\Sigma^l(V^*)$, we define their **symmetric product** to be the $(k+l)$-tensor $\alpha\beta$ (denoted by juxtaposition with no intervening product symbol) given by

$$\alpha\beta = \mathrm{Sym}(\alpha\otimes\beta)$$

More explicitly, the action of $\alpha\beta$ on vectors $v_1,\dots,v_{k+l}$ is given by

$$\alpha\beta(v_1,\dots,v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in\mathfrak{S}_{k+l}}\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$$

Proposition 3.1.2.2 (Properties of the Symmetric Product). *The symmetric product is symmetric, bilinear and associative.*

Proof. We only prove (a). We first verify that for any tensors $\omega_1\in T^k(V^*), \omega_2\in T^k(V^*)$,

$$\mathrm{Sym}(\mathrm{Sym}\omega_1\otimes\omega_2) = \mathrm{Sym}(\omega_1\otimes\mathrm{Sym}\omega_2) = \mathrm{Sym}(\omega_1\otimes\omega_2) \tag{3.1.2.1}$$

Indeed, if we embedd $\mathfrak{S}_k$ into $\mathfrak{S}_{k+l}$ in the obvious way, and denote the image of σ by $\widetilde{\sigma}$. Then

$$\begin{aligned}
\mathrm{Sym}(\mathrm{Sym}\omega_1\otimes\omega_2) &= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{Sym}((\sigma\cdot\omega_1)\otimes\omega_2)\\
&= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{1}{(k+l)!}\sum_{\tau\in\mathfrak{S}_{k+l}}\tau\cdot((\sigma\cdot\omega_1)\otimes\omega_2)\\
&= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{1}{(k+l)!}\sum_{\tau\in\mathfrak{S}_{k+l}}\tau(\widetilde{\sigma}\cdot(\omega_1\otimes\omega_2))\\
&= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{1}{(k+l)!}\sum_{\tau\in\mathfrak{S}_{k+l}}(\tau\widetilde{\sigma})\cdot(\omega_1\otimes\omega_2)\\
&= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{1}{(k+l)!}\sum_{\eta\in\mathfrak{S}_{k+l}}\eta\cdot(\omega_1\otimes\omega_2)\\
&= \frac{1}{(k+l)!}\sum_{\eta\in\mathfrak{S}_{k+l}}\eta\cdot(\omega_1\otimes\omega_2)\\
&= \mathrm{Sym}(\omega_1\otimes\omega_2).
\end{aligned}$$

The other equality is proved similarly.

From here it is easy to derive the associativity, in fact,

$$\alpha(\beta\gamma) = \mathrm{Sym}(\alpha\otimes(\beta\gamma)) = \mathrm{Sym}(\alpha\otimes\mathrm{Sym}(\beta\otimes\gamma)) = \mathrm{Sym}(\alpha\otimes\beta\otimes\gamma).$$

while

$$(\alpha\beta)\gamma = \mathrm{Sym}((\alpha\beta)\otimes\gamma) = \mathrm{Sym}(\mathrm{Sym}(\alpha\otimes\beta)\otimes\gamma)) = \mathrm{Sym}(\alpha\otimes\beta\otimes\gamma).$$

Now, the commutativity comes from the observation

$$\begin{aligned}
\alpha\beta(v_1,\dots,v_{k+l}) &= \frac{1}{(k+l)!}\sum_{\sigma\in\mathfrak{S}_{k+l}} \alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}) \\
&= \frac{1}{(k+l)!}\sum_{\sigma\in\mathfrak{S}_{k+l}} \beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)}) \\
&= \frac{1}{(k+l)!}\sum_{\sigma\in\mathfrak{S}_{k+l}} \beta(v_{\sigma\tau(1)},\dots,v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)},\dots,v_{\sigma\tau(k+l)}) \\
&= \tau\cdot(\beta\alpha)(v_1,\dots,v_{k+l}).
\end{aligned}$$

where τ is the permutation given by

$$\tau = \begin{pmatrix} 1 & \cdots & l & l+1 & \cdots & k+l \\ k+1 & \cdots & k+l & 1 & \cdots & k \end{pmatrix}$$

Since $\beta\alpha$ is a symmetric tensor, we get $\alpha\beta = \beta\alpha$. □

3.1.3 Tensors and tensor fields on manifolds

Now let M be a smooth manifold with or without boundary. We define the bundle of covariant k-tensors on M by

$$T^kT^*M = \coprod_{p\in M} T^k(T_p^*M)$$

Analogously, we define the bundle of contravariant k-tensors by

$$T^kTM = \coprod_{p\in M} T^k(T_pM)$$

and the bundle of mixed tensors of type (k,l) by

$$T^{(k,l)}TM = \coprod_{p\in M} T^{(k,l)}(T_pM)$$

There are natural identifications

$$\begin{aligned}
&T^{(0,0)}TM = T^0T^*M = T^0TM = M\times\mathbb{R}, \\
&T^{(0,1)}TM = T^1T^*M = T^*M, \quad T^{(1,0)}TM = T^1TM = TM, \\
&T^{(0,k)}TM = T^kT^*M, \quad T^{(k,0)}TM = T^kTM.
\end{aligned}$$

Proposition 3.1.3.1. *The bundles T^kT^*M, T^kTM and $T^{(k,l)}TM$ have natural structures as smooth vector bundles over M.*

Proof. The smooth structures and local trivializations are defined similar to Proposition **??** and **??**. □

Any one of these bundles is called a **tensor bundle** over M. (Thus, the tangent and cotangent bundles are special cases of tensor bundles.) A section of a tensor bundle is called a **tensor field on M**. A smooth tensor field is a section that is smooth in the usual sense of smooth sections of vector bundles. Using the identifications above, we see that contravariant 1-tensor fields are the same as vector fields, and covariant 1-tensor fields are covector fields. Because a 0-tensor is just a real number, a 0-tensor field is the same as a continuous real-valued function.

The spaces of smooth sections of these tensor bundles, $\Gamma(T^kT^*M)$, $\Gamma(T^kTM)$ and $\Gamma(T^{(k,l)}TM)$, are infinite-dimensional vector spaces over $\mathbb{R}$, and modules over $C^\infty(M)$. In any smooth local coordinates (x^i), sections of these bundles can be written (using the summation convention) as

$$A = \begin{cases} A_{i_1\dots i_k}dx^{i_1}\otimes\cdots\otimes dx^{i_k}, & A\in\Gamma(T^kT^*M); \\ A^{i_1\dots i_k}\dfrac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\dfrac{\partial}{\partial x^{i_k}}, & A\in\Gamma(T^kTM); \\ A^{i_1\dots i_k}_{j_1\dots j_l}\dfrac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\dfrac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}, & A\in\Gamma(T^{(k,l)}TM). \end{cases}$$

The functions $A_{i_1\dots i_k}$, $A^{i_1\dots i_k}$ or $A^{i_1\dots i_k}_{j_1\dots j_l}$ are called the **component functions** of A in the chosen coordinates. Because smooth covariant tensor fields occupy most of our attention, we adopt the following shorthand notation for the space of all smooth covariant k-tensor fields:

$$\mathcal{T}^k(M) = \Gamma(T^k T^* M).$$

Proposition 3.1.3.2 (Smoothness Criterion for Tensor Fields). *Let M be a smooth manifold with or without boundary, and let $A : M \to T^{(k,l)}TM$ be a rough section. The following are equivalent.*

(a) A is smooth.

(b) In every smooth coordinate chart, the component functions of A are smooth.

(c) Each point of M is contained in some coordinate chart in which A has smooth component functions.

(d) If $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\omega_1, \dots, \omega_l$ are covector fields, then the function

$$A(\omega_1, \dots, \omega_l, X_1, \dots, X_k) : M \to \mathbb{R}, \quad p \mapsto A_p(\omega_1|_p, \dots, \omega_l|_p, X_1|_p, \dots, X_k|_p)$$

is smooth.

(e) Whenever $X_1, \dots, X_k$, $\omega_1, \dots, \omega_l$ are smooth vector fields and covector fields defined on some open subset $U \subseteq M$, the function $A(\omega_1, \dots, \omega_l, X_1, \dots, X_k)$ is smooth on U.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ and $(a) \Rightarrow (d)$, $(a) \Rightarrow (e)$ are immediate. So we prove $(d) \Rightarrow (e)$ and $(e) \Rightarrow (a)$ to finish the proof.

Assume the condition of (d), then for any $X_1, \dots, X_k$ and $\omega_1, \dots, \omega_l$ defined on some open subset $U \subseteq M$, we can extend them to M by a bump function near a given point $p \in U$. Then by our hypothesis, $A(X_1, \dots, X_k)$ is smooth at p. Since p can be chosen arbitrarily, $A(X_1, \dots, X_k)$ is smooth on U.

For $(e) \Rightarrow (a)$, just choose coordinate vector fields and covector field in each chart of M. □

Proposition 3.1.3.3. *Suppose M is a smooth manifold with or without boundary, $A \in \mathcal{T}^k(M)$, $B \in \mathcal{T}^l(M)$ and $f \in C^\infty(M)$. Then fA and $A \otimes B$ are also smooth tensor fields, whose components in any smooth local coordinate chart are*

$$(fA)_{i_1\dots i_k} = f A_{i_1\dots i_k}, \quad (A \otimes B)_{i_1\dots i_{k+l}} = A_{i_1\dots i_k} B_{i_{k+1}\dots i_{k+l}}.$$

Proposition 3.1.3.2(d) shows that if A is a smooth covariant k-tensor field on M and $X_1, \dots, X_k$ are smooth vector fields, then $A(X_1, \dots, X_k)$ is a smooth realvalued function on M. Thus A induces a map

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ folds}} \to C^\infty(M)$$

It is easy to see that this map is multilinear over $\mathbb{R}$. In fact, more is true: it is multilinear over $C^\infty(M)$, which means that for $f, f' \in C^\infty(M)$ and $X_i, X_i' \in \mathfrak{X}(M)$, we have

$$A(X_1, \dots, fX_i + f'X_i', \dots, X_k) = fA(X_1, \dots, X_i, \dots, X_k) + f'A(X_1, \dots, X_i', \dots, X_k)$$

Lemma 3.1.3.4 (Tensor Characterization Lemma). *A map*

$$\mathcal{A} : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ folds}} \to C^\infty(M) \tag{3.1.3.1}$$

is induced by a smooth covariant k-tensor field as above if and only if it is multilinear over $C^\infty(M)$.

Proof. We already noted that if A is a smooth covariant k-tensor field, then the map $(X_1, \dots, X_k) \mapsto A(X_1, \dots, X_k)$ is multilinear over $C^\infty(M)$. To prove the converse, we proceed as in the proof of the bundle homomorphism characterization lemma (Lemma **??**).

Suppose, therefore, that A is a map as in (3.1.3.1), and assume that A is multilinear over $C^\infty(M)$. We show first that A acts locally. If X_i is a smooth vector field that vanishes on a neighborhood U of p, we can choose a bump function ψ supported in U such that $\psi(p) = 1$; then because $\psi X_i \equiv 0$ we have

$$0 = \mathcal{A}(X_1, \dots, \psi X_i, \dots, X_k) = \psi(p)\mathcal{A}_p(X_1, \dots, X_i, \dots, X_k)(p)$$

It follows as in the proof of Lemma **??** that the value of $\mathcal{A}(X_1,\ldots,\psi X_i,\ldots,X_k)$ at p depends only on the values of $X_1,\ldots,X_k$ in a neighborhood of p.

Next we show that $\mathcal{A}$ actually acts pointwise. If $X_i|_p = 0$, then in any coordinate chart centered at p we can write $X_i = X_i^j \partial/\partial x^j$, where the component functions X_i^j all vanish at p. By the extension lemma for vector fields, we can find global smooth vector fields E_j on M such that $E_j = \partial/\partial x^j$ in some neighborhood of p; and similarly the locally defined functions X_i^j can be extended to global smooth functions f_i^j on M that agree with X_i^j in a neighborhood of p. It follows from the multilinearity of $\mathcal{A}$ over $C^\infty(M)$ and the fact that $f_i^j E_j = X_i$ in a neighborhood of p that

$$\begin{aligned}\mathcal{A}(X_1,\ldots,X_i,\ldots,X_k)(p) &= \mathcal{A}(X_1,\ldots,f_i^j E_j,\ldots,X_k)(p)\\ &= f_i^j(p)\mathcal{A}(X_1,\ldots,E_j,\ldots,X_k)(p) = 0.\end{aligned}$$

It follows by linearity that $A(X_1,\ldots,X_k)$ depends only on the value of X_i at p.

Now we define a rough tensor field $A : M \to T^k T^* M$ by

$$A_p(v_1,\ldots,v_k) = \mathcal{A}(V_1,\ldots,V_k)(p)$$

for $p \in M$ and $v_1,\ldots,v_k \in T_pM$, where $V_1,\ldots,V_k$ are any extensions of $v_1,\ldots,v_k$ to smooth global vector fields on M. The discussion above shows that this is independent of the choices of extensions, and the resulting tensor field is smooth by Proposition 3.1.3.2(d). □

A **symmetric tensor field** on a manifold (with or without boundary) is simply a covariant tensor field whose value at each point is a symmetric tensor. The symmetric product of two or more tensor fields is defined pointwise, just like the tensor product. Thus, for example, if A and B are smooth covector fields, their symmetric product is the smooth 2-tensor field AB, which is given by

$$AB = \frac{1}{2}(A \otimes B + B \otimes A)$$

3.1.3.1 Pullbacks of tensor fields

Just like covector fields, covariant tensor fields can be pulled back by a smooth map to yield tensor fields on the domain. (This construction works only for covariant tensor fields, which is one reason why we focus most of our attention on the covariant case.)

Suppose $F : M \to N$ is a smooth map. For any point $p \in M$ and any k-tensor $\alpha \in T^k(T^*_{F(p)}M)$, we define a tensor $dF_p^*(\alpha)$, called the **pointwise pullback** of α by F at p, by

$$dF_p^*(\alpha)(v_1,\ldots,v_k) = \alpha\big(dF_p(v_1),\ldots,dF_p(v_k)\big)$$

for any $v_1,\ldots,v_k \in T_pM$. If A is a covariant k-tensor field on N, we define a rough k-tensor field F^*A on M, called the **pullback of $\boldsymbol{A}$ by $\boldsymbol{F}$**, by

$$(F^*A)_p = dF_p^*(A_{F(p)}).$$

This tensor field acts on vectors $v_1,\ldots,v_k \in T_pM$ by

$$(F^*A)_p(v_1,\ldots,v_k) = A_{F(p)}\big(dF_p(v_1),\ldots,dF_p(v_k)\big).$$

Proposition 3.1.3.5 (Properties of Tensor Pullbacks). *Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.*

(a) $F^*(fA) = (f \circ F)F^*A$.

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$.

(c) $F^*(A + B) = F^*A + F^*B$.

(d) *F^*B is a tensor field, and is smooth if B is smooth.*

(e) $(G \circ F)^* = F^* \circ G^*$.

(f) $\mathrm{id}^* = \mathrm{id}$.

Proof. For $p \in M$, we compute

$$(F^*(fA))_p = dF_p^*\big(f(F(p))A_{F(p)}\big) = f(F(p))dF^*(A_{F(p)}) = f \circ F(p)(F^*A)_p = \big((f \circ F)F^*A\big)_p$$

and

$$\begin{aligned} F^*(A \otimes B)(v_1, \dots, v_{k+l}) &= (A \otimes B)_{F(p)}\big(dF_p(v_1), \dots, dF_p(v_{k+l})\big) \\ &= A_{F(p)}\big(dF_p(v_1), \dots, dF_p(v_k)\big) B_{F(p)}\big(dF_p(v_{k+1}), \dots, dF_p(v_{k+l})\big) \\ &= (F^*A)_p(v_1, \dots, v_k)(F^*B)_p(v_{k+1}, \dots, _{k+l}) \\ &= (F^*A \otimes F^*B)_p(v_1, \dots, v_{k+l}) \end{aligned}$$

The statement (c) is proved similarly.

Now part (d) is an application of Proposition 3.1.3.2, and part (e), (f) follows easily. □

If f is a continuous real-valued function (i.e., a 0-tensor field) and B is a k-tensor field, then it is consistent with our definitions to interpret $f \otimes B$ as fB, and F^*f as $f \circ F$. With these interpretations, property (a) is really just a special case of (b).

The following proposition is an immediate consequence of Proposition 3.1.3.5.

Proposition 3.1.3.6. *Let $F : M \to N$ be smooth, and let B be a covariant k-tensor field on N. If $p \in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in a neighborhood of p:*

$$F^*(B_{i_1 \dots i_k} dy^{i_1} \otimes \dots \otimes dy^{i_k}) = (B_{i_1 \dots i_k} \circ F) d(y^{i_1} \circ F) \otimes \dots \otimes d(y^{i_k} \circ F)$$

Example 3.1.3.7 (Pullback of a Tensor Field). Let $M = \{(r, \theta) : r > 0, |\theta| < \pi/2\}$ and $N = \{(x, y) : x > 0\}$ and let $F : M \to \mathbb{R}^2$ be the smooth map $F(r, \theta) = (r\cos\theta, r\sin\theta)$. The pullback of the tensor field $A = x^{-2} dy \otimes dy$ by F can be computed easily by substituting $x = r\cos\theta$, $y = r\sin\theta$ and simplifying:

$$\begin{aligned} F^*A &= (r\cos\theta)^{-2} d(r\sin\theta) \otimes d(r\sin\theta) \\ &= (r\cos\theta)^{-2}(\sin\theta\, dr + r\cos\theta\, d\theta) \otimes (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r^{-2}\tan^2\theta\, dr \otimes dr + r^{-1}\tan\theta(d\theta \otimes dr + dr \otimes d\theta) + d\theta \otimes d\theta. \end{aligned}$$

3.1.3.2 Lie derivatives of tensor fields

The Lie derivative operation can be extended to tensor fields of arbitrary rank. As usual, we focus on covariant tensors; the analogous results for contravariant or mixed tensors require only minor modifications.

Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is its flow. (For simplicity, we discuss only the case $\partial M = \varnothing$ here, but these definitions and results carry over essentially unchanged to manifolds with boundary as long as V is tangent to the boundary, so that its flow exists by Theorem 1.2.5.11.) For any $p \in M$, if t is sufficiently close to zero, then θ_t is a diffeomorphism from a neighborhood of p to a neighborhood of $\theta_t(p)$, so $d(\theta_t)_p^*$ pulls back tensors at $\theta_t(p)$ to ones at p by the formula

$$d(\theta_t)_p^*(A_{\theta_t(p)})(v_1, \dots, v_k) = A_{\theta_t(p)}\big(d(\theta_t)_p(v_1), \dots, d(\theta_t)_p(v_k)\big)$$

Note that $d(\theta_t)_p^*(A_{\theta_t(p)})$ is just the value of the pullback tensor field θ_t^*A at p.

Given a smooth covariant tensor field A on M, we define the Lie derivative of A with respect to V, denoted by $\mathfrak{L}_V A$, by

$$(\mathfrak{L}_V A)_p = \frac{d}{dt}\bigg|_{t=0} (\theta_t^* A)_p = \lim_{t \to 0} \frac{d(\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t} \tag{3.1.3.2}$$

provided the derivative exists. Because the expression being differentiated lies in $T^k(T_p^*M)$ for all t, $(\mathfrak{L}_V A)_p$ makes sense as an element of $T^k(T_p^*M)$. The following lemma is an analogue of Lemma 1.2.6.1, and is proved in exactly the same way.

Lemma 3.1.3.8. *With M, V, and A as above, the derivative $(\mathfrak{L}_V A)_p$ exists for every $p \in M$ and defines $\mathfrak{L}_V A$ as a smooth tensor field on M.*

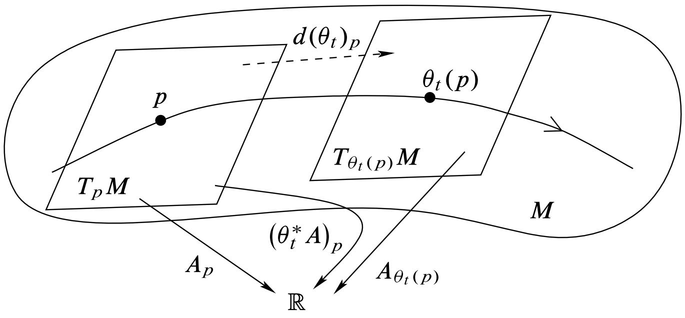


Figure 3.1: The Lie derivative of a tensor field.

Proposition 3.1.3.9. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose f is a smooth real-valued function (regarded as a 0-tensor field) on M, and A, B are smooth covariant tensor fields on M.*

(a) $\mathfrak{L}_V f = Vf$.

(b) $\mathfrak{L}_V(fA) = (Vf)A + f\mathfrak{L}_V A$.

(c) $\mathfrak{L}_V(A \otimes B) = (\mathfrak{L}_V A) \otimes B + A \otimes (\mathfrak{L}_V B)$.

(d) If $X_1, \ldots, X_k$ are smooth vector fields and A is a smooth k-tensor field,

$$\mathfrak{L}_V\big(A(X_1, \ldots, X_k)\big) = (\mathfrak{L}_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, \mathfrak{L}_V X_i, \ldots, X_k)$$

Proof. Let θ be the flow of V. For a real-valued function f, we can write

$$\theta_t^* f(p) = (f \circ \theta_t)(p) = f \circ \theta^{(p)}(t)$$

Thus the definition of $\mathfrak{L}_V f$ reduces to the ordinary derivative with respect to t of the composite function $f \circ \theta^{(p)}(t)$. Because $\theta^{(p)}(t)$ is an integral curve of V, it follows from Proposition **??** that

$$(\mathfrak{L}_V f)_p = \frac{d}{dt}\Big|_{t=0} f \circ \theta^{(p)} = df_p\big(\dot{\theta}^{(p)}(0)\big) = df_p(V_p) = Vf(p)$$

This proves (a).

The other assertions can be proved by the technique we used in Theorem 1.2.6.2: in a neighborhood of a regular point for V, if (u^i) are coordinates in which $V = \partial/\partial u^1$, then it follows immediately from the definition that $\mathfrak{L}_V$ acts on a tensor field simply by taking the partial derivative of its coefficients with respect to u^1, and (b)–(d) all follow from the ordinary product rule. The same relations hold on the support of V by continuity, and on the complement of the support because the flow of V is trivial there. □

One consequence of this proposition is the following formula expressing the Lie derivative of any smooth covariant tensor field in terms of Lie brackets and ordinary directional derivatives of functions, which allows us to compute Lie derivatives without first determining the flow.

Corollary 3.1.3.10. *If V is a smooth vector field and A is a smooth covariant k-tensor field, then for any smooth vector fields $X_1, \ldots, X_k$,*

$$(\mathfrak{L}_V A)(X_1, \ldots, X_k) = V\big(A(X_1, \ldots, X_k)\big) - \sum_{i=1}^{k} A(X_1, \ldots, [V, X_i], \ldots, X_k). \tag{3.1.3.3}$$

Proof. From Proposition 3.1.3.9 (d) we have

$$(\mathfrak{L}_V A)(X_1, \ldots, X_k) = \mathfrak{L}_V(A(X_1, \ldots, X_k)) - \sum_{i=1}^{k} A(X_1, \ldots, \mathfrak{L}_V X_i, \ldots, X_k).$$

Replace $\mathfrak{L}_V(A(X_1, \ldots, X_k))$ by $V\big(A(X_1, \ldots, X_k)\big)$ and $\mathfrak{L}_V X_i$ by $[V, X_i]$, we get the claim. □

Corollary 3.1.3.11. *If $f \in C^\infty(M)$, then $\mathfrak{L}_V(df) = d(\mathfrak{L}_V f)$.*

Proof. Using (3.1.3.3), for any $X \in \mathfrak{X}(M)$ we compute

$$\begin{aligned}(\mathfrak{L}_V df)(X) &= V\big(df(X)\big) - df([V,X]) = VXf - (VX - XV)f \\ &= XVf = X(\mathfrak{L}_V f) = d(\mathfrak{L}_V f)(X),\end{aligned}$$

thus the claim follows. □

One drawback of formula (3.1.3.3) is that in order to calculate what $\mathfrak{L}_V A$ does to vectors $v_1, \dots, v_k$ at a point $p \in M$, one must first extend them to vector fields in a neighborhood of p. But Corollary 3.1.3.11 lead to an easy method for computing Lie derivatives of smooth tensor fields in coordinates that avoids this problem, since any tensor field can be written locally as a linear combination of functions multiplied by tensor products of exact 1-forms. The next example illustrates the technique.

Example 3.1.3.12. Suppose A is an arbitrary smooth covariant 2-tensor field, and V is a smooth vector field. We compute the Lie derivative $\mathfrak{L}_V A$ in smooth local coordinates (x^i). First, we observe that

$$\mathfrak{L}_V dx^i = d(\mathfrak{L}_V x^i) = d(Vx^i) = dV^i.$$

Therefore,

$$\begin{aligned}\mathfrak{L}_V A &= \mathfrak{L}_V(A_{ij}\, dx^i \otimes dx^j) \\ &= \mathfrak{L}_V(A_{ij})\, dx^i \otimes dx^j + A_{ij}\mathfrak{L}_V(dx^i \otimes dx^j) \\ &= VA_{ij}\, dx^i \otimes dx^j + A_{ij}\mathfrak{L}_V(dx^i) \otimes dx^j + A_{ij}\, dx^i \otimes \mathfrak{L}_V(dx^j) \\ &= VA_{ij}\, dx^i \otimes dx^j + A_{ij}\, dV^i \otimes dx^j + A_{ij}\, dx^i \otimes dV^j \\ &= VA_{ij}\, dx^i \otimes dx^j + A_{ij}\frac{\partial V^i}{\partial x^k}\, dx^k \otimes dx^j + A_{ij}\frac{\partial V^j}{\partial x^k}\, dx^i \otimes dx^k \\ &= \Big(VA_{ij} + A_{kj}\frac{\partial V^k}{\partial x^i} + A_{ik}\frac{\partial V^k}{\partial x^j}\Big) dx^i \otimes dx^j.\end{aligned}$$

Recall that the Lie derivative of a vector field W with respect to V is zero if and only if W is invariant under the flow of V (see Theorem 1.2.7.1). It turns out that the Lie derivative of a covariant tensor field has exactly the same interpretation. If A is a smooth tensor field on M and θ is a flow on M, we say that A is **invariant under θ** if for each t, the map θ_t pulls A back to itself wherever it is defined; more precisely, this means

$$d(\theta_t)^*_p(A_{\theta_t(p)}) = A_p$$

for all (t, p) in the domain of θ. If θ is a global flow, this is equivalent to $\theta_t^* A = A$ for all $t \in \mathbb{R}$.

In order to prove the connection between Lie derivatives and invariance under flows, we need the following proposition, which shows how the Lie derivative can be used to compute t-derivatives at times other than $t = 0$. It is a generalization to tensor fields of Proposition 1.2.6.4.

Proposition 3.1.3.13. *Suppose M is a smooth manifold with or without boundary and $V \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume in addition that V is tangent to ∂M. Let θ be the flow of V. For any smooth covariant tensor field A and any (t_0, p) in the domain of θ,*

$$\frac{d}{dt}\Big|_{t=t_0}(\theta_t^* A)_p = \big(\theta_{t_0}^*(\mathfrak{L}_V A)\big)_p$$

Proof. After expanding the definitions of the pullbacks, we see that we have to prove

$$\frac{d}{dt}\Big|_{t=0} d(\theta_t)^*_p(A_{\theta_t(p)}) = d(\theta_{t_0})^*_p\big((\mathfrak{L}_V A)_{\theta_{t_0}(p)}\big)$$

Just as in the proof of Proposition 1.2.6.4, the change of variables $t = s + t_0$ yields

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} d(\theta_t)^*_p(A_{\theta_t(p)}) = \frac{d}{ds}\Big|_{s=0} d(\theta_{s+t_0})^*_p(A_{\theta_{s+t_0}(p)}) = \frac{d}{ds}\Big|_{s=0} d(\theta_{t_0})^*_p \circ d(\theta_s)^*_{\theta_{t_0}(p)}(A_{\theta_s(\theta_{t_0}(p))}) \\ &= d(\theta_{t_0})^*_p \frac{d}{ds}\Big|_{s=0} d(\theta_s)^*_{\theta_{t_0}(p)}(A_{\theta_s(\theta_{t_0}(p))}) = d(\theta_{t_0})^*_p\big((\mathfrak{L}_V A)_{\theta_{t_0}(p)}\big).\end{aligned}$$

□

Theorem 3.1.3.14. *Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $\mathfrak{L}_V A = 0$.*

Proof. This follows from the definition of $\mathfrak{L}_V A$. □

3.1.4 Exercise

Exercise 3.1.1. Let M be a smooth n-manifold, and let A be a smooth covariant k-tensor field on M. If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth charts on M, we can write

$$A = A_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \widetilde{A}_{j_1 \dots j_k} d\widetilde{x}^{j_1} \otimes \cdots \otimes d\widetilde{x}^{j_k}.$$

Compute the transformation law.

Proof. Recall the formulas

$$d\widetilde{x}^j = \frac{\partial \widetilde{x}^j}{\partial x^i} dx^i.$$

Thus

$$A = \widetilde{A}_{j_1 \dots j_k} d\widetilde{x}^{j_1} \otimes \cdots \otimes d\widetilde{x}^{j_k} = \widetilde{A}_{j_1 \dots j_k} \frac{\partial \widetilde{x}^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial \widetilde{x}^{j_k}}{\partial x^{i_k}} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

This gives the claim. □

Exercise 3.1.2. Suppose M is a smooth manifold, A is a smooth covariant tensor field on M, and $V, W \in \mathfrak{X}(M)$. Show that

$$\mathfrak{L}_V \mathfrak{L}_W A - \mathfrak{L}_W \mathfrak{L}_V A = \mathfrak{L}_{[V,W]} A.$$

Proof. Let $X_1, \dots, X_k \in \mathfrak{X}(M)$, then

$$\begin{aligned}
(\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) &= V\big(\mathfrak{L}_W A(X_1, \dots, X_k)\big) - \sum_{i=1}^k \mathfrak{L}_W A(X_1, \dots, [V, X_i], \dots, X_k) \\
&= VW\big(A(X_1, \dots, X_k)\big) - \sum_{i=1}^k V\big(A(X_1, \dots, [W, X_i], \dots, X_k)\big) \\
&- \sum_{i=1}^k W\big(A(X_1, \dots, [V, X_i], \dots, X_k)\big) + \sum_{i \neq j} A(X_1, \dots, [W, X_i], \dots, [V, X_j], \dots, X_k) \\
&+ \sum_{i=1}^k A\big(X_1, \dots, \big[W, [V, X_i]\big], \dots, X_k\big).
\end{aligned}$$

And thus

$$\begin{aligned}
(\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) - (\mathfrak{L}_W \mathfrak{L}_V A)(X_1, \dots, X_k) &= (VW - WV)\big(A(X_1, \dots, X_k)\big) \\
+ \sum_{i=1}^k A\big(X_1, \dots, \big[W, [V, X_i]\big], \dots, X_k\big) &- \sum_{i=1}^k A\big(X_1, \dots, \big[V, [W, X_i]\big], \dots, X_k\big).
\end{aligned}$$

Therefore, we compute using the Jacobi identity

$$\begin{aligned}
(\mathfrak{L}_{[V,W]} A)(X_1, \dots, X_k) &= (VW - WV)(A(X_1, \dots, X_k)) - \sum_{i=1}^k A(X_1, \dots, \big[[V, W], X_i\big], \dots, X_k) \\
&= (VW - WV)(A(X_1, \dots, X_k)) + \sum_{i=1}^k A(X_1, \dots, \big[[W, X_i], V\big] + \big[[X_i, V], W\big], \dots, X_k) \\
&= (VW - WV)(A(X_1, \dots, X_k)) + \sum_{i=1}^k A(X_1, \dots, \big[W, [V, X_i]\big] - \big[[V, W], X_i\big], \dots, X_k) \\
&= (\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) - (\mathfrak{L}_W \mathfrak{L}_V A)(X_1, \dots, X_k).
\end{aligned}$$

□

Exercise 3.1.3. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. Show that the Lie derivative operators on covariant tensor fields, $\mathfrak{L}_V : \mathcal{T}^k(M) \to \mathcal{T}^k(M)$ for $k \geq 0$, are uniquely characterized by the following properties:

(a) $\mathfrak{L}_V$ is linear over $\mathbb{R}$.

(b) $\mathfrak{L}_V f = Vf$ for $f \in \mathcal{T}^0(M) = C^\infty(M)$.

(c) $\mathfrak{L}_V(A \otimes B) = \mathfrak{L}_V A \otimes B + A \otimes \mathfrak{L}_V B$ for $A \in \mathcal{T}^k(M)$ and $B \in \mathcal{T}^l(M)$.

(d) $\mathfrak{L}_V(\omega(X)) = (\mathfrak{L}_V \omega)(X) + \omega([V, X])$ for $\omega \in \mathcal{T}^1(M), X \in \mathfrak{X}(M)$.

Proof. Let $\mathfrak{L}$ be an operator satisfying (a)–(d), we prove that $\mathfrak{L} = \mathfrak{L}_V$.

- First, by (b) and (d) we have for $\omega \in \mathcal{T}^1(M), X \in \mathfrak{X}(M)$:
$$(\mathfrak{L}\omega)(X) = V(\omega(X)) - \omega([V, X]) = (\mathfrak{L}_V \omega)(X)$$
- Then, by induction, we can prove
$$\mathfrak{L}(\omega_1 \otimes \cdots \otimes \omega_k) = \mathfrak{L}_V(\omega_1 \otimes \cdots \otimes \omega_k).$$
- By taking local charts, we verify that $\mathfrak{L} = \mathfrak{L}_V$.

□

3.2 Riemannian metrics

3.2.1 Riemannian manifolds

The most important examples of symmetric tensors on a vector space are inner products. Any inner product allows us to define lengths of vectors and angles between them, and thus to do Euclidean geometry.

Transferring these ideas to manifolds, we obtain one of the most important applications of tensors to differential geometry. Let M be a smooth manifold with or without boundary. A **Riemannian metric** on M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point. A **Riemannian manifold** is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M. One sometimes simply says M is a Riemannian manifold if M is understood to be endowed with a specific Riemannian metric. A Riemannian manifold with boundary is defined similarly.

If g is a Riemannian metric on M, then for each $p \in M$, the 2-tensor g_p is an inner product on T_pM. Because of this, we often use the notation $\langle v, w\rangle_g$ to denote the real number $g_p(v, w)$ for $v, w \in T_pM$.

In any smooth local coordinates (x^i), a Riemannian metric can be written
$$g = g_{ij}\, dx^i \otimes dx^j,$$
where (g_{ij}) is a symmetric positive definite matrix of smooth functions. The symmetry of g allows us to write g also in terms of symmetric products as follows:
$$g = g_{ij}\, dx^i \otimes dx^j = \frac{1}{2}(g_{ij}\, dx^i \otimes dx^j + g_{ji}\, dx^j \otimes dx^i) = \frac{1}{2} g_{ij}(dx^i \otimes dx^j + dx^j \otimes dx^i) = g_{ij}\, dx^i dx^j.$$

Example 3.2.1.1 (The Euclidean Metric). The simplest example of a Riemannian metric is the Euclidean metric $\bar{g}$ on $\mathbb{R}^n$, given in standard coordinates by
$$\bar{g} = \delta_{ij}\, dx^i \otimes dx^j.$$
It is common to abbreviate the symmetric product of a tensor α with itself by α^2, so the Euclidean metric can also be written
$$\bar{g} = (dx^1)^2 + \cdots + (dx^n)^2.$$

Example 3.2.1.2 (Product Metrics). If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, we can define a Riemannian metric $\widehat{g} = g \oplus \widetilde{g}$ on the product manifold $M \times \widetilde{M}$, called the product metric, as follows:
$$\widehat{g}\big((v, \widetilde{v}), (w, \widetilde{w})\big) = g(v, w) + \widetilde{g}(\widetilde{v}, \widetilde{w})$$
for any $(v, \widetilde{v}), (w, \widetilde{w}) \in T_pM \oplus T_q\widetilde{M}$. Given any local coordinates $(x^1, \ldots, x^n)$ for M and $(y^1, \ldots, y^m)$ for $\widetilde{M}$, we obtain local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ for $M \times \widetilde{M}$, and the product metric is represented locally by the block diagonal matrix
$$(\widehat{g}_{ij}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \widetilde{g}_{ij} \end{pmatrix}$$

Proposition 3.2.1.3 (Existence of Riemannian Metrics). *Every smooth manifold with or without boundary admits a Riemannian metric.*

Proof. Let M be a smooth manifold with or without boundary, and choose a covering of M by smooth coordinate charts $(U_\alpha, \varphi_\alpha)$. In each coordinate domain, there is a Riemannian metric $g_\alpha = \varphi_\alpha^* \bar{g}$, whose coordinate expression is $\delta_{ij}\, dx^i dx^j$. Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{U_\alpha\}$, and define

$$g = \sum_\alpha \psi_\alpha g_\alpha,$$

with each term interpreted to be zero outside $\operatorname{supp}(\psi_\alpha)$. By local finiteness, there are only finitely many nonzero terms in a neighborhood of each point, so this expression defines a smooth tensor field. It is obviously symmetric, so only positivity needs to be checked. If $v \in T_pM$ is any nonzero vector, then

$$g_p(v, v) = \sum_\alpha \psi_\alpha(p) g_\alpha|_p(v, v).$$

This sum is nonnegative, because each term is nonnegative. At least one of the functions ψ_α is strictly positive at p. Because $g_\alpha|_p(v, v) > 0$, it follows that $g_p(v, v) > 0$. □

Below are just a few of the geometric constructions that can be defined on a Riemannian manifold (M, g) with or without boundary.

- The length or norm of a tangent vector $v \in T_pM$ is defined to be

$$|v|_g = \langle v, v\rangle_g^{1/2} = g_p(v, v)^{1/2}$$

- The angle between two nonzero tangent vectors $v, w \in T_pM$ is the unique $\theta \in [0, \pi]$ satisfying

$$\cos\theta = \frac{\langle v, w\rangle_g}{|v|_g |w|_g}$$

- Tangent vectors $v, w \in T_pM$ are said to be orthogonal if $\langle v, w\rangle_g = 0$. This means either one or both vectors are zero, or the angle between them is $\pi/2$.

One highly useful tool for the study of Riemannian manifolds is orthonormal frames. Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. Just as we did for the case of $\mathbb{R}^n$ in Section 1.1, we say a local frame $(E_1, \ldots, E_n)$ for M on an open subset $U \subseteq M$ is an **orthonormal frame** if the vectors $(E_1|_p, \ldots, E_n|_p)$ form an orthonormal basis for TpM at each point $p \in U$, or equivalently if $\langle E_i, E_j\rangle_g = \delta_{ij}$, in which case g has the local expression

$$g = (\varepsilon^1)^2 + \cdots + (\varepsilon^n)^2.$$

where $(\varepsilon^i)^2$ denotes the symmetric product $(\varepsilon^i)^2 = \varepsilon^i \otimes \varepsilon^i$.

Example 3.2.1.4. The coordinate frame $(\partial/\partial x^i)$ is a global orthonormal frame for $\mathbb{R}^n$ with the Euclidean metric.

Example 3.2.1.5. The frame (E_1, E_2) on $\mathbb{R}^2 - \{0\}$ defined in Example 1.1.1.11 is a local orthonormal frame for $\mathbb{R}^2$. As we observed in Example 1.2.7.4, it is not a coordinate frame in any coordinates.

The next proposition is proved in just the same way as Lemma 1.1.1.12, with the Euclidean dot product replaced by the inner product $\langle\cdot,\cdot\rangle_g$.

Proposition 3.2.1.6. *Suppose (M, g) is a Riemannian manifold with or without boundary, and (X_j) is a smooth local frame for M over an open subset $U \subseteq M$. Then there is a smooth orthonormal frame (E_j) over U such that* $\operatorname{span}(E_1|_p, \ldots, E_j|_p) = \operatorname{span}(X_1|_p, \ldots, X_j|_p)$ *for each $1 \le j \le n$ and each $p \in U$.*

Proof. Applying the GramSchmidt algorithm to the vectors $(X_1|_p, \ldots, X_n|_p)$ at each $p \in U$, we obtain an ordered n-tuple of rough orthonormal vector fields $(E_1, \ldots, E_n)$ over U satisfying the span conditions. Because the vectors whose norms appear in the denominators of the GramSchmidt formulas are nowhere vanishing, it follows that each vector field E_j is smooth. The last statement of the proposition follows by applying this construction to any smooth local frame in a neighborhood of p. In particular, for every $p \in M$, there is a smooth orthonormal frame (E_j) defined on some neighborhood of p. □

Corollary 3.2.1.7 (Existence of Local Orthonormal Frames). *Let (M, g) be a Riemannian manifold with or without boundary. For each $p \in M$, there is a smooth orthonormal frame on a neighborhood of p.*

Observe that Corollary 3.2.1.7 does not show that there are smooth coordinates on a neighborhood of p for which the *coordinate frame* is orthonormal. In fact, this is possible only when the metric is flat, that is, locally isometric to the Euclidean metric.

For a Riemannian manifold (M, g) with or without boundary, we define the unit tangent bundle to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM = \{(p, v) \in TM : |v|_g = 1\}$$

Proposition 3.2.1.8 (Properties of the Unit Tangent Bundle). *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a smooth, properly embedded codimension-1 submanifold with boundary in TM, with $\partial(UTM) = \pi^{-1}(\partial M)$ (where $\pi : UTM \to M$ is the canonical projection). The unit tangent bundle is connected if and only if M is connected when $n > 1$, and compact if and only if M is compact.*

Proof. By the local orthonormal frames, we can replace M by $\mathbb{R}^n$. If U is open in $\mathbb{R}^n$, then the unit tangent bundle is diffeomorphic to $U \times S^{n-1}$, and therefore has codimension 1. It is properly embedded since S^{n-1} is properly embedded in $\mathbb{R}^n$. Similarly, if U is a half ball in $\mathbb{H}^n$, we can see that the tangent bundle is diffeomorphic to $U \times S^{n-1}$, and therefore has boundary charts. This also proves that $\partial(UTM) = \pi^{-1}(\partial M)$.

For the last claim, note that the projection $\pi : UTM \to M$ is surjective and open (Proposition **??**), so M is connect if UTM is, and compact if UTM is. Now we show the converse.

If M is compact, then for any $p \in M$ we can find a comapct subset V_p such that the interior of V_p is open and the tangent bundle of V_p is trivial. Then we can find $p_1, \dots, p_n \in M$ such that $M \subseteq \bigcup_{i=1}^n \operatorname{Int}(V_{p_i})$, and so

$$UTM = \bigcup_{i=1}^n \pi^{-1}(\operatorname{Int}(V_{p_i})) \subseteq \bigcup_{i=1}^n \pi^{-1}(V_{p_i}).$$

Since the tangent bundle of V_{p_i} is trivial, we have $\pi^{-1}(V_{p_i}) = V_{p_i} \times S^{n-1}$. This implies that UTM is compact since it is the union of finitely many compact subsets, so it is compact.

If M is connected and $n > 1$, then the fiber of UTM is diffeomorphic to S^{n-1} and so it connected. Assume that $UTM = U \cup V$ for two disjoint open subsets, then $M = \pi(E) = \pi(U) \cup \pi(V)$ is the union of the open subsets $\pi(U)$ and $\pi(V)$ wich are not disjoint since M is connected. Let $p \in \pi(U) \cap \pi(V)$, then there exists $v_1 \in U, v_2 \in V$ such that $\pi(v_1) = \pi(v_2) = p$. This means $E_p \cap U \neq \varnothing$ and $E_p \cap V \neq \varnothing$, so $E_p \cap U$ and $E_p \cap V$ is a partition of E_p. But E_p is connected, this is a contradiction. □

3.2.2 Methods for constructing riemannian metrics

3.2.2.1 Pullback metrics

Suppose M, N are smooth manifolds with or without boundary, g is a Riemannian metric on N, and $F : M \to N$ is smooth. The pullback F^*g is a smooth 2-tensor field on M. If it is positive definite, it is a Riemannian metric on M, called the pullback metric determined by F. The next proposition shows when this is the case.

Proposition 3.2.2.1 (Pullback Metric Criterion). *Suppose $F : M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a Riemannian metric on M if and only if F is a smooth immersion.*

Proof. F^*g is a Riemannian metric if and only if for all $p \in M$, $F^*g(v, v) = 0$ implies $v = 0$ for $v \in T_pM$, wich means $dF_p(v) = 0$ implies $v = 0$ for any $v \in T_pM$, which is to say F is a immersion. □

If the coordinate representation for an immersion is known, then the pullback metric is easy to compute using the usual algorithm for computing pullbacks.

Example 3.2.2.2. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(u, v) = (u \cos v, u \sin v, v)$$

It is a proper injective smooth immersion, and thus it is an embedding by Proposition **??**. The pullback metric $F^*\bar{g}$ can be computed by substituting the coordinate functions for F in place of x, y, z in the formula for $\bar{g}$:

$$\begin{aligned} F^*\bar{g} &= (du \cos v)^2 + (du \sin v)^2 + (dv)^2 \\ &= (\cos v\, du - u \sin v\, dv)^2 + (\sin v\, du + u \cos v\, dv)^2 + dv^2 \\ &= du^2 + u^2\, dv^2 + dv^2 = (u^2+1)^2\, du^2 + dv^2. \end{aligned}$$

By convention, when u is a real-valued function, the notation du^2 means the symmetric product $dudu$, not $d(u^2)$.

To transform a Riemannian metric under a change of coordinates, we use the same technique as we used for covector fields: think of the change of coordinates as the identity map expressed in terms of different coordinates for the domain and codomain, and use the formula of Corollary 3.1.3.6.

Example 3.2.2.3. To illustrate, we compute the coordinate expression for the Euclidean metric $\bar{g} = dx^2 + dy^2$ on $\mathbb{R}^2$ in polar coordinates. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and expanding, we obtain

$$\begin{aligned} \bar{g} = dx^2 + dy^2 &= [d(r\cos\theta)]^2 + [d(r\sin\theta)]^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 \\ &= dr^2 + r^2\, d\theta^2 \end{aligned}$$

If (M, g) and $(\widetilde{M}, \widetilde{g})$ are both Riemannian manifolds, a smooth map $F : M \to \widetilde{M}$ is called a **Riemannian isometry** if it is a diffeomorphism that satisfies $F^*\widetilde{g} = g$. More generally, F is called a **local isometry** if every point $p \in M$ has a neighborhood U such that $F|_U$ is an isometry of U onto an open subset of $\widetilde{M}$, or equivalently, if F is a local diffeomorphism satisfying $F^*\widetilde{g} = g$.

If there exists a Riemannian isometry between (M, g) and $(\widetilde{M}, \widetilde{g})$, we say that they are **isometric** as Riemannian manifolds. If each point of M has a neighborhood that is isometric to an open subset of $(\widetilde{M}, \widetilde{g})$, then we say that (M, g) is **locally isometric** to $(\widetilde{M}, \widetilde{g})$.

One such property is flatness. A Riemannian n-manifold (M, g) is said to be a **flat Riemannian manifold**, and g is a **flat metric**, if (M, g) is locally isometric to $(\mathbb{R}^n, \bar{g})$.

Proposition 3.2.2.4. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric Riemannian manifolds, then g is flat if and only if $\widetilde{g}$ is flat.*

The next theorem is the key to deciding whether a Riemannian metric is flat.

Theorem 3.2.2.5. *For a Riemannian manifold (M, g), the following are equivalent:*

(a) *g is flat.*

(b) *Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{ij} dx^i dx^j$.*

(c) *Each point of M is contained in the domain of a smooth coordinate chart in which the coordinate frame is orthonormal.*

(d) *Each point of M is contained in the domain of a commuting orthonormal frame.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy consequences of the definitions. The remaining implication, $(d) \Rightarrow (a)$, follows from the canonical form theorem for commuting frames: if (E_i) is a commuting orthonormal frame for g on an open subset $V \subseteq M$, then Theorem 1.2.7.5 implies that each $p \in V$ is contained in the domain of a smooth chart (U, φ) in which the coordinate frame is equal to (E_i). This means $\varphi_* E_i = \partial/\partial x^i$, so the diffeomorphism $\varphi : U \to \varphi(U)$ satisfies

$$\varphi^*\bar{g}(E_i, E_j) = \bar{g}(\varphi_* E_i, \varphi_* E_j) = \bar{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} = g(E_i, E_j)$$

Bilinearity then shows that $\varphi^*\bar{g} = g$, so φ is an isometry between $(U, \varphi|_U)$ and $\varphi(U)$ with the Euclidean metric. This shows that g is flat. ☐

3.2.2.2 Riemannian submanifolds

Pullback metrics are especially important for submanifolds. If (M, g) is a Riemannian manifold with or without boundary, every submanifold $S \subseteq M$ (immersed or embedded, with or without boundary) automatically inherits a pullback metric $\iota^* g$, where $\iota : S \hookrightarrow M$ is inclusion. In this setting, the pullback metric is also called the **induced metric** on S. By definition, this means for $v, w \in T_p S$ that

$$(\iota^* g)(v, w) = g(d\iota_p(v), d\iota_p(w)) = g(v, w)$$

because $d\iota_p : T_p S \to T_p M$ is our usual identification of $T_p S$ as a subspace of $T_p M$. Thus $\iota^* g$ is just the restriction of g to pairs of vectors tangent to S. With this metric, S is called a **Riemannian submanifold** (with or without boundary) of M.

Example 3.2.2.6. The metric $\mathring{g} = \iota^* \bar{g}$ induced on S^n by the usual inclusion $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is called the **round metric** or the **standard metric** on the sphere.

The next lemma describes one of the most important tools for studying Riemannian submanifolds. If $(\widetilde{M}, \widetilde{g})$ is an m-dimensional smooth Riemannian manifold and (M, g) is an n-dimensional submanifold (both with or without boundary), a local frame $(E_1, \dots, E_m)$ for $\widetilde{M}$ on an open subset $\widetilde{U} \subseteq \widetilde{M}$ is said to be **adapted to M** if the first n vector fields $(E_1, \dots, E_n)$ are tangent to M. In case $\widetilde{M}$ has empty boundary (so that slice coordinates are available), adapted local orthonormal frames are easy to find.

Proposition 3.2.2.7 (Existence of Adapted Orthonormal Frames). *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold (without boundary), and let (M, g) be an embedded smooth submanifold with or without boundary. Given $p \in M$, there exist a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame for $\widetilde{M}$ on $\widetilde{U}$ that is adapted to M.*

Proof. We can find a slice chart at p for M, and use the Gram-Schmidt algorithm to a coordinate frame in slice coordinates to produce a orthonormal frame. By the span property, we see that this frame is adapted to M. □

Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold and (M, g) is a smooth submanifold with or without boundary in $\widetilde{M}$. Given $p \in M$, a vector $v \in T_p \widetilde{M}$ is said to be **normal to M** if $\langle v, w \rangle = 0$ for every $w \in T_p M$. The space of all vectors normal to M at p is a subspace of $T_p \widetilde{M}$, called the **normal space at p** and denoted by $N_p M = (T_p M)^\perp$. At each $p \in M$, the ambient tangent space $T_p \widetilde{M}$ splits as an orthogonal direct sum $T_p \widetilde{M} = T_p M \oplus N_p M$. A section N of the ambient tangent bundle $T\widetilde{M}|_M$ is called a **normal vector field along M** if $N_p \in N_p M$ for each $p \in M$. The set

$$NM = \coprod_{p \in M} N_p M$$

is called the normal bundle of M.

Proposition 3.2.2.8 (The Normal Bundle to a Riemannian Submanifold). *If $\widetilde{M}$ is a Riemannian m-manifold and M is an immersed or embedded n-dimensional submanifold with or without boundary, then NM is a smooth rank-$(m-n)$ vector subbundle of the ambient tangent bundle $T\widetilde{M}|_M$. There are smooth bundle homomorphisms*

$$\pi^\top : T\widetilde{M}|_M \to TM, \quad \pi^\perp : T\widetilde{M}|_M \to NM$$

*called the **tangential** and **normal projections**, that for each $p \in M$ restrict to orthogonal projections from $T_p \widetilde{M}$ to $T_p M$ and $N_p M$, respectively.*

Proof. Given any point $p \in M$, Theorem **??** shows that there is a neighborhood U of p in M that is embedded in $\widetilde{M}$, and then Proposition 3.2.2.7 shows that there is a smooth orthonormal frame $(E_1, \dots, E_m)$ that is adapted to U on some neighborhood $\widetilde{U}$ of p in $\widetilde{M}$. This means that the restrictions of $(E_1, \dots, E_n)$ to $\widetilde{U} \cap U$ form a local orthonormal frame for M. Given such an adapted frame, the restrictions of the last $m - n$ vector fields $(E_{n+1}, \dots, E_m)$ to M form a smooth local frame for NM, so it follows from Lemma **??** that NM is a smooth subbundle.

The bundle homomorphisms $\pi^\top$ and $\pi^\perp$ are defined pointwise as orthogonal projections onto the tangent and normal spaces, respectively, which shows that they are uniquely defined. In terms of an adapted orthonormal frame, they can be written

$$\pi^\top(X^i E_i) = X^1 E_1 + \cdots + X^n E_n, \quad \pi^\perp(X^i E_i) = X^{n+1} E_{n+1} + \cdots + X^m E_m,$$

which shows that they are smooth. □

In case $\widetilde{M}$ is a manifold with boundary, the preceding constructions do not always work, because there is not a fully general construction of slice coordinates in that case. However, there is a satisfactory result in case the submanifold is the boundary itself, using boundary coordinates in place of slice coordinates.

Suppose (M, g) is a Riemannian manifold with boundary. We will always consider ∂M to be a Riemannian submanifold with the induced metric.

Proposition 3.2.2.9 (Existence of Outward-Pointing Normal). *If (M, g) is a smooth Riemannian manifold with boundary, the normal bundle to ∂M is a smooth rank-1 vector bundle over ∂M, and there is a unique smooth outward-pointing unit normal vector field along all of ∂M. By this, we see that $N(\partial M)$ is a rank-1 vector subbundle of $TM|_{\partial M}$.*

Proof. We can always construct a global smooth outward-pointing vector field by taking $-\partial/\partial x^n$ in boundary coordinates in a neighborhood of each boundary point, and gluing together with a partition of unity. By the Gram-Schmidt process, we may assume that this vector field is normal to ∂M. □

Computations on a submanifold $M \subseteq \widetilde{M}$ are usually carried out most conveniently in terms of a **smooth local parametrization**: this is a smooth map $X : U \to \widetilde{M}$, where U is an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$ in case M has a boundary), such that $X(U)$ is an open subset of M, and such that X, regarded as a map from U into M, is a diffeomorphism onto its image. Note that we can think of X either as a map into M or as a map into $\widetilde{M}$, both maps are typically denoted by the same symbol X. If we put $V = X(U) \subseteq \widetilde{M}$ and $\varphi = X^{-1} : V \to U$, then (V, φ) is a smooth coordinate chart on M. Suppose (M, g) is a Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ and $X : U \to \widetilde{M}$ is a smooth local parametrization of M. The coordinate representation of g in these coordinates is given by the following 2-tensor field on U:

$$\varphi_* g = X^* g = X^* \iota^* \widetilde{g} = (\iota \circ X)^* \widetilde{g}.$$

Since $\iota \circ X$ is just the map X itself, regarded as a map into $\widetilde{M}$, this is really just $X^*\widetilde{g}$. The simplicity of the formula for the pullback of a tensor field makes this expression exceedingly easy to compute, once a coordinate expression for $\widetilde{g}$ is known. For example, if M is an immersed n-dimensional Riemannian submanifold of $\mathbb{R}^m$ and $X : U \to \mathbb{R}^m$ is a smooth local parametrization of M, the induced metric on U is just

$$g = X^* \bar{g} = \sum_{i=1}^{m} (dX^i)^2 = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \frac{\partial X^i}{\partial u^j} du^j \Big)^2 = \sum_{i=1}^{m} \sum_{j,k=1}^{n} \frac{\partial X^i}{\partial u^i} \frac{\partial X^i}{\partial u^k} du^j du^k.$$

Example 3.2.2.10 (Induced Metrics in Graph Coordinates). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $S \subseteq \mathbb{R}^{n+1}$ be the graph of a smooth function $f : U \to \mathbb{R}$. The map $X : U \to \mathbb{R}^{n+1}$ given by $X(u^1, \dots, u^n) = (u^1, \dots, u^n, f(u))$ is a smooth global parametrization of S and the induced metric on S is given in graph coordinates by

$$X^* \bar{g} = X^* \big((dx^1)^2 + \cdots + (dx^{n+1})^2 \big) = (du^1)^2 + \cdots + (du^n)^2 + (df)^2$$

For example, the upper hemisphere of S^2 is parametrized by the map $X : \mathbb{B}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

In these coordinates, the round metric can be written

$$\mathring{g} = X^* \bar{g} = du^2 + dv^2 + \Big(\frac{u\,du + v\,dv}{\sqrt{1 - u^2 - v^2}} \Big)^2 = \frac{(1 - v^2)\,du^2 + (1 - u^2)\,dv^2 + 2uv\,du dv}{1 - u^2 - v^2}.$$

Example 3.2.2.11 (Induced Metrics on Surfaces of Revolution). Let C be an embedded one dimensional submanifold of the half-plane $\{(r, z) : r > 0\}$ and let S_C be the surface of revolution generated by C as described in Example **??**. To compute the induced metric on S_C, choose any smooth local parametrization $\gamma(t) = (a(t), b(t))$ for C, and note that the map $X(t, \theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$ yields a smooth local parametrization of S_C, provided that (t, θ) is restricted to a sufficiently small open subset of the plane. Thus we can compute

$$X^* \bar{g} = \big(da(t) \cos\theta \big)^2 + \big(da(t) \sin\theta \big)^2 + \big(db(t) \big)^2$$

$$= \big(a'(t)\cos\theta\,dt - a(t)\sin\theta\,d\theta\big)^2 + \big(a'(t)\sin\theta\,dt + a(t)\cos\theta\,d\theta\big)^2 + \big(b'(t)\,dt\big)^2$$
$$= \big(a'(t)^2 + b'(t)^2\big)dt^2 + a(t)^2 d\theta^2$$

In particular, if γ is a unit-speed curve, meaning that $|\gamma'(t)|^2 = 1$, this reduces to the simple formula $dt^2 + a(t)^2 d\theta^2$.

Here are some familiar examples of surfaces of revolution.

(a) The embedded torus is the surface of revolution generated by the circle $(r-2)^2 + z^2 = 1$. Using the unit-speed parametrization $\gamma(t) = (2 + \cos t + \sin t)$ for the circle, we obtain the formula $dt^2 + (2+\cos\theta)^2 d\theta^2$ for the induced metric.

(b) The unit sphere (minus the north and south poles) is a surface of revolution whose generating curve is the semicircle parametrized by $\gamma(t) = (\sin t, \cos t)$ for $0 < t < \pi$. The induced metric is $dt^2 + (\sin t)^2 d\theta^2$.

(c) The unit cylinder $x^2 + y^2 = 1$ is a surface of revolution whose generating curve is the vertical line parametrized by $\gamma(t) = (1, t)$ for $t \in \mathbb{R}$. The induced metric is $dt^2 + d\theta^2$.

Example 3.2.2.12 (The n-Torus as a Riemannian Submanifold). The smooth covering map $X : \mathbb{R}^n \to T^n$ defined by $X(u^1, \cdots, u^n) = (\cos u^1, \sin u^1, \cdots, \cos u^n, \sin u^n)$ restricts to a smooth local parametrization on any sufficiently small open subset of $\mathbb{R}^n$, and the induced metric is

$$X^*\bar{g} = d(\cos u^1)^2 + d(\sin u^1)^2 + \cdots + d(\cos u^n)^2 + d(\sin u^n)^2 = (du^1)^2 + \cdots + (du^n)^2.$$

Therefore the induced metric on T^n is flat.

3.2.2.3 Riemannian products

Next we consider products. If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product manifold $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by

$$g_{(p_1,p_2)}((v_1,v_2),(w_1,w_2)) = g_1|_{p_1}(v_1,w_1) + g_2|_{p_2}(v_2,w_2).$$

where (v_1, v_2) and (w_1, w_2) are elements of $T_{p_1}M_1 \oplus T_{p_2}M_2$, which is naturally identified with $T_{(p_1,p_2)}(M_1 \times M_2)$. Smooth local coordinates $(x^1, \ldots, x^n)$ for M_1 and $(x^{n+1}, \ldots, x^{n+m})$ for M_2 give coordinates $(x^1, \ldots, x^{n+m})$ for $M_1 \oplus M_2$. In terms of these coordinates, the product metric has the local expression $g = g_{ij}dx^i dx^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij}) = \begin{pmatrix} (g_1) & 0 \\ 0 & (g_2) \end{pmatrix}$$

Product metrics on products of three or more Riemannian manifolds are defined similarly.

Here is an important generalization of product metrics. Suppose (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds, and $f : M_1 \to \mathbb{R}^+$ is a strictly positive smooth function. The **warped product** $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ endowed with the Riemannian metric $g = g_1 \oplus f^2 g_2$, defined by

$$g_{(p_1,p_2)}((v_1,v_2),(w_1,w_2)) = g_1|_{p_1}(v_1,w_1) + f(p_1)^2 g_2|_{p_2}(v_2,w_2).$$

where $(v_1, v_2), (w_1, w_2) \in T_{p_1}M_1 \oplus T_{p_2}M_2$ as before. A wide variety of metrics can be constructed in this way; here are just a few examples.

Example 3.2.2.13. If we let ρ denote the standard coordinate function on $\mathbb{R}^+ \subseteq \mathbb{R}$, then the map $\Phi(\rho, \omega) = \rho\omega$ gives an isometry from the warped product $\mathbb{R}^+ \times_\rho S^{n-1}$ to $\mathbb{R}^n \setminus \{0\}$ with its Euclidean metric. To see this, we use the parametrization

$$X : (\rho, x^1, \ldots, x^{n-1}) \mapsto (\rho, x^1, \ldots, x^{n-1}, \sqrt{1 - |x|^2})$$

of $\mathbb{R}^+ \times S^{n-1}$ to compute the pull back metric $\Phi^*\bar{g}$ as follows:

$$(\Phi \circ X)^*\bar{g} = d(\rho x^1)^2 + \cdots + d(\rho x^{n-1})^2 + d(\rho\sqrt{1 - |x|^2})^2$$

$$
\begin{aligned}
&= (x^1\,d\rho + \rho\,dx^1)^2 + \cdots + (x^{n-1}\,d\rho + \rho\,dx^{n-1})^2 + \left(\sqrt{1-|x|^2}d\rho - \rho\frac{x^1dx^1 + \cdots + x^{n-1}dx^{n-1}}{\sqrt{1-|x|^2}}\right)^2 \\
&= d\rho^2 + \rho^2((dx^1)^2 + \cdots + (dx^{n-1})^2) + \rho^2\frac{(x^1dx^1 + \cdots + x^{n-1}dx^{n-1})^2}{1-|x|} \\
&\quad + 2\rho(x^1dx^1d\rho + \cdots + x^{n-1}dx^{n-1}d\rho) - 2\rho(x^1dx^1d\rho + \cdots + x^{n-1}dx^{n-1}d\rho) \\
&= d\rho^2 + \rho^2\left((dx^1)^2 + \cdots + (dx^{n-1})^2 + \frac{(x^1dx^1 + \cdots + x^{n-1}dx^{n-1})^2}{1-|x|}\right) \\
&= d\rho^2 + \rho^2\mathring{g}
\end{aligned}
$$

Example 3.2.2.14. Every surface of revolution can be expressed as a warped product, as follows. Let H be the half-plane $\{(r,z) : r > 0\}$, let $C \subseteq H$ be an embedded smooth 1-dimensional submanifold, and let $S_C \subseteq \mathbb{R}^3$ denote the corresponding surface of revolution as in Example 3.2.2.11. Endow C with the Riemannian metric induced from the Euclidean metric on H, and let S^1 be endowed with its standard metric. Let $f : C \to \mathbb{R}$ be the distance to the z-axis: $f(r,z) = r$. Then from Example 3.2.2.11 we see that S_C is isometric to the warped product $C \times_f S^1$.

3.2.2.4 Riemannian submersions

Unlike submanifolds and products of Riemannian manifolds, which automatically inherit Riemannian metrics of their own, quotients of Riemannian manifolds inherit Riemannian metrics only under very special circumstances. Now we see what those circumstances are.

Suppose $\widetilde{M}$ and M are smooth manifolds, $\pi : \widetilde{M} \to M$ is a smooth submersion, and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$. By the submersion level set theorem, each fiber $\widetilde{M}_y = \pi^{-1}(y)$ is a properly embedded smooth submanifold of $\widetilde{M}$. At each point $x \in \widetilde{M}$, we define two subspaces of the tangent space $T_x\widetilde{M}$ as follows: the **vertical tangent space** at x is

$$V_x = \ker d\pi_x = T_x(\widetilde{M}_{\pi(x)})$$

(that is, the tangent space to the fiber containing x), and the **horizontal tangent space** at x is its orthogonal complement:

$$H_x = (V_x)^{\perp}$$

Then the tangent space $T_x\widetilde{M}$ decomposes as an orthogonal direct sum $T_x\widetilde{M} = H_x \oplus V_x$. Note that the vertical space is well defined for every submersion, because it does not refer to the metric; but the horizontal space depends on the metric.

A vector field on $\widetilde{M}$ is said to be a **horizontal vector field** if its value at each point lies in the horizontal space at that point; a **vertical vector field** is defined similarly. Given a vector field X on M, a vector field $\widetilde{X}$ on $\widetilde{M}$ is called a **horizontal lift** of X if $\widetilde{X}$ is horizontal and π-related to X.

The next proposition is the principal tool for doing computations on Riemannian submersions.

Proposition 3.2.2.15 (Properties of Horizontal Vector Fields). *Let $\widetilde{M}$ and M be smooth manifolds, let $\pi : \widetilde{M} \to M$ be a smooth submersion, and let $\widetilde{g}$ be a Riemannian metric on $\widetilde{M}$.*

- *(a) Every smooth vector field W on $\widetilde{M}$ can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.*
- *(b) Every smooth vector field on M has a unique smooth horizontal lift to $\widetilde{M}$.*
- *(c) For every $x \in \widetilde{M}$ and $v \in H_x$, there is a vector field $X \in \mathfrak{X}(M)$ whose horizontal lift $\widetilde{X}$ satisfies $\widetilde{X}_x = v$.*

Proof. Let $p \in \widetilde{M}$ be arbitrary. Because π is a smooth submersion, the rank theorem shows that there exist smooth coordinate charts $(\widetilde{U}, (x^i))$ centered at p and $(U, (y^j))$ centered at $\pi(x)$ in which π has the coordinate representation

$$\pi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n)$$

where $m = \dim \widetilde{M}$ and $n = \dim M$. It follows that at each point $q \in \widetilde{U}$, the vertical space V_q is spanned by the vectors $(\partial/\partial x^{n+1}|_q, \ldots, \partial/\partial x^m|_q)$. (It probably will not be the case, however, that the horizontal space is spanned by the other n basis vectors.) If we apply the GramSchmidt algorithm to the ordered frame $(\partial/\partial x^{n+1}|, \ldots, \partial/\partial x^m, \partial/\partial x^1, \ldots, \partial/\partial x^n)$, we obtain a smooth orthonormal frame $(E_1, \ldots, E_m)$ on $\widetilde{U}$ such that V_q is spanned by $(E_1|_q, \ldots, E_{m-n}|_q)$ at each $q \in \widetilde{U}$. It follows that H_q is spanned by $(E_{n-m+1}|_q, \ldots, E_m|_q)$.

Now let $W \in \mathfrak{X}(\widetilde{M})$ be arbitrary. At each point $q \in \widetilde{M}$, W_q can be written uniquely as a sum of a vertical vector plus a horizontal vector, thus defining a decomposition $W = W^V + W^H$ into rough vertical and horizontal vector fields. To see that they are smooth, just note that in a neighborhood of each point we can express W in terms of a frame $(E_1, \ldots, E_m)$ of the type constructed above as $W = W^1 E_1 + \cdots + W^m E_m$ with smooth coefficients (W^i), and then it follows that

$$W^V = W^1 E_1 + \cdots + W^{m-n} E_{m-n}, \quad W^H = W^{n-m+1} E_{n-m+1} + \cdots + W^m E_m,$$

both of which are smooth.

Let $X \in \mathfrak{X}(M)$ be arbitrary. By the coordinate representation constructed above, we see that $d\pi$ is just the projection onto H_q, which is spanned by $(E_{m-n+1}, \ldots, E_m)$. Therefore we can use X to define a rough horizontal vector field on $\widetilde{M}$. From the local coordinate presentation we see that $\widetilde{X}$ is smooth.

For every $x \in \widetilde{M}$ and $v \in H_x$, we can choose coordinates as above, and construct a vector field on M such that $X_{\pi(x)} = d\pi(v)$. Then by the construction above the horizontal lift $\widetilde{X}$ satisfies $\widetilde{X}_x = v$. □

The fact that every horizontal vector at a point of $\widetilde{M}$ can be extended to a horizontal lift on all of $\widetilde{M}$ (part (c) of the preceding proposition) is highly useful for computations. It is important to be aware, though, that not every horizontal vector field on $\widetilde{M}$ is a horizontal lift, as the next example shows.

Example 3.2.2.16. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection map $\pi(x, y) = yx$, and let W be the smooth vector field $y\partial/\partial x$ on $\mathbb{R}^2$. Then W is horizontal, but there is no vector field on $\mathbb{R}$ whose horizontal lift is equal to W.

Now we can identify some quotients of Riemannian manifolds that inherit metrics of their own. Let us begin by describing what such a metric should look like. Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds, and $\pi : \widetilde{M} \to M$ is a smooth submersion. Then π is said to be a **Riemannian submersion** if for each $x \in \widetilde{M}$, the differential $d\pi_x$ restricts to a linear isometry from H_x onto $T_{\pi(x)}M$. In other words, $\widetilde{g}_x(v, w) = g_{\pi(x)}(d\pi_x(v), d\pi_x(w))$ whenever $v, w \in H_x$.

Example 3.2.2.17 (Riemannian Submersions).

(a) The projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a Riemannian submersion if $\mathbb{R}^{n+k}$ and $\mathbb{R}^n$ are both endowed with their Euclidean metrics.

(b) If M and N are Riemannian manifolds and $M \times N$ is endowed with the product metric, then both projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Riemannian submersions.

(c) If $M \times_f N$ is a warped product manifold, then the projection $\pi_M : M \times_f N \to M$ is a Riemannian submersion, but π_N typically is not.

Given a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and a surjective submersion $\pi : \widetilde{M} \to M$, it is almost never the case that there is a metric on M that makes π into a Riemannian submersion. It is not hard to see why: for this to be the case, whenever $p_1, p_2 \in \widetilde{M}$ are two points in the same fiber $\pi^{-1}(y)$, the linear maps $(d\pi_{p_i}|_{H_{p_i}})^{-1} : T_yM \to H_{p_i}$ both have to pull $\widetilde{g}$ back to the same inner product on T_yM. There is, however, an important special case in which there is such a metric. Suppose $\pi : \widetilde{M} \to M$ is a smooth surjective submersion, and G is a group acting on $\widetilde{M}$. We say that the action is **vertical** if every element $\varphi \in G$ takes each fiber to itself, meaning that $\pi(\varphi \cdot p) = \pi(p)$ for all $p \in M$. The action is **transitive on fibers** if for each $p, q \in M$ such that $\pi(p) = \pi(q)$, there exists $\varphi \in G$ such that $\varphi \cdot p = q$. If in addition $\widetilde{M}$ is endowed with a Riemannian metric, the action is said to be an **isometric action** or an **action by isometries**, and the metric is said to be **invariant under *G***, if the map $x \mapsto \varphi \cdot x$ is an isometry for each $\varphi \in G$. In that case, provided the action is effective (so that different elements of G define different isometries of $\widetilde{M}$), we can identify G with a subgroup of $\mathrm{Iso}(\widetilde{M}, g)$. Since an isometry is, in particular, a diffeomorphism, every isometric action is an action by diffeomorphisms.

Theorem 3.2.2.18. *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, let $\pi : \widetilde{M} \to M$ be a surjective smooth submersion, and let G be a group acting on $\widetilde{M}$. If the action is isometric, vertical, and transitive on fibers, then there is a unique Riemannian metric on M such that π is a Riemannian submersion.*

The next corollary describes one important situation to which the preceding theorem applies.

Corollary 3.2.2.19. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and G is a Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then the orbit space $M = \widetilde{M}/G$ has a unique smooth manifold structure and Riemannian metric such that $\pi : \widetilde{M} \to M$ is a Riemannian submersion.*

Proof. Under the given hypotheses, the quotient manifold theorem shows that M has a unique smooth manifold structure such that the quotient map $\widetilde{M} \to \widetilde{M}/G$ is a smooth submersion. It follows easily from the definitions in that case that the given action of G on $\widetilde{M}$ is vertical and transitive on fibers. Since the action is also isometric, Theorem 3.2.2.18 shows that M inherits a unique Riemannian metric making π into a Riemannian submersion. □

Here is an important example of a Riemannian metric defined in this way.

Example 3.2.2.20 (The Fubini-Study Metric). Let n be a positive integer, and consider the complex projective space $\mathbb{CP}^n$. The map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ sending each point in $\mathbb{C}^{n+!} \setminus \{0\}$ to its span is a surjective smooth submersion. Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ endowed with its Euclidean metric, we can view the unit sphere S^{2n+1} with its round metric $\mathring{g}$ as an embedded Riemannian submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$. Let $p : S^{2n+1} \to \mathbb{CP}^n$ denote the restriction of the map π. Then p is smooth, and it is surjective, because every 1-dimensional complex subspace contains elements of unit norm. We need to show that it is a submersion. Let $z_0 \in S^{2n+1}$ and set $\zeta_0 = p(z_0) \in \mathbb{CP}^n$. Since π is a smooth submersion, it has a smooth local section $\sigma : U \to \mathbb{C}^{n+1} \setminus \{0\}$ defined on a neighborhood U of ζ_0. Let $\widetilde{\sigma}$ be the radial projection of σ onto the sphere: $\widetilde{\sigma} = \sigma/|\sigma|$. Then $\widetilde{\sigma}$ is a local section of p. By Theorem **??** we see that p is a submersion.

Define an action of S^1 on S^{2n+1} by complex multiplication

$$\lambda \cdot (z^1, \ldots, z^{n+1}) = (\lambda z^1, \ldots, \lambda z^{n+1})$$

for $\lambda \in S^1$ (viewed as a complex number of norm 1) and $(z^1, \ldots, z^{n+1}) \in S^{2n+1}$. This is easily seen to be isometric, vertical, and transitive on fibers of p. By Theorem 3.2.2.18, therefore, there is a unique metric on $\mathbb{CP}^n$ such that the map $p : S^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion. This metric is called the **Fubini-Study metric**.

3.2.2.5 Riemannian coverings

Another important special case of Riemannian submersions occurs in the context of covering maps. Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds. A smooth covering map $\pi : \widetilde{M} \to M$ is called a **Riemannian covering** if it is a local isometry.

Proposition 3.2.2.21. *Suppose $\pi : \widetilde{M} \to M$ is a smooth normal covering map, and $\widetilde{g}$ is any metric on $\widetilde{M}$ that is invariant under all covering automorphisms. Then there is a unique metric g on M such that π is a Riemannian covering.*

Proof. Proposition **??** shows that π is a surjective smooth submersion. The automorphism group acts vertically by definition, and we know that it acts transitively on fibers when the covering is normal. It then follows from Theorem 3.2.2.18 that there is a unique metric g on M such that π is a Riemannian submersion. Since a Riemannian submersion is a local isometry, it follows that π is a Riemannian covering. □

Proposition 3.2.2.22. *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, and Γ is a discrete Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$. Then $\widetilde{M}/\Gamma$ has a unique Riemannian metric such that the quotient map $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$ is a normal Riemannian covering.*

Proof. Proposition 2.3.3.3 shows that π is a smooth normal covering map, and Proposition 3.2.2.21 shows that $\widetilde{M}/\Gamma$ has a unique Riemannian metric such that π is a Riemannian covering. □

Corollary 3.2.2.23. *Suppose $(\widetilde{M}, \widetilde{g})$ and (M, g) are connected Riemannian manifolds, $\pi : \widetilde{M} \to M$ is a normal Riemannian covering map, and $\Gamma = \mathrm{Aut}_\pi(\widetilde{M})$. Then M is isometric to $\widetilde{M}/\Gamma$.*

Proof. Proposition 2.3.3.2 shows that with the discrete topology, Γ is a discrete Lie group acting smoothly, freely, and properly on $\widetilde{M}$, and then Proposition 2.3.3.3 shows that $\widetilde{M}/\Gamma$ is a smooth manifold and the quotient map $q : \widetilde{M} \to \widetilde{M}/\Gamma$ is a smooth normal covering map. The fact that both π and q are normal coverings implies that Γ acts transitively on the fibers of both maps, so the two maps are constant on each other's fibers. Proposition **??** then implies that there is a diffeomorphism $F : M \to \widetilde{M}/\Gamma$ that satisfies $q \circ F = \pi$. Because both q and π are local isometries, F is too, and because it is bijective it is a global isometry. □

Example 3.2.2.24. The two-element group $\{\pm 1\}$ acts smoothly, freely, properly, and isometrically on S^n by multiplication. We know that the quotient space is diffeomorphic to the real projective space $\mathbb{RP}^n$ and the quotient map $q : S^n \to \mathbb{RP}^n$ is a smooth normal covering map. Because the action is isometric, Proposition 3.2.2.22 shows that there is a unique metric on $\mathbb{RP}^n$ such that q is a Riemannian covering.

Example 3.2.2.25 (The Open Möbius Band). The **open Möbius band** is the quotient space $M = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}^2$ by

$$n \cdot (x, y) = (x + n, (-1)^n y).$$

This action is smooth, free, proper, and isometric, and therefore M inherits a flat Riemannian metric such that the quotient map is a Riemannian covering.

3.2.3 Basic constructions on Riemannian manifolds

3.2.3.1 Raising and lowering indices

One elementary but important property of Riemannian metrics is that they allow us to convert vectors to covectors and vice versa. Given a Riemannian metric g on M, we define a bundle homomorphism $\widehat{g} : TM \to T^*M$ by setting

$$\widehat{g}(v)(w) = g_p(v, w)$$

for each $p \in M$ and each $v, w \in T_pM$. To see that this is a smooth bundle homomorphism, it is easiest to consider its action on smooth vector fields:

$$\widehat{g}(X)(Y) = g(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{X}(M).$$

Because $\widehat{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y, it follows from the tensor characterization lemma (Lemma 3.1.3.2) that $\widehat{g}(X)$ is a smooth covector field, and because $\widehat{g}(X)$ is linear over $C^\infty(M)$ as a function of X, this defines $\widehat{g}$ as a smooth bundle homomorphism by the bundle homomorphism characterization lemma (Lemma **??**). As usual, we use the same symbol for both the pointwise bundle homomorphism $\widehat{g} : TM \to T^*M$ and the linear map on sections $\widehat{g} : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.

Note that $\widehat{g}$ is injective at each point, because $\widehat{g}(v) = 0$ for some $v \in T_pM$ implies

$$0 = \widehat{g}(v)(v) = g(v, v)$$

which in turn implies $v = 0$. For dimensional reasons, therefore, $\widehat{g}$ is bijective, so it is a bundle isomorphism (Proposition **??**).

In any smooth coordinates (x^i), we can write $g = g_{ij}dx^i dx^j$. Thus, if X and Y are smooth vector fields, we have

$$\widehat{g}(X)(Y) = g_{ij}X^iY^j$$

which implies that the covector field $\widehat{g}(X)$ has the coordinate expression

$$\widehat{g}(X) = g_{ij}X^i dx^j$$

In other words, $\widehat{g}$ is the bundle homomorphism whose matrix with respect to coordinate frames for TM and T^*M is the same as the matrix of g itself.

It is customary to denote the components of the covector field $\widehat{g}(X)$ by

$$\widehat{g} = X_j dx^j, \quad \text{where } X_j = g_{ij}X^i.$$

Because of this, one says that $\widehat{g}(X)$ is obtained from X by lowering an index. The notation $X^\flat$ is frequently used for $\widehat{g}(X)$, because the symbol $\flat$ is used in musical notation to indicate that a tone is to be lowered.

The matrix of the inverse map $\widehat{g}^{-1} : T^*M \to TM$ is thus the inverse of (g_{ij}). We let (g^{ij}) denote the matrix-valued function whose value at $p \in M$ is the inverse of the matrix $(g_{ij}(p))$, so that

$$g^{ij}g_{jk} = g_{kj}g^{ji} = \delta^i_k$$

Because g_{ij} is a symmetric matrix, so is g^{ij}, as you can easily check. Thus for a covector field $\omega \in \mathfrak{X}^*(M)$, the vector field $\widehat{g}^{-1}(\omega)$ has the coordinate representation

$$\widehat{g}^{-1}(\omega) = \omega^i \frac{\partial}{\partial x^i}, \quad \text{where } \omega^i = \omega_j g^{ij}.$$

We use the notation $\sharp$ for $\widehat{g}^{-1}$, and say that $\omega^\sharp$ is obtained from ω by raising an index. Because the symbols $\flat$ and $\sharp$ are borrowed from musical notation, these two inverse isomorphisms are frequently called the **musical isomorphisms**. A handy mnemonic device for keeping the flat and sharp operations straight is to remember that the value of $\omega^\sharp$ at each point is a vector, which we visualize as a (sharp) arrow; while the value of $X^\flat$ is a covector, which we visualize by means of its (flat) level sets.

The most important use of the sharp operation is to reinstate the **gradient** as a vector field on Riemannian manifolds. For any smooth real-valued function f on a Riemannian manifold (M, g) with or without boundary, we define a vector field called the gradient of f by

$$\operatorname{grad} f := (df)^\sharp = \widehat{g}^{-1}(df).$$

Unraveling the definitions, we see that for any $X \in \mathfrak{X}(M)$, the gradient satisfies

$$\langle \operatorname{grad} f, X\rangle_g = \widehat{g}(\operatorname{grad} f)(X) = df(X) = Xf.$$

Thus grad f is the unique vector field that satisfies

$$\langle \operatorname{grad} f, X\rangle_g = Xf \quad \text{for} \quad X \in \mathfrak{X}(M),$$

or equivalently,

$$\langle \operatorname{grad} f, \cdot\rangle_g = df. \tag{3.2.3.1}$$

In smooth coordinates, grad f has the expression

$$\operatorname{grad} f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}.$$

In particular, this shows that gradf is smooth. On Rn with the Euclidean metric, this reduces to

$$\operatorname{grad} f = \delta^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}.$$

Thus our new definition of the gradient in this case coincides with the gradient from elementary calculus. In other coordinates, however, the gradient does not generally have the same form.

Example 3.2.3.1. Let us compute the gradient of a function $f \in C^\infty(\mathbb{R}^2)$ with respect to the Euclidean metric in polar coordinates. From Example 3.2.2.3 we see that the matrix of $\bar{g}$ in polar coordinates is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & r^2 \end{smallmatrix}\right)$, so its inverse matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & r^{-2} \end{smallmatrix}\right)$. Inserting this into the formula for the gradient, we obtain

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}.$$

The next proposition shows that the gradient has the same geometric interpretation on a Riemannian manifold as it does in Euclidean space. If f is a smooth real-valued function on a smooth manifold M, recall that a point $p \in M$ is called a **regular point** of f if $df_p \neq 0$, and a **critical point** of f otherwise; and a level set $f^{-1}(c)$ is called a regular level set if every point of $f^{-1}(c)$ is a regular point of f. Corollary **??** shows that each regular level set is an embedded smooth hypersurface in M.

Proposition 3.2.3.2. *Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $\mathcal{R} \subseteq M$ is the set of regular points of f. For each $c \in \mathbb{R}$, the set $M_c = f^{-1}(c) \cap \mathcal{R}$, if nonempty, is an embedded smooth hypersurface in M, and* grad f *is everywhere normal to M_c.*

Proof. From the definition of gradient we have

$$\langle \operatorname{grad} f|_p, v\rangle = df_p(v).$$

By Proposition **??** we see that $T_pM_c = \ker df$, therefore the claim holds. □

The flat and sharp operators can be applied to tensors of any rank, in any index position, to convert tensors from covariant to contravariant or vice versa. Formally, this operation is defined as follows: if F is any (k,l)-tensor and $i \in \{1,\dots,k+l\}$ is any covariant index position for F (meaning that the i-th argument is a vector, not a covector), we can form a new tensor $F^\sharp$ of type $(k+1, l-1)$ by setting

$$F^\sharp(\alpha_1,\dots,\alpha_{k+l}) = F(\alpha_1,\dots,\alpha_i^\sharp,\dots,\alpha_{k+l})$$

whenever $\alpha_1,\dots,\alpha_{k+l}$ are vectors or covectors as appropriate. In any local frame, the components of $F^\sharp$ are obtained by multiplying the components of F by g^{kl} and contracting one of the indices of g^{kl} with the i-th index of F. Similarly, if i is a contravariant index position, we can define a $(k-1, l+1)$-tensor $F^\flat$ by

$$F^\flat(\alpha_1,\dots,\alpha_{k+l}) = F(\alpha_1,\dots,\alpha_i^\flat,\dots,\alpha_{k+l})$$

In components, it is computed by multiplying by g_{kl} and contracting.

For example, if A is a mixed 3-tensor given in terms of a local frame by

$$A = A_{ik}^j \varepsilon^i \otimes E_j \otimes \varepsilon^k$$

we can lower its middle index to obtain a covariant 3-tensor $A^\flat$ with components

$$A_{ijk} = g_{jl}A_{ik}^l.$$

Another important application of the flat and sharp operators is to extend the trace operator to covariant tensors. If h is any covariant k-tensor field on a Riemannian manifold with $k \geq 2$, we can raise one of its indices (say the last one for definiteness) and obtain a $(1, k-1)$-tensor $h^\sharp$. The trace of $h^\sharp$ is thus a well-defined covariant $(k-2)$-tensor field. We define the trace of h with respect to g as

$$\operatorname{tr}_g h = \operatorname{tr}(h^\sharp).$$

Sometimes we may wish to raise an index other than the last, or to take the trace on a pair of indices other than the last covariant and contravariant ones. In each such case, we will say in words what is meant.

The most important case is that of a covariant 2-tensor field. In this case, $h^\sharp$ is a $(1,1)$-tensor field, which can equivalently be regarded as an endomorphism field, and $\operatorname{tr}_g h$ is just the ordinary trace of this endomorphism field. In terms of a basis, this is

$$\operatorname{tr}_g h = h_i^i = g^{ij}h_{ij}.$$

In particular, in an orthonormal frame this is the ordinary trace of the matrix (h_{ij}) (the sum of its diagonal entries); but if the frame is not orthonormal, then this trace is different from the ordinary trace.

If g is a Riemannian metric on M and (E_i) is a local frame on M, there is a potential ambiguity about what the expression (g^{ij}) represents: we have defined it to mean the inverse matrix of (g_{ij}), but one could also interpret it as the components of the contravariant 2-tensor field $g^{\sharp\sharp}$ obtained by raising both of the indices of g. But these two interpretations lead to the same result:

$$g^{\sharp\sharp} = g^{lk}g^{ki}g_{ij} = g^{lj},$$

so we do not distinguish them.

3.2.3.2 Inner products of tensors

A Riemannian metric yields, by definition, an inner product on tangent vectors at each point. Because of the musical isomorphisms between vectors and covectors, it is easy to carry the inner product over to covectors as well. Suppose g is a Riemannian metric on M, and $x \in M$. We can define an inner product on the cotangent space T_x^*M by

$$\langle \omega, \eta \rangle_g = \langle \omega^\sharp, \eta^\sharp \rangle_g$$

To see how to compute this, we just use the basis formula for the sharp operator, together with the relation $g_{kl}g^{ki} = g_{lk}g^{ki} = \delta^i_l$, to obtain

$$\begin{aligned}\langle \omega, \eta \rangle_g &= g_{kl}(g^{ki}\omega_i)(g^{lj}\eta_j) \\ &= \delta^i_l g^{lj}\omega_i\eta_j \\ &= g^{ij}\omega_j\eta_j\end{aligned}$$

In other words, the inner product on covectors is represented by the inverse matrix (g^{ij}). Using our conventions for raising and lowering indices, this can also be written

$$\langle \omega, \eta \rangle = \omega_i \eta^j = \omega^i \eta_j.$$

Now the following result is immediate from the definition.

Proposition 3.2.3.3. *Let (M, g) be a Riemannian manifold with or without boundary, let (E_i) be a local frame for M, and let (ε^i) be its dual coframe. Then the following are equivalent:*

(a) (E_i) is orthonormal.

(b) (ε^i) is orthonormal.

(c) $(\varepsilon^i)^\sharp = E_i$ for each i.

This construction can be extended to tensor bundles of any rank, as the following proposition shows. First a bit of terminology: if $E \to M$ is a smooth vector bundle, a **smooth fiber metric** on E is an inner product on each fiber E_p that varies smoothly, in the sense that for any (local) smooth sections σ, τ of E, the inner product (σ, τ) is a smooth function.

Proposition 3.2.3.4 (Inner Products of Tensors). *Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. There is a unique smooth fiber metric on each tensor bundle $T^{(k,l)}TM$ with the property that if $\alpha_1, \dots, \alpha_{k+l}$, $\beta_1, \dots, \beta_{k+l}$ are vector or covector fields as appropriate, then*

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_{k+l}, \beta_1 \otimes \cdots \otimes \beta_{k+l} \rangle = \langle \alpha_1, \beta_1 \rangle \cdots \langle \alpha_{k+l}, \beta_{k+l} \rangle. \tag{3.2.3.2}$$

With this inner product, if $(E_1, \dots, E_n)$ is a local orthonormal frame for TM and $(\varepsilon^1, \dots, \varepsilon^n)$ is the corresponding dual coframe, then the collection of tensor fields $E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}$ as all the indices range from 1 to n forms a local orthonormal frame for $T^{(k,l)}(T_pM)$. In terms of any (not necessarily orthonormal) frame, this fiber metric satisfies

$$\langle F, G \rangle = g_{i_1 r_1} \cdots g_{i_k r_k} g^{j_1 s_1} \cdots g^{j_l s_l} F^{i_1 \dots i_k}_{j_1 \dots j_l} G^{r_1 \dots r_k}_{s_1 \dots s_l}. \tag{3.2.3.3}$$

If F and G are both covariant, this can be written

$$\langle F, G \rangle = F_{j_1 \dots j_l} G^{j_1 \dots j_l},$$

where the last factor on the right represents the components of G with all of its indices raised:

$$G^{j_1 \dots j_l} = g^{j_1 s_1} \cdots g^{j_l s_l} G_{j_1 \dots j_l}.$$

Proof. The formula (3.2.3.2) is just the definition of the inner product. With this, all is clear. $\square$

3.2.4 The Riemannian distance function

One of the most important tools that a Riemannian metric gives us is the ability to define lengths of curves. Suppose (M, g) is a Riemannian manifold with or without boundary. If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, the length of γ is

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$$

Because $|\gamma'(t)|_g$ is continuous at all but finitely many values of t, and has well-defined limits from the left and right at those points, the integral is well defined.

Proposition 3.2.4.1. *suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Then $L_{\widetilde{g}}(F \circ \gamma) = L_g(\gamma)$ for every piecewise smooth curve segment γ in M.*

Proof. This follows from the observation

$$\begin{aligned} |(F \circ \gamma)'(t)|^2_{\widetilde{g}} &= |dF_{\gamma(t)}(\gamma'(t))|_{\widetilde{g}} = \widetilde{g}[dF_{\gamma(t)}(\gamma'(t)), dF_{\gamma(t)}(\gamma'(t))] \\ &= (F^*\widetilde{g})(\gamma'(t), \gamma'(t)) = g(\gamma'(t), \gamma'(t)) = |\gamma'(t)|^2_g \end{aligned}$$

since F is a local isometry. □

It is an extremely important fact that length is independent of parametrization in the following sense. In Section **??** we defined a reparametrization of a piecewise smooth curve segment $\gamma : [a, b] \to M$ to be a curve segment of the form $\widetilde{\gamma} = \gamma \circ \varphi$ where $\varphi : [c, d] \to [a, b]$ is a diffeomorphism. We also call them **admissible curves**.

Proposition 3.2.4.2 (Parameter Independence of Length). *Let (M, g) be a Riemannian manifold with or without boundary, and let $\gamma : [a, b] \to M$ be a piecewise smooth curve segment. If $\widetilde{\gamma}$ is a reparametrization of γ, then $L_g(\gamma) = L_g(\widetilde{\gamma})$.*

Proof. First suppose that γ is smooth, and $\varphi : [c, d] \to [a, b]$ is a diffeomorphism such that $\widetilde{\gamma} = \gamma \circ \varphi$. The fact that φ is a diffeomorphism implies that either $\varphi' > 0$ or $\varphi' < 0$ everywhere. Let us assume that $\varphi' > 0$. We have

$$\begin{aligned} L_g(\widetilde{\gamma}) &= \int_c^d |\widetilde{\gamma}'(t)|_g dt = \int_c^d |(\gamma \circ \varphi)'(t)|_g dt \\ &= \int_c^d |\varphi'(t)\gamma'(\varphi(t))|_g dt = \int_c^d |\gamma'(\varphi(t))|_g \varphi'(t) dt \\ &= \int_a^b |\gamma'(s) ds = L_g(\gamma). \end{aligned}$$

In the case $\varphi' < 0$, we just need to introduce two sign changes into the above calculation. The sign changes once when $\varphi'(t)$ is moved outside the absolute value signs, because $|\varphi'(t)| = -\varphi'(t)$. Then it changes again when we change variables, because φ reverses the direction of the integral. Since the two sign changes cancel each other, the result is the same.

If γ is only piecewise smooth, we just apply the same argument on each subinterval on which it is smooth. □

Suppose $\gamma : [a, b] \to M$ is an admissible curve. The **arc-length function** of γ is the function $s : [a, b] \to \mathbb{R}$ defined by

$$s(t) = L_g(\gamma|_{[a,t]}) = \int_a^t |\gamma'(u)|_g du.$$

It is continuous everywhere, and it follows from the fundamental theorem of calculus that it is smooth wherever γ is, with derivative $s'(t) = |\gamma'(t)|_g$. For this reason, if $\gamma : I \to M$ is any smooth curve (not necessarily a curve segment), we define the **speed** of γ at any time $t \in I$ to be the scalar $|\gamma'(t)|_g$. We say that γ is a **unit-speed curve** if $|\gamma'(t)|_g = 1$ for all t, and a **constant-speed curve** if $|\gamma'(t)|_g$ is constant. If γ is a piecewise smooth curve, we say that γ has unit speed if $|\gamma'(t)|_g = 1$ wherever γ is smooth.

If $\gamma : [a, b] \to M$ is a unit-speed admissible curve, then its arc-length function has the simple form $s(t) = t - a$. If, in addition, its parameter interval is of the form $[0, b]$ for some $b > 0$, then the arc-length function is $s(t) = t$. For this reason, a unit-speed admissible curve whose parameter interval is of the form $[0, b]$ is said to be **parametrized by arc length**.

Proposition 3.2.4.3. *Suppose (M, g) is a Riemannian manifold with or without boundary.*

(a) Every regular curve in M has a unit-speed forward reparametrization.

(b) Every admissible curve in M has a unique forward reparametrization by arc length.

Proof. Suppose $\gamma : I \to M$ is a regular curve. Choose an arbitrary $t_0 \in I$, and define $s : I \to \mathbb{R}$ by

$$s(t) = \int_{t_0}^{t} |\gamma'(u)|_g du.$$

Since $s'(t) = |\gamma'(t)|_g > 0$, it follows that s is a strictly increasing local diffeomorphism, and thus maps I diffeomorphically onto an interval $I' \subseteq \mathbb{R}$. If we let $\varphi = s^{-1} : I' \to I$, then $\widetilde{\gamma} := \gamma \circ \varphi$ is a forward reparametrization of γ, and the chain rule gives

$$|\widetilde{\gamma}'(u)|_g = |\varphi'(s)\gamma'(\varphi(s))|_g = \frac{1}{s'(\varphi(s))}|\gamma'(\varphi(s))|_g = 1.$$

Thus $\widetilde{\gamma}$ is a unit-speed reparametrization of γ.

Now let $\gamma : I \to M$ be an admissible curve. We prove the existence statement in part (b) by induction on the number of smooth segments in an admissible partition. If γ has only one smooth segment, then it is a regular curve segment, and (b) follows by applying (a) in the special case $I = [a, b]$ and choosing $t_0 = a$. Assuming that the result is true for admissible curves with k smooth segments, suppose γ has an admissible partition $(a_0, \dots, a_{k+1})$. The inductive hypothesis gives piecewise regular homeomorphisms $\varphi_1 : [0, c] \to [a, a_k]$ and $\varphi_2 : [0, d] \to [a_k, b]$ such that $\gamma \circ \varphi_1$ is an arc-length reparametrization of $\gamma|_{[a,a_k]}$ and $\gamma \circ \varphi_2$ is an arc-length reparametrization of $\gamma|_{[a_k,b]}$. If we define $\widetilde{\varphi} : [0, c + d] \to [a, b]$ by

$$\widetilde{\varphi}(s) = \begin{cases} \varphi_1(s) & s \in [0, c], \\ \varphi_2(s - c) & s \in [c, c + d], \end{cases}$$

then $\gamma \circ \psi$ is a reparametrization of γ by arc length.

To prove uniqueness, suppose that $\widetilde{\gamma} = \gamma \circ \varphi$ and $\widehat{\gamma} = \gamma \circ \psi$ are both forward reparametrizations of γ by arc length. Since both have the same length, it follows that φ and ψ both have the same parameter domain $[0, c]$, and thus both are piecewise regular homeomorphisms from $[0, c]$ to $[a, b]$. If we define $\eta = \varphi^{-1} \circ \psi$, then η is a piecewise regular increasing homeomorphism satisfying $\widehat{\gamma} = \gamma \circ \varphi \circ \eta = \widetilde{\gamma} \circ \eta$. The fact that both $\widetilde{\gamma}$ and $\widehat{\gamma}$ are of unit speed implies the following equality for all $s \in [0, c]$ except the finitely many values at which $\widetilde{\gamma}$ or η is not smooth:

$$1 = |\widehat{\gamma}'(s)|_g = |\widetilde{\gamma}'(\eta(s))\eta'(s)|_g = |\widetilde{\gamma}'(\eta(s))|_g\eta'(s) = \eta'(s).$$

Since η is continuous and $\eta(0) = 0$, it follows that $\eta(s) = s$ for all $s \in [0, c]$, and thus $\widetilde{\gamma} = \widehat{\gamma}$. □

Using curve segments as measuring tapes, we can define distances between points on a Riemannian manifold. Suppose (M, g) is a connected Riemannian manifold. (The theory is most straightforward when $\partial M = \varnothing$, so we assume that for the rest of this section.) For any $p, q \in M$, the **Riemannian distance** from p to q, denoted by $d_g(p, q)$, is defined to be the infimum of $L_g(\gamma)$ over all piecewise smooth curve segments γ from p to q. Because any pair of points in M can be joined by a piecewise smooth curve segment (Proposition **??**), this is well defined.

Proposition 3.2.4.4. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds and $F : M \to \widetilde{M}$ is a Riemannian isometry. Then*

$$d_g(p, q) = d_{\widetilde{g}}(F(p), F(q))$$

for all $p, q \in M$.

Proof. This follows from Proposition 3.2.4.1, since any curve on M or $\widetilde{M}$ can be mapped isometrically to the other manifodl. □

Remark 3.2.4.5. Note that unlike lengths of curves, Riemannian distances are not necessarily preserved by local isometries.

Example 3.2.4.6. In $(\mathbb{R}^n, \bar{g})$, It can be shown that any straight line segment is the shortest piecewise smooth curve segment between its endpoints. Therefore, the distance function $d_{\bar{g}}$ is equal to the usual Euclidean distance:

$$d_{\bar{g}}(x, y) = |x - y|.$$

We will see below that the Riemannian distance function turns M into a metric space whose topology is the same as the given manifold topology. The key is the following technical lemma, which shows that every Riemannian metric is locally comparable to the Euclidean metric in coordinates.

Lemma 3.2.4.7. *Let g be a Riemannian metric on an open subset $U \subseteq \mathbb{R}^n$. Given a compact subset $K \subseteq U$, there exist positive constants c, C such that for all $x \in K$ and all $v \in T_x\mathbb{R}^n$,*

$$c|v|_{\bar{g}} \leq |v|_g \leq C|v|_{\bar{g}} \tag{3.2.4.1}$$

Proof. For any compact subset $K \subseteq U$, let $L \subseteq T\mathbb{R}^n$ be the set

$$L = \{(x, v) \in T\mathbb{R}^n : x \in K, |v|_{\bar{g}} = 1\}$$

Under the canonical identification of $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, L is just the product set $K \times S^{n-1}$ and therefore is compact. Because the norm $|v|_g$ is continuous and strictly positive on L, there are positive constants c, C such that $c \leq |v|_g \leq C$ whenever $(x, v) \in L$. If $x \in K$ and v is a nonzero vector in $T_x\mathbb{R}^n$, let $\lambda = |v|_{\bar{g}}$. Then $(x, \lambda^{-1}v) \in L$, so by homogeneity of the norm,

$$|v|_g = \lambda|\lambda^{-1}v|_g \leq \lambda C = C|v|_{\bar{g}}$$

A similar computation shows that $|v|_g \geq c|v|_{\bar{g}}$. The same inequalities are trivially true when $v = 0$. □

Theorem 3.2.4.8 (Riemannian Manifolds as Metric Spaces). *Let (M, g) be a connected Riemannian manifold. With the Riemannian distance function, M is a metric space whose metric topology is the same as the original manifold topology.*

Proof. It is immediate from the definition that $d_g(p, q) \geq 0$. Because every constant curve segment has length zero, it follows that $d_g(p, p) = 0$; and $d_g(p, q) = d_g(q, p)$ follows from the fact that any curve segment from p to q can be reparametrized to go from q to p. Suppose γ_1 and γ_2 are piecewise smooth curve segments from p to q and q to r, respectively, and let γ be a piecewise smooth curve segment that first follows γ_1 and then follows γ_2 (reparametrized if necessary). Then

$$d_g(p, r) \leq L_g(\gamma) = L_g(\gamma_1) + L_g(\gamma_2).$$

Taking the infimum over all such γ_1 and γ_2, we find that $d_g(p, r) \leq d_g(p, q) + d_g(q, r)$. To complete the

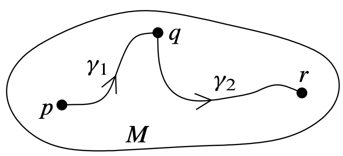


Figure 3.1: The triangle inequality.

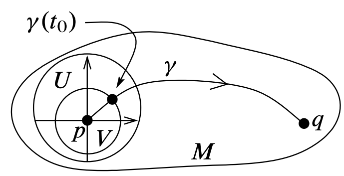


Figure 3.2: Positivity of d_g.

proof that (M, d_g) is a metric space, we need only show that $d_g(p, q) > 0$ if $p \neq q$. For this purpose, let $p, q \in M$ be distinct points, and let U be a smooth coordinate domain containing p but not q. Use the coordinate map as usual to identify U with an open subset in $\mathbb{R}^n$, and let $\bar{g}$ denote the Euclidean metric in these coordinates. If V is a regular coordinate ball of radius ε centered at p such that $\overline{V} \subseteq U$, Lemma 3.2.4.7 shows that there are positive constants c, C such that (3.2.4.1) is satisfied whenever $x \in \overline{V}$ and $v \in T_xM$. Then for any piecewise smooth curve segment γ lying entirely in $\overline{V}$, it follows that

$$cL_{\bar{g}}(\gamma) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma)$$

Suppose $\gamma : [a, b] \to M$ is a piecewise smooth curve segment from p to q. Let t_0 be the infimum of all $t \in [a, b]$ such that $\gamma(t) \notin \overline{V}$. It follows that $\gamma(t_0) \in \partial V$ by continuity, and $\gamma(t) \in \overline{V}$ for $a \leq t \leq t_0$. Thus

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq cL_{\bar{g}}(\gamma_{[a,t_0]}) \geq cd_{\bar{g}}(p, \gamma(t_0)) = c\varepsilon$$

Taking the infimum over all such γ, we conclude that $d_g(p,q) \geq c\varepsilon > 0$.

Finally, to show that the metric topology generated by d_g is the same as the given manifold topology on M, we need to show that the open subsets in the manifold topology are open in the metric topology, and vice versa.

Suppose, first, that $U \subseteq M$ is open in the manifold topology. Let $p \in U$, and let V be a regular coordinate ball of radius ε around p such that $\overline{V} \subseteq U$ as above. The argument in the previous paragraph shows that $d_g(p,q) \geq c\varepsilon$ whenever $q \notin \overline{V}$. The contrapositive of this statement is that $d_g(p,q) < c\varepsilon$ implies $q \in \overline{V} \subseteq U$, or in other words, the metric ball of radius $c\varepsilon$ around p is contained in U. This shows that U is open in the metric topology.

Conversely, suppose that W is open in the metric topology, and let $p \in W$. Let V be a regular coordinate ball of radius r around p, let $\bar{g}$ be the Euclidean metric on $\overline{V}$ determined by the given coordinates, and let c, C be positive constants such that (3.2.4.1) is satisfied for $v \in T_qM, q \in \overline{V}$. Let $\varepsilon < r$ be a positive number small enough that the metric ball around p of radius $C\varepsilon$ is contained in W, and let V_ε be the set of points in $\overline{V}$ whose Euclidean distance from p is less than ε. If $q \in V_\varepsilon$, let γ be the straight-line segment in coordinates from p to q. Using Lemma 3.2.4.7 as above, we conclude that

$$d_g(p,q) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma) < C\varepsilon$$

This shows that V_ε is contained in the metric ball of radius $C\varepsilon$ around p, and therefore in W. Since V_ε is a neighborhood of p in the manifold topology, this shows that W is open in the manifold topology as well. □

As a consequence of this theorem, all of the terminology of metric spaces can be carried over to connected Riemannian manifolds. Thus, a connected Riemannian manifold (M, g) is said to be **complete**, and g is said to be a **complete Riemannian metric**, if (M, d_g) is a complete metric space (i.e., if every Cauchy sequence in M converges to a point in M); and a subset $B \subseteq M$ is said to be **bounded** if there exists a constant K such that $d_g(x,y) \leq K$ for all $x, y \in B$; if this is the case, the **diameter** of A is the smallest such constant:

$$\operatorname{diam}(A) = \sup\{d_g(p,q) : p,q \in A\}$$

Since every compact metric space is bounded, every compact connected Riemannian manifold with or without boundary has finite diameter. (Note that the unit sphere with the Riemannian distance determined by the round metric has diameter π, not 2, since the Riemannian distance between antipodal points is π).

Recall that a topological space is said to be **metrizable** if it admits a distance function whose metric topology is the same as the given topology.

Corollary 3.2.4.9. *Every smooth manifold with or without boundary is metrizable.*

Proof. First suppose M is a smooth manifold without boundary, and choose any Riemannian metric g on M. If M is connected, Theorem 3.2.4.8 shows that M is metrizable. More generally, let $\{M_i\}$ be the connected components of M, and choose a point $p_i \in M_i$ for each i. For $x \in M_i$ and $y \in M_j$, define $d_g(x,y)$ as in Theorem 3.2.4.8 when $i = j$, and otherwise

$$d_g(x,y) = d_g(x,p_i) + 1 + d_g(p_j,y)$$

It is straightforward to check that this is a distance function that induces the given topology on M. Finally, if M has nonempty boundary, just embed M into its double, and note that a subspace of a metrizable topological space is always metrizable. □

3.2.5 Pseudo-Riemannian metrics

From the point of view of geometry, Riemannian metrics are by far the most important structures that manifolds carry. However, there is a generalization of Riemannian metrics that has become especially important because of its application to physics.

Before defining this generalization, we begin with some linear-algebraic preliminaries. Suppose V is a finite-dimensional vector space, and q is a symmetric covariant 2-tensor on V (also called a symmetric bilinear form). Just as for an inner product, there is a linear map $\widehat{q} : V \to V^*$ defined by

$$\widehat{q}(v)(w) = q(v,w)$$

for $v, w \in V$. We say that q is **nondegenerate** if $\widehat{q}$ is an isomorphism.

Lemma 3.2.5.1. *Suppose q is a symmetric covariant 2-tensor on a finite-dimensional vector space V. The following are equivalent:*

(a) q is nondegenerate.

(b) For every nonzero $v \in V$, there is some $w \in V$ such that $q(v, w) \neq 0$.

(c) If $q = q_{ij}\varepsilon^i\varepsilon^j$ in terms of some basis (ε^i) for V^*, then the matrix (q_{ij}) is invertible.

Proof. By dimension consideration, $\widehat{q}$ is an isomorphism if and only if it is injective, so (a) is equivalent to (b). Now if $q(v, w) = 0$, then in terms of matrix, we have $X^T(g_{ij})Y = 0$ for some vector X and every vector Y. This implies that $X^T(g_{ij}) = 0$, and thus $(g_{ij})X = 0$ since q is symmetric. This then implies that (g_{ij}) is not invertible, if $X \neq 0$. □

We use the term **scalar product** to denote any nondegenerate symmetric bilinear form on a finite-dimensional vector space V, and reserve the term inner product for the special case of a positive definite scalar product. A **scalar product space** is a finite-dimensional vector space endowed with a scalar product. When convenient, we will often use a notation like $\langle \cdot \cdot \rangle$ to denote a scalar product. We say that vectors $v, w \in V$ are orthogonal if $\langle v, w\rangle = 0$, just as in the case of an inner product. Given a vector $v \in V$, we define the **norm** of v to be $|v| = |\langle v, v\rangle|^{1/2}$. Note that in the indefinite case, it is possible for a nonzero vector to be orthogonal to itself, and thus to have norm zero.

If $S \subseteq V$ is any linear subspace, the set of vectors in V that are orthogonal to every vector in S is a linear subspace denoted by $S^\perp$.

Lemma 3.2.5.2. *Suppose (V, q) is a finite-dimensional scalar product space, and $S \subseteq V$ is a linear subspace.*

(a) $\dim S + \dim S^\perp = \dim V$.

(b) $(S^\perp)^\perp = S$.

Proof. Define a linear map $\Phi : V \to S^*$ by $\Phi(v) = \widehat{q}(v)|_S$. Note that $v \in \ker \Phi$ if and only if $q(v, x) = 0$ for all $x \in S$, so $\ker \Phi = S^\perp$. If $\varphi \in S^*$ is arbitrary, there is a covector $\widetilde{\varphi} \in V^*$ whose restriction to S is equal to φ. (For example, such a covector is easily constructed after choosing a basis for S and extending it to a basis for V.) Since $\widehat{q}$ is an isomorphism, there exists $v \in V$ such that $\widehat{q}(v) = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the ranknullity theorem, the dimension of $S^\perp = \ker \Phi$ is equal to $\dim V - \dim S^* = \dim V - \dim S$. This proves (a).

To prove (b), note that every $v \in S$ is orthogonal to every element of S by definition, so $S \subseteq (S^\perp)^\perp$. Because these finite-dimensional vector spaces have the same dimension by part (a), they are equal. □

An ordered k-tuple $(v_1, \ldots, v_k)$ of elements of V is said to be **orthonormal** if $|v_i| = 1$ for each i and $\langle v_i, v_j\rangle = 0$ for $i \neq j$, or equivalently, if $\langle v_i, v_j\rangle = \delta_{ij}$ for each i and j. We wish to prove that every scalar product space has an orthonormal basis. Note that the usual GramSchmidt algorithm does not always work in this situation, because the vectors that appear in the denominators might have vanishing norms. In order to get around this problem, we introduce the following definitions.

If (V, q) is a finite-dimensional scalar product space, a subspace $S \subseteq V$ is said to be nondegenerate if the restriction of q to $S \times S$ is nondegenerate. An ordered k-tuple of vectors $(v_1, \ldots, v_k)$ in V is said to be nondegenerate if for each $j = 1, \ldots, k$ the vectors $(v_1, \ldots, v_j)$ span a nondegenerate j-dimensional subspace of V. For example, every orthonormal basis is nondegenerate.

Lemma 3.2.5.3. *Suppose (V, q) is a finite-dimensional scalar product space, and $S \subseteq V$ is a linear subspace. The following are equivalent:*

(a) S is nondegenerate.

(b) $S^\perp$ is nondegenerate.

(c) $S \cap S^\perp = \{0\}$.

(d) $V = S \oplus S^\perp$.

Proof. Note that $q|_S$ is nondegenerate if and only if the set

$$\{x \in S : q(x, y) = 0 \forall y \in S\} = S \cap S^\perp$$

is $\{0\}$. Now since $(S^\perp)^\perp = S$, we have the equivalences (a), (b) and (c). Now, since $\dim S + \dim S^\perp = \dim V$, they are easily seen to be equivalent to (d). □

Lemma 3.2.5.4 (Completion of Nondegenerate Bases). *Suppose (V, q) is an n-dimensional scalar product space, and $(v_1, \ldots, v_k)$ is a nondegenerate k-tuple in V with $0 \le k < n$. Then there exist vectors $v_{k+1}, \ldots, v_n \in V$ such that $(v_1, \ldots, v_n)$ is a nondegenerate basis for V.*

Proof. Let $S = \operatorname{span}(v_1, \ldots, v_k) \subseteq V$. Because $k < n$, $S^\perp$ is a nontrivial subspace of V, and Lemma 3.2.5.3 shows that $S^\perp$ is nondegenerate and $V = S \oplus S^\perp$. By the nondegeneracy of $S^\perp$, there must be a vector in $S^\perp$ with nonzero length, because otherwise the polarization identity would imply that all inner products of pairs of elements of S would be zero. If $v_{k+1} \in S^\perp$ is any vector with nonzero length, then $(v_1, \ldots, v_{k+1})$ is easily seen to be a nondegenerate $(k+1)$-tuple. Repeating this argument for $v_{k+2}, \ldots, v_n$ completes the proof. □

Proposition 3.2.5.5 (GramSchmidt Algorithm for Scalar Products). *Suppose (V, q) is an n-dimensional scalar product space. If (v_i) is a nondegenerate basis for V, then there is an orthonormal basis (b_i) with the property that* $\operatorname{span}(b_1, \ldots, b_k) = \operatorname{span}(v_1, \ldots, v_k)$ *for each $k = 1, \ldots, n$.*

Proof. As in the positive definite case, the basis (b_i) is constructed recursively, starting with $b_1 = v_1/|v_1|$ and noting that the assumption that v_1 spans a nondegenerate subspace ensures that $|v_1| \neq 0$. At the inductive step, assuming we have constructed $(b_1, \ldots, b_k)$, we first set

$$z = v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, b_i \rangle}{\langle b_i, b_i \rangle} b_i.$$

Each denominator $\langle b_i, b_i \rangle$ is equal to ± 1, so this defines z as a nonzero element of V orthogonal to $b_1, \ldots, b_k$, and with the property that $\operatorname{span}(b_1, \ldots, b_k, z) = \operatorname{span}(v_1, \ldots, v_{k+1})$. If $\langle z, z \rangle = 0$, then z is orthogonal to $\operatorname{span}(v_1, \ldots, v_{k+1})$, contradicting the nondegeneracy assumption. Thus we can complete the inductive step by putting $b_{k+1} = z/|z|$. □

Corollary 3.2.5.6. *Suppose (v, q) is an n-dimensional scalar product space. There is a basis (β^i) for V^* with respect to which q has the expression*

$$q = (\beta^1)^2 + \cdots + (\beta^r)^2 - (\beta^{r+1})^2 - \cdots - (\beta^{r+s})^2, \tag{3.2.5.1}$$

for some nonnegative integers r, s with $r + s = n$.

Proof. Let (b_i) be an orthonormal basis for V, and let (β^i) be the dual basis for V^*. A computation shows that q has a basis expression of the form (3.2.5.1), but perhaps with the positive and negative terms in a different order. Reordering the basis so that the positive terms come first, we obtain (3.2.5.1). □

It turns out that the numbers r and s in (3.2.5.1) are independent of the choice of basis. The integer s in the expression (the number of negative terms) is called the **index** of q, and the ordered pair (r, s) is called the **signature** of q.

Now suppose M is a smooth manifold. A **pseudo-Riemannian metric** on M (called by some authors a semi-Riemannian metric) is a smooth symmetric 2-tensor field g that is nondegenerate at each point of M, and with the same signature everywhere. Every Riemannian metric is also a pseudo-Riemannian metric.

Proposition 3.2.5.7 (Orthonormal Frames for Pseudo-Riemannian Manifolds). *Let (M, g) be a pseudo-Riemannian manifold. For each $p \in M$, there exists a smooth orthonormal frame on a neighborhood of p in M.*

Example 3.2.5.8. Suppose (M_1, g_1) and (M_2, g_2) are pseudo-Riemannian manifolds of signatures (r_1, s_1) and (r_2, s_2), respectively. Then $(M_1 \times M_2, g_1 \oplus g_2)$ is a pseudo-Riemannian manifold of signature $(r_1 + r_2, s_1 + s_2)$.

For nonnegative integers r and s, we define the **pseudo-Euclidean space of signature (r, s)**, denoted by $\mathbb{R}^{r,s}$, to be the manifold $\mathbb{R}^{r+s}$, with standard coordinates denoted by $(\xi^1, \ldots, \xi^r, \tau^1, \ldots, \tau^s)$, and with the pseudo-Riemannian metric $\bar{q}^{(r,s)}$ defined by

$$\bar{q}^{(r,s)} = (d\xi^1)^2 + \cdots + (d\xi^r)^2 - (d\tau^1)^2 - \cdots - (d\tau^s)^2.$$

By far the most important pseudo-Riemannian metrics (other than the Riemannian ones) are the **Lorentz metrics**, which are pseudo-Riemannian metrics of index 1, and thus signature $(r, 1)$.

The pseudo-Euclidean metric $\bar{q}^{(r,1)}$ is a **Lorentz metric** called the **Minkowski metric**, and the Lorentz manifold $\mathbb{R}^{r,1}$ is called $(r+1)$-dimensional Minkowski space. If we denote standard coordinates on $\mathbb{R}^{r+1}$ by $(\xi^1, \dots, \xi^r, \tau)$ then the Minkowski metric is given by

$$\bar{q}^{(r,1)} = (d\xi^1)^2 + \cdots + (d\xi^r)^2 - (d\tau^1)^2.$$

Many, but not all, results from the theory of Riemannian metrics apply equally well to pseudo-Riemannian metrics. We will attempt to point out which results carry over directly to pseudo-Riemannian metrics, which ones can be adapted with minor modifications, and which ones do not carry over at all. As a rule of thumb, proofs that depend only on the nondegeneracy of the metric tensor, such as properties of the musical isomorphisms and existence and uniqueness of geodesics, work fine in the pseudo-Riemannian setting, while proofs that use positivity in an essential way, such as those involving lengths of curves, do not.

One notable result that does not carry over to the pseudo-Riemannian case is Proposition 3.2.4.8, about the existence of metrics. For example, the following result characterizes those manifolds that admit Lorentz metrics.

Theorem 3.2.5.9. *A smooth manifold M admits a Lorentz metric if and only if it admits a rank-1 tangent distribution (i.e., a rank-1 subbundle of TM).*

Proof. For sufficiency, assume that $D \subseteq TM$ is a 1-dimensional distribution, and choose any Riemannian metric g on M. Then, by Lemma **??**, locally it is possible to choose a g-orthonormal frame (E_i) and dual coframe (ε^i) such that E_1 spans D. Then the Lorentz metric $-(\varepsilon^1)^2 + (\varepsilon^2)^2 + \cdots + (\varepsilon^n)^2$ is independent of the choice of frame. □

3.2.5.1 Pseudo-Riemannian submanifolds

The theory of submanifolds is only slightly more complicated in the pseudo-Riemannian case. If $(\widetilde{M}, \widetilde{g})$ is a pseudo-Riemannian manifold, a smooth submanifold $\iota : M \hookrightarrow \widetilde{M}$ is called a **pseudo-Riemannian submanifold** of $\widetilde{M}$ if $\iota^*\widetilde{g}$ is nondegenerate with constant signature. If this is the case, we always consider M to be endowed with the induced pseudo-Riemannian metric $\iota^*\widetilde{g}$. In the special case in which $\iota^*\widetilde{g}$ is positive definite, M is called a **Riemannian submanifold**.

The nondegeneracy hypothesis is not automatically satisfied: for example, if $M \subseteq \mathbb{R}^{1,1}$ is the submanifold $\{(\xi, \tau) : \tau = \xi\}$ and $\iota : M \to \mathbb{R}^{1,1}$ is inclusion, then the pullback tensor $\iota^*\widetilde{g}$ is identically zero on M.

For hypersurfaces (submanifolds of codimension 1), the nondegeneracy condition is easy to check. If $M \subseteq \widetilde{M}$ is a smooth submanifold and $p \in M$, then a vector $v \in T_p\widetilde{M}$ is said to be **normal to M** if $\langle v, x\rangle = 0$ for all $x \in T_pM$, just as in the Riemannian case. The space of all normal vectors at p is a subspace of $T_p\widetilde{M}$ denoted by N_pM.

Proposition 3.2.5.10. *Suppose $(\widetilde{M}, \widetilde{g})$ is a pseudo-Riemannian manifold of signature (r, s). Let M be a smooth hypersurface in $\widetilde{M}$, and let $\iota : M \to \widetilde{M}$ be the inclusion map. Then the pullback tensor field $\iota^*\widetilde{g}$ is nondegenerate if and only if $\widetilde{g}(v, v) \neq 0$ for every $p \in M$ and every nonzero normal vector $v \in N_pM$. If $\widetilde{g}(v, v) > 0$ for every nonzero normal vector to M, then M is a pseudo-Riemannian submanifold of signature $(r-1, s)$; and if $\widetilde{g}(v, v) < 0$ for every such vector, then M is a pseudo-Riemannian submanifold of signature $(r, s-1)$.*

Proof. Given $p \in M$, Lemma 3.2.5.3 shows that T_pM is a nondegenerate subspace of $T_p\widetilde{M}$ if and only if the one-dimensional subspace $(T_pM)^\perp = N_pM$ is nondegenerate, which is the case if and only if every nonzero $v \in N_pM$ satisfies $\widetilde{g}(v, v) \neq 0$.

Now suppose $\widetilde{g}(v, v) > 0$ for every nonzero normal vector v. Let $p \in M$ be arbitrary, and let v be a nonzero element of N_pM. Writing $n = \dim \widetilde{M}$, we can complete v to a nondegenerate basis $v, w_2, \dots, w_n$ for T_pM by Lemma 3.2.5.4, and then use the GramSchmidt algorithm to find an orthonormal basis $b_1, \dots, b_n$ for $T_p\widetilde{M}$ such that $\operatorname{span}(b_1) = N_pM$. It follows that $\operatorname{span}(b_2, \dots, b_n) = T_pM$. If (β^j) is the dual basis to (b_i), then $\widetilde{g}_p$ has a basis representation of the form $(\beta^1)^2 \pm (\beta^2)^2 \pm \cdots \pm (\beta^n)^2$, with a total of r positive terms and s negative ones, and with a positive sign on the first term $(\beta^1)^2$. Therefore, $\iota^*\widetilde{g}_p = \pm(\beta^2)^2 \pm \cdots \pm (\beta^n)^2$ has signature $(r-1, s)$. The argument for the case $\widetilde{g}(v, v) < 0$ is similar. □

If (M, g) is a pseudo-Riemannian manifold and $f \in C^\infty(M)$, then the **gradient** of f is defined as the smooth vector field $\operatorname{grad} f = (df)^\sharp$ just as in the Riemannian case.

Proposition 3.2.5.11. *Suppose $(\widetilde{M}, \widetilde{g})$ is a pseudo-Riemannian manifold of signature (r,s), $f \in C^\infty(M)$, and $M = f^{-1}(c)$ for some $c \in \mathbb{R}$. If $\langle \operatorname{grad} f, \operatorname{grad} f\rangle > 0$ everywhere on M, then M is an embedded pseudo-Riemannian submanifold of M of signature $(r-1,s)$ and if $\langle \operatorname{grad} f, \operatorname{grad} f\rangle < 0$ everywhere on M, then M is an embedded pseudo-Riemannian submanifold of $\widetilde{M}$ of signature $(r, s-1)$. In either case,*$\operatorname{grad} f$ *is everywhere normal to M.*

Proof. By Proposition **??** and Proposition **??** we know that M is an embedded submanifold whose tangent space is every where tangent to $\operatorname{grad} f$. Now the result follows from Proposition 3.2.5.10. □

Proposition 3.2.5.12 (Pseudo-Riemannian Adapted Orthonormal Frames). *Suppose $(\widetilde{M}, \widetilde{g})$ is a pseudo-Riemannian manifold, and $M \subseteq \widetilde{M}$ is an embedded pseudo-Riemannian or Riemannian submanifold. For each $p \in M$, there exists a smooth orthonormal frame on a neighborhood of p in $\widetilde{M}$ that is adapted to M.*

Proof. Write $m = \dim \widetilde{M}$ and $n = \dim M$, and let $p \in M$ be arbitrary. Proposition 3.2.5.7 shows that there is a smooth orthonormal frame $(E_1, \ldots, E_n)$ for M on some neighborhood of p in M. Then by Lemma 3.2.5.4, we can find vectors $v_{n+1}, \ldots, v_m$ such that

$$(E_1|_p, \ldots, E_n|_p, v_{n+1}, \ldots, v_m)$$

is a nondegenerate basis for $T_p\widetilde{M}$. Now extend $v_{n+1}, \ldots, v_m$ arbitrarily to smooth vector fields $V_{m+1}, \ldots, V_m$ on a neighborhood of p in $\widetilde{M}$. By continuity, the ordered m-tuple

$$(E_1, \ldots, E_n, V_{n+1}, \ldots, V_m)$$

will be a nondegenerate frame for $\widetilde{M}$ in some (possibly smaller) neighborhood of p. Applying the GramSchmidt algorithm to this local frame yields a smooth local orthonormal frame $(E_1, \ldots, E_m)$ for $\widetilde{M}$ with the property that $(E_1, \ldots, E_n)$ restricts to a local orthonormal frame for M. □

The next corollary is proved in the same way as Proposition 3.2.2.8

Corollary 3.2.5.13 (Normal Bundle to a Pseudo-Riemannian Submanifold). *Suppose $(\widetilde{M}, \widetilde{g})$ is a pseudo-Riemannian manifold, and $M \subseteq \widetilde{M}$ is an embedded pseudo-Riemannian or Riemannian submanifold. The set of vectors normal to M is a smooth vector subbundle of $T\widetilde{M}|_M$, called the normal bundle to M.*

3.2.6 Exercise

Exercise 3.2.1. Suppose that E is a smooth vector bundle over a smooth manifold M with or without boundary, and $V \subseteq E$ is an open subset with the property that for each $p \in M$, the intersection of V with the fiber E_p is convex and nonempty. By a section of V, we mean a section of E whose image lies in V.

(a) Show that there exists a smooth global section of V.

(b) Suppose $\sigma : A \to V$ is a smooth section of V defined on a closed subset $A \subseteq M$. (This means that σ extends to a smooth section of V in a neighborhood of each point of A.) Show that there exists a smooth global section $\widetilde{\sigma}$ of V whose restriction to A is equal to σ. Show that if V contains the image of the zero section of E, then $\widetilde{\sigma}$ can be chosen to be supported in any predetermined neighborhood of A.

Proof. For $p \in M$, choose a neighborhood U contains p, with a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Let $v \in \mathbb{R}^k$ be a vector such that $(p, v) \in \Phi(V \cap \pi^{-1}(U))$. Since $V \cap \pi^{-1}(U)$ is open, by appropriately shrinking U, we can find an open subset $W \subseteq \mathbb{R}^k$ such that

$$(p, v) \in U \times W \subseteq \Phi(V \cap \pi^{-1}(U)).$$

In particular, $(q, v) \in \Phi(V \cap \pi^{-1}(U))$ for all $q \in U$, thus we can define $\sigma_p : U \to E$ by

$$\sigma_p(q) = \Phi^{-1}(q, v) \quad \text{for} \quad q \in U.$$

It follows that σ_p is a local section of V. Now the neighborhoods $\{U_p\}$ gives an open cover of M, by taking a partition of unity $\{\psi_p\}$ for the open cover $\{\psi_p\}$, we define

$$\eta = \sum_{p\in M} \psi_p \sigma_p.$$

Then at each point p_0, since $V \cap E_{p_0}$ is convex, we find $\eta(p_0) \in V$.

The second statement is the same as Lemma **??**. □

Exercise 3.2.2. Prove that every Riemannian 1-manifold is flat.

Proof. Let M be a Riemannian 1-manifold, then each point of M admits a commuting orthonormal frame. Thus by Proposition 3.2.2.5 M is flat. □

Exercise 3.2.3. Suppose V and W are finitely-dimensional real inner product spaces of the same dimension, and $F : V \to W$ is any map (not assumed to be linear or even continuous) that preserves the origin and all distances: $F(0) = 0$ and $|F(x) - F(y)| = |x - y|$ for all $x, y \in V$. Prove that F is a linear isometry.

First proof. First, inserting $y = 0$ in the second condition gives $|F(x)| = |x|$. Now let $t \in [0,1]$ and denote $r = |x - y|$. Then

$$\{tx + (1-t)y\} = \overline{B}(x, tr) \cap \overline{B}(y, (1-t)r).$$

Now since F is injective, $F(tx + (1-t)y)$ is the unique element of

$$F(\overline{B}(x, tr) \cap \overline{B}(y, (1-t)r)) = F(\overline{B}(x, tr)) \cap F(\overline{B}(y, (1-t)r)) = \overline{B}(F(x), tr) \cap \overline{B}(F(y), (1-t)r),$$

which is $tF(x) + (1-t)F(y)$. This then implies $F(x)$ is linear since $F(0) = 0$. □

Second proof. We have $|F(x)| = x$ for all $x \in V$. Now expand the equation $|F(x) - F(y)|^2 = |x - y|^2$, we get

$$|F(x)|^2 + |F(y)|^2 - 2(F(x), F(y)) = |x|^2 + |y|^2 - 2(x, y).$$

Therefore F preserves the inner products.

Now let $x, y \in V$, we want to prove $F(x + y) = F(x) + F(y)$ and $F(\lambda x) = \lambda x$ for $\lambda \in \mathbb{R}$. We note that for all $v \in V$,

$$(F(x+y), F(v)) = ((x+y), v) = (x, v) + (y, v) = (F(x) + F(y), F(v)).$$

Take v to be a orthonormal basis for V, this then implies $F(x + y) = F(x) + F(y)$. Similarly we can prove $F(\lambda x) = \lambda F(x)$, therefore F is linear. □

Exercise 3.2.4. Show that the shortest path between two points in Euclidean space is a straight line segment. More precisely, for $x, y \in \mathbb{R}^n$, let $\gamma : [0,1] \to \mathbb{R}^n$ be the curve segment $\gamma(t) = x + t(y - x)$, and show that any other piecewise smooth curve segment $\widetilde{\gamma}$ from x to y satisfies $L_{\bar{g}}(\widetilde{\gamma}) \geq L_{\bar{g}}(\gamma)$.

Proof. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a piecewise smooth curve segment such that $\gamma(a) = x$ and $\gamma(b) = y$. For any constant vector v with $|v| = 1$, by Cauchy-Schwarz inequality we have

$$(y - x) \cdot v = \left(\gamma(t) \cdot v\right)\Big|_{t=a}^{t=b} = \int_a^b \gamma'(t) \cdot v \, dt \leq \int_a^b |\gamma'(t)||v| dt = \int_a^b |\gamma'(t)| dt.$$

Set $v = (y - x)/|y - x|$, then

$$|y - x| \leq \int_a^b |\gamma'(t)| dt.$$

Now if the equality holds if and only if $\gamma'(t) = a(y - x)$ for some constant $a \in \mathbb{R}$. Hence if and only if $\gamma(t) = a(y - x)t + b$ with $a, b \in \mathbb{R}$. □

Exercise 3.2.5. Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric $\bar{g}$.

(a) Suppose $U, V \subseteq \mathbb{R}^n$ are connected open sets, $\varphi, \psi : U \to V$ are Riemannian isometries, and for some $p \in U$ they satisfy $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Show that $\varphi = \psi$.

(b) Show that the set of maps from $\mathbb{R}^n$ to itself given by the action of $E(n)$ on $\mathbb{R}^n$ described in Example 2.1.5.1 is the full group of Riemannian isometries of $(\mathbb{R}^n, \bar{g})$.

Proof. Let $x \in \mathbb{R}^n$, and let γ be a smooth curve in $\mathbb{R}^n$ from $\varphi(p)$ to $\varphi(x)$, then by Proposition 3.2.4.1,

$$L_{\bar{g}}(\gamma) = L_{\bar{g}}(\varphi^{-1} \circ \gamma) \geq L_{\bar{g}}(\sigma) = L_{\bar{g}}(\varphi \circ \sigma),$$

where $\sigma : [0,1] \to \mathbb{R}^n$ is the straight line from p to x and we use the result of Exercise 3.2.4. This implies that $\varphi \circ \sigma$ is the straight line from $\varphi(p)$ to $\varphi(x)$. Since the length of $\varphi \circ \sigma$ is fixed, $\varphi(x)$ is uniquely determined by its direction, that is, $(\varphi \circ \sigma)'(0)$. Since $d\varphi_p = d\psi_p$ we then conclude $\varphi = \psi$.

Let φ be any Riemannian isometry of $(\mathbb{R}^n, \bar{g})$. Since $d\varphi_0$ preserves the Euclidean dot product, it corresponds to a matrix $C \in \mathrm{O}(n)$. Let $(\varphi(0), C) \in E(n)$, then φ is represented by $(C, \varphi(0))$, in the sense that

$$\varphi(x) = (\varphi(0), C) \cdot x = \varphi(0) + Cx.$$

This gives the claim. $\square$

Exercise 3.2.6. Let (M, g) be a Riemannian manifold. A smooth vector field V on M is called a **Killing vector field** for g if the flow of V acts by isometries of g.

(a) Show that the set of all Killing vector fields on M constitutes a Lie subalgebra of $\mathfrak{X}(M)$.

(b) Show that a smooth vector field V on M is a Killing vector field if and only if it satisfies the following equation in each smooth local coordinate chart:

$$V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial V^k}{\partial x^i} + g_{ik} \frac{\partial V^k}{\partial x^j} = 0 \quad \text{for all } i, j.$$

Proof. By definition, V is a killing field if and only if g is invaraint under θ. Thus by Theorem 3.1.3.14, this is equivalent to $\mathfrak{L}_V g = 0$. Now by Exercise 3.1.2,

$$\mathfrak{L}_{[V,W]} g = \mathfrak{L}_V \mathfrak{L}_W g - \mathfrak{L}_W \mathfrak{L}_V g.$$

Thus part (a) follows.

For part (b), write $V = V^k \partial / \partial x^k$, we compute using Corollary 3.1.3.10:

$$\begin{aligned}
(\mathfrak{L}_V g)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) &= V(g_{ij}) - g\Big([V, \frac{\partial}{\partial x^i}], \frac{\partial}{\partial x^j}\Big) - g\Big(\frac{\partial}{\partial x^i}, [V, \frac{\partial}{\partial x^j}]\Big) \\
&= V^k \frac{\partial g_{ij}}{\partial x^k} + g\Big(\frac{\partial V^k}{\partial x^i}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\Big) + g\Big(\frac{\partial}{\partial x^i}, \frac{\partial V^k}{\partial x^j}\frac{\partial}{\partial x^k}\Big) \\
&= V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial V^k}{\partial x^i} + g_{ik} \frac{\partial V^k}{\partial x^j}.
\end{aligned}$$

$\square$

Exercise 3.2.7. Let $K \subseteq \mathfrak{X}(\mathbb{R}^n)$ denote the Lie algebra of Killing vector fields with respect to the Euclidean metric, and let $K_0 \subseteq K$ denote the subspace consisting of fields that vanish at the origin.

(a) Show that the map

$$V \mapsto \Big(\frac{\partial V^k}{\partial x^j}(0)\Big)$$

is an injective linear map from K_0 to $\mathfrak{o}(n)$.

(b) Show that the following vector fields form a basis for K:

$$\frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n; \qquad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad 1 \leq i < j \leq n.$$

Proof. If $g = \delta_{ij} dx^i dx^j$, then the equation becomes

$$\frac{\partial V^j}{\partial x^i} + \frac{\partial V^i}{\partial x^j} = 0$$

thus the image of the map is in $\mathfrak{o}(n)$. $\square$

Exercise 3.2.8. Suppose $g = f(t)dt^2$ is a Riemannian metric on $\mathbb{R}$. Show that g is complete if and only if both of the following improper integrals diverge:

$$\int_0^{+\infty} \sqrt{f(t)}dt, \quad \int_{-\infty}^0 \sqrt{f(t)}dt.$$

Exercise 3.2.9. Let (M, g) be a Riemannian manifold, let $f \in C^\infty(M)$, and let $p \in M$ be a regular point of f.

(a) Show that among all unit vectors $v \in T_pM$, the directional derivative vf is greatest when v points in the same direction as $\operatorname{grad} f|_p$, and the length of $\operatorname{grad} f|_p$ is equal to the value of the directional derivative in that direction.

(b) Show that $\operatorname{grad} f|_p$ is normal to the level set of f through p.

(c) Let $X \in \mathfrak{X}(M)$ be a nowhere-vanishing vector field. Prove that $X = \operatorname{grad} f$ if and only if $Xf \equiv |X|_g^2$ and X is orthogonal to the level sets of f at all regular points of f.

Proof. By (3.2.3.1) we have

$$\langle \operatorname{grad} f|_p, v\rangle_g = df_p(v) = vf.$$

Then by Cauchy-Schwarz inequality, we have

$$vf \le |v|_g \cdot \left|\operatorname{grad} f|_p\right|_g,$$

with the quality holds if and only if v is paralle to $\operatorname{grad} f|_p$.

Let $S = f^{-1}(f(p))$ be the level set of f through p, then by Proposition **??** we have

$$T_pS = \ker df_p = \ker(\widehat{g}(\operatorname{grad} f)_p) = \ker\left(\langle \operatorname{grad} f|_p, \cdot\rangle_g\right).$$

Thus $\operatorname{grad} f|_p$ is normal to T_pS.

For (c), one direction is clear. Conversely, assume that $X \in \mathfrak{X}(M)$ satisfies the conditions. Let S be a level set of f and $p \in S$, then $\dim(T_pS)^\perp = 1$ for $p \in S$. Since X and $\operatorname{grad} f$ are both contained in it, is follows that $X_p = \lambda \operatorname{grad} f|_p$ for some $\lambda \in \mathbb{R}$. From the condition we have

$$\lambda\langle X_p, X_p\rangle\langle \operatorname{grad} f|_p, X_p\rangle = Xf|_p = \langle X_p, X_p\rangle = \lambda^2\langle X_p, X_p\rangle.$$

Therefore $\lambda = 1$ since $X_p \neq 0$. Since X is non-vanishing, it follows that $X = \operatorname{grad} f$. ☐

Exercise 3.2.10. Let M be a compact smooth n-manifold, and suppose f is a smooth real-valued function on M that has only finitely many critical points $\{p_1, \dots, p_k\}$, with corresponding critical values $c_1, \dots, c_k$ labeled so that $c_1 \le \cdots \le c_k$. For any $a < b \in \mathbb{R}$, define

$$M_a = f^{-1}(a), \quad M_{(a,b)} = f^{-1}(a, b), \quad M_{[a,b]} = f^{-1}([a, b]).$$

If a and b are regular values, note that M_a and M_b are embedded hypersurfaces in M, $M_{(a,b)}$ is an open submanifold, and $M_{[a,b]}$ is a regular domain by Proposition **??**.

(a) Choose a Riemannian metric g on M, let X be the vector field $X = \operatorname{grad} f / |\operatorname{grad} f|_g^2$ on $M \setminus \{p_1, \dots, p_k\}$, and let θ denote the flow of X. Show that $f(\theta_t(p)) = f(p) + t$ whenever θ_t is defined.

(b) Let $[a, b] \times \mathbb{R}$ be a compact interval containing no critical values of f. Show that θ restricts to a diffeomorphism from $[0, b-a] \times M_a$ to $M_{[a,b]}$.

Proof. Fix a point $p \in M \setminus \{p_1, \dots, p_k\}$, we calculate the derivative of $f(\theta_t(p))$ with respect to t. We have

$$d(f \circ \theta_t(p)) = df \circ d(\theta_t(p)) = df(X_{\theta_t(p)}) = \langle \operatorname{grad} f|_{\theta_t(p)}, X_{\theta_t(p)}\rangle = 1.$$

Therefore

$$f(\theta_t(p)) = f(\theta_0(p)) + t = f(p) + t \tag{3.2.6.1}$$

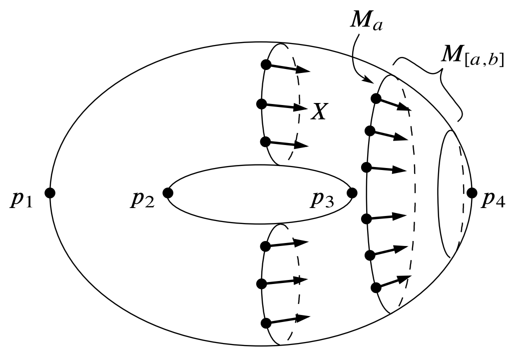


Figure 3.1: The setup for Exercise 3.2.10.

Now let $p \in M_a$, we observe that

$$f(\theta_t(p)) = f(p) + t = a + t.$$

Therefore θ restricts to a map $\theta|_{M_{[a,b]}} : [0, b-a] \times M_a \to M_{[a,b]}$. To show this is a diffeomorphism, we construct the inverse map of $\theta|_{M_{[a,b]}}$:

$$\psi : M_{[a,b]} \to [0, b-a] \times M_a, \quad p \mapsto (f(p) - a, \theta_{a-f(p)}(p)).$$

Note that since $M_{[a,b]}$ is compact (M is compact and $M_{[a,b]}$ is closed), the integral curve passing p is defined on $\mathbb{R}$ if it is contained in $M_{[a,b]}$ by the escape lemma (Lemma 1.2.3.12). But from equation (3.2.6.1), this is impossible. So it must pass the boundary of $M_{[a,b]}$, i.e., threr is $t_0 \in \mathbb{R}$ such that $\theta_t(p) \in M_a$. We can see $t_0 = a - f(p)$ from (3.2.6.1), so our definition of ψ is legal. Now it is easy to see that ψ is the inverse of $\theta|_{M_{[a,b]}}$, so we get the diffeomorphism. □

3.3 Differential forms

3.3.1 The alternating tensors

We continue to assume that V is a finite-dimensional real vector space. A covariant k-tensor α on V is said to be alternating (or antisymmetric or skew-symmetric) if it changes sign whenever two of its arguments are interchanged. This means that for all vectors $v_1, \ldots, v_k$ and every pair of distinct indices i, j it satisfies

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Alternating covariant k-tensors are also variously called **exterior forms**, **multicovectors**, or **k-covectors**. The subspace of all alternating covariant k-tensors on V is denoted by $\bigwedge^k(V^*) \subseteq T^k(V^*)$.

The next lemma gives two more useful characterizations of alternating tensors.

Lemma 3.3.1.1. *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

(a) α is alternating.

(b) $\alpha(v_1, \ldots, v_k) = 0$ whenever the k-tuple $v_1, \ldots, v_k$ is linearly dependent.

(c) α gives the value zero whenever two of its arguments are equal:

$$\alpha(v_1, \ldots, w, \ldots, w, v_k) = 0.$$

Proof. The implications $(a) \Rightarrow (c)$ and $(b) \Rightarrow (c)$ are immediate. We complete the proof by showing that (c) implies both (a) and (b).

Assume that α satisfies (c). For any vectors $v_1, \ldots, v_k$, the hypothesis implies

$$0 = \alpha(v_1, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, v_k)$$

$$= \alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) + (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Thus α is alternating. On the other hand, if $v_1, \ldots, v_k$ is a linearly dependent k-tuple, then one of the v_i's can be written as a linear combination of the others. For simplicity, let us assume that $v_k = \sum_{j=1}^{k-1} a^j v_j$. Then multilinearity of α implies

$$\alpha(v_1, \ldots, v_k) = \sum_{j=1}^{k-1} a^j \alpha(v_1, \ldots, v_{k-1}, v_j).$$

In each of these terms, α has two identical arguments, so every term is zero. $\square$

We now define a similar projection $\mathrm{Alt} : T^k(V^*) \to \bigwedge^k(V^*)$, called **alternation**, as follows:

$$\mathrm{Alt}\alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \sigma \cdot \alpha.$$

More explicitly, this means

$$\mathrm{Alt}\alpha(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\mathrm{sgn}\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Example 3.3.1.2. If α is any 1-tensor, then $\mathrm{Alt}\alpha = \alpha$. If α is a 2-tensor, then

$$(\mathrm{Alt}\alpha)(v, w) = \frac{1}{2}\big(\alpha(v, w) - \alpha(w, v)\big).$$

The next proposition is the analogue for alternating tensors of Proposition 3.1.2.1.

Proposition 3.3.1.3. *Let α be a covariant tensor on a finite-dimensional vector space.*

(a) $\mathrm{Alt}\alpha$ *is alternating.*

(b) $\mathrm{Alt}\alpha = \alpha$ *if and only if α is alternating.*

3.3.1.1 Elementary alternating tensors

For computations with alternating tensors, the following notation is exceedingly useful. Given a positive integer k, an ordered k-tuple $I = (i_1, \ldots, i_k)$ of positive integers is called a **multi-index of length k**. If I is such a multi-index and $\sigma \in \mathfrak{S}_k$ is a permutation, we write I_σ for the following multi-index:

$$I_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)}).$$

Note that $I_{\tau\sigma} = (I_\sigma)_\tau$ for $\sigma, \tau \in \mathfrak{S}_k$.

Let V be an n-dimensional vector space, and suppose $(\varepsilon^1, \ldots, \varepsilon^n)$ is any basis for V^*. We now define a collection of k-covectors on V that generalize the determinant function on $\mathbb{R}^n$. For each multi-index $I = (i_1, \ldots, i_k)$ of length k such that $1 \le i_1 < \cdots < i_k \le n$, define a covariant k-tensor

$$\varepsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix} \tag{3.3.1.1}$$

In other words, if V denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \ldots, v_k$ with respect to the basis (E_i) dual to (ε^i), then $\varepsilon^I(v_1, \ldots, v_k)$ is the determinant of the $k \times k$ submatrix consisting of rows $i_1, \ldots, i_k$ of V. Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k-tensor. We call ε^I an **elementary alternating tensor** or **elementary k-covector**.

Example 3.3.1.4. In terms of the standard dual basis (e^1, e^2, e^3) for $(\mathbb{R}^3)^*$, we have

$$e^{13}(v, w) = v^1 w^3 - w^1 v^3;$$

$$e^{123}(v, w, x) = \det(v, w, x).$$

In order to streamline computations with the elementary k-covectors, it is useful to extend the Kronecker delta notation in the following way. If I and J are multiindices of length k, we define

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}$$

so that

$$\delta^I_J = \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = I \text{ for some } \sigma \in \mathfrak{S}_k, \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I. \end{cases}$$

Lemma 3.3.1.5 (Properties of Elementary k-Covectors). *Let (E_i) be a basis for V, let (ε^i) be the dual basis for V^*, and let ε^I be as defined above.*

(a) If I has a repeated index, then $\varepsilon^I = 0$.

(b) If $J = I_\sigma$ for some $\sigma \in \mathfrak{S}_k$, then $\varepsilon^J = \operatorname{sgn}\sigma \varepsilon^I$.

(c) The result of evaluating ε^I on a sequence of basis vectors is

$$\varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J.$$

The significance of the elementary k-covectors is that they provide a convenient basis for $\bigwedge^k(V^*)$. Of course, the ε^I's are not all linearly independent, because some of them are zero and the ones corresponding to different permutations of the same multi-index are scalar multiples of each other. But as the next proposition shows, we can get a basis by restricting attention to an appropriate subset of multi-indices. A multi-index $I = (i_1, \dots, i_k)$ is said to be **increasing** if $i_1 < \cdots < i_k$. It is useful to use a primed summation sign to denote a sum over only increasing multi-indices, so that, for example,

$$\sum_I{}' \alpha_I \varepsilon^I := \sum_{i_1 < \cdots < i_k} \alpha_I \varepsilon^I.$$

Proposition 3.3.1.6 (A Basis for $\bigwedge^k(V^*)$). *Let V be an n-dimensional vector space. If (ε^i) is any basis for V^*, then for each positive integer $k \le n$, the collection of k-covectors*

$$\mathcal{E} := \{\varepsilon^I : I \text{ is an increasing multi-index of length } k\}$$

is a basis for $\bigwedge^k(V^*)$. Therefore,

$$\dim \bigwedge^k(V^*) = \binom{n}{k}.$$

If $k > n$, then $\dim \bigwedge^k(V^*) = 0$.

In particular, for an n-dimensional vector space V, this proposition implies that $\bigwedge^n(V^*)$ is 1-dimensional and is spanned by $\varepsilon^{1\dots n}$. By definition, this elementary n-covector acts on vectors $v_1, \dots, v_n$ by taking the determinant of their component matrix $V = (v^i_j)$. For example, on $\mathbb{R}^n$ with the standard basis, $e^{1\dots n}$ is precisely the determinant function.

One consequence of this is the following useful description of the behavior of an n-covector on an n-dimensional space under linear maps. Recall that if $T : V \to V$ is a linear map, the determinant of T is defined to be the determinant of the matrix representation of T with respect to any basis.

Proposition 3.3.1.7. *Suppose V is an n-dimensional vector space and $\omega \in \bigwedge^n(V^*)$. If $T : V \to V$ is any linear map and $v_1, \dots, v_n$ are arbitrary vectors in V, then*

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\omega(v_1, \dots, v_n). \tag{3.3.1.2}$$

Proof. Let (E_i) be any basis for V, and let (ε^i) be the dual basis. Let (T^j_i) denote the matrix of T with respect to this basis, and let $T_i = TE_i = T^j_i E_j$ be the i-th column of T. By Proposition 3.3.1.6, we can write $\omega = \lambda\, \varepsilon^{1\dots n}$ for some real number λ.

Since both sides of (3.3.1.2) are multilinear functions of $v_1, \ldots, v_n$, it suffices to verify the identity when the v_i's are basis vectors. Furthermore, since both sides are alternating, we only need to check the case $(v_1, \ldots, v_n) = (E_1, \ldots, E_n)$. In this case, the right-hand side of (3.3.1.2) is

$$(\det T)\lambda\, \varepsilon^{1\ldots n}(E_1, \ldots, E_n) = \lambda \det T.$$

On the other hand, the left-hand side reduces to

$$\omega(TE_1, \ldots, TE_n) = \lambda\, \varepsilon^{1\ldots n}(T_1, \ldots, T_n) = \lambda \det(T_i^j),$$

which is equal to the right-hand side. □

3.3.1.2 The wedge product

We continue with the assumption that V is a finite-dimensional real vector space. Given $\omega \in \bigwedge^k(V^*)$ and $\eta \in \bigwedge^l(V^*)$, we define their wedge product or exterior product to be the following $(k+l)$-covector:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta). \tag{3.3.1.3}$$

The mysterious coefficient is motivated by the simplicity of the statement of the following lemma.

Lemma 3.3.1.8. *Let V be an n-dimensional vector space and let $(\varepsilon^1, \ldots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \tag{3.3.1.4}$$

where $I = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is obtained by concatenating I and J.

Proof. By multilinearity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \ldots, E_{p_{k+l}}) = \varepsilon^{IJ}(E_{p_1}, \ldots, E_{p_{k+l}}) \tag{3.3.1.5}$$

for any sequence $(E_{p_1}, \ldots, E_{p_{k+l}})$ of basis vectors. We consider several cases.

- Case 1: $P = (p_1, \ldots, p_{k+l})$ has a repeated index. In this case, both sides of (3.3.1.5) are zero by Lemma 3.3.1.1(c).
- Case 2: P contains an index that does not appear in either I or J. In this case, the right-hand side is zero by Lemma 3.3.1.5(c). Similarly, each term in the expansion of the left-hand side involves either ε^I or ε^J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.
- Case 3: $P = IJ$ and P has no repeated indices. In this case, the right-hand side of (3.3.1.5) is equal to 1 by Lemma 3.3.1.5(c), so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned}\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \ldots, E_{p_{k+l}}) &= \frac{(k+l)!}{k!l!}\mathrm{Alt}(\varepsilon^I \otimes \varepsilon^J)(E_{p_1}, \ldots, E_{p_{k+l}}) \\ &= \frac{1}{k!l!}\sum_{\sigma \in \mathfrak{S}_{k+l}} (\mathrm{sgn}\sigma)\varepsilon^I(E_{p_{\sigma(1)}}, \ldots, E_{p_{\sigma(k)}})\varepsilon^J(E_{p_{\sigma(k+1)}}, \ldots, E_{p_{\sigma(k+l)}})\end{aligned}$$

  By Lemma 3.3.1.5 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must split into $\sigma = \tau\eta$, where $\tau \in \mathfrak{S}_k$ acts by permuting $\{1, \ldots, k\}$ and $\eta \in \mathfrak{S}_l$ acts by permuting $\{k+1, \ldots, k+l\}$. Since $\mathrm{sgn}(\tau\eta) = (\mathrm{sgn}\tau)(\mathrm{sgn}\eta)$, we have

$$\begin{aligned}&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \ldots, E_{p_{k+l}}) \\ &= \frac{1}{k!l!}\sum_{\tau \in \mathfrak{S}_k, \eta \in \mathfrak{S}_l} (\mathrm{sgn}\tau)(\mathrm{sgn}\eta)\varepsilon^I(E_{p_{\tau(1)}}, \ldots, E_{p_{\tau(k)}})\varepsilon^J(E_{p_{\eta(k+1)}}, \ldots, E_{p_{\eta(k+l)}}) \\ &= \Big(\frac{1}{k!}\sum_{\tau \in \mathfrak{S}_k}(\mathrm{sgn}\tau)\varepsilon^I(E_{p_{\tau(1)}}, \ldots, E_{p_{\tau(k)}})\Big)\Big(\frac{1}{l!}\sum_{\eta \in \mathfrak{S}_l}(\mathrm{sgn}\eta)\varepsilon^J(E_{p_{\eta(k+1)}}, \ldots, E_{p_{\eta(k+l)}})\Big) \\ &= (\mathrm{Alt}\varepsilon^I)(E_{p_1}, \ldots, E_{p_k})(\mathrm{Alt}\varepsilon^J)(E_{p_{k+1}}, \ldots, E_{p_{k+l}}) \\ &= \varepsilon^I(E_{p_1}, \ldots, E_{p_k})\varepsilon^J(E_{p_{k+1}}, \ldots, E_{p_{k+l}}) = 1.\end{aligned}$$

- Case 4: P is a permutation of IJ and has no repeated indices. In this case, applying a permutation to P brings us back to Case 3. Since the effect of the permutation is to multiply both sides of (3.3.1.5) by the same sign, the result holds in this case as well.

□

Proposition 3.3.1.9 (Properties of the Wedge Product). *Suppose $\omega, \omega', \eta, \eta'$ and ξ are multicovectors on a finite-dimensional vector space V.*

(a) Bilinearity: For $a, a' \in \mathbb{R}$,

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

(b) Associativity:

$$(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi).$$

(c) Anticommutativity: For $\omega \in \bigwedge^k(V^*)$ and $\eta \in \bigwedge^l(V^*)$,

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega. \tag{3.3.1.6}$$

(d) If (ε^i) is any basis for V^* and $I = (i_1, \ldots, i_k)$ is any multi-index, then

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I. \tag{3.3.1.7}$$

(e) For any covectors $i_1, \ldots, i_k$ and vectors $v_1, \ldots, v_k$

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) = \det\left(\omega^j(v_i)\right). \tag{3.3.1.8}$$

Proof. Bilinearity follows immediately from the definition, because the tensor product is bilinear and Alt is linear. To prove associativity, note that Lemma 3.3.1.8 gives

$$(\varepsilon^I \wedge \varepsilon^J)\varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge (\varepsilon^I \wedge \varepsilon^J).$$

The general case follows from bilinearity. Similarly, using Lemma 3.3.1.8 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\operatorname{sgn}\tau)\varepsilon^{JI} = (\operatorname{sgn}\tau)\varepsilon^J \wedge \varepsilon^I,$$

where τ is the permutation that sends IJ to JI. It is easy to check that $\operatorname{sgn}\tau = (-1)^{kl}$, because τ can be decomposed as a composition of kl transpositions. Anticommutativity then follows from bilinearity.

Part (d) is an immediate consequence of Lemma 3.3.1.8 and induction. To prove (e), we note that the special case in which each ω^j is one of the basis covectors ε^{i_j} reduces to (3.3.1.7). Since both sides of (3.3.1.8) are multilinear in $(\omega_1, \ldots, \omega_k)$, this suffices. □

Because of part (d) of this lemma, henceforth we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$ interchangeably.

A k-covector ω is said to be decomposable if it can be expressed in the form $\omega^1 \wedge \cdots \wedge \omega^k$, where $\omega^1, \ldots, \omega^k$ are covectors. It is important to be aware that not every k-covector is decomposable when $k > 1$; however, it follows from Propositions 3.3.1.6 and 3.3.1.9(d) that every k-covector can be written as a linear combination of decomposable ones.

The definition and computational properties of the wedge product can seem daunting at first sight. However, the only properties that you need to remember for most practical purposes are the ones expressed in the preceding proposition. In fact, these properties are more than enough to determine the wedge product uniquely, as the following proposition shows.

Proposition 3.3.1.10. *The wedge product is the unique associative, bilinear, and anticommutative map $\bigwedge^k(V^*) \times \bigwedge^k(V^*) \to \bigwedge^{k+l}(V^*)$ satisfying (3.3.1.7).*

For any n-dimensional vector space V, define a vector space $\bigwedge(V^*)$ by

$$\bigwedge(V^*) = \bigoplus_{k=0}^{n} \bigwedge^k(V^*).$$

It follows from Proposition 3.3.1.6 that $\dim \bigwedge(V^*) = 2^n$. Proposition 3.3.1.9 shows that the wedge product turns $\bigwedge(V^*)$ into an associative algebra, called the **exterior algebra** (or **Grassmann algebra**) of V. This algebra is not commutative, but it has a closely related property. An algebra A is said to be graded if it has a direct sum decomposition $A = \bigoplus_k A^k$ such that the product satisfies $A^k A^l \subseteq A^{k+l}$ for each k and l. A graded algebra is anticommutative if the product satisfies $ab = (-1)^{kl} ba$ for $a \in A^k$, $b \in A^l$. Proposition 3.3.1.9(c) shows that $\bigwedge(V^*)$ is an anticommutative graded algebra.

3.3.1.3 Interior multiplication

There is an important operation that relates vectors with alternating tensors. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v : \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$, called **interior multiplication by v**, as follows:

$$i_v\omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$

In other words, $i_v\omega$ is obtained from ω by inserting v into the first slot. By convention, we interpret $i_v\omega$ to be zero when ω is a 0-covector (i.e., a number). Another common notation is $v \lrcorner \omega = i_v\omega$.

Lemma 3.3.1.11. *Let V be a finite-dimensional vector space and $v \in V$*

(a) $i_v \circ i_v = 0$.

(b) If $\omega \in \bigwedge^k(V^*)$ and $\eta \in \bigwedge^l(V^*)$,

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta). \tag{3.3.1.9}$$

Proof. On k-covectors for $k \geq 2$, part (a) is immediate from the definition, because any alternating tensor gives zero when two of its arguments are identical. On 1-covectors and 0-covectors, it follows from the fact that $i_v \equiv 0$ on 0-covectors.

To prove (b), it suffices to consider the case in which both ω and η are decomposable, since every alternating tensor of positive rank can be written as a linear combination of decomposable ones. It is straightforward to verify that (b) follows in this special case from the following general formula for covectors $\omega^1, \dots, \omega^k$:

$$v \lrcorner (\omega^1 \wedge \cdots \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k \tag{3.3.1.10}$$

To prove this, let us write $v_1 = v$ and apply both sides to an arbitrary $(k-1)$-tuple of vectors $(v_2, \dots, v_k)$ then what we have to prove is

$$v \lrcorner (\omega^1 \wedge \cdots \omega^k)(v_1, \dots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1)(\omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k)(v_2, \dots, v_k). \tag{3.3.1.11}$$

The left-hand side of (3.3.1.11) is the determinant of the matrix V whose (i,j)-entry is $\omega^i(v_j)$. To simplify the right-hand side, let V^i_j denote the $(k-1) \times (k-1)$ submatrix of V obtained by deleting the i th row and j th column. Then the right-hand side of (3.3.1.11) is

$$\sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1) V^i_1.$$

This is just the expansion of $\det V$ by minors along the first column, and therefore is equal to $\det V$. $\square$

3.3.2 Differential forms on manifolds

Now we turn our attention to an n-dimensional smooth manifold M (with or without boundary). Recall that T^kT^*M is the bundle of covariant k-tensors on M. The subset of T^kT^*M consisting of alternating tensors is denoted by

$$\bigwedge^k T^*M = \prod_{p\in M} \bigwedge^k (T_p^*M).$$

A section of $\bigwedge^k T^*M$ is called a **differential k-form**, or just a **k-form**; this is a (continuous) tensor field whose value at each point is an alternating tensor. The integer k is called the degree of the form. We denote the vector space of smooth k-forms by

$$\Omega^k(M) := \Gamma(\bigwedge^k T^*M).$$

The wedge product of two differential forms is defined pointwise: $(\omega\wedge\eta)_p = \omega_p\wedge\eta_p$. Thus, the wedge product of a k-form with an l-form is a $(k+l)$-form. If f is a 0-form and ω is a k-form, we interpret the wedge product $f\wedge\omega$ to mean the ordinary product $f\omega$. If we define

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M),$$

then the wedge product turns $\Omega^*(M)$ into an associative, anticommutative graded algebra.

In any smooth chart, a k-form ω can be written locally as

$$\omega = \sum_I{}' \omega_I\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \sum_I{}' \omega_I\, dx^I.$$

Proposition 3.1.3.2 shows that ω is smooth on U if and only if the component functions ω^I are smooth. In terms of differential forms, the result of Lemma 3.3.1.5(c) translates to

$$dx^{i_1}\wedge\cdots\wedge dx^{i_k}\Big(\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_k}}\Big) = \delta^I_J.$$

Thus the component functions ω_I of ω are determined by

$$\omega_I = \omega\Big(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_k}}\Big).$$

Example 3.3.2.1. A 0-form is just a continuous real-valued function, and a 1-form is a covector field. On $\mathbb{R}^3$, some examples of smooth 2-forms are given by

$$\omega = (\sin xy)\, dy\wedge dz, \quad \eta = dx\wedge dy + dy\wedge dz + dz\wedge dx.$$

Every 3-form on $\mathbb{R}^3$ is a continuous real-valued function times $dx\wedge dy\wedge dz$.

If $F: M\to N$ is a smooth map and ω is a differential form on N, the pullback $F^*\omega$ is a differential form on M, defined as for any covariant tensor field:

$$(F^*\omega)_p(v_1,\dots,v_k) = \omega_p\big(dF_p(v_1),\dots,dF_p(v_k)\big).$$

Lemma 3.3.2.2. *Suppose $F: M\to N$ is smooth.*

(a) $F^*: \Omega^k(N)\to\Omega^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega\wedge\eta) = (F^*\omega)\wedge(F^*\eta)$.

(c) In any smooth chart,

$$F^*\Big(\sum_I{}' \omega_I\, dy^{i_1}\wedge\cdots\wedge dy^{i_k}\Big) = \sum_I{}' (\omega_I\circ F)\, d(y^{i_1}\circ F)\wedge\cdots\wedge d(y^{i_k}\circ F).$$

This lemma gives a computational rule for pullbacks of differential forms similar to the ones we developed for covector fields and arbitrary tensor fields earlier.

Example 3.3.2.3. Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$, and let ω be the 2-form $y\,dx \wedge dz + x\,dy^d z$ on $\mathbb{R}^3$. The pullback $F^*\omega$ is computed as follows:

$$\begin{aligned} F^*(y\,dx \wedge dz + x\,dy^d z) &= v\,du \wedge d(u^2 - v^2) + u\,dv \wedge d(u^2 - v^2) \\ &= v\,du \wedge (2u\,du - 2v\,dv) + u\,dv \wedge (2u\,du - 2v\,dv) \\ &= -2v^2\,du \wedge dv + 2u^2\,dv \wedge du \\ &= -2(u^2 + v^2)\,du \wedge dv. \end{aligned}$$

The same technique can also be used to compute the expression for a differential form in another smooth chart.

Example 3.3.2.4. Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Thinking of the transformation to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$dx \wedge dy = d(r\cos\theta) \wedge d(r\sin\theta) = (\cos\theta\,dr - r\sin\theta d\theta) \wedge (\sin\theta\,dr + r\cos\theta\,d\theta) = r\,dr \wedge d\theta.$$

The similarity between this formula and the formula for changing a double integral from Cartesian to polar coordinates is striking. The following proposition generalizes this.

Proposition 3.3.2.5 (Pullback Formula for Top-Degree Forms). *Let $F : M \to N$ be a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*(u\,dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det \partial F)\,dx^1 \wedge \cdots \wedge dx^n. \tag{3.3.2.1}$$

where ∂F represents the Jacobian matrix of F in these coordinates.

Proof. Because the fiber of $\bigwedge^n T^*M$ is spanned by $dx^1 \wedge \cdots \wedge dx^n$ at each point, it suffices to show that both sides of (3.3.2.1) give the same result when evaluated on $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. From Lemma 3.3.2.2 and Proposition 3.3.1.9(d), we see that

$$\begin{aligned} F^*(u\,dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F)\,dF^1 \wedge \cdots \wedge dF^n, \\ dF^1 \wedge \cdots \wedge dF^n\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) &= \det\left(\frac{\partial F^j}{\partial x^i}\right). \end{aligned}$$

Therefore, the left-hand side of (3.3.2.1) gives $(u \circ F)(\det \partial F)$ when applied to the element $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. On the other hand, the right-hand side gives the same thing, because we have

$$dx^1 \wedge \cdots \wedge dx^n(\partial/\partial x^1, \ldots, \partial/\partial x^n) = 1. \qquad \square$$

Corollary 3.3.2.6. *If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \widetilde{U}$:*

$$d\widetilde{x}^1 \wedge \cdots \wedge d\widetilde{x}^i = \det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n. \tag{3.3.2.2}$$

Interior multiplication also extends naturally to vector fields and differential forms, simply by letting it act pointwise: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$-form $X \lrcorner\, \omega = i_X\omega$ by

$$(X \lrcorner\, \omega)_X = X_p \lrcorner\, \omega_p.$$

The following results are clear.

Proposition 3.3.2.7. *Let X be a smooth vector field on M.*

(a) If ω is a smooth differential form, then $i_X\omega$ is smooth.

(b) The map $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is linear over $C^\infty(M)$, therefore corresponds to a smooth bundle homomorphism $i_X : \bigwedge^k T^*M \to \bigwedge^{k-1} T^*M$.

3.3.3 Exterior derivatives

For each smooth manifold M with or without boundary, we will show that there is a differential operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω. Thus, it will follow that a necessary condition for a smooth k-form ω to be equal to $d\omega$ for some $(k-1)$-form ω is that $d\omega = 0$.

The definition of d on Euclidean space is straightforward: if $\omega = \sum'_J \omega_J \, dx^J$ is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, we define its exterior derivative $d\omega$ to be the following $(k+1)$-form:

$$d\omega = \sum_J{}' d\omega_J \wedge dx^J, \tag{3.3.3.1}$$

where $d\omega_J$ is the differential of the function ω_J. In somewhat more detail, this is

$$d\Big(\sum_J{}' \omega_J \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\Big) = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

In order to transfer this definition to manifolds, we need to check that it satisfies the following properties.

Proposition 3.3.3.1 (Properties of the Exterior Derivative on $\mathbb{R}^n$).

(a) *d is linear over $\mathbb{R}$.*

(b) *If ω is a smooth k-form and η is a smooth l-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) *$d \circ d \equiv 0$.*

(d) *d commutes with pullbacks: if U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, V is an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$, $F : U \to V$ is a smooth map, and $\omega \in \Omega^k(M)$, then*

$$d(F^*\omega) = F^*(d\omega). \tag{3.3.3.2}$$

Proof. Linearity of d is an immediate consequence of the definition. To prove (b), by linearity it suffices to consider terms of the form $\omega = u \, dx^I \in \Omega^k(M)$ and $\eta = v \, dx^J \in \Omega^l(M)$ for smooth real-valued functions u and v. First, though, we need to know that d satisfies $d(u \, dx^I) = du \wedge dx^I$ for any multi-index I, not just increasing ones. If I has repeated indices, then clearly $d(u \, dx^I) = 0 = du \wedge dx^I$. If not, let σ be the permutation sending I to an increasing multi-index J. Then

$$d(u \, dx^I) = (\operatorname{sgn}\sigma) d(u \, dx^J) = (\operatorname{sgn}\sigma) du \wedge dx^J = du \wedge dx^I.$$

Using this, we compute

$$\begin{aligned} d(\omega \wedge \eta) &= d\big((u \, dx^I) \wedge (v \, dx^J)\big) = d(uv \, dx^I \wedge dx^J) = (v \, du + u \, dv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (v \, dx^J) + (-1)^k (u \, dx^I) \wedge (dv \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

We prove (c) first for the special case of a 0-form, which is just a real-valued function. In this case,

$$\begin{aligned} d(du) &= d\Big(\frac{\partial u}{\partial x^j} dx^j\Big) = \frac{\partial u^2}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ &= \sum_{i<j} \Big(\frac{\partial u^2}{\partial x^i \partial x^j} - \frac{\partial u^2}{\partial x^j \partial x^i}\Big) dx^i \wedge dx^j = 0. \end{aligned}$$

For the general case, we use the $k = 0$ case together with (b) to compute

$$d(d\omega) = d\Big(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\Big)$$

$$= \sum_J{}' d(d\omega_J) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J{}' \sum_{i=1}^k d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge d(dx^{j_i}) \wedge \cdots \wedge dx^{j_k} = 0.$$

Finally, to prove (d), again it suffices to consider $\omega = u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. For such a form, the left-hand side of (3.3.3.2) is

$$F^*\big(d(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k})\big) = F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d(u \circ F) \wedge d(x^{i_1} \circ F) \cdots \wedge d(x^{i_k} \circ F),$$

and the right-hand side is

$$d\big(F^*(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k})\big) = d\big((u \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)\big) = d(u \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

so they are equal. □

These results allow us to transplant the definition of the exterior derivative to manifolds.

Theorem 3.3.3.2 (Existence and Uniqueness of Exterior Differentiation). *Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called exterior differentiation, satisfying the following four properties:*

(i) *d is linear over $\mathbb{R}$.*

(ii) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) *$d \circ d \equiv 0$.*

(iv) *For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

In any smooth coordinate chart, d is given by (3.3.3.1).

Proof. First, we prove existence. Suppose $\omega \in \Omega^k(M)$. We wish to define $d\omega$ by means of the coordinate formula (3.3.3.1) in each chart; more precisely, this means that for each smooth chart (U, φ) for M, we wish to set

$$d\omega = \varphi^* d(\varphi^{-1*}\omega) \tag{3.3.3.3}$$

To see that this is well defined, we just note that for any other smooth chart (V, ψ), the map $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, so Proposition 3.3.3.1(d) implies

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) = d\big((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega\big).$$

Together with the fact that $(\varphi \circ \psi^{-1})^* = \psi^{-1*} \circ \varphi^*$, this implies $\varphi^* d(\varphi^{-1*}\omega) = \psi^* d(\psi^{-1*}\omega)$, so $d\omega$ is well defined. It satisfies (i)–(iv) by virtue of Proposition 3.3.3.1.

To prove uniqueness, suppose that d is any operator satisfying (i)–(iv). First we need to show that $d\omega$ is determined locally: if ω_1 and ω_2 are k-forms that agree on an open subset $U \subseteq M$, then $d\omega_1 = d\omega_2$ on U. To see this, let $p \in U$ be arbitrary, let $\eta = \omega_1 - \omega_2$, and let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero, so (i)–(iv) imply

$$0 = d(\psi\eta) = d\psi \wedge \eta + \psi \wedge d\eta.$$

Evaluating this at p and using the facts that $\psi(p) = 1$ and $d\psi_p = 0$, we conclude that $d\omega_1|p - d\omega_2|_p = d\eta|_p = 0$.

Now let $\omega \in \Omega^k(M)$ be arbitrary, and let (U, φ) be any smooth coordinate chart on M. We can write ω in coordinates as $\sum_I' \omega_I\, dx^I$ on U. For any $p \in U$, by means of a bump function we can construct global smooth functions $\widetilde{\omega}_I$ and $\widetilde{x}^i$ on M that agree with ω_I and x^i in a neighborhood of p. By virtue of (i)–(iv) together with the observation in the preceding paragraph, it follows that (3.3.3.1) holds at p. Since p was arbitrary, this d must be equal to the one we defined above. □

If $A = \bigoplus_k A^k$ is a graded algebra, a linear map $T : A \to A$ is said to be a map of degree m if $TA^k \subseteq A^{k+m}$ for each k. It is said to be an **antiderivation** if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x \in A^k$ and $y \in A^l$. The preceding theorem can be summarized by saying that the differential on functions extends uniquely to an antiderivation of $\Omega^*(M)$ of degree $+1$ whose square is zero.

Proposition 3.3.3.3. *Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. Then the interior multiplication $i_X : \Omega^*(M) \to \Omega^*(M)$ is an antiderivation of degree -1 whose square is zero.*

Proof. This follows from Lemma 3.3.1.11. □

Proposition 3.3.3.4 (Naturality of the Exterior Derivative). *If $F : M \to N$ is a smooth map, then for each k the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d: for all $\omega \in \Omega^k(N)$,*

$$F^*(d\omega) = d(F^*\omega) \tag{3.3.3.4}$$

Proof. If (U, φ) and (V, ψ) are smooth charts for M and N, respectively, we can apply Proposition 3.3.3.1(d) to the coordinate representation $\psi \circ F \circ \varphi^{-1}$. Using (3.3.3.3) twice, we compute as follows on

$$\begin{aligned} F^*(d\omega) &= F^*\psi^* d(\psi^{-1*}\omega) \\ &= \varphi^* \circ (\psi \circ F \circ \varphi^{-1}) d(\psi^{-1*}\omega) \\ &= \varphi^* d\big((\psi \circ F \circ \varphi^{-1})^* (\psi^{-1*}\omega)\big) \\ &= \varphi^* d(\varphi^{-1*} \circ F^*\omega) \\ &= d(F^*\omega). \end{aligned}$$

Therefore the claim follows. □

Extending the terminology we introduced for covector fields, we say that a smooth differential form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and exact if there exists a smooth $(k-1)$-form η on M such that $\omega = d\eta$. The fact that $d \circ d \equiv 0$ implies that every exact form is closed.

3.3.3.1 Exterior derivatives and vector calculus in $\mathbb{R}^3$

Example 3.3.3.5. Let us work out the exterior derivatives of arbitrary 1-forms and 2-forms on $\mathbb{R}^3$. Any smooth 1-form can be written

$$\omega = P\,dx + Q\,dy + R\,dz$$

for some smooth functions P, Q, R. Using (3.3.3.1) and the fact that the wedge product of any 1-form with itself is zero, we compute

$$\begin{aligned} d\omega =& dP \wedge dx + dQ \wedge dy + dR \wedge dz \\ =& \Big(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz\Big) \wedge dx + \Big(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz\Big) \wedge dy \\ &+ \Big(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz\Big) \wedge dz \\ =& \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx \wedge dy + \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big) dy \wedge dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big) dz \wedge dx. \end{aligned}$$

An arbitrary 2-form on $\mathbb{R}^3$ can be written

$$\eta = u\,dx \wedge dy + v\,dy \wedge dz + w\,dz \wedge dx.$$

A similar computation shows that

$$d\eta = \Big(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\Big) dx \wedge dy \wedge dz.$$

Recall the classical vector calculus operators on $\mathbb{R}^n$: the Euclidean **gradient** of a function $f \in C^\infty(\mathbb{R}^n)$ and the **divergence** of a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ are defined by

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}, \quad \operatorname{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}.$$

In addition, in the case $n = 3$, the curl of a vector field $X \in \mathbb{R}^3$ is defined by

$$\operatorname{curl} X = \Big(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\Big)\frac{\partial}{\partial x^1} + \Big(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\Big)\frac{\partial}{\partial x^2} + \Big(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\Big)\frac{\partial}{\partial x^3}.$$

It is interesting to note that the components of the 2-form $d\omega$ in the preceding example are exactly the components of the curl of the vector field with components (P, Q, R). Similarly, there is a strong analogy between the formula for $d\eta$ and the divergence of a vector field. These analogies can be made precise in the following way.

The Euclidean metric on $\mathbb{R}^3$ yields an index-lowering isomorphism $\flat : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$. Interior multiplication yields another map $\beta : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ as follows:

$$\beta(X) := X \lrcorner (dx \wedge dy \wedge dz) = X^1\, dy \wedge dz + X^2\, dz \wedge dx + X^3\, dx \wedge dy.$$

It is easy to check that β is linear over $C^\infty(\mathbb{R}^3)$, so it corresponds to a smooth bundle homomorphism from $T\mathbb{R}^3$ to $\bigwedge^2 T^*\mathbb{R}^3$. It is a bundle isomorphism because it is injective and both $T\mathbb{R}^3$ and $\bigwedge^2 T^*\mathbb{R}^3$ are bundles of rank 3. Similarly, we define a smooth bundle isomorphism

$$*(f) = f\, dx \wedge dy \wedge dz.$$

The relationships among all of these operators are summarized in the following diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

The desire to generalize these vector calculus operators from $\mathbb{R}^3$ to higher dimensions was one of the main motivations for developing the theory of differential forms. The curl, in particular, makes sense as an operator on vector fields only in dimension 3, whereas the exterior derivative expresses the same information but makes sense in all dimensions.

3.3.3.2 An invariant formula for the exterior derivative

In addition to the coordinate formula (3.3.3.1) that we used in the definition of d, there is another formula for d that is often useful, not least because it is manifestly coordinate-independent. The formula for 1-forms is by far the most important, and is the easiest to state and prove, so we begin with that. Note the similarity between this and the formula of Proposition **??**.

Proposition 3.3.3.6 (Exterior Derivative of a 1-Form). *For any smooth 1-form ω and smooth vector fields X and Y,*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{3.3.3.5}$$

Proof. Since any smooth 1-form can be expressed locally as a sum of terms of the form $u\, dv$ for smooth functions u and v, it suffices to consider that case. Suppose $\omega = u\, dv$, and X, Y are smooth vector fields. Then the left-hand side of (3.3.3.5) is

$$d(u\, dv)(X, Y) = du \wedge dv(X, Y) = du(X)dv(Y) - dv(X)du(Y) = XuYv - XvYu.$$

The right-hand side is

$$\begin{aligned}
&X(u\, dv(Y)) - Y(u\, dv(X)) - u\, dv([X, Y]) \\
&= X(uYv) - Y(uXv) - u(XY - YX)v \\
&= (XuYv + uXYv) - (YuXv + uYXv) - u(XY - YX)v \\
&= XuYv - YuXv.
\end{aligned}$$

This proves the claim. □

We will see some applications of (3.3.3.5) in later chapters. Here is our first one. It shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

Proposition 3.3.3.7. *Let M be a smooth n-manifold with or without boundary, let (E_i) be a smooth local frame for M, and let (ε^i) be the dual coframe. For each i, let b^i_{jk} denote the component functions of the exterior derivative of ε^i in this frame, and for each j, k, let c^i_{jk} be the component functions of the Lie bracket $[E_j, E_k]$:*

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\varepsilon^j \wedge \varepsilon^k, \quad [E_j, E_k] = c^i_{jk}E_i.$$

Then $b^i_{jk} = -c^i_{jk}$.

Proof. By (3.3.3.5), we have

$$b^i_{jk} = d\varepsilon^i(E_j, E_k) = E_j(\varepsilon^i(E_k)) - E_k(\varepsilon^i(E_j)) - \varepsilon^i([E_j, E_k]) = E_j(\delta^i_k) - E_k(\delta^i_j) - c^i_{jk} = -c^i_{jk}.$$

Since δ^i_k and δ^i_j are both constant. □

The generalization of (3.3.3.5) to higher-degree forms is more complicated.

Proposition 3.3.3.8 (Invariant Formula for the Exterior Derivative). *Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields $X_1, \dots, X_{k+1}$ on M,*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i-1}X_i\big(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}). \end{aligned}$$

where the hats indicate omitted arguments.

Proof. For this proof, let us denote the expression on the right by $D\omega(X_1, \dots, X_{k+1})$, and the two sums on the right-hand side by $I_1(X_1, \dots, X_{k+1})$ and $I_2(X_1, \dots, X_{k+1})$, respectively. Note that $D\omega$ is obviously multilinear over $\mathbb{R}$. We begin by showing that, like $d\omega$, it is actually multilinear over $C^\infty(M)$, which is to say that for $1 \le p \le k+1$ and $f \in C^\infty(M)$,

$$D\omega(X_1, \dots, fX_p, \dots, X_{k+1}) = fD\omega(X_1, \dots, X_p, \dots, X_{k+1}).$$

In the expansion of $I_1(X_1, \dots, fX_p, \dots, X_{k+1})$, f obviously factors out of the $i = p$ term. The other terms expand as follows:

$$\begin{aligned} \sum_{i\neq p}(-1)^{i-1}X_i\big(f\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) = &\sum_{i\neq p}(-1)^{i-1}fX_i\big(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) \\ &+ \sum_{i\neq p}(-1)^{i-1}(X_if)\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} I_1(X_1, \dots, fX_p, \dots, X_{k+1}) = &fI_1(X_1, \dots, fX_p, \dots, X_{k+1}) \\ &+ \sum_{i\neq p}(-1)^{i-1}(X_if)\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}). \end{aligned}$$

Consider next the expansion of I_2. Again, f factors out of all the terms in which $i \neq p$ and $j \neq p$. To expand the other terms, we use (1.1.3.3), which implies

$$\begin{aligned} [fX_p, X_j] &= f[X_p, X_j] - (X_jf)X_p, \\ [X_i, fX_p] &= f[X_i, X_p] + (X_if)X_p, \end{aligned}$$

Inserting these formulas into the $i = p$ and $j = p$ terms, we obtain

$$\begin{aligned} I_2(X_1, \dots, fX_p, \dots, X_{k+1}) = &fI_2(X_1, \dots, X_p, \dots, X_{k+1}) \\ &+ \sum_{p<j}(-1)^{p+j}\omega([fX_p, X_j], X_1, \dots, \widehat{X_p}, \dots, \widehat{X_j}, \dots, X_{k+1}) \end{aligned}$$

$$+ \sum_{i<p} (-1)^{i+p} \omega([X_i, fX_p], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_p}, \ldots, X_{k+1}).$$

Rearranging the arguments in these two sums so as to put X_p into its original position, we see that they exactly cancel the sum in I_1. This completes the proof that $D\omega$ is multilinear over $C^\infty(M)$, so it defines a smooth $(k+1)$-tensor field.

Since both $D\omega$ and $d\omega$ are smooth tensor fields, we can verify the equation $D\omega = d\omega$ in any frame. By multilinearity, it suffices to show that both sides give the same result on an arbitrary sequence of basis vectors in some chosen local frame. Let $(U, (x^i))$ be an arbitrary smooth chart on M. Because both $d\omega$ and $D\omega$ depend linearly on ω, we may assume that $\omega = u\, dx^I$ for some smooth function u and some increasing multi-index $I = (i_1, \ldots, i_k)$, so

$$d\omega = du \wedge dx^I = \sum_m \frac{\partial u}{\partial x^m} dx^m \wedge dx^I.$$

If $J = (j_1, \ldots, j_{k+1})$ is any multi-index of length $k+1$, it follows that

$$d\omega\Big(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_{k+1}}}\Big) = \sum_m \frac{\partial u}{\partial x^m} \delta^{mI}_J.$$

The only terms in this sum that can possibly be nonzero are those for which J has no repeated indices and m is equal to one of the indices in J, say $m = j_p$. In this case, it is easy to check that $\delta^{mI}_J = (-1)^{p-1}\delta^I_{\widehat{J}_p}$, where $\widehat{J}_p = (j_1, \ldots, \widehat{j_p}, \ldots, j_{k+1})$, so

$$d\omega\Big(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_{k+1}}}\Big) = \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J}_p}.$$

On the other hand, because all the Lie brackets are zero, we have

$$\begin{aligned} D\omega\Big(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_{k+1}}}\Big) &= \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial}{\partial x^{j_p}} \Big(u\, dx^I\Big(\frac{\partial}{\partial x^{j_1}}, \ldots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \ldots, \frac{\partial}{\partial x^{j_{k+1}}}\Big)\Big) \\ &= \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J}_p}. \end{aligned}$$

from whcih the claim follows. □

3.3.3.3 Lie derivatives of differential forms

Proposition 3.3.3.9. *Suppose M is a smooth manifold, $V \in \mathfrak{X}(M)$, and $\omega, \eta \in \Omega^*(M)$. Then*

$$\mathfrak{L}_V(\omega \wedge \eta) = (\mathfrak{L}_V\omega) \wedge \eta + \omega \wedge (\mathfrak{L}_V\eta).$$

Proof. Let θ be the flow of V, then by definition (3.1.3.2),

$$\begin{aligned} \mathfrak{L}_V(\omega \wedge \eta) &= \frac{d}{dt}\Big|_{t=0} \theta_t^*(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} \theta_t^*\omega \wedge (\theta_t^*\eta)|_{t=0} + (\theta_t^*\omega)|_{t=0} \wedge \frac{d}{dt}\Big|_{t=0} \theta_t^*\eta \\ &= \frac{d}{dt}\Big|_{t=0} \theta_t^*\omega \wedge \eta_p + \omega \wedge \frac{d}{dt}\Big|_{t=0} \theta_t^*\eta = (\mathfrak{L}_V\omega) \wedge \eta + \omega \wedge (\mathfrak{L}_V\eta), \end{aligned}$$

which is what we want. □

Proposition 3.3.3.10 (Cartan's Magic Formula). *On a smooth manifold M, for any smooth vector field V and any smooth differential form ω,*

$$\mathfrak{L}_V\omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega). \tag{3.3.3.6}$$

Proof. We prove that (3.3.3.6) holds for smooth k-forms by induction on k. We begin with a smooth 0-form f, in which case

$$V \lrcorner\, df + d(V \lrcorner\, f) = V \lrcorner\, df = df(V) = Vf = \mathfrak{L}_V f.$$

Now let $k \geq 1$, and suppose (3.3.3.6) has been proved for forms of degree less than k. Let ω be an arbitrary smooth k-form, written in smooth local coordinates as

$$\omega = \sum_I{}' \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Writing $u = x^{i_1}$ and $\beta = \omega_I \, dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, we see that each term in this sum can be written in the form $du \wedge \beta$, where u is a smooth function and β is a smooth $(k+1)$-form. Corollary 3.1.3.11 showed that $\mathfrak{L}_V(du) = d(\mathfrak{L}_V u) = d(Vu)$. Thus Proposition 3.3.3.9 and the induction hypothesis imply

$$\begin{aligned}
\mathfrak{L}_V(du \wedge \beta) &= (\mathfrak{L}_V du \wedge \beta) + du \wedge (\mathfrak{L}_V \beta) \\
&= d(Vu) \wedge \beta + du \wedge \big(V \lrcorner (d\beta) + d(V \lrcorner \beta)\big) \\
&= d(Vu) \wedge \beta + du \wedge V \lrcorner (d\beta) + du \wedge d(V \lrcorner \beta).
\end{aligned}$$

On the other hand, using the facts that both d and interior multiplication by V are antiderivations, and $V \lrcorner (du) = du(V) = Vu$, we compute

$$\begin{aligned}
&V \lrcorner d(du \wedge \beta) + d\big(V \lrcorner (du \wedge \beta)\big) \\
&\quad = V \lrcorner (-du \wedge d\beta) + d\big((Vu) \wedge \beta - du \wedge (V \lrcorner \beta)\big) \\
&\quad = V \lrcorner (-du \wedge d\beta) + d\big((Vu) \wedge \beta\big) - d\big(du \wedge (V \lrcorner \beta)\big) \\
&\quad = -(Vu)d\beta + du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + (Vu)d\beta + du \wedge d(V \lrcorner \beta) \\
&\quad = du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + du \wedge d(V \lrcorner \beta).
\end{aligned}$$

Thus the claim follows. □

Corollary 3.3.3.11 (The Lie Derivative Commutes with d). *If V is a smooth vector field and ω is a smooth differential form, then*

$$\mathfrak{L}_V(d\omega) = d(\mathfrak{L}_V \omega).$$

Proof. This follows from Cartan's formula and the fact that $d \circ d = 0$:

$$\begin{aligned}
\mathfrak{L}_V(d\omega) &= V \lrcorner d(d\omega) + d(V \lrcorner d\omega) = d(V \lrcorner d\omega), \\
d(\mathfrak{L}_V \omega) &= d\big(V \lrcorner d\omega + d(V \lrcorner \omega)\big) = d(V \lrcorner d\omega).
\end{aligned}$$

□

3.3.4 Exercise

Exercise 3.3.1. Show that covectors $\omega^1, \ldots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Proof. By Proposition 3.3.1.9, for 1-forms ω and η, we have $\omega \wedge \eta = -\eta \wedge \omega$, thus if $\omega^1, \ldots, \omega^k$ are linearly dependent, then $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Conversely, if $\omega^1, \ldots, \omega^k$ are linearly independent, let $E_1, \ldots, E_n$ be a basis of V. Then the matrix $(\omega^i(E_j))$ has full rank, thus

$$\omega^1 \wedge \cdots \wedge \omega^k(E_1, \ldots, E_k) = \det\big(\omega^i(E_j)\big) \neq 0,$$

which means $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$. □

Exercise 3.3.2 (Cartan's Lemma). Let M be a smooth n-manifold with or without boundary, and let $(\omega^1, \ldots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $(\omega^1|_p, \ldots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \ldots, \alpha^k$ on U such that

$$\sum_{i=1}^k \omega^i \wedge \alpha^i = 0.$$

show that each α^i can be written as a linear combination of $\omega^1, \ldots, \omega^k$ with smooth coefficients.

Proof. Since $(\omega^1, \dots, \omega^k)$ are linearly independent, the form

$$\omega := \omega^1 \wedge \cdots \wedge \omega^k \neq 0.$$

Wedge the given equation by $\omega_2 \wedge \cdots \wedge \omega^k$, we get

$$\omega \wedge \alpha^1 = 0.$$

Since $\omega \neq 0$, this implies α^1 is a linear combination of ω^i, and proceed similarly, we comclude that each α^i can be written as a linear combination of $\omega^1, \dots, \omega^k$. □

Exercise 3.3.3. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

(a) Compute ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota_{S^2}^* \omega$ to S^2, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota_{S^2}^* \omega$ is nowhere zero.

Proof. We first compute

$$\begin{aligned}
dx &= d(\rho \sin\varphi \cos\theta) = \sin\varphi \cos\theta\, d\rho + \rho \cos\varphi \cos\theta\, d\varphi - \rho \sin\varphi \sin\theta\, d\theta, \\
dy &= d(\rho \sin\varphi \sin\theta) = \sin\varphi \sin\theta\, d\rho + \rho \cos\varphi \sin\theta\, d\varphi + \rho \sin\varphi \cos\theta\, d\theta, \\
dz &= d(\rho \cos\varphi) = \cos\varphi\, d\rho - \rho \sin\varphi\, d\varphi.
\end{aligned}$$

Thus

$$\begin{aligned}
dy \wedge dz &= -\rho \sin\theta\, d\rho \wedge d\varphi + \rho^2 \sin^2\varphi \cos\theta\, d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \cos\theta\, d\theta \wedge d\rho \\
dz \wedge dx &= \rho \cos\theta\, d\rho \wedge d\varphi + \rho^2 \sin^2\varphi \sin\theta\, d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \sin\theta\, d\theta \wedge d\rho, \\
dx \wedge dy &= \rho^2 \sin\varphi \cos\varphi\, d\varphi \wedge d\theta - \rho \sin^2\varphi\, d\theta \wedge d\rho.
\end{aligned}$$

and then, plug this into ω we get

$$\omega = \rho^3 \sin\varphi\, d\varphi \wedge d\theta.$$

For (b), we have

$$d\omega = 3\, dx \wedge dy \wedge dz, \quad d\omega = (3\rho^2 \sin\varphi\, d\rho + \rho^3 \cos\varphi\, d\varphi) = 3\rho^2 \sin\varphi\, d\rho \wedge d\varphi \wedge d\theta.$$

and note that

$$dx \wedge dy \wedge dz = \rho^2 \sin\varphi\, d\rho \wedge d\varphi \wedge d\theta.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $F(\varphi, \theta) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$, then

$$\begin{aligned}
F^*(dy \wedge dz) &= \sin^2\varphi \cos\theta\, d\varphi \wedge d\theta, \\
F^*(dz \wedge dx) &= \sin^2\varphi \sin\theta\, d\varphi \wedge d\theta, \\
F^*(dx \wedge dy) &= \sin\varphi \cos\varphi\, d\varphi \wedge d\theta.
\end{aligned}$$

so that

$$F^*\omega = (\sin^3\varphi \cos^2\theta + \sin^3\varphi \sin^2\theta + \sin\varphi \cos^2\varphi)\, d\varphi \wedge d\theta = \sin\varphi\, d\varphi \wedge d\theta.$$

□

3.4 Orientations

3.4.1 Orientations of vector spaces

Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \ldots, E_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ for V are **consistently oriented** if the transition matrix (B_i^j), defined by

$$E_i = B_i^j \widetilde{E}_j.$$

has positive determinant.

If $\dim V = n$, we define an orientation for V as an equivalence class of ordered bases. If $(E_1, \ldots, E_n)$ is any ordered basis for V, we denote the orientation that it determines by $[E_1, \ldots, E_n]$, and the opposite orientation by $-[E_1, \ldots, E_n]$. A vector space together with a choice of orientation is called an **oriented vector space**. If V is oriented, then any ordered basis $(E_1, \ldots, E_n)$ that is in the given orientation is said to be oriented or **positively oriented**. Any basis that is not in the given orientation is said to be **negatively oriented**.

For the special case of a zero-dimensional vector space V, we define an orientation of V to be simply a choice of one of the numbers ± 1.

There is an important connection between orientations and alternating tensors, expressed in the following proposition.

Proposition 3.4.1.1. *Let V be a vector space of dimension n. Each nonzero element $\omega \in \bigwedge^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \ldots, E_n)$ such that $\omega(E_1, \ldots, E_n) > 0$; while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Two nonzero n-covectors determine the same orientation if and only if each is a positive multiple of the other.*

Proof. The 0-dimensional case is immediate, since a nonzero element of $\bigwedge^0(V^*)$ is just a nonzero real number. Thus we may assume $n \geq 1$. Let ω be a nonzero element of $\bigwedge^n(V^*)$, and let $\mathcal{O}_\omega$ denote the set of ordered bases on which ω gives positive values. We need to show that $\mathcal{O}_\omega$ is exactly one equivalence class.

Suppose (E_j) and $(\widetilde{E}_j)$ are any two ordered bases for V, and let $B : V \to V$ be the linear map sending E_j to $\widetilde{E}_j$. By Proposition 3.3.1.7,

$$\omega(\widetilde{E}_1, \ldots, \widetilde{E}_n) = (\det B)\omega(E_1, \ldots, E_n).$$

It follows that the basis $(\widetilde{E}_j)$ is consistently oriented with (E_j) if and only if $(E_1, \ldots, E_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily. □

If V is an oriented n-dimensional vector space and ω is an n-covector that determines the orientation of V as described in this proposition, we say that ω is a (positively) oriented n-covector. For example, the n-covector $e^1 \wedge \cdots \wedge e^n$ is positively oriented for the standard orientation on $\mathbb{R}^n$.

3.4.2 Orientations of manifolds

Let M be a smooth manifold with or without boundary. We define a **pointwise orientation** on M to be a choice of orientation of each tangent space. By itself, this is not a very useful concept, because the orientations of nearby points may have no relation to each other. For example, a pointwise orientation on $\mathbb{R}^n$ might switch randomly from point to point between the standard orientation and its opposite. In order for orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Let M be a smooth n-manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM defined on some open subset $U \subseteq M$, we say that (E_i) is **positively oriented** if $(E_1|_p, \ldots, E_n|_p)$ is a positively oriented basis for T_pM at each point $p \in U$. A **negatively oriented** frame is defined analogously.

A pointwise orientation is said to be **continuous** if every point of M is in the domain of an oriented local frame. (Recall that by definition the vector fields that make up a local frame are continuous.) An **orientation of M** is a continuous pointwise orientation. We say that M is **orientable** if there exists an orientation for it, and **nonorientable** if not. An **oriented manifold** is an ordered pair $(M, \mathcal{O})$, where

M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M, an **oriented manifold with boundary** is defined similarly. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

If M is zero-dimensional, this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

The next two propositions give ways of specifying orientations on manifolds that are more practical to use than the definition.

Proposition 3.4.2.1 (The Orientation Determined by an n-Form). *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Remark 3.4.2.2. Because of this proposition, if M is a smooth n-manifold with or without boundary, any nonvanishing n-form on M is called an **orientation form**. If M is oriented and ω is an orientation form determining the given orientation, we also say that ω is (**positively**) **oriented**. It is easy to check that if ω and $\widetilde{\omega}$ are two positively oriented smooth forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth realvalued function f. If M is a 0-manifold, a nonvanishing 0-form (i.e., real-valued function) assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Proof. Let ω be a nonvanishing n-form on M. Then ω defines a pointwise orientation by Proposition 3.4.1.1, so all we need to check is that it is continuous. This is trivially true when $n = 0$, so assume $n \geq 1$. Given $p \in M$, let (E_i) be any local frame on a connected neighborhood U of p, and let (ε^i) be the dual coframe. On U, the expression for ω in this frame is $\omega = f\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ for some continuous function f. The fact that ω is nonvanishing means that f is nonvanishing, and therefore

$$\omega(E_1, \ldots, E_n) = f \neq 0.$$

at all points of U. Since U is connected, it follows that this expression is either always positive or always negative on U, and therefore the given frame is either positively oriented or negatively oriented. If negatively, we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Thus, the pointwise orientation determined by ω is continuous.

Conversely, suppose M is oriented, and let $\bigwedge^n_+ T^*M \subseteq \bigwedge^n T^*M$ be the open subset consisting of positively oriented n-covectors at all points of M. At any point $p \in M$, the intersection of $\bigwedge^n_+ T^*M$ with the fiber $\bigwedge^n(T^*_pM)$ is an open half-line, and therefore convex. By the usual partition-of-unity argument (see Exercise 3.2.1), there exists a smooth global section of $\bigwedge^n_+ T^*M$ (i.e., a positively oriented smooth global n-form). □

Corollary 3.4.2.3. *Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, they are equal.*

A smooth coordinate chart on an oriented smooth manifold with or without boundary is said to be **(positively) oriented** if the coordinate frame $(\partial/\partial x^i)$ is positively oriented, and **negatively oriented** if the coordinate frame is negatively oriented. A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to be **consistently oriented** if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Proposition 3.4.2.4 (Orientation Determined by an Atlas). *Let M be a smooth positive-dimensional manifold with or without boundary. Given any consistently oriented smooth atlas for M, there is a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \varnothing$ or $\dim M > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.*

Proof. First, suppose M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map, which has positive determinant by hypothesis, so they determine the same pointwise orientation at each point. The orientation thus determined is continuous because each point is in the domain of an oriented coordinate frame.

Conversely, assume M is oriented and either $\partial M = \varnothing$ or $\dim M > 1$. Each point is in the domain of a smooth chart, and if the chart is negatively oriented, we can replace x_1 by $-x_1$ to obtain a new chart that is positively oriented. The fact that these charts all are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented atlas. (This does not work for boundary charts when $\dim M = 1$ because of our convention that the last coordinate is nonnegative in a boundary chart.) □

Proposition 3.4.2.5 (Product Orientations). *Suppose $M_1, \ldots, M_k$ are orientable smooth manifolds. There is a unique orientation on $M_1 \times \cdots \times M_k$, called the* ***product orientation****, with the following property: if for each $1 \leq i \leq k$, ω_i is an orientation form for the given orientation on M_i, then $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k$ is an orientation form for the product orientation.*

Proposition 3.4.2.6 (Orientations of Codimension-0 Submanifolds). *Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. Then the orientation of M restricts to an orientation of D. If ω is an orientation form for M, then $\iota^*\omega$ is an orientation form for D.*

Let M and N be oriented smooth manifolds with or without boundary, and suppose $F : M \to N$ is a local diffeomorphism. If M and N are positive-dimensional, we say that F is **orientation-preserving** if for each $p \in M$, the isomorphism dF_p takes oriented bases of T_pM to oriented bases of $T_{F(p)}N$, and **orientation-reversing** if it takes oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$.

If M and N are 0-manifolds, then F is orientation-preserving if for every $p \in M$, the points p and $F(p)$ have the same orientation; and it is orientation-reversing if they have the opposite orientation.

Proposition 3.4.2.7. *Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Then the following are equivalent.*

(a) F is orientation-preserving.

(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(c) For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.

Proof. This is a consequence of Proposition 3.4.2.4 and 3.3.1.7. □

Here is another important method for constructing orientations.

Proposition 3.4.2.8 (The Pullback Orientation). *Suppose M and N are smooth manifolds with or without boundary. If $F : M \to N$ is a local diffeomorphism and N is oriented, then M has a unique orientation, called the* ***pullback orientation induced by F****, such that F is orientation-preserving.*

Proof. For each $p \in M$, there is a unique orientation on T_pM that makes the isomorphism $dF_p : T_pM \to T_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M, and provided it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω for N and note that $F^*\omega$ is a smooth orientation form for M. □

In the situation of the preceding proposition, if $\mathcal{O}$ denotes the given orientation on N, the pullback orientation on M is denoted by $F^*\mathcal{O}$.

Proposition 3.4.2.9. *Suppose $F : M \to N$ and $G : N \to P$ are local diffeomorphisms and $\mathcal{O}$ is an orientation on P, then $(G \circ F)^*\mathcal{O} = F^*(G^*\mathcal{O})$.*

Recall that a smooth manifold is said to be *parallelizable* if it admits a smooth global frame.

Proposition 3.4.2.10. *Every parallelizable smooth manifold is orientable.*

Proof. Suppose M is parallelizable, and let $(E_1, \ldots, E_n)$ be a global smooth frame for M. Define a pointwise orientation on M by declaring the basis $(E_1|_p, \ldots, E_n|_p)$ to be positively oriented at each $p \in M$. This pointwise orientation is continuous, because every point of M is in the domain of the (global) oriented frame (E_i). □

Example 3.4.2.11. The preceding proposition implies that Euclidean space $\mathbb{R}^n$, the n-torus T^n, the spheres S^1, S^3, and S^7, and products of them are all orientable, because they are all parallelizable. Therefore, any codimension-0 submanifold of one of these manifolds is also orientable. Likewise, every Lie group is orientable because it is parallelizable.

In the case of Lie groups, we can say more. If G is a Lie group, an orientation of G is said to be **left-invariant** if L_g is orientation-preserving for every $g \in G$.

Proposition 3.4.2.12. *Every Lie group has precisely two left-invariant orientations, corresponding to the two orientations of its Lie algebra.*

3.4.2.1 Orientations of hypersurfaces

If M is an oriented smooth manifold and S is a smooth submanifold of M (with or without boundary), S might not inherit an orientation from M, even if S is embedded. Clearly, it is not sufficient to restrict an orientation form from M to S, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero. A useful example to consider is the Möbius band, which is not orientable, even though it can be embedded in $\mathbb{R}^3$.

Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is a smooth submanifold (immersed or embedded, with or without boundary). Recall that a vector field along S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N : S \to TM$ with the property that $N_p \in T_pM$ for each $p \in S$. For example, any vector field on M restricts to a vector field along S, but in general, not every vector field along S is of this form.

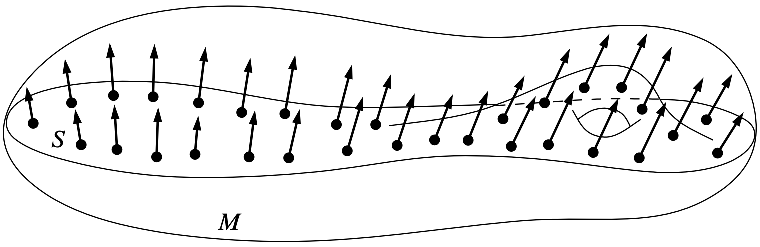


Figure 3.1: A vector field along a submanifold.

Proposition 3.4.2.13. *Suppose M is an oriented smooth n-manifold with or without boundary, S is an immersed hypersurface with or without boundary in M, and N is a vector field along S that is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \ldots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \ldots, E_{n-1})$ is an oriented basis for T_pM. If ω is an orientation form for M, then $\iota_S^*(N \lrcorner\, \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S : S \hookrightarrow M$ is inclusion.*

Remark 3.4.2.14. When $n = 1$, since S is a 0-manifold, this proposition should be interpreted as follows: at each point $p \in S$, we assign the orientation $+1$ to p if N_p is an oriented basis for T_pM, and -1 if N_p is negatively oriented. With this understanding, the proof below goes through in the $n = 1$ case without modification.

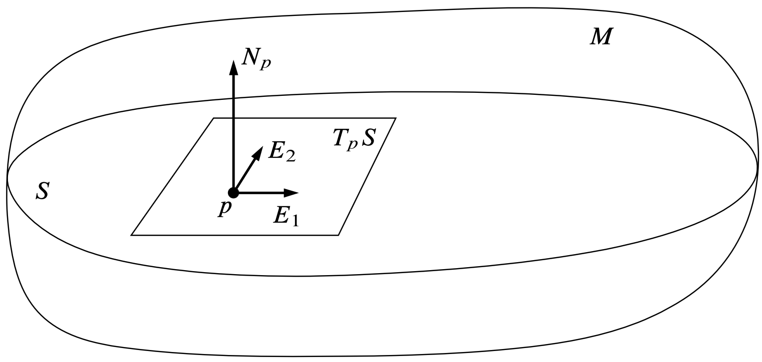


Figure 3.2: The orientation induced by a nowhere tangent vector field.

Proof. Let ω be an orientation form for M. Then $\sigma = \iota_S^*(N \lrcorner \omega)$ is an $(n-1)$-form on S. (Recall that the pullback ι_S^* is really just restriction to vectors tangent to S.) It will follow that σ is an orientation form for S if we can show that it never vanishes. Given any basis $(E_1, \ldots, E_{n-1})$ for T_pS, the fact that N is nowhere tangent to S implies that $(N_p, E_1, \ldots, E_{n-1})$ is a basis for T_pM. The fact that N is nonvanishing implies that

$$\sigma_p(E_1, \ldots, E_{n-1}) = \omega_p(N_p, E_1, \ldots, E_{n-1}) \neq 0.$$

Since $\sigma_p(E_1, \ldots, E_{n-1}) > 0$ if and only if $\omega_p(N_p, E_1, \ldots, E_{n-1}) > 0$, the orientation determined by σ is the one defined in the statement of the proposition. □

Example 3.4.2.15. The sphere S^n is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i \partial/\partial x^i$ is nowhere tangent, so this vector field induces an orientation on S^n. This shows that all spheres are orientable. We define the **standard orientation** of S^n to be the orientation determined by N. Unless otherwise specified, we always use this orientation. (The standard orientation on S^0 is the one that assigns the orientation $+1$ to the point $+1$ and -1 to $-1 \in S^0$.)

Not every hypersurface admits a nowhere tangent vector field. However, the following proposition gives a sufficient condition that holds in many cases.

Proposition 3.4.2.16. *Let M be an oriented smooth manifold, and suppose $S \subseteq M$ is a regular level set of a smooth function $f : M \to \mathbb{R}$. Then S is orientable.*

Proof. Choose any Riemannian metric on M, and let $N = \operatorname{grad} f|_S$. The hypotheses imply that N is a nowhere tangent vector field along S, so the result follows from Proposition 3.4.2.13. □

3.4.2.2 Boundary orientations

If M is a smooth manifold with boundary, then ∂M is an embedded hypersurface in M, and Exercise 1.1.2 showed that there is always a smooth outward-pointing vector field along ∂M. Because an outward-pointing vector field is nowhere tangent to ∂M, it determines an orientation on ∂M if M is oriented.

Proposition 3.4.2.17 (Induced Orientation on a Boundary). *Let M be an oriented smooth n-manifold with boundary, $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.*

Remark 3.4.2.18. The orientation on ∂M determined by any outward-pointing vector field is called the **induced orientation** or the **Stokes orientation** on ∂M.

Proof. Let $n = \dim M$, let ω be an orientation form for M, and let N be a smooth outward-pointing vector field along ∂M. The $(n-1)$-form $\iota_{\partial M}^*(N \lrcorner \omega)$ is an orientation form for ∂M by Proposition 3.4.2.13, so ∂M is orientable.

To show that this orientation is independent of the choice of N, let $p \in \partial M$ be arbitrary, and let (x^i) be smooth boundary coordinates for M on a neighborhood of p. If N and $\widetilde{N}$ are two different outward-pointing vector fields along ∂M, Proposition **??** shows that the last components $N^n(p)$ and $\widetilde{N}^n(p)$ are both negative. Both $(N_p, \partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ and $(\widetilde{N}_p, \partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ are bases for T_pM, and the transition matrix between them has determinant equal to $N^n(p)/\widetilde{N}^n(p) > 0$. Thus, both bases determine the same orientation for T_pM, so N and $\widetilde{N}$ determine the same orientation for $T_p\partial M$. (When $n = 1$, the bases in question are just N_p and $\widetilde{N}_p$, which determine the same orientation because they are both negative multiples of $\partial/\partial x^1|_p$.) □

Example 3.4.2.19. This proposition gives a simpler proof that S^n is orientable, because it is the boundary of the closed unit ball. The orientation thus induced on S^n is the standard one.

Example 3.4.2.20. Let us determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence $(x^1, \ldots, x^{n-1}, 0) \leftrightarrow (x^1, \ldots, x^{n-1})$. Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\partial\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}]$ is the standard orientation for $\mathbb{R}^n$. This orientation satisfies

$$[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}] = -[\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}]$$

$$= (-1)^n [\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n].$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even.

For many purposes, the most useful way of describing submanifolds is by means of local parametrizations. The next lemma gives a useful criterion for checking whether a local parametrization of a boundary is orientation-preserving.

Lemma 3.4.2.21. *Let M be an oriented smooth n-manifold with boundary. Suppose $U \subseteq \mathbb{R}^{n-1}$ is open, a, b are real numbers with $a < b$, and $F : (a, b] \times U \to M$ is a smooth embedding that restricts to an embedding of $\{b\} \times U$ to ∂M. Then the parametrization $f : U \to \partial M$ given by $f(x) = F(b, x)$ is orientation-preserving for ∂M if and only if F is orientation-preserving for M.*

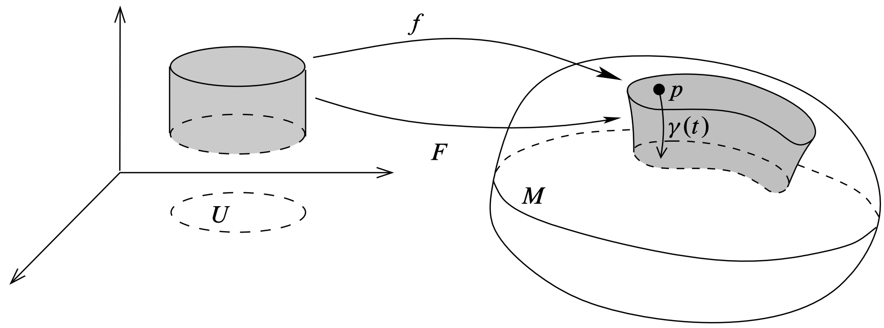


Figure 3.3: Orientation criterion for a boundary parametrization.

Proof. Let x be an arbitrary point of U, and let $p = f(x) = F(b, x) \in \partial M$. The hypothesis that F is an embedding means that the linear map $dF_{(b,x)} : (T_b\mathbb{R} \oplus T_x\mathbb{R}^{n-1}) \to T_pM$ is bijective. Since the restriction of $dF_{(b,x)}$ to $T_x\mathbb{R}^{n-1}$ is equal to $df_x : T_x\mathbb{R}^{n-1} \to T_p\partial M$ which is already injective, it follows that $dF(\partial/\partial s|_{(b,x)}) \notin T_p\partial M$ (where s denotes the coordinate on $(a, b]$).

Define a smooth curve $\gamma : [0, \varepsilon) \to M$ by

$$\gamma(t) = F(b - t, x).$$

This curve satisfies $\gamma(0) = p$ and $\gamma'(0) = -dF(\partial/\partial s|_{(b,x)})$. It follows that $-dF(\partial/\partial s|_{(b,x)})$ is inward-pointing, and therefore $dF(\partial/\partial s|_{(b,x)})$ is outward-pointing.

The definition of the induced orientation yields the following equivalences:

$$\begin{aligned}
&F \text{ is orientation-preserving for } M \\
&\iff \big(dF(\partial/\partial s), dF(\partial/\partial x^1), \ldots, dF(\partial/\partial x^{n-1})\big) \text{ is oriented for } TM \\
&\iff \big(dF(\partial/\partial x^1), \ldots, dF(\partial/\partial x^{n-1})\big) \text{ is oriented for } T\partial M \\
&\iff \big(df(\partial/\partial x^1), \ldots, df(\partial/\partial x^{n-1})\big) \text{ is oriented for } T\partial M \\
&\iff f \text{ is orientation-preserving for } M.
\end{aligned}$$

This proves the claim. □

Here is an illustration of how the lemma can be used.

Example 3.4.2.22. Spherical coordinates yield a smooth local parametrization of S^2 as follows. Let U be the open rectangle $(0, \pi) \times (0, 2\pi) \subseteq \mathbb{R}^2$, and let $X : U \to \mathbb{R}^3$ be the following map:

$$X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

We can check whether X preserves or reverses orientation by using the fact that it is the restriction of the 3-dimensional spherical coordinate parametrization $F : (0, 1] \times U \to \overline{B}^3$ defined by

$$F(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

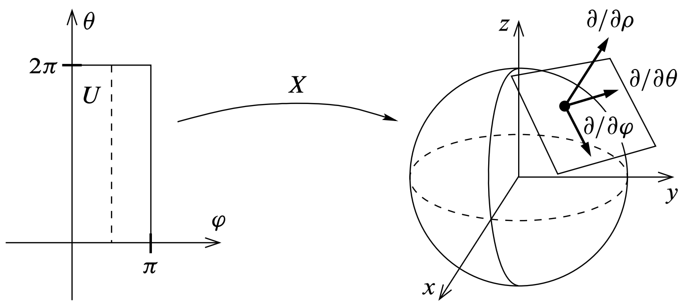


Figure 3.4: Parametrizing the sphere via spherical coordinates.

Because $F(1,\varphi,\theta) = X(\varphi,\theta)$, the hypotheses of Lemma 3.4.2.21 are satisfied. By direct computation, the Jacobian determinant of F is $\rho^2 \sin\varphi$, which is positive on $(0,1] \times U$. By virtue of Lemma 3.4.2.21, X is orientation-preserving.

3.4.3 The Riemannian volume form

Let (M,g) be an oriented Riemannian manifold of positive dimension. We know from Proposition 3.2.1.6 that there is a smooth orthonormal frame $(E_1,\dots,E_n)$ in a neighborhood of each point of M. By replacing E_1 by $-E_1$ if necessary, we can find an oriented orthonormal frame in a neighborhood of each point.

Proposition 3.4.3.1. *Suppose (M,g) is an oriented Riemannian n-manifold with or without boundary, and $n \geq 1$. There is a unique smooth orientation form $\omega_g \in \Omega^k(M)$, called the* ***Riemannian volume form****, that satisfies*

$$\omega_g(E_1,\dots,E_n) = 1 \tag{3.4.3.1}$$

for every local oriented orthonormal frame (E_i) for M.

Proof. Suppose first that such a form ω_g exists. If $(E_1,\dots,E_n)$ is any local oriented orthonormal frame on an open subset $U \subseteq M$ and $(\varepsilon^1,\dots,\varepsilon^n)$ is the dual coframe, we can write $\omega_g = f\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ on U. The condition (3.4.3.1) then reduces to $f = 1$, so

$$\omega_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n. \tag{3.4.3.2}$$

This proves that such a form is uniquely determined.

To prove existence, we would like to define ω_g in a neighborhood of each point by (3.4.3.2), so we need to check that this definition is independent of the choice of oriented orthonormal frame. If $(\widetilde{E}_1,\dots,\widetilde{E}_n)$ is another oriented orthonormal frame, with dual coframe $(\widetilde{\varepsilon}^1,\dots,\widetilde{\varepsilon}^n)$, let

$$\widetilde{\omega} = \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n.$$

We can write

$$\widetilde{E}_i = A_i^j E_j$$

for some matrix (A_i^j) of smooth functions. The fact that both frames are orthonormal means that $(A_i^j(p)) \in \mathrm{O}(n)$ for each p, so $\det(A_i^j) = \pm 1$, and the fact that the two frames are consistently oriented forces the positive sign. We compute by Proposition 3.3.1.7:

$$\omega_g(\widetilde{E}_1,\dots,\widetilde{E}_n) = \det\left(\varepsilon^j(\widetilde{E}_i)\right) = \det\left(A_i^j\right) = 1 = \widetilde{\omega}_g(\widetilde{E}_1,\dots,\widetilde{E}_n).$$

Thus $\omega_g = \widetilde{\omega}_g$, so defining ω_g in a neighborhood of each point by (3.4.3.2) with respect to some smooth oriented orthonormal frame yields a global n-form. The resulting form is clearly smooth and satisfies (3.4.3.1) for every oriented orthonormal frame. □

Proposition 3.4.3.2. *Suppose (M,g) and $(\widetilde{M},\widetilde{g})$ are positive-dimensional Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a orientation-preserving local isometry. Then that $F^*\omega_{\widetilde{g}} = \omega_g$.*

Proof. Since F is a orientation-preserving local isometry, it maps a local oriented orthonormal frame to a local oriented orthonormal frame. Then the claim follows immediately. □

Although the expression for the Riemannian volume form with respect to an oriented orthonormal frame is particularly simple, it is also useful to have an expression for it in coordinates.

Proposition 3.4.3.3. *Let (M, g) be an oriented Riemannian n-manifold with or without boundary within ≥ 1. In any oriented smooth coordinates (x^i), the Riemannian volume form has the local coordinate expression*

$$\omega_g = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^n,$$

where $G = \det(g_{ij})$ with g_{ij} the components of g in these coordinates.

Proof. Let $(U, (x^i))$ be an oriented smooth chart, and let $p \in M$. In these coordinates, $\omega_g = f\, dx^1 \wedge \cdots \wedge dx^n$ for some positive coefficient function f. To compute f, let (E_i) be any smooth oriented orthonormal frame defined on a neighborhood of p, and let (ε^i) be the dual coframe. If we write the coordinate frame in terms of the orthonormal frame as

$$\frac{\partial}{\partial x^i} = A_i^j E_j,$$

then we can compute

$$f = \omega_g\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = \det\left(\varepsilon^j\left(\frac{\partial}{\partial x^i}\right)\right) = \det(A_i^j).$$

On the other hand, observe that

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_g = \langle A_i^k E_k, A_j^l E_l \rangle_g = A_i^k A_j^l \langle E_k, E_l \rangle_g = \sum_k A_i^k A_j^k.$$

This last expression is the (i, j)-entry of the matrix product $A^T A$, where $A = (A_i^j)$. Thus,

$$G := \det(g_{ij}) = \det(A^T A) = (\det A)^2,$$

Since both frames $(\partial/\partial x^i)$ and (E_i) are oriented, f must be positive. So it follows that $f = \sqrt{G}$. □

3.4.3.1 Hypersurfaces in Riemannian manifolds

Let (M, g) be an oriented Riemannian manifold with or without boundary, and suppose $S \subseteq M$ is an immersed hypersurface with or without boundary. Any unit normal vector field along S is nowhere tangent to S, so it determines an orientation of S by Proposition 3.4.2.13. The next proposition gives a simple formula for the volume form of the induced metric on S with respect to this orientation.

Proposition 3.4.3.4. *Let (M, g) be an oriented Riemannian manifold with or without boundary, let $S \subseteq M$ be an immersed hypersurface with or without boundary, and let $\widetilde{g}$ denote the induced metric on S. Suppose N is a smooth unit normal vector field along S. With respect to the orientation of S determined by N, the volume form of $(S, \widetilde{g})$ is given by*

$$\omega_{\widetilde{g}} = \iota_S^*(N \lrcorner \omega_g).$$

Proof. By Proposition 3.4.2.13, the $(n-1)$-form $\iota_S^*(N \lrcorner \omega_g)$ is an orientation form for S. To prove that it is the volume form for the induced Riemannian metric, we need only show that it gives the value 1 whenever it is applied to an oriented orthonormal frame for S. Thus, let $(E_1, \ldots, E_{n-1})$ be such a frame. At each point $p \in S$, the basis $(N_p, E_1|_p, \ldots, E_{n-1}|_p)$ is orthonormal, and is oriented for T_pM (this is the definition of the orientation determined by N). Thus

$$\iota_S^*(N \lrcorner \omega_g)(E_1, \ldots, E_{n-1}) = \omega_g(N, E_1, \ldots, E_{n-1}) = 1,$$

which proves the result. □

The result of Proposition 3.4.3.4 takes on particular importance in the case of a Riemannian manifold with boundary, because of the following proposition.

Proposition 3.4.3.5. *Suppose M is any Riemannian manifold with boundary. There is a unique smooth outward-pointing unit normal vector field N along ∂M.*

Proof. First, we prove uniqueness. At any point $p \in \partial M$, the subspace $(T_p\partial M)^\perp \subseteq T_pM$ is 1-dimensional, so there are exactly two unit vectors at p that are normal to ∂M. Since any unit normal vector N is nowhere tangent to ∂M, it must have nonzero x^n-component in any smooth boundary chart. Thus, exactly one of the two choices of unit normal has negative x^n-component, which is equivalent to being outward-pointing.

To prove existence, let $f : M \to \mathbb{R}$ be a boundary defining function (Proposition 5.43), and let N be the restriction to ∂M of the unit vector field $-\operatorname{grad} f/|\operatorname{grad} f|_g$. Because $df \neq 0$ at points of ∂M, N is well-defined and smooth on ∂M. Then N is normal to ∂M by Exercise 3.2.9, and outward pointing by Proposition **??**, because

$$Nf = \frac{-\langle \operatorname{grad} f, \operatorname{grad} f\rangle_g}{|\operatorname{grad} f|_g} = -|\operatorname{grad} f|_g < 0.$$

□

The next corollary is immediate.

Corollary 3.4.3.6. *If (M, g) is an oriented Riemannian manifold with boundary and $\tilde{g}$ is the induced Riemannian metric on ∂M, then the volume form of $\tilde{g}$ is*

$$\omega_{\tilde{g}} = \iota_{\partial M}^*(N \lrcorner\, \omega_g),$$

where N is the outward unit normal vector field along ∂M.

3.4.4 Orientations and covering maps

Although it is often easy to prove that a given smooth manifold is orientable by constructing an orientation for it, proving that a manifold is not orientable can be much trickier. The theory of covering spaces provides one of the most useful techniques for doing so. In this part, we explore the close relationship between orientability and covering maps.

Our first result is a simple application of pullback orientations.

Proposition 3.4.4.1. *If $\pi : E \to M$ is a smooth covering map and M is orientable, then E is also orientable.*

Proposition 3.4.4.2. *Because a covering map is a local diffeomorphism, this follows immediately from Proposition 3.4.2.8.*

The next theorem is more interesting. If G is a Lie group acting smoothly on a smooth manifold E (on the left, say), we say the action is an **orientation-preserving action** if for each $g \in G$, the diffeomorphism $x \mapsto g \cdot x$ is orientation-preserving.

Theorem 3.4.4.3. *Suppose E is a connected, oriented, smooth manifold with or without boundary, and $\pi : E \to M$ is a smooth normal covering map. Then M is orientable if and only if the action of $\operatorname{Aut}_\pi(E)$ on E is orientation-preserving.*

Proof. Let $\mathcal{O}_E$ denote the given orientation on E. First suppose M is orientable, and let q be an arbitrary point in E. Because M is connected, it has exactly two orientations, and one of them has the property that $d\pi_q : T_qE \to T_{\pi(q)}M$ is orientation-preserving. Call that orientation $\mathcal{O}_M$. The pullback orientation $\pi^*\mathcal{O}_M$ agrees with the given orientation at q, so it must be equal to $\mathcal{O}_E$ by Corollary 3.4.2.3. Suppose $\varphi \in \operatorname{Aut}_\pi(E)$. The fact that $\pi \circ \varphi = \pi$ implies that

$$\varphi^*\mathcal{O}_E = \varphi^*(\pi^*\mathcal{O}_M) = (\pi \circ \varphi)^*\mathcal{O}_M = \pi^*\mathcal{O}_M = \mathcal{O}_E.$$

Thus, φ is orientation-preserving.

Conversely, suppose the action of $\operatorname{Aut}_\pi(E)$ is orientation-preserving, and let $p \in M$. If $U \subseteq M$ is any evenly covered neighborhood of p, there is a smooth section $\sigma : U \to E$, which induces an orientation $\sigma^*\mathcal{O}_E$ on U. Suppose $\sigma_1 : U \to E$ is any other smooth local section over U. Because π is a normal covering, $\operatorname{Aut}_\pi(E)$ acts transitively on each fiber of π, so there is a covering automorphism φ such that $\sigma_1(p) = \varphi(\sigma(p))$. Then $\varphi \circ \sigma$ is a local section of π that agrees with σ_1 at p, and thus $\sigma_1 = \varphi \circ \sigma$ on all of U by Theorem **??**. Because φ is orientation-preserving, $\sigma_1^*\mathcal{O}_E = \sigma^*\varphi^*\mathcal{O}_E = \sigma^*\mathcal{O}_E$, so the orientations induced by σ and σ_1 are equal. Thus, we can define a global orientation $\mathcal{O}_M$ on M by defining it on each evenly covered open subset to be the pullback orientation induced by any local section; the argument above shows that the orientations so defined agree where they overlap. □

Here are two applications of the preceding theorem.

Example 3.4.4.4 (Orientability of Projective Spaces). For $n \geq 1$, consider the smooth covering map $q : S^n \to \mathbb{RP}^n$. The only nontrivial covering automorphism of q is the antipodal map $\alpha(x) = -x$. It can be easily shown that α is orientation-preserving if and only if n is odd, so it follows that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Example 3.4.4.5 (The Möbius Bundle and the Möbius Band). Let E be the total space of the Möbius bundle (*Example* **??**). The quotient map $q : \mathbb{R}^2 \to E$ used to define E is a smooth normal covering map, and the covering automorphism group is isomorphic to $\mathbb{Z}$, acting on $\mathbb{R}^2$ by

$$n(x, y) = (x + n, (-1)^n y)$$

For n odd, the diffeomorphism $(x, y) \mapsto n \cdot (x, y)$ of $\mathbb{R}^2$ pulls back the orientation form $dx \wedge dy$ to $-dx \wedge dy$, so the action of $\mathrm{Aut}_q(E)$ is not orientation-preserving. Thus, Theorem 3.4.4.3 shows that E is not orientable.

For each $r > 0$, the image under q of the rectangle $[0, 1] \times [-r, r]$ is a Möbius band M_r. Because q restricts to a smooth covering map from $\mathbb{R} \times [-r, r]$, the same argument shows that a Möbius band is not orientable either.

3.4.4.1 The orientation covering

Next we show that every nonorientable smooth manifold M has an orientable two-sheeted covering manifold. The fiber over a point $p \in M$ will correspond to the two orientations of T_pM.

In order to handle the orientable and nonorientable cases in a uniform way, it is useful to expand our definition of covering maps slightly, by allowing covering spaces that are not connected. If N and M are topological spaces, let us say that a map $\pi : N \to M$ is a **generalized covering map** if it satisfies all of the requirements for a covering map except that N might not be connected: this means that N is locally path-connected, π is surjective and continuous, and each point $p \in M$ has a neighborhood that is evenly covered by π. If in addition N and M are smooth manifolds with or without boundary and π is a local diffeomorphism, we say it is a **generalized smooth covering map**.

Lemma 3.4.4.6. *Suppose N and M are topological spaces and $\pi : N \to M$ is a generalized covering map. If M is connected, then the restriction of π to each component of N is a covering map.*

Proof. Suppose W is a component of N. If U is any open subset of M that is evenly covered by π, then each component of $\pi^{-1}(U)$ is connected and therefore contained in a single component of N. It follows that $(\pi|_W)^{-1}(U) = \pi^{-1}(U) \cap W$ is either the empty set or a nonempty disjoint union of components of $\pi^{-1}(U)$, each of which is mapped homeomorphically onto U by $\pi|_W$. In particular, this means that each point in $\pi(W)$ has a neighborhood that is evenly covered by $\pi|_W$.

To complete the proof, we just need to show that $\pi|_W$ is surjective. Because π is a local homeomorphism, $\pi(W)$ is an open subset of M. On the other hand, if $p \in M - \pi(W)$, and U is a neighborhood of p that is evenly covered by π, then the discussion in the preceding paragraph shows that $(\pi|_W)^{-1}(U) = \varnothing$, which implies that $U \subseteq M - \pi(W)$. Therefore, $\pi(W)$ is closed in M. Because W is not empty, $\pi(W)$ is all of M. □

Let M be a connected, smooth, positive-dimensional manifold with or without boundary, and let $\widehat{M}$ denote the set of orientations of all tangent spaces to M:

$$\widehat{M} = \{(p, \mathcal{O}_p) : p \in M \text{ and } \mathcal{O}_p \text{ is an orientation of } T_pM\}.$$

Define the projection $\widehat{\pi} : \widehat{M} \to M$ by sending an orientation of T_pM to the point p itself: $\widehat{\pi}(p, \mathcal{O}_p) = p$. Since each tangent space has exactly two orientations, each fiber of this map has cardinality 2. The map $\widehat{\pi} : \widehat{M} \to M$ is called the **orientation covering** of M.

Proposition 3.4.4.7 (Properties of the Orientation Covering). *Suppose M is a connected, smooth, positive-dimensional manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering. Then $\widehat{M}$ can be given the structure of a smooth, oriented manifold with or without boundary, with the following properties:*

(a) $\widehat{\pi} : \widehat{M} \to M$ *is a generalized smooth covering map.*

(b) A connected open subset $U \subseteq M$ is evenly covered by $\widehat{\pi}$ if and only if U is orientable.

(c) *If $U \subseteq M$ is an evenly covered open subset, then every orientation of U is the pullback orientation induced by a local section of $\widehat{\pi}$ over U.*

Proof. We first topologize $\widehat{M}$ by defining a basis for it. For each pair $(U, \mathcal{O})$, where U is an open subset of M and $\mathcal{O}$ is an orientation on U, define a subset $\widehat{U}_{\mathcal{O}} \subseteq \widehat{M}$ as follows:

$$\widehat{U}_{\mathcal{O}} = \{(p, \mathcal{O}_p) \in \widehat{M} : p \in U \text{ and } \mathcal{O}_p \text{ is the orientation of } T_pM \text{ determined by } \mathcal{O}\}.$$

We will show that the collection of all subsets of the form $\widehat{U}_{\mathcal{O}}$ is a basis for a topology on $\widehat{M}$.

- Given an arbitrary point $(p, \mathcal{O}_p) \in \widehat{M}$, let U be an orientable neighborhood of p in M, and let $\mathcal{O}$ be an orientation on it. After replacing $\mathcal{O}$ by $-\mathcal{O}$ if necessary, we may assume that the given orientation $\mathcal{O}_p$ is same as the orientation of T_pM determined by $\mathcal{O}$. It follows that $(p, \mathcal{O}_p) \in \widehat{U}_{\mathcal{O}}$, so the collection of all sets of the form $\widehat{U}_{\mathcal{O}}$ covers $\widehat{M}$.

- If $\widehat{U}_{\mathcal{O}}$ and $\widehat{U}'_{\mathcal{O}'}$ are two such sets and $(p, \mathcal{O}_p)$ is a point in their intersection, then $\mathcal{O}_p$ is the orientation of T_pM determined by both $\mathcal{O}$ and $\mathcal{O}'$. If V is the component of $U \cap U'$ containing p, then the restricted orientations $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ agree at p and therefore are identical by Proposition 3.4.2.3, so it follows that $\widehat{U}_{\mathcal{O}} \cap \widehat{U}'_{\mathcal{O}'}$ contains the basis set $\widehat{V}_{\mathcal{O}|_V}$.

Thus, we have defined a topology on $\widehat{M}$.

Note that for each orientable open subset $U \subseteq M$ and each orientation $\mathcal{O}$ of U, $\widehat{\pi}$ maps the basis set $\widehat{U}_{\mathcal{O}}$ bijectively onto U. Because the orientable open subsets form a basis for the topology of M, this implies that $\widehat{\pi}$ restricts to a homeomorphism from $\widehat{U}_{\mathcal{O}}$ to U. In particular, $\widehat{\pi}$ is a local homeomorphism.

Next we show that with this topology, $\widehat{\pi}$ is a generalized covering map. Suppose $U \subseteq M$ is an orientable connected open subset and $\mathcal{O}$ is an orientation for U. Then $\widehat{\pi}^{-1}(U)$ is the disjoint union of open subsets $\widehat{U}_{\mathcal{O}}$ and $\widehat{U}_{-\mathcal{O}}$, and $\widehat{\pi}$ restricts to a homeomorphism from each of these sets to U. Thus, each such set U is evenly covered, and it follows that $\widehat{\pi}$ is a generalized covering map. By Lemma 3.4.4.6, $\widehat{\pi}$ restricts to an ordinary covering map on each component of $\widehat{M}$, and so Proposition **??** shows that each such component is a topological n-manifold with or without boundary and has a unique smooth structure making $\widehat{\pi}$ into a smooth covering map. These smooth structures combine to give a smooth structure on all of $\widehat{M}$. This completes the proof of (a).

Next we give $\widehat{M}$ an orientation. Let $\widehat{p} = (p, \mathcal{O}_p)$ be a point in $\widehat{M}$. By definition, $\mathcal{O}_p$ is an orientation of T_pM, so we can give $T_{\widehat{p}}\widehat{M}$ the unique orientation $\widehat{\mathcal{O}}_{\widehat{p}}$ such that $d\widehat{\pi}_{\widehat{p}} : T_{\widehat{p}}\widehat{M} \to T_pM$ is orientation-preserving. This defines a pointwise orientation $\widehat{\mathcal{O}}$ on $\widehat{M}$. On each basis open subset $\widehat{U}_{\mathcal{O}}$, the orientation $\widehat{\mathcal{O}}$ agrees with the pullback orientation induced from $(U, \mathcal{O})$ by (the restriction of) $\widehat{\pi}$, so it is continuous.

Next we prove (b). We showed earlier that every orientable connected open subset of M is evenly covered by $\widehat{\pi}$. Conversely, if $U \subseteq M$ is any evenly covered open subset, then there is a smooth local section $\sigma : U \to \widehat{M}$ by Proposition **??**, which pulls $\widehat{\mathcal{O}}$ back to an orientation on U by Proposition 3.4.2.8.

Finally, to prove (c), assume $U \subseteq M$ is evenly covered and therefore orientable. Given any orientation $\mathcal{O}$ of U, define a section $\sigma : U \to M$ by setting $\sigma(p) = (p, \mathcal{O}_p)$. To see that σ is continuous, suppose $\widehat{U}'_{\mathcal{O}'}$ is any basis open subset of $\widehat{M}$. Then for each component V of $U \cap U'$, the restricted orientations $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ must either agree or disagree on all of V, so $\sigma^{-1}(\widehat{U}'_{\mathcal{O}'})$ is a union of such components and therefore open. □

Theorem 3.4.4.8 (Orientation Covering Theorem). *Suppose M is a connected smooth manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering.*

(a) *If M is orientable, then $\widehat{M}$ has exactly two components, and the restriction of $\widehat{\pi}$ to each component is a diffeomorphism onto M.*

(b) *If M is nonorientable, then $\widehat{M}$ is connected, and $\widehat{\pi}$ is a two-sheeted smooth covering map.*

Proof. If M is orientable, then Proposition 3.4.4.7(b) shows that M is evenly covered by $\widehat{\pi}$, which means that $\widehat{M}$ has two components, each mapped diffeomorphically onto M.

Now assume M is nonorientable. We show first that $\widehat{M}$ is connected. Let W be a component of $\widehat{M}$. Lemma 3.4.4.6 shows that $\widehat{\pi}|_W$ is a covering map, so its fibers all have the same cardinality. Because the fibers of $\widehat{\pi}$ have cardinality 2 and W is not empty, the fibers of $\widehat{\pi}|_W$ must have cardinality 1 or 2. If the cardinality were 1, then $\widehat{\pi}|_W$ would be an injective smooth covering map and thus a diffeomorphism, and its inverse would be a smooth section of $\widehat{\pi}$, which would induce an orientation on M. Thus, the cardinality must be 2, which implies that $W = \widehat{M}$. Because $\widehat{M}$ is connected, $\widehat{\pi}$ is a covering map by Lemma 3.4.4.6, and because it is a local diffeomorphism it is a smooth covering map. □

The orientation covering is sometimes called the **oriented double covering of M**. There are other ways of constructing it besides the one we have given here, but as the next theorem shows, the specific details of the construction do not matter, because they all yield isomorphic covering manifolds.

Theorem 3.4.4.9 (Characteristic property of the orientation covering). *Let M be a connected nonorientable smooth manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering. If X is any oriented smooth manifold with or without boundary, and $F : X \to M$ is any local diffeomorphism, then there exists a unique orientation-preserving local diffeomorphism $\widehat{F} : X \to \widehat{M}$ such that $\widehat{\pi} \circ \widehat{F} = F$:*

$$\begin{tikzcd} & \widehat{M} \arrow[d, "\widehat{\pi}"] \\ X \arrow[ur, dashed, "\widehat{F}"] \arrow[r, "F"'] & M \end{tikzcd}$$

Proof. At each point $p = F(x) \in M$, since F is a local diffeomorphism, we get a unique orientation of T_pM, which we denote by $\mathcal{O}_p$. Then we can define

$$\widehat{F}(x) = (F(x), \mathcal{O}_{F(x)}).$$

It is easy to check that $\widehat{\pi} \circ \widehat{F} = F$. Since $\widehat{\pi}$ is also a local diffeomorphism, it follows that $\widehat{F}$ is also a local diffeomorphism. The orientation-preserving statement is clear from the definition. □

Theorem 3.4.4.10 (Uniqueness of the Orientation Covering). *Let M be a nonorientable connected smooth manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering. If $\widetilde{M}$ is an oriented smooth manifold with or without boundary that admits a two-sheeted smooth covering map $\widetilde{\pi} : \widetilde{M} \to M$, then there exists a unique orientation-preserving diffeomorphism $\varphi : \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \widetilde{\pi}$.*

By invoking a little more covering space theory, we obtain the following sufficient topological condition for orientability.

Theorem 3.4.4.11. *Let M be a connected smooth manifold with or without boundary, and suppose the fundamental group of M has no subgroup of index 2. Then M is orientable. In particular, if M is simply connected then it is orientable.*

Proof. Suppose M is not orientable, and let $\widehat{\pi} : \widehat{M} \to M$ be its orientation covering, which is an honest covering map in this case. Choose any point $q \in \widehat{M}$, and let $p = \widehat{\pi}(q) \in M$. Let $\alpha : \widehat{M} \to \widehat{M}$ be the map that interchanges the two points in each fiber of $\widehat{\pi}$. To prove that α is smooth, suppose $U \subseteq M$ is any evenly covered open subset and $U_0, U_1 \subseteq \widehat{M}$ are the two components of $\widehat{\pi}^{-1}(U)$. Since $\widehat{\pi}$ restricts to a diffeomorphism from each component onto U, we can write $\alpha|_{U_0} = (\widehat{\pi}|_{U_1})^{-1} \circ (\widehat{\pi}|_{U_0})$, which is smooth. Similarly, $\alpha|_{U_1}$ is also smooth. Since the collection of all such sets U_0, U_1 is an open covering of $\widehat{M}$, it follows that α is smooth, and it is a covering automorphism because it satisfies $\widehat{\pi} \circ \alpha = \widehat{\pi}$. In fact, since a covering automorphism is determined by what it does to one point, α is the unique nontrivial element of the automorphism group $\mathrm{Aut}_{\widehat{\pi}}(\widehat{M})$, which is therefore equal to the two-element group $\{\mathrm{id}_{\widehat{M}}, \alpha\}$. Because the automorphism group acts transitively on fibers, $\widehat{\pi}$ is a normal covering map. A fundamental result in the theory of covering spaces is that

$$\mathrm{Aut}_{\widehat{\pi}}(\widehat{M}) \cong \frac{\pi_1(M, p)}{\widehat{\pi}^*\big(\pi_1(\widehat{M}, q)\big)}.$$

Therefore, $\pi_1(M, p)$ has an index 2 subgroup. □

3.4.5 Exercise

Exercise 3.4.1. Suppose$M$ is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable.

Proof. Use Proposition 3.4.2.3. □

Exercise 3.4.2. Suppose M and N are oriented smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Show that if M is connected, then F is either orientation-preserving or orientation-reversing.

Exercise 3.4.3. Suppose $n \geq 1$, and let $\alpha : S^n \to S^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.

Proof. Consider the orientation form on $\overline{B}^{n+1}$:

$$\omega = dx^1 \wedge \cdots \wedge dx^{n+1}.$$

The induced orientation on S^n is then given by

$$\omega_{S^n} = \iota^*(N \lrcorner\, \omega),$$

where $N = x^i \partial/\partial x^i$. Let $F : \overline{B}^{n+1} \to \overline{B}^{n+1}$ be the map $F(x) = -x$, then $F_* \omega = (-1)^{n+1}\omega$. Since the map α preserves that vector field N, and the orientation of S^n is induced by N, we conclude that α is orientation-preserving if and only if F is orientation-preserving. □

Exercise 3.4.4. Let θ be a smooth flow on an oriented smooth manifold with or without boundary. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined.

Proof. The map $\theta : \mathbb{R} \to \mathrm{Diff}(M)$ is a smooth group homomorphism, and hence its image is connected. Since $\theta_0 = \mathrm{id}_M$, this curve lies in the orientation preserving component of $\mathrm{Diff}(M)$. □

Exercise 3.4.5. Let M be a smooth manifold with or without boundary. Show that the total spaces of TM and T^*M are orientable.

Proof. We prove for TM. By Proposition **??**, the transition map between charts (U, φ) and (V, ψ) of M is

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = (\widetilde{x}^1(x), \ldots, \widetilde{x}^n(x), v^i \frac{\partial \widetilde{x}^1}{\partial x^i}(x), \ldots, v^i \frac{\partial \widetilde{x}^n}{\partial x^i}(x)).$$

The Jacobian matrix of this map is

$$\partial(\widetilde{\psi} \circ \widetilde{\varphi}^{-1})(x, v) = \begin{pmatrix} \dfrac{\partial \widetilde{x}^j}{\partial x^i}(x) & 0 \\ * & \dfrac{\partial \widetilde{x}^j}{\partial x^i}(x) \end{pmatrix}.$$

Thus

$$\det \partial(\widetilde{\psi} \circ \widetilde{\varphi}^{-1}) = \left(\det \partial(\psi \circ \varphi^{-1})\right)^2,$$

which, by Proposition 3.4.2.4, implies TM is orientable. □

Exercise 3.4.6. Suppose M is an oriented Riemannian manifold with or without boundary, and $S \subseteq M$ is an oriented smooth hypersurface with or without boundary. Show that there is a unique smooth unit normal vector field along S that determines the given orientation of S.

Exercise 3.4.7. Suppose M is an orientable Riemannian manifold, and $S \subseteq M$ is an immersed or embedded submanifold with or without boundary. Prove the following statements.

(a) If S has trivial normal bundle, then S is orientable.

(b) If S is an orientable hypersurface, then S has trivial normal bundle.

Proof. Let S has codimension k. If S has trivial normal bundle, then there is an diffeomorphism $NS \to S \times \mathbb{R}^k$. Thus we have global frames $(\sigma_1, \ldots, \sigma_k)$ of NS. By Proposition 1.1.1.10, at each point p we can find local frames $(\sigma_{k+1}, \ldots, \sigma_n)$ such that $(\sigma_1, \ldots, \sigma_n)$ is an oriented frame for T_pM. Thus S is orientable.

If S is an orientable hypersurface, then there is a unit normal vector field along S. Since this is a global frame for NS, we get a global trivialization for NS. □

Exercise 3.4.8. Show that every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point.

Exercise 3.4.9. Let M be a connected smooth 1-manifold. Show that M is diffeomorphic to either $\mathbb{R}$ or S^1, as follows:

(a) First, do the case in which M is orientable by showing that M admits a nonvanishing smooth vector field.

(b) Now let M be arbitrary, and prove that M is orientable by showing that its universal covering manifold is diffeomorphic to $\mathbb{R}$.

Conclude that the smooth structures on both $\mathbb{R}$ and S^1 are unique up to diffeomorphism.

Proof. The only 1-manifolds are $\mathbb{R}$ and S^1, thus orientable. □

Exercise 3.4.10. Show that every connected smooth 1-manifold with nonempty boundary is diffeomorphic to either $[0, \infty)$ or $[0, 1]$.

Exercise 3.4.11. Let M be a nonorientable embedded hypersurface in $\mathbb{R}^n$, and let NM be its normal bundle with projection $\pi_{NM} : NM \to M$. Show that the set

$$W = \{(x, v) \in NM : |v| = 1\}$$

is an embedded submanifold of NM, and the restriction of π_{NM} to W is a smooth covering map isomorphic to the orientation covering of M.

Proof. We can show that W is a orientable manifold, and the map $\pi_{NM} : W \to M$ is a 2-sheet covering. Thus there is a diffeomorphism from W to $\widehat{M}$ by Theorem 3.4.4.10. □

Exercise 3.4.12. Let E be the total space of the Möbius bundle as in Example **??**. Show that the orientation covering of E is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Proof. Let $S^1 \times \mathbb{R}$ be embedded into $\mathbb{R}^3$, we can apply the antipodal map α on it to obtain E. This gives a 2-sheet cover of E, and is the orientation covering by Theorem 3.4.4.10. □

3.5 Integration on manifolds

3.5.1 Integration of differential forms

In this part we will apply the integration theory on manifolds. In particular, we need the Lebesgue integration in $\mathbb{R}^n$. Let $D \subseteq \mathbb{R}^n$ be a measurable set, and let ω be a continuous n-form on $\overline{D}$. Any such form can be written as $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ for some continuous function $f : \overline{D} \to \mathbb{R}$. We define the **integral of ω over D** to be

$$\int_D \omega = \int_D f\, dV.$$

This can be written more suggestively as

$$\int_D f\, dx^1 \wedge \cdots \wedge dx^n = \int_D f\, dx^1 \cdots dx^n.$$

Somewhat more generally, let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and suppose ω is a compactly supported n-form on U. We define

$$\int_U \omega = \int_D \omega,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any measurable set containing $\operatorname{supp}(\omega)$, and ω is extended to be zero on the complement of its support. It is easy to check that this definition does not depend on what domain D is chosen.

Like the definition of the integral of a 1-form over an interval, our definition of the integral of an n-form might look like a trick of notation. The next proposition shows why it is natural.

Proposition 3.5.1.1. *Suppose U, V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : U \to V$ is an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then*

$$\int_V \omega = \pm \int_U G^* \omega.$$

with the positive sign if G is orientation-preserving, and the negative sign otherwise.

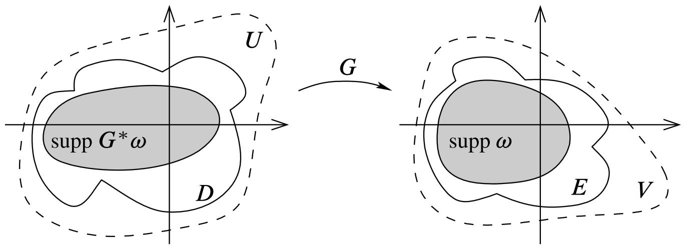


Figure 3.1: Diffeomorphism invariance of the integral of a form on an open subset.

Proof. Let us use $(y^1, \ldots, y^n)$ to denote standard coordinates on V, and $(x^1, \ldots, x^n)$ to denote those on U. Suppose first that G is orientation-preserving. With $\omega = f\, dy^1 \wedge \cdots \wedge dy^n$, the change of variables formula together with formula (3.3.2.1) for pullbacks of n-forms yields

$$\begin{aligned} \int_V \omega = \int_V f\, dV = \int_U (f \circ G) |\det \partial G|\, dV = \int_U (f \circ G)(\det \partial G)\, dV \\ = \int_U (f \circ G)(\det \partial G)\, dx^1 \wedge \cdots \wedge dx^n = \int_U G^* \omega. \end{aligned}$$

If G is orientation-reversing, the same computation holds except that a negative sign is introduced when the absolute value signs are removed. $\square$

3.5.1.1 Integration on manifolds

Let M be an oriented smooth nmanifold with or without boundary, and let ω be an n-form on M. Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented. We define the integral of ω over M to be

$$\int_M \omega = \pm \int_{\varphi(U)} \varphi_* \omega, \tag{3.5.1.1}$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. Since $\varphi_* \omega$ is a

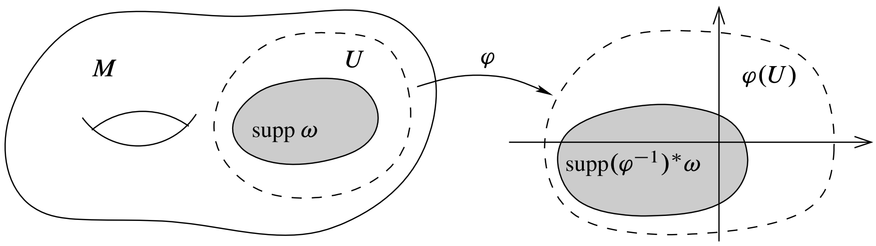


Figure 3.2: The integral of a form over a manifold.

compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as discussed above.

Proposition 3.5.1.2. *With ω as above, $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains* supp(ω)*.*

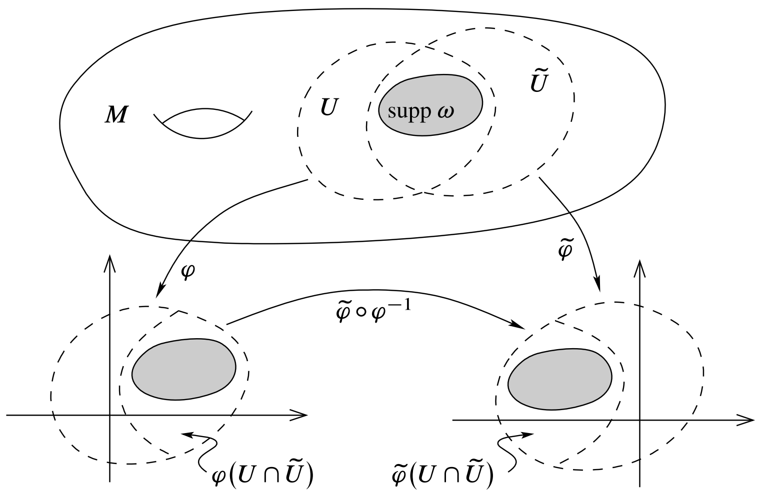


Figure 3.3: Coordinate independence of the integral.

Proof. Suppose (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$ are two smooth charts such that $\mathrm{supp}(\omega) \subseteq U \cap \widetilde{U}$. If both charts are positively oriented or both are negatively oriented, then $\widetilde{\varphi} \circ \varphi^{-1}$ is an orientation-preserving diffeomorphism from $\varphi(U \cap \widetilde{U})$ to $\widetilde{\varphi}(U \cap \widetilde{U})$, so Proposition 3.5.1.1 implies that

$$\begin{aligned}\int_{\widetilde{\varphi}(\widetilde{U})} \widetilde{\varphi}_* \omega &= \int_{\widetilde{\varphi}(U \cap \widetilde{U})} \widetilde{\varphi}_* \omega = \int_{\varphi(U \cap \widetilde{U})} (\widetilde{\varphi} \circ \varphi^{-1})^* \widetilde{\varphi}_* \omega \\ &= \int_{\varphi(U \cap \widetilde{U})} (\varphi^{-1})^* \circ \widetilde{\varphi}^* \circ \widetilde{\varphi}_* \omega = \int_{\varphi(U)} \varphi_* \omega.\end{aligned}$$

If the charts are oppositely oriented, then the two definitions given by (3.5.1.1) have opposite signs, but this is compensated by the fact that $\widetilde{\varphi} \circ \varphi^{-1}$ is orientationreversing, so Proposition 3.5.1.1 introduces an extra negative sign into the computation above. In either case, the two definitions of $\int_M \omega$ agree. □

To integrate over an entire manifold, we combine this definition with a partition of unity. Suppose M is an oriented smooth n-manifold with or without boundary, and ω is a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\mathrm{supp}(\omega)$ by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. Define the integral of ω over M to be

$$\int_M \omega = \sum_i \int_{U_i} \psi_i \omega. \tag{3.5.1.2}$$

(The reason we allow for negatively oriented charts is that it may not be possible to find positively oriented boundary charts on a 1-manifold with boundary, as noted in the proof of Proposition 3.4.2.4.) Since for each i, the n-form $\psi_i \omega$ is compactly supported in U_i, each of the terms in this sum is well defined according to our discussion above. To show that the integral is well defined, we need only examine the dependence on the open cover and the partition of unity.

Proposition 3.5.1.3. *The definition of $\int_M \omega$ given above does not depend on the choice of open cover or partition of unity.*

Proof. Suppose $\{\widetilde{U}_j\}$ is another finite open cover of $\mathrm{supp}(\omega)$ by domains of positively or negatively oriented smooth charts, and $\{\widetilde{\psi}_j\}$ is a subordinate smooth partition of unity. For each i, we compute

$$\int_M \psi_i \omega = \int_M \Big(\sum_j \widetilde{\psi}_j\Big) \psi_i \omega = \sum_j \int_M \widetilde{\psi}_j \psi_i \omega.$$

Summing over i, we obtain

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \widetilde{\psi}_j \psi_i \omega.$$

Observe that each term in this last sum is the integral of a form that is compactly supported in a single smooth chart (e.g., in U_i), so by Proposition 3.5.1.2 each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \widetilde{\psi}_j\omega$, shows that

$$\sum_j \int_M \widetilde{\psi}_j\omega = \sum_{ij} \widetilde{\psi}_j\psi_i\omega.$$

Thus, both definitions yield the same value for $\int_M \omega$. □

As usual, we have a special definition in the zero-dimensional case. The integral of a compactly supported 0-form (i.e., a real-valued function) f over an oriented 0-manifold M is defined to be the sum

$$\int_M f = \sum_{p\in M} \pm f(p).$$

where we take the positive sign at points where the orientation is positive and the negative sign at points where it is negative. The assumption that f is compactly supported implies that there are only finitely many nonzero terms in this sum.

If $S \subseteq M$ is an oriented immersed k-dimensional submanifold (with or without boundary), and ω is a k-form on M whose restriction to S is compactly supported, we interpret $\int_S \omega$ to mean $\int_S \iota_S^*\omega$, where $\iota_S : S \hookrightarrow M$ is inclusion. In particular, if M is a compact, oriented, smooth n-manifold with boundary and ω is an $(n-1)$-form on M, we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $\iota_{\partial M}^*\omega$ over ∂M, where ∂M is always understood to have the induced orientation.

Remark 3.5.1.4. It is worth remarking that it is possible to extend the definition of the integral to some noncompactly supported forms, and such integrals are important in many applications. However, in such cases the resulting multiple integrals are improper, so one must pay close attention to convergence issues. For the purposes we have in mind, the cases we have described here are quite sufficient.

Proposition 3.5.1.5 (Properties of Integrals of Forms). *Suppose M and N are nonempty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.*

(a) Linearity: If $a, b \in \mathbb{R}$, then
$$\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta.$$

(b) Orientation reversal: If $-M$ denotes M with the opposite orientation, then
$$\int_{-M} \omega = -\int_M \omega.$$

(c) Positivity: If ω is a positively oriented orientation form, then $\int_M \omega > 0$.

(d) Diffeomorphism invariance: If $F : M \to N$ is an orientation-preserving or orientation-reversing diffeomorphism, then
$$\int_M F^*\omega = \begin{cases} \displaystyle\int_N \omega & \text{if } F \text{ is orientation-preserving,} \\ \displaystyle -\int_N \omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Proof. Suppose ω is a positively oriented orientation form for M. This means that if (U, φ) is a positively oriented smooth chart, then $\varphi_*\omega$ is a positive function times $dx^1 \wedge \cdots \wedge dx^n$, and for a negatively oriented chart it is a negative function times the same form. Therefore, each term in the sum defining $\int_M \omega$ is nonnegative, with at least one strictly positive term, thus proving (c).

To prove (d), it suffices to assume that ω is compactly supported in a single positively or negatively oriented smooth chart, because any compactly supported n-form on N can be written as a finite sum of such forms by means of a partition of unity. Thus, suppose (U, φ) is a positively oriented smooth chart on N whose domain contains the support of ω. When F is orientation-preserving, it is easy to check that $(F^{-1}(U), \varphi \circ F)$ is an oriented smooth chart on M whose domain contains the support of $F^*\omega$, and the result then follows immediately from Proposition 3.5.1.2. The cases in which the chart is negatively oriented or F is orientation-reversing then follow from this result together with (b). □

Although the definition of the integral of a form based on partitions of unity is very convenient for theoretical purposes, it is useless for doing actual computations. It is generally quite difficult to write down a smooth partition of unity explicitly, and even when one can be written down, one would have to be exceptionally lucky to be able to compute the resulting integrals.

For computational purposes, it is much more convenient to *chop up* the manifold into a finite number of pieces whose boundaries are sets of measure zero, and compute the integral on each piece separately by means of local parametrizations. One way to do this is described below.

Proposition 3.5.1.6 (Integration Over Parametrizations). *Let M be an oriented smooth n-manifold with or without boundary, and ω be a compactly supported n-form on M. Suppose that $D_1, \ldots, D_k$ are open sets in $\mathbb{R}^n$, and for $1 \leq i \leq k$, we are given smooth maps $F_i : D_i \to M$ satisfying*

(i) *F_i restricts to an orientation-preserving diffeomorphism from D_i onto an open subset $W_i \subseteq M$.*

(ii) *$D_i \cap D_j = \varnothing$ whenever $i \neq j$.*

(iii) *∂D_i has Lebesgue measure zero for each i.*

(iv) *$\operatorname{supp}(\omega) \subseteq \bigcup_{i=1}^k \overline{W}_i$.*

Then

$$\int_M \omega = \sum_{i=1}^k \int_{D_i} F_i^* \omega. \tag{3.5.1.3}$$

Proof. As in the preceding proof, it suffices to assume that ω is supported in the domain of a single oriented smooth chart (U, φ). In fact, by restricting to sufficiently nice charts, we may assume that U is precompact, $Y = \varphi(U)$ is open in $\mathbb{R}^n$ or $\mathbb{H}^n$, and φ extends to a diffeomorphism from $\overline{U}$ to $\overline{Y}$.

For each i, define open subsets $A_i \subseteq D_i, B_i \subseteq W_i$, and $C_i \subseteq Y$ by

$$A_i = F_i^{-1}(U \cap W_i), \quad B_i = U \cap W_i = F_i(A_i), \quad C_i = \varphi(B_i)\varphi \circ F_i(A_i).$$

Because $\overline{D}_i$ is compact, it is straightforward to check that $\partial W_i \subseteq F_i(\partial D_i)$, and therefore ∂W_i has measure

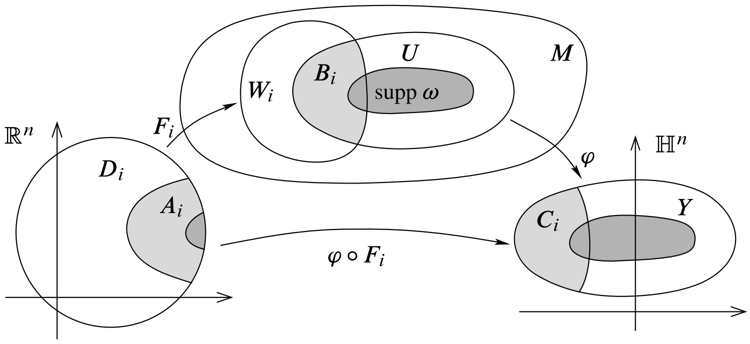


Figure 3.4: Integrating over parametrizations.

zero in M, and $\partial C_i = \varphi(\partial B_i)$ has measure zero in $\mathbb{R}^n$.

The support of $\varphi_* \omega$ is contained in $\bigcup_{i=1}^k \overline{C}_i$, and any two of these sets intersect only on their boundaries, which have measure zero. Thus

$$\int_M \omega = \int_Y \varphi_* \omega = \sum_{i=1}^k \int_{C_i} \varphi_* \omega.$$

The proof is completed by applying Proposition 3.5.1.1 to each term above, using the diffeomorphism $\varphi \circ F_i : A_i \to C_i$:

$$\int_{C_i} \varphi_* \omega = \int_{A_i} (\varphi \circ F_i)^* \varphi_* \omega = \int_{A_i} F_i^* \omega = \int_{D_i} F_i^* \omega.$$

Summing over i, we obtain (3.5.1.3). □

Example 3.5.1.7. Let us use this technique to compute the integral of a 2-form over S^2, oriented as the boundary of $\overline{B}^3$. Let ω be the following 2-form on $\mathbb{R}^3$

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

Let D be the open rectangle $(0, \pi) \times (0, 2\pi)$, and let $F : D \to S^2$ be the spherical coordinate parametrization $F(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\theta)$. Example 3.4.2.22 showed that $F|_D$ is orientation-preserving, so it satisfies the hypotheses of Proposition 3.5.1.6. Note that

$$\begin{aligned}
F^*dx &= \cos\varphi\cos\theta\,d\varphi - \sin\varphi\sin\theta\,d\theta,\\
F^*dy &= \cos\varphi\sin\theta\,d\varphi + \sin\varphi\cos\theta\,d\theta,\\
F^*dz &= -\sin\varphi\,d\varphi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{S^2}\omega &= \int_D \big(-\sin^3\varphi\cos^2\theta\,d\theta\wedge d\varphi + \sin^3\varphi\sin^2\theta\,d\varphi\wedge d\theta\\
&\qquad + \cos^3\varphi\sin\theta\cos^2\theta\,d\varphi\wedge d\theta - \cos^2\varphi\sin\varphi\sin^2\theta\,d\theta\wedge d\varphi\big)\\
&= \int_D \sin\varphi\,d\varphi\wedge d\theta = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi d\theta = 4\pi.
\end{aligned}$$

Remark 3.5.1.8. It is worth remarking that the hypotheses of Proposition 3.5.1.6 can be relaxed somewhat. The requirement that each map F_i be smooth on $\overline{D}_i$ is included to ensure that the boundaries of the image sets W_i have measure zero and that the pullback forms $F^*\omega$ are continuous on $\overline{D}_i$. Provided the open subsets W_i together fill up all of M except for a set of measure zero, we can allow maps F_i that do not extend smoothly to the boundary, by interpreting the resulting integrals of unbounded forms either as improper Riemann integrals or as Lebesgue integrals. For example, if the closed upper hemisphere of S^2 is parametrized by the map $F : \overline{B}^2 \to S^2$ given by $F(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, then F is continuous but not smooth up to the boundary, but the conclusion of the proposition still holds.

3.5.1.2 Integration on Lie groups

Let G be a Lie group. A covariant tensor field A on G is said to be **left-invariant** if $L_g^* A = A$ for all $g \in G$.

Proposition 3.5.1.9. *Let G be a compact Lie group endowed with a left-invariant orientation. Then G has a unique positively oriented left-invariant n-form ω_G with the property that $\int_G \omega_G = 1$.*

Proof. If $\dim G = 0$, we just let ω_G be the constant function $1/k$, where k is the cardinality of G. Otherwise, let $E_1, \ldots, E_n$ be a left-invariant global frame on G (i.e., a basis for the Lie algebra of G). By replacing E_1 with $-E_1$ if necessary, we may assume that this frame is positively oriented. Let $\varepsilon^1, \ldots, \varepsilon^n$ be the dual coframe. Left invariance of E_j implies that

$$(L_g^*)(E_j) = \varepsilon^i\big(d(L_g)(E_j)\big) = \varepsilon^i(E_j) = \delta^i_j,$$

which shows that $L_g^*\varepsilon^i = \varepsilon^i$, so ε^i is left-invariant.

Let $\omega_G = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. Then

$$L_g^*\omega_G = L_g^*\varepsilon^1 \wedge \cdots \wedge L_g^*\varepsilon^n = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \omega_G,$$

so ω_G is left-invariant as well. Because $\omega_G(E_1, \ldots, E_n) > 0$, ω_G is an orientation form for the given orientation. Clearly, any positive constant multiple of ω_G is also a left-invariant orientation form. Conversely, if $\widetilde{\omega}_G$ is any other left-invariant orientation form, we can write $\widetilde{\omega}_G|_e = c\omega_G|_e$ for some positive number c. Using left-invariance, we find that

$$\widetilde{\omega}_G|_g = L^*_{g^{-1}}\widetilde{\omega}_G|_e = cL^*_{g^{-1}}\omega_G|_e = c\omega_G|_g,$$

which proves that $\widetilde{\omega}_G$ is a positive constant multiple of ω_G.

Since G is compact and oriented, $\int_G \omega_G$ is a positive real number, so we can define $\widetilde{\omega}_G = (\int_G \omega_G)^{-1}\omega_G$. Clearly, $\widetilde{\omega}_G$ is the unique positively oriented left-invariant orientation form with integral 1. ☐

Remark 3.5.1.10. The orientation form whose existence is asserted in this proposition is called the **Haar volume form** on G. Similarly, the map $f \mapsto \int_G f\omega_G$ is called the **Haar integral**. Observe that the proof above did not use the fact that G was compact until the last paragraph; thus every Lie group has a left-invariant orientation form that is uniquely defined up to a constant multiple. It is only in the compact case, however, that we can use the volume normalization to single out a unique one.

3.5.2 Stokes's theorem

Theorem 3.5.2.1 (Stokes's Theorem). *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{3.5.2.1}$$

Remark 3.5.2.2. The statement of this theorem is concise and elegant, but it requires a bit of interpretation. First, as usual, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota_{\partial M}^{*}\omega$. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as zero. When M is 1-dimensional, the right-hand integral is really just a finite sum.

Proof. We begin with a very special case: suppose M is the upper half-space $\mathbb{H}^n$ itself. Then because ω has compact support, there is a number $R > 0$ such that $\operatorname{supp}(\omega)$ is contained in the rectangle $A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$. We can write ω in standard coordinates as

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x_j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus we compute

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n} \int_A (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^1 \cdots dx^n. \end{aligned}$$

We can change the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$\begin{aligned} &\sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^1 \cdots dx^n \\ &\qquad = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &\qquad = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \Big[\omega_i(x)\Big]\Big|_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0. \end{aligned}$$

because we have chosen R large enough that $\omega = 0$ when $x^i = R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$\int_{\mathbb{H}^n} d\omega = (-1)^{n-1} \int_{-R}^R \int_{-R}^R \cdots \int_0^R \frac{\partial \omega_n}{\partial x^n}(x) \, dx^n dx^1 \cdots dx^{n-1}$$

$$\begin{aligned} &= (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \Big[\omega_n(x)\Big]\Big|_{x^n=0}^{x^n=R} dx^n dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x^1,\dots,x^{n-1},0)\, dx^1 \wedge \cdots \wedge dx^{n-1}. \end{aligned} \tag{3.5.2.2}$$

because $\omega_n = R$ when $x^n = R$.

To compare this to the other side of (3.5.2.1), we compute as follows:

$$\int_{\partial \mathbb{H}^n} \omega = \sum_i \int_{A \cap \partial \mathbb{H}^n} \omega_i(x^1,\dots,x^{n-1},0)\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Because x^n vanishes on $\partial \mathbb{H}^n$, the pullback of dx^n to the boundary is identically zero. Thus, the only term above that is nonzero is the one for which $i = n$, which becomes

$$\int_{\partial \mathbb{H}^n} \omega = \int_{A \cap \partial \mathbb{H}^n} \omega_n(x^1,\dots,x^{n-1},0)\, dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Taking into account the fact that the coordinates $(x^1,\dots,x^{n-1})$ are positively oriented for $\partial \mathbb{H}^n$ when n is even and negatively oriented when n is odd (Example 3.4.2.20), we find that this is equal to (3.5.2.2).

Next we consider another special case: $M = \mathbb{R}^n$. In this case, the support of ω is contained in a cube of the form $A = [-R, R]^n$. Exactly the same computation goes through, except that in this case the $i = n$ term vanishes like all the others, so the left-hand side of (3.5.2.1) is zero. Since M has empty boundary in this case, the right-hand side is zero as well.

Now let M be an arbitrary smooth manifold with boundary, but consider an $(n-1)$-form ω that is compactly supported in the domain of a single positively or negatively oriented smooth chart (U, φ). Assuming that φ is a positively oriented boundary chart, the definition yields

$$\int_M d\omega = \int_{\mathbb{H}^n} \varphi_* d\omega = \int_{\mathbb{H}^n} d(\varphi_* \omega).$$

By the computation above, this is equal to

$$\int_{\partial \mathbb{H}^n} \varphi_* \omega. \tag{3.5.2.3}$$

where $\partial \mathbb{H}^n$ is given the induced orientation. Since $d\varphi$ takes outward-pointing vectors on ∂M to outward-pointing vectors on $\mathbb{H}^n$ (by Proposition **??**), it follows that $\varphi|_{U \cap \partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U) \cap \partial \mathbb{H}^n$, and thus (3.5.2.3) is equal to $\int_{\partial M} \omega$. For a negatively oriented smooth boundary chart, the same argument applies with an additional negative sign on each side of the equation. For an interior chart, we get the the same computations with $\mathbb{H}^n$ replaced by $\mathbb{R}^n$. This proves the theorem in this case.

Finally, let ω be an arbitrary compactly supported smooth $(n-1)$-form. Choosing a cover of $\operatorname{supp}(\omega)$ by finitely many domains of positively or negatively oriented smooth charts $\{U_i\}$, and choosing a subordinate smooth partition of unity $\{\psi_i\}$, we can apply the preceding argument to $\psi_i \omega$ for each i and obtain

$$\begin{aligned} \int_{\partial M} \omega &= \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i\, \omega + \psi_i\, d\omega \\ &= \int_M d\Big(\sum_i \psi_i\Big)\omega + \int_M \Big(\sum_i \psi_i\Big) d\omega = \int_M d\omega. \end{aligned}$$

because $\sum_i \psi_i \equiv 1$. □

Example 3.5.2.3. Let M be a smooth manifold and suppose $\gamma : [a, b] \to M$ is a smooth embedding, so that $S = \gamma([a, b])$ is an embedded 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(M)$, Stokes's theorem says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* f = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)).$$

Two special cases of Stokes's theorem arise so frequently that they are worthy of special note. The proofs are immediate.

Corollary 3.5.2.4 (Integrals of Exact Forms). *If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:*

$$\int_M d\omega = 0 \quad \text{if } \partial M = \varnothing.$$

Corollary 3.5.2.5 (Integrals of Closed Forms over Boundaries). *Suppose M is a compact oriented smooth manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:*

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0.$$

These results have the following extremely useful applications to submanifolds.

Corollary 3.5.2.6. *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an oriented compact smooth k-dimensional submanifold (without boundary), and ω is a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:*

(a) ω is not exact on M.

(b) S is not the boundary of an oriented compact smooth submanifold with boundary in M.

Example 3.5.2.7. It follows from the computation of Example **??** that the closed 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

has nonzero integral over S^1. We already observed that ω is not exact on $\mathbb{R}^2 - \{0\}$. The preceding corollary tells us in addition that S^1 is not the boundary of a compact regular domain in $\mathbb{R}^2 - \{0\}$.

The following classical result is an easy application of Stokes's theorem.

Theorem 3.5.2.8 (Green's Theorem). *Suppose D is a compact regular domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P\,dx + Q\,dy.$$

3.5.3 Manifolds with corners

In many applications of Stokes's theorem it is necessary to deal with geometric objects such as triangles, squares, or cubes that are topological manifolds with boundary, but are not smooth manifolds with boundary because they have *corners*. It is easy to generalize Stokes's theorem to this setting, and we do so in this section.

Let $\overline{\mathbb{R}}^n_+$ denote the subset of $\mathbb{R}^n$ where all of the coordinates are nonnegative:

$$\overline{\mathbb{R}}^n_+ = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 \geq 0, \ldots, x^n \geq 0\}.$$

This space is the model for the type of corners we are concerned with.

Proposition 3.5.3.1. *The set $\overline{\mathbb{R}}^n_+$ is homeomorphic to the upper half-space $\mathbb{H}^n$.*

Proof. The homeomorphism $\varphi : \overline{\mathbb{R}}^n_+ \to \mathbb{H}^n$ is given by

$$\varphi(x^1, \ldots, x^n) = (e^{x^1}, \ldots, e^{x^{n-1}}, x^n).$$

□

Suppose M is a topological n-manifold with boundary. A **chart with corners** for M is a pair (U, φ), where $U \subseteq M$ is open and φ is a homeomorphism from U to a (relatively) open subset $\widehat{U} \subseteq \overline{\mathbb{R}}^n_+$. Two charts with corners (U, φ), (V, ψ) are smoothly compatible if the composite map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. (As usual, this means that it admits a smooth extension in an open neighborhood of each point.)

A **smooth structure** with corners on a topological manifold with boundary is a maximal collection of smoothly compatible interior charts and charts with corners whose domains cover M. A topological manifold with boundary together with a smooth structure with corners is called a **smooth manifold with corners**. Any chart with corners in the given smooth structure with corners is called a **smooth chart with corners** for M.

Example 3.5.3.2. Any closed rectangle in $\mathbb{R}^n$ is a smooth n-manifold with corners.

The **boundary** of $\overline{\mathbb{R}}^n_+$ in $\mathbb{R}^n$ is the set of points at which at least one coordinate vanishes. The points in $\overline{\mathbb{R}}^n_+$ at which more than one coordinate vanishes are called its **corner points**. For example, the corner points of $\overline{\mathbb{R}}^3_+$ are the origin together with all the points on the positive x-, y-, and z-axes.

Proposition 3.5.3.3 (Invariance of Corner Points). *Let M be a smooth n-manifold with corners, $n \geq 2$, and let $p \in M$. If $\varphi(p)$ is a corner point for some smooth chart with corners (U, φ), then the same is true for every such chart whose domain contains p.*

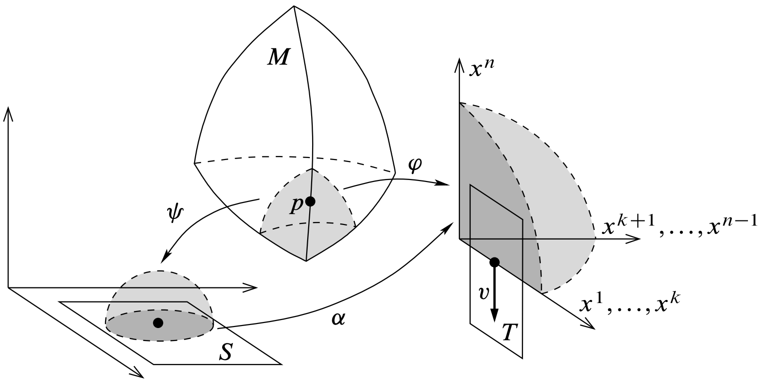


Figure 3.1: Invariance of corner points.

Proof. Suppose (U, φ) and (V, ψ) are two smooth charts with corners such that $\varphi(p)$ is a corner point but $\psi(p)$ is not. To simplify notation, let us assume without loss of generality that $\varphi(p)$ has coordinates $(x^1, \ldots, x^k, 0, \ldots, 0)$ with $k \leq n-2$. Then $\psi(V)$ contains an open subset of some $(n-1)$-dimensional linear subspace $S \subseteq \mathbb{R}^n$, with $\psi(p) \in S$. (If $\psi(p) \in \partial\overline{\mathbb{R}}^n_+$, take S to be the unique subspace defined by an equation of the form $x^i = 0$ that contains $\psi(p)$. If $\psi(p)$ is an interior point, any $(n-1)$-dimensional subspace containing $\psi(p)$ will do.)

Let $S' = S \cap \psi(U \cap V)$, and let $\alpha : S' \to \mathbb{R}^n$ be the restriction of $\varphi \circ \psi^{-1}$ to S'. Because $\varphi \circ \psi^{-1}$ is a diffeomorphism from $\psi(U \cap V)$ to $\varphi(U \cap V)$, it follows that $\psi \circ \varphi^{-1} \circ \alpha$ is the identity of S', and therefore $d\alpha_{\psi(p)}$ is an injective linear map. Let $T = d\alpha_{\psi(p)}(T_{\psi(p)}S) \subseteq \mathbb{R}^n$. Because T is $(n-1)$-dimensional, it must contain a vector v such that one of the last two components, v^{n-1} or v^n, is nonzero (otherwise, T would be contained in a codimension-2 subspace). Renumbering the coordinates and replacing v by $-v$ if necessary, we may assume that $v^n < 0$.

Now let $\gamma : (-\varepsilon, \varepsilon) \to S$ be a smooth curve such that $\gamma(0) = p$ and $d\alpha(\gamma'(0)) = v$. Then $\alpha \circ \gamma(t)$ has negative x^n coordinate for small $t > 0$, which contradicts the fact that α takes its values in $\overline{\mathbb{R}}^n_+$. $\square$

If M is a smooth manifold with corners, a point $p \in M$ is called a corner point if $\varphi(p)$ is a corner point in $\overline{\mathbb{R}}^n_+$ with respect to some (and hence every) smooth chart with corners $(U, \varphi(p))$. Similarly, p is called a boundary point if $\varphi(p) \in \partial\overline{\mathbb{R}}^n_+$ with respect to some (hence every) such chart. For example, the set of corner points of the unit cube $[0,1]^3 \subseteq \mathbb{R}^3$ is the union of its eight vertices and twelve edges.

Every smooth manifold with or without boundary is also a smooth manifold with corners (but with no corner points). Conversely, a smooth manifold with corners is a smooth manifold with boundary if and only if it has no corner points. The boundary of a smooth manifold with corners, however, is in general not a smooth manifold with corners (e.g., think of the boundary of a cube). In fact, even the boundary of $\overline{\mathbb{R}}^n_+$ itself is not a smooth manifold with corners. It is, however, a union of finitely many such: $\partial\overline{\mathbb{R}}^n_+ = H_1 \cup \cdots \cup H_n$, where

$$H_i = \{(x^1, \ldots, x^n) \in \overline{\mathbb{R}}^n_+ : x^i = 0\} \tag{3.5.3.1}$$

is an $(n-1)$-dimensional smooth manifold with corners contained in the subspace defined by $x^i = 0$.

The usual flora and fauna of smooth manifoldssmooth maps, partitions of unity, tangent vectors, covectors, tensors, differential forms, orientations, and integrals of differential formscan be defined on smooth manifolds with corners in exactly the same way as we have done for smooth manifolds and smooth manifolds with boundary, using smooth charts with corners in place of smooth boundary charts.

In addition, for Stokes's theorem we need to integrate a differential form over the boundary of a smooth manifold with corners. Since the boundary is not itself a smooth manifold with corners, this requires a separate (albeit routine) definition.

Let M be an oriented smooth n-manifold with corners, and suppose ω is an $(n-1)$-form on ∂M that is compactly supported in the domain of a single chart with corners (U, φ). We define the integral of ω over ∂M by

$$\int_{\partial M} \omega = \sum_{i=1}^{n} \int_{H_i} \varphi_* \omega,$$

where H_i, defined by (3.5.3.1), is given the induced orientation as part of the boundary of the set where $x^i \geq 0$. In other words, we simply integrate ω in coordinates over the codimension-1 portion of the boundary.

Finally, if ω is an arbitrary compactly supported $(n-1)$-form on M, we define the integral of ω over ∂M by piecing together with a partition of unity just as in the case of a manifold with boundary.

In practice, of course, one does not evaluate such integrals by using partitions of unity. Instead, one chops up the boundary into pieces that can be parametrized by open subsets, just as for ordinary manifolds with or without boundary. The following proposition is an analogue of Proposition 3.5.1.6.

Proposition 3.5.3.4. *The statement of Proposition 3.5.1.6 is true if M is replaced by the boundary of a oriented smooth $(n+1)$-manifold with corners.*

Example 3.5.3.5. Let $I \times I = [0,1]^2 \subseteq \mathbb{R}^2$ be the unit square in $\mathbb{R}^2$, and suppose ω is a 1-form on $\partial(I \times I)$. Then it is not hard to check that the maps $F_i : I \to I \times I$ given by

$$\begin{aligned} F_1(t) &= (t, 0), & F_2(t) &= (1, t), \\ F_3(t) &= (1-t, 1), & F_4(t) &= (0, -t). \end{aligned} \tag{3.5.3.2}$$

satisfy the hypotheses of Proposition 3.5.1.6. Therefore,

$$\int_{\partial(I \times I)} \omega = \int_{F_1} \omega + \int_{F_2} \omega + \int_{F_3} \omega + \int_{F_4} \omega.$$

Theorem 3.5.3.6 (Stokes's Theorem on Manifolds with Corners). *Let M be an oriented smooth n-manifold with corners, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. The proof is nearly identical to the proof of Stokes's theorem proper, so we just indicate where changes need to be made. By means of smooth charts and a partition of unity, we may reduce the theorem to the case in which either $M = \mathbb{R}^n, \mathbb{H}^n$ or $M = \overline{\mathbb{R}}^n_+$. The $\mathbb{R}^n$ and $\mathbb{H}^n$ case are treated just as before. In the case of a chart with corners, ω is supported in some cube $[0, R]^n$, and we calculate exactly as in the proof of Theorem 3.5.2.1:

$$\begin{aligned} \int_{\overline{\mathbb{R}}^n_+} d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^1 \cdots dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \cdots \int_0^R \Big[\omega_i(x)\Big]\Big|_{x^i=0}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^{n} (-1)^{i} \int_0^R \cdots \int_0^R \omega_i(x^1, \dots, 0, \dots, x^n)\, dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^{n} \int_{H_i} \omega = \int_{\partial \overline{\mathbb{R}}^n_+} \omega. \end{aligned}$$

(The factor $(-1)^i$ disappeared because the induced orientation on H_i is $(-1)^i$ times that of the standard coordinates $(x^1, \dots, \widehat{x^i}, \dots, x^n)$) This completes the proof. ∎

The preceding theorem has the following important application.

Theorem 3.5.3.7. *Suppose M is a smooth manifold and $\gamma_0, \gamma_1 : [a,b] \to M$ are path-homotopic piecewise smooth curve segments. For every closed 1-form ω on M,*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. By means of an affine reparametrization, we may as well assume for simplicity that $[a,b] = [0,1]$. Assume first that γ_0 and γ_1 are smooth. By Theorem **??**, γ_0 and γ_1 are smoothly homotopic relative to $\{0,1\}$. Let $H : I \times I \to M$ be such a smooth homotopy. Since ω is closed, we have

$$\int_{I\times I} d(H^*\omega) = \int_{I\times I} H^* d\omega = 0.$$

On the other hand, $I \times I$ is a smooth manifold with corners, so Stokes's theorem implies

$$0 = \int_{I\times I} d(H^*\omega) = \int_{\partial(I\times I)} H^*\omega.$$

Using the parametrization of $\partial(I \times I)$ given in Example 3.5.3.5 together with Proposition **??**, we obtain

$$\begin{aligned} 0 &= \int_{\partial(I\times I)} H^*\omega = \int_{\partial(I\times I)} H^*\omega = \int_{F_1} H^*\omega + \int_{F_2} H^*\omega + \int_{F_3} H^*\omega + \int_{F_4} H^*\omega \\ &= \int_{H\circ F_1} \omega + \int_{H\circ F_2} \omega + \int_{H\circ F_3} \omega + \int_{H\circ F_4} \omega. \end{aligned}$$

The fact that H is a homotopy relative to $\{0,1\}$ means that $H \circ F_2$ and $H \circ F_4$ are constant maps, and therefore the second and fourth terms above are zero. The theorem then follows from the facts that $H \circ F_1 = \gamma_0$ and $H \circ F_3$ is a backward reparametrization of γ_1.

Next we consider the general case of piecewise smooth curves.We cannot simply apply the preceding result on each subinterval where γ_0 and γ_1 are smooth, because the restricted curves may not start and end at the same points. Instead, we prove the following more general claim: Let $\gamma_0, \gamma_1 : [0,1] \to M$ be piecewise smooth curve segments (not necessarily with the same endpoints), and suppose $H : I \times I \to M$ is any homotopy between them. Define curve segments $\sigma_0, \sigma_1 : I \to M$ by

$$\sigma_0(t) = H(0,t), \quad \sigma_1(t) = H(1,t),$$

and let $\widetilde{\sigma}_0, \widetilde{\sigma}_1$ be any smooth curve segments that are path-homotopic to σ_0, σ_1 respectively. Then

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = \int_{\widetilde{\sigma}_1} \omega - \int_{\widetilde{\sigma}_0} \omega. \tag{3.5.3.3}$$

When specialized to the case in which γ_0 and γ_1 are path-homotopic, this implies the theorem, because γ_0 and γ_1 are constant maps in that case.

Since γ_0 and γ_1 are piecewise smooth, there are only finitely many points $\{a_1, \ldots, a_m\}$ in $[0,1]$ at which either γ_0 or γ_1 is not smooth. We prove the claim by induction on the number m of such points. When $m = 0$, both curves are smooth, and by Theorem **??** we may replace the given homotopy H by a smooth homotopy $\widetilde{H}$. Recall from the proof of Theorem **??** that the smooth homotopy $\widetilde{H}$ can actually be taken to be homotopic to H relative to $I \times \{0\} \cup I \times \{1\}$. Thus, for $i = 0,1$, the curve $\widetilde{\sigma}_i = \widetilde{H}(i,t)$ is a smooth curve segment that is path-homotopic to σ_i. In this setting, (3.5.3.3) just reduces to the integration formula of Example 3.5.3.5. Note that the integrals over $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ do not depend on which smooth curves pathhomotopic to σ_0 and σ_1 are chosen, by the smooth case proved above.

Now let γ_0, γ_1 be homotopic piecewise smooth curves with m nonsmooth points $\{a_1, \ldots, a_m\}$, and suppose the claim is true for curves with fewer than m such points. For $i = 0,1$, let γ_i' be the restriction of γ_i to $[0, a_m]$, and let γ_i'' be its restriction to $[a_m, 1]$. Let $\sigma : I \to M$ be the curve segment $\sigma(t) = H(a_m, t)$, and let $\widetilde{\sigma}$ by any smooth curve segment that is path-homotopic to σ. Then, since γ_i' and γ_i'' have fewer than m nonsmooth points, the inductive hypothesis implies

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = \Big(\int_{\gamma_1'} \omega + \int_{\gamma_1''} \omega \Big) - \Big(\int_{\gamma_0'} \omega + \int_{\gamma_0''} \omega \Big)$$

$$
\begin{aligned}
&= \Big(\int_{\gamma_1'} \omega - \int_{\gamma_0'} \omega \Big) + \Big(\int_{\gamma_1''} \omega + \int_{\gamma_1''} \omega \Big) \\
&= \Big(\int_{\widetilde{\sigma}} \omega - \int_{\widetilde{\sigma}_0} \omega \Big) + \Big(\int_{\widetilde{\sigma}_1} \omega + \int_{\widetilde{\sigma}} \omega \Big) \\
&= \int_{\widetilde{\sigma}_1} \omega - \int_{\widetilde{\sigma}_0} \omega.
\end{aligned}
$$

□

Corollary 3.5.3.8. *On a simply connected smooth manifold, every closed 1-form is exact.*

Proof. Suppose M is simply connected and ω is a closed 1-form on M. Since every piecewise smooth closed curve segment in M is path-homotopic to a constant curve, the preceding theorem shows that the integral of ω over every such curve is equal to 0. Thus, ω is conservative and therefore exact. □

3.5.4 Integration on Riemannian manifolds

3.5.4.1 Integration of functions on Riemannian manifolds

Suppose (M, g) is an oriented Riemannian manifold with or without boundary, and let ω_g denote its Riemannian volume form. If f is a compactly supported continuous real-valued function on M, then $f\omega_g$ is a compactly supported n-form, so we can define the **integral of f over M** to be $\int_M f\omega_g$. If M itself is compact, we define the **volume of M** by $\mathrm{Vol}(M) = \int_M \omega_g$.

Because of these definitions, the Riemannian volume form is often denoted by dV_g (or dA_g or ds_g in the 2-dimensional or 1-dimensional case, respectively). Then the integral of f over M is written $\int_M f\, dV_g$, and the volume of M as $\int_M dV_g$. Be warned, however, that this notation is not meant to imply that the volume form is the exterior derivative of an $(n-1)$-form; in fact, as we will see when we study de Rham cohomology, this is never the case on a compact manifold. You should just interpret dV_g as a notational convenience.

Proposition 3.5.4.1. *Let (M, g) be a nonempty oriented Riemannian manifold with or without boundary, and suppose f is a compactly supported continuous realvalued function on M satisfying $f \geq 0$. Then $\int_M f\, dV_g \geq 0$, with equality if and only if $f \equiv 0$.*

Proof. If f is supported in the domain of a single oriented smooth chart (U, φ), then Proposition 3.4.3.1 shows that

$$\int_M f\, dV_g = \int_M f\sqrt{G}\, dx^1 \cdots dx^n \geq 0.$$

The same inequality holds in a negatively oriented chart because the negative sign from the chart cancels the negative sign in the expression for dV_g. The general case follows from this one, because $\int_M f\, dV_g$ is equal to a sum of terms like $\int_M \psi f\, dV_g$, where each integrand if is nonnegative and supported in a single smooth chart. If in addition f is positive somewhere, then it is positive on a nonempty open subset by continuity, so at least one of the integrals in this sum is positive. On the other hand, if f is identically zero, then clearly $\int_M f\, dV_g = 0$. □

In the same way, we can prove the following result.

Proposition 3.5.4.2. *Suppose (M, g) is an oriented Riemannian manifold and $f : M \to \mathbb{R}$ is continuous and compactly supported. Then*

$$\Big| \int_M f\, dV_g \Big| \leq \int_M |f|\, dV_g.$$

3.5.4.2 The divergence theorem

Let (M, g) be an oriented Riemannian n-manifold (with or without boundary). We can generalize the classical divergence operator to this setting as follows. Multiplication by the Riemannian volume form defines a smooth bundle isomorphism $* : C^\infty(M) \to \Omega^n(M)$:

$$*f := f\, dV_g.$$

In addition, as we did in the case of $\mathbb{R}^3$, we define a smooth bundle isomorphism $\beta : \mathfrak{X}(M) \to \Omega^{n-1}(M)$ as follows:

$$\beta(X) = X \lrcorner dV_g. \tag{3.5.4.1}$$

We need the following technical lemma.

Lemma 3.5.4.3. *Let (M, g) be an oriented Riemannian manifold with or without boundary. Suppose $S \subseteq M$ is an immersed hypersurface with the orientation determined by a unit normal vector field N, and $\widetilde{g}$ is the induced metric on S. If X is any vector field along S, then*

$$\iota_S^*(\beta(X)) = \langle X, N\rangle_g dV_{\widetilde{g}}. \tag{3.5.4.2}$$

Proof. Define two vector fields X and X along S by

$$X^\perp = \langle X, N\rangle_g N, \quad X^\top = X - X^\perp.$$

Then $X = X^\perp + X^\top$, where $X^\perp$ is normal to S and $X^\top$ is tangent to it. Using this decomposition,

$$\beta(X) = X^\perp \lrcorner dV_g + X^\top \lrcorner dV_g.$$

Now pull back to S. Proposition 3.4.3.4 shows that the first term simplifies to

$$\iota_S^*(X^\perp \lrcorner dV_g) = \langle X, N\rangle_g \iota_S^*(N \lrcorner dV_g) = \langle X, N\rangle_g dV_{\widetilde{g}}.$$

Thus (3.5.4.2) will be proved if we can show that $\iota_S^*(X^\top \lrcorner dV_g) = 0$. If $X_1, \ldots, X_{n-1}$ are any vectors tangent to S, then

$$(X^\top \lrcorner dV_g)(X_1, \ldots, X_{n-1}) = dV_g(X, X_1, \ldots, X_{n-1}) = 0,$$

because any n-tuple of vectors in an $(n-1)$-dimensional vector space is linearly dependent. □

Now we define the **divergence operator** $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ by

$$\operatorname{div} X = *^{-1} d(\beta(X)) = *^{-1} d(X \lrcorner dV_g). \tag{3.5.4.3}$$

or equivalently,

$$d(X \lrcorner dV_g) = (\operatorname{div} X) dV_g. \tag{3.5.4.4}$$

Even if M is nonorientable, in a neighborhood of each point we can choose an orientation and define the divergence by (3.5.4.4), and then note that reversing the orientation changes the sign of dV_g on both sides of the equation, so $\operatorname{div} X$ is well defined, independently of the choice of orientation. In this way, we can define the divergence operator on any Riemannian manifold with or without boundary, by requiring that it satisfy (3.5.4.4) for any choice of orientation in a neighborhood of each point.

The next theorem is a fundamental result about vector fields on Riemannian manifolds. In the special case of a compact regular domain in $\mathbb{R}^3$, it is often referred to as **Gauss's theorem**.

Theorem 3.5.4.4 (The Divergence Theorem). *Let (M, g) be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,*

$$\int_M (\operatorname{div} X)\, dV_g = \int_{\partial M} \langle N, X\rangle_g \, dV_{\widetilde{g}}.$$

where N is the outward-pointing unit normal vector field along ∂M and $\widetilde{g}$ is the induced Riemannian metric on ∂M.

Proof. By Stokes's theorem,

$$\int_M (\operatorname{div} X)\, dV_g = \int_M d(\beta(X)) = \int_{\partial M} \iota_{\partial M}^* \beta(X).$$

The divergence theorem then follows from Lemma 3.5.4.3. □

The term divergence is used because of the following geometric interpretation. A smooth flow θ on M is said to be volume-preserving if for every compact regular domain D, we have $\mathrm{Vol}(\theta_t(D)) = \mathrm{Vol}(D)$ whenever the domain of θ_t contains D. It is called **volume-increasing**, **volume-decreasing**, **volume-nonincreasing**, or **volume-nondecreasing** if for every such D, $\mathrm{Vol}(\theta_t(D))$ is strictly increasing, strictly decreasing, nonincreasing, or nondecreasing, respectively, as a function of t. Note that the properties of flow domains ensure that if D is contained in the domain of θ_t for some t, then the same is true for all times between 0 and t.

The next proposition shows that the divergence of a vector field can be interpreted as a measure of the tendency of its flow to spread out.

Proposition 3.5.4.5 (Geometric Interpretation of the Divergence). *Let M be an oriented Riemannian manifold, let $X \in \mathfrak{X}(M)$, and let θ be the flow of X. Then θ is*

(a) volume-preserving if and only if $\mathrm{div} X = 0$ *everywhere on M.*

(b) volume-nondecreasing if and only if $\mathrm{div} X = 0$ *everywhere on M.*

(c) volume-nonincreasing if and only if $\mathrm{div} X \geq 0$ *everywhere on M.*

(d) volume-increasing if and only if $\mathrm{div} X > 0$ *on a dense subset of M.*

(e) volume-decreasing if and only if $\mathrm{div} X < 0$ *on a dense subset of M.*

Proof. First we establish some preliminary results. For each $t \in \mathbb{R}$, let M_t be the domain of θ_t. If D is a compact regular domain contained in M_t, then θ_t is an orientation-preserving diffeomorphism from D to $\theta_t(D)$ by the result of Exercise 3.4.4, so

$$\mathrm{Vol}(\theta_t(D)) = \int_{\theta_t(D)} dV_g = \int_D \theta_t^* dV_g.$$

Because the integrand on the right depends smoothly on (t, p) in the domain of θ, we can differentiate this expression with respect to t by differentiating under the integral sign.

Using Cartan's magic formula for the Lie derivative of the Riemannian volume form, we obtain

$$\mathfrak{L}_X dV_g = X \lrcorner\, d(dV_g) + d(X \lrcorner\, dV_g) = (\mathrm{div} X) dV_g.$$

because $d(dV_g)$ is an $(n+1)$-form on an n-manifold. Then Proposition 3.1.3.13 implies

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0} \mathrm{Vol}(\theta_t(D)) &= \int_D \frac{\partial}{\partial t}\Big|_{t=t_0} (\theta_t^* dV_g) = \int_D \theta_{t_0}^* (\mathfrak{L}_X dV_g) \\ &= \int_D \theta_{t_0}^* ((\mathrm{div} X) dV_g) = \int_{\theta_{t_0}(D)} (\mathrm{div} X) dV_g.\end{aligned}$$

From this, the claim is easily proved. □

Using the divergence operator, we can define another important operator. The **Laplacian** (or **Laplace-Beltrami operator**) is the linear operator $\Delta : C^\infty(M) \to C^\infty(M)$ defined by

$$\Delta f = \mathrm{div}(\mathrm{grad}\, f). \tag{3.5.4.5}$$

The next proposition gives alternative formulas for these operators.

Proposition 3.5.4.6. *Let (M, g) be a Riemannian manifold with or without boundary, and let (x^i) be any smooth local coordinates on an open set $U \subseteq M$. The coordinate representations of the divergence and Laplacian are as follows:*

$$\mathrm{div}(X^i \frac{\partial}{\partial x^i}) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G} X^i\right) \tag{3.5.4.6}$$

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x^j}\right). \tag{3.5.4.7}$$

where $G = \det(g_{ij})$ is the determinant of the component matrix of g in these coordinates.

Proof. By Proposition 3.4.3.3 2e have

$$X \lrcorner dV_g = X \lrcorner \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{G} \sum_{i=1}^{n} (-1)^{n-1} X^i \, dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Therefore

$$d(X \lrcorner dV_g) = \frac{\partial}{\partial x^i}\left(\sqrt{G} X^i\right) dx^1 \wedge \cdots dx^n = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G} X^i\right) dV_g,$$

and so (3.5.4.6) holds. The second formula follows from the expression of grad f. □

3.5.4.3 Surface integrals

The original theorem that bears the name of Stokes concerned surface integrals of vector fields over surfaces in $\mathbb{R}^3$. Using the version of Stokes's theorem that we have proved, we cam generalize this to surfaces in Riemannian 3-manifolds.

Let (M, g) be an oriented Riemannian 3-manifold. Define the curl operator, denoted by curl : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$\operatorname{curl} X = \beta^{-1} d(X^\flat),$$

where $\beta : \mathfrak{X}(M) \to \Omega^2(M)$ is defined in (3.5.4.1). Unwinding the definitions, we see that this is equivalent to

$$(\operatorname{curl} X) \lrcorner dV_g = d(X^\flat).$$

The operators div, grad, and curl on an oriented Riemannian 3-manifold M are related by the following commutative diagram

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{id}"] & \mathfrak{X}(M) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(M) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"] & C^\infty(M) \arrow[d, "*"] \\
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M)
\end{tikzcd}$$

The curl operator is defined only in dimension 3 because it is only in that case that $\bigwedge^2 T^*M$ is isomorphic to TM via the map β.

Now suppose $S \subseteq M$ is a compact 2-dimensional submanifold with or without boundary, and N is a smooth unit normal vector field along S. Let dA denote the Riemannian volume form on S with respect to the induced metric $\iota_S^* g$ and the orientation determined by N, so that $dA = \iota_S^*(N \lrcorner dV_g)$ by Proposition 3.4.3.4. For any smooth vector field X defined on M, the surface integral of X over S (with respect to the given choice of unit normal field) is defined as

$$\int_S \langle X, N \rangle_g \, dA.$$

The next result, in the special case in which $M = \mathbb{R}^3$, is the theorem usually referred to as Stokes's theorem in multivariable calculus texts.

Theorem 3.5.4.7 (Stokes's Theorem for Surface Integrals). *Suppose M is an oriented Riemannian 3-manifold with or without boundary, and S is a compact oriented 2-dimensional smooth submanifold with boundary in M. For any smooth vector field X on M,*

$$\int_S \langle \operatorname{curl} X, N \rangle_g \, dA = \int_{\partial S} \langle X, T \rangle_g \, ds,$$

where N is the smooth unit normal vector field along S that determines its orientation, ds is the Riemannian volume form for ∂S (with respect to the metric and orientation induced from S), and T is the unique positively oriented unit tangent vector field on ∂S.

Proof. By the defining of the curl and the result of Lemma 3.5.4.3, we get

$$\langle \operatorname{curl} X, N \rangle_g \, dA = \iota_S^* \beta(\operatorname{curl} X) = \iota_S^* d(X^\flat).$$

Thus we only need to show

$$\iota_{\partial S}^* X^\flat = \langle X, T\rangle_g ds.$$

To prove this, we note that $\iota_{\partial S}^* X^\flat$ is a smooth 1-form on a 1-manifold, and thus must be equal to f ds for some smooth function f on ∂S. To evaluate f, we note that $ds(T) = 1$, and so the definition of $X^\flat$ yields

$$f = f ds(T) = X^\flat(T) = \langle X, T\rangle_g.$$

Thus the general version of Stokes's theorem gives the claim. □

3.5.4.4 Densities

In the theory of integration of differential forms, the crucial place where orientations entered the picture was in our proof of the diffeomorphism-invariance of the integral (Proposition 3.5.1.1), because the transformation law for an n-form on an n-manifold under a change of coordinates involves the Jacobian determinant of the transition map, while the transformation law for integrals involves the absolute value of the determinant. We had to restrict attention to orientation-preserving diffeomorphisms so that we could freely remove the absolute value signs. In this part we define objects whose transformation law involves the absolute value of the determinant, so that we no longer have this sign problem.

We begin, as always, in the linear-algebraic setting. Let V be an n-dimensional vector space. A **density** on V is a function

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ folds}} \to \mathbb{R}$$

satisfying the following condition: if $T : V \to V$ is any linear map, then

$$\mu(Tv_1, \ldots, Tv_n) = |\det T|\mu(v_1, \ldots, v_n).$$

Observe that a density is not a tensor, because it is not linear over $\mathbb{R}$ in any of its arguments. Let $\mathfrak{D}(V)$ denote the set of all densities on V.

Proposition 3.5.4.8 (Properties of Densities). *Let V be a vector space of dimension n.*

(a) *$\mathfrak{D}(V)$ is a vector space under the obvious vector operations:*

$$(c_1\mu_1 + c_2\mu_2)(v_1, \ldots, v_n) = c_1\mu_1(v_1, \ldots, v_n) + c_2\mu_2(v_1, \ldots, v_n).$$

(b) *If $\mu_1, \mu_2 \in \mathfrak{D}(V)$ and $\mu_1(E_1, \ldots, E_n) = \mu_2(E_1, \ldots, E_n)$ for some basis (E_i) of V, then $\mu_1 = \mu_2$.*

(c) *If $\omega \in \bigwedge^n(V^*)$, the map $|\omega|$ defined by*

$$|\omega|(v_1, \ldots, v_n) = |\omega(v_1, \ldots, v_n)|$$

is a density.

(d) *$\mathfrak{D}(V)$ is 1-dimensional, spanned by $|\omega|$ for any nonzero $\omega \in \bigwedge^n(V^*)$.*

Proof. For part (b), suppose μ_1 and μ_2 give the same value when applied to $(E_1, \ldots, E_n)$. If $v_1, \ldots, v_n$ are arbitrary vectors in V, let $T : V \to V$ be the unique linear map that takes E_i to v_i for $i = 1, \ldots, n$. It follows that

$$\mu_1(v_1, \ldots, v_n) = |\det T|\mu_1(E_1, \ldots, E_n) = |\det T|\mu_2(E_1, \ldots, E_n) = \mu_2(v_1, \ldots, v_n).$$

Therefore $\mu_1 = \mu_2$.

Part (c) follows from Proposition 3.3.1.7. Finally, to prove (d), suppose ω is any nonzero element of $\bigwedge^n(V^*)$. If μ is an arbitrary element of $\mathfrak{D}(V)$, it suffices to show that $\mu = c|\omega|$ for some $c \in \mathbb{R}$. Let (E_i) be a basis for V, and define $a, b \in \mathbb{R}$ by

$$a = |\omega|(E_1, \ldots, E_n) = |\omega(E_1, \ldots, E_n)|, \quad b = \mu(E_1, \ldots, E_n).$$

Because $\omega \neq 0$, it follows that $a \neq 0$. Thus, μ and $(b/a)|\omega|$ give the same result when applied to $(E_1, \ldots, E_n)$, so they are equal by part (b). □

A **positive density** on V is a density μ satisfying $\mu(v_1,\dots,v_n) > 0$ whenever $(v_1,\dots,v_n)$ is a linearly independent n-tuple. A **negative density** is defined similarly. If ω is a nonzero element of $\bigwedge^n(V^*)$, then it is clear that $|\omega|$ is a positive density; more generally, a density $c|\omega|$ is positive, negative, or zero if and only if c has the same property. Thus, each density on V is either positive, negative, or zero, and the set of positive densities is a convex subset of $\mathfrak{D}$ (namely, a half-line).

Now let M be a smooth manifold with or without boundary. The set

$$\mathfrak{D}M = \coprod_{p\in M} \mathfrak{D}(T_pM)$$

is called the density bundle of M. Let $\pi : \mathfrak{D}M \to M$ be the natural projection map taking each element of $\mathfrak{D}(T_pM)$ to p.

Proposition 3.5.4.9. *If M is a smooth manifold with or without boundary, its density bundle is a smooth line bundle over M.*

Proof. We will construct local trivializations and use the vector bundle chart lemma (Lemma **??**). Let (U, x^i) be any smooth coordinate chart on M; and let $\omega = dx^1 \wedge \cdots \wedge dx^n$. Proposition 3.5.4.8 shows that $|\omega_p|$ is a basis for $\mathfrak{D}(T_pM)$ at each point $p \in U$. Therefore, the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}$ given by

$$\Phi(c|\omega_p|) = (p, c)$$

is a bijection.

Now suppose $(\widetilde{U}, \widetilde{x}^j)$ is another smooth chart with $U \cap \widetilde{U} \neq \varnothing$. Let $\widetilde{\omega} = d\widetilde{x}^1 \wedge \cdots \wedge d\widetilde{x}^n$, and define $\widetilde{\Phi} : \pi^{-1}(\widetilde{U} \to \widetilde{U} \times \mathbb{R})$ correspondingly:

$$\widetilde{\Phi}(c|\widetilde{\omega}_p|) = (p, c).$$

It follows from the transformation law (3.3.2.2) for n-forms under changes of coordinates that

$$\begin{aligned}\Phi \circ \widetilde{\Phi}^{-1}(p, c) = \Phi(c|\widetilde{\omega}_p|) &= \Phi\Big(c\Big|\det\Big(\frac{\partial \widetilde{x}^j}{\partial x_i}\Big)\Big||\omega_p|\Big)\\ &= \Big(p, c\Big|\det\Big(\frac{\partial \widetilde{x}^j}{\partial x_i}\Big)\Big|\Big)\end{aligned}$$

Thus, the hypotheses of Lemma **??** are satisfied, with the transition functions equal to $c|\det(\partial\widetilde{x}^j/\partial x^i)|$. □

If M is a smooth n-manifold with or without boundary, a section of $\mathfrak{D}M$ is called a **density** on M. If μ is a density and f is a continuous real-valued function, then $f\mu$ is again a density, which is smooth if both f and μ are. A density on M is said to be positive or negative if its value at each point has that property. Any nonvanishing n-form ω determines a positive density $|\omega|$, defined by $|\omega|_p := |\omega_p|$ for each $p \in M$. If ω is a nonvanishing n-form on an open subset $U \subseteq M$; then any density μ on U can be written $\mu = f|\omega|$ for some real-valued function f.

One important fact about densities is that every smooth manifold admits a global smooth positive density, without any orientability assumptions.

Proposition 3.5.4.10. *If M is a smooth manifold with or without boundary, there exists a smooth positive density on M.*

Proof. Because the set of positive elements of $\mathfrak{D}M$ is an open subset whose intersection with each fiber is convex, the usual partition of unity argument (Exercise 3.2.1) allows us to piece together local positive densities to obtain a global smooth positive density. □

It is important to understand that this proposition works because positivity of a density is a well-defined property, independent of any choices of coordinates or orientations. There is no corresponding existence result for orientation forms because without a choice of orientation, there is no way to decide which n-forms are positive.

Under smooth maps, densities pull back in the same way as differential forms. If $F : M \to N$ is a smooth map between n-manifolds (with or without boundary) and μ is a density on N, we define a density $F^*\mu$ on M by

$$(F^*\mu)_p(v_1,\dots,v_n) = \mu_{F(p)}(dF_p(v_1),\dots,dF_p(v_n)).$$

Proposition 3.5.4.11. *Let $G : P \to M$ and $F : M \to N$ be smooth maps between n-manifolds with or without boundary, and let μ be a density on N.*

(a) For any $f \in C^\infty(N)$, $F^*(f\mu) = (f \circ F)F^*\mu$.

(b) If ω is an n-form on N, then $F^*|\omega| = |F^*\omega|$.

(c) If μ is smooth, then $F^*\mu$ is a smooth density on M.

(d) $F(F \circ G)^*\mu = G^*(F^*\mu)$.

The next result shows how to compute the pullback of a density in coordinates. It is an analogue for densities of Proposition 3.3.2.5.

Proposition 3.5.4.12. *Suppose $F : M \to N$ is a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*(u|dy^1 \wedge \cdots \wedge dy^n|) = (u \circ F)|\det \partial F||dx^1 \wedge \cdots dx^n|,$$

where ∂F represents the matrix of partial derivatives of F in these coordinates.

Now we turn to integration. As we did with forms, we begin by defining integrals of densities on subsets of $\mathbb{R}^n$. If $D \subseteq \mathbb{R}^n$ is an open set and μ is a density on $\overline{D}$, we can write $\mu = f\, dx^1 \wedge \cdots \wedge dx^n$ for some uniquely determined continuous function $f : \overline{D} \to \mathbb{R}$. We define the integral of μ over D by

$$\int_D \mu = \int_D f dV,$$

or more suggestively,

$$\int_D f|dx^1 \wedge \cdots \wedge dx^n| = \int_D f dx^1 \wedge dx^n.$$

Similarly, if U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and μ is compactly supported in U, we define

$$\int_U \mu = \int_D \mu$$

where D is any open set containing the support of μ. The key fact is that this is diffeomorphism-invariant.

Proposition 3.5.4.13. *Suppose U and V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $F : U \to V$ is a diffeomorphism. If μ is a compactly supported density on V, then*

$$\int_V \mu = \int_U F^*\mu.$$

Proof. The proof is essentially identical to that of Proposition 3.5.1.1. □

Now let M be a smooth n-manifold (with or without boundary). If μ is a density on M whose support is contained in the domain of a single smooth chart (U, φ), the integral of μ over M is defined as

$$\int_M \mu = \int_{\varphi(U)} \varphi_*\mu.$$

This is extended to arbitrary densities μ by setting

$$\int_M \mu = \sum_i \int_M \psi_i\mu$$

where $\{\psi_i\}$ is a smooth partition of unity subordinate to an open cover of M by smooth charts. The fact that this is independent of the choices of coordinates or partition of unity follows just as in the case of forms.

The following proposition is proved in the same way as Proposition 3.5.1.5.

Proposition 3.5.4.14 (Properties of Integrals of Densities). *Suppose M and N are smooth n-manifolds with or without boundary, and μ, η are compactly supported densities on M.*

- *If $a, b \in \mathbb{R}$, then*
$$\int_M (a\mu + b\eta) = a\int_M \mu + b\int_M \eta.$$
- *If μ is a positive density, then $\int_M \mu > 0$.*
- *If $F : N \to M$ is a diffeomorphism, then $\int_M \mu = \int_N F^*\mu$.*

Densities are particularly useful on Riemannian manifolds

Proposition 3.5.4.15 (The Riemannian Density). *Let (M, g) be a Riemannian manifold with or without boundary. There is a unique smooth positive density μ_g on M, called the **Riemannian density**, with the property that*
$$\mu_g(E_1, \dots, E_n) = 1$$
for any local orthonormal frame (E_i).

Proof. Uniqueness is immediate, because any two densities that agree on a basis must be equal. Given any point $p \in M$; let U be a connected smooth coordinate neighborhood of p. Since U is diffeomorphic to an open subset of Euclidean space, it is orientable. Any choice of orientation of U uniquely determines a Riemannian volume form ω_g on U, with the property that $\omega_g(E_1, \dots, E_n) = 1$ for any oriented orthonormal frame. If we put $\mu_g = |\omega_g|$, it follows easily that μ_g is a smooth positive density on U satisfying the condition. If U and V are two overlapping smooth coordinate neighborhoods, the two definitions of μ_g agree where they overlap by uniqueness, so this defines μ_g globally. □

Proposition 3.5.4.16. *Let (M, g) be an oriented Riemannian manifold with or without boundary and let μ_g be its Riemannian volume form.*

- *The Riemannian density of M is given by μ_g.*
- *For any compactly supported continuous function $f : M \to \mathbb{R}$, we have*
$$\int_M f\mu_g = \int_M f\omega_g.$$

Proof. For an oriented Riemannian manifold, the volume form ω_g is nonzero, so the claims are obvious. □

Proposition 3.5.4.17. *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Then $F^*\mu_{\widetilde{g}} = \mu_g$.*

Theorem 3.5.4.18 (The Divergence Theorem in the Nonorientable Case). *Suppose (M, g) is a nonorientable Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,*
$$\int_M (\mathrm{div} X)\mu_g = \int_{\partial M} \langle X, N\rangle_g \mu_{\widetilde{g}},$$
where N is the outward-pointing unit normal vector field along ∂M, $\widetilde{g}$ is the induced Riemannian metric on ∂M, and μ_g, $\mu_{\widetilde{g}}$ are the Riemannian densities of g and $\widetilde{g}$, respectively.

3.5.5 Exercise

Exercise 3.5.1. Suppose E and M are smooth n-manifolds with or without boundary, and $\pi : E \to M$ is a smooth k-sheeted covering map or generalized covering map. Show that if E and M are oriented and π is orientation-preserving, then $\int_E \pi^*\omega = k\int_M \omega$ for any compactly supported n-form ω on M.

Proof. Let $\{(U_i, \varphi)\}$ be an evenly covered positively oriented family of M, and choose a partition of unity $\{\psi_i\}$ subordinate to U_i. Then the integral of ω is defined to be
$$\int_M \omega = \sum_i \int_{U_i} \psi_i \omega.$$
On each U_i, we may assume the image of $\varphi_i : U_i \to \widehat{U}_i$ is an open disk in $\mathbb{R}^n$. If we write $\pi^{-1} = \bigcup_{j=1}^k V_{ij}$ such that $\pi|_{V_{ij}} : V_{ij} \to U_i$ is a diffeomorphism, then since $\widehat{U}_i$ is simply-connected, we have k liftings

$\widetilde{\varphi}_{ij}$ such that $\pi \circ \widetilde{\varphi}_{ij} = \varphi_i$ for all j. Then we can see that each of the $\widetilde{\varphi}_{ij}$ is an orientation-preserving diffeomorphism from $\widehat{U}_i$ to V_{ij}. Thus we get

$$\int_{U_i} \psi_i \omega = \int_{V_{ij}} \pi^*(\psi_i \omega).$$

This then implies the desired result since $\{\pi^* \psi_i\}$ is a partition subordinate to $\{V_{ij}\}$. $\square$

Exercise 3.5.2. Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a retraction of M onto its boundary.

Exercise 3.5.3. Suppose M and N are oriented, compact, connected, smooth manifolds, and $F : M \to N$ are homotopic diffeomorphisms. Show that F and G are either both orientation-preserving or both orientation-reversing.

Proof. By Theorem **??** we can choose a smooth homotopy $H : M \times I \to N$ such that $H(x,0) = F, H(x,1) = N$. Let ω be a orientation form on N, then

$$\int_{M\times I} d(H^*\omega) = \int_{M\times I} H^*(d\omega) = 0.$$

On the other hand, by stokes's theorem,

$$\int_{M\times I} d(H^*\omega) = \int_{\partial(M\times I)} H^*\omega = \int_{M\times\{1\}} H^*\omega - \int_{M\times\{0\}} H^*\omega = \int_M G^*\omega - \int_M F^*\omega.$$

Thus F and G are either both orientation-preserving or both orientation-reversing. $\square$

Exercise 3.5.4. Show that any finite product $M_1 \times \cdots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

Exercise 3.5.5. Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(x-2)^2 + y^2 = 1$ around the z-axis, with its induced Riemannian metric and with the orientation determined by the outward unit normal.

(a) Compute the surface area of D.

(b) Compute the integral over D of the function $f(x,y,z) = z^2 + 1$.

(c) Compute the integral over D of the 2-form $\omega = z\, dx \wedge dy$.

Exercise 3.5.6. Let (M, g) be a compact Riemannian manifold with boundary, let $\widetilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.

(a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M), X \in \mathfrak{X}(M)$:

$$\operatorname{div}(fX) = f\operatorname{div}X + \langle \operatorname{grad} f, X\rangle_g.$$

(b) Prove the following integration by parts formula:

$$\int_M \langle \operatorname{grad} f, X\rangle_g \, dV_g = \int_{\partial M} f\langle X, N\rangle_g \, dV_{\widetilde{g}} - \int_M (f\operatorname{div}X)\, dV_g.$$

Proof. By definition,

$$\begin{aligned}\operatorname{div}(fX)dV_g &= d\big((fX) \lrcorner\, dV_g\big) = dv(f(X \lrcorner\, dV_g)v) \\ &= df \wedge (X \lrcorner\, dV_g) + f \wedge d(X \lrcorner\, dV_g) = df \wedge (X \lrcorner\, dV_g) + f\operatorname{div}X.\end{aligned}$$

So we only need to check $df \wedge (X \lrcorner\, dV_g) = \langle \operatorname{grad} f, X\rangle_g = df(X)$. This can be done in local coordinates.

The second claim follows from

$$\int_{\partial M} f\langle X, N\rangle_g \, dV_{\widetilde{g}} = \int_{\partial M} \langle fX, N\rangle_g \, dV_{\widetilde{g}} = \int_M \operatorname{div}(fX)\, dV_g.$$

$\square$

Exercise 3.5.7. Let (M, g) be a Riemannian manifold with or without boundary. A function $u \in C^\infty(M)$ is said to be **harmonic** if $\Delta u = 0$.

(a) Suppose M is compact, and prove **Green's identities**:

$$\int_M u\Delta v \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} uNv \, dV_{\tilde{g}},$$

$$\int_M (u\Delta v - v\Delta u) \, dV_g = \int_{\partial M} (vNu - uNv) \, dV_{\tilde{g}},$$

where $\tilde{g}$ denotes the induced Riemannian metric on ∂M, and N is the outward unit normal vector field along ∂M.

(b) Show that if M is compact and connected and $\partial M = \varnothing$, the only harmonic functions on M are the constants.

(c) Show that if M is compact and connected, $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, then $u \equiv v$.

Proof. By Exercise 3.5.6:

$$\begin{aligned}\int_M u\Delta v \, dV_g &= -\int_M u \operatorname{div}(\operatorname{grad} v) = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} u\langle \operatorname{grad} v, N\rangle_g \, dV_{\tilde{g}} \\ &= \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} uNv \, dV_{\tilde{g}}.\end{aligned}$$

and therefore,

$$\int_M v\Delta u \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} vNu \, dV_{\tilde{g}}.$$

Thus

$$\int_M (u\Delta v - v\Delta u) \, dV_g = \int_{\partial M} (vNu - uNv) \, dV_{\tilde{g}}.$$

Now if $\Delta u = 0$, then

$$\int_M \langle \operatorname{grad} f, \operatorname{grad} u\rangle_g \, dV_g = 0$$

for any $f \in C^\infty(M)$. In particular, $\int_M \langle \operatorname{grad} u, \operatorname{grad} u\rangle_g \, dV_g = 0$, which implies $du = 0$. Now the second claim follows from the first, by applying on $u - v$. □

Exercise 3.5.8. Let (M, g) be a compact connected Riemannian manifold without boundary, and let Δ be its geometric Laplacian. A real number λ is called an eigenvalue of Δ if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$. In this case, u is called an eigenfunction corresponding to λ.

(a) Prove that 0 is an eigenvalue of Δ, and that all other eigenvalues are strictly positive.

(b) Prove that if u and v are eigenfunctions corresponding to distinct eigenvalues, then $\int_M uv \, dV_g = 0$.

Proof. The constant functions are eigenfunction corresponding to 0. Also, let $\Delta u = \lambda u$ with $\lambda \neq 0$, then

$$\lambda \int_M u^2 \, dV_g = \int_M u\Delta u \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g.$$

Thus $\lambda > 0$.

If $\Delta u = \lambda u$, $\Delta v = \mu v$ with $\lambda \neq \mu$, then by Exercise 3.5.7,

$$0 = \int_M (u\Delta v - v\Delta u) \, dV_g = (\mu - \lambda) \int_M uv \, dV_g,$$

which implies $\int_M uv \, dV_g = 0$. □

Exercise 3.5.9. Let M be a compact connected Riemannian n-manifold with nonempty boundary. A real number λ is called a **Dirichlet eigenvalue** for M if there exists a smooth real-valued nonzero function u on M such that $\Delta u = \lambda u$ and $u|_{\partial M} = 0$. Similarly, λ is called a **Neumann eigenvalue** if there exists such a function u satisfying $\Delta u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.

(b) Show that every Dirichlet eigenvalue is strictly positive.

(b) Show that 0 is a Neumann eigenvalue, and all other Neumann eigenvalues are strictly positive.

Proof. For Dirichlet eigenvalues, we can assume $\partial M = \varnothing$, since $u|_{\partial M} = 0$. Then the claim follows from the previous exercise.

Any constant function f satisfies $Nf|_{\partial M} = 0$ and $\Delta f = 0$, thus 0 is a Neumann eigenvalue. Now let $\Delta u = \lambda u$ with $Nu|_{\partial M} = 0$, then by Exercise 3.5.7

$$\lambda \int_M u^2 \, dV_g = \int_M u \Delta u \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} u \rangle_g \, dV_g.$$

Thus $\lambda > 0$. □

Exercise 3.5.10 (Dirichlet's Principle). Suppose M is a compact connected Riemannian n-manifold with nonempty boundary. Prove that a function $u \in C^\infty(M)$ is harmonic if and only if it minimizes $\int_M |\operatorname{grad} u|_g^2 \, dV_g$ among all smooth functions with the same boundary values.

Proof. If $\Delta u = 0$, then for any function $f \in C^\infty(M)$ that vanishes on ∂M,

$$\begin{aligned}
\int_M |\operatorname{grad}(u + \varepsilon f)|_g^2 \, dV_g &= \int_M |\operatorname{grad} u|_g^2 \, dV_g + 2\varepsilon \int_M \langle \operatorname{grad} f, \operatorname{grad} u \rangle_g \, dV_g + \varepsilon^2 \int_M |\operatorname{grad} f|_g^2 \, dV_g \\
&= \int_M |\operatorname{grad} u|_g^2 \, dV_g + \varepsilon^2 \int_M |\operatorname{grad} f|_g^2 \, dV_g + 2\varepsilon \Big(\int_M f \Delta u \, dV_g + \int_{\partial M} f N u \, dV_{\tilde{g}} \Big) \\
&= \int_M |\operatorname{grad} u|_g^2 \, dV_g + \varepsilon^2 \int_M |\operatorname{grad} f|_g^2 \, dV_g.
\end{aligned}$$

Thus u minimizes $\int_M |\operatorname{grad} u|_g^2 \, dV_g$. □

Chapter 4

Symplectic geometry

4.1 Symplectic manifolds

4.1.1 Symplectic tensors

A 2-covector ω on a finite-dimensional vector space V is said to be **nondegenerate** if the linear map $\widehat{\omega} : V \to V^*$ defined by $\widehat{\omega}(v) = v \lrcorner \omega$ is invertible for every nonzero $v \in V$.

Proposition 4.1.1.1. *The following are equivalent for* 2*-covector* ω *on a finite-dimensional vector space* V*:*

(a) ω *is nondegenerate.*

(b) *For each nonzero* $v \in V$*, there exists* $w \in V$ *such that* $\omega(v, w) \neq 0$*.*

(c) *In terms of some (hence every) basis, the matrix* (ω_{ij}) *representing* ω *is nonsingular.*

Proof. The nondegenerate condition is to say: for each nonzero $v \in V$, $v \lrcorner \omega \neq 0$, which is exactly (b). Now let $(E_1, \ldots, E_n)$ be a basis of V, and $W = (\omega_{ij})$ be the matrix. Then for $v = v^i E_i$ and $w = w^j E_j$,

$$\omega(v, w) = \omega_{ij} v^i w^j.$$

Thus

$$\omega(v, w) = 0 \text{ for all } w \in V \iff \omega_{ij} v^i = 0 \text{ for all } j \iff v \text{ is a solution of } W^T X = 0.$$

so ω is nondegenerate if and only if W is nonsingular. $\square$

A nondegenerate 2-covector is called a **symplectic tensor**. A vector space V endowed with a specific symplectic tensor is called a **symplectic vector space**.

Example 4.1.1.2. Let V be a $2n$-dimensional vector space. Let $(A_1, \ldots, A_n, B_1, \ldots, B_n)$ be a basis of V and $(\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^n)$ denote the corresponding dual basis for V^*, and let $\omega \in \bigwedge^2(V^*)$ be the 2-covector defined by

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i. \tag{4.1.1.1}$$

Note that the action of ω on basis vectors is given by

$$\omega(A_i, A_j) = \omega(B_i, B_j) = 0, \quad \omega(A_i, B_j) = -\omega(B_j, A_i) = \delta_{ij}. \tag{4.1.1.2}$$

and the matrix (ω_{ij}) representing ω is

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Thus ω is nondegenerate, and so is a symplectic tensor.

If (V, ω) is a symplectic vector space and $S \subseteq V$ is any linear subspace, we define the **symplectic complement** of S, denoted by $S^\perp$, to be the subspace

$$S^\perp = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in S\}.$$

As the notation suggests, the symplectic complement is analogous to the orthogonal complement in an inner product space. Just as in the inner product case, the dimension of $S^\perp$ is the codimension of S, as the next lemma shows.

Lemma 4.1.1.3. *Let (V, ω) be a symplectic vector space. For any linear subspace $S \subseteq V$, we have $\dim S + \dim S^\perp = \dim V$.*

Proof. Let $S \subseteq V$ be a subspace, and define a linear map $\Phi : V \to S^*$ by $\Phi(v) = (v \lrcorner \omega)|_S$, or equivalently

$$\Phi(v)(w) = \omega(v, w).$$

Suppose φ is an arbitrary element of S^*, and let $\widetilde{\varphi} \in V^*$ be any extension of φ to a linear functional on all of V. Since the map $\widehat{\omega} : V \to V^*$ is an isomorphism, there exists $v \in V$ such that $v \lrcorner \omega = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the rank-nullity law, $S^\perp = \ker \Phi$ has dimension equal to $\dim V - \dim S^* = \dim V - \dim S$. $\square$

Proposition 4.1.1.4. *Let (V, ω) be a symplectic vector space and $S \subseteq V$ be a linear subspace. Then $(S^\perp)^\perp = S$.*

Proof. By definition, since $\omega(v, w) = -\omega(w, v)$, we have $S \subseteq (S^\perp)^\perp$. Now note that

$$\dim(S^\perp)^\perp = \dim V - \dim S^\perp = \dim S,$$

thus $(S^\perp)^\perp = S$. $\square$

Symplectic complements differ from orthogonal complements in one important respect: although it is always true that $S \cap S^\perp = 0$ in an inner product space, this need not be true in a symplectic vector space. Indeed, if S is 1-dimensional, the fact that ω is alternating forces $\omega(v, v) = 0$ for every $v \in S$, so $S = S^\perp$. Carrying this idea a little further, a linear subspace $S \subseteq V$ is said to be

- **symplectic** if $S \cap S^\perp = \{0\}$.
- **isotropic** if $S \subseteq S^\perp$.
- **coisotropic** if $S \supseteq S^\perp$.
- **Lagrangian** if $S = S^\perp$.

Proposition 4.1.1.5. *Let (V, ω) be a symplectic vector space and $S \subseteq V$ be a linear subspace. Then*

(a) *S is symplectic if and only if $S^\perp$ is symplectic.*

(b) *S is symplectic if and only if $\omega|_S$ is nondegenerate.*

(c) *S is isotropic if and only if $\omega|_S = 0$.*

(d) *S is coisotropic if and only if $S^\perp$ is isotropic.*

(e) *S is Lagrangian if and only if $\omega|_S = 0$ and $\dim S = \frac{1}{2} \dim V$.*

Proof. Since $(S^\perp)^\perp = S$, part (a) and (d) are immediate. Next, we note that

$$v \in S \cap S^\perp \iff \omega(v, w) = 0 \text{ for all } w \in S.$$

Thus S is symplectic if and only if $\omega|_S$ is nondegenerate, and S is isotropic if and only if $\omega|_S = 0$.

Finally, S is Lagrangian means S and $S^\perp$ are both isotropic, which implies $\omega|_S = 0$ and $\dim S = \frac{1}{2} \dim V$. Conversely, if S is isotropic and $\dim S = \frac{1}{2} \dim V$, then $S \subseteq S^\perp$ with $\dim S = \dim S^\perp$. Thus $S = S^\perp$ and S is Lagrangian. $\square$

The symplectic tensor ω defined in Example 4.1.1.2 turns out to be the prototype of all symplectic tensors, as the next proposition shows. This can be viewed as a symplectic version of the Gram-Schmidt algorithm.

Proposition 4.1.1.6 (Canonical Form for a Symplectic Tensor). *Let ω be a symplectic tensor on an m-dimensional vector space V. Then V has even dimension $m = 2n$, and there exists a basis for V in which ω has the form (4.1.1.1).*

Proof. The tensor ω has the form (4.1.1.1) with respect to a basis $(A_1, \dots, A_n, B_1, \dots, B_n)$ if and only if its action on basis vectors is given by (4.1.1.2).We prove the theorem by induction on $m = \dim V$ by showing that there is a basis with this property.

For $m = 0$ there is nothing to prove. Suppose (V, ω) is a symplectic vector space of dimension $m = 1$, and assume that the proposition is true for all symplectic vector spaces of dimension less than m. Let A_1 be any nonzero vector in V. Since ω is nondegenerate, there exists $B_1 \in V$ such that $\omega(A_1, B_1) \neq 0$. Multiplying B_1 by a constant if necessary, we may assume that $\omega(A_1, B_1) = 1$. Because ω is alternating, B_1 cannot be a multiple of A_1, so the set $\{A_1, B_1\}$ is linearly independent, and hence $\dim V \geq 2$.

Let $S \subseteq V$ be the span of $\{A_1, B_1\}$. Then $\dim S^\perp = \dim m - 2$ by Lemma 4.1.1.3. Since $\omega|_S$ is nondegenerate, by Proposition 4.1.1.5 it follows that S is symplectic, and thus $S^\perp$ is also symplectic. By induction, $S^\perp$ is even-dimensional and there is a basis $(A_2, \dots, A_n, B_2, \dots, B_n)$ for $S^\perp$ such that (4.1.1.2) is satisfied for $2 \leq i, j \leq n$. It follows easily that $(A_1, \dots, A_n, B_1, \dots, B_n)$ is the required basis for V. □

Because of this, if (V, ω) is a symplectic vector space, a basis $(A_1, \dots, A_n, B_1, \dots, B_n)$ for V is called a **symplectic basis** if (4.1.1.2) holds, which is equivalent to ω being given by (4.1.1.1) in terms of the dual basis. The proposition then says that every symplectic vector space has a symplectic basis.

This leads to another useful criterion for 2-covector to be nondegenerate. For an alternating tensor ω, the notation ω^k denotes the k-fold wedge product $\omega \wedge \cdots \wedge \omega$.

Proposition 4.1.1.7. *Suppose V is a $2n$-dimensional vector space and $\omega \in \bigwedge^2(V^*)$. Then ω is a symplectic tensor if and only if $\omega^n \neq 0$.*

Proof. Suppose first that ω is a symplectic tensor. Let (A_i, B_i) be a symplectic basis for V, and write $\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i$ in terms of the dual coframe. Then we compute

$$\omega^n = \sum_I{}' \alpha^{i_1} \wedge \beta^{i_1} \wedge \cdots \wedge \alpha^{i_n} \wedge \beta^{i_n} = n!(\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n) \neq 0.$$

Conversely, suppose ω is degenerate. Then there is a nonzero vector $v \in V$ such that $v \lrcorner \omega = \widehat{\omega}(v) = 0$. Since interior multiplication by v is an antiderivation, by induction we can show

$$v \lrcorner (\omega^n) = n(v \lrcorner \omega) \wedge \omega^{n-1} = 0.$$

We can extend v to a basis $(E_1, \dots, E_{2n})$ for V with $E_1 = v$, and then $\omega^n(E_1, \dots, E_{2n}) = 0$, which implies $\omega^n = 0$. □

4.1.2 Symplectic structures on manifolds

Now let us turn to a smooth manifold M. A **nondegenerate 2-form** on M is a 2-form ω such that ω_p is a nondegenerate 2-covector for each $p \in M$. A **symplectic form** on M is a closed nondegenerate 2-form. A smooth manifold endowed with a specific choice of symplectic form is called a **symplectic manifold**. A choice of symplectic form is also sometimes called a **symplectic structure**.

If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, a diffeomorphism $F : M_1 \to M_2$ satisfying $F^*\omega_2 = \omega_1$ is called a **symplectomorphism**. The study of properties of symplectic manifolds that are invariant under symplectomorphisms is known as symplectic geometry or symplectic topology.

Example 4.1.2.1 (Examples of Symplectic Manifolds).

(a) With standard coordinates on $\mathbb{R}^{2n}$ denoted by $(x^1, \dots, x^n, y^1, \dots, y^n)$, the 2-form

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i$$

is symplectic: it is obviously closed, and it is nondegenerate because its value at each point is the symplectic tensor of Example 4.1.1.2. This is called the **standard symplectic form** on $\mathbb{R}^{2n}$.

(b) Suppose Σ is any orientable smooth 2-manifold and ω is a nonvanishing smooth 2-form on ω. Then ω is closed because $d\omega$ is a 3-form on a 2-manifold. Moreover, in two dimensions every nonvanishing 2-form is nondegenerate (by Proposition 4.1.1.7), so (Σ, ω) is a symplectic manifold.

Suppose (M, ω) is a symplectic manifold. An (immersed or embedded) submanifold $S \subseteq M$ is said to be a **symplectic**, **isotropic**, **coisotropic**, or **Lagrangian** submanifold if T_pS (thought of as a subspace of T_pM) has the corresponding property at each point $p \in S$. More generally, a smooth immersion (or embedding) $F : N \to M$ is said to have one of these properties if the subspace $dF_p(T_pN) \subseteq T_{F(p)}M$ has the corresponding property for every $p \in N$. Thus a submanifold is symplectic (isotropic, etc.) if and only if its inclusion map has the same property.

Proposition 4.1.2.2. *Suppose (M, ω) is a symplectic manifold and $F : N \to M$ is a smooth immersion. Then F is isotropic if and only if $F^*\omega = 0$, and F is symplectic if and only if $F^*\omega$ is a symplectic form.*

Proof. By Proposition 4.1.1.5, F is isotropic means $\omega|_{dF(TN)} = 0$, which means for all $v, w \in T_pN$ we have

$$0 = \omega(dF_p(v), dF_p(w)) = F^*\omega(v, w).$$

So Therefore F is isotropic if and only if $F^*\omega = 0$ on N. Similarly, by Proposition 4.1.1.5 we can show that F is symplectic if and only if $F^*\omega$ is nondegenerate on N, which is to say $F^*\omega$ is a symplectic form. ◻

4.1.2.1 The canonical symplectic form on the cotangent bundle The most important symplectic manifolds are total spaces of cotangent bundles, which carry canonical symplectic structures that we now define. First, there is a natural 1-form τ on the total space of T^*M, called the **tautological 1-form**, defined as follows. A point in T^*M is a covector $\varphi \in T_q^*M$ for some $q \in M$, we denote such a point by the notation (q, φ). We define $\tau \in \Omega^1(T^*M)$ (a 1-form on the total space of T^*M) by

$$\tau_{(q,\varphi)} = d\pi^*_{(q,\varphi)}\varphi.$$

where $\pi : T^*M \to M$ is the projection. In other words, the value of τ at (q, φ) is the pullback with respect to π of the covector φ itself. If v is a tangent vector in $T_{(q,\varphi)}(T^*M)$, then

$$\tau_{(q,\varphi)}(v) = \varphi\big(d\pi_{(q,\varphi)}(v)\big). \tag{4.1.2.1}$$

Proposition 4.1.2.3. *Let M be a smooth manifold. The tautological 1-form τ is smooth, and $\omega = -d\tau$ is a symplectic form on the total space of T^*M.*

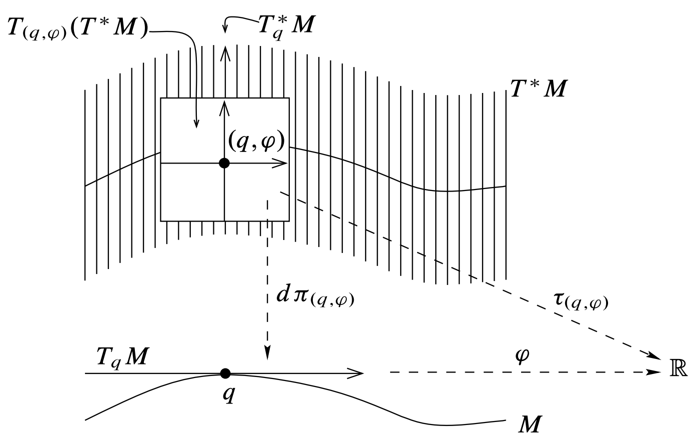


Figure 4.1: The tautological 1-form on T^*M.

Proof. Let (x^i) be smooth coordinates on M, and let (x^i, ξ_i) denote the corresponding natural coordinates on T^*M. Recall that the coordinates of $(q, \varphi) \in T^*M$ are defined to be (x^i, ξ_i), where (x^i) is the coordinate representation of q, and $\xi_i dx^i$ is the coordinate representation of φ. In terms of these coordinates, the projection $\pi : T^*M \to M$ has the coordinate expression $\pi(x, \xi) = x$. This implies that $d\pi^*(dx^i) = dx^i$, so the coordinate expression for τ is

$$\tau_{(x,\xi)} = d\pi^*_{(x,\xi)}(\xi_i dx^i) = \xi_i dx^i.$$

It follows immediately that τ is smooth, because its component functions in these coordinates are linear.
Let $\omega = -d\tau \in \Omega^2(T^*M)$. Clearly, ω is closed, because it is exact. Moreover, in natural coordinates,

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\xi_i.$$

Under the identification of an open subset of T^*M with an open subset of $\mathbb{R}^{2n}$ by means of these coordinates, ω corresponds to the standard symplectic form on $\mathbb{R}^{2n}$. It follows that ω is symplectic. □

The symplectic form defined in this proposition is called the canonical symplectic form on T^*M. One of its many uses is in giving the following somewhat more geometric interpretation of what it means for a 1-form to be closed.

Proposition 4.1.2.4. *Let M be a smooth manifold, and let σ be a smooth 1-form on M. Thought of as a smooth map from M to T^*M, σ is a smooth embedding, and σ is closed if and only if its image $\sigma(M)$ is a Lagrangian submanifold of T^*M.*

Proof. Throughout this proof we need to remember that $\sigma : M \to T^*M$ is playing two roles: on the one hand, it is a 1-form on M, and on the other hand, it is a smooth map between manifolds. Since they are literally the same map, we do not use different notations to distinguish between them.
In terms of smooth local coordinates (x^i) for M and corresponding natural coordinates (x^i, ξ_i) for T^*M, the map $\sigma : M \to T^*M$ has the coordinate representation

$$\sigma(x^1, \dots, x^n) = (x^1, \dots, x^n, \sigma_1(x), \dots, \sigma_n(x)),$$

where $\sigma_i dx^i$ is the coordinate representation of σ as a 1-form. It follows immediately that σ is a smooth immersion, and it is injective because $\pi \circ \sigma = \mathrm{id}_M$. To show that it is an embedding, it suffices by **??** to show that it is a proper map. This in turn follows from the fact that π is a left inverse for σ, by **??**.
Because $\sigma(M)$ is n-dimensional, it is Lagrangian if and only if it is isotropic, which is the case if and only if $\sigma^*\omega = 0$ (Proposition 4.1.2.2). The pullback of the tautological form τ under σ is

$$\sigma^*\tau = \sigma^*(\xi_i dx^i) = \sigma_i dx^i = \sigma.$$

This can also be seen somewhat more invariantly from the computation

$$(\sigma^*\tau)_p(v) = \tau_{\sigma(p)}\big(d\sigma_p(v)\big) = \sigma_p\big(d\pi_{\sigma(p)} \circ d\sigma_p(v)\big) = \sigma_p\big(d(\pi \circ \sigma)_p(v)\big) = \sigma_p(v).$$

which follows from the definition of τ and the fact that $\pi \circ \sigma = \mathrm{id}_M$. Therefore,

$$\sigma^*\omega = -\sigma^* d\tau = -d(\sigma^*\tau) = -d\sigma.$$

It follows that σ is a Lagrangian embedding if and only if $d\sigma = 0$. □

Proposition 4.1.2.5. *Let M be a smooth manifold, and let S be an embedded Lagrangian submanifold of the total space of T^*M.*

(a) *If S is transverse to the fiber of T^*M at a point $q \in T^*M$, then there exist a neighborhood V of q in S and a neighborhood U of $\pi(q)$ in M such that V is the image of a smooth closed 1-form defined on U.*

(b) *S is the image of a globally defined smooth closed 1-form on M if and only if S intersects each fiber transversely in exactly one point.*

Proof. If S is transverse to the fiber of T^*M at a point $q \in T^*M$, then $d(\pi)_q : T_qS \to T_{\pi(q)}M$ is an isomorphism. Therefore $\pi|_S$ restricts to a diffeomorphism from a neighborhood V of q in S to a neighborhood U of $\pi(q)$. Then V is the graph of $\sigma = \rho \circ (\pi)^{-1}$, where ρ is the projection to the fiber.
If S intersects each fiber transversely in exactly one point, then the projection $\pi|_S$ is bijective and a submersion, hence a diffeomorphism. Therefore $(\pi|_S)^{-1}$ is a closed 1-form whose image is S. □

4.1.3 The Darboux theorem

Our next theorem is one of the most fundamental results in symplectic geometry. It is a nonlinear analogue of the canonical form for a symplectic tensor given in Proposition 4.1.1.6. It illustrates the most dramatic difference between symplectic structures and Riemannian metrics: unlike the Riemannian case, there is no local obstruction to a symplectic structure being locally equivalent to the standard flat model.

Theorem 4.1.3.1 (Darboux). *Let (M, ω)be a $2n$-dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ centered at p in which ω has the coordinate representation*

$$\omega = \sum_{i=1}^{n} x^i \wedge y^i. \tag{4.1.3.1}$$

Any coordinates satisfying (4.1.3.1) theorem are called **Darboux coordinates**, **symplectic coordinates**, or **canonical coordinates**. Obviously, the standard coordinates $(x^1, \ldots, ^n, y^1, \ldots, y^n)$ on $\mathbb{R}^{2n}$ are Darboux coordinates. The proof of Proposition 4.1.2.3 showed that the natural coordinates (x^i, ξ_i) are Darboux coordinates for T^*M with its canonical symplectic structure.

First, recall that Proposition 3.1.3.13 shows how to use Lie derivatives to compute the derivative of a tensor field under a flow. We need the following generalization of that result to the case of time-dependent flows.

Proposition 4.1.3.2. *Let M be a smooth manifold. Suppose $V : J \times M \to TM$ is a smooth time-dependent vector field and $\psi : \mathcal{E} \to M$ is its time-dependent flow. For any smooth covariant tensor field A on M and any $(t_1, t_0, p) \in \mathcal{E}$,*

$$\frac{d}{dt}\bigg|_{t=t_1} (\psi^*_{t,t_0} A)_p = \left(\psi^*_{t_1,t_0}(\mathfrak{L}_{V_{t_1}} A)\right)_p. \tag{4.1.3.2}$$

Proof. First, assume $t_1 = t_0$. In this case, ψ_{t_0,t_1} is the identity map of M. so we need to prove

$$\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} A)_p = (\mathfrak{L}_{V_{t_1}} A)_p. \tag{4.1.3.3}$$

We begin with the special case in which $A = f$ is a smooth 0-tensor field:

$$\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} f)(p) = \frac{\partial}{\partial t}\bigg|_{t=t_0} (f(\psi(t, t_0, p))) = V(t_0, \psi(t_0, t_0, p))f = (\mathfrak{L}_{V_{t_0}} f)(p).$$

Next consider an exact 1-form $A = df$. In any smooth local coordinates (x^i), the function $\psi^*_{t,t_0} f(x) = f(\psi(t, t_0, x))$ depends smoothly on all $n+1$ variables $(t, x^1, \ldots, x^n)$. Thus, the operator ∂/∂ commutes with each of the partial derivatives $\partial/\partial x^i$ when applied to $\psi^*_{t,t_0} f$. In particular, this means that the exterior derivative operator d commutes with $\partial/\partial t$, and so by Corollary 3.3.3.11,

$$\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} df)(p) = \frac{\partial}{\partial t}\bigg|_{t=t_0} d(\psi^*_{t,t_0} f)_p = d\left(\frac{\partial}{\partial t}\bigg|_{t=t_0} (\psi^*_{t,t_0} f)\right)_p = d(\mathfrak{L}_{V_{t_0}} f)_p = (\mathfrak{L}_{V_{t_0}} df)_p.$$

Thus, the result is proved for 0-tensors and for exact 1-forms.

Now suppose that $A = B \otimes C$ for some smooth covariant tensor fields B and C, and assume that the proposition is true for B and C. (We include the possibility that B or C has rank 0, in which case the tensor product is just ordinary multiplication.) By the product rule for Lie derivatives (Proposition 3.1.3.9(c)), the right-hand side of (4.1.3.3) satisfies

$$(\mathfrak{L}_{V_{t_0}} (B \otimes C))_p = (\mathfrak{L}_{V_{t_0}} B)_p \otimes C_p + B_p \otimes (\mathfrak{L}_{V_{t_0}} C)_p.$$

On the other hand, by an argument entirely analogous to that in the proof of Proposition 3.1.3.9, the left-hand side satisfies a similar product rule:

$$\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} (B \otimes C))_p = \left(\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} B)\right)_p \otimes C_p + B_p \otimes \left(\frac{d}{dt}\bigg|_{t=t_0} (\psi^*_{t,t_0} C)\right)_p.$$

This shows that (4.1.3.3) holds for $A = B \otimes C$, provided it holds for B and C. The case of arbitrary tensor fields now follows by induction, using the fact that any smooth covariant tensor field can be written locally as a sum of tensor fields of the form $A = f\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$.

To handle arbitrary t_1, we use Theorem 1.2.8.1(d), which shows that $\psi_{t,t_0} = \psi_{t,t_1} \circ \psi_{t_1,t_0}$ wherever the right-hand side is defined. Therefore, because the linear map $d(\psi_{t_1,t_0})^*_p$ does not depend on t,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_1} (\psi^*_{t,t_0} A)_p &= \frac{d}{dt}\Big|_{t=t_1} d(\psi_{t,t_0})^*_p (A_{\psi_{t,t_0}(p)}) = \frac{d}{dt}\Big|_{t=t_1} d(\psi_{t_1,t_0})^*_p \circ d(\psi_{t,t_1})^*_{\psi_{t_1,t_0}(p)} (A_{\psi_{t,t_0}(p)}) \\ &= d(\psi_{t_1,t_0})^*_p \Big(\frac{d}{dt}\Big|_{t=t_1} d(\psi_{t,t_1})^*_{\psi_{t_1,t_0}(p)} (A_{\psi_{t,t_1} \circ \psi_{t_1,t_0}(p)}) \Big) = \big(\psi^*_{t_1,t_0}(\mathfrak{L}_{V_{t_1}} A)\big)_p.\end{aligned}$$

This finishes the proof. □

A **smooth time-dependent tensor field** on a smooth manifold M is a smooth map $A : J \times M \to T^k T^* M$, where $J \subseteq \mathbb{R}$ is an interval, satisfying $A(t,p) \in T^k(T^*_p M)$ for each $(t,p) \in J \times M$.

Proposition 4.1.3.3. *Let M be a smooth manifold and $J \subseteq \mathbb{R}$ be an open interval. Suppose $V : J \times M \to TM$ is a smooth time-dependent vector field on M, $\psi : \mathcal{E} \to M$ is its time-dependent flow, and $A : J \times M \to T^k T^* M$ is a smooth time-dependent tensor field on M. Then for any $(t_1, t_0, p) \in \mathcal{E}$,*

$$\frac{d}{dt}\Big|_{t=t_1} (\psi^*_{t,t_0} A_t)_p = \Big(\psi^*_{t_1,t_0}\Big(\mathfrak{L}_{V_{t_1}} A_{t_1} + \frac{d}{dt}\Big|_{t=t_1} A_t\Big)\Big)_p. \tag{4.1.3.4}$$

Proof. For sufficiently small $\varepsilon > 0$, consider the smooth map $F : (t_1 - \varepsilon, t_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \to T^k(T^*_p M)$ defined by

$$F(u,v) = (\psi^*_{u,t_0} A_v)_p = d(\psi_{u,t_0})^*_p (A_v|_{\psi_{u,t_0}(p)}).$$

Since F takes its values in the finite-dimensional vector space $T^k(T^*_p M)$, we can apply the chain rule together with Proposition 4.1.3.2 to conclude that

$$\frac{d}{dt}\Big|_{t=t_1} F(t,t) = \frac{\partial F}{\partial u}(t_1,t_1) + \frac{\partial F}{\partial v}(t_1,t_1) = (\psi^*_{t_1,t_0}(\mathfrak{L}_{V_{t_1}} A_{t_1}))_p + \frac{\partial}{\partial v}\Big|_{t=t_1} d(\psi^*_{t_1,t_0})_p (A_v|_{\psi_{u,t_0}(p)}).$$

As in the proof of Proposition 4.1.3.2, the linear map $d(\psi^*_{t_1,t_0})_p$ commutes with $\partial/\partial v$, yielding (4.1.3.4). □

Proof of the Darboux theorem. Let ω_0 denote the given symplectic form on M, and let $p_0 \in M$ be arbitrary. The theorem will be proved if we can find a smooth coordinate chart (U_0, φ) centered at p_0 such that $\varphi^* \omega_0 = \omega_1$, where $\omega_1 = \sum_{i=1}^n dx^i \wedge dy^i$ is the standard symplectic form on $\mathbb{R}^{2n}$. Since this is a local question, by choosing smooth coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ in a neighborhood of p_0, we may replace M with an open ball $U \subseteq \mathbb{R}^{2n}$. Proposition 4.1.1.6 shows that we can arrange by a linear change of coordinates that $\omega_0|_{p_0} = \omega_1|_{p_0}$.

Let $\eta = \omega_1 - \omega_0$. Because ω is closed, the Poincaré lemma shows that we can find a smooth 1-form α on U such that $d\alpha = -\eta$. By subtracting a constant-coefficient (and thus closed) 1-form from α, we may assume without loss of generality that $\alpha_{p_0} = 0$.

For each $t \in \mathbb{R}$, define a closed 2-form ω_t on U by

$$\omega_t = \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1.$$

Let J be a bounded open interval containing $[0,1]$. Because $\omega_t|_{p_0} = \omega_0|_{p_0}$ is nondegenerate for all t, a simple compactness argument shows that there is some neighborhood $U_1 \subseteq U$ of p_0 such that ω_t is nondegenerate on U_1 for all $t \in \bar{J}$. Because of this nondegeneracy, the smooth bundle homomorphism $\widehat{\omega}_t : TU_1 \to T^* U_1$ defined by $\widehat{\omega}_t(X) = X \lrcorner\, \omega_t$ is an isomorphism for each $t \in \bar{J}$.

Define a smooth time-dependent vector field $V : J \times U_1 \to TU_1$ by $V_t = \widehat{\omega}_t^{-1} \alpha$, or equivalently

$$V_t \lrcorner\, \omega_t = \alpha.$$

Our assumption that $\alpha_{p_0} = 0$ implies that $V_t|_{p_0} = 0$ for each t. If $\psi : \mathcal{E} \to U_1$ denotes the time-dependent flow of V, it follows that $\psi(t, 0, p_0) = p_0$ for all $t \in J$, so $J \times \{0\} \times \{p_0\} \subseteq \mathcal{E}$. Because E is open in $J \times J \times M$ and $[0,1]$ is compact, there is some neighborhood U_0 of p_0 such that $[0,1] \times \{0\} \times U_0 \subseteq \mathcal{E}$.

For each $t_1 \in [0,1]$, it follows from Proposition 4.1.3.3 that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_1} \psi_{t,0}^* \omega_t &= \psi_{t_1,0}^*\Big(\mathfrak{L}_{V_{t_1}}\omega_{t_1} + \frac{d}{dt}\Big|_{t=t_1} \omega_t\Big) \\ &= \psi_{t_1,0}^*(V_{t_1} \lrcorner\, d\omega_{t_1} + d(V_{t_1} \lrcorner\, \omega_t) + \eta) \\ &= \psi_{t_1,0}^*(V_{t_1} \lrcorner\, 0 + d\alpha + \eta) = 0. \end{aligned}$$

Therefore, $\psi_{t,0}^*\omega_t = \psi_{0,0}^*\omega_0 = \omega_0$ for all t. In particular, $\psi_{1,0}^*\omega_1 = \omega_0$. It follows from Theorem 1.2.8.1(c) that $\psi_{1,0}^*$ is a diffeomorphism onto its image, so it is a coordinate map. Because $\psi_{1,0}(p_0) = p_0$, these coordinate are centered at p_0. □

4.1.4 Hamiltonian vector fields

One of the most useful constructions on symplectic manifolds is a symplectic analogue of the gradient, defined as follows. Suppose (M, ω) is a symplectic manifold. For any smooth function $f \in C^\infty(M)$, we define the **Hamiltonian vector field of** f to be the smooth vector field X_f defined by

$$X_f = \widehat{\omega}^{-1}(df),$$

where $\widehat{\omega} : TM \to T^*M$ is the bundle isomorphism determined by ω. Equivalently,

$$X_f \lrcorner\, \omega = df.$$

or for any vector field Y,

$$\omega(X_f, Y) = df(Y) = Yf.$$

In any Darboux coordinates, X_f can be computed explicitly as follows. Writing

$$X_f = \sum_{i=1}^n \Big(a^i \frac{\partial}{\partial x^i} - b^i \frac{\partial}{\partial y^i}\Big)$$

for some coefficient functions (a^i, b^i) to be determined, we compute

$$X_f \lrcorner\, \omega = \sum_{j=1}^n \Big(a^j \frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial y^j}\Big) \lrcorner \sum_{i=1}^n dx^i \wedge dy^i = \sum_{i=1}^n (a^i dy^i - b^i dx^i).$$

On the other hand,

$$df = \sum_{i=1}^n \Big(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial x^i} dy^i\Big).$$

Setting these two expressions equal to each other, we find that $a^i = \partial f / \partial y^i$ and $b^i = -\partial f / \partial x^i$, which yields the following formula for the Hamiltonian vector field of f in Darboux coordinates:

$$X_f = \sum_{i=1}^n \Big(\frac{\partial f}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^i}\Big). \tag{4.1.4.1}$$

This formula holds, in particular, on $\mathbb{R}^{2n}$ with its standard symplectic form.

Although the definition of the Hamiltonian vector field is formally analogous to that of the gradient on a Riemannian manifold, they differ from gradients in some very significant ways, as the next proposition shows.

Proposition 4.1.4.1 (Properties of Hamiltonian Vector Fields). *Let (M, ω) be a symplectic manifold and let $f \in C^\infty(M)$.*

(a) *f is constant along each integral curve of X_f.*

(b) *At each regular point of f, the Hamiltonian vector field X_f is tangent to the level set of f.*

Proof. Both assertions follow from the fact that

$$X_f f = df(X_f) = \omega(X_f, X_f) = 0$$

because ω is alternating. □

A smooth vector field X on M is said to be **symplectic** if ω is invariant under the flow of X. It is said to be **Hamiltonian** (or **globally Hamiltonian**) if there exists a function $f \in C^\infty(M)$ such that $X = X_f$, and **locally Hamiltonian** if each point p has a neighborhood on which X is Hamiltonian. Clearly, every globally Hamiltonian vector field is locally Hamiltonian.

Proposition 4.1.4.2 (Hamiltonian and Symplectic Vector Fields). *Let (M, ω) be a symplectic manifold. A smooth vector field on M is symplectic if and only if it is locally Hamiltonian. Every locally Hamiltonian vector field on M is globally Hamiltonian if and only if $H^1_{dR}(M) = 0$.*

Proof. By Theorem 3.1.3.14, a smooth vector field X is symplectic if and only if $\mathfrak{L}_X\omega = 0$. Using Cartan's magic formula, we compute

$$\mathfrak{L}_X\omega = d(X \lrcorner \omega) + X \lrcorner d\omega = d(X \lrcorner \omega). \tag{4.1.4.2}$$

Therefore, X is symplectic if and only if the 1-form $X \lrcorner \omega$ is closed. On the one hand, if X is locally Hamiltonian, then in a neighborhood of each point there is a real-valued function f such that $X = X_f$, so $X \lrcorner \omega = df$, which is certainly closed. Conversely, if X is symplectic, then by the Poincaré lemma each point $p \in M$ has a neighborhood U on which the closed 1-form $X \lrcorner \omega$ is exact. This means there is a smooth real-valued function f defined on U such that $X \lrcorner \omega = df$; because ω is nondegenerate, this implies that $X = X_f$ on U.

Now suppose M is a smooth manifold with $H^1_{dR}(M) = 0$. If X is a locally Hamiltonian vector field, then it is symplectic, so (4.1.4.2) shows that $X \lrcorner \omega$ is closed. The hypothesis then implies that there is a function $f \in C^\infty(M)$ such that $X \lrcorner \omega = df$. This means that $X = X_f$, so X is globally Hamiltonian. Conversely, suppose every locally Hamiltonian vector field is globally Hamiltonian. Let η be a closed 1-form, and let X be the vector field $X = \widehat{\omega}^{-1}\eta$. Then (4.1.4.2) shows that $\mathfrak{L}_X\omega = 0$, so X is symplectic and therefore locally Hamiltonian. By hypothesis, there is a global smooth real-valued function f such that $X = X_f$, and then unwinding the definitions, we find that $\eta = df$. □

A symplectic manifold (M, ω) together with a smooth function $H \in C^\infty(M)$ is called a **Hamiltonian system**. The function H is called the **Hamiltonian** of the system; the flow of the Hamiltonian vector field X_H is called its **Hamiltonian flow**, and the integral curves of X_H are called the **trajectories** or **orbits** of the system. In Darboux coordinates, formula (4.1.3.1) implies that the orbits are those curves $\gamma(t) = (x^i(t), y^i(t))$ that satisfy

$$\begin{cases} \dot{x}^i(t) = \dfrac{\partial H}{\partial y^i}(x(t), y(t)), \\ \dot{y}^i(t) = -\dfrac{\partial H}{\partial y^i}(x(t), y(t)). \end{cases} \tag{4.1.4.3}$$

These are called **Hamilton's equations**. Hamiltonian systems play a central role in classical mechanics. We illustrate how they arise with a simple example.

Example 4.1.4.3 (The n-Body Problem). Consider n physical particles moving in space, and suppose their masses are $m_1, \ldots, m_n$. For many purposes, an effective model of such a system is obtained by idealizing the particles as points in $\mathbb{R}^3$, which we denote by $\mathbf{q}_1, \ldots, \mathbf{q}_n$. Writing the coordinates of $\mathbf{q}_k$ at time t as $(q_k^1(t), q_k^2(t), q_k^3(t))$, we can represent the evolution of the system over time by a curve in $\mathbb{R}^{3n}$:

$$q(t) = (q_1^1(t), q_1^2(t), q_1^3(t), \ldots, q_n^1(t), q_n^2(t), q_n^3(t)).$$

The **collision set** is the subset $\mathcal{C} \subseteq \mathbb{R}^{3n}$ where two or more particles occupy the same position in space:

$$\mathcal{C} = \{q \in \mathbb{R}^{3n} : \mathbf{q}_k = \mathbf{q}_l \text{ for some } k \neq l\}.$$

We consider only motions with no collisions, so we are interested in curves in the open subset $Q = \mathbb{R}^{3n} \setminus \mathcal{C}$.

Suppose the particles are acted upon by forces that depend only on the positions of all the particles in the system. (A typical example is gravitational forces.) If we denote the components of the net force on the kth particle by $(F_k^1(q), F_k^2(q), F_k^3(q))$, then Newton's second law of motion asserts that the particles' motion satisfies $m_k\ddot{\mathbf{q}}_k = \mathbf{F}_k(\mathbf{q}(t))$ for each k, which translates into the $3n \times 3n$ system of second-order ODEs

$$\begin{cases} m_k\ddot{q}_k^1(t) = F_k^1(q(t)), \\ m_k\ddot{q}_k^2(t) = F_k^2(q(t)), \\ m_k\ddot{q}_k^3(t) = F_k^3(q(t)), \end{cases}$$

for $k = 1, \ldots, n$.

This can be written in a more compact form if we relabel the $3n$ position coordinates as $q(t) = (q^1(t), \ldots, q^{3n}(t))$ and the $3n$ components of the forces as $F(q) = (F_1(q), \ldots, F_{3n}(q))$, and let $M = (M_{ij})$ denote the $3n \times 3n$ diagonal matrix $\operatorname{diag}(m_1, m_1, m_1, \ldots, m_n, m_n, m_n)$. Then Newton's second law can be written

$$M_{ij}\ddot{q}^j(t) = F_i(q(t)). \tag{4.1.4.4}$$

We assume that the forces depend smoothly on q, so we can interpret $F(q) = (F_1(q), \ldots, F_{3n}(q))$ as the components of a smooth covector field on Q. We assume further that the forces are conservative, which by **??** is equivalent to the existence of a smooth function $V \in C^\infty(Q)$ (called the potential energy of the system) such that $F = -dV$.

Under the physically reasonable assumption that all of the masses are positive, the matrix M is positive definite, and thus can be interpreted as a (constant-coefficient) Riemannian metric on Q. It therefore defines a smooth bundle isomorphism $\widehat{M} : TQ \to T^*Q$. If we denote the natural coordinates on TQ by (q^i, v^i) and those on T^*Q by (q^i, p_i), then $M(v, w) = M_{ij}v^i w^j$, and $\widehat{M}$ has the coordinate representation

$$(q^i, p_i) = \widehat{M}(q^i, v^i) = (q^i, M_{ij}v^j).$$

If $q'(t) = (\dot{q}^1(t), \ldots, \dot{q}^{3n}(t))$ is the velocity vector of the system of particles at time t, then the covector $p(t) = \widehat{M}(q'(t))$ is given by the formula

$$p_i(t) = M_{ij}\dot{q}^j(t). \tag{4.1.4.5}$$

To give this equation a physical interpretation, we can revert to our original labeling of the coordinates and write

$$p(t) = (p_1^1(t), p_1^2(t), p_1^3(t), \ldots, p_n^1(t), p_n^2(t), p_n^3(t)),$$

and then $\mathbf{p}_k(t) = (p_k^1(t), p_k^2(t), p_k^3(t)) = m_k \dot{\mathbf{q}}_k(t)$ is interpreted as the momentum of the k-th particle at time t.

Using (4.1.4.5), we see that a curve $q(t)$ in Q satisfies Newton's second law (4.1.4.4) if and only if the curve $\gamma(t) = (q(t), p(t))$ in T^*Q satisfies the first-order system of ODEs

$$\begin{cases} \dot{q}^i(t) = M^{ij}p_j(t), \\ \dot{p}^i(t) = -\dfrac{\partial V}{\partial q^i}(q(t)), \end{cases} \tag{4.1.4.6}$$

where (M^{ij}) is the inverse of the matrix of (M_{ij}). Define a function $E \in C^\infty(T^*Q)$, called the **total energy** of the system, by

$$E(q, p) = V(q) + K(p),$$

where V is the potential energy introduced above, and K is the **kinetic energy**, defined by

$$K(p) = \frac{1}{2}M^{ij}p_i p_j.$$

Since (q^i, p_i) are Darboux coordinates on T^*Q, a comparison of (4.1.4.6) with (4.1.4.3) shows that (4.1.4.6) is precisely Hamilton's equations for the Hamiltonian flow of E. The fact that E is constant along the trajectories of its own Hamiltonian flow is known as the **law of conservation of energy**.

4.1.4.1 Poisson brackets Hamiltonian vector fields allow us to define an operation on real-valued functions on a symplectic manifoldM similar to the Lie bracket of vector fields. Given $f, g \in C^\infty(M)$, we define their Poisson bracket $\{f, g\} \in C^\infty(M)$ by any of the following equivalent formulas:

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g f. \tag{4.1.4.7}$$

Two functions are said to **Poisson commute** if their Poisson bracket is zero.

The geometric interpretation of the Poisson bracket is evident from the characterization $\{f, g\} = X_g f$: it is a measure of the rate of change of f along the Hamiltonian flow of g. In particular, f and g are Poisson commute if and only if f is constant along the Hamiltonian flow of g.

Using (4.1.4.1), we can readily compute the Poisson bracket of two functions f, g in Darboux coordinates:

$$\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}.$$

Proposition 4.1.4.4 (Properties of the Poisson Bracket). *Suppose (M, ω) is a symplectic manifold, and $f, g \in C^\infty(M)$.*

(a) *Bilinearity: $\{f, g\}$ is linear over $\mathbb{R}$ in f and g.*

(b) *Antisymmetry: $\{f, g\} = -\{g, f\}$.*

(c) *Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.*

(d) *Leibniz rule: $\{f, gh\} = \{f, g\}h + \{f, h\}g$.*

(e) $X_{\{f,g\}} = -[X_f, X_g]$.

Proof. Parts (a) and (b) are obvious from the characterization $\{f, g\} = \omega(X_f, X_g)$ together with the fact that $X_f = \widehat{\omega}^{-1}(df)$ depends linearly on f. Because of the nondegeneracy of ω, to prove (e), it suffices to show that the following holds for every vector field Y:

$$\omega(X_{\{f,g\}}, Y) + \omega([X_f, X_g], Y) = 0. \tag{4.1.4.8}$$

On the one hand, note that $\omega(X_{\{f,g\}}, Y) = d(\{f, g\})(Y) = Y\{f, g\} = YX_g f$. On the other hand, because Hamiltonian vector fields are symplectic, the Lie derivative formula of Corollary 3.1.3.10 yields

$$\begin{aligned} 0 = (\mathfrak{L}_{X_g}\omega)(X_f, Y) &= X_g(\omega(X_f, Y)) - \omega([X_g, X_f], Y) - \omega(X_f, [X_g, Y]) \\ &= X_g(df(Y)) + \omega([X_f, X_g], Y) - df([X_g, Y]) \\ &= X_g Y f + \omega([X_f, X_g], Y) - [X_g, Y]f \\ &= \omega([X_f, X_g], Y) + YX_g f. \end{aligned}$$

Therefore (e) follows, and (c) follows from (e), (b), and (4.1.4.7):

$$\begin{aligned} \{f, \{g, h\}\} &= X_{\{g,h\}} f = -[X_g, X_h] f = -X_g X_h f + X_h X_g f \\ &= -X_g\{f, h\} + X_h\{f, g\} = -\{\{f, h\}, g\} + \{\{f, g\}, h\} \\ &= -\{g, \{h, f\}\} - \{h, \{f, g\}\}. \end{aligned}$$

Finally, we note that by Leibniz rule:

$$\{gh, f\} = X_f(gh) = (X_f g)h + g(X_f h) = \{g, f\}h + g\{h, f\}$$

from which (d) follows. □

Corollary 4.1.4.5. *If (M, ω) is a symplectic manifold, the vector space $C^\infty(M)$ is a Lie algebra under the Poisson bracket.*

If (M, ω, H) is a Hamiltonian system, any function $f \in C^\infty(M)$ that is constant on every integral curve of X_H is called a **conserved quantity** of the system. Conserved quantities turn out to be deeply related to symmetries, as we now show.

A smooth vector field V on M is called an **infinitesimal symmetry** of (M, ω, H) if both ω and H are invariant under the flow of V.

Proposition 4.1.4.6. *Let (M, ω, H) be a Hamiltonian system.*

(a) *A function $f \in C^\infty(M)$ is a conserved quantity if and only if $\{f, H\} = 0$.*

(b) *The infinitesimal symmetries of (M, ω, H) are precisely the symplectic vector fields V that satisfy $VH = 0$.*

(c) *If θ is the flow of an infinitesimal symmetry and γ is a trajectory of the system, then for each $s \in \mathbb{R}$, $\theta_s \circ \gamma$ is also a trajectory on its domain of definition.*

Proof. Since the Possion bracket $\{f, H\}$ measures the rate of change of f along the Hamiltonian flow of H, it is clear that f is conserved if and only if $\{f, H\} = 0$. This proves (a). For (b), ω is invairant under the flow of V if and only if V is symplectic, and H is invairant under the flow of V if and only if $VH = 0$. Therefore (b) is clear.

Finally, for (c), γ is a trajectory of the system means $\gamma'(t) = X_H|_{\gamma(t)}$. Since θ is a flow of an infinitesimal symmetry, ω and H are invairant under V. This then implies

$$(\theta_s \circ \gamma)'(t) = d(\theta_s)_{\gamma(t)}(\gamma'(t)) = d(\theta_s)_{\gamma(t)}(V_{\gamma(t)}) = V_{\theta_s \circ \gamma(t)}.$$

Therefore $\theta_s \circ \gamma$ is still a trajectory of the system. □

The following theorem, first proved (in a somewhat different form) by Emmy Noether in 1918, has had a profound influence on both physics and mathematics. It shows that for many Hamiltonian systems, there is a one-to-one correspondence between conserved quantities (modulo constants) and infinitesimal symmetries.

Theorem 4.1.4.7 (Noether's Theorem). *Let (M, ω, H) be a Hamiltonian system. If f is any conserved quantity, then its Hamiltonian vector field is an infinitesimal symmetry. Conversely, if $H^1_{dR}(M) = 0$, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to addition of a function that is constant on each component of M.*

Proof. Suppose f is a conserved quantity. Then Proposition 4.1.4.6 shows that $\{f, H\} = 0$. This in turn implies that $X_f H = \{H, f\} = 0$, so H is constant along the flow of X_f. Since ω is invariant along the flow of any Hamiltonian vector field by Proposition 4.1.4.2, this shows that X_f is an infinitesimal symmetry.

Now suppose that M is a smooth manifold with $H^1_{dR}(M) = 0$. Let V be an infinitesimal symmetry of (M, ω, H). Then V is symplectic by definition, and globally Hamiltonian by Proposition 4.1.4.2. Writing $V = X_f$, the fact that H is constant along the flow of V implies that $\{H, f\} = X_f H = VH = 0$, so f is a conserved quantity. If $\widetilde{f}$ is any other function that satisfies $X_{\widetilde{f}} = X_f$, then $d(\widetilde{f} - f) = \widehat{\omega}(X_{\widetilde{f}} - X_f) = 0$, so $\widetilde{f} - f$ must be constant on each component of M. □

There is one conserved quantity that every Hamiltonian system possesses: the Hamiltonian H itself. The infinitesimal symmetry corresponding to it, of course, generates the Hamiltonian flow of the system, which describes how the system evolves over time. Since H is typically interpreted as the total energy of the system, one usually says that the symmetry corresponding to conservation of energy is "translation in the time variable".

4.1.4.2 Hamiltonian Flowouts Hamiltonian vector fields are powerful tools for constructing isotropic and Lagrangian submanifolds. Because Lagrangian submanifolds of T^*M correspond to closed 1-forms (Proposition 4.1.2.4), which in turn correspond locally to differentials of functions, such constructions have numerous applications in PDE theory.

Theorem 4.1.4.8 (Hamiltonian Flowout Theorem). *Suppose (M, ω) is a symplectic manifold, $H \in C^\infty(M)$, Γ is an embedded isotropic submanifold of M that is contained in a single level set of H, and the Hamiltonian vector field X_H is nowhere tangent to Γ. If $\mathcal{S}$ is a flowout from Γ along X_H, then $\mathcal{S}$ is also isotropic and contained in the same level set of H.*

Proof. Let θ be the flow of X_H. Recall from Theorem 1.2.4.1 that the flowout is parametrized by the restriction of θ to a neighborhood $\mathcal{O}_\delta$ of $\{0\} \times \Gamma$ in $\mathbb{R} \times \Gamma$. First consider a point $p \in \Gamma \subseteq \mathcal{S}$. If we choose a basis $E_1, \dots, E_k$ for $T_p\Gamma$, then $T_p\mathcal{S}$ is spanned by $(X_H|_p, E_1, \dots, E_k)$. The assumption that Γ is isotropic implies that $\omega_p(E_i, E_j) = 0$ for all i and j. On the other hand, by definition of the Hamiltonian vector field,

$$\omega_p(X_H|_p, E_j) = dH_p(E_j) = 0.$$

because E_j is tangent to Γ, which is contained in a level set of H. This shows that the restriction of ω to $T_p\mathcal{S}$ is zero when $p \in \Gamma$.

Any other point $p;' \in \mathcal{S}$ is of the form $p' = \theta_t(p)$ for some $(t, p) \in \mathcal{O}_\delta$. Because θ_t is a local diffeomorphism that maps a neighborhood of p in $\mathcal{S}$ to a neighborhood of p' in $\mathcal{S}$, its differential takes $T_p\mathcal{S}$ isomorphically onto $T_{p'}\mathcal{S}$. Thus, for any vectors $v, w \in T_{p'}\mathcal{S}$, there are vectors $\widehat{v}, \widehat{w}$ such that $d(\theta_t)_p(\widehat{v}) = v$ and $d(\theta_t)_p(\widehat{w}) = w$. Moreover, because X_H is a symplectic vector field, its flow preserves ω. Therefore,

$$\omega_{p'}(v, w) = \omega_{p'}(d(\theta_t)_p(\widehat{v}), d(\theta_t)_p(\widehat{w})) = (\theta_t^*\omega)_p(\widehat{v}, \widehat{w}) = \omega_p(\widehat{v}, \widehat{w}) = 0.$$

It follows that $\mathcal{S}$ is isotropic. By Proposition 4.1.4.1, H is constant along each integral curve of X_H, so $\mathcal{S}$ is contained in the same level set of H as Γ. □

4.1.5 Coadjoint orbits

Coadjoint orbits arise in a natural way for any given Lie group G. The group G operates via a coadjoint representation Ad^* on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, by the formula

$$\langle \mathrm{Ad}^*_g \mu, Y \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}} Y \rangle$$

for $g \in G$, $\mu \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$. The orbits of this action are called **coadjoint orbits** and can (under known conditions) be made into symplectic manifolds. More precisely, the goal of this subsection will be to discuss the following theorem of Kostant and Souriau:

Theorem 4.1.5.1. *Let G be a Lie group with $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = \{0\}$ for $\mathfrak{g} = \mathfrak{Lie}(G)$. Then there is (up to covering) a one-to-one correspondence between the symplectic manifolds with transitive G-operation and G-orbits in $\mathfrak{g}^*$.*

For what follows, we fix G to be a Lie group and $\mathfrak{g}$ its Lie algebra, which we identify with T_eG or with the space of all left-invariant vector fields X on G. Analogously, $\mathfrak{g}^* = T_e^*G$ is identified with the space of left-invariant differential forms of degree 1, and from this it then follows that the space $\Omega_l^q(G)$ of left-invariant q-forms can be identified with $\bigwedge^q\mathfrak{g}^*$, the space of alternating q-forms on $\mathfrak{g}$. Via this identification of $\Omega_l^q(G)$ with $\bigwedge^q\mathfrak{g}^*$, the operators of exterior differentiation on $\Omega_l^q(G)$ are exactly the coboundary operators δ defined in **??** (Lie algebra cohomology). Thus, in particular, the space $Z^2(\mathfrak{g})$ of 2-cycles in $\bigwedge^2\mathfrak{g}^*$ can be identified with the space of left-invariant closed differentials on G.

The coadjoint representation on $\Omega_l^q(G)$ can now be easily described as follows: Recall that the coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ is the dual of the adjoint representation of Ad, which is induced by inner automorphisms $C : G \to \mathrm{Aut}(G)$ of G. For $q > 1$, this induces a representation

$$\mathrm{Ad}^* : G \to \mathrm{Aut}(\textstyle\bigwedge^q\mathfrak{g}^*).$$

In those cases where $\bigwedge^q\mathfrak{g}^*$ is identified with the space of left invariant q-forms ω on G, we get the formula

$$\mathrm{Ad}^*_{g_0}\omega = (C_{g_0})^*\omega = (R_{g_0})^*\omega$$

where L is the right translation on G.

Unlike Kirillov, we do not study the coadjoint orbits in $\mathfrak{g}^*$ directly, that is, the G-orbits in the coadjoint representation in $\mathfrak{g}^*$

$$G \cdot \mu = \{\mathrm{Ad}^*_{g_0}\mu \mid g_0 \in G\}$$

for $\mu \in \mathfrak{g}^*$, but rather study the orbits in $\bigwedge^2\mathfrak{g}^*$. We begin with some notation. Let (M, ω) be a symplectic manifold on which G operates on the left as smooth maps

$$G \times M \to M, \quad (g, m) \mapsto gm.$$

Then we may consider two maps

$$\begin{aligned} \phi_g &: M \to M, \quad m \mapsto gm \\ \theta^m &: G \to M, \quad g \mapsto gm. \end{aligned}$$

In the situation where ω is fixed by each ϕ_g, $g \in G$, we call this given operation of G on M a symplectic operation.

For the right translation R and the left translation L we clearly have

$$\phi_g \circ \theta^{(m)} = \theta^{(m)} \circ L_g, \quad \theta^{(gm)} = \theta^{(m)} \circ L_g.$$

With the help of $\theta^{(m)}$, forms can be pulled back from M to G; in particular, the closed form ω on M induces a closed form on G. More precisely, we have the following statement:

Theorem 4.1.5.2. (a) *A symplectic group operation $G \times M \to M$ defines a map*

$$\Psi : M \to Z^2(\mathfrak{g}), \quad m \mapsto (\theta^{(m)})^*\omega$$

with

$$\Psi(gm) = \mathrm{Ad}^*_g(\Psi(m)). \tag{4.1.5.1}$$

(b) *$\Psi(M)$ is a union of G-orbits in $Z^2(\mathfrak{g})$.*

Proof. We note that $(\theta^{(m)})^*\omega$ is a 2-form on G and is left-invariant as we have

$$(L_g)^*(\theta^{(m)})^*\omega = (\theta^{(m)} \circ L_g)^*\omega = (\phi_g \circ \theta^{(m)})^*\omega = (\theta^{(m)})^* \circ (L_g)^*\omega = (\theta^{(m)})^*\omega.$$

To see that Ψ is a G-morphism, we note that

$$\Psi(gm) = (\theta^{(gm)})^*\omega = (\theta^{(m)} \circ R_g)^*\omega = (R_g)^*(\theta^{(m)})^*\omega = (R_g)^*\Psi(m) = \mathrm{Ad}^*(g)\Psi(m).$$

From the fact that $\Phi(m) = (\theta^{(m)})^*\omega$, we see that $d\Psi(m) = 0$; this proves (a), and (b) follows from (a). Finally, if the action of G is transitive on M, then it is clear that $\Psi(M)$ consists of a single orbit in $Z^2(\mathfrak{g})$. □

We now consider the inverse question: for $\omega \in Z^2(\mathfrak{g})$ and a given G-orbit $G \cdot \omega$ in $Z^2(\mathfrak{g})$, are there a symplectic manifold M and a map $\Psi : M \to Z^2(\mathfrak{g})$ as above with $G \cdot \omega = \Psi(M)$? This gives us cause to ask whether M can be constructed as a homogeneous space of the form $M = G/H$, where H is a closed Lie subgoup of G. To find such H, the difficulty will be in showing that a symplectic form $\overline{\omega}$ on G/H can be defined, so that for the given closed $\omega \in Z^2(\mathfrak{g})$ we have $\pi^*\overline{\omega} = \omega$, where $\pi : G \to G/H$ is the natural projection. For this, let $\omega \in Z^2(\mathfrak{g})$ be a closed form on G and set

$$\mathfrak{h}_\omega = \{X \in \mathfrak{g} \mid X \lrcorner\, \omega = 0\}.$$

Lemma 4.1.5.3. *$\mathfrak{h}_\omega$ is a subalgebra and $\mathfrak{h}_\omega = \{0\}$ when ω is non-degenerate.*

Proof. The last statement follows immediately from the definition of an inner product $i_X\omega$. To prove the first statement, we note that for $Z \in \mathfrak{g}$, in terms of the formula (3.1.3.3), we have

$$0 = \mathfrak{L}_Y(\omega(X,Z)) = (\mathfrak{L}_Y\omega)(X,Z) + \omega([Y,X],Z) + \omega(X,[Y,Z])$$

as $\omega(X,Z)$ is constant. On the other hand, by Cartan's formula (3.3.3.6), we have

$$\mathfrak{L}_Y\omega = i_Y d\omega + d(i_Y\omega) = 0.$$

These together imply that $\omega([Y,X],Z) = 0$, so $[X,Y] \in \mathfrak{h}_\omega$. □

By Theorem 2.2.3.5, there then exists a subgroup H_ω of G such that $\mathfrak{Lie}(H_\omega) = \mathfrak{h}_\omega$. This gives the desired H_ω:

Theorem 4.1.5.4. *Suppose that H_ω is closed, then there exists a symplectic form $\overline{\omega}$ on $M_\omega := G/H_\omega$ with $\omega = \pi^*\overline{\omega}$, where $\pi : G \to G/H_\omega = M_\omega$ is the canonical projection.*

Proof. Since H_ω is closed, Theorem 2.3.4.2 tells us that M_ω is a smooth manifold. More precisely, we define a distribution D by $D_e = \mathfrak{h}_\omega$ for the identity element $e \in G$, and $D_g = (L_g)_*\mathfrak{h}_\omega$ for $g \in G$ with left translation L_g. From Lemma 4.1.5.3, it immediately follows that D is involutive, so every point $g \in G$ has an integral manifold N_g. These are clearly given by $N_g = gH_\omega$, and by Frobenius' theorem, N_g at every $g \in G$ can be locally written as

$$N_g = \{x_i \text{ is constant for } i = 1,\dots,k\}, \quad k = n - \dim(h_\omega).$$

The tangent space T_xN_g then has a basis $(\partial/\partial_{x_{k+1}},\dots,\partial/\partial x_n)$. Now the given closed 2-form ω on G has the form

$$\omega = \sum_{i<j} a_{ij}(x)dx_i \wedge dx_j.$$

It follows from the construction of $\mathfrak{h}_\omega$ that $i_X\omega = 0$ for $X \in \mathfrak{h}_\omega$, thus, in particular, for $X = \partial/\partial x_i$, $i = k+1,\dots,n$. This and the fact that $d\omega = 0$ show that ω, in these coordinates, ω can be dependent only on $x_1,\dots,x_k$, so

$$\omega = \sum_{i<j}^{k} a_{ij}(x_1,\dots,x_k)dx_i \wedge dx_j.$$

With this, we are essentially done. The integral manifolds N_a of D are the fibers of the projection $\pi : G \to M_\omega$ and M_ω is described in the local coordinates $(x_1,\dots,x_k)$. For

$$\overline{\omega} = \sum_{i<j}^{k} a_{ij}(x_1,\dots,x_k)dx_i \wedge dx_j$$

we naturally have $\pi^*\overline{\omega} = \omega$. It is now routine to see that in this manner also a global form $\overline{\omega}$ on M_ω, with the desired properties can be given. □

With this last theorem, we can now answer the question proposed above.

Proposition 4.1.5.5. *For a given orbit $G \cdot \omega$ in $Z^2(\mathfrak{g})$, H_ω and (since under the presumptions H_ω is closed) the manifold M_ω are fixed. Then $G \cdot \omega$ is the image of M_ω under the map Ψ. Thus*

$$\Psi(M_\omega) = G \cdot \omega.$$

Proof. When $m_0 = eH_\omega$ is interpreted as a point of M_ω, we have, for $g \in G$, with the canonical projection $\pi : G \to M_\omega$,

$$\pi(g) = gH_\omega = gm_0 = \theta^{(m_0)}(g),$$

thus $\pi = \theta^{(m_0)}$. When we further define $\Psi : M_\omega \to Z^2(\mathfrak{g})$ as in Theorem 4.1.5.2, we get

$$\Psi(m_0) = \pi^* \overline{\omega} = \omega.$$

Since Ψ is a G-morphism, we conclude that $\Psi(M_\omega) = G \cdot \omega$, since G operates transitively on M_ω. $\square$