

Analysis

Longer

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Chapter 1

Tensors and forms on manifolds

1.1 Tensors

1.1.1 Multilinear algebra

Suppose V_1, \dots, V_k , and W are vector spaces. A map $F : V_1 \times \dots \times V_k \rightarrow W$ is said to be **multilinear** if it is linear as a function of each variable. Let us write $\mathcal{L}(V_1, \dots, V_k; W)$ for the set of all multilinear maps from $V_1 \times \dots \times V_k$ to W . It is a vector space under the usual operations of pointwise addition and scalar multiplication.

Example 1.1.1 (Some Familiar Multilinear Functions).

- (a) The inner product in \mathbb{R}^n is a scalar-valued bilinear function of two vectors, used to compute lengths of vectors and angles between them.
- (b) The cross product in \mathbb{R}^3 is a vector-valued bilinear function of two vectors, used to compute areas of parallelograms and to find a third vector orthogonal to two given ones.
- (c) The determinant is a real-valued multilinear function of n vectors in \mathbb{R}^n , used to detect linear independence and to compute the volume of the parallelepiped spanned by the vectors.
- (d) The bracket in a Lie algebra \mathfrak{g} is a \mathfrak{g} -valued bilinear function of two elements of \mathfrak{g} .

The next example is probably not as familiar, but it is extremely important

Example 1.1.2 (Tensor Products of Covectors). Suppose V is a vector space, and $\omega, \eta \in V^*$. Define a function $\omega \otimes \eta : V \times V \rightarrow \mathbb{R}$ by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$$

where the product on the right is just ordinary multiplication of real numbers. The linearity of ω and η guarantees that $\omega \otimes \eta$ is a bilinear function of v_1 and v_2 , so it is an element of $\mathcal{L}(V_1, V_2; \mathbb{R})$. For example, if (e^1, e^2) denotes the standard dual basis for $(\mathbb{R}^2)^*$, then $e^1 \otimes e^2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is the bilinear function

$$e^1 \otimes e^2((w, x), (y, z)) = wz$$

The last example can be generalized to arbitrary multilinear functions as follows: let V_1, \dots, V_k and W_1, \dots, W_l be real vector spaces, and suppose $F \in \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$ and $G \in \mathcal{L}(W_1, \dots, W_l; \mathbb{R})$. Define a function

$$F \otimes G : V_1 \times \dots \times V_k \times W_1 \times \dots \times W_l \rightarrow \mathbb{R}$$

by

$$F \otimes G(v_1, \dots, v_k, w_1, \dots, w_l) = F(v_1, \dots, v_k)G(w_1, \dots, w_l)$$

It follows from the multilinearity of F and G that $F \otimes G$ depends linearly on each argument v_i or w_j separately, so $F \otimes G$ is an element of $\mathcal{L}(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$, called the **tensor product of F and G** .

Proposition 1.1.3. *The tensor product operation is bilinear and associative: $F \otimes G$ depends bilinearly on F and G , and $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.*

Because of the result of the preceding proposition, we can write tensor products of multilinear functions unambiguously without parentheses. For example, if $\omega^j \in V_j^*$ for $1 \leq j \leq k$, then $\omega^1 \otimes \dots \otimes \omega^k$ is the multilinear function given by

$$\omega^1 \otimes \dots \otimes \omega^k(v_1, \dots, v_k) = \omega^1(v_1) \dots \omega^k(v_k)$$

The tensor product operation is important in part because of its role in the following proposition.

Proposition 1.1.4 (A Basis for the Space of Multilinear Functions). *Let V_1, \dots, V_k be real vector spaces of dimensions n_1, \dots, n_k , respectively. For each $j \in \{1, \dots, k\}$, let $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ be a basis for V_j , and let $(\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j})$ be the corresponding dual basis for V_j^* . Then the set*

$$\mathcal{B} = \{\varepsilon_{(1)}^{i_1} \otimes \dots \otimes \varepsilon_{(k)}^{i_k} : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$$

is a basis for $\mathcal{L}(V_1, \dots, V_k; \mathbb{R})$, which therefore has dimension equal to $n_1 \dots n_k$.

Proof. Suppose $F \in \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$ is arbitrary. For each ordered k -tuple $i = (i_1, \dots, i_k)$ of integers with $1 \leq i_j \leq n_j$, define a number $F_{i_1 \dots i_k}$ by

$$F_{i_1 \dots i_k} = F(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}).$$

Then

$$F = F_{i_1 \dots i_k} \varepsilon_{(1)}^{i_1} \otimes \dots \otimes \varepsilon_{(k)}^{i_k}.$$

Also, \mathcal{B} is linearly independent by the observation

$$\varepsilon_{(1)}^{i_1} \otimes \dots \otimes \varepsilon_{(k)}^{i_k}(E_{i'_1}^{(1)}, \dots, E_{i'_k}^{(k)}) = \delta_{i'_1 \dots i'_k}^{i_1 \dots i_k}.$$

□

Abstract tensor products of vector spaces

For any set S , a **formal linear combination** of elements of S is a function $f : S \rightarrow \mathbb{R}$ such that $f(s) = 0$ for all but finitely many $s \in S$. The **free vector space** on S , denoted by $F(S)$, is the set of all formal linear combinations of elements of S . Under pointwise addition and scalar multiplication, $F(S)$ becomes a vector space over \mathbb{R} .

For each element $x \in S$, there is a function δ_x that takes the value 1 on x and zero on all other elements of S ; typically we identify this function with x itself, and thus think of S as a subset of $F(S)$. Then we see S is a basis of $F(S)$.

Proposition 1.1.5 (Characteristic Property of the Free Vector Space). *For any set S and any vector space W , every map $A : S \rightarrow W$ has a unique extension to a linear map $\tilde{A} : F(S) \rightarrow W$:*

$$\begin{array}{ccc} F(S) & \xrightarrow{\tilde{A}} & W \\ \uparrow & \nearrow A & \\ S & & \end{array}$$

Now let V_1, \dots, V_k be real vector spaces. Consider the free vector space $F(V_1 \times \dots \times V_k)$, let R is the subspace of $F(V_1 \times \dots \times V_k)$ spanned by all elements of the following forms:

$$\begin{aligned} & (v_1, \dots, av_i, \dots, v_k) - a(v_1, \dots, v_k) \\ & (v_1, \dots, v_i + v'_i, \dots, v_k) - (v_1, \dots, v_i, \dots, v_k) - (v_1, \dots, v'_i, \dots, v_k) \end{aligned}$$

with $v_j, v'_j \in V_j, i \in \{1, \dots, k\}$, and $a \in \mathbb{R}$.

We define the **tensor product** of V_1, \dots, V_k to be

$$V_1 \otimes \dots \otimes V_k = F(V_1 \times \dots \times V_k) / R$$

and let $\pi : F(V_1 \times \dots \times V_k) \rightarrow V_1 \otimes \dots \otimes V_k$ be the natural projection. The equivalence class of an element (v_1, \dots, v_k) in $V_1 \otimes \dots \otimes V_k$ is denoted by

$$v_1 \otimes \dots \otimes v_k = \pi(v_1, \dots, v_k)$$

and is called the **tensor product** of v_1, \dots, v_k .

Proposition 1.1.6 (Characteristic Property of the Tensor Product Space). *Let V_1, \dots, V_k be finite-dimensional real vector spaces. If $A : V_1 \times \dots \times V_k \rightarrow X$ any multilinear map into a vector space X , then there is a unique linear map $\tilde{A} : V_1 \otimes \dots \otimes V_k \rightarrow X$ such that the following diagram commutes:*

$$\begin{array}{ccc} V_1 \times \dots \times V_k & \xrightarrow{A} & X \\ \downarrow \pi & \nearrow \tilde{A} & \\ V_1 \otimes \dots \otimes V_k & & \end{array}$$

where π is the map $\pi(v_1, \dots, v_k) = v_1 \otimes \dots \otimes v_k$.

or prod basis

Proposition 1.1.7 (A Basis for the Tensor Product Space). *Suppose V_1, \dots, V_k are real vector spaces of dimensions n_1, \dots, n_k , respectively. For each $1 \leq j \leq k$, suppose $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ is a basis for V_j . Then the set*

$$C = \{E_{i_1}^{(1)} \otimes \dots \otimes E_{i_k}^{(k)} : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$$

Proposition 1.1.8 (Associativity of Tensor Product Spaces). *Let V_1, V_2, V_3 be finite-dimensional real vector spaces. There are unique isomorphisms*

$$V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$$

under which elements of the forms $v_1 \otimes (v_2 \otimes v_3)$ and $(v_1 \otimes v_2) \otimes v_3$ all correspond.

The connection between tensor products in this abstract setting and the more concrete tensor products of multilinear functionals that we defined earlier is based on the following proposition.

Proposition 1.1.9. *If V_1, \dots, V_k are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$$

Proof. First, define a map $\Phi : V_1^* \times \cdots \times V_k^* \rightarrow \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$ by

$$\Phi(\omega^1, \dots, \omega^k)(v_1, \dots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$$

The expression on the right depends linearly on each v_i , so $\Phi(\omega^1, \dots, \omega^k)$ is indeed an element of the space $\mathcal{L}(V_1, \dots, V_k; \mathbb{R})$. It is easy to check that Φ is multilinear as a function of $\omega^1, \dots, \omega^k$, so by the characteristic property it descends uniquely to a linear map $\tilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \rightarrow \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$, which satisfies

$$\tilde{\Phi}(\omega^1 \otimes \cdots \otimes \omega^k)(v_1, \dots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k).$$

It follows immediately from the definition that $\tilde{\Phi}$ takes abstract tensor products to tensor products of covectors. It also takes the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis for $\mathcal{L}(V_1, \dots, V_k; \mathbb{R})$, so it is an isomorphism. \square

Using this canonical isomorphism, we henceforth use the notation $V_1^* \otimes \cdots \otimes V_k^*$ to denote either the abstract tensor product space or the space $\mathcal{L}(V_1, \dots, V_k; \mathbb{R})$, focusing on whichever interpretation is more convenient for the problem at hand. Since we are assuming the vector spaces are all finite-dimensional, we can also identify each V_j with its second dual space V_j^{**} , and thereby obtain another canonical identification

$$V_1 \otimes \cdots \otimes V_k \cong \mathcal{L}(V_1^*, \dots, V_k^*; \mathbb{R})$$

Covariant and contravariant tensors on a vector space

Let V be a finite-dimensional real vector space. If k is a positive integer, a **covariant k -tensor on V** is an element of the k -fold tensor product $V^* \otimes \cdots \otimes V^*$, which we typically think of as a real-valued multilinear function of k elements of V :

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ folds}} \rightarrow \mathbb{R}$$

The number k is called the **rank of α** . A 0-tensor is, by convention, just a real number. We denote the vector space of all covariant k -tensors on V by the shorthand notation

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ folds}}$$

Let us look at some examples.

Example 1.1.10 (Covariant Tensors). Let V be a finite-dimensional vector space.

- (a) Every linear functional $\omega : V \rightarrow \mathbb{R}$ is multilinear, so a covariant 1-tensor is just a covector. Thus, $T^1(V^*)$ is equal to V^* .
- (b) A covariant 2-tensor on V is a real-valued bilinear function of two vectors, also called a **bilinear form**. One example is the inner product on \mathbb{R}^n . More generally, every inner product is a covariant 2-tensor.
- (c) The determinant, thought of as a function of n vectors, is a covariant n -tensor on \mathbb{R}^n .

For some purposes, it is important to generalize the notion of covariant tensors as follows. For any finite-dimensional real vector space V , we define the space of contravariant tensors on V of rank k to be the vector space

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ folds}}$$

In particular, $T^1(V) = V$, and by convention $T^0(V) = \mathbb{R}$. Because we are assuming that V is finite-dimensional, it is possible to identify this space with the set of multilinear functionals of k covectors:

$$T^k(V) \cong \{\text{multilinear functions } \alpha : V^* \times \cdots \times V^* \rightarrow \mathbb{R}\}$$

Even more generally, for any nonnegative integers k, l , we define the space of **mixed tensors on V of type k, l** as

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ folds}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ folds}}$$

Some of these spaces are identical:

$$\begin{aligned} T^{(0,0)}(V) &= T^0(V) = T^0(V^*) = \mathbb{R} \\ T^{(1,0)} &= T^1(V) = V, \quad T^{(0,1)}(V) = T^1(V^*) = V^* \\ T^{(k,0)} &= T^k(V), \quad T^{(0,k)}(V) = T^k(V^*) \end{aligned}$$

When V is finite-dimensional, any choice of basis for V automatically yields bases for all of the tensor spaces over V . The following corollary follows immediately from Proposition 1.1.7.

Corollary 1.1.11. *Let V be an n -dimensional real vector space. Suppose (E_i) is any basis for V and (ε^j) is the dual basis for V^* . Then the basis for the tensor spaces $T^{(k,l)}(V)$ is given by*

$$\mathcal{B} = \{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \leq i_1, \dots, i_k, j_1, \dots, j_l \leq n\}$$

Therefore, $\dim T^{(k,l)}(V) = n^{k+l}$.

In particular, once a basis is chosen for V , every covariant k -tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

For example, $T^2(V^*)$ is the space of bilinear forms on V , and every bilinear form can be written as $\beta = \beta_{ij} \varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}) .

A less obvious, but extremely important, identification is the following:

$$T^{(1,1)}(V) \cong \text{End}(V)$$

where $\text{End}(V)$ denotes the space of linear maps from V to itself (also called the endomorphisms of V). This is a special case of the following proposition.

ensor end iso

Proposition 1.1.12. *Let V be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between $T^{(k+1,l)}(V)$ and the space of multilinear maps*

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ folds}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ folds}} \rightarrow V$$

Proof. Consider the map $\Phi : \mathcal{L}(V^* \times \cdots \times V^* \times V \times \cdots \times V; V)$ defined by letting ΦA be the

$(k+1, l+1)$ tensor defined by

$$\Phi A(\alpha_1, \dots, \alpha_{k+1}, \beta_1, \dots, \beta_{l+1}) = \alpha_{k+1} A(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l)$$

where $\alpha_i \in V^*, \beta_j \in V$. It is clear that this is an isomorphism. \square

We can use the result of Proposition [1.1.12](#) to define a natural operation called **trace** or **contraction**, which lowers the rank of a tensor by 2. In one special case, it is easy to describe: the operator $\text{tr} : T^{(1,1)}(V) \rightarrow \mathbb{R}$ is just the trace of F when it is regarded as an endomorphism of V , or in other words the sum of the diagonal entries of any matrix representation of F . Since the trace of a linear endomorphism is basis-independent, this is well defined. More generally, we define $\text{tr} : T^{(k+1, l+1)}(V) \rightarrow T^{(k, l)}(V)$ by letting $(\text{tr} F)(\omega^1, \dots, \omega^k, v_1, \dots, v_l)$ be the trace of the $(1, 1)$ -tensor

$$F(\omega^1, \dots, \omega^k, \cdot, v_1, \dots, v_l, \cdot) \in T^{(1,1)}(V).$$

In terms of a basis, the components of $\text{tr} F$ are

$$(\text{tr} F)_{j_1 \dots j_l}^{i_1 \dots i_k} = F_{j_1 \dots j_{l+1}}^{i_1 \dots i_{k+1}}$$

In other words, just set the last upper and lower indices equal and sum.

1.1.2 Symmetric tensors

Let V be a finite-dimensional vector space. A covariant k -tensor α on V is said to be symmetric if its value is unchanged by interchanging any pair of arguments:

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

whenever $1 \leq i < j \leq k$.

The set of symmetric covariant k -tensors is a linear subspace of the space $T^k(V^*)$ of all covariant k -tensors on V ; we denote this subspace by $\Sigma^k(V^*)$. There is a natural projection from $T^k(V^*)$ to $\Sigma^k(V^*)$ defined as follows. First, let \mathfrak{S}_k denote the symmetric group on k elements. Let \mathfrak{S}_k act on $T^k(V^*)$ by

$$\sigma \cdot \alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

We define a projection $\text{Sym} : T^k(V^*) \rightarrow \Sigma^k(V^*)$ called symmetrization by

$$\text{Sym} \alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \cdot \alpha$$

More explicitly, this means that

$$\text{Sym} \alpha(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Proposition 1.1.13 (Properties of Symmetrization). *Let α be a covariant tensor on a finite-dimensional vector space.*

(a) $\text{Sym} \alpha$ is symmetric.

(b) $\text{Sym} \alpha = \alpha$ if and only if α is symmetric.

Proof. Suppose $\alpha \in T^k(V^*)$. If $\tau \in \mathfrak{S}_k$ is any permutation, then

$$\tau \cdot \text{Sym} \alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \tau \cdot (\sigma \cdot \alpha) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\tau \sigma) \cdot \alpha = \frac{1}{k!} \sum_{\eta \in \mathfrak{S}_k} \eta \cdot \alpha = \text{Sym} \alpha.$$

This shows that $\text{Sym}\alpha$ is symmetric.

If α is symmetric, then it follows that $\sigma \cdot \alpha = \alpha$ for every $\sigma \in \mathfrak{S}_k$, so it follows immediately that $\text{Sym}\alpha = \alpha$. On the other hand, if $\text{Sym}\alpha = \alpha$, then α is symmetric because part (a) shows that $\text{Sym}\alpha$ is. \square

If α and β are symmetric tensors on V , then $\alpha \otimes \beta$ is not symmetric in general. However, using the symmetrization operator, it is possible to define a new product that takes a pair of symmetric tensors and yields another symmetric tensor. If $\alpha \in \Sigma^k(V^*)$ and $\beta \in \Sigma^l(V^*)$, we define their **symmetric product** to be the $(k+l)$ -tensor $\alpha\beta$ (denoted by juxtaposition with no intervening product symbol) given by

$$\alpha\beta = \text{Sym}(\alpha \otimes \beta)$$

More explicitly, the action of $\alpha\beta$ on vectors v_1, \dots, v_{k+l} is given by

$$\alpha\beta(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

Proposition 1.1.14 (Properties of the Symmetric Product). *The symmetric product is symmetric, bilinear and associative.*

Proof. We only prove (a). We first verify that for any tensors $\omega_1 \in T^k(V^*)$, $\omega_2 \in T^l(V^*)$,

$$\text{Sym}(\text{Sym}\omega_1 \otimes \omega_2) = \text{Sym}(\omega_1 \otimes \text{Sym}\omega_2) = \text{Sym}(\omega_1 \otimes \omega_2) \quad (1.1) \quad \boxed{\text{symmetric product}}$$

Indeed, if we embed \mathfrak{S}_k into \mathfrak{S}_{k+l} in the obvious way, and denote the image of σ by $\tilde{\sigma}$. Then

$$\begin{aligned} \text{Sym}(\text{Sym}\omega_1 \otimes \omega_2) &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \text{Sym}((\sigma \cdot \omega_1) \otimes \omega_2) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{(k+l)!} \sum_{\tau \in \mathfrak{S}_{k+l}} \tau \cdot ((\sigma \cdot \omega_1) \otimes \omega_2) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{(k+l)!} \sum_{\tau \in \mathfrak{S}_{k+l}} \tau(\tilde{\sigma} \cdot (\omega_1 \otimes \omega_2)) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{(k+l)!} \sum_{\tau \in \mathfrak{S}_{k+l}} (\tau\tilde{\sigma}) \cdot (\omega_1 \otimes \omega_2) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{(k+l)!} \sum_{\eta \in \mathfrak{S}_{k+l}} \eta \cdot (\omega_1 \otimes \omega_2) \\ &= \frac{1}{(k+l)!} \sum_{\eta \in \mathfrak{S}_{k+l}} \eta \cdot (\omega_1 \otimes \omega_2) \\ &= \text{Sym}(\omega_1 \otimes \omega_2). \end{aligned}$$

The other equality is proved similarly.

From here it is easy to derive the associativity, in fact,

$$\alpha(\beta\gamma) = \text{Sym}(\alpha \otimes (\beta\gamma)) = \text{Sym}(\alpha \otimes \text{Sym}(\beta \otimes \gamma)) = \text{Sym}(\alpha \otimes \beta \otimes \gamma).$$

while

$$(\alpha\beta)\gamma = \text{Sym}((\alpha\beta) \otimes \gamma) = \text{Sym}(\text{Sym}(\alpha \otimes \beta) \otimes \gamma) = \text{Sym}(\alpha \otimes \beta \otimes \gamma).$$

Now, the commutativity comes from the observation

$$\begin{aligned}
 \alpha\beta(v_1, \dots, v_{k+l}) &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\
 &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
 &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(k+l)}) \\
 &= \tau \cdot (\beta\alpha)(v_1, \dots, v_{k+l}).
 \end{aligned}$$

where τ is the permutation given by

$$\tau = \begin{pmatrix} 1 & \cdots & l & l+1 & \cdots & k+l \\ k+1 & \cdots & k+l & 1 & \cdots & k \end{pmatrix}$$

Since $\beta\alpha$ is a symmetric tensor, we get $\alpha\beta = \beta\alpha$. \square

1.1.3 Tensors and tensor fields on manifolds

Now let M be a smooth manifold with or without boundary. We define the bundle of covariant k -tensors on M by

$$T^k T^* M = \bigsqcup_{p \in M} T^k(T_p^* M)$$

Analogously, we define the bundle of contravariant k -tensors by

$$T^k T M = \bigsqcup_{p \in M} T^k(T_p M)$$

and the bundle of mixed tensors of type (k, l) by

$$T^{(k,l)} T M = \bigsqcup_{p \in M} T^{(k,l)}(T_p M)$$

There are natural identifications

$$\begin{aligned}
 T^{(0,0)} T M &= T^0 T^* M = T^0 T M = M \times \mathbb{R}, \\
 T^{(0,1)} T M &= T^1 T^* M = T^* M, \quad T^{(1,0)} T M = T^1 T M = T M, \\
 T^{(0,k)} T M &= T^k T^* M, \quad T^{(k,0)} T M = T^k T M.
 \end{aligned}$$

Proposition 1.1.15. *The bundles $T^k T^* M$, $T^k T M$ and $T^{(k,l)} T M$ have natural structures as smooth vector bundles over M .*

Proof. The smooth structures and local trivializations are defined similar to Proposition 1.1.14. \square

Any one of these bundles is called a **tensor bundle** over M . (Thus, the tangent and cotangent bundles are special cases of tensor bundles.) A section of a tensor bundle is called a **tensor field on M** . A smooth tensor field is a section that is smooth in the usual sense of smooth sections of vector bundles. Using the identifications above, we see that contravariant 1-tensor fields are the same as vector fields, and covariant 1-tensor fields are covector fields. Because a 0-tensor is just a real number, a 0-tensor field is the same as a continuous real-valued function.

The spaces of smooth sections of these tensor bundles, $\Gamma(T^k T^* M)$, $\Gamma(T^k TM)$ and $\Gamma(T^{(k,l)} TM)$, are infinite-dimensional vector spaces over \mathbb{R} , and modules over $C^\infty(M)$. In any smooth local coordinates (x^i) , sections of these bundles can be written (using the summation convention) as

$$A = \begin{cases} A_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}, & A \in \Gamma(T^k T^* M); \\ A^{i_1 \dots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}}, & A \in \Gamma(T^k TM); \\ A_{j_1 \dots j_l}^{i_1 \dots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}, & A \in \Gamma(T^{(k,l)} TM). \end{cases}$$

The functions $A_{i_1 \dots i_k}$, $A^{i_1 \dots i_k}$ or $A_{j_1 \dots j_l}^{i_1 \dots i_k}$ are called the **component functions** of A in the chosen coordinates. Because smooth covariant tensor fields occupy most of our attention, we adopt the following shorthand notation for the space of all smooth covariant k -tensor fields:

$$\mathcal{T}^k(M) = \Gamma(T^k T^* M).$$

smooth crit

Proposition 1.1.16 (Smoothness Criterion for Tensor Fields). *Let M be a smooth manifold with or without boundary, and let $A : M \rightarrow T^{(k,l)} TM$ be a rough section. The following are equivalent.*

- (a) A is smooth.
- (b) In every smooth coordinate chart, the component functions of A are smooth.
- (c) Each point of M is contained in some coordinate chart in which A has smooth component functions.
- (d) If $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\omega_1, \dots, \omega_l$ are covector fields, then the function
$$A(\omega_1, \dots, \omega_l, X_1, \dots, X_k) : M \rightarrow \mathbb{R}, \quad p \mapsto A_p(\omega_1|_p, \dots, \omega_l|_p, X_1|_p, \dots, X_k|_p)$$
is smooth.
- (e) Whenever $X_1, \dots, X_k, \omega_1, \dots, \omega_l$ are smooth vector fields and covector fields defined on some open subset $U \subseteq M$, the function $A(\omega_1, \dots, \omega_l, X_1, \dots, X_k)$ is smooth on U .

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ and $(a) \Rightarrow (d), (a) \Rightarrow (e)$ are immediate. So we prove $(d) \Rightarrow (e)$ and $(e) \Rightarrow (a)$ to finish the proof.

Assume the condition of (d), then for any X_1, \dots, X_k and $\omega_1, \dots, \omega_l$ defined on some open subset $U \subseteq M$, we can extend them to M by a bump function near a given point $p \in U$. Then by our hypothesis, $A(X_1, \dots, X_k)$ is smooth at p . Since p can be chosen arbitrarily, $A(X_1, \dots, X_k)$ is smooth on U .

For $(e) \Rightarrow (a)$, just choose coordinate vector fields and covector field in each chart of M . □

Proposition 1.1.17. *Suppose M is a smooth manifold with or without boundary, $A \in \mathcal{T}^k(M)$, $B \in \mathcal{T}^l(M)$ and $f \in C^\infty(M)$. Then fA and $A \otimes B$ are also smooth tensor fields, whose components in any smooth local coordinate chart are*

$$(fA)_{i_1 \dots i_k} = fA_{i_1 \dots i_k}, \quad (A \otimes B)_{i_1 \dots i_{k+l}} = A_{i_1 \dots i_k} B_{i_{k+1} \dots i_{k+l}}.$$

Proposition [1.1.16\(d\)](#) ^{tensor field smooth crit} shows that if A is a smooth covariant k -tensor field on M and X_1, \dots, X_k are smooth vector fields, then $A(X_1, \dots, X_k)$ is a smooth realvalued function on

M . Thus A induces a map

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ folds}} \rightarrow C^\infty(M)$$

It is easy to see that this map is multilinear over \mathbb{R} . In fact, more is true: it is multilinear over $C^\infty(M)$, which means that for $f, f' \in C^\infty(M)$ and $X_i, X'_i \in \mathfrak{X}(M)$, we have

$$A(X_1, \dots, fX_i + f'X'_i, \dots, X_k) = fA(X_1, \dots, X_i, \dots, X_k) + f'A(X_1, \dots, X'_i, \dots, X_k)$$

tensor char

Lemma 1.1.18 (Tensor Characterization Lemma). *A map*

$$\mathcal{A} : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ folds}} \rightarrow C^\infty(M) \quad (1.2) \quad \text{tensor char-1}$$

is induced by a smooth covariant k -tensor field as above if and only if it is multilinear over $C^\infty(M)$.

Proof. We already noted that if A is a smooth covariant k -tensor field, then the map $(X_1, \dots, X_k) \mapsto A(X_1, \dots, X_k)$ is multilinear over $C^\infty(M)$. To prove the converse, we proceed as in the proof of the bundle homomorphism characterization lemma (Lemma [1.1.17](#)). vector bundle homo char

Suppose, therefore, that A is a map as in (1.2), and assume that A is multilinear over $C^\infty(M)$. We show first that A acts locally. If X_i is a smooth vector field that vanishes on a neighborhood U of p , we can choose a bump function ψ supported in U such that $\psi(p) = 1$; then because $\psi X_i \equiv 0$ we have

$$0 = \mathcal{A}(X_1, \dots, \psi X_i, \dots, X_k) = \psi(p) \mathcal{A}_p(X_1, \dots, X_i, \dots, X_k)(p)$$

It follows as in the proof of Lemma [1.1.17](#) that the value of $\mathcal{A}(X_1, \dots, \psi X_i, \dots, X_k)$ at p depends only on the values of X_1, \dots, X_k in a neighborhood of p . vector bundle homo char

Next we show that \mathcal{A} actually acts pointwise. If $X_i|_p = 0$, then in any coordinate chart centered at p we can write $X_i = X_i^j \partial/\partial x^j$, where the component functions X_i^j all vanish at p . By the extension lemma for vector fields, we can find global smooth vector fields E_j on M such that $E_j = \partial/\partial x^j$ in some neighborhood of p ; and similarly the locally defined functions X_i^j can be extended to global smooth functions f_i^j on M that agree with X_i^j in a neighborhood of p . It follows from the multilinearity of \mathcal{A} over $C^\infty(M)$ and the fact that $f_i^j E_j = X_i$ in a neighborhood of p that

$$\begin{aligned} \mathcal{A}(X_1, \dots, X_i, \dots, X_k)(p) &= \mathcal{A}(X_1, \dots, f_i^j E_j, \dots, X_k)(p) \\ &= f_i^j(p) \mathcal{A}(X_1, \dots, E_j, \dots, X_k)(p) = 0. \end{aligned}$$

It follows by linearity that $\mathcal{A}(X_1, \dots, X_k)$ depends only on the value of X_i at p .

Now we define a rough tensor field $A : M \rightarrow T^k T^*M$ by

$$A_p(v_1, \dots, v_k) = \mathcal{A}(V_1, \dots, V_k)(p)$$

for $p \in M$ and $v_1, \dots, v_k \in T_p M$, where V_1, \dots, V_k are any extensions of v_1, \dots, v_k to smooth global vector fields on M . The discussion above shows that this is independent of the choices of extensions, and the resulting tensor field is smooth by Proposition [1.1.16\(d\)](#). tensor field smooth crit \square

A **symmetric tensor field** on a manifold (with or without boundary) is simply a covariant tensor field whose value at each point is a symmetric tensor. The symmetric product of two or more tensor fields is defined pointwise, just like the tensor product. Thus, for example, if A and B are smooth covector fields, their symmetric product is the smooth 2-tensor field AB ,

which is given by

$$AB = \frac{1}{2}(A \otimes B + B \otimes A)$$

Pullbacks of tensor fields

Just like covector fields, covariant tensor fields can be pulled back by a smooth map to yield tensor fields on the domain. (This construction works only for covariant tensor fields, which is one reason why we focus most of our attention on the covariant case.)

Suppose $F : M \rightarrow N$ is a smooth map. For any point $p \in M$ and any k -tensor $\alpha \in T^k(T_{F(p)}^*M)$, we define a tensor $dF_p^*(\alpha)$, called the **pointwise pullback** of α by F at p , by

$$dF_p^*(\alpha)(v_1, \dots, v_k) = \alpha(dF_p(v_1), \dots, dF_p(v_k))$$

for any $v_1, \dots, v_k \in T_pM$. If A is a covariant k -tensor field on N , we define a rough k -tensor field F^*A on M , called the **pullback of A by F** , by

$$(F^*A)_p = dF_p^*(A_{F(p)}).$$

This tensor field acts on vectors $v_1, \dots, v_k \in T_pM$ by

$$(F^*A)_p(v_1, \dots, v_k) = A_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

all back prop

Proposition 1.1.19 (Properties of Tensor Pullbacks). Suppose $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth maps, A and B are covariant tensor fields on N , and f is a real-valued function on N .

- (a) $F^*(fA) = (f \circ F)F^*A$.
- (b) $F^*(A \otimes B) = F^*A \otimes F^*B$.
- (c) $F^*(A + B) = F^*A + F^*B$.
- (d) F^*B is a tensor field, and is smooth if B is smooth.
- (e) $(G \circ F)^* = F^* \circ G^*$.
- (f) $\text{id}^* = \text{id}$.

Proof. For $p \in M$, we compute

$$(F^*(fA))_p = dF_p^*(f(F(p))A_{F(p)}) = f(F(p))dF_p^*(A_{F(p)}) = f \circ F(p)(F^*A)_p = ((f \circ F)F^*A)_p$$

and

$$\begin{aligned} F^*(A \otimes B)(v_1, \dots, v_{k+l}) &= (A \otimes B)_{F(p)}(dF_p(v_1), \dots, dF_p(v_{k+l})) \\ &= A_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))B_{F(p)}(dF_p(v_{k+1}), \dots, dF_p(v_{k+l})) \\ &= (F^*A)_p(v_1, \dots, v_k)(F^*B)_p(v_{k+1}, \dots, v_{k+l}) \\ &= (F^*A \otimes F^*B)_p(v_1, \dots, v_{k+l}) \end{aligned}$$

The statement (c) is proved similarly.

Now part (d) is an application of Proposition [1.1.16](#), and part (e), (f) follows easily. \square

If f is a continuous real-valued function (i.e., a 0-tensor field) and B is a k -tensor field, then it is consistent with our definitions to interpret $f \otimes B$ as fB , and F^*f as $f \circ F$. With these interpretations, property (a) is really just a special case of (b).

The following proposition is an immediate consequence of Proposition [1.1.19](#). tensor pull back prop

back formula

Proposition 1.1.20. Let $F : M \rightarrow N$ be smooth, and let B be a covariant k -tensor field on N . If $p \in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in a neighborhood of p :

$$F^*(B_{i_1 \dots i_k} dy^{i_1} \otimes \dots \otimes dy^{i_k}) = (B_{i_1 \dots i_k} \circ F) d(y^{i_1} \circ F) \otimes \dots \otimes d(y^{i_k} \circ F)$$

Example 1.1.21 (Pullback of a Tensor Field). Let $M = \{(r, \theta) : r > 0, |\theta| < \pi/2\}$ and $N = \{(x, y) : x > 0\}$ and let $F : M \rightarrow \mathbb{R}^2$ be the smooth map $F(r, \theta) = (r \cos \theta, r \sin \theta)$. The pullback of the tensor field $A = x^{-2} dy \otimes dy$ by F can be computed easily by substituting $x = r \cos \theta$, $y = r \sin \theta$ and simplifying:

$$\begin{aligned} F^*A &= (r \cos \theta)^{-2} d(r \sin \theta) \otimes d(r \sin \theta) \\ &= (r \cos \theta)^{-2} (\sin \theta dr + r \cos \theta d\theta) \otimes (\sin \theta dr + r \cos \theta d\theta) \\ &= r^{-2} \tan^2 \theta dr \otimes dr + r^{-1} \tan \theta (d\theta \otimes dr + dr \otimes d\theta) + d\theta \otimes d\theta. \end{aligned}$$

Lie derivatives of tensor fields

The Lie derivative operation can be extended to tensor fields of arbitrary rank. As usual, we focus on covariant tensors; the analogous results for contravariant or mixed tensors require only minor modifications.

Suppose M is a smooth manifold, V is a smooth vector field on M , and θ is its flow. (For simplicity, we discuss only the case $\partial M = \emptyset$ here, but these definitions and results carry over essentially unchanged to manifolds with boundary as long as V is tangent to the boundary, so that its flow exists by Theorem ??.) For any $p \in M$, if t is sufficiently close to zero, then θ_t is a diffeomorphism from a neighborhood of p to a neighborhood of $\theta_t(p)$, so $d(\theta_t)_p^*$ pulls back tensors at $\theta_t(p)$ to ones at p by the formula

$$d(\theta_t)_p^*(A_{\theta_t(p)})(v_1, \dots, v_k) = A_{\theta_t(p)}(d(\theta_t)_p(v_1), \dots, d(\theta_t)_p(v_k))$$

Note that $d(\theta_t)_p^*(A_{\theta_t(p)})$ is just the value of the pullback tensor field θ_t^*A at p .

Given a smooth covariant tensor field A on M , we define the Lie derivative of A with respect to V , denoted by $\mathfrak{L}_V A$, by

$$(\mathfrak{L}_V A)_p = \left. \frac{d}{dt} \right|_{t=0} (\theta_t^* A)_p = \lim_{t \rightarrow 0} \frac{d(\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t} \quad (1.3) \quad \text{tensor Lie de}$$

provided the derivative exists. Because the expression being differentiated lies in $T^k(T_p^*M)$ for all t , $(\mathfrak{L}_V A)_p$ makes sense as an element of $T^k(T_p^*M)$. The following lemma is an analogue of Lemma ??, and is proved in exactly the same way.

Lemma 1.1.22. With M, V , and A as above, the derivative $(\mathfrak{L}_V A)_p$ exists for every $p \in M$ and defines $\mathfrak{L}_V A$ as a smooth tensor field on M .

ivative prop

Proposition 1.1.23. Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. Suppose f is a smooth real-valued function (regarded as a 0-tensor field) on M , and A, B are smooth covariant tensor fields on M .

- (a) $\mathfrak{L}_V f = Vf$.
- (b) $\mathfrak{L}_V(fA) = (Vf)A + f\mathfrak{L}_V A$.
- (c) $\mathfrak{L}_V(A \otimes B) = (\mathfrak{L}_V A) \otimes B + A \otimes (\mathfrak{L}_V B)$.

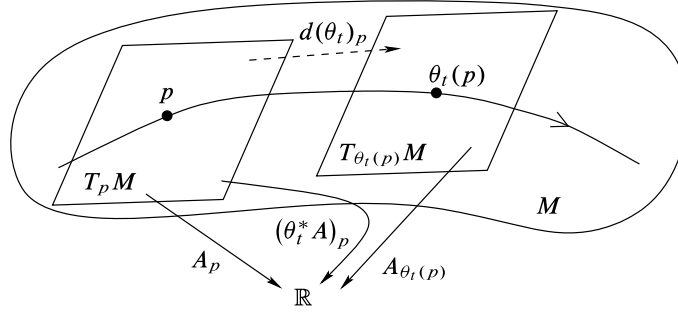


Figure 1.1: The Lie derivative of a tensor field.

(d) If X_1, \dots, X_k are smooth vector fields and A is a smooth k -tensor field,

$$\mathfrak{L}_V(A(X_1, \dots, X_k)) = (\mathfrak{L}_V A)(X_1, \dots, X_k) + \sum_{i=1}^k A(X_1, \dots, \mathfrak{L}_V X_i, \dots, X_k)$$

Proof. Let θ be the flow of V . For a real-valued function f , we can write

$$\theta_t^* f(p) = (f \circ \theta_t)(p) = f \circ \theta^{(p)}(t)$$

Thus the definition of $\mathfrak{L}_V f$ reduces to the ordinary derivative with respect to t of the composite function $f \circ \theta^{(p)}(t)$. Because $\theta^{(p)}(t)$ is an integral curve of V , it follows from Proposition 1.1.23 that

$$(\mathfrak{L}_V f)_p = \left. \frac{d}{dt} \right|_{t=0} f \circ \theta^{(p)} = df_p(\dot{\theta}^{(p)}(0)) = df_p(V_p) = Vf(p)$$

This proves (a).

The other assertions can be proved by the technique we used in Theorem 1.1.22: in a neighborhood of a regular point for V , if (u^i) are coordinates in which $V = \partial/\partial u^1$, then it follows immediately from the definition that \mathfrak{L}_V acts on a tensor field simply by taking the partial derivative of its coefficients with respect to u^1 , and (b)–(d) all follow from the ordinary product rule. The same relations hold on the support of V by continuity, and on the complement of the support because the flow of V is trivial there. \square

One consequence of this proposition is the following formula expressing the Lie derivative of any smooth covariant tensor field in terms of Lie brackets and ordinary directional derivatives of functions, which allows us to compute Lie derivatives without first determining the flow.

Corollary 1.1.24. *If V is a smooth vector field and A is a smooth covariant k -tensor field, then for any smooth vector fields X_1, \dots, X_k ,*

$$(\mathfrak{L}_V A)(X_1, \dots, X_k) = V(A(X_1, \dots, X_k)) - \sum_{i=1}^k A(X_1, \dots, [V, X_i], \dots, X_k). \quad (1.4)$$

Proof. From Proposition 1.1.23(d) we have

$$(\mathfrak{L}_V A)(X_1, \dots, X_k) = \mathfrak{L}_V(A(X_1, \dots, X_k)) - \sum_{i=1}^k A(X_1, \dots, \mathfrak{L}_V X_i, \dots, X_k).$$

Replace $\mathfrak{L}_V(A(X_1, \dots, X_k))$ by $V(A(X_1, \dots, X_k))$ and $\mathfrak{L}_V X_i$ by $[V, X_i]$, we get the claim. \square

vector Lie diff

Corollary 1.1.25. *If $f \in C^\infty(M)$, then $\mathfrak{L}_V(df) = d(\mathfrak{L}_V f)$.*

Proof. Using (1.4), for any $X \in \mathfrak{X}(M)$ we compute

$$\begin{aligned} (\mathfrak{L}_V df)(X) &= V(df(X)) - df([V, X]) = VXf - (VX - XV)f \\ &= XVf = X(\mathfrak{L}_V f) = d(\mathfrak{L}_V f)(X), \end{aligned}$$

thus the claim follows. \square

One drawback of formula (1.4) is that in order to calculate what $\mathfrak{L}_V A$ does to vectors v_1, \dots, v_k at a point $p \in M$, one must first extend them to vector fields in a neighborhood of p . But Corollary 1.1.25 lead to an easy method for computing Lie derivatives of smooth tensor fields in coordinates that avoids this problem, since any tensor field can be written locally as a linear combination of functions multiplied by tensor products of exact 1-forms. The next example illustrates the technique.

Example 1.1.26. Suppose A is an arbitrary smooth covariant 2-tensor field, and V is a smooth vector field. We compute the Lie derivative $\mathfrak{L}_V A$ in smooth local coordinates (x^i) . First, we observe that

$$\mathfrak{L}_V dx^i = d(\mathfrak{L}_V x^i) = d(Vx^i) = dV^i.$$

Therefore,

$$\begin{aligned} \mathfrak{L}_V A &= \mathfrak{L}_V (A_{ij} dx^i \otimes dx^j) \\ &= \mathfrak{L}_V (A_{ij}) dx^i \otimes dx^j + A_{ij} \mathfrak{L}_V (dx^i \otimes dx^j) \\ &= VA_{ij} dx^i \otimes dx^j + A_{ij} \mathfrak{L}_V (dx^i) \otimes dx^j + A_{ij} dx^i \otimes \mathfrak{L}_V (dx^j) \\ &= VA_{ij} dx^i \otimes dx^j + A_{ij} dV^i \otimes dx^j + A_{ij} dx^i \otimes dV^j \\ &= VA_{ij} dx^i \otimes dx^j + A_{ij} \frac{\partial V^i}{\partial x^k} dx^k \otimes dx^j + A_{ij} \frac{\partial V^j}{\partial x^k} dx^i \otimes dx^k \\ &= \left(VA_{ij} + A_{kj} \frac{\partial V^k}{\partial x^i} + A_{ik} \frac{\partial V^k}{\partial x^j} \right) dx^i \otimes dx^j. \end{aligned}$$

Recall that the Lie derivative of a vector field W with respect to V is zero if and only if W is invariant under the flow of V (see Theorem ??). It turns out that the Lie derivative of a covariant tensor field has exactly the same interpretation. If A is a smooth tensor field on M and θ is a flow on M , we say that A is **invariant under θ** if for each t , the map θ_t pulls A back to itself wherever it is defined; more precisely, this means

$$d(\theta_t)_p^*(A_{\theta_t(p)}) = A_p$$

for all (t, p) in the domain of θ . If θ is a global flow, this is equivalent to $\theta_t^* A = A$ for all $t \in \mathbb{R}$.

In order to prove the connection between Lie derivatives and invariance under flows, we need the following proposition, which shows how the Lie derivative can be used to compute t -derivatives at times other than $t = 0$. It is a generalization to tensor fields of Proposition ??.

er tensor t_0

Proposition 1.1.27. *Suppose M is a smooth manifold with or without boundary and $V \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume in addition that V is tangent to ∂M . Let θ be the flow of V . For any smooth covariant tensor field A and any (t_0, p) in the domain of θ ,*

$$\left. \frac{d}{dt} \right|_{t=t_0} (\theta_t^* A)_p = (\theta_{t_0}^* (\mathfrak{L}_V A))_p$$

Proof. After expanding the definitions of the pullbacks, we see that we have to prove

$$\frac{d}{dt}\Big|_{t=0} d(\theta_t)_p^*(A_{\theta_t(p)}) = d(\theta_{t_0})_p^*((\mathfrak{L}_V A)_{\theta_{t_0}(p)})$$

Just as in the proof of Proposition 1.1.27, the change of variables $t = s + t_0$ yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} d(\theta_t)_p^*(A_{\theta_t(p)}) &= \frac{d}{ds}\Big|_{s=0} d(\theta_{s+t_0})_p^*(A_{\theta_{s+t_0}(p)}) = \frac{d}{ds}\Big|_{s=0} d(\theta_{t_0})_p^* \circ d(\theta_s)_{\theta_{t_0}(p)}^*(A_{\theta_s(\theta_{t_0}(p))}) \\ &= d(\theta_{t_0})_p^* \frac{d}{ds}\Big|_{s=0} d(\theta_s)_{\theta_{t_0}(p)}^*(A_{\theta_s(\theta_{t_0}(p))}) = d(\theta_{t_0})_p^*((\mathfrak{L}_V A)_{\theta_{t_0}(p)}). \end{aligned}$$

□

ant flow iff

Theorem 1.1.28. Let M be a smooth manifold and let $V \in \mathfrak{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $\mathfrak{L}_V A = 0$.

Proof. This follows from the definition of $\mathfrak{L}_V A$. □

1.1.4 Exercise

Exercise 1.1.1. Let M be a smooth n -manifold, and let A be a smooth covariant k -tensor field on M . If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^j))$ are overlapping smooth charts on M , we can write

$$A = A_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k} = \tilde{A}_{j_1 \dots j_k} d\tilde{x}^{j_1} \otimes \dots \otimes d\tilde{x}^{j_k}.$$

Compute the transformation law.

Proof. Recall the formulas

$$d\tilde{x}^j = \frac{\partial \tilde{x}^j}{\partial x^i} dx^i.$$

Thus

$$A = \tilde{A}_{j_1 \dots j_k} d\tilde{x}^{j_1} \otimes \dots \otimes d\tilde{x}^{j_k} = \tilde{A}_{j_1 \dots j_k} \frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}} \dots \frac{\partial \tilde{x}^{j_k}}{\partial x^{i_k}} dx^{i_1} \otimes \dots \otimes dx^{i_k}.$$

This gives the claim. □

ative bracket

Exercise 1.1.2. Suppose M is a smooth manifold, A is a smooth covariant tensor field on M , and $V, W \in \mathfrak{X}(M)$. Show that

$$\mathfrak{L}_V \mathfrak{L}_W A - \mathfrak{L}_W \mathfrak{L}_V A = \mathfrak{L}_{[V, W]} A.$$

Proof. Let $X_1, \dots, X_k \in \mathfrak{X}(M)$, then

$$\begin{aligned} (\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) &= V(\mathfrak{L}_W A(X_1, \dots, X_k)) - \sum_{i=1}^k \mathfrak{L}_W A(X_1, \dots, [V, X_i], \dots, X_k) \\ &= VW(A(X_1, \dots, X_k)) - \sum_{i=1}^k V(A(X_1, \dots, [W, X_i], \dots, X_k)) \\ &\quad - \sum_{i=1}^k W(A(X_1, \dots, [V, X_i], \dots, X_k)) + \sum_{i \neq j} A(X_1, \dots, [W, X_i], \dots, [V, X_j], \dots, X_k) \\ &\quad + \sum_{i=1}^k A(X_1, \dots, [W, [V, X_i]], \dots, X_k). \end{aligned}$$

And thus

$$\begin{aligned} (\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) - (\mathfrak{L}_W \mathfrak{L}_V A)(X_1, \dots, X_k) &= (VW - WV)(A(X_1, \dots, X_k)) \\ &+ \sum_{i=1}^k A(X_1, \dots, [W, [V, X_i]], \dots, X_k) - \sum_{i=1}^k A(X_1, \dots, [V, [W, X_i]], \dots, X_k). \end{aligned}$$

Therefore, we compute using the Jacobi identity

$$\begin{aligned} (\mathfrak{L}_{[V, W]} A)(X_1, \dots, X_k) &= (VW - WV)(A(X_1, \dots, X_k)) - \sum_{i=1}^k A(X_1, \dots, [[V, W], X_i], \dots, X_k) \\ &= (VW - WV)(A(X_1, \dots, X_k)) + \sum_{i=1}^k A(X_1, \dots, [[W, X_i], V] + [[X_i, V], W], \dots, X_k) \\ &= (VW - WV)(A(X_1, \dots, X_k)) + \sum_{i=1}^k A(X_1, \dots, [W, [V, X_i]] - [[V, W], X_i], \dots, X_k) \\ &= (\mathfrak{L}_V \mathfrak{L}_W A)(X_1, \dots, X_k) - (\mathfrak{L}_W \mathfrak{L}_V A)(X_1, \dots, X_k). \end{aligned}$$

□

Exercise 1.1.3. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. Show that the Lie derivative operators on covariant tensor fields, $\mathfrak{L}_V : \mathcal{T}^k(M) \rightarrow \mathcal{T}^k(M)$ for $k \geq 0$, are uniquely characterized by the following properties:

- (a) \mathfrak{L}_V is linear over \mathbb{R} .
- (b) $\mathfrak{L}_V f = Vf$ for $f \in \mathcal{T}^0(M) = C^\infty(M)$.
- (c) $\mathfrak{L}_V(A \otimes B) = \mathfrak{L}_V A \otimes B + A \otimes \mathfrak{L}_V B$ for $A \in \mathcal{T}^k(M)$ and $B \in \mathcal{T}^l(M)$.
- (d) $\mathfrak{L}_V(\omega(X)) = (\mathfrak{L}_V \omega)(X) + \omega([V, X])$ for $\omega \in \mathcal{T}^1(M)$, $X \in \mathfrak{X}(M)$.

Proof. Let \mathfrak{L} be an operator satisfying (a)–(d), we prove that $\mathfrak{L} = \mathfrak{L}_V$.

- First, by (b) and (d) we have for $\omega \in \mathcal{T}^1(M)$, $X \in \mathfrak{X}(M)$:

$$(\mathfrak{L}\omega)(X) = V(\omega(X)) - \omega([V, X]) = (\mathfrak{L}_V \omega)(X)$$

- Then, by induction, we can prove

$$\mathfrak{L}(\omega_1 \otimes \dots \otimes \omega_k) = \mathfrak{L}_V(\omega_1 \otimes \dots \otimes \omega_k).$$

- By taking local charts, we verify that $\mathfrak{L} = \mathfrak{L}_V$.

□

1.2 Riemannian metrics

1.2.1 Riemannian manifolds

The most important examples of symmetric tensors on a vector space are inner products. Any inner product allows us to define lengths of vectors and angles between them, and thus to do Euclidean geometry.

Transferring these ideas to manifolds, we obtain one of the most important applications of tensors to differential geometry. Let M be a smooth manifold with or without boundary. A **Riemannian metric** on M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point. A **Riemannian manifold** is a pair (M, g) , where M is a smooth manifold and g is a Riemannian metric on M . One sometimes simply says M is a Riemannian manifold if M is understood to be endowed with a specific Riemannian metric. A Riemannian manifold with boundary is defined similarly.

If g is a Riemannian metric on M , then for each $p \in M$, the 2-tensor g_p is an inner product on $T_p M$. Because of this, we often use the notation $\langle v, w \rangle_g$ to denote the real number $g_p(v, w)$ for $v, w \in T_p M$.

In any smooth local coordinates (x^i) , a Riemannian metric can be written

$$g = g_{ij} dx^i \otimes dx^j,$$

where (g_{ij}) is a symmetric positive definite matrix of smooth functions. The symmetry of g allows us to write g also in terms of symmetric products as follows:

$$g = g_{ij} dx^i \otimes dx^j = \frac{1}{2} (g_{ij} dx^i \otimes dx^j + g_{ji} dx^j \otimes dx^i) = \frac{1}{2} g_{ij} (dx^i \otimes dx^j + dx^j \otimes dx^i) = g_{ij} dx^i dx^j.$$

Example 1.2.1 (The Euclidean Metric). The simplest example of a Riemannian metric is the Euclidean metric \bar{g} on \mathbb{R}^n , given in standard coordinates by

$$\bar{g} = \delta_{ij} dx^i \otimes dx^j.$$

It is common to abbreviate the symmetric product of a tensor α with itself by α^2 , so the Euclidean metric can also be written

$$\bar{g} = (dx^1)^2 + \cdots + (dx^n)^2.$$

Example 1.2.2 (Product Metrics). If (M, g) and (\tilde{M}, \tilde{g}) are Riemannian manifolds, we can define a Riemannian metric $\widehat{g} = g \oplus \tilde{g}$ on the product manifold $M \times \tilde{M}$, called the product metric, as follows:

$$\widehat{g}((v, \tilde{v}), (w, \tilde{w})) = g(v, w) + \tilde{g}(\tilde{v}, \tilde{w})$$

for any $(v, \tilde{v}), (w, \tilde{w}) \in T_p M \oplus T_q \tilde{M}$. Given any local coordinates (x^1, \dots, x^n) for M and (y^1, \dots, y^m) for \tilde{M} , we obtain local coordinates $(x^1, \dots, x^n, y^1, \dots, y^m)$ for $M \times \tilde{M}$, and the product metric is represented locally by the block diagonal matrix

$$(\widehat{g}_{ij}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \tilde{g}_{ij} \end{pmatrix}$$

Proposition 1.2.3 (Existence of Riemannian Metrics). *Every smooth manifold with or without boundary admits a Riemannian metric.*

Proof. Let M be a smooth manifold with or without boundary, and choose a covering of M by smooth coordinate charts $(U_\alpha, \varphi_\alpha)$. In each coordinate domain, there is a Riemannian metric $g_\alpha = \varphi_\alpha^* \bar{g}$, whose coordinate expression is $\delta_{ij} dx^i dx^j$. Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{U_\alpha\}$, and define

$$g = \sum_\alpha \psi_\alpha g_\alpha,$$

with each term interpreted to be zero outside $\text{supp}(\psi_\alpha)$. By local finiteness, there are only finitely many nonzero terms in a neighborhood of each point, so this expression defines a

smooth tensor field. It is obviously symmetric, so only positivity needs to be checked. If $v \in T_p M$ is any nonzero vector, then

$$g_p(v, v) = \sum_{\alpha} \psi_{\alpha}(p) g_{\alpha}|_p(v, v).$$

This sum is nonnegative, because each term is nonnegative. At least one of the functions ψ_{α} is strictly positive at p . Because $g_{\alpha}|_p(v, v) > 0$, it follows that $g_p(v, v) > 0$. \square

Below are just a few of the geometric constructions that can be defined on a Riemannian manifold (M, g) with or without boundary.

- The length or norm of a tangent vector $v \in T_p M$ is defined to be

$$|v|_g = \langle v, v \rangle_g^{1/2} = g_p(v, v)^{1/2}$$

- The angle between two nonzero tangent vectors $v, w \in T_p M$ is the unique $\theta \in [0, \pi]$ satisfying

$$\cos \theta = \frac{\langle v, w \rangle_g}{|v|_g |w|_g}$$

- Tangent vectors $v, w \in T_p M$ are said to be orthogonal if $\langle v, w \rangle_g = 0$. This means either one or both vectors are zero, or the angle between them is $\pi/2$.

One highly useful tool for the study of Riemannian manifolds is orthonormal frames. Let (M, g) be an n -dimensional Riemannian manifold with or without boundary. Just as we did for the case of \mathbb{R}^n in Section [1.1](#), we say a local frame (E_1, \dots, E_n) for M on an open subset $U \subseteq M$ is an **orthonormal frame** if the vectors $(E_1|_p, \dots, E_n|_p)$ form an orthonormal basis for $T_p M$ at each point $p \in U$, or equivalently if $\langle E_i, E_j \rangle_g = \delta_{ij}$, in which case g has the local expression

$$g = (\varepsilon^1)^2 + \dots + (\varepsilon^n)^2.$$

where $(\varepsilon^i)^2$ denotes the symmetric product $(\varepsilon^i)^2 = \varepsilon^i \otimes \varepsilon^i$.

Example 1.2.4. The coordinate frame $(\partial/\partial x^i)$ is a global orthonormal frame for \mathbb{R}^n with the Euclidean metric.

Example 1.2.5. The frame (E_1, E_2) on $\mathbb{R}^2 - \{0\}$ defined in Example [1.1](#) is a local orthonormal frame for \mathbb{R}^2 . As we observed in Example [1.1](#), it is not a coordinate frame in any coordinates.

The next proposition is proved in just the same way as Lemma [1.1](#), with the Euclidean dot product replaced by the inner product $\langle \cdot, \cdot \rangle_g$.

Proposition 1.2.6. Suppose (M, g) is a Riemannian manifold with or without boundary, and (X_j) is a smooth local frame for M over an open subset $U \subseteq M$. Then there is a smooth orthonormal frame (E_j) over U such that $\text{span}(E_1|_p, \dots, E_j|_p) = \text{span}(X_1|_p, \dots, X_j|_p)$ for each $1 \leq j \leq n$ and each $p \in U$.

Proof. Applying the GramSchmidt algorithm to the vectors $(X_1|_p, \dots, X_n|_p)$ at each $p \in U$, we obtain an ordered n -tuple of rough orthonormal vector fields (E_1, \dots, E_n) over U satisfying the span conditions. Because the vectors whose norms appear in the denominators of the GramSchmidt formulas are nowhere vanishing, it follows that each vector field E_j is

smooth. The last statement of the proposition follows by applying this construction to any smooth local frame in a neighborhood of p . In particular, for every $p \in M$, there is a smooth orthonormal frame (E_j) defined on some neighborhood of p . \square

Corollary 1.2.7 (Existence of Local Orthonormal Frames). *Let (M, g) be a Riemannian manifold with or without boundary. For each $p \in M$, there is a smooth orthonormal frame on a neighborhood of p .*

Observe that Corollary 1.2.7 does not show that there are smooth coordinates on a neighborhood of p for which the coordinate frame is orthonormal. In fact, this is possible only when the metric is flat, that is, locally isometric to the Euclidean metric.

For a Riemannian manifold (M, g) with or without boundary, we define the unit tangent bundle to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM = \{(p, v) \in TM : |v|_g = 1\}$$

Proposition 1.2.8 (Properties of the Unit Tangent Bundle). *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a smooth, properly embedded codimension-1 submanifold with boundary in TM , with $\partial(UTM) = \pi^{-1}(\partial M)$ (where $\pi : UTM \rightarrow M$ is the canonical projection). The unit tangent bundle is connected if and only if M is connected when $n > 1$, and compact if and only if M is compact.*

Proof. By the local orthonormal frames, we can replace M by \mathbb{R}^n . If U is open in \mathbb{R}^n , then the unit tangent bundle is diffeomorphic to $U \times S^{n-1}$, and therefore has codimension 1. It is properly embedded since S^{n-1} is properly embedded in \mathbb{R}^n . Similarly, if U is a half ball in \mathbb{H}^n , we can see that the tangent bundle is diffeomorphic to $U \times S^{n-1}$, and therefore has boundary charts. This also proves that $\partial(UTM) = \pi^{-1}(\partial M)$.

For the last claim, note that the projection $\pi : UTM \rightarrow M$ is surjective and open (Proposition 1.2.7), so M is connect if UTM is, and compact if UTM is. Now we show the converse.

If M is compact, then for any $p \in M$ we can find a compact subset V_p such that the interior of V_p is open and the tangent bundle of V_p is trivial. Then we can find $p_1, \dots, p_n \in M$ such that $M \subseteq \bigcup_{i=1}^n \text{Int}(V_{p_i})$, and so

$$UTM = \bigcup_{i=1}^n \pi^{-1}(\text{Int}(V_{p_i})) \subseteq \bigcup_{i=1}^n \pi^{-1}(V_{p_i}).$$

Since the tangent bundle of V_{p_i} is trivial, we have $\pi^{-1}(V_{p_i}) = V_{p_i} \times S^{n-1}$. This implies that UTM is compact since it is the union of finitely many compact subsets, so it is compact.

If M is connected and $n > 1$, then the fiber of UTM is diffeomorphic to S^{n-1} and so it is connected. Assume that $UTM = U \cup V$ for two disjoint open subsets, then $M = \pi(E) = \pi(U) \cup \pi(V)$ is the union of the open subsets $\pi(U)$ and $\pi(V)$ which are not disjoint since M is connected. Let $p \in \pi(U) \cap \pi(V)$, then there exists $v_1 \in U$, $v_2 \in V$ such that $\pi(v_1) = \pi(v_2) = p$. This means $E_p \cap U \neq \emptyset$ and $E_p \cap V \neq \emptyset$, so $E_p \cap U$ and $E_p \cap V$ is a partition of E_p . But E_p is connected, this is a contradiction. \square

1.2.2 Methods for constructing riemannian metrics

Pullback metrics

Suppose M, N are smooth manifolds with or without boundary, g is a Riemannian metric on N , and $F : M \rightarrow N$ is smooth. The pullback F^*g is a smooth 2-tensor field on M . If it is

positive definite, it is a Riemannian metric on M , called the pullback metric determined by F . The next proposition shows when this is the case.

Proposition 1.2.9 (Pullback Metric Criterion). *Suppose $F : M \rightarrow N$ is a smooth map and g is a Riemannian metric on N . Then F^*g is a Riemannian metric on M if and only if F is a smooth immersion.*

Proof. F^*g is a Riemannian metric if and only if for all $p \in M$, $F^*g(v, v) = 0$ implies $v = 0$ for $v \in T_pM$, which means $dF_p(v) = 0$ implies $v = 0$ for any $v \in T_pM$, which is to say F is a immersion. \square

If the coordinate representation for an immersion is known, then the pullback metric is easy to compute using the usual algorithm for computing pullbacks.

Example 1.2.10. Consider the smooth map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$F(u, v) = (u \cos v, u \sin v, v)$$

It is a proper injective smooth immersion, and thus it is an embedding by Proposition 1.2.9. The pullback metric $F^*\bar{g}$ can be computed by substituting the coordinate functions for F in place of x, y, z in the formula for \bar{g} : |smooth embedd if

$$\begin{aligned} F^*\bar{g} &= (du \cos v)^2 + (du \sin v)^2 + (dv)^2 \\ &= (\cos v du - u \sin v dv)^2 + (\sin v du + u \cos v dv)^2 + dv^2 \\ &= du^2 + u^2 dv^2 + dv^2 = (u^2 + 1) du^2 + dv^2. \end{aligned}$$

By convention, when u is a real-valued function, the notation du^2 means the symmetric product $dudu$, not $d(u^2)$.

To transform a Riemannian metric under a change of coordinates, we use the same technique as we used for covector fields: think of the change of coordinates as the identity map expressed in terms of different coordinates for the domain and codomain, and use the formula of Corollary 1.1.20. |tensor pullback formula

Example 1.2.11. To illustrate, we compute the coordinate expression for the Euclidean metric $\bar{g} = dx^2 + dy^2$ on \mathbb{R}^2 in polar coordinates. Substituting $x = r \cos \theta$ and $y = r \sin \theta$ and expanding, we obtain metric polar

$$\begin{aligned} \bar{g} &= dx^2 + dy^2 = [d(r \cos \theta)]^2 + [d(r \sin \theta)]^2 \\ &= (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2 \\ &= dr^2 + r^2 d\theta^2 \end{aligned}$$

If (M, g) and (\tilde{M}, \tilde{g}) are both Riemannian manifolds, a smooth map $F : M \rightarrow \tilde{M}$ is called a **Riemannian isometry** if it is a diffeomorphism that satisfies $F^*\tilde{g} = g$. More generally, F is called a **local isometry** if every point $p \in M$ has a neighborhood U such that $F|_U$ is an isometry of U onto an open subset of \tilde{M} , or equivalently, if F is a local diffeomorphism satisfying $F^*\tilde{g} = g$.

If there exists a Riemannian isometry between (M, g) and (\tilde{M}, \tilde{g}) , we say that they are **isometric** as Riemannian manifolds. If each point of M has a neighborhood that is isometric to an open subset of (\tilde{M}, \tilde{g}) , then we say that (M, g) is **locally isometric** to (\tilde{M}, \tilde{g}) .

One such property is flatness. A Riemannian n -manifold (M, g) is said to be a **flat Riemannian manifold**, and g is a **flat metric**, if (M, g) is locally isometric to (\mathbb{R}^n, \bar{g}) .

Proposition 1.2.12. Suppose (M, g) and (\tilde{M}, \tilde{g}) are isometric Riemannian manifolds, then g is flat if and only if \tilde{g} is flat.

The next theorem is the key to deciding whether a Riemannian metric is flat.

mann flat iff

Theorem 1.2.13. For a Riemannian manifold (M, g) , the following are equivalent:

- (a) g is flat.
- (b) Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{ij} dx^i dx^j$.
- (c) Each point of M is contained in the domain of a smooth coordinate chart in which the coordinate frame is orthonormal.
- (d) Each point of M is contained in the domain of a commuting orthonormal frame.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are easy consequences of the definitions. The remaining implication, $(d) \Rightarrow (a)$, follows from the canonical form theorem for commuting frames: if (E_i) is a commuting orthonormal frame for g on an open subset $V \subseteq M$, then Theorem ?? implies that each $p \in V$ is contained in the domain of a smooth chart (U, φ) in which the coordinate frame is equal to (E_i) . This means $\varphi_* E_i = \partial / \partial x^i$, so the diffeomorphism $\varphi : U \rightarrow \varphi(U)$ satisfies

$$\varphi^* \tilde{g}(E_i, E_j) = \tilde{g}(\varphi_* E_i, \varphi_* E_j) = \tilde{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} = g(E_i, E_j)$$

Bilinearity then shows that $\varphi^* \tilde{g} = g$, so φ is an isometry between $(U, \varphi|_U)$ and $\varphi(U)$ with the Euclidean metric. This shows that g is flat. \square

Riemannian submanifolds

Pullback metrics are especially important for submanifolds. If (M, g) is a Riemannian manifold with or without boundary, every submanifold $S \subseteq M$ (immersed or embedded, with or without boundary) automatically inherits a pullback metric $\iota^* g$, where $\iota : S \hookrightarrow M$ is inclusion. In this setting, the pullback metric is also called the **induced metric** on S . By definition, this means for $v, w \in T_p S$ that

$$(\iota^* g)(v, w) = g(d\iota_p(v), d\iota_p(w)) = g(v, w)$$

because $d\iota_p : T_p S \rightarrow T_p M$ is our usual identification of $T_p S$ as a subspace of $T_p M$. Thus $\iota^* g$ is just the restriction of g to pairs of vectors tangent to S . With this metric, S is called a **Riemannian submanifold** (with or without boundary) of M .

Example 1.2.14. The metric $\hat{g} = \iota^* \tilde{g}$ induced on S^n by the usual inclusion $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is called the **round metric** or the **standard metric** on the sphere.

The next lemma describes one of the most important tools for studying Riemannian submanifolds. If (\tilde{M}, \tilde{g}) is an m -dimensional smooth Riemannian manifold and (M, g) is an n -dimensional submanifold (both with or without boundary), a local frame (E_1, \dots, E_m) for \tilde{M} on an open subset $\tilde{U} \subseteq \tilde{M}$ is said to be **adapted to M** if the first n vector fields (E_1, \dots, E_n) are tangent to M . In case \tilde{M} has empty boundary (so that slice coordinates are available), adapted local orthonormal frames are easy to find.

embed adapted

Proposition 1.2.15 (Existence of Adapted Orthonormal Frames). *Let (\tilde{M}, \tilde{g}) be a Riemannian manifold (without boundary), and let (M, g) be an embedded smooth submanifold with or without boundary. Given $p \in M$, there exist a neighborhood \tilde{U} of p in \tilde{M} and a smooth orthonormal frame for \tilde{M} on \tilde{U} that is adapted to M .*

Proof. We can find a slice chart at p for M , and use the Gram-Schmidt algorithm to a coordinate frame in slice coordinates to produce a orthonormal frame. By the span property, we see that this frame is adapted to M . \square

Suppose (\tilde{M}, \tilde{g}) is a Riemannian manifold and (M, g) is a smooth submanifold with or without boundary in \tilde{M} . Given $p \in M$, a vector $v \in T_p \tilde{M}$ is said to be **normal to M** if $\langle v, w \rangle = 0$ for every $w \in T_p M$. The space of all vectors normal to M at p is a subspace of $T_p \tilde{M}$, called the **normal space at p** and denoted by $N_p M = (T_p M)^\perp$. At each $p \in M$, the ambient tangent space $T_p \tilde{M}$ splits as an orthogonal direct sum $T_p \tilde{M} = T_p M \oplus N_p M$. A section N of the ambient tangent bundle $T\tilde{M}|_M$ is called a **normal vector field along M** if $N_p \in N_p M$ for each $p \in M$. The set

$$NM = \bigsqcup_{p \in M} N_p M$$

is called the normal bundle of M .

normal bundle

Proposition 1.2.16 (The Normal Bundle to a Riemannian Submanifold). *If \tilde{M} is a Riemannian m -manifold and M is an immersed or embedded n -dimensional submanifold with or without boundary, then NM is a smooth rank- $(m - n)$ vector subbundle of the ambient tangent bundle $T\tilde{M}|_M$. There are smooth bundle homomorphisms*

$$\pi^\top : T\tilde{M}|_M \rightarrow TM, \quad \pi^\perp : T\tilde{M}|_M \rightarrow NM$$

*called the **tangential** and **normal projections**, that for each $p \in M$ restrict to orthogonal projections from $T_p \tilde{M}$ to $T_p M$ and $N_p M$, respectively.*

Proof. Given any point $p \in M$, Theorem ?? shows that there is a neighborhood U of p in M that is embedded in \tilde{M} , and then Proposition 1.2.15 shows that there is a smooth orthonormal frame (E_1, \dots, E_m) that is adapted to U on some neighborhood \tilde{U} of p in \tilde{M} . This means that the restrictions of (E_1, \dots, E_n) to $\tilde{U} \cap U$ form a local orthonormal frame for M . Given such an adapted frame, the restrictions of the last $m - n$ vector fields (E_{n+1}, \dots, E_m) to M form a smooth local frame for NM , so it follows from Lemma ?? that NM is a smooth subbundle.

The bundle homomorphisms π^\top and π^\perp are defined pointwise as orthogonal projections onto the tangent and normal spaces, respectively, which shows that they are uniquely defined. In terms of an adapted orthonormal frame, they can be written

$$\pi^\top(X^i E_i) = X^1 E_1 + \dots + X^n E_n, \quad \pi^\perp(X^i E_i) = X^{n+1} E_{n+1} + \dots + X^m E_m,$$

which shows that they are smooth. \square

In case \tilde{M} is a manifold with boundary, the preceding constructions do not always work, because there is not a fully general construction of slice coordinates in that case. However, there is a satisfactory result in case the submanifold is the boundary itself, using boundary coordinates in place of slice coordinates.

Suppose (M, g) is a Riemannian manifold with boundary. We will always consider ∂M to be a Riemannian submanifold with the induced metric.

outward normal

Proposition 1.2.17 (Existence of Outward-Pointing Normal). *If (M, g) is a smooth Riemannian manifold with boundary, the normal bundle to ∂M is a smooth rank-1 vector bundle over ∂M , and there is a unique smooth outward-pointing unit normal vector field along all of ∂M . By this, we see that $N(\partial M)$ is a rank-1 vector subbundle of $TM|_{\partial M}$.*

Proof. We can always construct a global smooth outward-pointing vector field by taking $-\partial/\partial x^n$ in boundary coordinates in a neighborhood of each boundary point, and gluing together with a partition of unity. By the Gram-Schmidt process, we may assume that this vector field is normal to ∂M . \square

Computations on a submanifold $M \subseteq \tilde{M}$ are usually carried out most conveniently in terms of a **smooth local parametrization**: this is a smooth map $X : U \rightarrow \tilde{M}$, where U is an open subset of \mathbb{R}^n (or \mathbb{H}^n in case M has a boundary), such that $X(U)$ is an open subset of M , and such that X , regarded as a map from U into M , is a diffeomorphism onto its image. Note that we can think of X either as a map into M or as a map into \tilde{M} , both maps are typically denoted by the same symbol X . If we put $V = X(U) \subseteq \tilde{M}$ and $\varphi = X^{-1} : V \rightarrow U$, then (V, φ) is a smooth coordinate chart on M . Suppose (M, g) is a Riemannian submanifold of (\tilde{M}, \tilde{g}) and $X : U \rightarrow \tilde{M}$ is a smooth local parametrization of M . The coordinate representation of g in these coordinates is given by the following 2-tensor field on U :

$$\varphi_* g = X^* g = X^* \iota^* \tilde{g} = (\iota \circ X)^* \tilde{g}.$$

Since $\iota \circ X$ is just the map X itself, regarded as a map into \tilde{M} , this is really just $X^* \tilde{g}$. The simplicity of the formula for the pullback of a tensor field makes this expression exceedingly easy to compute, once a coordinate expression for \tilde{g} is known. For example, if M is an immersed n -dimensional Riemannian submanifold of \mathbb{R}^m and $X : U \rightarrow \mathbb{R}^m$ is a smooth local parametrization of M , the induced metric on U is just

$$g = X^* \tilde{g} = \sum_{i=1}^m (dX^i)^2 = \sum_{i=1}^m \left(\sum_{j=1}^n \frac{\partial X^i}{\partial u^j} du^j \right)^2 = \sum_{i=1}^m \sum_{j,k=1}^n \frac{\partial X^i}{\partial u^j} \frac{\partial X^i}{\partial u^k} du^j du^k.$$

Example 1.2.18 (Induced Metrics in Graph Coordinates). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $S \subseteq \mathbb{R}^{n+1}$ be the graph of a smooth function $f : U \rightarrow \mathbb{R}$. The map $X : U \rightarrow \mathbb{R}^{n+1}$ given by $X(u^1, \dots, u^n) = (u^1, \dots, u^n, f(u))$ is a smooth global parametrization of S and the induced metric on S is given in graph coordinates by

$$X^* \tilde{g} = X^* ((dx^1)^2 + \dots + (dx^{n+1})^2) = (du^1)^2 + \dots + (du^n)^2 + (df)^2$$

For example, the upper hemisphere of S^2 is parametrized by the map $X : \mathbb{B}^2 \rightarrow \mathbb{R}^3$ given by

$$X(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

In these coordinates, the round metric can be written

$$\dot{g} = X^* \tilde{g} = du^2 + dv^2 + \left(\frac{u du + v dv}{\sqrt{1 - u^2 - v^2}} \right)^2 = \frac{(1 - v^2) du^2 + (1 - u^2) dv^2 + 2uv du dv}{1 - u^2 - v^2}.$$

induced metric

Example 1.2.19 (Induced Metrics on Surfaces of Revolution). Let C be an embedded one dimensional submanifold of the half-plane $\{(r, z) : r > 0\}$ and let S_C be the surface of revolution generated by C as described in Example 1.2.18. To compute the induced metric on S_C , choose any smooth local parametrization $\gamma(t) = (a(t), b(t))$ for C , and note that the map $X(t, \theta) = (a(t) \cos \theta, a(t) \sin \theta, b(t))$ yields a smooth local parametrization of S_C , provided

that (t, θ) is restricted to a sufficiently small open subset of the plane. Thus we can compute

$$\begin{aligned} X^* \bar{g} &= (da(t) \cos \theta)^2 + (da(t) \sin \theta)^2 + (db(t))^2 \\ &= (a'(t) \cos \theta dt - a(t) \sin \theta d\theta)^2 + (a'(t) \sin \theta dt + a(t) \cos \theta d\theta)^2 + (b'(t) dt)^2 \\ &= (a'(t)^2 + b'(t)^2) dt^2 + a(t)^2 d\theta^2 \end{aligned}$$

In particular, if γ is a unit-speed curve, meaning that $|\gamma'(t)|^2 = 1$, this reduces to the simple formula $dt^2 + a(t)^2 d\theta^2$.

Here are some familiar examples of surfaces of revolution.

- (a) The embedded torus is the surface of revolution generated by the circle $(r - 2)^2 + z^2 = 1$. Using the unit-speed parametrization $\gamma(t) = (2 + \cos t, \sin t)$ for the circle, we obtain the formula $dt^2 + (2 + \cos \theta)^2 d\theta^2$ for the induced metric.
- (b) The unit sphere (minus the north and south poles) is a surface of revolution whose generating curve is the semicircle parametrized by $\gamma(t) = (\sin t, \cos t)$ for $0 < t < \pi$. The induced metric is $dt^2 + (\sin t)^2 d\theta^2$.
- (c) The unit cylinder $x^2 + y^2 = 1$ is a surface of revolution whose generating curve is the vertical line parametrized by $\gamma(t) = (1, t)$ for $t \in \mathbb{R}$. The induced metric is $dt^2 + d\theta^2$.

Example 1.2.20 (The n -Torus as a Riemannian Submanifold). The smooth covering map $X : \mathbb{R}^n \rightarrow T^n$ defined by $X(u^1, \dots, u^n) = (\cos u^1, \sin u^1, \dots, \cos u^n, \sin u^n)$ restricts to a smooth local parametrization on any sufficiently small open subset of \mathbb{R}^n , and the induced metric is

$$X^* \bar{g} = d(\cos u^1)^2 + d(\sin u^1)^2 + \dots + d(\cos u^n)^2 + d(\sin u^n)^2 = (du^1)^2 + \dots + (du^n)^2.$$

Therefore the induced metric on T^n is flat.

Riemannian products

Next we consider products. If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product manifold $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by

$$g_{(p_1, p_2)}((v_1, v_2), (w_1, w_2)) = g_1|_{p_1}(v_1, w_1) + g_2|_{p_2}(v_2, w_2).$$

where (v_1, v_2) and (w_1, w_2) are elements of $T_{p_1} M_1 \oplus T_{p_2} M_2$, which is naturally identified with $T_{(p_1, p_2)}(M_1 \times M_2)$. Smooth local coordinates (x^1, \dots, x^n) for M_1 and $(x^{n+1}, \dots, x^{n+m})$ for M_2 give coordinates (x^1, \dots, x^{n+m}) for $M_1 \oplus M_2$. In terms of these coordinates, the product metric has the local expression $g = g_{ij} dx^i dx^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij}) = \begin{pmatrix} (g_1) & 0 \\ 0 & (g_2) \end{pmatrix}$$

Product metrics on products of three or more Riemannian manifolds are defined similarly.

Here is an important generalization of product metrics. Suppose (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds, and $f : M_1 \rightarrow \mathbb{R}^+$ is a strictly positive smooth function. The **warped product** $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ endowed with the Riemannian metric $g = g_1 \oplus f^2 g_2$, defined by

$$g_{(p_1, p_2)}((v_1, v_2), (w_1, w_2)) = g_1|_{p_1}(v_1, w_1) + f(p_1)^2 g_2|_{p_2}(v_2, w_2).$$

where $(v_1, v_2), (w_1, w_2) \in T_{p_1}M_1 \oplus T_{p_2}M_2$ as before. A wide variety of metrics can be constructed in this way; here are just a few examples.

Example 1.2.21. If we let ρ denote the standard coordinate function on $\mathbb{R}^+ \subseteq \mathbb{R}$, then the map $\Phi(\rho, \omega) = \rho\omega$ gives an isometry from the warped product $\mathbb{R}^+ \times_\rho S^{n-1}$ to $\mathbb{R}^n \setminus \{0\}$ with its Euclidean metric. To see this, we use the parametrization

$$X : (\rho, x^1, \dots, x^{n-1}) \mapsto (\rho, x^1, \dots, x^{n-1}, \sqrt{1 - |x|^2})$$

of $\mathbb{R}^+ \times S^{n-1}$ to compute the pull back metric $\Phi^*\bar{g}$ as follows:

$$\begin{aligned} (\Phi \circ X)^*\bar{g} &= d(\rho x^1)^2 + \dots + d(\rho x^{n-1})^2 + d(\rho \sqrt{1 - |x|^2})^2 \\ &= (x^1 d\rho + \rho dx^1)^2 + \dots + (x^{n-1} d\rho + \rho dx^{n-1})^2 + \left(\sqrt{1 - |x|^2} d\rho - \rho \frac{x^1 dx^1 + \dots + x^{n-1} dx^{n-1}}{\sqrt{1 - |x|^2}} \right)^2 \\ &= d\rho^2 + \rho^2((dx^1)^2 + \dots + (dx^{n-1})^2) + \rho^2 \frac{(x^1 dx^1 + \dots + x^{n-1} dx^{n-1})^2}{1 - |x|^2} \\ &\quad + 2\rho(x^1 dx^1 d\rho + \dots + x^{n-1} dx^{n-1} d\rho) - 2\rho(x^1 dx^1 d\rho + \dots + x^{n-1} dx^{n-1} d\rho) \\ &= d\rho^2 + \rho^2 \left((dx^1)^2 + \dots + (dx^{n-1})^2 + \frac{(x^1 dx^1 + \dots + x^{n-1} dx^{n-1})^2}{1 - |x|^2} \right) \\ &= d\rho^2 + \rho^2 \dot{g} \end{aligned}$$

Example 1.2.22. Every surface of revolution can be expressed as a warped product, as follows. Let H be the half-plane $\{(r, z) : r > 0\}$, let $C \subseteq H$ be an embedded smooth 1-dimensional submanifold, and let $S_C \subseteq \mathbb{R}^3$ denote the corresponding surface of revolution as in Example 1.2.19. Endow C with the Riemannian metric induced from the Euclidean metric on H , and let S^1 be endowed with its standard metric. Let $f : C \rightarrow \mathbb{R}$ be the distance to the z -axis: $f(r, z) = r$. Then from Example 1.2.19 we see that S_C is isometric to the warped product $C \times_f S^1$.

Riemannian submersions

Unlike submanifolds and products of Riemannian manifolds, which automatically inherit Riemannian metrics of their own, quotients of Riemannian manifolds inherit Riemannian metrics only under very special circumstances. Now we see what those circumstances are.

Suppose \tilde{M} and M are smooth manifolds, $\pi : \tilde{M} \rightarrow M$ is a smooth submersion, and \tilde{g} is a Riemannian metric on \tilde{M} . By the submersion level set theorem, each fiber $\tilde{M}_y = \pi^{-1}(y)$ is a properly embedded smooth submanifold of \tilde{M} . At each point $x \in \tilde{M}$, we define two subspaces of the tangent space $T_x \tilde{M}$ as follows: the **vertical tangent space** at x is

$$V_x = \ker d\pi_x = T_x(\tilde{M}_{\pi(x)})$$

(that is, the tangent space to the fiber containing x), and the **horizontal tangent space** at x is its orthogonal complement:

$$H_x = (V_x)^\perp$$

Then the tangent space $T_x \tilde{M}$ decomposes as an orthogonal direct sum $T_x \tilde{M} = H_x \oplus V_x$. Note that the vertical space is well defined for every submersion, because it does not refer to the metric; but the horizontal space depends on the metric.

A vector field on \tilde{M} is said to be a **horizontal vector field** if its value at each point lies in the horizontal space at that point; a **vertical vector field** is defined similarly. Given a vector

field X on M , a vector field \tilde{X} on \tilde{M} is called a **horizontal lift** of X if \tilde{X} is horizontal and π -related to X .

The next proposition is the principal tool for doing computations on Riemannian submersions.

vector field

Proposition 1.2.23 (Properties of Horizontal Vector Fields). *Let \tilde{M} and M be smooth manifolds, let $\pi : \tilde{M} \rightarrow M$ be a smooth submersion, and let \tilde{g} be a Riemannian metric on \tilde{M} .*

- (a) *Every smooth vector field W on \tilde{M} can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.*
- (b) *Every smooth vector field on M has a unique smooth horizontal lift to \tilde{M} .*
- (c) *For every $x \in \tilde{M}$ and $v \in H_x$, there is a vector field $X \in \mathfrak{X}(M)$ whose horizontal lift \tilde{X} satisfies $\tilde{X}_x = v$.*

Proof. Let $p \in \tilde{M}$ be arbitrary. Because π is a smooth submersion, the rank theorem shows that there exist smooth coordinate charts $(\tilde{U}, (x^i))$ centered at p and $(U, (y^j))$ centered at $\pi(p)$ in which π has the coordinate representation

$$\pi(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n)$$

where $m = \dim \tilde{M}$ and $n = \dim M$. It follows that at each point $q \in \tilde{U}$, the vertical space V_q is spanned by the vectors $(\partial/\partial x^{n+1}|_q, \dots, \partial/\partial x^m|_q)$. (It probably will not be the case, however, that the horizontal space is spanned by the other n basis vectors.) If we apply the Gram-Schmidt algorithm to the ordered frame $(\partial/\partial x^{n+1}|_q, \dots, \partial/\partial x^m|_q, \partial/\partial x^1|_q, \dots, \partial/\partial x^n|_q)$, we obtain a smooth orthonormal frame (E_1, \dots, E_m) on \tilde{U} such that V_q is spanned by $(E_1|_q, \dots, E_{m-n}|_q)$ at each $q \in \tilde{U}$. It follows that H_q is spanned by $(E_{n-m+1}|_q, \dots, E_m|_q)$.

Now let $W \in \mathfrak{X}(\tilde{M})$ be arbitrary. At each point $q \in \tilde{M}$, W_q can be written uniquely as a sum of a vertical vector plus a horizontal vector, thus defining a decomposition $W = W^V + W^H$ into rough vertical and horizontal vector fields. To see that they are smooth, just note that in a neighborhood of each point we can express W in terms of a frame (E_1, \dots, E_m) of the type constructed above as $W = W^1 E_1 + \dots + W^m E_m$ with smooth coefficients (W^i) , and then it follows that

$$W^V = W^1 E_1 + \dots + W^{m-n} E_{m-n}, \quad W^H = W^{n-m+1} E_{n-m+1} + \dots + W^m E_m,$$

both of which are smooth.

Let $X \in \mathfrak{X}(M)$ be arbitrary. By the coordinate representation constructed above, we see that $d\pi$ is just the projection onto H_q , which is spanned by (E_{n-m+1}, \dots, E_m) . Therefore we can use X to define a rough horizontal vector field on \tilde{M} . From the local coordinate presentation we see that \tilde{X} is smooth.

For every $x \in \tilde{M}$ and $v \in H_x$, we can choose coordinates as above, and construct a vector field on M such that $X_{\pi(x)} = d\pi(v)$. Then by the construction above the horizontal lift \tilde{X} satisfies $\tilde{X}_x = v$. \square

The fact that every horizontal vector at a point of \tilde{M} can be extended to a horizontal lift on all of \tilde{M} (part (c) of the preceding proposition) is highly useful for computations. It is important to be aware, though, that not every horizontal vector field on \tilde{M} is a horizontal lift, as the next example shows.

Example 1.2.24. Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection map $\pi(x, y) = yx$, and let W be the smooth vector field $y\partial/\partial x$ on \mathbb{R}^2 . Then W is horizontal, but there is no vector field on \mathbb{R} whose horizontal lift is equal to W .

Now we can identify some quotients of Riemannian manifolds that inherit metrics of their own. Let us begin by describing what such a metric should look like. Suppose (\tilde{M}, \tilde{g}) and (M, g) are Riemannian manifolds, and $\pi : \tilde{M} \rightarrow M$ is a smooth submersion. Then π is said to be a **Riemannian submersion** if for each $x \in \tilde{M}$, the differential $d\pi_x$ restricts to a linear isometry from H_x onto $T_{\pi(x)}M$. In other words, $\tilde{g}_x(v, w) = g_{\pi(x)}(d\pi_x(v), d\pi_x(w))$ whenever $v, w \in H_x$.

Example 1.2.25 (Riemannian Submersions).

- (a) The projection $\pi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ onto the first n coordinates is a Riemannian submersion if \mathbb{R}^{n+k} and \mathbb{R}^n are both endowed with their Euclidean metrics.
- (b) If M and N are Riemannian manifolds and $M \times N$ is endowed with the product metric, then both projections $\pi_M : M \times N \rightarrow M$ and $\pi_N : M \times N \rightarrow N$ are Riemannian submersions.
- (c) If $M \times_f N$ is a warped product manifold, then the projection $\pi_M : M \times_f N \rightarrow M$ is a Riemannian submersion, but π_N typically is not.

Given a Riemannian manifold (\tilde{M}, \tilde{g}) and a surjective submersion $\pi : \tilde{M} \rightarrow M$, it is almost never the case that there is a metric on M that makes π into a Riemannian submersion. It is not hard to see why: for this to be the case, whenever $p_1, p_2 \in \tilde{M}$ are two points in the same fiber $\pi^{-1}(y)$, the linear maps $(d\pi_{p_i}|_{H_{p_i}})^{-1} : T_y M \rightarrow H_{p_i}$ both have to pull \tilde{g} back to the same inner product on $T_y M$. There is, however, an important special case in which there is such a metric. Suppose $\pi : \tilde{M} \rightarrow M$ is a smooth surjective submersion, and G is a group acting on \tilde{M} . We say that the action is **vertical** if every element $\varphi \in G$ takes each fiber to itself, meaning that $\pi(\varphi \cdot p) = \pi(p)$ for all $p \in \tilde{M}$. The action is **transitive on fibers** if for each $p, q \in \tilde{M}$ such that $\pi(p) = \pi(q)$, there exists $\varphi \in G$ such that $\varphi \cdot p = q$. If in addition \tilde{M} is endowed with a Riemannian metric, the action is said to be an **isometric action** or an **action by isometries**, and the metric is said to be **invariant under G** , if the map $x \mapsto \varphi \cdot x$ is an isometry for each $\varphi \in G$. In that case, provided the action is effective (so that different elements of G define different isometries of \tilde{M}), we can identify G with a subgroup of $\text{Iso}(\tilde{M}, \tilde{g})$. Since an isometry is, in particular, a diffeomorphism, every isometric action is an action by diffeomorphisms.

Induce metric

Theorem 1.2.26. Let (\tilde{M}, \tilde{g}) be a Riemannian manifold, let $\pi : \tilde{M} \rightarrow M$ be a surjective smooth submersion, and let G be a group acting on \tilde{M} . If the action is isometric, vertical, and transitive on fibers, then there is a unique Riemannian metric on M such that π is a Riemannian submersion.

The next corollary describes one important situation to which the preceding theorem applies.

Induce metric

Corollary 1.2.27. Suppose (\tilde{M}, \tilde{g}) is a Riemannian manifold, and G is a Lie group acting smoothly, freely, properly, and isometrically on \tilde{M} . Then the orbit space $M = \tilde{M}/G$ has a unique smooth manifold structure and Riemannian metric such that $\pi : \tilde{M} \rightarrow M$ is a Riemannian submersion.

Proof. Under the given hypotheses, the quotient manifold theorem shows that M has a unique smooth manifold structure such that the quotient map $\tilde{M} \rightarrow \tilde{M}/G$ is a smooth submersion. It follows easily from the definitions in that case that the given action of G on \tilde{M} is vertical and transitive on fibers. Since the action is also isometric, Theorem 1.2.26 shows that \tilde{M} inherits a unique Riemannian metric making π into a Riemannian submersion. \square

Here is an important example of a Riemannian metric defined in this way.

Example 1.2.28 (The Fubini-Study Metric). Let n be a positive integer, and consider the complex projective space \mathbb{CP}^n . The map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$ sending each point in $\mathbb{C}^{n+1} \setminus \{0\}$ to its span is a surjective smooth submersion. Identifying \mathbb{C}^{n+1} with \mathbb{R}^{2n+2} endowed with its Euclidean metric, we can view the unit sphere S^{2n+1} with its round metric \dot{g} as an embedded Riemannian submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$. Let $p : S^{2n+1} \rightarrow \mathbb{CP}^n$ denote the restriction of the map π . Then p is smooth, and it is surjective, because every 1-dimensional complex subspace contains elements of unit norm. We need to show that it is a submersion. Let $z_0 \in S^{2n+1}$ and set $\zeta_0 = p(z_0) \in \mathbb{CP}^n$. Since π is a smooth submersion, it has a smooth local section $\sigma : U \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ defined on a neighborhood U of ζ_0 . Let $\tilde{\sigma}$ be the radial projection of σ onto the sphere: $\tilde{\sigma} = \sigma/|\sigma|$. Then $\tilde{\sigma}$ is a local section of p . By Theorem 1.2.29 we see that p is a submersion.

Define an action of S^1 on S^{2n+1} by complex multiplication

$$\lambda \cdot (z^1, \dots, z^{n+1}) = (\lambda z^1, \dots, \lambda z^{n+1})$$

for $\lambda \in S^1$ (viewed as a complex number of norm 1) and $(z^1, \dots, z^{n+1}) \in S^{2n+1}$. This is easily seen to be isometric, vertical, and transitive on fibers of p . By Theorem 1.2.26, therefore, there is a unique metric on \mathbb{CP}^n such that the map $p : S^{2n+1} \rightarrow \mathbb{CP}^n$ is a Riemannian submersion. This metric is called the **Fubini-Study metric**.

Riemannian coverings

Another important special case of Riemannian submersions occurs in the context of covering maps. Suppose (\tilde{M}, \tilde{g}) and (M, g) are Riemannian manifolds. A smooth covering map $\pi : \tilde{M} \rightarrow M$ is called a **Riemannian covering** if it is a local isometry.

Proposition 1.2.29. Suppose $\pi : \tilde{M} \rightarrow M$ is a smooth normal covering map, and \tilde{g} is any metric on \tilde{M} that is invariant under all covering automorphisms. Then there is a unique metric g on M such that π is a Riemannian covering.

Proof. Proposition 1.2.29 shows that π is a surjective smooth submersion. The automorphism group acts vertically by definition, and we know that it acts transitively on fibers when the covering is normal. It then follows from Theorem 1.2.26 that there is a unique metric g on M such that π is a Riemannian submersion. Since a Riemannian submersion is a local isometry, it follows that π is a Riemannian covering. \square

Proposition 1.2.30. Suppose (\tilde{M}, \tilde{g}) is a Riemannian manifold, and Γ is a discrete Lie group acting smoothly, freely, properly, and isometrically on \tilde{M} . Then \tilde{M}/Γ has a unique Riemannian metric such that the quotient map $\pi : \tilde{M} \rightarrow \tilde{M}/\Gamma$ is a normal Riemannian covering.

Proof. Proposition 1.2.30 shows that π is a smooth normal covering map, and Proposition 1.2.29 shows that \tilde{M}/Γ has a unique Riemannian metric such that π is a Riemannian covering. \square

Corollary 1.2.31. Suppose (\tilde{M}, \tilde{g}) and (M, g) are connected Riemannian manifolds, $\pi : \tilde{M} \rightarrow M$ is a normal Riemannian covering map, and $\Gamma = \text{Aut}_\pi(\tilde{M})$. Then M is isometric to \tilde{M}/Γ .

Proof. Proposition ?? shows that with the discrete topology, Γ is a discrete Lie group acting smoothly, freely, and properly on \tilde{M} , and then Proposition ?? shows that \tilde{M}/Γ is a smooth manifold and the quotient map $q : \tilde{M} \rightarrow \tilde{M}/\Gamma$ is a smooth normal covering map. The fact that both π and q are normal coverings implies that Γ acts transitively on the fibers of both maps, so the two maps are constant on each other's fibers. Proposition ?? then implies that there is a diffeomorphism $F : M \rightarrow \tilde{M}/\Gamma$ that satisfies $q \circ F = \pi$. Because both q and π are local isometries, F is too, and because it is bijective it is a global isometry. \square

Example 1.2.32. The two-element group $\{\pm 1\}$ acts smoothly, freely, properly, and isometrically on S^n by multiplication. We know that the quotient space is diffeomorphic to the real projective space \mathbb{RP}^n and the quotient map $q : S^n \rightarrow \mathbb{RP}^n$ is a smooth normal covering map. Because the action is isometric, Proposition 1.2.30 shows that there is a unique metric on \mathbb{RP}^n such that q is a Riemannian covering.

Example 1.2.33 (The Open Möbius Band). The open Möbius band is the quotient space $M = \mathbb{R}^2/\mathbb{Z}$, where \mathbb{Z} acts on \mathbb{R}^2 by

$$n \cdot (x, y) = (x + n, (-1)^n y).$$

This action is smooth, free, proper, and isometric, and therefore M inherits a flat Riemannian metric such that the quotient map is a Riemannian covering.

1.2.3 Basic constructions on Riemannian manifolds

Raising and lowering indices

One elementary but important property of Riemannian metrics is that they allow us to convert vectors to covectors and vice versa. Given a Riemannian metric g on M , we define a bundle homomorphism $\hat{g} : TM \rightarrow T^*M$ by setting

$$\hat{g}(v)(w) = g_p(v, w)$$

for each $p \in M$ and each $v, w \in T_p M$. To see that this is a smooth bundle homomorphism, it is easiest to consider its action on smooth vector fields:

$$\hat{g}(X)(Y) = g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Because $\hat{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y , it follows from the tensor characterization lemma (Lemma 1.1.16) that $\hat{g}(X)$ is a smooth covector field, and because $\hat{g}(X)$ is linear over $C^\infty(M)$ as a function of X , this defines \hat{g} as a smooth bundle homomorphism by the bundle homomorphism characterization lemma (Lemma ??). As usual, we use the same symbol for both the pointwise bundle homomorphism $\hat{g} : TM \rightarrow T^*M$ and the linear map on sections $\hat{g} : \mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$.

Note that \hat{g} is injective at each point, because $\hat{g}(v) = 0$ for some $v \in T_p M$ implies

$$0 = \hat{g}(v)(v) = g(v, v)$$

which in turn implies $v = 0$. For dimensional reasons, therefore, \hat{g} is bijective, so it is a bundle isomorphism (Proposition ??).

In any smooth coordinates (x^i) , we can write $g = g_{ij}dx^i dx^j$. Thus, if X and Y are smooth vector fields, we have

$$\widehat{g}(X)(Y) = g_{ij}X^i Y^j$$

which implies that the covector field $\widehat{g}(X)$ has the coordinate expression

$$\widehat{g}(X) = g_{ij}X^i dx^j$$

In other words, \widehat{g} is the bundle homomorphism whose matrix with respect to coordinate frames for TM and T^*M is the same as the matrix of g itself.

It is customary to denote the components of the covector field $\widehat{g}(X)$ by

$$\widehat{g} = X_j dx^j, \quad \text{where } X_j = g_{ij}X^i.$$

Because of this, one says that $\widehat{g}(X)$ is obtained from X by lowering an index. The notation X^\flat is frequently used for $\widehat{g}(X)$, because the symbol \flat is used in musical notation to indicate that a tone is to be lowered.

The matrix of the inverse map $\widehat{g}^{-1} : T^*M \rightarrow TM$ is thus the inverse of (g_{ij}) . We let (g^{ij}) denote the matrix-valued function whose value at $p \in M$ is the inverse of the matrix $(g_{ij}(p))$, so that

$$g^{ij}g_{jk} = g_{kj}g^{ji} = \delta_k^i$$

Because g_{ij} is a symmetric matrix, so is g^{ij} , as you can easily check. Thus for a covector field $\omega \in \mathfrak{X}^*(M)$, the vector field $\widehat{g}^{-1}(\omega)$ has the coordinate representation

$$\widehat{g}^{-1}(\omega) = \omega^i \frac{\partial}{\partial x^i}, \quad \text{where } \omega^i = \omega_j g^{ij}.$$

We use the notation \sharp for \widehat{g}^{-1} , and say that ω^\sharp is obtained from ω by raising an index. Because the symbols \flat and \sharp are borrowed from musical notation, these two inverse isomorphisms are frequently called the **musical isomorphisms**. A handy mnemonic device for keeping the flat and sharp operations straight is to remember that the value of ω^\sharp at each point is a vector, which we visualize as a (sharp) arrow; while the value of X^\flat is a covector, which we visualize by means of its (flat) level sets.

The most important use of the sharp operation is to reinstate the **gradient** as a vector field on Riemannian manifolds. For any smooth real-valued function f on a Riemannian manifold (M, g) with or without boundary, we define a vector field called the gradient of f by

$$\text{grad } f := (df)^\sharp = \widehat{g}^{-1}(df).$$

Unraveling the definitions, we see that for any $X \in \mathfrak{X}(M)$, the gradient satisfies

$$\langle \text{grad } f, X \rangle_g = \widehat{g}(\text{grad } f)(X) = df(X) = Xf.$$

Thus $\text{grad } f$ is the unique vector field that satisfies

$$\langle \text{grad } f, X \rangle_g = Xf \quad \text{for } X \in \mathfrak{X}(M),$$

or equivalently,

$$\langle \text{grad } f, \cdot \rangle_g = df. \tag{2.5} \quad \boxed{\text{Riemann grad}}$$

In smooth coordinates, $\text{grad } f$ has the expression

$$\text{grad } f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

In particular, this shows that $\text{grad} f$ is smooth. On \mathbb{R}^n with the Euclidean metric, this reduces to

$$\text{grad} f = \delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Thus our new definition of the gradient in this case coincides with the gradient from elementary calculus. In other coordinates, however, the gradient does not generally have the same form.

Example 1.2.34. Let us compute the gradient of a function $f \in C^\infty(\mathbb{R}^2)$ with respect to the Euclidean metric in polar coordinates. From Example 1.2.11 we see that the matrix of \bar{g} in polar coordinates is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, so its inverse matrix is $\begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$. Inserting this into the formula for the gradient, we obtain

$$\text{grad} f = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

The next proposition shows that the gradient has the same geometric interpretation on a Riemannian manifold as it does in Euclidean space. If f is a smooth real-valued function on a smooth manifold M , recall that a point $p \in M$ is called a **regular point** of f if $df_p \neq 0$, and a **critical point** of f otherwise; and a level set $f^{-1}(c)$ is called a regular level set if every point of $f^{-1}(c)$ is a regular point of f . Corollary 1.2.33 shows that each regular level set is an embedded smooth hypersurface in M .

Proposition 1.2.35. Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $\mathcal{R} \subseteq M$ is the set of regular points of f . For each $c \in \mathbb{R}$, the set $M_c = f^{-1}(c) \cap \mathcal{R}$, if nonempty, is an embedded smooth hypersurface in M , and $\text{grad} f$ is everywhere normal to M_c .

Proof. From the definition of gradient we have

$$\langle \text{grad} f|_p, v \rangle = df_p(v).$$

By Proposition 1.2.33 we see that $T_p M_c = \ker df_p$, therefore the claim holds. \square

The flat and sharp operators can be applied to tensors of any rank, in any index position, to convert tensors from covariant to contravariant or vice versa. Formally, this operation is defined as follows: if F is any (k, l) -tensor and $i \in \{1, \dots, k + l\}$ is any covariant index position for F (meaning that the i -th argument is a vector, not a covector), we can form a new tensor F^\sharp of type $(k + 1, l - 1)$ by setting

$$F^\sharp(\alpha_1, \dots, \alpha_{k+l}) = F(\alpha_1, \dots, \alpha_i^\sharp, \dots, \alpha_{k+l})$$

whenever $\alpha_1, \dots, \alpha_{k+l}$ are vectors or covectors as appropriate. In any local frame, the components of F^\sharp are obtained by multiplying the components of F by g^{kl} and contracting one of the indices of g^{kl} with the i -th index of F . Similarly, if i is a contravariant index position, we can define a $(k - 1, l + 1)$ -tensor F^\flat by

$$F^\flat(\alpha_1, \dots, \alpha_{k+l}) = F(\alpha_1, \dots, \alpha_i^\flat, \dots, \alpha_{k+l})$$

In components, it is computed by multiplying by g_{kl} and contracting.

For example, if A is a mixed 3-tensor given in terms of a local frame by

$$A = A_{ik}^j \varepsilon^i \otimes E_j \otimes \varepsilon^k$$

we can lower its middle index to obtain a covariant 3-tensor A^b with components

$$A_{ijk} = g_{jl} A_{ik}^l.$$

Another important application of the flat and sharp operators is to extend the trace operator to covariant tensors. If h is any covariant k -tensor field on a Riemannian manifold with $k \geq 2$, we can raise one of its indices (say the last one for definiteness) and obtain a $(1, k-1)$ -tensor h^\sharp . The trace of h^\sharp is thus a well-defined covariant $(k-2)$ -tensor field. We define the trace of h with respect to g as

$$\mathrm{tr}_g h = \mathrm{tr}(h^\sharp).$$

Sometimes we may wish to raise an index other than the last, or to take the trace on a pair of indices other than the last covariant and contravariant ones. In each such case, we will say in words what is meant.

The most important case is that of a covariant 2-tensor field. In this case, h^\sharp is a $(1, 1)$ -tensor field, which can equivalently be regarded as an endomorphism field, and $\mathrm{tr}_g h$ is just the ordinary trace of this endomorphism field. In terms of a basis, this is

$$\mathrm{tr}_g h = h_i^i = g^{ij} h_{ij}.$$

In particular, in an orthonormal frame this is the ordinary trace of the matrix (h_{ij}) (the sum of its diagonal entries); but if the frame is not orthonormal, then this trace is different from the ordinary trace.

If g is a Riemannian metric on M and (E_i) is a local frame on M , there is a potential ambiguity about what the expression (g^{ij}) represents: we have defined it to mean the inverse matrix of (g_{ij}) , but one could also interpret it as the components of the contravariant 2-tensor field $g^{\sharp\sharp}$ obtained by raising both of the indices of g . But these two interpretations lead to the same result:

$$g^{\sharp\sharp} = g^{lk} g^{ki} g_{ij} = g^{lj},$$

so we do not distinguish them.

Inner products of tensors

A Riemannian metric yields, by definition, an inner product on tangent vectors at each point. Because of the musical isomorphisms between vectors and covectors, it is easy to carry the inner product over to covectors as well. Suppose g is a Riemannian metric on M , and $x \in M$. We can define an inner product on the cotangent space T_x^*M by

$$\langle \omega, \eta \rangle_g = \langle \omega^\sharp, \eta^\sharp \rangle_g$$

To see how to compute this, we just use the basis formula for the sharp operator, together with the relation $g_{kl} g^{ki} = g_{lk} g^{ki} = \delta_l^i$, to obtain

$$\begin{aligned} \langle \omega, \eta \rangle_g &= g_{kl} (g^{ki} \omega_i) (g^{lj} \eta_j) \\ &= \delta_l^i g^{lj} \omega_i \eta_j \\ &= g^{ij} \omega_j \eta_i \end{aligned}$$

In other words, the inner product on covectors is represented by the inverse matrix (g^{ij}) . Using our conventions for raising and lowering indices, this can also be written

$$\langle \omega, \eta \rangle = \omega_i \eta^i = \omega^i \eta_i.$$

Now the following result is immediate from the definition.

Proposition 1.2.36. *Let (M, g) be a Riemannian manifold with or without boundary, let (E_i) be a local frame for M , and let (ε^i) be its dual coframe. Then the following are equivalent:*

- (a) (E_i) is orthonormal.
- (b) (ε^i) is orthonormal.
- (c) $(\varepsilon^i)^\# = E_i$ for each i .

This construction can be extended to tensor bundles of any rank, as the following proposition shows. First a bit of terminology: if $E \rightarrow M$ is a smooth vector bundle, a **smooth fiber metric** on E is an inner product on each fiber E_p that varies smoothly, in the sense that for any (local) smooth sections σ, τ of E , the inner product $\langle \sigma, \tau \rangle$ is a smooth function.

Proposition 1.2.37 (Inner Products of Tensors). *Let (M, g) be an n -dimensional Riemannian manifold with or without boundary. There is a unique smooth fiber metric on each tensor bundle $T^{(k,l)}TM$ with the property that if $\alpha_1, \dots, \alpha_{k+l}, \beta_1, \dots, \beta_{k+l}$ are vector or covector fields as appropriate, then*

$$\langle \alpha_1 \otimes \dots \otimes \alpha_{k+l}, \beta_1 \otimes \dots \otimes \beta_{k+l} \rangle = \langle \alpha_1, \beta_1 \rangle \dots \langle \alpha_{k+l}, \beta_{k+l} \rangle. \quad (2.6)$$

Riemann tensor

With this inner product, if (E_1, \dots, E_n) is a local orthonormal frame for TM and $(\varepsilon^1, \dots, \varepsilon^n)$ is the corresponding dual coframe, then the collection of tensor fields $E_{i_1} \otimes \dots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \dots \otimes \varepsilon^{j_l}$ as all the indices range from 1 to n forms a local orthonormal frame for $T^{(k,l)}(T_p M)$. In terms of any (not necessarily orthonormal) frame, this fiber metric satisfies

$$\langle F, G \rangle = g_{i_1 r_1} \dots g_{i_k r_k} g^{j_1 s_1} \dots g^{j_l s_l} F_{j_1 \dots j_l}^{i_1 \dots i_k} G_{s_1 \dots s_l}^{r_1 \dots r_k}. \quad (2.7)$$

Riemann tensor

If F and G are both covariant, this can be written

$$\langle F, G \rangle = F_{j_1 \dots j_l} G^{j_1 \dots j_l},$$

where the last factor on the right represents the components of G with all of its indices raised:

$$G^{j_1 \dots j_l} = g^{j_1 s_1} \dots g^{j_l s_l} G_{j_1 \dots j_l}.$$

Proof. The formula [\(2.6\)](#) is just the definition of the inner product. With this, all is clear. \square

1.2.4 The Riemannian distance function

One of the most important tools that a Riemannian metric gives us is the ability to define lengths of curves. Suppose (M, g) is a Riemannian manifold with or without boundary. If $\gamma : [a, b] \rightarrow M$ is a piecewise smooth curve segment, the length of γ is

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$$

Because $|\gamma'(t)|_g$ is continuous at all but finitely many values of t , and has well-defined limits from the left and right at those points, the integral is well defined.

Proposition 1.2.38. *suppose (M, g) and (\tilde{M}, \tilde{g}) are Riemannian manifolds with or without boundary, and $F : M \rightarrow \tilde{M}$ is a local isometry. Then $L_{\tilde{g}}(F \circ \gamma) = L_g(\gamma)$ for every piecewise smooth curve segment γ in M .*

ometry length

Proof. This follows from the observation

$$\begin{aligned} |(F \circ \gamma)'(t)|_g^2 &= |dF_{\gamma(t)}(\gamma'(t))|_{\tilde{g}}^2 = \tilde{g}[dF_{\gamma(t)}(\gamma'(t)), dF_{\gamma(t)}(\gamma'(t))] \\ &= (F^*\tilde{g})(\gamma'(t), \gamma'(t)) = g(\gamma'(t), \gamma'(t)) = |\gamma'(t)|_g^2 \end{aligned}$$

since F is a local isometry. \square

It is an extremely important fact that length is independent of parametrization in the following sense. In Section 1.2.39 we defined a reparametrization of a piecewise smooth curve segment $\gamma : [a, b] \rightarrow M$ to be a curve segment of the form $\tilde{\gamma} = \gamma \circ \varphi$ where $\varphi : [c, d] \rightarrow [a, b]$ is a diffeomorphism. We also call them **admissible curves**.

Proposition 1.2.39 (Parameter Independence of Length). *Let (M, g) be a Riemannian manifold with or without boundary, and let $\gamma : [a, b] \rightarrow M$ be a piecewise smooth curve segment. If $\tilde{\gamma}$ is a reparametrization of γ , then $L_g(\gamma) = L_g(\tilde{\gamma})$.*

Proof. First suppose that γ is smooth, and $\varphi : [c, d] \rightarrow [a, b]$ is a diffeomorphism such that $\tilde{\gamma} = \gamma \circ \varphi$. The fact that φ is a diffeomorphism implies that either $\varphi' > 0$ or $\varphi' < 0$ everywhere. Let us assume that $\varphi' > 0$. We have

$$\begin{aligned} L_g(\tilde{\gamma}) &= \int_c^d |\tilde{\gamma}'(t)|_g dt = \int_c^d |(\gamma \circ \varphi)'(t)|_g dt \\ &= \int_c^d |\varphi'(t)\gamma'(\varphi(t))|_g dt = \int_c^d |\gamma'(\varphi(t))|_g \varphi'(t) dt \\ &= \int_a^b |\gamma'(s)|_g ds = L_g(\gamma). \end{aligned}$$

In the case $\varphi' < 0$, we just need to introduce two sign changes into the above calculation. The sign changes once when $\varphi'(t)$ is moved outside the absolute value signs, because $|\varphi'(t)| = -\varphi'(t)$. Then it changes again when we change variables, because φ reverses the direction of the integral. Since the two sign changes cancel each other, the result is the same.

If γ is only piecewise smooth, we just apply the same argument on each subinterval on which it is smooth. \square

Suppose $\gamma : [a, b] \rightarrow M$ is an admissible curve. The **arc-length function** of γ is the function $s : [a, b] \rightarrow \mathbb{R}$ defined by

$$s(t) = L_g(\gamma|_{[a, t]}) = \int_a^t |\gamma'(u)|_g du.$$

It is continuous everywhere, and it follows from the fundamental theorem of calculus that it is smooth wherever γ is, with derivative $s'(t) = |\gamma'(t)|_g$. For this reason, if $\gamma : I \rightarrow M$ is any smooth curve (not necessarily a curve segment), we define the **speed** of γ at any time $t \in I$ to be the scalar $|\gamma'(t)|_g$. We say that γ is a **unit-speed curve** if $|\gamma'(t)|_g = 1$ for all t , and a **constant-speed curve** if $|\gamma'(t)|_g$ is constant. If γ is a piecewise smooth curve, we say that γ has unit speed if $|\gamma'(t)|_g = 1$ wherever γ is smooth.

If $\gamma : [a, b] \rightarrow M$ is a unit-speed admissible curve, then its arc-length function has the simple form $s(t) = t - a$. If, in addition, its parameter interval is of the form $[0, b]$ for some $b > 0$, then the arc-length function is $s(t) = t$. For this reason, a unit-speed admissible curve whose parameter interval is of the form $[0, b]$ is said to be **parametrized by arc length**.

Proposition 1.2.40. *Suppose (M, g) is a Riemannian manifold with or without boundary.*

- (a) Every regular curve in M has a unit-speed forward reparametrization.
- (b) Every admissible curve in M has a unique forward reparametrization by arc length.

Proof. Suppose $\gamma : I \rightarrow M$ is a regular curve. Choose an arbitrary $t_0 \in I$, and define $s : I \rightarrow \mathbb{R}$ by

$$s(t) = \int_{t_0}^t |\gamma'(u)|_g du.$$

Since $s'(t) = |\gamma'(t)|_g > 0$, it follows that s is a strictly increasing local diffeomorphism, and thus maps I diffeomorphically onto an interval $I' \subseteq \mathbb{R}$. If we let $\varphi = s^{-1} : I' \rightarrow I$, then $\tilde{\gamma} := \gamma \circ \varphi$ is a forward reparametrization of γ , and the chain rule gives

$$|\tilde{\gamma}'(u)|_g = |\varphi'(s)\gamma'(\varphi(s))|_g = \frac{1}{s'(\varphi(s))} |\gamma'(\varphi(s))|_g = 1.$$

Thus $\tilde{\gamma}$ is a unit-speed reparametrization of γ .

Now let $\gamma : I \rightarrow M$ be an admissible curve. We prove the existence statement in part (b) by induction on the number of smooth segments in an admissible partition. If γ has only one smooth segment, then it is a regular curve segment, and (b) follows by applying (a) in the special case $I = [a, b]$ and choosing $t_0 = a$. Assuming that the result is true for admissible curves with k smooth segments, suppose γ has an admissible partition (a_0, \dots, a_{k+1}) . The inductive hypothesis gives piecewise regular homeomorphisms $\varphi_1 : [0, c] \rightarrow [a, a_k]$ and $\varphi_2 : [0, d] \rightarrow [a_k, b]$ such that $\gamma \circ \varphi_1$ is an arc-length reparametrization of $\gamma|_{[a, a_k]}$ and $\gamma \circ \varphi_2$ is an arc-length reparametrization of $\gamma|_{[a_k, b]}$. If we define $\tilde{\varphi} : [0, c + d] \rightarrow [a, b]$ by

$$\tilde{\varphi}(s) = \begin{cases} \varphi_1(s) & s \in [0, c], \\ \varphi_2(s - c) & s \in [c, c + d], \end{cases}$$

then $\gamma \circ \tilde{\varphi}$ is a reparametrization of γ by arc length.

To prove uniqueness, suppose that $\tilde{\gamma} = \gamma \circ \varphi$ and $\hat{\gamma} = \gamma \circ \psi$ are both forward reparametrizations of γ by arc length. Since both have the same length, it follows that φ and ψ both have the same parameter domain $[0, c]$, and thus both are piecewise regular homeomorphisms from $[0, c]$ to $[a, b]$. If we define $\eta = \varphi^{-1} \circ \psi$, then η is a piecewise regular increasing homeomorphism satisfying $\hat{\gamma} = \gamma \circ \varphi \circ \eta = \tilde{\gamma} \circ \eta$. The fact that both $\tilde{\gamma}$ and $\hat{\gamma}$ are of unit speed implies the following equality for all $s \in [0, c]$ except the finitely many values at which $\tilde{\gamma}$ or η is not smooth:

$$1 = |\hat{\gamma}'(s)|_g = |\tilde{\gamma}'(\eta(s))\eta'(s)|_g = |\tilde{\gamma}'(\eta(s))|_g \eta'(s) = \eta'(s).$$

Since η is continuous and $\eta(0) = 0$, it follows that $\eta(s) = s$ for all $s \in [0, c]$, and thus $\tilde{\gamma} = \hat{\gamma}$. \square

Using curve segments as measuring tapes, we can define distances between points on a Riemannian manifold. Suppose (M, g) is a connected Riemannian manifold. (The theory is most straightforward when $\partial M = \emptyset$, so we assume that for the rest of this section.) For any $p, q \in M$, the **Riemannian distance** from p to q , denoted by $d_g(p, q)$, is defined to be the infimum of $L_g(\gamma)$ over all piecewise smooth curve segments γ from p to q . Because any pair of points in M can be joined by a piecewise smooth curve segment (Proposition 1.2.40), this is well defined.

Proposition 1.2.41. Suppose (M, g) and (\tilde{M}, \tilde{g}) are connected Riemannian manifolds and $F : M \rightarrow$

\tilde{M} is a Riemannian isometry. Then

$$d_g(p, q) = d_{\tilde{g}}(F(p), F(q))$$

for all $p, q \in M$.

Proof. This follows from Proposition [1.2.38](#) ^{Riemann isometry length}, since any curve on M or \tilde{M} can be mapped isometrically to the other manifold. \square

Remark 1.2.1. Note that unlike lengths of curves, Riemannian distances are not necessarily preserved by local isometries.

Example 1.2.42. In (\mathbb{R}^n, \bar{g}) , It can be shown that any straight line segment is the shortest piecewise smooth curve segment between its endpoints. Therefore, the distance function $d_{\bar{g}}$ is equal to the usual Euclidean distance:

$$d_{\bar{g}}(x, y) = |x - y|.$$

We will see below that the Riemannian distance function turns M into a metric space whose topology is the same as the given manifold topology. The key is the following technical lemma, which shows that every Riemannian metric is locally comparable to the Euclidean metric in coordinates.

an metric lem

Lemma 1.2.43. Let g be a Riemannian metric on an open subset $U \subseteq \mathbb{R}^n$. Given a compact subset $K \subseteq U$, there exist positive constants c, C such that for all $x \in K$ and all $v \in T_x \mathbb{R}^n$,

$$c|v|_{\bar{g}} \leq |v|_g \leq C|v|_{\bar{g}} \quad (2.8)$$

Riemann metri

Proof. For any compact subset $K \subseteq U$, let $L \subseteq T\mathbb{R}^n$ be the set

$$L = \{(x, v) \in T\mathbb{R}^n : x \in K, |v|_{\bar{g}} = 1\}$$

Under the canonical identification of $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, L is just the product set $K \times S^{n-1}$ and therefore is compact. Because the norm $|v|_g$ is continuous and strictly positive on L , there are positive constants c, C such that $c \leq |v|_g \leq C$ whenever $(x, v) \in L$. If $x \in K$ and v is a nonzero vector in $T_x \mathbb{R}^n$, let $\lambda = |v|_{\bar{g}}$. Then $(x, \lambda^{-1}v) \in L$, so by homogeneity of the norm,

$$|v|_g = \lambda |\lambda^{-1}v|_g \leq \lambda C = C|v|_{\bar{g}}$$

A similar computation shows that $|v|_g \geq c|v|_{\bar{g}}$. The same inequalities are trivially true when $v = 0$. \square

metric space

Theorem 1.2.44 (Riemannian Manifolds as Metric Spaces). Let (M, g) be a connected Riemannian manifold. With the Riemannian distance function, M is a metric space whose metric topology is the same as the original manifold topology.

Proof. It is immediate from the definition that $d_g(p, q) \geq 0$. Because every constant curve segment has length zero, it follows that $d_g(p, p) = 0$; and $d_g(p, q) = d_g(q, p)$ follows from the fact that any curve segment from p to q can be reparametrized to go from q to p . Suppose γ_1 and γ_2 are piecewise smooth curve segments from p to q and q to r , respectively, and let γ be a piecewise smooth curve segment that first follows γ_1 and then follows γ_2 (reparametrized if necessary). Then

$$d_g(p, r) \leq L_g(\gamma) = L_g(\gamma_1) + L_g(\gamma_2).$$

Taking the infimum over all such γ_1 and γ_2 , we find that $d_g(p, r) \leq d_g(p, q) + d_g(q, r)$. To

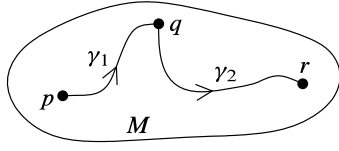
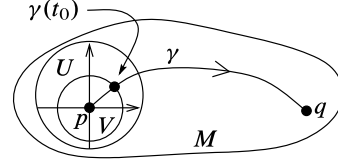


Figure 1.1: The triangle inequality.

Figure 1.2: Positivity of d_g .

complete the proof that (M, d_g) is a metric space, we need only show that $d_g(p, q) > 0$ if $p \neq q$. For this purpose, let $p, q \in M$ be distinct points, and let U be a smooth coordinate domain containing p but not q . Use the coordinate map as usual to identify U with an open subset in \mathbb{R}^n , and let \bar{g} denote the Euclidean metric in these coordinates. If V is a regular coordinate ball of radius ε centered at p such that $\bar{V} \subseteq U$, Lemma 1.2.43 shows that there are positive constants c, C such that (2.8) is satisfied whenever $x \in \bar{V}$ and $v \in T_x M$. Then for any piecewise smooth curve segment γ lying entirely in \bar{V} , it follows that

$$cL_{\bar{g}}(\gamma) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma)$$

Suppose $\gamma : [a, b] \rightarrow M$ is a piecewise smooth curve segment from p to q . Let t_0 be the infimum of all $t \in [a, b]$ such that $\gamma(t) \notin \bar{V}$. It follows that $\gamma(t_0) \in \partial V$ by continuity, and $\gamma(t) \in \bar{V}$ for $a \leq t \leq t_0$. Thus

$$L_g(\gamma) \geq L_g(\gamma|_{[a, t_0]}) \geq cL_{\bar{g}}(\gamma|_{[a, t_0]}) \geq cd_{\bar{g}}(p, \gamma(t_0)) = c\varepsilon$$

Taking the infimum over all such γ , we conclude that $d_g(p, q) \geq c\varepsilon > 0$.

Finally, to show that the metric topology generated by d_g is the same as the given manifold topology on M , we need to show that the open subsets in the manifold topology are open in the metric topology, and vice versa.

Suppose, first, that $U \subseteq M$ is open in the manifold topology. Let $p \in U$, and let V be a regular coordinate ball of radius ε around p such that $\bar{V} \subseteq U$ as above. The argument in the previous paragraph shows that $d_g(p, q) \geq c\varepsilon$ whenever $q \notin \bar{V}$. The contrapositive of this statement is that $d_g(p, q) < c\varepsilon$ implies $q \in \bar{V} \subseteq U$, or in other words, the metric ball of radius $c\varepsilon$ around p is contained in U . This shows that U is open in the metric topology.

Conversely, suppose that W is open in the metric topology, and let $p \in W$. Let V be a regular coordinate ball of radius r around p , let \bar{g} be the Euclidean metric on \bar{V} determined by the given coordinates, and let c, C be positive constants such that (2.8) is satisfied for $v \in T_q M, q \in \bar{V}$. Let $\varepsilon < r$ be a positive number small enough that the metric ball around p of radius $C\varepsilon$ is contained in W , and let V_ε be the set of points in \bar{V} whose Euclidean distance from p is less than ε . If $q \in V_\varepsilon$, let γ be the straight-line segment in coordinates from p to q . Using Lemma 1.2.43 as above, we conclude that

$$d_g(p, q) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma) < C\varepsilon$$

This shows that V_ε is contained in the metric ball of radius $C\varepsilon$ around p , and therefore in W . Since V_ε is a neighborhood of p in the manifold topology, this shows that W is open in the manifold topology as well. \square

As a consequence of this theorem, all of the terminology of metric spaces can be carried over to connected Riemannian manifolds. Thus, a connected Riemannian manifold (M, g) is said to be **complete**, and g is said to be a **complete Riemannian metric**, if (M, d_g) is a

complete metric space (i.e., if every Cauchy sequence in M converges to a point in M); and a subset $B \subseteq M$ is said to be **bounded** if there exists a constant K such that $d_g(x, y) \leq K$ for all $x, y \in B$; if this is the case, the **diameter** of A is the smallest such constant:

$$\text{diam}(A) = \sup\{d_g(p, q) : p, q \in A\}$$

Since every compact metric space is bounded, every compact connected Riemannian manifold with or without boundary has finite diameter. (Note that the unit sphere with the Riemannian distance determined by the round metric has diameter π , not 2, since the Riemannian distance between antipodal points is π).

Recall that a topological space is said to be **metrizable** if it admits a distance function whose metric topology is the same as the given topology.

Corollary 1.2.45. *Every smooth manifold with or without boundary is metrizable.*

Proof. First suppose M is a smooth manifold without boundary, and choose any Riemannian metric g on M . If M is connected, Theorem 1.2.44 shows that M is metrizable. More generally, let $\{M_i\}$ be the connected components of M , and choose a point $p_i \in M_i$ for each i . For $x \in M_i$ and $y \in M_j$, define $d_g(x, y)$ as in Theorem 1.2.44 when $i = j$, and otherwise

$$d_g(x, y) = d_g(x, p_i) + 1 + d_g(p_j, y)$$

It is straightforward to check that this is a distance function that induces the given topology on M . Finally, if M has nonempty boundary, just embed M into its double, and note that a subspace of a metrizable topological space is always metrizable. \square

1.2.5 Pseudo-Riemannian metrics

From the point of view of geometry, Riemannian metrics are by far the most important structures that manifolds carry. However, there is a generalization of Riemannian metrics that has become especially important because of its application to physics.

Before defining this generalization, we begin with some linear-algebraic preliminaries. Suppose V is a finite-dimensional vector space, and q is a symmetric covariant 2-tensor on V (also called a symmetric bilinear form). Just as for an inner product, there is a linear map $\widehat{q} : V \rightarrow V^*$ defined by

$$\widehat{q}(v)(w) = q(v, w)$$

for $v, w \in V$. We say that q is **nondegenerate** if \widehat{q} is an isomorphism.

Lemma 1.2.46. *Suppose q is a symmetric covariant 2-tensor on a finite-dimensional vector space V . The following are equivalent:*

- (a) q is nondegenerate.
- (b) For every nonzero $v \in V$, there is some $w \in V$ such that $q(v, w) \neq 0$.
- (c) If $q = q_{ij}\varepsilon^i\varepsilon^j$ in terms of some basis (ε^i) for V^* , then the matrix (q_{ij}) is invertible.

Proof. By dimension consideration, \widehat{q} is an isomorphism if and only if it is injective, so (a) is equivalent to (b). Now if $q(v, w) = 0$, then in terms of matrix, we have $X^T(g_{ij})Y = 0$ for some vector X and every vector Y . This implies that $X^T(g_{ij}) = 0$, and thus $(g_{ij})X = 0$ since q is symmetric. This then implies that (g_{ij}) is not invertible, if $X \neq 0$. \square

We use the term **scalar product** to denote any nondegenerate symmetric bilinear form on a finite-dimensional vector space V , and reserve the term inner product for the special case of a positive definite scalar product. A **scalar product space** is a finite-dimensional vector space endowed with a scalar product. When convenient, we will often use a notation like $\langle \cdot \cdot \rangle$ to denote a scalar product. We say that vectors $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$, just as in the case of an inner product. Given a vector $v \in V$, we define the **norm** of v to be $|v| = |\langle v, v \rangle|^{1/2}$. Note that in the indefinite case, it is possible for a nonzero vector to be orthogonal to itself, and thus to have norm zero.

If $S \subseteq V$ is any linear subspace, the set of vectors in V that are orthogonal to every vector in S is a linear subspace denoted by S^\perp .

Lemma 1.2.47. *Suppose (V, q) is a finite-dimensional scalar product space, and $S \subseteq V$ is a linear subspace.*

- (a) $\dim S + \dim S^\perp = \dim V$.
- (b) $(S^\perp)^\perp = S$.

Proof. Define a linear map $\Phi : V \rightarrow S^*$ by $\Phi(v) = \widehat{q}(v)|_S$. Note that $v \in \ker \Phi$ if and only if $q(v, x) = 0$ for all $x \in S$, so $\ker \Phi = S^\perp$. If $\varphi \in S^*$ is arbitrary, there is a covector $\widetilde{\varphi} \in V^*$ whose restriction to S is equal to φ . (For example, such a covector is easily constructed after choosing a basis for S and extending it to a basis for V .) Since \widehat{q} is an isomorphism, there exists $v \in V$ such that $\widehat{q}(v) = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the ranknullity theorem, the dimension of $S^\perp = \ker \Phi$ is equal to $\dim V - \dim S^* = \dim V - \dim S$. This proves (a).

To prove (b), note that every $v \in S$ is orthogonal to every element of S by definition, so $S \subseteq (S^\perp)^\perp$. Because these finite-dimensional vector spaces have the same dimension by part (a), they are equal. \square

An ordered k -tuple (v_1, \dots, v_k) of elements of V is said to be **orthonormal** if $|v_i| = 1$ for each i and $\langle v_i, v_j \rangle = 0$ for $i \neq j$, or equivalently, if $\langle v_i, v_j \rangle = \delta_{ij}$ for each i and j . We wish to prove that every scalar product space has an orthonormal basis. Note that the usual Gram-Schmidt algorithm does not always work in this situation, because the vectors that appear in the denominators might have vanishing norms. In order to get around this problem, we introduce the following definitions.

If (V, q) is a finite-dimensional scalar product space, a subspace $S \subseteq V$ is said to be **nondegenerate** if the restriction of q to $S \times S$ is nondegenerate. An ordered k -tuple of vectors (v_1, \dots, v_k) in V is said to be **nondegenerate** if for each $j = 1, \dots, k$ the vectors (v_1, \dots, v_j) span a nondegenerate j -dimensional subspace of V . For example, every orthonormal basis is nondegenerate.

nondegenerate

Lemma 1.2.48. *Suppose (V, q) is a finite-dimensional scalar product space, and $S \subseteq V$ is a linear subspace. The following are equivalent:*

- (a) S is nondegenerate.
- (b) S^\perp is nondegenerate.
- (c) $S \cap S^\perp = \{0\}$.

(d) $V = S \oplus S^\perp$.

Proof. Note that $q|_S$ is nondegenerate if and only if the set

$$\{x \in S : q(x, y) = 0 \forall y \in S\} = S \cap S^\perp$$

is $\{0\}$. Now since $(S^\perp)^\perp = S$, we have the equivalences (a), (b) and (c). Now, since $\dim S + \dim S^\perp = \dim V$, they are easily seen to be equivalent to (d). \square

generate base

Lemma 1.2.49 (Completion of Nondegenerate Bases). Suppose (V, q) is an n -dimensional scalar product space, and (v_1, \dots, v_k) is a nondegenerate k -tuple in V with $0 \leq k < n$. Then there exist vectors $v_{k+1}, \dots, v_n \in V$ such that (v_1, \dots, v_n) is a nondegenerate basis for V .

Proof. Let $S = \text{span}(v_1, \dots, v_k) \subseteq V$. Because $k < n$, S^\perp is a nontrivial subspace of V , and Lemma 1.2.48 shows that S^\perp is nondegenerate and $V = S \oplus S^\perp$. By the nondegeneracy of S^\perp , there must be a vector in S^\perp with nonzero length, because otherwise the polarization identity would imply that all inner products of pairs of elements of S would be zero. If $v_{k+1} \in S^\perp$ is any vector with nonzero length, then (v_1, \dots, v_{k+1}) is easily seen to be a nondegenerate $(k+1)$ -tuple. Repeating this argument for v_{k+2}, \dots, v_n completes the proof. \square

Proposition 1.2.50 (GramSchmidt Algorithm for Scalar Products). Suppose (V, q) is an n -dimensional scalar product space. If (v_i) is a nondegenerate basis for V , then there is an orthonormal basis (b_i) with the property that $\text{span}(b_1, \dots, b_k) = \text{span}(v_1, \dots, v_k)$ for each $k = 1, \dots, n$.

Proof. As in the positive definite case, the basis (b_i) is constructed recursively, starting with $b_1 = v_1/|v_1|$ and noting that the assumption that v_1 spans a nondegenerate subspace ensures that $|v_1| \neq 0$. At the inductive step, assuming we have constructed (b_1, \dots, b_k) , we first set

$$z = v_{k+1} - \sum_{i=1}^k \frac{\langle v_{k+1}, b_i \rangle}{\langle b_i, b_i \rangle} b_i.$$

Each denominator $\langle b_i, b_i \rangle$ is equal to ± 1 , so this defines z as a nonzero element of V orthogonal to b_1, \dots, b_k , and with the property that $\text{span}(b_1, \dots, b_k, z) = \text{span}(v_1, \dots, v_{k+1})$. If $\langle z, z \rangle = 0$, then z is orthogonal to $\text{span}(v_1, \dots, v_{k+1})$, contradicting the nondegeneracy assumption. Thus we can complete the inductive step by putting $b_{k+1} = z/|z|$. \square

Corollary 1.2.51. Suppose (v, q) is an n -dimensional scalar product space. There is a basis (β^i) for V^* with respect to which q has the expression

$$q = (\beta^1)^2 + \dots + (\beta^r)^2 - (\beta^{r+1})^2 - \dots - (\beta^{r+s})^2, \quad (2.9)$$

scalar prod

for some nonnegative integers r, s with $r + s = n$.

Proof. Let (b_i) be an orthonormal basis for V , and let (β^i) be the dual basis for V^* . A computation shows that q has a basis expression of the form (2.9), but perhaps with the positive and negative terms in a different order. Reordering the basis so that the positive terms come first, we obtain (2.9). \square

It turns out that the numbers r and s in (2.9) are independent of the choice of basis. The integer s in the expression (the number of negative terms) is called the **index** of q , and the ordered pair (r, s) is called the **signature** of q .

Now suppose M is a smooth manifold. A **pseudo-Riemannian metric** on M (called by some authors a semi-Riemannian metric) is a smooth symmetric 2-tensor field g that is non-degenerate at each point of M , and with the same signature everywhere. Every Riemannian metric is also a pseudo-Riemannian metric.

normal frame

Proposition 1.2.52 (Orthonormal Frames for Pseudo-Riemannian Manifolds). *Let (M, g) be a pseudo-Riemannian manifold. For each $p \in M$, there exists a smooth orthonormal frame on a neighborhood of p in M .*

Example 1.2.53. Suppose (M_1, g_1) and (M_2, g_2) are pseudo-Riemannian manifolds of signatures (r_1, s_1) and (r_2, s_2) , respectively. Then $(M_1 \times M_2, g_1 \oplus g_2)$ is a pseudo-Riemannian manifold of signature $(r_1 + r_2, s_1 + s_2)$.

For nonnegative integers r and s , we define the **pseudo-Euclidean space of signature r, s** , denoted by $\mathbb{R}^{r,s}$, to be the manifold \mathbb{R}^{r+s} , with standard coordinates denoted by $(\xi^1, \dots, \xi^r, \tau^1, \dots, \tau^s)$, and with the pseudo-Riemannian metric $\bar{q}^{(r,s)}$ defined by

$$\bar{q}^{(r,s)} = (d\xi^1)^2 + \dots + (d\xi^r)^2 - (d\tau^1)^2 - \dots - (d\tau^s)^2.$$

By far the most important pseudo-Riemannian metrics (other than the Riemannian ones) are the **Lorentz metrics**, which are pseudo-Riemannian metrics of index 1, and thus signature $(r, 1)$.

The pseudo-Euclidean metric $\bar{q}^{(r,1)}$ is a **Lorentz metric** called the **Minkowski metric**, and the Lorentz manifold $\mathbb{R}^{r,1}$ is called $(r+1)$ -dimensional Minkowski space. If we denote standard coordinates on \mathbb{R}^{r+1} by $(\xi^1, \dots, \xi^r, \tau)$ then the Minkowski metric is given by

$$\bar{q}^{(r,1)} = (d\xi^1)^2 + \dots + (d\xi^r)^2 - (d\tau)^2.$$

Many, but not all, results from the theory of Riemannian metrics apply equally well to pseudo-Riemannian metrics. We will attempt to point out which results carry over directly to pseudo-Riemannian metrics, which ones can be adapted with minor modifications, and which ones do not carry over at all. As a rule of thumb, proofs that depend only on the nondegeneracy of the metric tensor, such as properties of the musical isomorphisms and existence and uniqueness of geodesics, work fine in the pseudo-Riemannian setting, while proofs that use positivity in an essential way, such as those involving lengths of curves, do not.

One notable result that does not carry over to the pseudo-Riemannian case is Proposition 1.2.44, about the existence of metrics. For example, the following result characterizes those manifolds that admit Lorentz metrics.

orenta metric

Theorem 1.2.54. *A smooth manifold M admits a Lorentz metric if and only if it admits a rank-1 tangent distribution (i.e., a rank-1 subbundle of TM).*

Proof. For sufficiency, assume that $D \subseteq TM$ is a 1-dimensional distribution, and choose any Riemannian metric g on M . Then, by Lemma 1.2.44, locally it is possible to choose a g-orthonormal frame (E_i) and dual coframe (ε^i) such that E_1 spans D . Then the Lorentz metric $-(\varepsilon^1)^2 + (\varepsilon^2)^2 + \dots + (\varepsilon^n)^2$ is independent of the choice of frame. \square

Pseudo-Riemannian submanifolds

The theory of submanifolds is only slightly more complicated in the pseudo-Riemannian case. If (\tilde{M}, \tilde{g}) is a pseudo-Riemannian manifold, a smooth submanifold $\iota : M \hookrightarrow \tilde{M}$ is called

a **pseudo-Riemannian submanifold** of \tilde{M} if $\iota^*\tilde{g}$ is nondegenerate with constant signature. If this is the case, we always consider M to be endowed with the induced pseudo-Riemannian metric $\iota^*\tilde{g}$. In the special case in which $\iota^*\tilde{g}$ is positive definite, M is called a **Riemannian submanifold**.

The nondegeneracy hypothesis is not automatically satisfied: for example, if $M \subseteq \mathbb{R}^{1,1}$ is the submanifold $\{(\xi, \tau) : \tau = \xi\}$ and $\iota : M \rightarrow \mathbb{R}^{1,1}$ is inclusion, then the pullback tensor $\iota^*\tilde{g}$ is identically zero on M .

For hypersurfaces (submanifolds of codimension 1), the nondegeneracy condition is easy to check. If $M \subseteq \tilde{M}$ is a smooth submanifold and $p \in M$, then a vector $v \in T_p\tilde{M}$ is said to be **normal to M** if $\langle v, x \rangle = 0$ for all $x \in T_pM$, just as in the Riemannian case. The space of all normal vectors at p is a subspace of $T_p\tilde{M}$ denoted by N_pM .

Proposition 1.2.55. *Suppose (\tilde{M}, \tilde{g}) is a pseudo-Riemannian manifold of signature (r, s) . Let M be a smooth hypersurface in \tilde{M} , and let $\iota : M \rightarrow \tilde{M}$ be the inclusion map. Then the pullback tensor field $\iota^*\tilde{g}$ is nondegenerate if and only if $\tilde{g}(v, v) \neq 0$ for every $p \in M$ and every nonzero normal vector $v \in N_pM$. If $\tilde{g}(v, v) > 0$ for every nonzero normal vector to M , then M is a pseudo-Riemannian submanifold of signature $(r - 1, s)$; and if $\tilde{g}(v, v) < 0$ for every such vector, then M is a pseudo-Riemannian submanifold of signature $(r, s - 1)$.*

Proof. Given $p \in M$, Lemma 1.2.48 shows that T_pM is a nondegenerate subspace of $T_p\tilde{M}$ if and only if the one-dimensional subspace $(T_pM)^\perp = N_pM$ is nondegenerate, which is the case if and only if every nonzero $v \in N_pM$ satisfies $\tilde{g}(v, v) \neq 0$.

Now suppose $\tilde{g}(v, v) > 0$ for every nonzero normal vector v . Let $p \in M$ be arbitrary, and let v be a nonzero element of N_pM . Writing $n = \dim \tilde{M}$, we can complete v to a nondegenerate basis v, w_2, \dots, w_n for $T_p\tilde{M}$ by Lemma 1.2.49, and then use the Gram-Schmidt algorithm to find an orthonormal basis b_1, \dots, b_n for $T_p\tilde{M}$ such that $\text{span}(b_1) = N_pM$. It follows that $\text{span}(b_2, \dots, b_n) = T_pM$. If (β^j) is the dual basis to (b_i) , then \tilde{g}_p has a basis representation of the form $(\beta^1)^2 \pm (\beta^2)^2 \pm \dots \pm (\beta^n)^2$, with a total of r positive terms and s negative ones, and with a positive sign on the first term $(\beta^1)^2$. Therefore, $\iota^*\tilde{g}_p = \pm(\beta^2)^2 \pm \dots \pm (\beta^n)^2$ has signature $(r - 1, s)$. The argument for the case $\tilde{g}(v, v) < 0$ is similar. \square

If (M, g) is a pseudo-Riemannian manifold and $f \in C^\infty(M)$, then the **gradient** of f is defined as the smooth vector field $\text{grad } f = (df)^\sharp$ just as in the Riemannian case.

Proposition 1.2.56. *Suppose (\tilde{M}, \tilde{g}) is a pseudo-Riemannian manifold of signature (r, s) , $f \in C^\infty(M)$, and $M = f^{-1}(c)$ for some $c \in \mathbb{R}$. If $\langle \text{grad } f, \text{grad } f \rangle > 0$ everywhere on M , then M is an embedded pseudo-Riemannian submanifold of \tilde{M} of signature $(r - 1, s)$ and if $\langle \text{grad } f, \text{grad } f \rangle < 0$ everywhere on M , then M is an embedded pseudo-Riemannian submanifold of \tilde{M} of signature $(r, s - 1)$. In either case, $\text{grad } f$ is everywhere normal to M .*

Proof. By Proposition 1.2.48 and Proposition 1.2.49 we know that M is an embedded submanifold whose tangent space is everywhere tangent to $\text{grad } f$. Now the result follows from Proposition 1.2.55. \square

Proposition 1.2.57 (Pseudo-Riemannian Adapted Orthonormal Frames). *Suppose (\tilde{M}, \tilde{g}) is a pseudo-Riemannian manifold, and $M \subseteq \tilde{M}$ is an embedded pseudo-Riemannian or Riemannian submanifold. For each $p \in M$, there exists a smooth orthonormal frame on a neighborhood of p in \tilde{M} that is adapted to M .*

Proof. Write $m = \dim \tilde{M}$ and $n = \dim M$, and let $p \in M$ be arbitrary. Proposition [1.2.52](#) shows that there is a smooth orthonormal frame (E_1, \dots, E_n) for M on some neighborhood of p in M . Then by Lemma [1.2.49](#), we can find vectors v_{n+1}, \dots, v_m such that

$$(E_1|_p, \dots, E_n|_p, v_{n+1}, \dots, v_m)$$

is a nondegenerate basis for $T_p \tilde{M}$. Now extend v_{n+1}, \dots, v_m arbitrarily to smooth vector fields V_{n+1}, \dots, V_m on a neighborhood of p in \tilde{M} . By continuity, the ordered m -tuple

$$(E_1, \dots, E_n, V_{n+1}, \dots, V_m)$$

will be a nondegenerate frame for \tilde{M} in some (possibly smaller) neighborhood of p . Applying the GramSchmidt algorithm to this local frame yields a smooth local orthonormal frame (E_1, \dots, E_m) for \tilde{M} with the property that (E_1, \dots, E_n) restricts to a local orthonormal frame for M . \square

The next corollary is proved in the same way as Proposition [1.2.16](#).

Corollary 1.2.58 (Normal Bundle to a Pseudo-Riemannian Submanifold). *Suppose (\tilde{M}, \tilde{g}) is a pseudo-Riemannian manifold, and $M \subseteq \tilde{M}$ is an embedded pseudo-Riemannian or Riemannian submanifold. The set of vectors normal to M is a smooth vector subbundle of $T\tilde{M}|_M$, called the normal bundle to M .*

1.2.6 Exercise

Exercise 1.2.1. Suppose that E is a smooth vector bundle over a smooth manifold M with or without boundary, and $V \subseteq E$ is an open subset with the property that for each $p \in M$, the intersection of V with the fiber E_p is convex and nonempty. By a section of V , we mean a section of E whose image lies in V .

- (a) Show that there exists a smooth global section of V .
- (b) Suppose $\sigma : A \rightarrow V$ is a smooth section of V defined on a closed subset $A \subseteq M$. (This means that σ extends to a smooth section of V in a neighborhood of each point of A .) Show that there exists a smooth global section $\tilde{\sigma}$ of V whose restriction to A is equal to σ . Show that if V contains the image of the zero section of E , then $\tilde{\sigma}$ can be chosen to be supported in any predetermined neighborhood of A .

Proof. For $p \in M$, choose a neighborhood U contains p , with a local trivialization $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$. Let $v \in \mathbb{R}^k$ be a vector such that $(p, v) \in \Phi(V \cap \pi^{-1}(U))$. Since $V \cap \pi^{-1}(U)$ is open, by appropriately shrinking U , we can find an open subset $W \subseteq \mathbb{R}^k$ such that

$$(p, v) \in U \times W \subseteq \Phi(V \cap \pi^{-1}(U)).$$

In particular, $(q, v) \in \Phi(V \cap \pi^{-1}(U))$ for all $q \in U$, thus we can define $\sigma_p : U \rightarrow E$ by

$$\sigma_p(q) = \Phi^{-1}(q, v) \quad \text{for } q \in U.$$

It follows that σ_p is a local section of V . Now the neighborhoods $\{U_p\}$ gives an open cover of M , by taking a partition of unity $\{\psi_p\}$ for the open cover $\{\psi_p\}$, we define

$$\eta = \sum_{p \in M} \psi_p \sigma_p.$$

Then at each point p_0 , since $V \cap E_{p_0}$ is convex, we find $\eta(p_0) \in V$.

The second statement is the same as Lemma [1.2.13](#). □

Exercise 1.2.2. Prove that every Riemannian 1-manifold is flat.

Proof. Let M be a Riemannian 1-manifold, then each point of M admits a commuting orthonormal frame. Thus by Proposition [1.2.13](#) M is flat. □

try is linear

Exercise 1.2.3. Suppose V and W are finitely-dimensional real inner product spaces of the same dimension, and $F : V \rightarrow W$ is any map (not assumed to be linear or even continuous) that preserves the origin and all distances: $F(0) = 0$ and $|F(x) - F(y)| = |x - y|$ for all $x, y \in V$. Prove that F is a linear isometry.

First proof. First, inserting $y = 0$ in the second condition gives $|F(x)| = |x|$. Now let $t \in [0, 1]$ and denote $r = |x - y|$. Then

$$\{tx + (1 - t)y\} = \bar{B}(x, tr) \cap \bar{B}(y, (1 - t)r).$$

Now since F is injective, $F(tx + (1 - t)y)$ is the unique element of

$$F(\bar{B}(x, tr) \cap \bar{B}(y, (1 - t)r)) = F(\bar{B}(x, tr)) \cap F(\bar{B}(y, (1 - t)r)) = \bar{B}(F(x), tr) \cap \bar{B}(F(y), (1 - t)r),$$

which is $tF(x) + (1 - t)F(y)$. This then implies $F(x)$ is linear since $F(0) = 0$. □

Second proof. We have $|F(x)| = |x|$ for all $x \in V$. Now expand the equation $|F(x) - F(y)|^2 = |x - y|^2$, we get

$$|F(x)|^2 + |F(y)|^2 - 2(F(x), F(y)) = |x|^2 + |y|^2 - 2(x, y).$$

Therefore F preserves the inner products.

Now let $x, y \in V$, we want to prove $F(x + y) = F(x) + F(y)$ and $F(\lambda x) = \lambda x$ for $\lambda \in \mathbb{R}$. We note that for all $v \in V$,

$$(F(x + y), F(v)) = ((x + y), v) = (x, v) + (y, v) = (F(x) + F(y), F(v)).$$

Take v to be a orthonormal basis for V , this then implies $F(x + y) = F(x) + F(y)$. Similarly we can prove $F(\lambda x) = \lambda F(x)$, therefore F is linear. □

shortest path

Exercise 1.2.4. Show that the shortest path between two points in Euclidean space is a straight line segment. More precisely, for $x, y \in \mathbb{R}^n$, let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ be the curve segment $\gamma(t) = x + t(y - x)$, and show that any other piecewise smooth curve segment $\tilde{\gamma}$ from x to y satisfies $L_{\bar{g}}(\tilde{\gamma}) \geq L_{\bar{g}}(\gamma)$.

Proof. Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a piecewise smooth curve segment such that $\gamma(a) = x$ and $\gamma(b) = y$. For any constant vector v with $|v| = 1$, by Cauchy-Schwarz inequality we have

$$(y - x) \cdot v = (\gamma(t) \cdot v) \Big|_{t=a}^{t=b} = \int_a^b \gamma'(t) \cdot v \, dt \leq \int_a^b |\gamma'(t)| |v| \, dt = \int_a^b |\gamma'(t)| \, dt.$$

Set $v = (y - x)/|y - x|$, then

$$|y - x| \leq \int_a^b |\gamma'(t)| \, dt.$$

Now if the equality holds if and only if $\gamma'(t) = a(y - x)$ for some constant $a \in \mathbb{R}$. Hence if and only if $\gamma(t) = a(y - x)t + b$ with $a, b \in \mathbb{R}$. □

ometry group

Exercise 1.2.5. Consider \mathbb{R}^n as a Riemannian manifold with the Euclidean metric \bar{g} .

- (a) Suppose $U, V \subseteq \mathbb{R}^n$ are connected open sets, $\varphi, \psi : U \rightarrow V$ are Riemannian isometries, and for some $p \in U$ they satisfy $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Show that $\varphi = \psi$.
- (b) Show that the set of maps from \mathbb{R}^n to itself given by the action of $E(n)$ on \mathbb{R}^n described in Example 1.1.2 is the full group of Riemannian isometries of (\mathbb{R}^n, \bar{g}) .

Proof. Let $x \in \mathbb{R}^n$, and let γ be a smooth curve in \mathbb{R}^n from $\varphi(p)$ to $\varphi(x)$, then by Proposition 1.2.38, Riemann isometry length

$$L_{\bar{g}}(\gamma) = L_{\bar{g}}(\varphi^{-1} \circ \gamma) \geq L_{\bar{g}}(\sigma) = L_{\bar{g}}(\varphi \circ \sigma),$$

where $\sigma : [0, 1] \rightarrow \mathbb{R}^n$ is the straight line from p to x and we use the result of Exercise 1.2.4, Euclidean shortest p. This implies that $\varphi \circ \sigma$ is the straight line from $\varphi(p)$ to $\varphi(x)$. Since the length of $\varphi \circ \sigma$ is fixed, $\varphi(x)$ is uniquely determined by its direction, that is, $(\varphi \circ \sigma)'(0)$. Since $d\varphi_p = d\psi_p$ we then conclude $\varphi = \psi$.

Let φ be any Riemannian isometry of (\mathbb{R}^n, \bar{g}) . Since $d\varphi_0$ preserves the Euclidean dot product, it corresponds to a matrix $C \in O(n)$. Let $(\varphi(0), C) \in E(n)$, then φ is represented by $(C, \varphi(0))$, in the sense that

$$\varphi(x) = (\varphi(0), C) \cdot x = \varphi(0) + Cx.$$

This gives the claim. \square

Exercise 1.2.6. Let (M, g) be a Riemannian manifold. A smooth vector field V on M is called a **Killing vector field** for g if the flow of V acts by isometries of g .

- (a) Show that the set of all Killing vector fields on M constitutes a Lie subalgebra of $\mathfrak{X}(M)$.
- (b) Show that a smooth vector field V on M is a Killing vector field if and only if it satisfies the following equation in each smooth local coordinate chart:

$$V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial V^k}{\partial x^i} + g_{ik} \frac{\partial V^k}{\partial x^j} = 0 \quad \text{for all } i, j.$$

Proof. By definition, V is a killing field if and only if g is invariant under θ . Thus by Theorem 1.1.28, tensor field invariant flow iff this is equivalent to $\mathfrak{L}_V g = 0$. Now by Exercise 1.1.2, tensor field Lie derivative bracket

$$\mathfrak{L}_{[V, W]} g = \mathfrak{L}_V \mathfrak{L}_W g - \mathfrak{L}_W \mathfrak{L}_V g.$$

Thus part (a) follows.

For part (b), write $V = V^k \partial / \partial x^k$, we compute using Corollary 1.1.24, Lie derivative tensor

$$\begin{aligned} (\mathfrak{L}_V g) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) &= V(g_{ij}) - g \left([V, \frac{\partial}{\partial x^i}], \frac{\partial}{\partial x^j} \right) - g \left(\frac{\partial}{\partial x^i}, [V, \frac{\partial}{\partial x^j}] \right) \\ &= V^k \frac{\partial g_{ij}}{\partial x^k} + g \left(\frac{\partial V^k}{\partial x^i} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right) + g \left(\frac{\partial}{\partial x^i}, \frac{\partial V^k}{\partial x^j} \frac{\partial}{\partial x^k} \right) \\ &= V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial V^k}{\partial x^i} + g_{ik} \frac{\partial V^k}{\partial x^j}. \end{aligned}$$

\square

Exercise 1.2.7. Let $K \subseteq \mathfrak{X}(\mathbb{R}^n)$ denote the Lie algebra of Killing vector fields with respect to the Euclidean metric, and let $K_0 \subseteq K$ denote the subspace consisting of fields that vanish at the origin.

(a) Show that the map

$$V \mapsto \left(\frac{\partial V^k}{\partial x^j}(0) \right)$$

is an injective linear map from K_0 to $\mathfrak{o}(n)$.

(b) Show that the following vector fields form a basis for K :

$$\frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n; \quad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad 1 \leq i < j \leq n.$$

Proof. If $g = \delta_{ij} dx^i dx^j$, then the equation becomes

$$\frac{\partial V^j}{\partial x^i} + \frac{\partial V^i}{\partial x^j} = 0$$

thus the image of the map is in $\mathfrak{o}(n)$. \square

Exercise 1.2.8. Suppose $g = f(t)dt^2$ is a Riemannian metric on \mathbb{R} . Show that g is complete if and only if both of the following improper integrals diverge:

$$\int_0^{+\infty} \sqrt{f(t)} dt, \quad \int_{-\infty}^0 \sqrt{f(t)} dt.$$

ann grad prop

Exercise 1.2.9. Let (M, g) be a Riemannian manifold, let $f \in C^\infty(M)$, and let $p \in M$ be a regular point of f .

- (a) Show that among all unit vectors $v \in T_p M$, the directional derivative $v f$ is greatest when v points in the same direction as $\text{grad } f|_p$, and the length of $\text{grad } f|_p$ is equal to the value of the directional derivative in that direction.
- (b) Show that $\text{grad } f|_p$ is normal to the level set of f through p .
- (c) Let $X \in \mathfrak{X}(M)$ be a nowhere-vanishing vector field. Prove that $X = \text{grad } f$ if and only if $Xf \equiv |X|_g^2$ and X is orthogonal to the level sets of f at all regular points of f .

Proof. By Riemann grad def (2.5) we have

$$\langle \text{grad } f|_p, v \rangle_g = df_p(v) = v f.$$

Then by Cauchy-Schwarz inequality, we have

$$v f \leq |v|_g \cdot |\text{grad } f|_p|_g,$$

with the quality holds if and only if v is parallel to $\text{grad } f|_p$.

Let $S = f^{-1}(f(p))$ be the level set of f through p , then by Proposition tangent space level set 1.2.7 we have

$$T_p S = \ker df_p = \ker(\widehat{g}(\text{grad } f)_p) = \ker(\langle \text{grad } f|_p, \cdot \rangle_g).$$

Thus $\text{grad } f|_p$ is normal to $T_p S$.

For (c), one direction is clear. Conversely, assume that $X \in \mathfrak{X}(M)$ satisfies the conditions. Let S be a level set of f and $p \in S$, then $\dim(T_p S)^\perp = 1$ for $p \in S$. Since X and $\text{grad } f$ are both contained in it, it follows that $X_p = \lambda \text{grad } f|_p$ for some $\lambda \in \mathbb{R}$. From the condition we have

$$\lambda \langle X_p, X_p \rangle \langle \text{grad } f|_p, X_p \rangle = X f|_p = \langle X_p, X_p \rangle = \lambda^2 \langle X_p, X_p \rangle.$$

Therefore $\lambda = 1$ since $X_p \neq 0$. Since X is non-vanishing, it follows that $X = \text{grad } f$. \square

Morse exercise

Exercise 1.2.10. Let M be a compact smooth n -manifold, and suppose f is a smooth real-valued function on M that has only finitely many critical points $\{p_1, \dots, p_k\}$, with corresponding critical values c_1, \dots, c_k labeled so that $c_1 \leq \dots \leq c_k$. For any $a < b \in \mathbb{R}$, define

$$M_a = f^{-1}(a), \quad M_{(a,b)} = f^{-1}(a, b), \quad M_{[a,b]} = f^{-1}([a, b]).$$

If a and b are regular values, note that M_a and M_b are embedded hypersurfaces in M , $M_{(a,b)}$ is an open submanifold, and $M_{[a,b]}$ is a regular domain by Proposition 7.7.

- Choose a Riemannian metric g on M , let X be the vector field $X = \text{grad } f / |\text{grad } f|_g^2$ on $M \setminus \{p_1, \dots, p_k\}$, and let θ denote the flow of X . Show that $f(\theta_t(p)) = f(p) + t$ whenever θ_t is defined.
- Let $[a, b] \times \mathbb{R}$ be a compact interval containing no critical values of f . Show that θ restricts to a diffeomorphism from $[0, b - a] \times M_a$ to $M_{[a,b]}$.

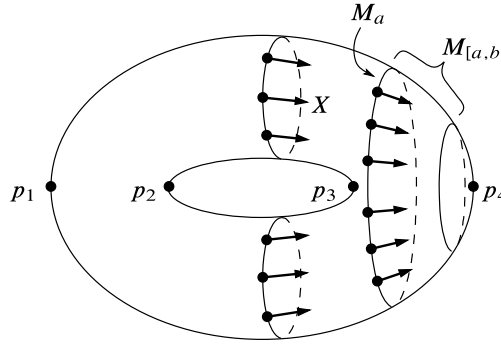


Figure 1.3: The setup for Exercise 1.2.10. Morse exercise

Proof. Fix a point $p \in M \setminus \{p_1, \dots, p_k\}$, we calculate the derivative of $f(\theta_t(p))$ with respect to t . We have

$$d(f \circ \theta_t(p)) = df \circ d(\theta_t(p)) = df(X_{\theta_t(p)}) = \langle \text{grad } f|_{\theta_t(p)}, X_{\theta_t(p)} \rangle = 1.$$

Therefore

$$f(\theta_t(p)) = f(\theta_0(p)) + t = f(p) + t \tag{2.10} \quad \text{Morse exercise-1}$$

Now let $p \in M_a$, we observe that

$$f(\theta_t(p)) = f(p) + t = a + t.$$

Therefore θ restricts to a map $\theta|_{M_{[a,b]}} : [0, b - a] \times M_a \rightarrow M_{[a,b]}$. To show this is a diffeomorphism, we construct the inverse map of $\theta|_{M_{[a,b]}}$:

$$\psi : M_{[a,b]} \rightarrow [0, b - a] \times M_a, \quad p \mapsto (f(p) - a, \theta_{a-f(p)}(p)).$$

Note that since $M_{[a,b]}$ is compact (M is compact and $M_{[a,b]}$ is closed), the integral curve passing p is defined on \mathbb{R} if it is contained in $M_{[a,b]}$ by the escape lemma (Lemma 7.7). But from equation (2.10), this is impossible. So it must pass the boundary of $M_{[a,b]}$, i.e., there is $t_0 \in \mathbb{R}$ such that $\theta_{t_0}(p) \in M_a$. We can see $t_0 = a - f(p)$ from (2.10), so our definition of ψ is legal. Now it is easy to see that ψ is the inverse of $\theta|_{M_{[a,b]}}$, so we get the diffeomorphism. \square vector flow escape lemma

1.3 Differential forms

1.3.1 The alternating tensors

We continue to assume that V is a finite-dimensional real vector space. A covariant k -tensor α on V is said to be **alternating** (or **antisymmetric** or **skew-symmetric**) if it changes sign whenever two of its arguments are interchanged. This means that for all vectors v_1, \dots, v_k and every pair of distinct indices i, j it satisfies

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Alternating covariant k -tensors are also variously called **exterior forms**, **multicovectors**, or **k -covectors**. The subspace of all alternating covariant k -tensors on V is denoted by $\wedge^k(V^*) \subseteq T^k(V^*)$.

The next lemma gives two more useful characterizations of alternating tensors.

alt tensor lem

Lemma 1.3.1. *Let α be a covariant k -tensor on a finite-dimensional vector space V . The following are equivalent:*

- (a) α is alternating.
- (b) $\alpha(v_1, \dots, v_k) = 0$ whenever the k -tuple v_1, \dots, v_k is linearly dependent.
- (c) α gives the value zero whenever two of its arguments are equal:

$$\alpha(v_1, \dots, w, \dots, w, v_k) = 0.$$

Proof. The implications (a) \Rightarrow (c) and (b) \Rightarrow (c) are immediate. We complete the proof by showing that (c) implies both (a) and (b).

Assume that α satisfies (c). For any vectors v_1, \dots, v_k , the hypothesis implies

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \end{aligned}$$

Thus α is alternating. On the other hand, if v_1, \dots, v_k is a linearly dependent k -tuple, then one of the v_i 's can be written as a linear combination of the others. For simplicity, let us assume that $v_k = \sum_{j=1}^{k-1} a^j v_j$. Then multilinearity of α implies

$$\alpha(v_1, \dots, v_k) = \sum_{j=1}^{k-1} a^j \alpha(v_1, \dots, v_{k-1}, v_j).$$

In each of these terms, α has two identical arguments, so every term is zero. \square

We now define a similar projection $\text{Alt} : T^k(V^*) \rightarrow \wedge^k(V^*)$, called **alternation**, as follows:

$$\text{Alt}\alpha = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \sigma \cdot \alpha.$$

More explicitly, this means

$$\text{Alt}\alpha(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\text{sgn}\sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Example 1.3.2. If α is any 1-tensor, then $\text{Alt}\alpha = \alpha$. If α is a 2-tensor, then

$$(\text{Alt}\alpha)(v, w) = \frac{1}{2}(\alpha(v, w) - \alpha(w, v)).$$

The next proposition is the analogue for alternating tensors of Proposition [1.1.13](#) symmetrization prop.

Proposition 1.3.3. *Let α be a covariant tensor on a finite-dimensional vector space.*

- (a) *$\text{Alt}\alpha$ is alternating.*
- (b) *$\text{Alt}\alpha = \alpha$ if and only if α is alternating.*

Elementary alternating tensors

For computations with alternating tensors, the following notation is exceedingly useful. Given a positive integer k , an ordered k -tuple $I = (i_1, \dots, i_k)$ of positive integers is called a **multi-index of length k** . If I is such a multi-index and $\sigma \in \mathfrak{S}_k$ is a permutation, we write I_σ for the following multi-index:

$$I_\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)}).$$

Note that $I_{\tau\sigma} = (I_\sigma)_\tau$ for $\sigma, \tau \in \mathfrak{S}_k$.

Let V be an n -dimensional vector space, and suppose $(\varepsilon^1, \dots, \varepsilon^n)$ is any basis for V^* . We now define a collection of k -covectors on V that generalize the determinant function on \mathbb{R}^n . For each multi-index $I = (i_1, \dots, i_k)$ of length k such that $1 \leq i_1 < \dots < i_k \leq n$, define a covariant k -tensor

$$\varepsilon^I(v_1, \dots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix} \quad (3.11) \quad \text{elementary alt}$$

In other words, if V denotes the $n \times k$ matrix whose columns are the components of the vectors v_1, \dots, v_k with respect to the basis (E_i) dual to (ε^i) , then $\varepsilon^I(v_1, \dots, v_k)$ is the determinant of the $k \times k$ submatrix consisting of rows i_1, \dots, i_k of V . Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k -tensor. We call ε^I an **elementary alternating tensor** or **elementary k -covector**.

Example 1.3.4. In terms of the standard dual basis (e^1, e^2, e^3) for $(\mathbb{R}^3)^*$, we have

$$\begin{aligned} e^{13}(v, w) &= v^1 w^3 - w^1 v^3; \\ e^{123}(v, w, x) &= \det(v, w, x). \end{aligned}$$

In order to streamline computations with the elementary k -covectors, it is useful to extend the Kronecker delta notation in the following way. If I and J are multiindices of length k , we define

$$\delta_J^I = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_k}^{i_1} \\ \vdots & \ddots & \vdots \\ \delta_{j_1}^{i_k} & \cdots & \delta_{j_k}^{i_k} \end{pmatrix}$$

so that

$$\delta_J^I = \begin{cases} \text{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = I \text{ for some } \sigma \in \mathfrak{S}_k, \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I. \end{cases}$$

Lemma 1.3.5 (Properties of Elementary k -Covectors). *Let (E_i) be a basis for V , let (ε^i) be the dual basis for V^* , and let ε^I be as defined above.*

- (a) If I has a repeated index, then $\varepsilon^I = 0$.
- (b) If $J = I_\sigma$ for some $\sigma \in \mathfrak{S}_k$, then $\varepsilon^J = \text{sgn } \sigma \varepsilon^I$.
- (c) The result of evaluating ε^I on a sequence of basis vectors is

$$\varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I.$$

The significance of the elementary k -covectors is that they provide a convenient basis for $\bigwedge^k(V^*)$. Of course, the ε^I 's are not all linearly independent, because some of them are zero and the ones corresponding to different permutations of the same multi-index are scalar multiples of each other. But as the next proposition shows, we can get a basis by restricting attention to an appropriate subset of multi-indices. A multi-index $I = (i_1, \dots, i_k)$ is said to be **increasing** if $i_1 < \dots < i_k$. It is useful to use a primed summation sign to denote a sum over only increasing multi-indices, so that, for example,

$$\sum_I' \alpha_I \varepsilon^I := \sum_{i_1 < \dots < i_k} \alpha_I \varepsilon^I.$$

tensor basis

Proposition 1.3.6 (A Basis for $\bigwedge^k V^*$). Let V be an n -dimensional vector space. If (ε^i) is any basis for V^* , then for each positive integer $k \leq n$, the collection of k -covectors

$$\mathcal{E} := \{\varepsilon^I : I \text{ is an increasing multi-index of length } k\}$$

is a basis for $\bigwedge^k(V^*)$. Therefore,

$$\dim \bigwedge^k(V^*) = \binom{n}{k}.$$

If $k > n$, then $\dim \bigwedge^k(V^*) = 0$.

In particular, for an n -dimensional vector space V , this proposition implies that $\bigwedge^n(V^*)$ is 1-dimensional and is spanned by $\varepsilon^{1\dots n}$. By definition, this elementary n -covector acts on vectors v_1, \dots, v_n by taking the determinant of their component matrix $V = (v_j^i)$. For example, on \mathbb{R}^n with the standard basis, $\varepsilon^{1\dots n}$ is precisely the determinant function.

One consequence of this is the following useful description of the behavior of an n -covector on an n -dimensional space under linear maps. Recall that if $T : V \rightarrow V$ is a linear map, the determinant of T is defined to be the determinant of the matrix representation of T with respect to any basis.

or linear map

Proposition 1.3.7. Suppose V is an n -dimensional vector space and $\omega \in \bigwedge^n(V^*)$. If $T : V \rightarrow V$ is any linear map and v_1, \dots, v_n are arbitrary vectors in V , then

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\omega(v_1, \dots, v_n). \quad (3.12) \quad \text{alt } n \text{ tensor}$$

Proof. Let (E_i) be any basis for V , and let (ε^i) be the dual basis. Let (T_i^j) denote the matrix of T with respect to this basis, and let $T_i = TE_i = T_i^j E_j$ be the i -th column of T . By Proposition 1.3.6, we can write $\omega = \lambda \varepsilon^{1\dots n}$ for some real number λ .

Since both sides of (3.12) are multilinear functions of v_1, \dots, v_n , it suffices to verify the identity when the v_i 's are basis vectors. Furthermore, since both sides are alternating, we only need to check the case $(v_1, \dots, v_n) = (E_1, \dots, E_n)$. In this case, the right-hand side of (3.12) is

$$(\det T)\lambda \varepsilon^{1\dots n}(E_1, \dots, E_n) = \lambda \det T.$$

On the other hand, the left-hand side reduces to

$$\omega(TE_1, \dots, TE_n) = \lambda \varepsilon^{1\dots n}(T_1, \dots, T_n) = \lambda \det(T_i^j),$$

which is equal to the right-hand side. \square

The wedge product

We continue with the assumption that V is a finite-dimensional real vector space. Given $\omega \in \wedge^k(V^*)$ and $\eta \in \wedge^l(V^*)$, we define their wedge product or exterior product to be the following $(k+l)$ -covector:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta). \quad (3.13)$$

wedge product

The mysterious coefficient is motivated by the simplicity of the statement of the following lemma.

dge prod form

Lemma 1.3.8. *Let V be an n -dimensional vector space and let (e^1, \dots, e^n) be a basis for V^* . For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \quad (3.14)$$

wedge product

where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$ is obtained by concatenating I and J .

Proof. By multilinearity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \varepsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}}) \quad (3.15)$$

wedge product

for any sequence $(E_{p_1}, \dots, E_{p_{k+l}})$ of basis vectors. We consider several cases.

- Case 1: $P = (p_1, \dots, p_{k+l})$ has a repeated index. In this case, both sides of (3.15) are zero by Lemma 1.3.1(c).
- Case 2: P contains an index that does not appear in either I or J . In this case, the right-hand side is zero by Lemma 1.3.5(c). Similarly, each term in the expansion of the left-hand side involves either ε^I or ε^J evaluated on a sequence of basis vectors that is not a permutation of I or J , respectively, so the left-hand side is also zero.
- Case 3: $P = IJ$ and P has no repeated indices. In this case, the right-hand side of (3.15) is equal to 1 by Lemma 1.3.5(c), so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned} \varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) &= \frac{(k+l)!}{k!l!} \text{Alt}(\varepsilon^I \otimes \varepsilon^J)(E_{p_1}, \dots, E_{p_{k+l}}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} (\text{sgn} \sigma) \varepsilon^I(E_{p_{\sigma(1)}}, \dots, E_{p_{\sigma(k)}}) \varepsilon^J(E_{p_{\sigma(k+1)}}, \dots, E_{p_{\sigma(k+l)}}) \end{aligned}$$

By Lemma 1.3.5 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must split into $\sigma = \tau\eta$, where $\tau \in \mathfrak{S}_k$ acts by permuting $\{1, \dots, k\}$ and $\eta \in \mathfrak{S}_l$ acts by permuting $\{k+1, \dots, k+l\}$. Since $\text{sgn}(\tau\eta) = (\text{sgn} \tau)(\text{sgn} \eta)$, we have

$$\begin{aligned} &\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) \\ &= \frac{1}{k!l!} \sum_{\tau \in \mathfrak{S}_k, \eta \in \mathfrak{S}_l} (\text{sgn} \tau)(\text{sgn} \eta) \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}) \varepsilon^J(E_{p_{\eta(k+1)}}, \dots, E_{p_{\eta(k+l)}}) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} (\text{sgn} \tau) \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}) \right) \left(\frac{1}{l!} \sum_{\eta \in \mathfrak{S}_l} (\text{sgn} \eta) \varepsilon^J(E_{p_{\eta(k+1)}}, \dots, E_{p_{\eta(k+l)}}) \right) \\
&= (\text{Alt} \varepsilon^I)(E_{p_1}, \dots, E_{p_k}) (\text{Alt} \varepsilon^J)(E_{p_{k+1}}, \dots, E_{p_{k+l}}) \\
&= \varepsilon^I(E_{p_1}, \dots, E_{p_k}) \varepsilon^J(E_{p_{k+1}}, \dots, E_{p_{k+l}}) = 1.
\end{aligned}$$

- Case 4: P is a permutation of IJ and has no repeated indices. In this case, applying a permutation to P brings us back to Case 3. Since the effect of the permutation is to multiply both sides of (3.15) by the same sign, the result holds in this case as well.

□

dge prod prop

Proposition 1.3.9 (Properties of the Wedge Product). Suppose $\omega, \omega', \eta, \eta'$ and ξ are multivectors on a finite-dimensional vector space V .

(a) *Bilinearity:* For $a, a' \in \mathbb{R}$,

$$\begin{aligned}
(a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta), \\
\eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega').
\end{aligned}$$

(b) *Associativity:*

$$(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi).$$

(c) *Anticommutativity:* For $\omega \in \wedge^k(V^*)$ and $\eta \in \wedge^l(V^*)$,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega. \quad (3.16)$$

wedge prod pr

(d) If (ε^i) is any basis for V^* and $I = (i_1, \dots, i_k)$ is any multi-index, then

$$\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k} = \varepsilon^I. \quad (3.17)$$

wedge prod pr

(e) For any covectors i_1, \dots, i_k and vectors v_1, \dots, v_k

$$\omega_1 \wedge \dots \wedge \omega_k(v_1, \dots, v_k) = \det(\omega^j(v_i)). \quad (3.18)$$

wedge prod pr

Proof. Bilinearity follows immediately from the definition, because the tensor product is bilinear and Alt is linear. To prove associativity, note that Lemma 1.3.8 gives

$$(\varepsilon^I \wedge \varepsilon^J) \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K).$$

The general case follows from bilinearity. Similarly, using Lemma 1.3.8 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\text{sgn} \tau) \varepsilon^{JI} = (\text{sgn} \tau) \varepsilon^J \wedge \varepsilon^I,$$

where τ is the permutation that sends IJ to JI . It is easy to check that $\text{sgn} \tau = (-1)^{kl}$, because τ can be decomposed as a composition of kl transpositions. Anticommutativity then follows from bilinearity.

Part (d) is an immediate consequence of Lemma 1.3.8 and induction. To prove (e), we note that the special case in which each ω^j is one of the basis covectors ε^{ij} reduces to (3.17). Since both sides of (3.18) are multilinear in $(\omega_1, \dots, \omega_k)$, this suffices. □

Because of part (d) of this lemma, henceforth we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}$ interchangeably.

A k -covector ω is said to be decomposable if it can be expressed in the form $\omega^1 \wedge \dots \wedge \omega^k$, where $\omega^1, \dots, \omega^k$ are covectors. It is important to be aware that not every k -covector

is decomposable when $k > 1$; however, it follows from Propositions [1.3.6](#) and [1.3.9\(d\)](#) that every k -covector can be written as a linear combination of decomposable ones.

The definition and computational properties of the wedge product can seem daunting at first sight. However, the only properties that you need to remember for most practical purposes are the ones expressed in the preceding proposition. In fact, these properties are more than enough to determine the wedge product uniquely, as the following proposition shows.

Proposition 1.3.10. *The wedge product is the unique associative, bilinear, and anticommutative map $\wedge^k(V^*) \times \wedge^l(V^*) \rightarrow \wedge^{k+l}(V^*)$ satisfying [\(3.17\)](#).*

For any n -dimensional vector space V , define a vector space $\wedge(V^*)$ by

$$\wedge(V^*) = \bigoplus_{k=0}^n \wedge^k(V^*).$$

It follows from Proposition [1.3.6](#) that $\dim \wedge(V^*) = 2^n$. Proposition [1.3.9](#) shows that the wedge product turns $\wedge(V^*)$ into an associative algebra, called the **exterior algebra** (or **Grassmann algebra**) of V . This algebra is not commutative, but it has a closely related property. An algebra A is said to be graded if it has a direct sum decomposition $A = \bigoplus_k A^k$ such that the product satisfies $A^k A^l \subseteq A^{k+l}$ for each k and l . A graded algebra is anticommutative if the product satisfies $ab = (-1)^{kl}ba$ for $a \in A^k$, $b \in A^l$. Proposition [1.3.9\(c\)](#) shows that $\wedge(V^*)$ is an anticommutative graded algebra.

Interior multiplication

There is an important operation that relates vectors with alternating tensors. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v : \wedge^k(V^*) \rightarrow \wedge^{k-1}(V^*)$, called **interior multiplication by v** , as follows:

$$i_v \omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$

In other words, $i_v \omega$ is obtained from ω by inserting v into the first slot. By convention, we interpret $i_v \omega$ to be zero when ω is a 0-covector (i.e., a number). Another common notation is $v \lrcorner \omega = i_v \omega$.

Lemma 1.3.11. *Let V be a finite-dimensional vector space and $v \in V$*

(a) $i_v \circ i_v = 0$.

(b) If $\omega \in \wedge^k(V^*)$ and $\eta \in \wedge^l(V^*)$,

$$i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^k \omega \wedge (i_v \eta). \quad (3.19)$$

Proof. On k -covectors for $k \geq 2$, part (a) is immediate from the definition, because any alternating tensor gives zero when two of its arguments are identical. On 1-covectors and 0-covectors, it follows from the fact that $i_v \equiv 0$ on 0-covectors.

To prove (b), it suffices to consider the case in which both ω and η are decomposable, since every alternating tensor of positive rank can be written as a linear combination of decomposable ones. It is straightforward to verify that (b) follows in this special case from the

following general formula for covectors $\omega^1, \dots, \omega^k$:

$$v \lrcorner (\omega^1 \wedge \dots \wedge \omega^k) = \sum_{i=1}^k (-1)^{i-1} \omega^i(v) \omega^1 \wedge \dots \wedge \widehat{\omega^i} \wedge \dots \wedge \omega^k \quad (3.20) \quad \text{interior mult}$$

To prove this, let us write $v_1 = v$ and apply both sides to an arbitrary $(k-1)$ -tuple of vectors (v_2, \dots, v_k) then what we have to prove is

$$v \lrcorner (\omega^1 \wedge \dots \wedge \omega^k)(v_1, \dots, v_k) = \sum_{i=1}^k (-1)^{i-1} \omega^i(v_1) (\omega^1 \wedge \dots \wedge \widehat{\omega^i} \wedge \dots \wedge \omega^k)(v_2, \dots, v_k). \quad (3.21) \quad \text{interior mult}$$

The left-hand side of (3.21) is the interior multiplication-3 determinant of the matrix V whose (i, j) -entry is $\omega^i(v_j)$. To simplify the right-hand side, let V_j^i denote the $(k-1) \times (k-1)$ submatrix of V obtained by deleting the i th row and j th column. Then the right-hand side of (3.21) is interior multiplication-3

$$\sum_{i=1}^k (-1)^{i-1} \omega^i(v_1) V_1^i.$$

This is just the expansion of $\det V$ by minors along the first column, and therefore is equal to $\det V$. \square

1.3.2 Differential forms on manifolds

Now we turn our attention to an n -dimensional smooth manifold M (with or without boundary). Recall that $T^k T^* M$ is the bundle of covariant k -tensors on M . The subset of $T^k T^* M$ consisting of alternating tensors is denoted by

$$\bigwedge^k T^* M = \prod_{p \in M} \bigwedge^k (T_p^* M).$$

A section of $\bigwedge^k T^* M$ is called a **differential k -form**, or just a **k -form**; this is a (continuous) tensor field whose value at each point is an alternating tensor. The integer k is called the degree of the form. We denote the vector space of smooth k -forms by

$$\Omega^k(M) := \Gamma(\bigwedge^k T^* M).$$

The wedge product of two differential forms is defined pointwise: $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Thus, the wedge product of a k -form with an l -form is a $(k+l)$ -form. If f is a 0-form and ω is a k -form, we interpret the wedge product $f \wedge \omega$ to mean the ordinary product $f\omega$. If we define

$$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M),$$

then the wedge product turns $\Omega^*(M)$ into an associative, anticommutative graded algebra.

In any smooth chart, a k -form ω can be written locally as

$$\omega = \sum_I \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_I \omega_I dx^I.$$

Proposition tensor field smooth crit 1.1.16 shows that ω is smooth on U if and only if the component functions ω^I are smooth. In terms of differential forms, the result of Lemma alt ele prop 1.3.5(c) translates to

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \delta_J^I.$$

Thus the component functions ω_I of ω are determined by

$$\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right).$$

Example 1.3.12. A 0-form is just a continuous real-valued function, and a 1-form is a covector field. On \mathbb{R}^3 , some examples of smooth 2-forms are given by

$$\omega = (\sin xy) dy \wedge dz, \quad \eta = dx \wedge dy + dy \wedge dz + dz \wedge dx.$$

Every 3-form on \mathbb{R}^3 is a continuous real-valued function times $dx \wedge dy \wedge dz$.

If $F : M \rightarrow N$ is a smooth map and ω is a differential form on N , the pullback $F^*\omega$ is a differential form on M , defined as for any covariant tensor field:

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_p(dF_p(v_1), \dots, dF_p(v_k)).$$

back formula

Lemma 1.3.13. Suppose $F : M \rightarrow N$ is smooth.

(a) $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ is linear over \mathbb{R} .

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart,

$$F^*\left(\sum_I \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F).$$

This lemma gives a computational rule for pullbacks of differential forms similar to the ones we developed for covector fields and arbitrary tensor fields earlier.

Example 1.3.14. Define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $F(u, v) = (u, v, u^2 - v^2)$, and let ω be the 2-form $y dx \wedge dz + x dy \wedge dz$ on \mathbb{R}^3 . The pullback $F^*\omega$ is computed as follows:

$$\begin{aligned} F^*(y dx \wedge dz + x dy \wedge dz) &= v du \wedge d(u^2 - v^2) + u dv \wedge d(u^2 - v^2) \\ &= v du \wedge (2u du - 2v dv) + u dv \wedge (2u du - 2v dv) \\ &= -2v^2 du \wedge dv + 2u^2 dv \wedge du \\ &= -2(u^2 + v^2) du \wedge dv. \end{aligned}$$

The same technique can also be used to compute the expression for a differential form in another smooth chart.

Example 1.3.15. Let $\omega = dx \wedge dy$ on \mathbb{R}^2 . Thinking of the transformation to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$dx \wedge dy = d(r \cos \theta) \wedge d(r \sin \theta) = (\cos \theta dr - r \sin \theta d\theta) \wedge (\sin \theta dr + r \cos \theta d\theta) = r dr \wedge d\theta.$$

The similarity between this formula and the formula for changing a double integral from Cartesian to polar coordinates is striking. The following proposition generalizes this.

back n form

Proposition 1.3.16 (Pullback Formula for Top-Degree Forms). Let $F : M \rightarrow N$ be a smooth map between n -manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open

subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V , then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det \partial F) dx^1 \wedge \cdots \wedge dx^n. \quad (3.22) \quad \boxed{\text{pull back n f}}$$

where ∂F represents the Jacobian matrix of F in these coordinates.

Proof. Because the fiber of $\wedge^n T^*M$ is spanned by $dx^1 \wedge \cdots \wedge dx^n$ at each point, it suffices to show that both sides of (3.22) give the same result when evaluated on $(\partial/\partial x^1, \dots, \partial/\partial x^n)$. From Lemma 1.3.13, $\boxed{\text{pull back n form-1}}$

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n.$$

Proposition 1.3.9(d) $\boxed{\text{wedge prod prop}}$ shows that

$$dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) = \det \left(\frac{\partial F^j}{\partial x^i} \right).$$

Therefore, the left-hand side of (3.22) $\boxed{\text{pull back n form-1}}$ gives $(u \circ F)(\det \partial F)$ when applied to $(\partial/\partial x^1, \dots, \partial/\partial x^n)$.

On the other hand, the right-hand side gives the same thing, because $dx^1 \wedge \cdots \wedge dx^n (\partial/\partial x^1, \dots, \partial/\partial x^n) = 1$. \square

Corollary 1.3.17. *If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^j))$ are overlapping smooth coordinate charts on M , then the following identity holds on $U \cap \tilde{U}$:*

$$d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^i = \det \left(\frac{\partial \tilde{x}^j}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n. \quad (3.23) \quad \boxed{\text{form n transi}}$$

Interior multiplication also extends naturally to vector fields and differential forms, simply by letting it act pointwise: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$ -form $X \lrcorner \omega = i_X \omega$ by

$$(X \lrcorner \omega)_X = X_p \lrcorner \omega_p.$$

The following results are clear.

Proposition 1.3.18. *Let X be a smooth vector field on M .*

- (a) *If ω is a smooth differential form, then $i_X \omega$ is smooth.*
- (b) *The map $i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is linear over $C^\infty(M)$, therefore corresponds to a smooth bundle homomorphism $i_X : \wedge^k T^*M \rightarrow \wedge^{k-1} T^*M$.*

1.3.3 Exterior derivatives

For each smooth manifold M with or without boundary, we will show that there is a differential operator $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω . Thus, it will follow that a necessary condition for a smooth k -form ω to be equal to $d\omega$ for some $(k-1)$ -form ω is that $d\omega = 0$.

The definition of d on Euclidean space is straightforward: if $\omega = \sum'_J \omega_J dx^J$ is a smooth k -form on an open subset $U \subseteq \mathbb{R}^n$ or \mathbb{H}^n , we define its exterior derivative $d\omega$ to be the following $(k+1)$ -form:

$$d\omega = \sum'_J d\omega_J \wedge dx^J, \quad (3.24) \quad \boxed{\text{ext der R^n}}$$

where $d\omega_J$ is the differential of the function ω_J . In somewhat more detail, this is

$$d\left(\sum_J \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

In order to transfer this definition to manifolds, we need to check that it satisfies the following properties.

Proposition 1.3.19 (Properties of the Exterior Derivative on \mathbb{R}^n).

(a) d is linear over \mathbb{R} .

(b) If ω is a smooth k -form and η is a smooth l -form on an open subset $U \subseteq \mathbb{R}^n$ or \mathbb{H}^n , then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d \equiv 0$.

(d) d commutes with pullbacks: if U is an open subset of \mathbb{R}^n or \mathbb{H}^n , V is an open subset of \mathbb{R}^m or \mathbb{H}^m , $F : U \rightarrow V$ is a smooth map, and $\omega \in \Omega^k(M)$, then

$$d(F^*\omega) = F^*(d\omega). \quad (3.25) \quad \text{ext der } \mathbb{R}^{n-1}$$

Proof. Linearity of d is an immediate consequence of the definition. To prove (b), by linearity it suffices to consider terms of the form $\omega = u dx^I \in \Omega^k(M)$ and $\eta = v dx^J \in \Omega^l(M)$ for smooth real-valued functions u and v . First, though, we need to know that d satisfies $d(u dx^I) = du \wedge dx^I$ for any multi-index I , not just increasing ones. If I has repeated indices, then clearly $d(u dx^I) = 0 = du \wedge dx^I$. If not, let σ be the permutation sending I to an increasing multi-index J . Then

$$d(u dx^I) = (\text{sgn } \sigma) d(u dx^J) = (\text{sgn } \sigma) du \wedge dx^J = du \wedge dx^I.$$

Using this, we compute

$$\begin{aligned} d(\omega \wedge \eta) &= d((u dx^I) \wedge (v dx^J)) = d(uv dx^I \wedge dx^J) = (v du + u dv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (v dx^J) + (-1)^k (u dx^I) \wedge (dv \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

We prove (c) first for the special case of a 0-form, which is just a real-valued function. In this case,

$$\begin{aligned} d(du) &= d\left(\frac{\partial u}{\partial x^j} dx^j\right) = \frac{\partial u^2}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ &= \sum_{i < j} \left(\frac{\partial u^2}{\partial x^i \partial x^j} - \frac{\partial u^2}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0. \end{aligned}$$

For the general case, we use the $k = 0$ case together with (b) to compute

$$\begin{aligned} d(d\omega) &= d\left(\sum_J d\omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \\ &= \sum_J d(d\omega_J) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J \sum_{i=1}^k d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge d(dx^{j_i}) \wedge \cdots \wedge dx^{j_k} = 0. \end{aligned}$$

Finally, to prove (d), again it suffices to consider $\omega = u dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. For such a form, the left-hand side of (3.25) is

$$F^*(d(u dx^{i_1} \wedge \cdots \wedge dx^{i_k})) = F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F),$$

and the right-hand side is

$$d(F^*(u dx^{i_1} \wedge \cdots \wedge dx^{i_k})) = d((u \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) = d(u \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

so they are equal. \square

These results allow us to transplant the definition of the exterior derivative to manifolds.

Theorem 1.3.20 (Existence and Uniqueness of Exterior Differentiation). *Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ for all k , called exterior differentiation, satisfying the following four properties:*

(i) d is linear over \mathbb{R} .

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d \circ d \equiv 0$.

(iv) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f , given by $df(X) = Xf$.

In any smooth coordinate chart, d is given by (3.24).

Proof. First, we prove existence. Suppose $\omega \in \Omega^k(M)$. We wish to define $d\omega$ by means of the coordinate formula (3.24) in each chart; more precisely, this means that for each smooth chart (U, φ) for M , we wish to set

$$d\omega = \varphi^* d(\varphi^{-1*} \omega) \quad (3.26) \quad \boxed{\text{ext der def}}$$

To see that this is well defined, we just note that for any other smooth chart (V, ψ) , the map $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of \mathbb{R}^n or \mathbb{H}^n , so Proposition 1.3.19(d) implies

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*} \omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*} \omega).$$

Together with the fact that $(\varphi \circ \psi^{-1})^* = \psi^{-1*} \circ \varphi^*$, this implies $\varphi^* d(\varphi^{-1*} \omega) = \psi^* d(\psi^{-1*} \omega)$, so $d\omega$ is well defined. It satisfies (i)–(iv) by virtue of Proposition 1.3.19.

To prove uniqueness, suppose that d is any operator satisfying (i)–(iv). First we need to show that $d\omega$ is determined locally: if ω_1 and ω_2 are k -forms that agree on an open subset $U \subseteq M$, then $d\omega_1 = d\omega_2$ on U . To see this, let $p \in U$ be arbitrary, let $\eta = \omega_1 - \omega_2$, and let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U . Then $\psi\eta$ is identically zero, so (i)–(iv) imply

$$0 = d(\psi\eta) = d\psi \wedge \eta + \psi \wedge d\eta.$$

Evaluating this at p and using the facts that $\psi(p) = 1$ and $d\psi_p = 0$, we conclude that $d\omega_1|_p = d\omega_2|_p = d\eta|_p = 0$.

Now let $\omega \in \Omega^k(M)$ be arbitrary, and let (U, φ) be any smooth coordinate chart on M . We can write ω in coordinates as $\sum_I' \omega_I dx^I$ on U . For any $p \in U$, by means of a bump function we can construct global smooth functions $\tilde{\omega}_I$ and \tilde{x}^i on M that agree with ω_I and x^i in a neighborhood of p . By virtue of (i)–(iv) together with the observation in the preceding

paragraph, it follows that $\overset{\text{ext der } \mathbb{R}^n}{(3.24)}$ holds at p . Since p was arbitrary, this d must be equal to the one we defined above. \square

If $A = \bigoplus_k A^k$ is a graded algebra, a linear map $T : A \rightarrow A$ is said to be a map of degree m if $TA^k \subseteq A^{k+m}$ for each k . It is said to be an **antiderivation** if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x \in A^k$ and $y \in A^l$. The preceding theorem can be summarized by saying that the differential on functions extends uniquely to an antiderivation of $\Omega^*(M)$ of degree +1 whose square is zero.

Proposition 1.3.21. *Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. Then the interior multiplication $i_X : \Omega^*(M) \rightarrow \Omega^*(M)$ is an antiderivation of degree -1 whose square is zero.*

Proof. This follows from Lemma $\overset{\text{interior multiplication lem}}{1.3.11}$. \square

Proposition 1.3.22 (Naturality of the Exterior Derivative). *If $F : M \rightarrow N$ is a smooth map, then for each k the pullback map $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ commutes with d : for all $\omega \in \Omega^k(N)$,*

$$F^*(d\omega) = d(F^*\omega) \quad (3.27)$$

$\overset{\text{ext der natur}}{\text{ext der natur}}$

Proof. If (U, φ) and (V, ψ) are smooth charts for M and N , respectively, we can apply Proposition $\overset{\text{ext der } \mathbb{R}^n \text{ prop}}{1.3.19(d)}$ to the coordinate representation $\psi \circ F \circ \varphi^{-1}$. Using $\overset{\text{ext der def}}{(3.26)}$ twice, we compute as follows on

$$\begin{aligned} F^*(d\omega) &= F^*\psi^*d(\psi^{-1*}\omega) \\ &= \varphi^* \circ (\psi \circ F \circ \varphi^{-1})d(\psi^{-1*}\omega) \\ &= \varphi^*d((\psi \circ F \circ \varphi^{-1})^*(\psi^{-1*}\omega)) \\ &= \varphi^*d(\varphi^{-1*} \circ F^*\omega) \\ &= d(F^*\omega). \end{aligned}$$

Therefore the claim follows. \square

Extending the terminology we introduced for covector fields, we say that a smooth differential form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and exact if there exists a smooth $(k-1)$ -form η on M such that $\omega = d\eta$. The fact that $d \circ d \equiv 0$ implies that every exact form is closed.

Exterior derivatives and vector calculus in \mathbb{R}^3

Example 1.3.23. Let us work out the exterior derivatives of arbitrary 1-forms and 2-forms on \mathbb{R}^3 . Any smooth 1-form can be written

$$\omega = P dx + Q dy + R dz$$

for some smooth functions P, Q, R . Using $\overset{\text{ext der } \mathbb{R}^n}{(3.24)}$ and the fact that the wedge product of any 1-form with itself is zero, we compute

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\ &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz \right) \wedge dy \\ &\quad + \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz \right) \wedge dz \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx. \end{aligned}$$

An arbitrary 2-form on \mathbb{R}^3 can be written

$$\eta = u \, dx \wedge dy + v \, dy \wedge dz + w \, dz \wedge dx.$$

A similar computation shows that

$$d\eta = \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) dx \wedge dy \wedge dz.$$

Recall the classical vector calculus operators on \mathbb{R}^n : the Euclidean **gradient** of a function $f \in C^\infty(\mathbb{R}^n)$ and the **divergence** of a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ are defined by

$$\text{grad } f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}, \quad \text{div } X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}.$$

In addition, in the case $n = 3$, the curl of a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ is defined by

$$\text{curl } X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1} \right) \frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \frac{\partial}{\partial x^3}.$$

It is interesting to note that the components of the 2-form $d\omega$ in the preceding example are exactly the components of the curl of the vector field with components (P, Q, R) . Similarly, there is a strong analogy between the formula for $d\eta$ and the divergence of a vector field. These analogies can be made precise in the following way.

The Euclidean metric on \mathbb{R}^3 yields an index-lowering isomorphism $\flat : \mathfrak{X}(\mathbb{R}^3) \rightarrow \Omega^1(\mathbb{R}^3)$. Interior multiplication yields another map $\beta : \mathfrak{X}(\mathbb{R}^3) \rightarrow \Omega^2(\mathbb{R}^3)$ as follows:

$$\beta(X) := X \lrcorner (dx \wedge dy \wedge dz) = X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy.$$

It is easy to check that β is linear over $C^\infty(\mathbb{R}^3)$, so it corresponds to a smooth bundle homomorphism from $T\mathbb{R}^3$ to $\wedge^2 T^*\mathbb{R}^3$. It is a bundle isomorphism because it is injective and both $T\mathbb{R}^3$ and $\wedge^2 T^*\mathbb{R}^3$ are bundles of rank 3. Similarly, we define a smooth bundle isomorphism

$$*(f) = f \, dx \wedge dy \wedge dz.$$

The relationships among all of these operators are summarized in the following diagram:

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \downarrow \text{id} & & \downarrow \flat & & \downarrow \beta & & \downarrow * \\ \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \end{array}$$

The desire to generalize these vector calculus operators from \mathbb{R}^3 to higher dimensions was one of the main motivations for developing the theory of differential forms. The curl, in particular, makes sense as an operator on vector fields only in dimension 3, whereas the exterior derivative expresses the same information but makes sense in all dimensions.

An invariant formula for the exterior derivative

In addition to the coordinate formula ^(3.24) $\text{ext_der } \mathbb{R}^n$ that we used in the definition of d , there is another formula for d that is often useful, not least because it is manifestly coordinate-independent. The formula for 1-forms is by far the most important, and is the easiest to state and prove, so we begin with that. Note the similarity between this and the formula of Proposition 7.7. ^{Covector closed iff}

Proposition 1.3.24 (Exterior Derivative of a 1-Form). *For any smooth 1-form ω and smooth vector fields X and Y ,*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \quad (3.28) \quad \boxed{\text{ext der 1 form}}$$

Proof. Since any smooth 1-form can be expressed locally as a sum of terms of the form $u dv$ for smooth functions u and v , it suffices to consider that case. Suppose $\omega = u dv$, and X, Y are smooth vector fields. Then the left-hand side of (3.28) is $\boxed{\text{ext der 1 form}}$

$$d(u dv)(X, Y) = du \wedge dv(X, Y) = du(X)dv(Y) - dv(X)du(Y) = XuYv - XvYu.$$

The right-hand side is

$$\begin{aligned} & X(u dv(Y)) - Y(u dv(X)) - u dv([X, Y]) \\ &= X(uYv) - Y(uXv) - u(XY - YX)v \\ &= (XuYv + uXYv) - (YuXv + uYXv) - u(XY - YX)v \\ &= XuYv - YuXv. \end{aligned}$$

This proves the claim. □

We will see some applications of $\boxed{\text{ext der 1 form}}$ in later chapters. Here is our first one. It shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

Proposition 1.3.25. *Let M be a smooth n -manifold with or without boundary, let (E_i) be a smooth local frame for M , and let (ε^i) be the dual coframe. For each i , let b^i_{jk} denote the component functions of the exterior derivative of ε^i in this frame, and for each j, k , let c^i_{jk} be the component functions of the Lie bracket $[E_j, E_k]$:*

$$d\varepsilon^i = \sum_{j < k} b^i_{jk} \varepsilon^j \wedge \varepsilon^k, \quad [E_j, E_k] = c^i_{jk} E_i.$$

Then $b^i_{jk} = -c^i_{jk}$.

Proof. By $\boxed{\text{ext der 1 form}}$ we have

$$b^i_{jk} = d\varepsilon^i(E_j, E_k) = E_j(\varepsilon^i(E_k)) - E_k(\varepsilon^i(E_j)) - \varepsilon^i([E_j, E_k]) = E_j(\delta^i_k) - E_k(\delta^i_j) - c^i_{jk} = -c^i_{jk}.$$

Since δ^i_k and δ^i_j are both constant. □

The generalization of $\boxed{\text{ext der 1 form}}$ to higher-degree forms is more complicated.

Proposition 1.3.26 (Invariant Formula for the Exterior Derivative). *Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields X_1, \dots, X_{k+1} on M ,*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

where the hats indicate omitted arguments.

Proof. For this proof, let us denote the expression on the right by $D\omega(X_1, \dots, X_{k+1})$, and the two sums on the right-hand side by $I_1(X_1, \dots, X_{k+1})$ and $I_2(X_1, \dots, X_{k+1})$, respectively. Note that $D\omega$ is obviously multilinear over \mathbb{R} . We begin by showing that, like $d\omega$, it is actually multilinear over $C^\infty(M)$, which is to say that for $1 \leq p \leq k+1$ and $f \in C^\infty(M)$,

$$D\omega(X_1, \dots, fX_p, \dots, X_{k+1}) = fD\omega(X_1, \dots, X_p, \dots, X_{k+1}).$$

In the expansion of $I_1(X_1, \dots, fX_p, \dots, X_{k+1})$, f obviously factors out of the $i = p$ term. The other terms expand as follows:

$$\begin{aligned} \sum_{i \neq p} (-1)^{i-1} X_i(f\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) &= \sum_{i \neq p} (-1)^{i-1} fX_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i \neq p} (-1)^{i-1} (X_i f)\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} I_1(X_1, \dots, fX_p, \dots, X_{k+1}) &= fI_1(X_1, \dots, X_p, \dots, X_{k+1}) \\ &\quad + \sum_{i \neq p} (-1)^{i-1} (X_i f)\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}). \end{aligned}$$

Consider next the expansion of I_2 . Again, f factors out of all the terms in which $i \neq p$ and $j \neq p$. To expand the other terms, we use the bracket two product rule, which implies

$$\begin{aligned} [fX_p, X_j] &= f[X_p, X_j] - (X_j f)X_p, \\ [X_i, fX_p] &= f[X_i, X_p] + (X_i f)X_p, \end{aligned}$$

Inserting these formulas into the $i = p$ and $j = p$ terms, we obtain

$$\begin{aligned} I_2(X_1, \dots, fX_p, \dots, X_{k+1}) &= fI_2(X_1, \dots, X_p, \dots, X_{k+1}) \\ &\quad + \sum_{p < j} (-1)^{p+j} \omega([fX_p, X_j], X_1, \dots, \widehat{X}_p, \dots, \widehat{X}_j, \dots, X_{k+1}) \\ &\quad + \sum_{i < p} (-1)^{i+p} \omega([X_i, fX_p], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_p, \dots, X_{k+1}). \end{aligned}$$

Rearranging the arguments in these two sums so as to put X_p into its original position, we see that they exactly cancel the sum in I_1 . This completes the proof that $D\omega$ is multilinear over $C^\infty(M)$, so it defines a smooth $(k+1)$ -tensor field.

Since both $D\omega$ and $d\omega$ are smooth tensor fields, we can verify the equation $D\omega = d\omega$ in any frame. By multilinearity, it suffices to show that both sides give the same result on an arbitrary sequence of basis vectors in some chosen local frame. Let $(U, (x^i))$ be an arbitrary smooth chart on M . Because both $d\omega$ and $D\omega$ depend linearly on ω , we may assume that $\omega = u dx^I$ for some smooth function u and some increasing multi-index $I = (i_1, \dots, i_k)$, so

$$d\omega = du \wedge dx^I = \sum_m \frac{\partial u}{\partial x^m} dx^m \wedge dx^I.$$

If $J = (j_1, \dots, j_{k+1})$ is any multi-index of length $k+1$, it follows that

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_m \frac{\partial u}{\partial x^m} \delta_J^{mI}.$$

The only terms in this sum that can possibly be nonzero are those for which J has no repeated indices and m is equal to one of the indices in J , say $m = j_p$. In this case, it is easy to check

that $\delta_J^{mI} = (-1)^{p-1} \delta_{\widehat{J}_p}^I$, where $\widehat{J}_p = (j_1, \dots, \widehat{j_p}, \dots, j_{k+1})$, so

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta_{\widehat{J}_p}^I.$$

On the other hand, because all the Lie brackets are zero, we have

$$\begin{aligned} D\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) &= \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial}{\partial x^{j_p}} \left(u dx^I \left(\frac{\partial}{\partial x^{j_1}}, \dots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}} \right) \right) \\ &= \sum_{p=1}^{k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta_{\widehat{J}_p}^I. \end{aligned}$$

from which the claim follows. \square

Lie derivatives of differential forms

wedge Lie der

Proposition 1.3.27. Suppose M is a smooth manifold, $V \in \mathfrak{X}(M)$, and $\omega, \eta \in \Omega^*(M)$. Then

$$\mathfrak{L}_V(\omega \wedge \eta) = (\mathfrak{L}_V \omega) \wedge \eta + \omega \wedge (\mathfrak{L}_V \eta).$$

Proof. Let θ be the flow of V , then by definition [\(1.3\)](#), ^{tensor Lie def}

$$\begin{aligned} \mathfrak{L}_V(\omega \wedge \eta) &= \frac{d}{dt} \Big|_{t=0} \theta_t^*(\omega \wedge \eta) = \frac{d}{dt} \Big|_{t=0} \theta_t^* \omega \wedge (\theta_t^* \eta) \Big|_{t=0} + (\theta_t^* \omega) \Big|_{t=0} \wedge \frac{d}{dt} \Big|_{t=0} \theta_t^* \eta \\ &= \frac{d}{dt} \Big|_{t=0} \theta_t^* \omega \wedge \eta + \omega \wedge \frac{d}{dt} \Big|_{t=0} \theta_t^* \eta = (\mathfrak{L}_V \omega) \wedge \eta + \omega \wedge (\mathfrak{L}_V \eta), \end{aligned}$$

which is what we want. \square

Proposition 1.3.28 (Cartan's Magic Formula). On a smooth manifold M , for any smooth vector field V and any smooth differential form ω ,

$$\mathfrak{L}_V \omega = V \lrcorner d\omega + d(V \lrcorner \omega). \quad (3.29) \quad \text{Cartan magic}$$

Proof. We prove that [\(3.29\)](#) ^{Cartan magic} holds for smooth k -forms by induction on k . We begin with a smooth 0-form f , in which case

$$V \lrcorner df + d(V \lrcorner f) = V \lrcorner df = df(V) = Vf = \mathfrak{L}_V f.$$

Now let $k \geq 1$, and suppose [\(3.29\)](#) ^{Cartan magic} has been proved for forms of degree less than k . Let ω be an arbitrary smooth k -form, written in smooth local coordinates as

$$\omega = \sum_I \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Writing $u = x^{i_1}$ and $\beta = \omega_I dx^{i_2} \wedge \dots \wedge dx^{i_k}$, we see that each term in this sum can be written in the form $du \wedge \beta$, where u is a smooth function and β is a smooth $(k+1)$ -form. Corollary [1.1.25](#) ^{covector Lie diff} showed that $\mathfrak{L}_V(du) = d(\mathfrak{L}_V u) = d(Vu)$. Thus Proposition [1.3.27](#) ^{wedge Lie der} and the induction hypothesis imply

$$\begin{aligned} \mathfrak{L}_V(du \wedge \beta) &= (\mathfrak{L}_V du) \wedge \beta + du \wedge (\mathfrak{L}_V \beta) \\ &= d(Vu) \wedge \beta + du \wedge (V \lrcorner (d\beta) + d(V \lrcorner \beta)) \\ &= d(Vu) \wedge \beta + du \wedge V \lrcorner (d\beta) + du \wedge d(V \lrcorner \beta). \end{aligned}$$

On the other hand, using the facts that both d and interior multiplication by V are antiderivations, and $V \lrcorner (du) = du(V) = Vu$, we compute

$$\begin{aligned}
 & V \lrcorner d(du \wedge \beta) + d(V \lrcorner (du \wedge \beta)) \\
 &= V \lrcorner (-du \wedge d\beta) + d((Vu) \wedge \beta - du \wedge (V \lrcorner \beta)) \\
 &= V \lrcorner (-du \wedge d\beta) + d((Vu) \wedge \beta) - d(du \wedge (V \lrcorner \beta)) \\
 &= -(Vu)d\beta + du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + (Vu)d\beta + du \wedge d(V \lrcorner \beta) \\
 &= du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + du \wedge d(V \lrcorner \beta).
 \end{aligned}$$

Thus the claim follows. \square

der commute

Corollary 1.3.29 (The Lie Derivative Commutes with d). *If V is a smooth vector field and ω is a smooth differential form, then*

$$\mathfrak{L}_V(d\omega) = d(\mathfrak{L}_V\omega).$$

Proof. This follows from Cartan's formula and the fact that $d \circ d = 0$:

$$\begin{aligned}
 \mathfrak{L}_V(d\omega) &= V \lrcorner d(d\omega) + d(V \lrcorner d\omega) = d(V \lrcorner d\omega), \\
 d(\mathfrak{L}_V\omega) &= d(V \lrcorner d\omega + d(V \lrcorner \omega)) = d(V \lrcorner d\omega).
 \end{aligned}$$

\square

1.3.4 Exercise

Exercise 1.3.1. Show that covectors $\omega^1, \dots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \dots \wedge \omega^k = 0$.

Proof. By Proposition 1.3.9, for 1-forms ω and η , we have $\omega \wedge \eta = -\eta \wedge \omega$, thus if $\omega^1, \dots, \omega^k$ are linearly dependent, then $\omega^1 \wedge \dots \wedge \omega^k = 0$.

Conversely, if $\omega^1, \dots, \omega^k$ are linearly independent, let E_1, \dots, E_n be a basis of V . Then the matrix $(\omega^i(E_j))$ has full rank, thus

$$\omega^1 \wedge \dots \wedge \omega^k(E_1, \dots, E_k) = \det(\omega^i(E_j)) \neq 0,$$

which means $\omega^1 \wedge \dots \wedge \omega^k \neq 0$. \square

Exercise 1.3.2 (Cartan's Lemma). Let M be a smooth n -manifold with or without boundary, and let $(\omega^1, \dots, \omega^k)$ be an ordered k -tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $(\omega^1|_p, \dots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \dots, \alpha^k$ on U such that

$$\sum_{i=1}^k \omega^i \wedge \alpha^i = 0.$$

show that each α^i can be written as a linear combination of $\omega^1, \dots, \omega^k$ with smooth coefficients.

Proof. Since $(\omega^1, \dots, \omega^k)$ are linearly independent, the form

$$\omega := \omega^1 \wedge \dots \wedge \omega^k \neq 0.$$

Wedge the given equation by ω , we get

$$\omega \wedge \alpha^1 = 0.$$

Since $\omega \neq 0$, this implies α^1 is a linear combination of ω^i , and proceed similarly, we conclude that each α^i can be written as a linear combination of $\omega^1, \dots, \omega^k$. \square

Exercise 1.3.3. Define a 2-form ω on \mathbb{R}^3 by

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

(a) Compute ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).$$

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota_{S^2}^* \omega$ to S^2 , using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota_{S^2}^* \omega$ is nowhere zero.

Proof. We first compute

$$\begin{aligned} dx &= d(\rho \sin \varphi \cos \theta) = \sin \varphi \cos \theta \, d\rho + \rho \cos \varphi \cos \theta \, d\varphi - \rho \sin \varphi \sin \theta \, d\theta, \\ dy &= d(\rho \sin \varphi \sin \theta) = \sin \varphi \sin \theta \, d\rho + \rho \cos \varphi \sin \theta \, d\varphi + \rho \sin \varphi \cos \theta \, d\theta, \\ dz &= d(\rho \cos \varphi) = \cos \varphi \, d\rho - \rho \sin \varphi \, d\varphi. \end{aligned}$$

Thus

$$\begin{aligned} dy \wedge dz &= -\rho \sin \theta \, d\rho \wedge d\varphi + \rho^2 \sin^2 \varphi \cos \theta \, d\varphi \wedge d\theta + \rho \sin \varphi \cos \varphi \cos \theta \, d\theta \wedge d\rho, \\ dz \wedge dx &= \rho \cos \theta \, d\rho \wedge d\varphi + \rho^2 \sin^2 \varphi \sin \theta \, d\varphi \wedge d\theta + \rho \sin \varphi \cos \varphi \sin \theta \, d\theta \wedge d\rho, \\ dx \wedge dy &= \rho^2 \sin \varphi \cos \varphi \, d\varphi \wedge d\theta - \rho \sin^2 \varphi \, d\theta \wedge d\rho. \end{aligned}$$

and then, plug this into ω we get

$$\omega = \rho^3 \sin \varphi \, d\varphi \wedge d\theta.$$

For (b), we have

$$d\omega = 3 \, dx \wedge dy \wedge dz, \quad d\omega = (3\rho^2 \sin \varphi \, d\rho + \rho^3 \cos \varphi \, d\varphi) = 3\rho^2 \sin \varphi \, d\rho \wedge d\varphi \wedge d\theta.$$

and note that

$$dx \wedge dy \wedge dz = \rho^2 \sin \varphi \, d\rho \wedge d\varphi \wedge d\theta.$$

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the map $F(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$, then

$$\begin{aligned} F^*(dy \wedge dz) &= \sin^2 \varphi \cos \theta \, d\varphi \wedge d\theta, \\ F^*(dz \wedge dx) &= \sin^2 \varphi \sin \theta \, d\varphi \wedge d\theta, \\ F^*(dx \wedge dy) &= \sin \varphi \cos \varphi \, d\varphi \wedge d\theta. \end{aligned}$$

so that

$$F^* \omega = (\sin^3 \varphi \cos^2 \theta + \sin^3 \varphi \sin^2 \theta + \sin \varphi \cos^2 \varphi) \, d\varphi \wedge d\theta = \sin \varphi \, d\varphi \wedge d\theta.$$

\square

1.4 Orientations

1.4.1 Orientations of vector spaces

Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases (E_1, \dots, E_n) and $(\tilde{E}_1, \dots, \tilde{E}_n)$ for V are **consistently oriented** if the transition matrix (B_i^j) , defined by

$$E_i = B_i^j \tilde{E}_j,$$

has positive determinant.

If $\dim V = n$, we define an orientation for V as an equivalence class of ordered bases. If (E_1, \dots, E_n) is any ordered basis for V , we denote the orientation that it determines by $[E_1, \dots, E_n]$, and the opposite orientation by $-[E_1, \dots, E_n]$. A vector space together with a choice of orientation is called an **oriented vector space**. If V is oriented, then any ordered basis (E_1, \dots, E_n) that is in the given orientation is said to be oriented or **positively oriented**. Any basis that is not in the given orientation is said to be **negatively oriented**.

For the special case of a zero-dimensional vector space V , we define an orientation of V to be simply a choice of one of the numbers ± 1 .

There is an important connection between orientations and alternating tensors, expressed in the following proposition.

or space form

Proposition 1.4.1. *Let V be a vector space of dimension n . Each nonzero element $\omega \in \wedge^n(V^*)$ determines an orientation O_ω of V as follows: if $n \geq 1$, then O_ω is the set of ordered bases (E_1, \dots, E_n) such that $\omega(E_1, \dots, E_n) > 0$; while if $n = 0$, then O_ω is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Two nonzero n -covectors determine the same orientation if and only if each is a positive multiple of the other.*

Proof. The 0-dimensional case is immediate, since a nonzero element of $\wedge^0(V^*)$ is just a nonzero real number. Thus we may assume $n \geq 1$. Let ω be a nonzero element of $\wedge^n(V^*)$, and let O_ω denote the set of ordered bases on which ω gives positive values. We need to show that O_ω is exactly one equivalence class.

Suppose (E_j) and (\tilde{E}_j) are any two ordered bases for V , and let $B : V \rightarrow V$ be the linear map sending E_j to \tilde{E}_j . By Proposition 1.3.7, $\omega(\tilde{E}_1, \dots, \tilde{E}_n) = (\det B)\omega(E_1, \dots, E_n)$.

$$\omega(\tilde{E}_1, \dots, \tilde{E}_n) = (\det B)\omega(E_1, \dots, E_n).$$

It follows that the basis (\tilde{E}_j) is consistently oriented with (E_j) if and only if (E_1, \dots, E_n) and $(\tilde{E}_1, \dots, \tilde{E}_n)$ have the same sign, which is the same as saying that O_ω is one equivalence class. The last statement then follows easily. \square

If V is an oriented n -dimensional vector space and ω is an n -covector that determines the orientation of V as described in this proposition, we say that ω is a (positively) oriented n -covector. For example, the n -covector $e^1 \wedge \dots \wedge e^n$ is positively oriented for the standard orientation on \mathbb{R}^n .

1.4.2 Orientations of manifolds

Let M be a smooth manifold with or without boundary. We define a **pointwise orientation** on M to be a choice of orientation of each tangent space. By itself, this is not a very useful concept, because the orientations of nearby points may have no relation to each other. For example, a pointwise orientation on \mathbb{R}^n might switch randomly from point to point between

the standard orientation and its opposite. In order for orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Let M be a smooth n -manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM defined on some open subset $U \subseteq M$, we say that (E_i) is **positively oriented** if $(E_1|_p, \dots, E_n|_p)$ is a positively oriented basis for T_pM at each point $p \in U$. A **negatively oriented** frame is defined analogously.

A pointwise orientation is said to be **continuous** if every point of M is in the domain of an oriented local frame. (Recall that by definition the vector fields that make up a local frame are continuous.) An **orientation of M** is a continuous pointwise orientation. We say that M is **orientable** if there exists an orientation for it, and **nonorientable** if not. An **oriented manifold** is an ordered pair (M, O) , where M is an orientable smooth manifold and O is a choice of orientation for M , an **oriented manifold with boundary** is defined similarly. For each $p \in M$, the orientation of T_pM determined by O is denoted by O_p .

If M is zero-dimensional, this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

The next two propositions give ways of specifying orientations on manifolds that are more practical to use than the definition.

ation n form

Proposition 1.4.2 (The Orientation Determined by an n -Form). *Let M be a smooth n -manifold with or without boundary. Any nonvanishing n -form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n -form on M that is positively oriented at each point.*

Remark 1.4.1. Because of this proposition, if M is a smooth n -manifold with or without boundary, any nonvanishing n -form on M is called an **orientation form**. If M is oriented and ω is an orientation form determining the given orientation, we also say that ω is **(positively) oriented**. It is easy to check that if ω and $\tilde{\omega}$ are two positively oriented smooth forms on M , then $\tilde{\omega} = f\omega$ for some strictly positive smooth realvalued function f . If M is a 0-manifold, a nonvanishing 0-form (i.e., real-valued function) assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Proof. Let ω be a nonvanishing n -form on M . Then ω defines a pointwise orientation by Proposition 1.4.1, so all we need to check is that it is continuous. This is trivially true when $n = 0$, so assume $n \geq 1$. Given $p \in M$, let (E_i) be any local frame on a connected neighborhood U of p , and let (ε^i) be the dual coframe. On U , the expression for ω in this frame is $\omega = f\varepsilon^1 \wedge \dots \wedge \varepsilon^n$ for some continuous function f . The fact that ω is nonvanishing means that f is nonvanishing, and therefore

$$\omega(E_1, \dots, E_n) = f \neq 0.$$

at all points of U . Since U is connected, it follows that this expression is either always positive or always negative on U , and therefore the given frame is either positively oriented or negatively oriented. If negatively, we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Thus, the pointwise orientation determined by ω is continuous.

Conversely, suppose M is oriented, and let $\bigwedge_+^n T^*M \subseteq \bigwedge^n T^*M$ be the open subset consisting of positively oriented n -covectors at all points of M . At any point $p \in M$, the intersection of $\bigwedge_+^n T^*M$ with the fiber $\bigwedge^n(T_p^*M)$ is an open half-line, and therefore convex. By the

usual partition-of-unity argument (see Exercise [1.2.1](#)), there exists a smooth global section of $\wedge_+^n T^*M$ (i.e., a positively oriented smooth global n -form). \square

on connected

Corollary 1.4.3. *Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, they are equal.*

A smooth coordinate chart on an oriented smooth manifold with or without boundary is said to be **(positively) oriented** if the coordinate frame $(\partial/\partial x^i)$ is positively oriented, and **negatively oriented** if the coordinate frame is negatively oriented. A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to be **consistently oriented** if for each α, β , the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

tion by atlas

Proposition 1.4.4 (Orientation Determined by an Atlas). *Let M be a smooth positive-dimensional manifold with or without boundary. Given any consistently oriented smooth atlas for M , there is a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \emptyset$ or $\dim M > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M .*

Proof. First, suppose M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map, which has positive determinant by hypothesis, so they determine the same pointwise orientation at each point. The orientation thus determined is continuous because each point is in the domain of an oriented coordinate frame.

Conversely, assume M is oriented and either $\partial M = \emptyset$ or $\dim M > 1$. Each point is in the domain of a smooth chart, and if the chart is negatively oriented, we can replace x_1 by $-x_1$ to obtain a new chart that is positively oriented. The fact that these charts all are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented atlas. (This does not work for boundary charts when $\dim M = 1$ because of our convention that the last coordinate is nonnegative in a boundary chart.) \square

Proposition 1.4.5 (Product Orientations). *Suppose M_1, \dots, M_k are orientable smooth manifolds. There is a unique orientation on $M_1 \times \dots \times M_k$, called the **product orientation**, with the following property: if for each $1 \leq i \leq k$, ω_i is an orientation form for the given orientation on M_i , then $\pi_1^* \omega_1 \wedge \dots \wedge \pi_k^* \omega_k$ is an orientation form for the product orientation.*

Proposition 1.4.6 (Orientations of Codimension-0 Submanifolds). *Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. Then the orientation of M restricts to an orientation of D . If ω is an orientation form for M , then $i^* \omega$ is an orientation form for D .*

Let M and N be oriented smooth manifolds with or without boundary, and suppose $F : M \rightarrow N$ is a local diffeomorphism. If M and N are positive-dimensional, we say that F is **orientation-preserving** if for each $p \in M$, the isomorphism dF_p takes oriented bases of $T_p M$ to oriented bases of $T_{F(p)} N$, and **orientation-reversing** if it takes oriented bases of $T_p M$ to negatively oriented bases of $T_{F(p)} N$.

If M and N are 0-manifolds, then F is orientation-preserving if for every $p \in M$, the points p and $F(p)$ have the same orientation; and it is orientation-reversing if they have the opposite orientation.

Proposition 1.4.7. *Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \rightarrow N$ is a local diffeomorphism. Then the following are equivalent.*

- (a) F is orientation-preserving.
- (b) With respect to any oriented smooth charts for M and N , the Jacobian matrix of F has positive determinant.
- (c) For any positively oriented orientation form ω for N , the form $F^*\omega$ is positively oriented for M .

Proof. This is a consequence of Proposition 1.4.4 and 1.3.7. □

Here is another important method for constructing orientations.

on pull back

Proposition 1.4.8 (The Pullback Orientation). *Suppose M and N are smooth manifolds with or without boundary. If $F : M \rightarrow N$ is a local diffeomorphism and N is oriented, then M has a unique orientation, called the **pullback orientation induced by F** , such that F is orientation-preserving.*

Proof. For each $p \in M$, there is a unique orientation on $T_p M$ that makes the isomorphism $dF_p : T_p M \rightarrow T_{F(p)} N$ orientation-preserving. This defines a pointwise orientation on M , and provided it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω for N and note that $F^*\omega$ is a smooth orientation form for M . □

In the situation of the preceding proposition, if O denotes the given orientation on N , the pullback orientation on M is denoted by F^*O .

Proposition 1.4.9. *Suppose $F : M \rightarrow N$ and $G : N \rightarrow P$ are local diffeomorphisms and O is an orientation on P , then $(G \circ F)^*O = F^*(G^*O)$.*

Recall that a smooth manifold is said to be *parallelizable* if it admits a smooth global frame.

Proposition 1.4.10. *Every parallelizable smooth manifold is orientable.*

Proof. Suppose M is parallelizable, and let (E_1, \dots, E_n) be a global smooth frame for M . Define a pointwise orientation on M by declaring the basis $(E_1|_p, \dots, E_n|_p)$ to be positively oriented at each $p \in M$. This pointwise orientation is continuous, because every point of M is in the domain of the (global) oriented frame (E_i) . □

Example 1.4.11. The preceding proposition implies that Euclidean space \mathbb{R}^n , the n -torus T^n , the spheres S^1 , S^3 , and S^7 , and products of them are all orientable, because they are all parallelizable. Therefore, any codimension-0 submanifold of one of these manifolds is also orientable. Likewise, every Lie group is orientable because it is parallelizable.

In the case of Lie groups, we can say more. If G is a Lie group, an orientation of G is said to be **left-invariant** if L_g is orientation-preserving for every $g \in G$.

Proposition 1.4.12. *Every Lie group has precisely two left-invariant orientations, corresponding to the two orientations of its Lie algebra.*

Orientations of hypersurfaces

If M is an oriented smooth manifold and S is a smooth submanifold of M (with or without boundary), S might not inherit an orientation from M , even if S is embedded. Clearly, it is not sufficient to restrict an orientation form from M to S , since the restriction of an n -form to a manifold of lower dimension must necessarily be zero. A useful example to consider is the Möbius band, which is not orientable, even though it can be embedded in \mathbb{R}^3 .

Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is a smooth submanifold (immersed or embedded, with or without boundary). Recall that a vector field along S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N : S \rightarrow TM$ with the property that $N_p \in T_p M$ for each $p \in S$. For example, any vector field on M restricts to a vector field along S , but in general, not every vector field along S is of this form.

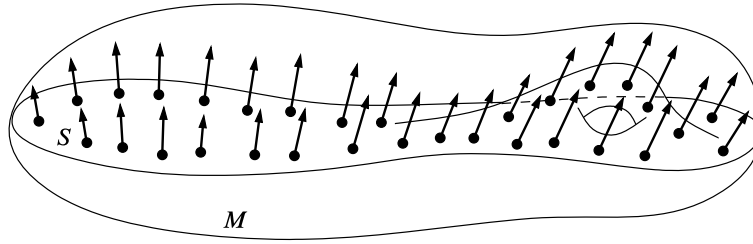


Figure 1.1: A vector field along a submanifold.

hypersurface

Proposition 1.4.13. *Suppose M is an oriented smooth n -manifold with or without boundary, S is an immersed hypersurface with or without boundary in M , and N is a vector field along S that is nowhere tangent to S . Then S has a unique orientation such that for each $p \in S$, (E_1, \dots, E_{n-1}) is an oriented basis for $T_p S$ if and only if $(N_p, E_1, \dots, E_{n-1})$ is an oriented basis for $T_p M$. If ω is an orientation form for M , then $\iota_S^*(N \lrcorner \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S : S \hookrightarrow M$ is inclusion.*

Remark 1.4.2. When $n = 1$, since S is a 0-manifold, this proposition should be interpreted as follows: at each point $p \in S$, we assign the orientation $+1$ to p if N_p is an oriented basis for $T_p M$, and -1 if N_p is negatively oriented. With this understanding, the proof below goes through in the $n = 1$ case without modification.

Proof. Let ω be an orientation form for M . Then $\sigma = \iota_S^*(N \lrcorner \omega)$ is an $(n-1)$ -form on S . (Recall that the pullback ι_S^* is really just restriction to vectors tangent to S .) It will follow that σ is an orientation form for S if we can show that it never vanishes. Given any basis (E_1, \dots, E_{n-1}) for $T_p S$, the fact that N is nowhere tangent to S implies that $(N_p, E_1, \dots, E_{n-1})$ is a basis for $T_p M$. The fact that N is nonvanishing implies that

$$\sigma_p(E_1, \dots, E_{n-1}) = \omega_p(N_p, E_1, \dots, E_{n-1}) \neq 0.$$

Since $\sigma_p(E_1, \dots, E_{n-1}) > 0$ if and only if $\omega_p(N_p, E_1, \dots, E_{n-1}) > 0$, the orientation determined by σ is the one defined in the statement of the proposition. \square

Example 1.4.14. The sphere S^n is a hypersurface in \mathbb{R}^{n+1} , to which the vector field $N = x^i \partial / \partial x^i$ is nowhere tangent, so this vector field induces an orientation on S^n . This shows that all

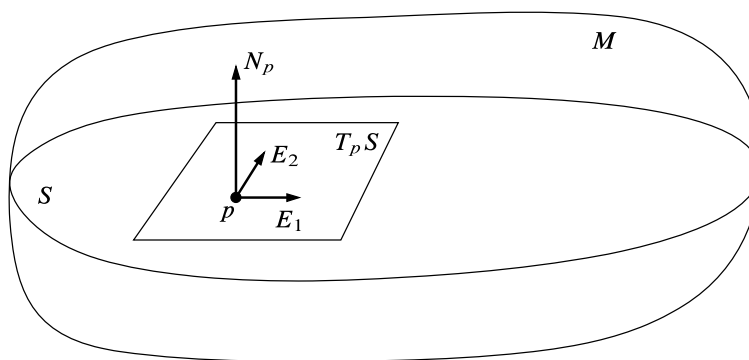


Figure 1.2: The orientation induced by a nowhere tangent vector field.

spheres are orientable. We define the **standard orientation** of S^n to be the orientation determined by N . Unless otherwise specified, we always use this orientation. (The standard orientation on S^0 is the one that assigns the orientation $+1$ to the point $+1$ and -1 to $-1 \in S^0$.)

Not every hypersurface admits a nowhere tangent vector field. However, the following proposition gives a sufficient condition that holds in many cases.

Proposition 1.4.15. *Let M be an oriented smooth manifold, and suppose $S \subseteq M$ is a regular level set of a smooth function $f : M \rightarrow \mathbb{R}$. Then S is orientable.*

Proof. Choose any Riemannian metric on M , and let $N = \text{grad } f|_S$. The hypotheses imply that N is a nowhere tangent vector field along S , so the result follows from Proposition 1.4.13. \square

Boundary orientations

If M is a smooth manifold with boundary, then ∂M is an embedded hypersurface in M , and Exercise ?? showed that there is always a smooth outward-pointing vector field along ∂M . Because an outward-pointing vector field is nowhere tangent to ∂M , it determines an orientation on ∂M if M is oriented.

Proposition 1.4.16 (Induced Orientation on a Boundary). *Let M be an oriented smooth n -manifold with boundary, $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M .*

Remark 1.4.3. The orientation on ∂M determined by any outward-pointing vector field is called the **induced orientation** or the **Stokes orientation** on ∂M .

Proof. Let $n = \dim M$, let ω be an orientation form for M , and let N be a smooth outward-pointing vector field along ∂M . The $(n-1)$ -form $\iota_{\partial M}^*(N \lrcorner \omega)$ is an orientation form for ∂M by Proposition 1.4.13, so ∂M is orientable.

To show that this orientation is independent of the choice of N , let $p \in \partial M$ be arbitrary, and let (x^i) be smooth boundary coordinates for M on a neighborhood of p . If N and \tilde{N} are two different outward-pointing vector fields along ∂M , Proposition ?? shows that the last components $N^n(p)$ and $\tilde{N}^n(p)$ are both negative. Both $(N_p, \partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p)$ and $(\tilde{N}_p, \partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p)$ are bases for $T_p M$, and the transition matrix between them has

determinant equal to $N^n(p)/\tilde{N}^n(p) > 0$. Thus, both bases determine the same orientation for $T_p M$, so N and \tilde{N} determine the same orientation for $T_p \partial M$. (When $n = 1$, the bases in question are just N_p and \tilde{N}_p , which determine the same orientation because they are both negative multiples of $\partial/\partial x^1|_p$.) \square

Example 1.4.17. This proposition gives a simpler proof that S^n is orientable, because it is the boundary of the closed unit ball. The orientation thus induced on S^n is the standard one.

on induce H^n

Example 1.4.18. Let us determine the induced orientation on $\partial \mathbb{H}^n$ when \mathbb{H}^n itself has the standard orientation inherited from \mathbb{R}^n . We can identify $\partial \mathbb{H}^n$ with \mathbb{R}^{n-1} under the correspondence $(x^1, \dots, x^{n-1}, 0) \leftrightarrow (x^1, \dots, x^{n-1})$. Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\partial \mathbb{H}^n$, the standard coordinate frame for \mathbb{R}^{n-1} is positively oriented for $\partial \mathbb{H}^n$ if and only if $[-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}]$ is the standard orientation for \mathbb{R}^n . This orientation satisfies

$$\begin{aligned} [-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}] &= -[\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}] \\ &= (-1)^n [\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n]. \end{aligned}$$

Thus, the induced orientation on $\partial \mathbb{H}^n$ is equal to the standard orientation on \mathbb{R}^{n-1} when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial \mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even.

For many purposes, the most useful way of describing submanifolds is by means of local parametrizations. The next lemma gives a useful criterion for checking whether a local parametrization of a boundary is orientation-preserving.

preserve lem

Lemma 1.4.19. Let M be an oriented smooth n -manifold with boundary. Suppose $U \subseteq \mathbb{R}^{n-1}$ is open, a, b are real numbers with $a < b$, and $F : (a, b] \times U \rightarrow M$ is a smooth embedding that restricts to an embedding of $\{b\} \times U$ to ∂M . Then the parametrization $f : U \rightarrow \partial M$ given by $f(x) = F(b, x)$ is orientation-preserving for ∂M if and only if F is orientation-preserving for M .

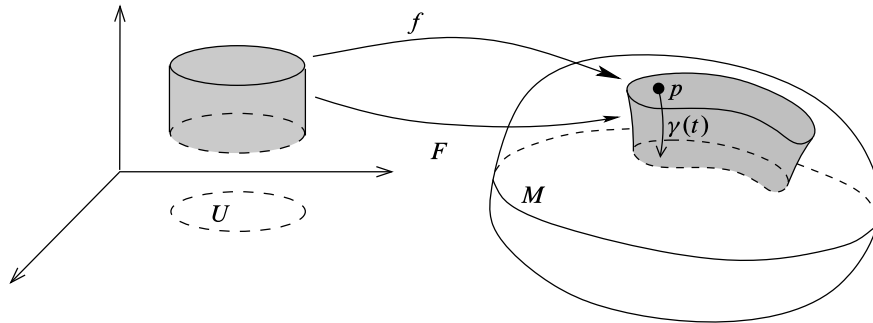


Figure 1.3: Orientation criterion for a boundary parametrization.

Proof. Let x be an arbitrary point of U , and let $p = f(x) = F(b, x) \in \partial M$. The hypothesis that F is an embedding means that the linear map $dF_{(b,x)} : (T_b \mathbb{R} \oplus T_x \mathbb{R}^{n-1}) \rightarrow T_p M$ is bijective. Since the restriction of $dF_{(b,x)}$ to $T_x \mathbb{R}^{n-1}$ is equal to $df_x : T_x \mathbb{R}^{n-1} \rightarrow T_p \partial M$ which is already injective, it follows that $dF(\partial/\partial s|_{(b,x)}) \notin T_p \partial M$ (where s denotes the coordinate on $(a, b]$).

Define a smooth curve $\gamma : [0, \varepsilon) \rightarrow M$ by

$$\gamma(t) = F(b - t, x).$$

This curve satisfies $\gamma(0) = p$ and $\gamma'(0) = -dF(\partial/\partial s|_{(b,x)})$. It follows that $-dF(\partial/\partial s|_{(b,x)})$ is inward-pointing, and therefore $dF(\partial/\partial s|_{(b,x)})$ is outward-pointing.

The definition of the induced orientation yields the following equivalences:

$$\begin{aligned} &F \text{ is orientation-preserving for } M \\ \iff &(dF(\partial/\partial s), dF(\partial/\partial x^1), \dots, dF(\partial/\partial x^{n-1})) \text{ is oriented for } TM \\ \iff &(dF(\partial/\partial x^1), \dots, dF(\partial/\partial x^{n-1})) \text{ is oriented for } T\partial M \\ \iff &(df(\partial/\partial x^1), \dots, df(\partial/\partial x^{n-1})) \text{ is oriented for } T\partial M \\ \iff &f \text{ is orientation-preserving for } M. \end{aligned}$$

This proves the claim. \square

Here is an illustration of how the lemma can be used.

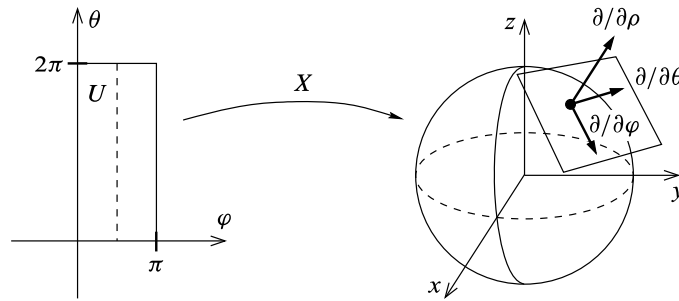


Figure 1.4: Parametrizing the sphere via spherical coordinates.

ation para eg

Example 1.4.20. Spherical coordinates yield a smooth local parametrization of S^2 as follows. Let U be the open rectangle $(0, \pi) \times (0, 2\pi) \subseteq \mathbb{R}^2$, and let $X : U \rightarrow \mathbb{R}^3$ be the following map:

$$X(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi).$$

We can check whether X preserves or reverses orientation by using the fact that it is the restriction of the 3-dimensional spherical coordinate parametrization $F : (0, 1] \times U \rightarrow \bar{B}^3$ defined by

$$F(\rho, \varphi, \theta) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).$$

Because $F(1, \varphi, \theta) = X(\varphi, \theta)$, the hypotheses of Lemma [1.4.19](#) are satisfied. By direct computation, the Jacobian determinant of F is $\rho^2 \sin \varphi$, which is positive on $(0, 1] \times U$. By virtue of Lemma [1.4.19](#), X is orientation-preserving.

1.4.3 The Riemannian volume form

Let (M, g) be an oriented Riemannian manifold of positive dimension. We know from Proposition [1.2.6](#) that there is a smooth orthonormal frame (E_1, \dots, E_n) in a neighborhood of each point of M . By replacing E_1 by $-E_1$ if necessary, we can find an oriented orthonormal frame in a neighborhood of each point.

n volume form

Proposition 1.4.21. Suppose (M, g) is an oriented Riemannian n -manifold with or without boundary, and $n \geq 1$. There is a unique smooth orientation form $\omega_g \in \Omega^n(M)$, called the **Riemannian**

volume form, that satisfies

$$\omega_g(E_1, \dots, E_n) = 1 \quad (4.30) \quad \text{Riemannian volume form}$$

for every local oriented orthonormal frame (E_i) for M .

Proof. Suppose first that such a form ω_g exists. If (E_1, \dots, E_n) is any local oriented orthonormal frame on an open subset $U \subseteq M$ and $(\varepsilon^1, \dots, \varepsilon^n)$ is the dual coframe, we can write $\omega_g = f \varepsilon^1 \wedge \dots \wedge \varepsilon^n$ on U . The condition (4.30) then reduces to $f = 1$, so

$$\omega_g = \varepsilon^1 \wedge \dots \wedge \varepsilon^n. \quad (4.31) \quad \text{Riemannian volume form}$$

This proves that such a form is uniquely determined.

To prove existence, we would like to define ω_g in a neighborhood of each point by (4.31), so we need to check that this definition is independent of the choice of oriented orthonormal frame. If $(\tilde{E}_1, \dots, \tilde{E}_n)$ is another oriented orthonormal frame, with dual coframe $(\tilde{\varepsilon}^1, \dots, \tilde{\varepsilon}^n)$, let

$$\tilde{\omega} = \tilde{\varepsilon}^1 \wedge \dots \wedge \tilde{\varepsilon}^n.$$

We can write

$$\tilde{E}_i = A_i^j E_j$$

for some matrix (A_i^j) of smooth functions. The fact that both frames are orthonormal means that $(A_i^j(p)) \in O(n)$ for each p , so $\det(A_i^j) = \pm 1$, and the fact that the two frames are consistently oriented forces the positive sign. We compute by Proposition 1.3.7:

$$\omega_g(\tilde{E}_1, \dots, \tilde{E}_n) = \det(\varepsilon^j(\tilde{E}_i)) = \det(A_i^j) = 1 = \tilde{\omega}_g(\tilde{E}_1, \dots, \tilde{E}_n).$$

Thus $\omega_g = \tilde{\omega}_g$, so defining ω_g in a neighborhood of each point by (4.31) with respect to some smooth oriented orthonormal frame yields a global n -form. The resulting form is clearly smooth and satisfies (4.30) for every oriented orthonormal frame. \square

Proposition 1.4.22. Suppose (M, g) and (\tilde{M}, \tilde{g}) are positive-dimensional Riemannian manifolds with or without boundary, and $F : M \rightarrow \tilde{M}$ is a orientation-preserving local isometry. Then that $F^* \omega_{\tilde{g}} = \omega_g$.

Proof. Since F is a orientation-preserving local isometry, it maps a local oriented orthonormal frame to a local oriented orthonormal frame. Then the claim follows immediately. \square

Although the expression for the Riemannian volume form with respect to an oriented orthonormal frame is particularly simple, it is also useful to have an expression for it in coordinates.

Proposition 1.4.23. Let (M, g) be an oriented Riemannian n -manifold with or without boundary with $n \geq 1$. In any oriented smooth coordinates (x^i) , the Riemannian volume form has the local coordinate expression

$$\omega_g = \sqrt{G} dx^1 \wedge \dots \wedge dx^n,$$

where $G = \det(g_{ij})$ with g_{ij} the components of g in these coordinates.

Proof. Let $(U, (x^i))$ be an oriented smooth chart, and let $p \in M$. In these coordinates, $\omega_g = f dx^1 \wedge \dots \wedge dx^n$ for some positive coefficient function f . To compute f , let (E_i) be any smooth

oriented orthonormal frame defined on a neighborhood of p , and let (ε^i) be the dual coframe. If we write the coordinate frame in terms of the orthonormal frame as

$$\frac{\partial}{\partial x^i} = A_i^j E_j,$$

then we can compute

$$f = \omega_g\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \det\left(\varepsilon^j\left(\frac{\partial}{\partial x^i}\right)\right) = \det(A_i^j).$$

On the other hand, observe that

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_g = \langle A_i^k E_k, A_j^l E_l \rangle_g = A_i^k A_j^l \langle E_k, E_l \rangle_g = \sum_k A_i^k A_j^k.$$

This last expression is the (i, j) -entry of the matrix product $A^T A$, where $A = (A_i^j)$. Thus,

$$G := \det(g_{ij}) = \det(A^T A) = (\det A)^2,$$

Since both frames $(\partial/\partial x^i)$ and (E_i) are oriented, f must be positive. So it follows that $f = \sqrt{G}$. \square

Hypersurfaces in Riemannian manifolds

Let (M, g) be an oriented Riemannian manifold with or without boundary, and suppose $S \subseteq M$ is an immersed hypersurface with or without boundary. Any unit normal vector field along S is nowhere tangent to S , so it determines an orientation of S by Proposition 1.4.13. The next proposition gives a simple formula for the volume form of the induced metric on S with respect to this orientation.

Proposition 1.4.24. *Let (M, g) be an oriented Riemannian manifold with or without boundary, let $S \subseteq M$ be an immersed hypersurface with or without boundary, and let \tilde{g} denote the induced metric on S . Suppose N is a smooth unit normal vector field along S . With respect to the orientation of S determined by N , the volume form of (S, \tilde{g}) is given by*

$$\omega_{\tilde{g}} = \iota_S^*(N \lrcorner \omega_g).$$

Proof. By Proposition 1.4.13, the $(n-1)$ -form $\iota_S^*(N \lrcorner \omega_g)$ is an orientation form for S . To prove that it is the volume form for the induced Riemannian metric, we need only show that it gives the value 1 whenever it is applied to an oriented orthonormal frame for S . Thus, let (E_1, \dots, E_{n-1}) be such a frame. At each point $p \in S$, the basis $(N_p, E_1|_p, \dots, E_{n-1}|_p)$ is orthonormal, and is oriented for $T_p M$ (this is the definition of the orientation determined by N). Thus

$$\iota_S^*(N \lrcorner \omega_g)(E_1, \dots, E_{n-1}) = \omega_g(N, E_1, \dots, E_{n-1}) = 1,$$

which proves the result. \square

The result of Proposition 1.4.24 takes on particular importance in the case of a Riemannian manifold with boundary, because of the following proposition.

Proposition 1.4.25. *Suppose M is any Riemannian manifold with boundary. There is a unique smooth outward-pointing unit normal vector field N along ∂M .*

Proof. First, we prove uniqueness. At any point $p \in \partial M$, the subspace $(T_p \partial M)^\perp \subseteq T_p M$ is 1-dimensional, so there are exactly two unit vectors at p that are normal to ∂M . Since any

unit normal vector N is nowhere tangent to ∂M , it must have nonzero x^n -component in any smooth boundary chart. Thus, exactly one of the two choices of unit normal has negative x^n -component, which is equivalent to being outward-pointing.

To prove existence, let $f : M \rightarrow \mathbb{R}$ be a boundary defining function (Proposition 5.43), and let N be the restriction to ∂M of the unit vector field $-\text{grad } f / |\text{grad } f|_g$. Because $df \neq 0$ at points of ∂M , N is well-defined and smooth on ∂M . Then N is normal to ∂M by Exercise [1.2.9](#), and outward pointing by Proposition [1.4.27](#), because Riemann grad prop
inward derivative crit

$$Nf = \frac{-\langle \text{grad } f, \text{grad } f \rangle_g}{|\text{grad } f|_g} = -|\text{grad } f|_g < 0.$$

□

The next corollary is immediate.

Corollary 1.4.26. *If (M, g) is an oriented Riemannian manifold with boundary and \tilde{g} is the induced Riemannian metric on ∂M , then the volume form of \tilde{g} is*

$$\omega_{\tilde{g}} = \iota_{\partial M}^*(N \lrcorner \omega_g),$$

where N is the outward unit normal vector field along ∂M .

1.4.4 Orientations and covering maps

Although it is often easy to prove that a given smooth manifold is orientable by constructing an orientation for it, proving that a manifold is not orientable can be much trickier. The theory of covering spaces provides one of the most useful techniques for doing so. In this part, we explore the close relationship between orientability and covering maps.

Our first result is a simple application of pullback orientations.

Proposition 1.4.27. *If $\pi : E \rightarrow M$ is a smooth covering map and M is orientable, then E is also orientable.*

Proposition 1.4.28. *Because a covering map is a local diffeomorphism, this follows immediately from Proposition [1.4.8](#).* orientation pull back

The next theorem is more interesting. If G is a Lie group acting smoothly on a smooth manifold E (on the left, say), we say the action is an **orientation-preserving action** if for each $g \in G$, the diffeomorphism $x \mapsto g \cdot x$ is orientation-preserving.

Theorem 1.4.29. *Suppose E is a connected, oriented, smooth manifold with or without boundary, and $\pi : E \rightarrow M$ is a smooth normal covering map. Then M is orientable if and only if the action of $\text{Aut}_\pi(E)$ on E is orientation-preserving.* covering thm

Proof. Let O_E denote the given orientation on E . First suppose M is orientable, and let q be an arbitrary point in E . Because M is connected, it has exactly two orientations, and one of them has the property that $d\pi_q : T_q E \rightarrow T_{\pi(q)} M$ is orientation-preserving. Call that orientation O_M . The pullback orientation $\pi^* O_M$ agrees with the given orientation at q , so it must be equal to O_E by Corollary [1.4.3](#). Suppose $\varphi \in \text{Aut}_\pi(E)$. The fact that $\pi \circ \varphi = \pi$ implies that orientation connected

$$\varphi^* O_E = \varphi^*(\pi^* O_M) = (\pi \circ \varphi)^* O_M = \pi^* O_M = O_E.$$

Thus, φ is orientation-preserving.

Conversely, suppose the action of $\text{Aut}_\pi(E)$ is orientation-preserving, and let $p \in M$. If $U \subseteq M$ is any evenly covered neighborhood of p , there is a smooth section $\sigma : U \rightarrow E$, which induces an orientation σ^*O_E on U . Suppose $\sigma_1 : U \rightarrow E$ is any other smooth local section over U . Because π is a normal covering, $\text{Aut}_\pi(E)$ acts transitively on each fiber of π , so there is a covering automorphism φ such that $\sigma_1(p) = \varphi(\sigma(p))$. Then $\varphi \circ \sigma$ is a local section of π that agrees with σ_1 at p , and thus $\sigma_1 = \varphi \circ \sigma$ on all of U by Theorem 1.4.29 (local section unique). Because φ is orientation-preserving, $\sigma_1^*O_E = \sigma^*\varphi^*O_E = \sigma^*O_E$, so the orientations induced by σ and σ_1 are equal. Thus, we can define a global orientation O_M on M by defining it on each evenly covered open subset to be the pullback orientation induced by any local section; the argument above shows that the orientations so defined agree where they overlap. \square

Here are two applications of the preceding theorem.

Example 1.4.30 (Orientability of Projective Spaces). For $n \geq 1$, consider the smooth covering map $q : S^n \rightarrow \mathbb{RP}^n$. The only nontrivial covering automorphism of q is the antipodal map $\alpha(x) = -x$. It can be easily shown that α is orientation-preserving if and only if n is odd, so it follows that \mathbb{RP}^n is orientable if and only if n is odd.

Example 1.4.31 (The Möbius Bundle and the Möbius Band). Let E be the total space of the Möbius bundle (Example 1.4.28). The quotient map $q : \mathbb{R}^2 \rightarrow E$ used to define E is a smooth normal covering map, and the covering automorphism group is isomorphic to \mathbb{Z} , acting on \mathbb{R}^2 by

$$n(x, y) = (x + n, (-1)^n y)$$

For n odd, the diffeomorphism $(x, y) \mapsto n \cdot (x, y)$ of \mathbb{R}^2 pulls back the orientation form $dx \wedge dy$ to $-dx \wedge dy$, so the action of $\text{Aut}_q(E)$ is not orientation-preserving. Thus, Theorem 1.4.29 shows that E is not orientable.

For each $r > 0$, the image under q of the rectangle $[0, 1] \times [-r, r]$ is a Möbius band M_r . Because q restricts to a smooth covering map from $\mathbb{R} \times [-r, r]$, the same argument shows that a Möbius band is not orientable either.

The orientation covering

Next we show that every nonorientable smooth manifold M has an orientable two-sheeted covering manifold. The fiber over a point $p \in M$ will correspond to the two orientations of $T_p M$.

In order to handle the orientable and nonorientable cases in a uniform way, it is useful to expand our definition of covering maps slightly, by allowing covering spaces that are not connected. If N and M are topological spaces, let us say that a map $\pi : N \rightarrow M$ is a **generalized covering map** if it satisfies all of the requirements for a covering map except that N might not be connected: this means that N is locally path-connected, π is surjective and continuous, and each point $p \in M$ has a neighborhood that is evenly covered by π . If in addition N and M are smooth manifolds with or without boundary and π is a local diffeomorphism, we say it is a **generalized smooth covering map**.

Lemma 1.4.32. Suppose N and M are topological spaces and $\pi : N \rightarrow M$ is a generalized covering map. If M is connected, then the restriction of π to each component of N is a covering map.

Proof. Suppose W is a component of N . If U is any open subset of M that is evenly covered by π , then each component of $\pi^{-1}(U)$ is connected and therefore contained in a single component of N . It follows that $(\pi|_W)^{-1}(U) = \pi^{-1}(U) \cap W$ is either the empty set or a nonempty disjoint union of components of $\pi^{-1}(U)$, each of which is mapped homeomorphically onto U by $\pi|_W$. In particular, this means that each point in $\pi(W)$ has a neighborhood that is evenly covered by $\pi|_W$.

To complete the proof, we just need to show that $\pi|_W$ is surjective. Because π is a local homeomorphism, $\pi(W)$ is an open subset of M . On the other hand, if $p \in M - \pi(W)$, and U is a neighborhood of p that is evenly covered by π , then the discussion in the preceding paragraph shows that $(\pi|_W)^{-1}(U) = \emptyset$, which implies that $U \subseteq M - \pi(W)$. Therefore, $\pi(W)$ is closed in M . Because W is not empty, $\pi(W)$ is all of M . \square

Let M be a connected, smooth, positive-dimensional manifold with or without boundary, and let \widehat{M} denote the set of orientations of all tangent spaces to M :

$$\widehat{M} = \{(p, O_p) : p \in M \text{ and } O_p \text{ is an orientation of } T_p M\}.$$

Define the projection $\widehat{\pi} : \widehat{M} \rightarrow M$ by sending an orientation of $T_p M$ to the point p itself: $\widehat{\pi}(p, O_p) = p$. Since each tangent space has exactly two orientations, each fiber of this map has cardinality 2. The map $\widehat{\pi} : \widehat{M} \rightarrow M$ is called the **orientation covering** of M .

on cover prop

Proposition 1.4.33 (Properties of the Orientation Covering). *Suppose M is a connected, smooth, positive-dimensional manifold with or without boundary, and let $\widehat{\pi} : \widehat{M} \rightarrow M$ be its orientation covering. Then \widehat{M} can be given the structure of a smooth, oriented manifold with or without boundary, with the following properties:*

- (a) $\widehat{\pi} : \widehat{M} \rightarrow M$ is a generalized smooth covering map.
- (b) A connected open subset $U \subseteq M$ is evenly covered by $\widehat{\pi}$ if and only if U is orientable.
- (c) If $U \subseteq M$ is an evenly covered open subset, then every orientation of U is the pullback orientation induced by a local section of $\widehat{\pi}$ over U .

Proof. We first topologize \widehat{M} by defining a basis for it. For each pair (U, O) , where U is an open subset of M and O is an orientation on U , define a subset $\widehat{U}_O \subseteq \widehat{M}$ as follows:

$$\widehat{U}_O = \{(p, O_p) \in \widehat{M} : p \in U \text{ and } O_p \text{ is the orientation of } T_p M \text{ determined by } O\}.$$

We will show that the collection of all subsets of the form \widehat{U}_O is a basis for a topology on \widehat{M} .

- Given an arbitrary point $(p, O_p) \in \widehat{M}$, let U be an orientable neighborhood of p in M , and let O be an orientation on it. After replacing O by $-O$ if necessary, we may assume that the given orientation O_p is same as the orientation of $T_p M$ determined by O . It follows that $(p, O_p) \in \widehat{U}_O$, so the collection of all sets of the form \widehat{U}_O covers \widehat{M} .
- If \widehat{U}_O and $\widehat{U}'_{O'}$ are two such sets and (p, O_p) is a point in their intersection, then O_p is the orientation of $T_p M$ determined by both O and O' . If V is the component of $U \cap U'$ containing p , then the restricted orientations $O|_V$ and $O'|_V$ agree at p and therefore are identical by Proposition 1.4.3, so it follows that $\widehat{U}_O \cap \widehat{U}'_{O'}$ contains the basis set $\widehat{V}_{O|_V}$.

Thus, we have defined a topology on \widehat{M} .

Note that for each orientable open subset $U \subseteq M$ and each orientation O of U , $\hat{\pi}$ maps the basis set \hat{U}_O bijectively onto U . Because the orientable open subsets form a basis for the topology of M , this implies that $\hat{\pi}$ restricts to a homeomorphism from \hat{U}_O to U . In particular, $\hat{\pi}$ is a local homeomorphism.

Next we show that with this topology, $\hat{\pi}$ is a generalized covering map. Suppose $U \subseteq M$ is an orientable connected open subset and O is an orientation for U . Then $\hat{\pi}^{-1}(U)$ is the disjoint union of open subsets \hat{U}_O and \hat{U}_{-O} , and $\hat{\pi}$ restricts to a homeomorphism from each of these sets to U . Thus, each such set U is evenly covered, and it follows that $\hat{\pi}$ is a generalized covering map. By Lemma 1.4.32, $\hat{\pi}$ restricts to an ordinary covering map on each component of \hat{M} , and so Proposition 1.4.33 shows that each such component is a topological n -manifold with or without boundary and has a unique smooth structure making $\hat{\pi}$ into a smooth covering map. These smooth structures combine to give a smooth structure on all of \hat{M} . This completes the proof of (a).

Next we give \hat{M} an orientation. Let $\hat{p} = (p, O_p)$ be a point in \hat{M} . By definition, O_p is an orientation of $T_p M$, so we can give $T_{\hat{p}} \hat{M}$ the unique orientation $\hat{O}_{\hat{p}}$ such that $d\hat{\pi}_{\hat{p}} : T_{\hat{p}} \hat{M} \rightarrow T_p M$ is orientation-preserving. This defines a pointwise orientation \hat{O} on \hat{M} . On each basis open subset \hat{U}_O , the orientation \hat{O} agrees with the pullback orientation induced from (U, O) by (the restriction of) $\hat{\pi}$, so it is continuous.

Next we prove (b). We showed earlier that every orientable connected open subset of M is evenly covered by $\hat{\pi}$. Conversely, if $U \subseteq M$ is any evenly covered open subset, then there is a smooth local section $\sigma : U \rightarrow \hat{M}$ by Proposition 1.4.33, which pulls O back to an orientation on U by Proposition 1.4.8.

Finally, to prove (c), assume $U \subseteq M$ is evenly covered and therefore orientable. Given any orientation O of U , define a section $\sigma : U \rightarrow \hat{M}$ by setting $\sigma(p) = (p, O_p)$. To see that σ is continuous, suppose \hat{U}'_O is any basis open subset of \hat{M} . Then for each component V of $U \cap U'$, the restricted orientations $O|_V$ and $O'|_V$ must either agree or disagree on all of V , so $\sigma^{-1}(\hat{U}'_O)$ is a union of such components and therefore open. \square

Theorem 1.4.34 (Orientation Covering Theorem). *Suppose M is a connected smooth manifold with or without boundary, and let $\hat{\pi} : \hat{M} \rightarrow M$ be its orientation covering.*

- (a) *If M is orientable, then \hat{M} has exactly two components, and the restriction of $\hat{\pi}$ to each component is a diffeomorphism onto M .*
- (b) *If M is nonorientable, then \hat{M} is connected, and $\hat{\pi}$ is a two-sheeted smooth covering map.*

Proof. If M is orientable, then Proposition 1.4.33(b) shows that \hat{M} is evenly covered by $\hat{\pi}$, which means that \hat{M} has two components, each mapped diffeomorphically onto M .

Now assume M is nonorientable. We show first that \hat{M} is connected. Let W be a component of \hat{M} . Lemma 1.4.32 shows that $\hat{\pi}|_W$ is a covering map, so its fibers all have the same cardinality. Because the fibers of $\hat{\pi}$ have cardinality 2 and W is not empty, the fibers of $\hat{\pi}|_W$ must have cardinality 1 or 2. If the cardinality were 1, then $\hat{\pi}|_W$ would be an injective smooth covering map and thus a diffeomorphism, and its inverse would be a smooth section of $\hat{\pi}$, which would induce an orientation on M . Thus, the cardinality must be 2, which implies that $W = \hat{M}$. Because \hat{M} is connected, $\hat{\pi}$ is a covering map by Lemma 1.4.32, and because it is a local diffeomorphism it is a smooth covering map. \square

The orientation covering is sometimes called the **oriented double covering** of M . There are other ways of constructing it besides the one we have given here, but as the next theorem shows, the specific details of the construction do not matter, because they all yield isomorphic covering manifolds.

on cover char

Theorem 1.4.35 (Characteristic property of the orientation covering). *Let M be a connected nonorientable smooth manifold with or without boundary, and let $\hat{\pi} : \hat{M} \rightarrow M$ be its orientation covering. If X is any oriented smooth manifold with or without boundary, and $F : X \rightarrow M$ is any local diffeomorphism, then there exists a unique orientation-preserving local diffeomorphism $\hat{F} : X \rightarrow \hat{M}$ such that $\hat{\pi} \circ \hat{F} = F$:*

$$\begin{array}{ccc} & & \hat{M} \\ & \nearrow \hat{F} & \downarrow \hat{\pi} \\ X & \xrightarrow{F} & M \end{array}$$

Proof. At each point $p = F(x) \in M$, since F is a local diffeomorphism, we get a unique orientation of $T_p M$, which we denote by O_p . Then we can define

$$\hat{F}(x) = (F(x), O_{F(x)}).$$

It is easy to check that $\hat{\pi} \circ \hat{F} = F$. Since $\hat{\pi}$ is also a local diffeomorphism, it follows that \hat{F} is also a local diffeomorphism. The orientation-preserving statement is clear from the definition. \square

cover unique

Theorem 1.4.36 (Uniqueness of the Orientation Covering). *Let M be a nonorientable connected smooth manifold with or without boundary, and let $\hat{\pi} : \hat{M} \rightarrow M$ be its orientation covering. If \tilde{M} is an oriented smooth manifold with or without boundary that admits a two-sheeted smooth covering map $\tilde{\pi} : \tilde{M} \rightarrow M$, then there exists a unique orientation-preserving diffeomorphism $\varphi : \tilde{M} \rightarrow \hat{M}$ such that $\hat{\pi} \circ \varphi = \tilde{\pi}$.*

By invoking a little more covering space theory, we obtain the following sufficient topological condition for orientability.

Theorem 1.4.37. *Let M be a connected smooth manifold with or without boundary, and suppose the fundamental group of M has no subgroup of index 2. Then M is orientable. In particular, if M is simply connected then it is orientable.*

Proof. Suppose M is not orientable, and let $\hat{\pi} : \hat{M} \rightarrow M$ be its orientation covering, which is an honest covering map in this case. Choose any point $q \in \hat{M}$, and let $p = \hat{\pi}(q) \in M$. Let $\alpha : \hat{M} \rightarrow \hat{M}$ be the map that interchanges the two points in each fiber of $\hat{\pi}$. To prove that α is smooth, suppose $U \subseteq M$ is any evenly covered open subset and $U_0, U_1 \subseteq \hat{M}$ are the two components of $\hat{\pi}^{-1}(U)$. Since $\hat{\pi}$ restricts to a diffeomorphism from each component onto U , we can write $\alpha|_{U_0} = (\hat{\pi}|_{U_1})^{-1} \circ (\hat{\pi}|_{U_0})$, which is smooth. Similarly, $\alpha|_{U_1}$ is also smooth. Since the collection of all such sets U_0, U_1 is an open covering of \hat{M} , it follows that α is smooth, and it is a covering automorphism because it satisfies $\hat{\pi} \circ \alpha = \hat{\pi}$. In fact, since a covering automorphism is determined by what it does to one point, α is the unique nontrivial element of the automorphism group $\text{Aut}_{\hat{\pi}}(\hat{M})$, which is therefore equal to the two-element group $\{\text{id}_{\hat{M}}, \alpha\}$. Because the automorphism group acts transitively on fibers, $\hat{\pi}$ is a normal covering

map. A fundamental result in the theory of covering spaces is that

$$\text{Aut}_{\widehat{\pi}}(\widehat{M}) \cong \frac{\pi_1(M, p)}{\widehat{\pi}^*(\pi_1(\widehat{M}, q))}.$$

Therefore, $\pi_1(M, p)$ has an index 2 subgroup. \square

1.4.5 Exercise

Exercise 1.4.1. Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable.

Proof. Use Proposition [1.4.3](#). \square

Exercise 1.4.2. Suppose M and N are oriented smooth manifolds with or without boundary, and $F : M \rightarrow N$ is a local diffeomorphism. Show that if M is connected, then F is either orientation-preserving or orientation-reversing.

Exercise 1.4.3. Suppose $n \geq 1$, and let $\alpha : S^n \rightarrow S^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.

Proof. Consider the orientation form on \bar{B}^{n+1} :

$$\omega = dx^1 \wedge \cdots \wedge dx^{n+1}.$$

The induced orientation on S^n is then given by

$$\omega_{S^n} = \iota^*(N \lrcorner \omega),$$

where $N = x^i \partial / \partial x^i$. Let $F : \bar{B}^{n+1} \rightarrow \bar{B}^{n+1}$ be the map $F(x) = -x$, then $F_*\omega = (-1)^{n+1}\omega$. Since the map α preserves that vector field N , and the orientation of S^n is induced by N , we conclude that α is orientation-preserving if and only if F is orientation-preserving. \square

Exercise 1.4.4. Let θ be a smooth flow on an oriented smooth manifold with or without boundary. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined.

Proof. The map $\theta : \mathbb{R} \rightarrow \text{Diff}(M)$ is a smooth group homomorphism, and hence its image is connected. Since $\theta_0 = \text{id}_M$, this curve lies in the orientation preserving component of $\text{Diff}(M)$. \square

Exercise 1.4.5. Let M be a smooth manifold with or without boundary. Show that the total spaces of TM and T^*M are orientable.

Proof. We prove for TM . By Proposition [1.4.3](#), the transition map between charts (U, φ) and (V, ψ) of M is

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = (\widetilde{x}^1(x), \dots, \widetilde{x}^n(x), v^i \frac{\partial \widetilde{x}^1}{\partial x^i}(x), \dots, v^i \frac{\partial \widetilde{x}^n}{\partial x^i}(x)).$$

The Jacobian matrix of this map is

$$\partial(\widetilde{\psi} \circ \widetilde{\varphi}^{-1})(x, v) = \begin{pmatrix} \frac{\partial \widetilde{x}^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial \widetilde{x}^j}{\partial x^i}(x) \end{pmatrix}.$$

Thus

$$\det \partial(\tilde{\psi} \circ \tilde{\varphi}^{-1}) = (\det \partial(\psi \circ \varphi^{-1}))^2,$$

which, by Proposition [1.4.4](#), implies TM is orientable. \square

Exercise 1.4.6. Suppose M is an oriented Riemannian manifold with or without boundary, and $S \subseteq M$ is an oriented smooth hypersurface with or without boundary. Show that there is a unique smooth unit normal vector field along S that determines the given orientation of S .

Exercise 1.4.7. Suppose M is an orientable Riemannian manifold, and $S \subseteq M$ is an immersed or embedded submanifold with or without boundary. Prove the following statements.

- (a) If S has trivial normal bundle, then S is orientable.
- (b) If S is an orientable hypersurface, then S has trivial normal bundle.

Proof. Let S has codimension k . If S has trivial normal bundle, then there is an diffeomorphism $NS \rightarrow S \times \mathbb{R}^k$. Thus we have global frames $(\sigma_1, \dots, \sigma_k)$ of NS . By Proposition [1.4.4](#), at each point p we can find local frames $(\sigma_{k+1}, \dots, \sigma_n)$ such that $(\sigma_1, \dots, \sigma_n)$ is an oriented frame for $T_p M$. Thus S is orientable. \square

If S is an orientable hypersurface, then there is a unit normal vector field along S . Since this is a global frame for NS , we get a global trivialization for NS . \square

Exercise 1.4.8. Show that every orientation-reversing diffeomorphism of \mathbb{R} has a fixed point.

Exercise 1.4.9. Let M be a connected smooth 1-manifold. Show that M is diffeomorphic to either \mathbb{R} or S^1 , as follows:

- (a) First, do the case in which M is orientable by showing that M admits a nonvanishing smooth vector field.
- (b) Now let M be arbitrary, and prove that M is orientable by showing that its universal covering manifold is diffeomorphic to \mathbb{R} .

Conclude that the smooth structures on both \mathbb{R} and S^1 are unique up to diffeomorphism.

Proof. The only 1-manifolds are \mathbb{R} and S^1 , thus orientable. \square

Exercise 1.4.10. Show that every connected smooth 1-manifold with nonempty boundary is diffeomorphic to either $[0, \infty)$ or $[0, 1]$.

Exercise 1.4.11. Let M be a nonorientable embedded hypersurface in \mathbb{R}^n , and let NM be its normal bundle with projection $\pi_{NM} : NM \rightarrow M$. Show that the set

$$W = \{(x, v) \in NM : |v| = 1\}$$

is an embedded submanifold of NM , and the restriction of π_{NM} to W is a smooth covering map isomorphic to the orientation covering of M .

Proof. We can show that W is a orientable manifold, and the map $\pi_{NM} : W \rightarrow M$ is a 2-sheet covering. Thus there is a diffeomorphism from W to \hat{M} by Theorem [1.4.36](#). \square

Exercise 1.4.12. Let E be the total space of the Möbius bundle as in Example [1.4.35](#). Show that the orientation covering of E is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. \square

Proof. Let $S^1 \times \mathbb{R}$ be embedded into \mathbb{R}^3 , we can apply the antipodal map α on it to obtain E . This gives a 2-sheet cover of E , and is the orientation covering by Theorem [1.4.36](#). \square

1.5 Integration on manifolds

1.5.1 Integration of differential forms

In this part we will apply the integration theory on manifolds. In particular, we need the Lebesgue integration in \mathbb{R}^n . Let $D \subseteq \mathbb{R}^n$ be a measurable set, and let ω be a continuous n -form on \bar{D} . Any such form can be written as $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some continuous function $f : \bar{D} \rightarrow \mathbb{R}$. We define the **integral of ω over D** to be

$$\int_D \omega = \int_D f dV.$$

This can be written more suggestively as

$$\int_D f dx^1 \wedge \cdots \wedge dx^n = \int_D f dx^1 \cdots dx^n.$$

Somewhat more generally, let U be an open subset of \mathbb{R}^n or \mathbb{H}^n , and suppose ω is a compactly supported n -form on U . We define

$$\int_U \omega = \int_D \omega,$$

where $D \subseteq \mathbb{R}^n$ or \mathbb{H}^n is any measurable set containing $\text{supp}(\omega)$, and ω is extended to be zero on the complement of its support. It is easy to check that this definition does not depend on what domain D is chosen.

Like the definition of the integral of a 1-form over an interval, our definition of the integral of an n -form might look like a trick of notation. The next proposition shows why it is natural.

Proposition 1.5.1. *Suppose U, V are open subsets of \mathbb{R}^n or \mathbb{H}^n , and $G : U \rightarrow V$ is an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n -form on V , then*

$$\int_V \omega = \pm \int_U G^* \omega.$$

with the positive sign if G is orientation-preserving, and the negative sign otherwise.

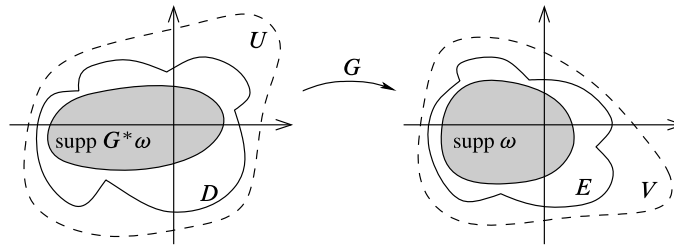


Figure 1.1: Diffeomorphism invariance of the integral of a form on an open subset.

Proof. Let us use (y^1, \dots, y^n) to denote standard coordinates on V , and (x^1, \dots, x^n) to denote those on U . Suppose first that G is orientation-preserving. With $\omega = f dy^1 \wedge \cdots \wedge dy^n$, the change of variables formula together with formula (3.22) for pullbacks of n -forms yields

$$\int_V \omega = \int_V f dV = \int_U (f \circ G) |\det \partial G| dV = \int_U (f \circ G) (\det \partial G) dV$$

$$= \int_U (f \circ G)(\det \partial G) dx^1 \wedge \cdots \wedge dx^n = \int_U G^* \omega.$$

If G is orientation-reversing, the same computation holds except that a negative sign is introduced when the absolute value signs are removed. \square

Integration on manifolds

Let M be an oriented smooth n -manifold with or without boundary, and let ω be an n -form on M . Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented. We define the integral of ω over M to be

$$\int_M \omega = \pm \int_{\varphi(U)} \varphi_* \omega, \quad (5.32)$$

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with the positive sign for a positively oriented chart, and the negative sign otherwise. Since

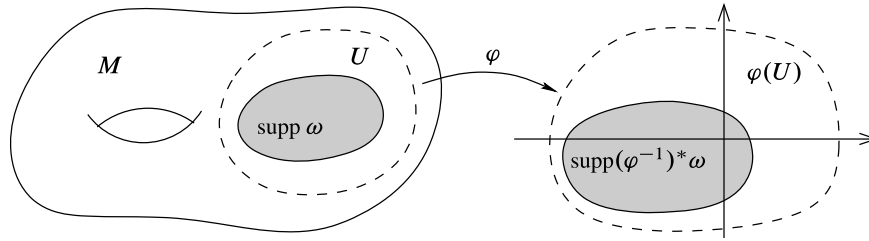


Figure 1.2: The integral of a form over a manifold.

$\varphi_* \omega$ is a compactly supported n -form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or \mathbb{H}^n , its integral is defined as discussed above.

orm one chart

Proposition 1.5.2. *With ω as above, $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains $\text{supp}(\omega)$.*

Proof. Suppose (U, φ) and $(\tilde{U}, \tilde{\varphi})$ are two smooth charts such that $\text{supp}(\omega) \subseteq U \cap \tilde{U}$. If both charts are positively oriented or both are negatively oriented, then $\tilde{\varphi} \circ \varphi^{-1}$ is an orientation-preserving diffeomorphism from $\varphi(U \cap \tilde{U})$ to $\tilde{\varphi}(U \cap \tilde{U})$, so Proposition 1.5.1 implies that

$$\begin{aligned} \int_{\tilde{\varphi}(\tilde{U})} \tilde{\varphi}_* \omega &= \int_{\tilde{\varphi}(U \cap \tilde{U})} \tilde{\varphi}_* \omega = \int_{\varphi(U \cap \tilde{U})} (\tilde{\varphi} \circ \varphi^{-1})^* \tilde{\varphi}_* \omega \\ &= \int_{\varphi(U \cap \tilde{U})} (\varphi^{-1})^* \circ \tilde{\varphi}^* \circ \tilde{\varphi}_* \omega = \int_{\varphi(U)} \varphi_* \omega. \end{aligned}$$

If the charts are oppositely oriented, then the two definitions given by (5.32) have opposite signs, but this is compensated by the fact that $\tilde{\varphi} \circ \varphi^{-1}$ is orientation-reversing, so Proposition 1.5.1 introduces an extra negative sign into the computation above. In either case, the two definitions of $\int_M \omega$ agree. \square

To integrate over an entire manifold, we combine this definition with a partition of unity. Suppose M is an oriented smooth n -manifold with or without boundary, and ω is a compactly supported n -form on M . Let $\{U_i\}$ be a finite open cover of $\text{supp}(\omega)$ by domains of positively

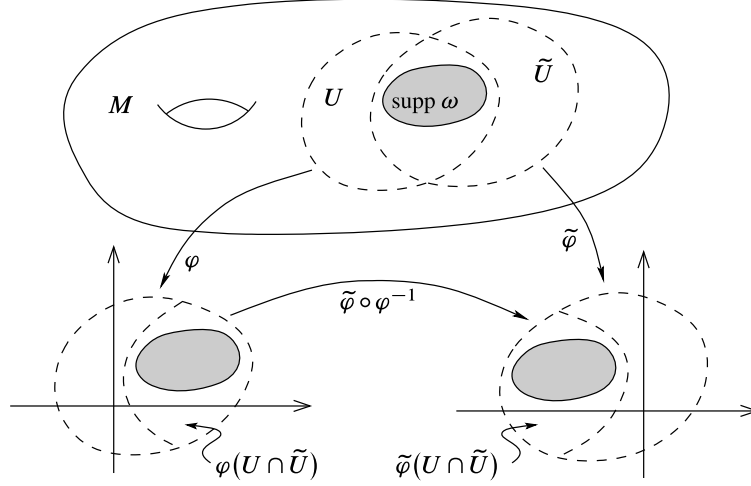


Figure 1.3: Coordinate independence of the integral.

or negatively oriented smooth charts, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. Define the integral of ω over M to be

$$\int_M \omega = \sum_i \int_{U_i} \psi_i \omega. \quad (5.33) \quad \boxed{\text{int mani def-}}$$

(The reason we allow for negatively oriented charts is that it may not be possible to find positively oriented boundary charts on a 1-manifold with boundary, as noted in the proof of Proposition [1.4.4](#).) ^{orientation by atlas} Since for each i , the n -form $\psi_i \omega$ is compactly supported in U_i , each of the terms in this sum is well defined according to our discussion above. To show that the integral is well defined, we need only examine the dependence on the open cover and the partition of unity.

Proposition 1.5.3. *The definition of $\int_M \omega$ given above does not depend on the choice of open cover or partition of unity.*

Proof. Suppose $\{\tilde{U}_j\}$ is another finite open cover of $\text{supp}(\omega)$ by domains of positively or negatively oriented smooth charts, and $\{\tilde{\psi}_j\}$ is a subordinate smooth partition of unity. For each i , we compute

$$\int_M \psi_i \omega = \int_M \left(\sum_j \tilde{\psi}_j \right) \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \psi_i \omega.$$

Summing over i , we obtain

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \tilde{\psi}_j \psi_i \omega.$$

Observe that each term in this last sum is the integral of a form that is compactly supported in a single smooth chart (e.g., in U_i), so by Proposition [1.5.2](#) ^{integrate from one chart} each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \tilde{\psi}_j \omega$, shows that

$$\sum_j \int_M \tilde{\psi}_j \omega = \sum_{i,j} \int_M \tilde{\psi}_j \psi_i \omega.$$

Thus, both definitions yield the same value for $\int_M \omega$. \square

As usual, we have a special definition in the zero-dimensional case. The integral of a compactly supported 0-form (i.e., a real-valued function) f over an oriented 0-manifold M is defined to be the sum

$$\int_M f = \sum_{p \in M} \pm f(p).$$

where we take the positive sign at points where the orientation is positive and the negative sign at points where it is negative. The assumption that f is compactly supported implies that there are only finitely many nonzero terms in this sum.

If $S \subseteq M$ is an oriented immersed k -dimensional submanifold (with or without boundary), and ω is a k -form on M whose restriction to S is compactly supported, we interpret $\int_S \omega$ to mean $\int_S \iota_S^* \omega$, where $\iota_S : S \hookrightarrow M$ is inclusion. In particular, if M is a compact, oriented, smooth n -manifold with boundary and ω is an $(n-1)$ -form on M , we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $\iota_{\partial M}^* \omega$ over ∂M , where ∂M is always understood to have the induced orientation.

Remark 1.5.1. It is worth remarking that it is possible to extend the definition of the integral to some noncompactly supported forms, and such integrals are important in many applications. However, in such cases the resulting multiple integrals are improper, so one must pay close attention to convergence issues. For the purposes we have in mind, the cases we have described here are quite sufficient.

nt form prop

Proposition 1.5.4 (Properties of Integrals of Forms). Suppose M and N are nonempty oriented smooth n -manifolds with or without boundary, and ω, η are compactly supported n -forms on M .

(a) *Linearity:* If $a, b \in \mathbb{R}$, then

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

(b) *Orientation reversal:* If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = - \int_M \omega.$$

(c) *Positivity:* If ω is a positively oriented orientation form, then $\int_M \omega > 0$.

(d) *Diffeomorphism invariance:* If $F : M \rightarrow N$ is an orientation-preserving or orientation-reversing diffeomorphism, then

$$\int_M F^* \omega = \begin{cases} \int_N \omega & \text{if } F \text{ is orientation-preserving,} \\ - \int_N \omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Proof. Suppose ω is a positively oriented orientation form for M . This means that if (U, φ) is a positively oriented smooth chart, then $\varphi_* \omega$ is a positive function times $dx^1 \wedge \cdots \wedge dx^n$, and for a negatively oriented chart it is a negative function times the same form. Therefore, each term in the sum defining $\int_M \omega$ is nonnegative, with at least one strictly positive term, thus proving (c).

To prove (d), it suffices to assume that ω is compactly supported in a single positively or negatively oriented smooth chart, because any compactly supported n -form on N can be written as a finite sum of such forms by means of a partition of unity. Thus, suppose (U, φ) is a positively oriented smooth chart on N whose domain contains the support of ω . When F is orientation-preserving, it is easy to check that $(F^{-1}(U), \varphi \circ F)$ is an oriented smooth chart on M whose domain contains the support of $F^*\omega$, and the result then follows immediately from Proposition 1.5.2. The cases in which the chart is negatively oriented or F is orientation-reversing then follow from this result together with (b). \square

Although the definition of the integral of a form based on partitions of unity is very convenient for theoretical purposes, it is useless for doing actual computations. It is generally quite difficult to write down a smooth partition of unity explicitly, and even when one can be written down, one would have to be exceptionally lucky to be able to compute the resulting integrals.

For computational purposes, it is much more convenient to *chop up* the manifold into a finite number of pieces whose boundaries are sets of measure zero, and compute the integral on each piece separately by means of local parametrizations. One way to do this is described below.

Proposition 1.5.5 (Integration Over Parametrizations). *Let M be an oriented smooth n -manifold with or without boundary, and ω be a compactly supported n -form on M . Suppose that D_1, \dots, D_k are open sets in \mathbb{R}^n , and for $1 \leq i \leq k$, we are given smooth maps $F_i : D_i \rightarrow M$ satisfying*

- (i) F_i restricts to an orientation-preserving diffeomorphism from D_i onto an open subset $W_i \subseteq M$.
- (ii) $D_i \cap D_j = \emptyset$ whenever $i \neq j$.
- (iii) ∂D_i has Lebesgue measure zero for each i .
- (iv) $\text{supp}(\omega) \subseteq \bigcup_{i=1}^k \overline{W}_i$.

Then

$$\int_M \omega = \sum_{i=1}^k \int_{D_i} F_i^* \omega. \quad (5.34)$$

Proof. As in the preceding proof, it suffices to assume that ω is supported in the domain of a single oriented smooth chart (U, φ) . In fact, by restricting to sufficiently nice charts, we may assume that U is precompact, $Y = \varphi(U)$ is open in \mathbb{R}^n or \mathbb{H}^n , and φ extends to a diffeomorphism from \overline{U} to \overline{Y} .

For each i , define open subsets $A_i \subseteq D_i$, $B_i \subseteq W_i$, and $C_i \subseteq Y$ by

$$A_i = F_i^{-1}(U \cap W_i), \quad B_i = U \cap W_i = F_i(A_i), \quad C_i = \varphi(B_i) \varphi \circ F_i(A_i).$$

Because \overline{D}_i is compact, it is straightforward to check that $\partial W_i \subseteq F_i(\partial D_i)$, and therefore ∂W_i has measure zero in M , and $\partial C_i = \varphi(\partial B_i)$ has measure zero in \mathbb{R}^n .

The support of $\varphi_*\omega$ is contained in $\bigcup_{i=1}^k \overline{C}_i$, and any two of these sets intersect only on their boundaries, which have measure zero. Thus

$$\int_M \omega = \int_Y \varphi_*\omega = \sum_{i=1}^k \int_{C_i} \varphi_*\omega.$$

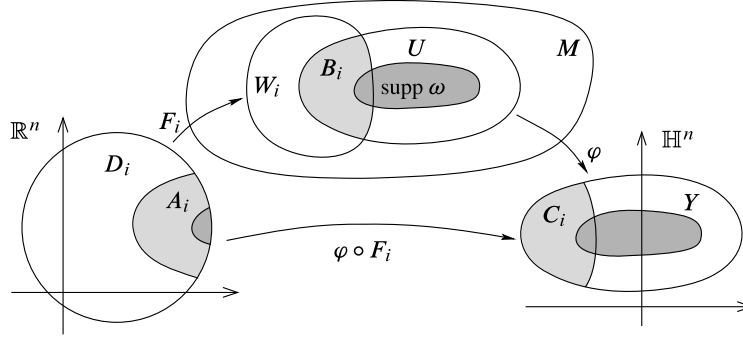


Figure 1.4: Integrating over parametrizations.

The proof is completed by applying Proposition [1.5.1](#) to each term above, using the diffeomorphism $\varphi \circ F_i : A_i \rightarrow C_i$:

$$\int_{C_i} \varphi_* \omega = \int_{A_i} (\varphi \circ F_i)^* \varphi_* \omega = \int_{A_i} F_i^* \omega = \int_{D_i} F_i^* \omega.$$

Summing over i , we obtain [\(5.34\)](#). □

Example 1.5.6. Let us use this technique to compute the integral of a 2-form over S^2 , oriented as the boundary of \bar{B}^3 . Let ω be the following 2-form on \mathbb{R}^3

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

Let D be the open rectangle $(0, \pi) \times (0, 2\pi)$, and let $F : D \rightarrow S^2$ be the spherical coordinate parametrization $F(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$. Example [1.4.20](#) showed that $F|_D$ is orientation-preserving, so it satisfies the hypotheses of Proposition [1.5.5](#). Note that

$$F^* dx = \cos \varphi \cos \theta \, d\varphi - \sin \varphi \sin \theta \, d\theta,$$

$$F^* dy = \cos \varphi \sin \theta \, d\varphi + \sin \varphi \cos \theta \, d\theta,$$

$$F^* dz = -\sin \varphi \, d\varphi.$$

Therefore,

$$\begin{aligned} \int_{S^2} \omega &= \int_D (-\sin^3 \varphi \cos^2 \theta \, d\theta \wedge d\varphi + \sin^3 \varphi \sin^2 \theta \, d\varphi \wedge d\theta \\ &\quad + \cos^3 \varphi \sin \theta \cos^2 \theta \, d\varphi \wedge d\theta - \cos^2 \varphi \sin \varphi \sin^2 \theta \, d\theta \wedge d\varphi) \\ &= \int_D \sin \varphi \, d\varphi \wedge d\theta = \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta = 4\pi. \end{aligned}$$

Remark 1.5.2. It is worth remarking that the hypotheses of Proposition [1.5.5](#) can be relaxed somewhat. The requirement that each map F_i be smooth on \bar{D}_i is included to ensure that the boundaries of the image sets W_i have measure zero and that the pullback forms $F_i^* \omega$ are continuous on \bar{D}_i . Provided the open subsets W_i together fill up all of M except for a set of measure zero, we can allow maps F_i that do not extend smoothly to the boundary, by interpreting the resulting integrals of unbounded forms either as improper Riemann integrals or as Lebesgue integrals. For example, if the closed upper hemisphere of S^2 is parametrized by the map $F : \bar{B}^2 \rightarrow S^2$ given by $F(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, then F is continuous but not smooth up to the boundary, but the conclusion of the proposition still holds.

Integration on Lie groups

Let G be a Lie group. A covariant tensor field A on G is said to be **left-invariant** if $L_g^* A = A$ for all $g \in G$.

Proposition 1.5.7. *Let G be a compact Lie group endowed with a left-invariant orientation. Then G has a unique positively oriented left-invariant n -form ω_G with the property that $\int_G \omega_G = 1$.*

Proof. If $\dim G = 0$, we just let ω_G be the constant function $1/k$, where k is the cardinality of G . Otherwise, let E_1, \dots, E_n be a left-invariant global frame on G (i.e., a basis for the Lie algebra of G). By replacing E_1 with $-E_1$ if necessary, we may assume that this frame is positively oriented. Let $\varepsilon^1, \dots, \varepsilon^n$ be the dual coframe. Left invariance of E_j implies that

$$(L_g^*)(E_j) = \varepsilon^i(d(L_g)(E_j)) = \varepsilon^i(E_j) = \delta_j^i,$$

which shows that $L_g^* \varepsilon^i = \varepsilon^i$, so ε^i is left-invariant.

Let $\omega_G = \varepsilon^1 \wedge \dots \wedge \varepsilon^n$. Then

$$L_g^* \omega_G = L_g^* \varepsilon^1 \wedge \dots \wedge L_g^* \varepsilon^n = \varepsilon^1 \wedge \dots \wedge \varepsilon^n = \omega_G,$$

so ω_G is left-invariant as well. Because $\omega_G(E_1, \dots, E_n) > 0$, ω_G is an orientation form for the given orientation. Clearly, any positive constant multiple of ω_G is also a left-invariant orientation form. Conversely, if $\tilde{\omega}_G$ is any other left-invariant orientation form, we can write $\tilde{\omega}_G|_e = c\omega_G|_e$ for some positive number c . Using left-invariance, we find that

$$\tilde{\omega}_G|_g = L_{g^{-1}}^* \tilde{\omega}_G|_e = c L_{g^{-1}}^* \omega_G|_e = c\omega_G|_g,$$

which proves that $\tilde{\omega}_G$ is a positive constant multiple of ω_G .

Since G is compact and oriented, $\int_G \omega_G$ is a positive real number, so we can define $\tilde{\omega}_G = (\int_G \omega_G)^{-1} \omega_G$. Clearly, $\tilde{\omega}_G$ is the unique positively oriented left-invariant orientation form with integral 1. \square

Remark 1.5.3. The orientation form whose existence is asserted in this proposition is called the **Haar volume form** on G . Similarly, the map $f \mapsto \int_G f \omega_G$ is called the **Haar integral**. Observe that the proof above did not use the fact that G was compact until the last paragraph; thus every Lie group has a left-invariant orientation form that is uniquely defined up to a constant multiple. It is only in the compact case, however, that we can use the volume normalization to single out a unique one.

1.5.2 Stokes's theorem

Stokes

Theorem 1.5.8 (Stokes's Theorem). *Let M be an oriented smooth n -manifold with boundary, and let ω be a compactly supported smooth $(n-1)$ -form on M . Then*

$$\int_M d\omega = \int_{\partial M} \omega. \quad (5.35) \quad \text{Stokes-1}$$

Remark 1.5.4. The statement of this theorem is concise and elegant, but it requires a bit of interpretation. First, as usual, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota_{\partial M}^* \omega$. If $\partial M = \emptyset$, then the right-hand side is to be interpreted as zero. When M is 1-dimensional, the right-hand integral is really just a finite sum.

Proof. We begin with a very special case: suppose M is the upper half-space \mathbb{H}^n itself. Then because ω has compact support, there is a number $R > 0$ such that $\text{supp}(\omega)$ is contained in the rectangle $A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$. We can write ω in standard coordinates as

$$\omega = \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x_j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus we compute

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^n \int_A (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n. \end{aligned}$$

We can change the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$\begin{aligned} &\sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \left[\omega_i(x) \right] \Big|_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0. \end{aligned}$$

because we have chosen R large enough that $\omega = 0$ when $x^i = R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= (-1)^{n-1} \int_{-R}^R \int_{-R}^R \cdots \int_0^R \frac{\partial \omega_n}{\partial x^n}(x) dx^n dx^1 \cdots dx^{n-1} \\ &= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \left[\omega_n(x) \right] \Big|_{x^n=0}^{x^n=R} dx^n dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}. \end{aligned} \tag{5.36} \quad \boxed{\text{Stokes-2}}$$

because $\omega_n = R$ when $x^n = R$.

To compare this to the other side of [\(5.35\)](#), we compute as follows:

$$\int_{\partial \mathbb{H}^n} \omega = \sum_i \int_{A \cap \partial \mathbb{H}^n} \omega_i(x^1, \dots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Because x^n vanishes on $\partial \mathbb{H}^n$, the pullback of dx^n to the boundary is identically zero. Thus,

the only term above that is nonzero is the one for which $i = n$, which becomes

$$\int_{\partial \mathbb{H}^n} \omega = \int_{A \cap \partial \mathbb{H}^n} \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \wedge \dots \wedge dx^{n-1}.$$

Taking into account the fact that the coordinates (x^1, \dots, x^{n-1}) are positively oriented for $\partial \mathbb{H}^n$ when n is even and negatively oriented when n is odd (Example 1.4.18), we find that this is equal to (Stokes-2) (5.36).

Next we consider another special case: $M = \mathbb{R}^n$. In this case, the support of ω is contained in a cube of the form $A = [-R, R]^n$. Exactly the same computation goes through, except that in this case the $i = n$ term vanishes like all the others, so the left-hand side of (Stokes-1) (5.35) is zero. Since M has empty boundary in this case, the right-hand side is zero as well.

Now let M be an arbitrary smooth manifold with boundary, but consider an $(n-1)$ -form ω that is compactly supported in the domain of a single positively or negatively oriented smooth chart (U, φ) . Assuming that φ is a positively oriented boundary chart, the definition yields

$$\int_M d\omega = \int_{\mathbb{H}^n} \varphi_* d\omega = \int_{\mathbb{H}^n} d(\varphi_* \omega).$$

By the computation above, this is equal to

$$\int_{\partial \mathbb{H}^n} \varphi_* \omega. \quad (5.37) \quad \text{Stokes-3}$$

where $\partial \mathbb{H}^n$ is given the induced orientation. Since $d\varphi$ takes outward-pointing vectors on ∂M to outward-pointing vectors on \mathbb{H}^n (by Proposition 1.4.17), it follows that $\varphi|_{U \cap \partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U) \cap \partial \mathbb{H}^n$, and thus (Stokes-3) (5.37) is equal to $\int_{\partial M} \omega$. For a negatively oriented smooth boundary chart, the same argument applies with an additional negative sign on each side of the equation. For an interior chart, we get the the same computations with \mathbb{H}^n replaced by \mathbb{R}^n . This proves the theorem in this case.

Finally, let ω be an arbitrary compactly supported smooth $(n-1)$ -form. Choosing a cover of $\text{supp}(\omega)$ by finitely many domains of positively or negatively oriented smooth charts $\{U_i\}$, and choosing a subordinate smooth partition of unity $\{\psi_i\}$, we can apply the preceding argument to $\psi_i \omega$ for each i and obtain

$$\begin{aligned} \int_{\partial M} \omega &= \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \omega + \psi_i d\omega \\ &= \int_M d\left(\sum_i \psi_i\right) \omega + \int_M \left(\sum_i \psi_i\right) d\omega = \int_M d\omega. \end{aligned}$$

because $\sum_i \psi_i \equiv 1$. □

Example 1.5.9. Let M be a smooth manifold and suppose $\gamma : [a, b] \rightarrow M$ is a smooth embedding, so that $S = \gamma([a, b])$ is an embedded 1-submanifold with boundary in M . If we give S the orientation such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(M)$, Stokes's theorem says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* f = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)).$$

Two special cases of Stokes's theorem arise so frequently that they are worthy of special note. The proofs are immediate.

at exact form

Corollary 1.5.10 (Integrals of Exact Forms). *If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:*

$$\int_M d\omega = 0 \quad \text{if } \partial M = \emptyset.$$

closed form

Corollary 1.5.11 (Integrals of Closed Forms over Boundaries). *Suppose M is a compact oriented smooth manifold with boundary. If ω is a closed form on M , then the integral of ω over ∂M is zero:*

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0.$$

These results have the following extremely useful applications to submanifolds.

Corollary 1.5.12. *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an oriented compact smooth k -dimensional submanifold (without boundary), and ω is a closed k -form on M . If $\int_S \omega \neq 0$, then both of the following are true:*

- (a) ω is not exact on M .
- (b) S is not the boundary of an oriented compact smooth submanifold with boundary in M .

Example 1.5.13. It follows from the computation of Example 1.5.12 that the closed 1-form $\omega = \frac{x dy - y dx}{x^2 + y^2}$ has nonzero integral over S^1 . We already observed that ω is not exact on $\mathbb{R}^2 - \{0\}$. The preceding corollary tells us in addition that S^1 is not the boundary of a compact regular domain in $\mathbb{R}^2 - \{0\}$.

The following classical result is an easy application of Stokes's theorem.

Theorem 1.5.14 (Green's Theorem). *Suppose D is a compact regular domain in \mathbb{R}^2 , and P, Q are smooth real-valued functions on D . Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} P dx + Q dy.$$

1.5.3 Manifolds with corners

In many applications of Stokes's theorem it is necessary to deal with geometric objects such as triangles, squares, or cubes that are topological manifolds with boundary, but are not smooth manifolds with boundary because they have *corners*. It is easy to generalize Stokes's theorem to this setting, and we do so in this section.

Let $\bar{\mathbb{R}}_+^n$ denote the subset of \mathbb{R}^n where all of the coordinates are nonnegative:

$$\bar{\mathbb{R}}_+^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^1 \geq 0, \dots, x^n \geq 0\}.$$

This space is the model for the type of corners we are concerned with.

Proposition 1.5.15. *The set $\bar{\mathbb{R}}_+^n$ is homeomorphic to the upper half-space \mathbb{H}^n .*

Proof. The homeomorphism $\varphi : \bar{\mathbb{R}}_+^n \rightarrow \mathbb{H}^n$ is given by

$$\varphi(x^1, \dots, x^n) = (e^{x^1}, \dots, e^{x^{n-1}}, x^n).$$

□

Suppose M is a topological n -manifold with boundary. A **chart with corners** for M is a pair (U, φ) , where $U \subseteq M$ is open and φ is a homeomorphism from U to a (relatively) open subset $\widehat{U} \subseteq \overline{\mathbb{R}}_+^n$. Two charts with corners (U, φ) , (V, ψ) are smoothly compatible if the composite map $\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$ is smooth. (As usual, this means that it admits a smooth extension in an open neighborhood of each point.)

A **smooth structure** with corners on a topological manifold with boundary is a maximal collection of smoothly compatible interior charts and charts with corners whose domains cover M . A topological manifold with boundary together with a smooth structure with corners is called a **smooth manifold with corners**. Any chart with corners in the given smooth structure with corners is called a **smooth chart with corners** for M .

Example 1.5.16. Any closed rectangle in \mathbb{R}^n is a smooth n -manifold with corners.

The **boundary** of $\overline{\mathbb{R}}_+^n$ in \mathbb{R}^n is the set of points at which at least one coordinate vanishes. The points in $\overline{\mathbb{R}}_+^n$ at which more than one coordinate vanishes are called its **corner points**. For example, the corner points of $\overline{\mathbb{R}}_+^3$ are the origin together with all the points on the positive x -, y -, and z -axes.

Proposition 1.5.17 (Invariance of Corner Points). *Let M be a smooth n -manifold with corners, $n \geq 2$, and let $p \in M$. If $\varphi(p)$ is a corner point for some smooth chart with corners (U, φ) , then the same is true for every such chart whose domain contains p .*

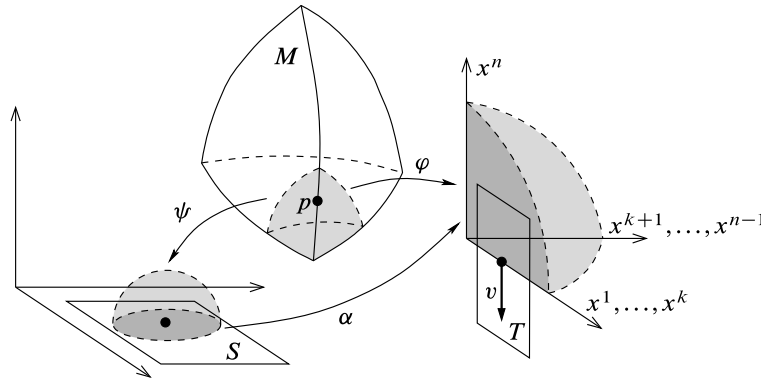


Figure 1.5: Invariance of corner points.

Proof. Suppose (U, φ) and (V, ψ) are two smooth charts with corners such that $\varphi(p)$ is a corner point but $\psi(p)$ is not. To simplify notation, let us assume without loss of generality that $\varphi(p)$ has coordinates $(x^1, \dots, x^k, 0, \dots, 0)$ with $k \leq n - 2$. Then $\psi(V)$ contains an open subset of some $(n - 1)$ -dimensional linear subspace $S \subseteq \mathbb{R}^n$, with $\psi(p) \in S$. (If $\psi(p) \in \partial \overline{\mathbb{R}}_+^n$, take S to be the unique subspace defined by an equation of the form $x^i = 0$ that contains $\psi(p)$. If $\psi(p)$ is an interior point, any $(n - 1)$ -dimensional subspace containing $\psi(p)$ will do.)

Let $S' = S \cap \psi(U \cap V)$, and let $\alpha : S' \rightarrow \mathbb{R}^n$ be the restriction of $\varphi \circ \psi^{-1}$ to S' . Because $\varphi \circ \psi^{-1}$ is a diffeomorphism from $\psi(U \cap V)$ to $\varphi(U \cap V)$, it follows that $\psi \circ \varphi^{-1} \circ \alpha$ is the identity of S' , and therefore $d\alpha_{\psi(p)}$ is an injective linear map. Let $T = d\alpha_{\psi(p)}(T_{\psi(p)}S) \subseteq \mathbb{R}^n$. Because T is $(n - 1)$ -dimensional, it must contain a vector v such that one of the last two

components, v^{n-1} or v^n , is nonzero (otherwise, T would be contained in a codimension-2 subspace). Renumbering the coordinates and replacing v by $-v$ if necessary, we may assume that $v^n < 0$.

Now let $\gamma : (-\varepsilon, \varepsilon) \rightarrow S$ be a smooth curve such that $\gamma(0) = p$ and $d\alpha(\gamma'(0)) = v$. Then $\alpha \circ \gamma(t)$ has negative x^n coordinate for small $t > 0$, which contradicts the fact that α takes its values in $\bar{\mathbb{R}}_+^n$. \square

If M is a smooth manifold with corners, a point $p \in M$ is called a corner point if $\varphi(p)$ is a corner point in $\bar{\mathbb{R}}_+^n$ with respect to some (and hence every) smooth chart with corners $(U, \varphi(p))$. Similarly, p is called a boundary point if $\varphi(p) \in \partial\bar{\mathbb{R}}_+^n$ with respect to some (hence every) such chart. For example, the set of corner points of the unit cube $[0, 1]^3 \subseteq \mathbb{R}^3$ is the union of its eight vertices and twelve edges.

Every smooth manifold with or without boundary is also a smooth manifold with corners (but with no corner points). Conversely, a smooth manifold with corners is a smooth manifold with boundary if and only if it has no corner points. The boundary of a smooth manifold with corners, however, is in general not a smooth manifold with corners (e.g., think of the boundary of a cube). In fact, even the boundary of $\bar{\mathbb{R}}_+^n$ itself is not a smooth manifold with corners. It is, however, a union of finitely many such: $\partial\bar{\mathbb{R}}_+^n = H_1 \cup \cdots \cup H_n$, where

$$H_i = \{(x^1, \dots, x^n) \in \bar{\mathbb{R}}_+^n : x^i = 0\} \quad (5.38)$$

mani corner-1

is an $(n-1)$ -dimensional smooth manifold with corners contained in the subspace defined by $x^i = 0$.

The usual flora and fauna of smooth manifoldssmooth maps, partitions of unity, tangent vectors, covectors, tensors, differential forms, orientations, and integrals of differential formscan be defined on smooth manifolds with corners in exactly the same way as we have done for smooth manifolds and smooth manifolds with boundary, using smooth charts with corners in place of smooth boundary charts.

In addition, for Stokes's theorem we need to integrate a differential form over the boundary of a smooth manifold with corners. Since the boundary is not itself a smooth manifold with corners, this requires a separate (albeit routine) definition.

Let M be an oriented smooth n -manifold with corners, and suppose ω is an $(n-1)$ -form on ∂M that is compactly supported in the domain of a single chart with corners (U, φ) . We define the integral of ω over ∂M by

$$\int_{\partial M} \omega = \sum_{i=1}^n \int_{H_i} \varphi_* \omega,$$

where H_i , defined by (5.38), is given the induced orientation as part of the boundary of the set where $x^i \geq 0$. In other words, we simply integrate ω in coordinates over the codimension-1 portion of the boundary.

Finally, if ω is an arbitrary compactly supported $(n-1)$ -form on M , we define the integral of ω over ∂M by piecing together with a partition of unity just as in the case of a manifold with boundary.

In practice, of course, one does not evaluate such integrals by using partitions of unity. Instead, one chops up the boundary into pieces that can be parametrized by open subsets, just as for ordinary manifolds with or without boundary. The following proposition is an analogue of Proposition 1.5.5.

int_mani_para

Proposition 1.5.18. *The statement of Proposition [1.5.5](#) is true if M is replaced by the boundary of a oriented smooth $(n + 1)$ -manifold with corners.*

Example 1.5.19. Let $I \times I = [0, 1]^2 \subseteq \mathbb{R}^2$ be the unit square in \mathbb{R}^2 , and suppose ω is a 1-form on $\partial(I \times I)$. Then it is not hard to check that the maps $F_i : I \rightarrow I \times I$ given by

$$\begin{aligned} F_1(t) &= (t, 0), & F_2(t) &= (1, t), \\ F_3(t) &= (1 - t, 1), & F_4(t) &= (0, -t). \end{aligned} \quad (5.39)$$

satisfy the hypotheses of Proposition [1.5.5](#). Therefore,

$$\int_{\partial(I \times I)} \omega = \int_{F_1} \omega + \int_{F_2} \omega + \int_{F_3} \omega + \int_{F_4} \omega.$$

Theorem 1.5.20 (Stokes's Theorem on Manifolds with Corners). *Let M be an oriented smooth n -manifold with corners, and let ω be a compactly supported smooth $(n - 1)$ -form on M . Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. The proof is nearly identical to the proof of Stokes's theorem proper, so we just indicate where changes need to be made. By means of smooth charts and a partition of unity, we may reduce the theorem to the case in which either $M = \mathbb{R}^n, \mathbb{H}^n$ or $M = \bar{\mathbb{R}}_+^n$. The \mathbb{R}^n and \mathbb{H}^n case are treated just as before. In the case of a chart with corners, ω is supported in some cube $[0, R]^n$, and we calculate exactly as in the proof of Theorem [1.5.8](#):

$$\begin{aligned} \int_{\bar{\mathbb{R}}_+^n} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x) dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \left[\omega_i(x) \right]_{x^i=0}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^n (-1)^i \int_0^R \cdots \int_0^R \omega_i(x^1, \dots, 0, \dots, x^n) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \sum_{i=1}^n \int_{H_i} \omega = \int_{\partial \bar{\mathbb{R}}_+^n} \omega. \end{aligned}$$

(The factor $(-1)^i$ disappeared because the induced orientation on H_i is $(-1)^i$ times that of the standard coordinates $(x^1, \dots, \widehat{x^i}, \dots, x^n)$) This completes the proof. \square

The preceding theorem has the following important application.

Theorem 1.5.21. *Suppose M is a smooth manifold and $\gamma_0, \gamma_1 : [a, b] \rightarrow M$ are path-homotopic piecewise smooth curve segments. For every closed 1-form ω on M ,*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. By means of an affine reparametrization, we may as well assume for simplicity that $[a, b] = [0, 1]$. Assume first that γ_0 and γ_1 are smooth. By Theorem [1.5.8](#), γ_0 and γ_1 are smoothly

homotopic relative to $\{0, 1\}$. Let $H : I \times I \rightarrow M$ be such a smooth homotopy. Since ω is closed, we have

$$\int_{I \times I} d(H^* \omega) = \int_{I \times I} H^* d\omega = 0.$$

On the other hand, $I \times I$ is a smooth manifold with corners, so Stokes's theorem implies

$$0 = \int_{I \times I} d(H^* \omega) = \int_{\partial(I \times I)} H^* \omega.$$

Using the parametrization of $\partial(I \times I)$ given in Example [1.5.19](#) together with Proposition [1.5.17](#), we obtain

$$\begin{aligned} 0 &= \int_{\partial(I \times I)} H^* \omega = \int_{\partial(I \times I)} H^* \omega = \int_{F_1} H^* \omega + \int_{F_2} H^* \omega + \int_{F_3} H^* \omega + \int_{F_4} H^* \omega \\ &= \int_{H \circ F_1} \omega + \int_{H \circ F_2} \omega + \int_{H \circ F_3} \omega + \int_{H \circ F_4} \omega. \end{aligned}$$

The fact that H is a homotopy relative to $\{0, 1\}$ means that $H \circ F_2$ and $H \circ F_4$ are constant maps, and therefore the second and fourth terms above are zero. The theorem then follows from the facts that $H \circ F_1 = \gamma_0$ and $H \circ F_3$ is a backward reparametrization of γ_1 .

Next we consider the general case of piecewise smooth curves. We cannot simply apply the preceding result on each subinterval where γ_0 and γ_1 are smooth, because the restricted curves may not start and end at the same points. Instead, we prove the following more general claim: Let $\gamma_0, \gamma_1 : [0, 1] \rightarrow M$ be piecewise smooth curve segments (not necessarily with the same endpoints), and suppose $H : I \times I \rightarrow M$ is any homotopy between them. Define curve segments $\sigma_0, \sigma_1 : I \rightarrow M$ by

$$\sigma_0(t) = H(0, t), \quad \sigma_1(t) = H(1, t),$$

and let $\tilde{\sigma}_0, \tilde{\sigma}_1$ be any smooth curve segments that are path-homotopic to σ_0, σ_1 respectively. Then

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = \int_{\tilde{\sigma}_1} \omega - \int_{\tilde{\sigma}_0} \omega. \quad (5.40) \quad \text{int homotopy-}$$

When specialized to the case in which γ_0 and γ_1 are path-homotopic, this implies the theorem, because γ_0 and γ_1 are constant maps in that case.

Since γ_0 and γ_1 are piecewise smooth, there are only finitely many points $\{a_1, \dots, a_m\}$ in $[0, 1]$ at which either γ_0 or γ_1 is not smooth. We prove the claim by induction on the number m of such points. When $m = 0$, both curves are smooth, and by Theorem [1.5.17](#) we may replace the given homotopy H by a smooth homotopy \tilde{H} . Recall from the proof of Theorem [1.5.17](#) that the smooth homotopy \tilde{H} can actually be taken to be homotopic to H relative to $I \times \{0\} \cup I \times \{1\}$. Thus, for $i = 0, 1$, the curve $\tilde{\sigma}_i = \tilde{H}(i, t)$ is a smooth curve segment that is path-homotopic to σ_i . In this setting, (5.40) just reduces to the integration formula of Example [1.5.19](#). Note that the integrals over $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ do not depend on which smooth curves pathhomotopic to σ_0 and σ_1 are chosen, by the smooth case proved above.

Now let γ_0, γ_1 be homotopic piecewise smooth curves with m nonsmooth points $\{a_1, \dots, a_m\}$, and suppose the claim is true for curves with fewer than m such points. For $i = 0, 1$, let γ'_i be the restriction of γ_i to $[0, a_m]$, and let γ''_i be its restriction to $[a_m, 1]$. Let $\sigma : I \rightarrow M$ be the curve segment $\sigma(t) = H(a_m, t)$, and let $\tilde{\sigma}$ be any smooth curve segment that is path-homotopic to σ . Then, since γ'_i and γ''_i have fewer than m nonsmooth points, the inductive

hypothesis implies

$$\begin{aligned}
 \int_{\gamma_1} \omega - \int_{\gamma_0} \omega &= \left(\int_{\gamma'_1} \omega + \int_{\gamma''_1} \omega \right) - \left(\int_{\gamma'_0} \omega + \int_{\gamma''_0} \omega \right) \\
 &= \left(\int_{\gamma'_1} \omega - \int_{\gamma'_0} \omega \right) + \left(\int_{\gamma''_1} \omega + \int_{\gamma''_0} \omega \right) \\
 &= \left(\int_{\tilde{\sigma}} \omega - \int_{\tilde{\sigma}_0} \omega \right) + \left(\int_{\tilde{\sigma}_1} \omega + \int_{\tilde{\sigma}} \omega \right) \\
 &= \int_{\tilde{\sigma}_1} \omega - \int_{\tilde{\sigma}_0} \omega.
 \end{aligned}$$

□

y con 1 form

Corollary 1.5.22. *On a simply connected smooth manifold, every closed 1-form is exact.*

Proof. Suppose M is simply connected and ω is a closed 1-form on M . Since every piecewise smooth closed curve segment in M is path-homotopic to a constant curve, the preceding theorem shows that the integral of ω over every such curve is equal to 0. Thus, ω is conservative and therefore exact. □

1.5.4 Integration on Riemannian manifolds

Integration of functions on Riemannian manifolds

Suppose (M, g) is an oriented Riemannian manifold with or without boundary, and let ω_g denote its Riemannian volume form. If f is a compactly supported continuous real-valued function on M , then $f\omega_g$ is a compactly supported n -form, so we can define the **integral of f over M** to be $\int_M f\omega_g$. If M itself is compact, we define the **volume of M** by $\text{Vol}(M) = \int_M \omega_g$.

Because of these definitions, the Riemannian volume form is often denoted by dV_g (or dA_g or ds_g in the 2-dimensional or 1-dimensional case, respectively). Then the integral of f over M is written $\int_M f dV_g$, and the volume of M as $\int_M dV_g$. Be warned, however, that this notation is not meant to imply that the volume form is the exterior derivative of an $(n-1)$ -form; in fact, as we will see when we study de Rham cohomology, this is never the case on a compact manifold. You should just interpret dV_g as a notational convenience.

Proposition 1.5.23. *Let (M, g) be a nonempty oriented Riemannian manifold with or without boundary, and suppose f is a compactly supported continuous realvalued function on M satisfying $f \geq 0$. Then $\int_M f dV_g \geq 0$, with equality if and only if $f \equiv 0$.*

Proof. If f is supported in the domain of a single oriented smooth chart (U, φ) , then Proposition 1.4.21 shows that

$$\int_M f dV_g = \int_M f \sqrt{G} dx^1 \cdots dx^n \geq 0.$$

The same inequality holds in a negatively oriented chart because the negative sign from the chart cancels the negative sign in the expression for dV_g . The general case follows from this one, because $\int_M f dV_g$ is equal to a sum of terms like $\int_M \psi f dV_g$, where each integrand is nonnegative and supported in a single smooth chart. If in addition f is positive somewhere, then it is positive on a nonempty open subset by continuity, so at least one of the integrals in this sum is positive. On the other hand, if f is identically zero, then clearly $\int_M f dV_g = 0$. □

In the same way, we can prove the following result.

Proposition 1.5.24. *Suppose (M, g) is an oriented Riemannian manifold and $f : M \rightarrow \mathbb{R}$ is continuous and compactly supported. Then*

$$\left| \int_M f dV_g \right| \leq \int_M |f| dV_g.$$

The divergence theorem

Let (M, g) be an oriented Riemannian n -manifold (with or without boundary). We can generalize the classical divergence operator to this setting as follows. Multiplication by the Riemannian volume form defines a smooth bundle isomorphism $*$: $C^\infty(M) \rightarrow \Omega^n(M)$:

$$*f := f dV_g.$$

In addition, as we did in the case of \mathbb{R}^3 , we define a smooth bundle isomorphism $\beta : \mathfrak{X}(M) \rightarrow \Omega^{n-1}(M)$ as follows:

$$\beta(X) = X \lrcorner dV_g. \quad (5.41) \quad \text{int prod Riem}$$

We need the following technical lemma.

ergence lemma

Lemma 1.5.25. *Let (M, g) be an oriented Riemannian manifold with or without boundary. Suppose $S \subseteq M$ is an immersed hypersurface with the orientation determined by a unit normal vector field N , and \tilde{g} is the induced metric on S . If X is any vector field along S , then*

$$\iota_S^*(\beta(X)) = \langle X, N \rangle_g dV_{\tilde{g}}. \quad (5.42) \quad \text{div lem-1}$$

Proof. Define two vector fields X and X along S by

$$X^\perp = \langle X, N \rangle_g N, \quad X^\top = X - X^\perp.$$

Then $X = X^\perp + X^\top$, where X^\perp is normal to S and X^\top is tangent to it. Using this decomposition,

$$\beta(X) = X^\perp \lrcorner dV_g + X^\top \lrcorner dV_g.$$

Now pull back to S . Proposition [1.4.24](#) Riemann volume form hyper shows that the first term simplifies to

$$\iota_S^*(X^\perp \lrcorner dV_g) = \langle X, N \rangle_g \iota_S^*(N \lrcorner dV_g) = \langle X, N \rangle_g dV_{\tilde{g}}.$$

Thus [\(5.42\)](#) div lem-1 will be proved if we can show that $\iota_S^*(X^\top \lrcorner dV_g) = 0$. If X_1, \dots, X_{n-1} are any vectors tangent to S , then

$$(X^\top \lrcorner dV_g)(X_1, \dots, X_{n-1}) = dV_g(X, X_1, \dots, X_{n-1}) = 0,$$

because any n -tuple of vectors in an $(n-1)$ -dimensional vector space is linearly dependent. \square

Now we define the **divergence operator** $\text{div} : \mathfrak{X}(M) \rightarrow C^\infty(M)$ by

$$\text{div} X = *^{-1} d(\beta(X)) = *^{-1} d(X \lrcorner dV_g). \quad (5.43) \quad \text{div def-1}$$

or equivalently,

$$d(X \lrcorner dV_g) = (\text{div} X) dV_g. \quad (5.44) \quad \text{div def-2}$$

Even if M is nonorientable, in a neighborhood of each point we can choose an orientation and define the divergence by [\(5.44\)](#) div def-2, and then note that reversing the orientation changes the sign

of dV_g on both sides of the equation, so $\operatorname{div} X$ is well defined, independently of the choice of orientation. In this way, we can define the divergence operator on any Riemannian manifold with or without boundary, by requiring that it satisfy $(\operatorname{div} X)_{\text{def-2}} = \int_M \langle N, X \rangle_g dV_{\tilde{g}}$ for any choice of orientation in a neighborhood of each point.

The next theorem is a fundamental result about vector fields on Riemannian manifolds. In the special case of a compact regular domain in \mathbb{R}^3 , it is often referred to as **Gauss's theorem**.

Theorem 1.5.26 (The Divergence Theorem). *Let (M, g) be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M ,*

$$\int_M (\operatorname{div} X) dV_g = \int_{\partial M} \langle N, X \rangle_g dV_{\tilde{g}}.$$

where N is the outward-pointing unit normal vector field along ∂M and \tilde{g} is the induced Riemannian metric on ∂M .

Proof. By Stokes's theorem,

$$\int_M (\operatorname{div} X) dV_g = \int_M d(\beta(X)) = \int_{\partial M} \iota_{\partial M}^* \beta(X).$$

The divergence theorem then follows from Lemma [1.5.25](#). □

The term divergence is used because of the following geometric interpretation. A smooth flow θ on M is said to be volume-preserving if for every compact regular domain D , we have $\operatorname{Vol}(\theta_t(D)) = \operatorname{Vol}(D)$ whenever the domain of θ_t contains D . It is called **volume-increasing**, **volume-decreasing**, **volume-nonincreasing**, or **volume-nondecreasing** if for every such D , $\operatorname{Vol}(\theta_t(D))$ is strictly increasing, strictly decreasing, nonincreasing, or nondecreasing, respectively, as a function of t . Note that the properties of flow domains ensure that if D is contained in the domain of θ_t for some t , then the same is true for all times between 0 and t .

The next proposition shows that the divergence of a vector field can be interpreted as a measure of the tendency of its flow to spread out.

interpretation

Proposition 1.5.27 (Geometric Interpretation of the Divergence). *Let M be an oriented Riemannian manifold, let $X \in \mathfrak{X}(M)$, and let θ be the flow of X . Then θ is*

- (a) *volume-preserving if and only if $\operatorname{div} X = 0$ everywhere on M .*
- (b) *volume-nondecreasing if and only if $\operatorname{div} X \geq 0$ everywhere on M .*
- (c) *volume-nonincreasing if and only if $\operatorname{div} X \leq 0$ everywhere on M .*
- (d) *volume-increasing if and only if $\operatorname{div} X > 0$ on a dense subset of M .*
- (e) *volume-decreasing if and only if $\operatorname{div} X < 0$ on a dense subset of M .*

Proof. First we establish some preliminary results. For each $t \in \mathbb{R}$, let M_t be the domain of θ_t . If D is a compact regular domain contained in M_t , then θ_t is an orientation-preserving diffeomorphism from D to $\theta_t(D)$ by the result of Exercise [1.4.4](#), so

$$\operatorname{Vol}(\theta_t(D)) = \int_{\theta_t(D)} dV_g = \int_D \theta_t^* dV_g.$$

Because the integrand on the right depends smoothly on (t, p) in the domain of θ , we can differentiate this expression with respect to t by differentiating under the integral sign.

Using Cartan's magic formula for the Lie derivative of the Riemannian volume form, we obtain

$$\mathfrak{L}_X dV_g = X \lrcorner d(dV_g) + d(X \lrcorner dV_g) = (\operatorname{div} X) dV_g.$$

because $d(dV_g)$ is an $(n+1)$ -form on an n -manifold. Then Proposition [1.1.27](#) Lie der tensor t_0 implies

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} \operatorname{Vol}(\theta_t(D)) &= \int_D \frac{\partial}{\partial t} \Big|_{t=t_0} (\theta_t^* dV_g) = \int_D \theta_{t_0}^* (\mathfrak{L}_X dV_g) \\ &= \int_D \theta_{t_0}^* ((\operatorname{div} X) dV_g) = \int_{\theta_{t_0}(D)} (\operatorname{div} X) dV_g. \end{aligned}$$

From this, the claim is easily proved. \square

Using the divergence operator, we can define another important operator. The **Laplacian** (or **LaplaceBeltrami operator**) is the linear operator $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ defined by

$$\Delta f = \operatorname{div}(\operatorname{grad} f). \quad (5.45) \quad \text{Laplacian def}$$

The next proposition gives alternative formulas for these operators.

Proposition 1.5.28. *Let (M, g) be a Riemannian manifold with or without boundary, and let (x^i) be any smooth local coordinates on an open set $U \subseteq M$. The coordinate representations of the divergence and Laplacian are as follows:*

$$\operatorname{div}(X^i \frac{\partial}{\partial x^i}) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} X^i) \quad (5.46) \quad \text{div coordinate}$$

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{ij} \frac{\partial f}{\partial x^j}). \quad (5.47) \quad \text{Laplacian coordinate}$$

where $G = \det(g_{ij})$ is the determinant of the component matrix of g in these coordinates.

Proof. By Proposition [1.4.23](#) Riemann volume form formula we have

$$X \lrcorner dV_g = X \lrcorner \sqrt{G} dx^1 \wedge \cdots \wedge dx^n = \sqrt{G} \sum_{i=1}^n (-1)^{n-1} X^i dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Therefore

$$d(X \lrcorner dV_g) = \frac{\partial}{\partial x^i} (\sqrt{G} X^i) dx^1 \wedge \cdots \wedge dx^n = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} X^i) dV_g,$$

and so [\(5.46\)](#) div coordinate holds. The second formula follows from the expression of $\operatorname{grad} f$. \square

Surface integrals

The original theorem that bears the name of Stokes concerned surface integrals of vector fields over surfaces in \mathbb{R}^3 . Using the version of Stokes's theorem that we have proved, we can generalize this to surfaces in Riemannian 3-manifolds.

Let (M, g) be an oriented Riemannian 3-manifold. Define the curl operator, denoted by $\operatorname{curl} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$\operatorname{curl} X = \beta^{-1} d(X^\flat),$$

where $\beta : \mathfrak{X}(M) \rightarrow \Omega^2(M)$ is defined in (5.41). Unwinding the definitions, we see that this is equivalent to

$$(\text{curl } X) \lrcorner dV_g = d(X^\flat).$$

The operators div , grad , and curl on an oriented Riemannian 3-manifold M are related by the following commutative diagram

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{\text{grad}} & \mathfrak{X}(M) & \xrightarrow{\text{curl}} & \mathfrak{X}(M) & \xrightarrow{\text{div}} & C^\infty(M) \\ \downarrow \text{id} & & \downarrow \flat & & \downarrow \beta & & \downarrow * \\ \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M) \end{array}$$

The curl operator is defined only in dimension 3 because it is only in that case that $\wedge^2 T^*M$ is isomorphic to TM via the map β .

Now suppose $S \subseteq M$ is a compact 2-dimensional submanifold with or without boundary, and N is a smooth unit normal vector field along S . Let dA denote the Riemannian volume form on S with respect to the induced metric ι_S^*g and the orientation determined by N , so that $dA = \iota_S^*(N \lrcorner dV_g)$ by Proposition 1.4.24. For any smooth vector field X defined on M , the surface integral of X over S (with respect to the given choice of unit normal field) is defined as

$$\int_S \langle X, N \rangle_g dA.$$

The next result, in the special case in which $M = \mathbb{R}^3$, is the theorem usually referred to as Stokes's theorem in multivariable calculus texts.

Theorem 1.5.29 (Stokes's Theorem for Surface Integrals). *Suppose M is an oriented Riemannian 3-manifold with or without boundary, and S is a compact oriented 2-dimensional smooth submanifold with boundary in M . For any smooth vector field X on M ,*

$$\int_S \langle \text{curl } X, N \rangle_g dA = \int_{\partial S} \langle X, T \rangle_g ds,$$

where N is the smooth unit normal vector field along S that determines its orientation, ds is the Riemannian volume form for ∂S (with respect to the metric and orientation induced from S), and T is the unique positively oriented unit tangent vector field on ∂S .

Proof. By the defining of the curl and the result of Lemma 1.5.25, we get

$$\langle \text{curl } X, N \rangle_g dA = \iota_S^* \beta(\text{curl } X) = \iota_S^* d(X^\flat).$$

Thus we only need to show

$$\iota_{\partial S}^* X^\flat = \langle X, T \rangle_g ds.$$

To prove this, we note that $\iota_{\partial S}^* X^\flat$ is a smooth 1-form on a 1-manifold, and thus must be equal to $f ds$ for some smooth function f on ∂S . To evaluate f , we note that $ds(T) = 1$, and so the definition of X^\flat yields

$$f = f ds(T) = X^\flat(T) = \langle X, T \rangle_g.$$

Thus the general version of Stokes's theorem gives the claim. \square

Densities

In the theory of integration of differential forms, the crucial place where orientations entered the picture was in our proof of the diffeomorphism-invariance of the integral (Proposition 1.5.1), because the transformation law for an n -form on an n -manifold under a change of coordinates involves the Jacobian determinant of the transition map, while the transformation law for integrals involves the absolute value of the determinant. We had to restrict attention to orientation-preserving diffeomorphisms so that we could freely remove the absolute value signs. In this part we define objects whose transformation law involves the absolute value of the determinant, so that we no longer have this sign problem.

We begin, as always, in the linear-algebraic setting. Let V be an n -dimensional vector space. A **density** on V is a function

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ folds}} \rightarrow \mathbb{R}$$

satisfying the following condition: if $T : V \rightarrow V$ is any linear map, then

$$\mu(Tv_1, \dots, Tv_n) = |\det T| \mu(v_1, \dots, v_n).$$

Observe that a density is not a tensor, because it is not linear over \mathbb{R} in any of its arguments. Let $\mathfrak{D}(V)$ denote the set of all densities on V .

density prop

Proposition 1.5.30 (Properties of Densities). *Let V be a vector space of dimension n .*

(a) $\mathfrak{D}(V)$ is a vector space under the obvious vector operations:

$$(c_1\mu_1 + c_2\mu_2)(v_1, \dots, v_n) = c_1\mu_1(v_1, \dots, v_n) + c_2\mu_2(v_1, \dots, v_n).$$

(b) If $\mu_1, \mu_2 \in \mathfrak{D}(V)$ and $\mu_1(E_1, \dots, E_n) = \mu_2(E_1, \dots, E_n)$ for some basis (E_i) of V , then $\mu_1 = \mu_2$.

(c) If $\omega \in \wedge^n(V^*)$, the map $|\omega|$ defined by

$$|\omega|(v_1, \dots, v_n) = |\omega(v_1, \dots, v_n)|$$

is a density.

(d) $\mathfrak{D}(V)$ is 1-dimensional, spanned by $|\omega|$ for any nonzero $\omega \in \wedge^n(V^*)$.

Proof. For part (b), suppose μ_1 and μ_2 give the same value when applied to (E_1, \dots, E_n) . If v_1, \dots, v_n are arbitrary vectors in V , let $T : V \rightarrow V$ be the unique linear map that takes E_i to v_i for $i = 1, \dots, n$. It follows that

$$\mu_1(v_1, \dots, v_n) = |\det T| \mu_1(E_1, \dots, E_n) = |\det T| \mu_2(E_1, \dots, E_n) = \mu_2(v_1, \dots, v_n).$$

Therefore $\mu_1 = \mu_2$.

Part (c) follows from Proposition 1.3.7. Finally, to prove (d), suppose ω is any nonzero element of $\wedge^n(V^*)$. If μ is an arbitrary element of $\mathfrak{D}(V)$, it suffices to show that $\mu = c|\omega|$ for some $c \in \mathbb{R}$. Let (E_i) be a basis for V , and define $a, b \in \mathbb{R}$ by

$$a = |\omega|(E_1, \dots, E_n) = |\omega(E_1, \dots, E_n)|, \quad b = \mu(E_1, \dots, E_n).$$

Because $\omega \neq 0$, it follows that $a \neq 0$. Thus, μ and $(b/a)|\omega|$ give the same result when applied to (E_1, \dots, E_n) , so they are equal by part (b). \square

A **positive density** on V is a density μ satisfying $\mu(v_1, \dots, v_n) > 0$ whenever (v_1, \dots, v_n) is a linearly independent n -tuple. A **negative density** is defined similarly. If ω is a nonzero element of $\wedge^n(V^*)$, then it is clear that $|\omega|$ is a positive density; more generally, a density $c|\omega|$ is positive, negative, or zero if and only if c has the same property. Thus, each density on V is either positive, negative, or zero, and the set of positive densities is a convex subset of \mathfrak{D} (namely, a half-line).

Now let M be a smooth manifold with or without boundary. The set

$$\mathfrak{D}M = \bigsqcup_{p \in M} \mathfrak{D}(T_p M)$$

is called the density bundle of M . Let $\pi : \mathfrak{D}M \rightarrow M$ be the natural projection map taking each element of $\mathfrak{D}(T_p M)$ to p .

Proposition 1.5.31. *If M is a smooth manifold with or without boundary, its density bundle is a smooth line bundle over M .*

Proof. We will construct local trivializations and use the vector bundle chart lemma (Lemma 1.5.29).

Let (U, x^i) be any smooth coordinate chart on M ; and let $\omega = dx^1 \wedge \dots \wedge dx^n$. Proposition 1.5.30 shows that $|\omega_p|$ is a basis for $\mathfrak{D}(T_p M)$ at each point $p \in U$. Therefore, the map $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}$ given by

$$\Phi(c|\omega_p|) = (p, c)$$

is a bijection.

Now suppose (\tilde{U}, \tilde{x}^j) is another smooth chart with $U \cap \tilde{U} \neq \emptyset$. Let $\tilde{\omega} = d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n$, and define $\tilde{\Phi} : \pi^{-1}(\tilde{U}) \rightarrow \tilde{U} \times \mathbb{R}$ correspondingly:

$$\tilde{\Phi}(c|\tilde{\omega}_p|) = (p, c).$$

It follows from the transformation law (3.23) for n -forms under changes of coordinates that

$$\begin{aligned} \Phi \circ \tilde{\Phi}^{-1}(p, c) &= \Phi(c|\tilde{\omega}_p|) = \Phi\left(c \left| \det \left(\frac{\partial \tilde{x}^j}{\partial x^i} \right) \right| |\omega_p|\right) \\ &= \left(p, c \left| \det \left(\frac{\partial \tilde{x}^j}{\partial x^i} \right) \right| \right) \end{aligned}$$

Thus, the hypotheses of Lemma 1.5.29 are satisfied, with the transition functions equal to $c|\det(\partial \tilde{x}^j / \partial x^i)|$. \square

If M is a smooth n -manifold with or without boundary, a section of $\mathfrak{D}M$ is called a **density** on M . If μ is a density and f is a continuous real-valued function, then $f\mu$ is again a density, which is smooth if both f and μ are. A density on M is said to be positive or negative if its value at each point has that property. Any nonvanishing n -form ω determines a positive density $|\omega|$, defined by $|\omega|_p := |\omega_p|$ for each $p \in M$. If ω is a nonvanishing n -form on an open subset $U \subseteq M$; then any density μ on U can be written $\mu = f|\omega|$ for some real-valued function f .

One important fact about densities is that every smooth manifold admits a global smooth positive density, without any orientability assumptions.

Proposition 1.5.32. *If M is a smooth manifold with or without boundary, there exists a smooth positive density on M .*

Proof. Because the set of positive elements of $\mathfrak{D}M$ is an open subset whose intersection with each fiber is convex, the usual partition of unity argument (Exercise 1.2.1) allows us to piece together local positive densities to obtain a global smooth positive density. \square

It is important to understand that this proposition works because positivity of a density is a well-defined property, independent of any choices of coordinates or orientations. There is no corresponding existence result for orientation forms because without a choice of orientation, there is no way to decide which n -forms are positive.

Under smooth maps, densities pull back in the same way as differential forms. If $F : M \rightarrow N$ is a smooth map between n -manifolds (with or without boundary) and μ is a density on N , we define a density $F^*\mu$ on M by

$$(F^*\mu)_p(v_1, \dots, v_n) = \mu_{F(p)}(dF_p(v_1), \dots, dF_p(v_n)).$$

all back prop

Proposition 1.5.33. *Let $G : P \rightarrow M$ and $F : M \rightarrow N$ be smooth maps between n -manifolds with or without boundary, and let μ be a density on N .*

- (a) *For any $f \in C^\infty(N)$, $F^*(f\mu) = (f \circ F)F^*\mu$.*
- (b) *If ω is an n -form on N , then $F^*|\omega| = |F^*\omega|$.*
- (c) *If μ is smooth, then $F^*\mu$ is a smooth density on M .*
- (d) *$F(F \circ G)^*\mu = G^*(F^*\mu)$.*

The next result shows how to compute the pullback of a density in coordinates. It is an analogue for densities of Proposition 1.3.16.

Proposition 1.5.34. *Suppose $F : M \rightarrow N$ is a smooth map between n -manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V , then the following holds on $U \cap F^{-1}(V)$:*

$$F^*(u|dy^1 \wedge \dots \wedge dy^n|) = (u \circ F)|\det \partial F||dx^1 \wedge \dots \wedge dx^n|,$$

where ∂F represents the matrix of partial derivatives of F in these coordinates.

Now we turn to integration. As we did with forms, we begin by defining integrals of densities on subsets of \mathbb{R}^n . If $D \subseteq \mathbb{R}^n$ is an open set and μ is a density on \bar{D} , we can write $\mu = f dx^1 \wedge \dots \wedge dx^n$ for some uniquely determined continuous function $f : \bar{D} \rightarrow \mathbb{R}$. We define the integral of μ over D by

$$\int_D \mu = \int_D f dV,$$

or more suggestively,

$$\int_D f |dx^1 \wedge \dots \wedge dx^n| = \int_D f dx^1 \wedge dx^n.$$

Similarly, if U is an open subset of \mathbb{R}^n or \mathbb{H}^n and μ is compactly supported in U , we define

$$\int_U \mu = \int_D \mu$$

where D is any open set containing the support of μ . The key fact is that this is diffeomorphism-invariant.

Proposition 1.5.35. *Suppose U and V are open subsets of \mathbb{R}^n or \mathbb{H}^n , and $F : U \rightarrow V$ is a diffeomorphism. If μ is a compactly supported density on V , then*

$$\int_V \mu = \int_U F^* \mu.$$

Proof. The proof is essentially identical to that of Proposition [1.5.1](#). □

Now let M be a smooth n -manifold (with or without boundary). If μ is a density on M whose support is contained in the domain of a single smooth chart (U, φ) , the integral of μ over M is defined as

$$\int_M \mu = \int_{\varphi(U)} \varphi_* \mu.$$

This is extended to arbitrary densities μ by setting

$$\int_M \mu = \sum_i \int_M \psi_i \mu$$

where $\{\psi_i\}$ is a smooth partition of unity subordinate to an open cover of M by smooth charts. The fact that this is independent of the choices of coordinates or partition of unity follows just as in the case of forms.

The following proposition is proved in the same way as Proposition [1.5.4](#).

Proposition 1.5.36 (Properties of Integrals of Densities). *Suppose M and N are smooth n -manifolds with or without boundary, and μ, η are compactly supported densities on M .*

- If $a, b \in \mathbb{R}$, then

$$\int_M (a\mu + b\eta) = a \int_M \mu + b \int_M \eta.$$

- If μ is a positive density, then $\int_M \mu > 0$.
- If $F : N \rightarrow M$ is a diffeomorphism, then $\int_M \mu = \int_N F^* \mu$.

Densities are particularly useful on Riemannian manifolds

Proposition 1.5.37 (The Riemannian Density). *Let (M, g) be a Riemannian manifold with or without boundary. There is a unique smooth positive density μ_g on M , called the **Riemannian density**, with the property that*

$$\mu_g(E_1, \dots, E_n) = 1$$

for any local orthonormal frame (E_i) .

Proof. Uniqueness is immediate, because any two densities that agree on a basis must be equal. Given any point $p \in M$; let U be a connected smooth coordinate neighborhood of p . Since U is diffeomorphic to an open subset of Euclidean space, it is orientable. Any choice of orientation of U uniquely determines a Riemannian volume form ω_g on U , with the property that $\omega_g(E_1, \dots, E_n) = 1$ for any oriented orthonormal frame. If we put $\mu_g = |\omega_g|$, it follows easily that μ_g is a smooth positive density on U satisfying the condition. If U and V are two overlapping smooth coordinate neighborhoods, the two definitions of μ_g agree where they overlap by uniqueness, so this defines μ_g globally. □

Proposition 1.5.38. *Let (M, g) be an oriented Riemannian manifold with or without boundary and let μ_g be its Riemannian volume form.*

- *The Riemannian density of M is given by μ_g .*
- *For any compactly supported continuous function $f : M \rightarrow \mathbb{R}$, we have*

$$\int_M f \mu_g = \int_M f \omega_g.$$

Proof. For an oriented Riemannian manifold, the volume form ω_g is nonzero, so the claims are obvious. \square

Proposition 1.5.39. *Suppose (M, g) and (\tilde{M}, \tilde{g}) are Riemannian manifolds with or without boundary, and $F : M \rightarrow \tilde{M}$ is a local isometry. Then $F^* \mu_{\tilde{g}} = \mu_g$.*

Theorem 1.5.40 (The Divergence Theorem in the Nonorientable Case). *Suppose (M, g) is a nonorientable Riemannian manifold with boundary. For any compactly supported smooth vector field X on M ,*

$$\int_M (\operatorname{div} X) \mu_g = \int_{\partial M} \langle X, N \rangle_g \mu_{\tilde{g}},$$

where N is the outward-pointing unit normal vector field along ∂M , \tilde{g} is the induced Riemannian metric on ∂M , and $\mu_g, \mu_{\tilde{g}}$ are the Riemannian densities of g and \tilde{g} , respectively.

1.5.5 Exercise

ring relation

Exercise 1.5.1. Suppose E and M are smooth n -manifolds with or without boundary, and $\pi : E \rightarrow M$ is a smooth k -sheeted covering map or generalized covering map. Show that if E and M are oriented and π is orientation-preserving, then $\int_E \pi^* \omega = k \int_M \omega$ for any compactly supported n -form ω on M .

Proof. Let $\{(U_i, \varphi)\}$ be an evenly covered positively oriented family of M , and choose a partition of unity $\{\psi_i\}$ subordinate to U_i . Then the integral of ω is defined to be

$$\int_M \omega = \sum_i \int_{U_i} \psi_i \omega.$$

On each U_i , we may assume the image of $\varphi_i : U_i \rightarrow \widehat{U}_i$ is an open disk in \mathbb{R}^n . If we write $\pi^{-1} = \bigcup_{j=1}^k V_{ij}$ such that $\pi|_{V_{ij}} : V_{ij} \rightarrow U_i$ is a diffeomorphism, then since \widehat{U}_i is simply-connected, we have k liftings $\tilde{\varphi}_{ij}$ such that $\pi \circ \tilde{\varphi}_{ij} = \varphi_i$ for all j . Then we can see that each of the $\tilde{\varphi}_{ij}$ is an orientation-preserving diffeomorphism from \widehat{U}_i to V_{ij} . Thus we get

$$\int_{U_i} \psi_i \omega = \int_{V_{ij}} \pi^*(\psi_i \omega).$$

This then implies the desired result since $\{\pi^* \psi_i\}$ is a partition subordinate to $\{V_{ij}\}$. \square

Exercise 1.5.2. Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a retraction of M onto its boundary.

Exercise 1.5.3. Suppose M and N are oriented, compact, connected, smooth manifolds, and $F : M \rightarrow N$ are homotopic diffeomorphisms. Show that F and G are either both orientation-preserving or both orientation-reversing.

Proof. By Theorem 1.5.3, we can choose a smooth homotopy $H : M \times I \rightarrow N$ such that $H(x, 0) = F$, $H(x, 1) = G$. Let ω be an orientation form on N , then

$$\int_{M \times I} d(H^* \omega) = \int_{M \times I} H^*(d\omega) = 0.$$

On the other hand, by Stokes's theorem,

$$\int_{M \times I} d(H^* \omega) = \int_{\partial(M \times I)} H^* \omega = \int_{M \times \{1\}} H^* \omega - \int_{M \times \{0\}} H^* \omega = \int_M G^* \omega - \int_M F^* \omega.$$

Thus F and G are either both orientation-preserving or both orientation-reversing. \square

Exercise 1.5.4. Show that any finite product $M_1 \times \cdots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

Exercise 1.5.5. Let D denote the torus of revolution in \mathbb{R}^3 obtained by revolving the circle $(x-2)^2 + y^2 = 1$ around the z -axis, with its induced Riemannian metric and with the orientation determined by the outward unit normal.

- (a) Compute the surface area of D .
- (b) Compute the integral over D of the function $f(x, y, z) = z^2 + 1$.
- (c) Compute the integral over D of the 2-form $\omega = z \, dx \wedge dy$.

(fX) exercise

Exercise 1.5.6. Let (M, g) be a compact Riemannian manifold with boundary, let \tilde{g} denote the induced Riemannian metric on ∂M , and let N be the outward unit normal vector field along ∂M .

- (a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$:

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X \rangle_g.$$

- (b) Prove the following integration by parts formula:

$$\int_M \langle \operatorname{grad} f, X \rangle_g \, dV_g = \int_{\partial M} f \langle X, N \rangle_g \, dV_{\tilde{g}} - \int_M (f \operatorname{div} X) \, dV_g.$$

Proof. By definition,

$$\begin{aligned} \operatorname{div}(fX) \, dV_g &= d((fX) \lrcorner dV_g) = dv(f(X \lrcorner dV_g)v) \\ &= df \wedge (X \lrcorner dV_g) + f \wedge d(X \lrcorner dV_g) = df \wedge (X \lrcorner dV_g) + f \operatorname{div} X. \end{aligned}$$

So we only need to check $df \wedge (X \lrcorner dV_g) = \langle \operatorname{grad} f, X \rangle_g \, dV_g = df(X) \, dV_g$. This can be done in local coordinates.

The second claim follows from

$$\int_{\partial M} f \langle X, N \rangle_g \, dV_{\tilde{g}} = \int_{\partial M} \langle fX, N \rangle_g \, dV_{\tilde{g}} = \int_M \operatorname{div}(fX) \, dV_g.$$

\square

onic exercise

Exercise 1.5.7. Let (M, g) be a Riemannian manifold with or without boundary. A function $u \in C^\infty(M)$ is said to be **harmonic** if $\Delta u = 0$.

(a) Suppose M is compact, and prove **Green's identities**:

$$\begin{aligned}\int_M u \Delta v \, dV_g &= \int_M \langle \text{grad } u, \text{grad } v \rangle_g \, dV_g - \int_{\partial M} u N v \, dV_{\tilde{g}}, \\ \int_M (u \Delta v - v \Delta u) \, dV_g &= \int_{\partial M} (v N u - u N v) \, dV_{\tilde{g}},\end{aligned}$$

where \tilde{g} denotes the induced Riemannian metric on ∂M , and N is the outward unit normal vector field along ∂M .

(b) Show that if M is compact and connected and $\partial M = \emptyset$, the only harmonic functions on M are the constants.

(c) Show that if M is compact and connected, $\partial M \neq \emptyset$, and u, v are harmonic functions on M whose restrictions to ∂M agree, then $u \equiv v$.

Proof. By Exercise [1.5.6](#) ^{div(fX) exercise}.

$$\begin{aligned}\int_M u \Delta v \, dV_g &= - \int_M u \operatorname{div}(\text{grad } v) = \int_M \langle \text{grad } u, \text{grad } v \rangle_g \, dV_g - \int_{\partial M} u \langle \text{grad } v, N \rangle_g \, dV_{\tilde{g}} \\ &= \int_M \langle \text{grad } u, \text{grad } v \rangle_g \, dV_g - \int_{\partial M} u N v \, dV_{\tilde{g}}.\end{aligned}$$

and therefore,

$$\int_M v \Delta u \, dV_g = \int_M \langle \text{grad } u, \text{grad } v \rangle_g \, dV_g - \int_{\partial M} v N u \, dV_{\tilde{g}}.$$

Thus

$$\int_M (u \Delta v - v \Delta u) \, dV_g = \int_{\partial M} (v N u - u N v) \, dV_{\tilde{g}}.$$

Now if $\Delta u = 0$, then

$$\int_M \langle \text{grad } f, \text{grad } u \rangle_g \, dV_g = 0$$

for any $f \in C^\infty(M)$. In particular, $\int_M \langle \text{grad } u, \text{grad } u \rangle_g \, dV_g = 0$, which implies $du = 0$. Now the second claim follows from the first, by applying on $u - v$. \square

Exercise 1.5.8. Let (M, g) be a compact connected Riemannian manifold without boundary, and let Δ be its geometric Laplacian. A real number λ is called an eigenvalue of Δ if there exists a smooth real-valued function u on M , not identically zero, such that $\Delta u = \lambda u$. In this case, u is called an eigenfunction corresponding to λ .

(a) Prove that 0 is an eigenvalue of Δ , and that all other eigenvalues are strictly positive.

(b) Prove that if u and v are eigenfunctions corresponding to distinct eigenvalues, then $\int_M uv \, dV_g = 0$.

Proof. The constant functions are eigenfunction corresponding to 0. Also, let $\Delta u = \lambda u$ with $\lambda \neq 0$, then

$$\lambda \int_M u^2 \, dV_g = \int_M u \Delta u \, dV_g = \int_M \langle \text{grad } u, \text{grad } u \rangle_g \, dV_g.$$

Thus $\lambda > 0$.

If $\Delta u = \lambda u$, $\Delta v = \mu v$ with $\lambda \neq \mu$, then by Exercise [1.5.7](#), ^{harmonic exercise}

$$0 = \int_M (u\Delta v - v\Delta u) dV_g = (\mu - \lambda) \int_M uv dV_g,$$

which implies $\int_M uv dV_g = 0$. □

Exercise 1.5.9. Let M be a compact connected Riemannian n -manifold with nonempty boundary. A real number λ is called a **Dirichlet eigenvalue** for M if there exists a smooth real-valued nonzero function u on M such that $\Delta u = \lambda u$ and $u|_{\partial M} = 0$. Similarly, λ is called a **Neumann eigenvalue** if there exists such a function u satisfying $\Delta u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.

(b) Show that every Dirichlet eigenvalue is strictly positive.

(b) Show that 0 is a Neumann eigenvalue, and all other Neumann eigenvalues are strictly positive.

Proof. For Dirichlet eigenvalues, we can assume $\partial M = \emptyset$, since $u|_{\partial M} = 0$. Then the claim follows from the previous exercise.

Any constant function f satisfies $Nf|_{\partial M} = 0$ and $\Delta f = 0$, thus 0 is a Neumann eigenvalue. Now let $\Delta u = \lambda u$ with $Nu|_{\partial M} = 0$, then by Exercise [1.5.7](#), ^{harmonic exercise}

$$\lambda \int_M u^2 dV_g = \int_M u\Delta u dV_g = \int_M \langle \text{grad } u, \text{grad } u \rangle_g dV_g.$$

Thus $\lambda > 0$. □

Exercise 1.5.10 (Dirichlet's Principle). Suppose M is a compact connected Riemannian n -manifold with nonempty boundary. Prove that a function $u \in C^\infty(M)$ is harmonic if and only if it minimizes $\int_M |\text{grad } u|_g^2 dV_g$ among all smooth functions with the same boundary values.

Proof. If $\Delta u = 0$, then for any function $f \in C^\infty(M)$ that vanishes on ∂M ,

$$\begin{aligned} \int_M |\text{grad } (u + \varepsilon f)|_g^2 dV_g &= \int_M |\text{grad } u|_g^2 dV_g + 2\varepsilon \int_M \langle \text{grad } f, \text{grad } u \rangle_g dV_g + \varepsilon^2 \int_M |\text{grad } f|_g^2 dV_g \\ &= \int_M |\text{grad } u|_g^2 dV_g + \varepsilon^2 \int_M |\text{grad } f|_g^2 dV_g + 2\varepsilon \left(\int_M f\Delta u dV_g + \int_{\partial M} fNu dV_g \right) \\ &= \int_M |\text{grad } u|_g^2 dV_g + \varepsilon^2 \int_M |\text{grad } f|_g^2 dV_g. \end{aligned}$$

Thus u minimizes $\int_M |\text{grad } u|_g^2 dV_g$. □

Chapter 2

Complex manifolds

2.1 Local theory of complex vector spaces

2.1.1 Complex and Hermitian Structures

Let us now come back to an Euclidian vector space $(V, \langle \cdot, \cdot \rangle)$ with a compatible almost complex structure J . The **fundamental form** associated to $(V, \langle \cdot, \cdot \rangle, J)$ is the form ω defined by

$$\omega(v, w) = \langle Jv, w \rangle = -\langle v, Jw \rangle.$$

Since $\langle \cdot, \cdot \rangle$ is compatible with J , it is easy to see ω is alternating:

$$\omega(v, w) = \langle Jv, w \rangle = \langle J^2v, Jw \rangle = -\langle v, Jw \rangle = -\omega(w, v).$$

Moreover, we note that ω is invariant under J :

$$(J\omega)(v, w) = \omega(Jv, Jw) = \langle J^2v, Jw \rangle = \langle Jv, w \rangle = \omega(v, w)$$

so that $\omega \in \bigwedge^{1,1}V^*$. We also note that two of the three structures $\{\langle \cdot, \cdot \rangle, J, \omega\}$ determine the remaining one.

Following a standard procedure, the scalar product and the fundamental form are encoded by a natural hermitian form.

ence Hermitian

Proposition 2.1.1. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space endowed with a compatible complex structure J . Then the form $(\cdot, \cdot) := \langle \cdot, \cdot \rangle - i\omega$ is a positive Hermitian form on (V, J) .*

Proof. The form (\cdot, \cdot) is clearly \mathbb{R} -linear and we have $(v, v) = \langle v, v \rangle > 0$ for nonzero $v \in V$. Moreover, $(v, w) = \overline{(w, v)}$ and

$$\begin{aligned} (Jv, w) &= \langle Jv, w \rangle - i\omega(Jv, w) = \langle Jv, w \rangle + i\langle v, w \rangle = i(\langle v, w \rangle - i\langle Jv, w \rangle) \\ &= i(\langle v, w \rangle - i\omega(v, w)) = i(v, w). \end{aligned}$$

This shows that (v, w) is sesquilinear, so it is a Hermitian form. \square

We can also consider the extension of the scalar product $\langle \cdot, \cdot \rangle$ to a positive definite hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $V_{\mathbb{C}}$. This is defined by

$$\langle v \otimes \lambda, w \otimes \mu \rangle_{\mathbb{C}} = \lambda \bar{\mu} \langle v, w \rangle$$

for $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$.

position of $V_{\mathbb{C}}$

Proposition 2.1.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space endowed with a compatible complex structure J . Then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition with respect to the Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$.*

Proof. Let $v - iJv \in V^{1,0}$ and $w + iJw \in V^{0,1}$ with $v, w \in V$. We then compute that

$$\begin{aligned} \langle v - iJv, w + iJw \rangle_{\mathbb{C}} &= \langle v, w \rangle_{\mathbb{C}} + \langle v, iJw \rangle_{\mathbb{C}} - \langle iJv, w \rangle_{\mathbb{C}} - \langle iJv, iJw \rangle_{\mathbb{C}} \\ &= \langle v, w \rangle - i\langle v, Jw \rangle - i\langle Jv, w \rangle - \langle Jv, Jw \rangle = 0. \end{aligned}$$

This proves the claim, since $V^{1,0}$ is generated by $v - iJv$, and $V^{0,1}$ is generated by $w + iJw$. \square

It turn out that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ coincides with (\cdot, \cdot) , as the following proposition shows.

form relation

Proposition 2.1.3. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space endowed with a compatible complex structure J . Then under the canonical isomorphism $(V, J) \cong (V^{1,0}, i)$, we have $\frac{1}{2}(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{\mathbb{C}}|_{V^{1,0}}$.*

Proof. The natural isomorphism is given by $v \mapsto \frac{1}{2}(v - iJv)$, so by the definition of (\cdot, \cdot) , for $v, w \in V$ we have

$$\begin{aligned} \langle v - iJv, w - iJw \rangle_{\mathbb{C}} &= \langle v, w \rangle + i\langle v, Jw \rangle - i\langle Jv, w \rangle + \langle Jv, Jw \rangle \\ &= 2\langle v, w \rangle + 2i\langle v, Jw \rangle = 2(v, w). \end{aligned}$$

This proves the assertion. \square

Often, it is useful to do calculations in coordinates. Let us see how the above products can be expressed explicitly once suitable basis have been chosen. Let z_1, \dots, z_n be a \mathbb{C} -basis of $V^{1,0}$. Write $z_i = \frac{1}{2}(x_i - iJx_i)$ with $x_i \in V$, then we have seen that $x_1, y_1 = Jx_1, \dots, x_n, y_n = Jx_n$ is a \mathbb{R} -basis of V and x_1, \dots, x_n is a \mathbb{C} -basis for (V, J) . The Hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $V^{1,0}$ with respect to the basis z_i is given by an Hermitian matrix, say $\frac{1}{2}(h_{ij})$. Concretely,

$$\langle \sum_{i=1}^n a_i z_i, \sum_{j=1}^n b_j z_j \rangle_{\mathbb{C}} = \frac{1}{2} \sum_{i,j=1}^n h_{ij} a_i \bar{b}_j.$$

By Proposition [2.1.3](#), the Hermitian form (\cdot, \cdot) on (V, J) with respect to the basis x_i is given by (h_{ij}) . We want to compute the fundamental form ω in this case, with respect to the basis x_i, y_i .

class express

Proposition 2.1.4. *The fundamental form ω on (V, J) is given by*

$$\omega = - \sum_{i < j} (\operatorname{Im} h_{ij})(x^i \wedge x^j + y^i \wedge y^j) + \sum_{i,j=1}^n (\operatorname{Re} h_{ij}) x^i \wedge y^j \quad (1.1) \quad \text{almost complex}$$

with respect to the basis x_i, y_i . Moreover, the extension of ω on $V_{\mathbb{C}}$ can be written as

$$\omega = \frac{i}{2} \sum_{i,j=1}^n h_{ij} z^i \wedge \bar{z}^j. \quad (1.2) \quad \text{almost complex}$$

Proof. By the definition of (\cdot, \cdot) , we have $\omega = -\operatorname{Im}(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}} = \operatorname{Re}(\cdot, \cdot)$. Hence,

$$\omega(x_i, x_j) = \omega(y_i, y_j) = -\operatorname{Im} h_{ij}, \quad \omega(x_i, y_j) = \operatorname{Re} h_{ij}.$$

This then gives the expression of ω as

$$\omega = - \sum_{i < j} \operatorname{Im} h_{ij} (x^i \wedge x^j + y^i \wedge y^j) + \sum_{i,j=1}^n \operatorname{Re} h_{ij} x^i \wedge y^j.$$

To see the expression (1.2), we note that

$$z^i \wedge \bar{z}^j = (x^i + iy^i) \wedge (x^j - iy^j) = (x^i \wedge x^j + y^i \wedge y^j) - i(x^i \wedge y^j + x^j \wedge y^i);$$

so (1.2) is equal to

$$\begin{aligned} \frac{i}{2} \sum_{i,j=1}^n h_{ij} z^i \wedge \bar{z}^j &= \sum_{i,j=1}^n \left[\frac{i}{2} \operatorname{Re} h_{ij} - \frac{1}{2} \operatorname{Im} h_{ij} \right] [(x^i \wedge x^j + y^i \wedge y^j) - i(x^i \wedge y^j + x^j \wedge y^i)] \\ &= \sum_{i,j} \frac{1}{2} \operatorname{Re} h_{ij} (x^i \wedge y^j + x^j \wedge y^i) - \sum_{i,j} \frac{1}{2} \operatorname{Im} h_{ij} (x^i \wedge x^j + y^i \wedge y^j) \\ &\quad + \sum_{i,j} \frac{i}{2} \operatorname{Re} h_{ij} (x^i \wedge x^j + y^i \wedge y^j) + \sum_{i,j} \frac{i}{2} \operatorname{Im} h_{ij} (x^i \wedge y^j + x^j \wedge y^i). \end{aligned} \quad (1.3)$$

On the other hand, we note that since (h_{ij}) is a Hermitian matrix, we have

$$\operatorname{Re} h_{ij} = \operatorname{Re} h_{ji}, \quad \operatorname{Im} h_{ij} = -\operatorname{Im} h_{ji}.$$

With these relations, we see the first two terms in the last summand of (1.3) add up to ω , and the last two terms are zero. This completes the proof. \square

Example 2.1.5. If x_i, y_i is an orthonormal basis for V with respect to $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^n (x^i \otimes x^i + y^i \otimes y^i)$$

then by Proposition 2.1.4, we have

$$\omega = \frac{i}{2} \sum_{i=1}^n z^i \wedge \bar{z}^i = \sum_{i=1}^n x^i \wedge y^i.$$

With the fundamental class ω , we now define the Lefschetz operator $L : \Lambda^* V_{\mathbb{C}}^* \rightarrow \Lambda^* V_{\mathbb{C}}^*$ by $\alpha \mapsto \omega \wedge \alpha$. It is clear that L is the \mathbb{C} -linear extension of the real operator $\Lambda^* V \rightarrow \Lambda^* V$ given by $\alpha \mapsto \omega \wedge \alpha$, and since ω is of bidegree $(1, 1)$, L is of bidegree $(1, 1)$, i.e.,

$$L(\Lambda^{p,q} V^*) \subseteq \Lambda^{p+1,q+1} V^*.$$

We shall show that the power

$$L^k : \Lambda^k V^* \rightarrow \Lambda^{2n-k} V^*$$

is an isomorphism for all $k \leq n$, where we write $\dim_{\mathbb{R}} V = 2n$. An elementary proof can be given by choosing a basis, but it is slightly cumbersome. A more elegant but less elementary argument, using $\mathfrak{sl}(2, \mathbb{C})$ -representation theory, will be given below.

The Lefschetz operator comes along with its dual Λ . In order to define and to describe Λ we need to recall the Hodge $*$ -operator on a real vector space. Let $(V, \langle \cdot, \cdot \rangle)$ be an oriented euclidian vector space of dimension d , then $\langle \cdot, \cdot \rangle$ can be extended to all the exterior powers $\Lambda^k V$. Explicitly, if $e_1, \dots, e_d \in V$ is an orthonormal basis of V , then $e^I \in \Lambda^k V$ is an orthonormal basis of $\Lambda^k V$, where $I = \{i_1 < \dots < i_k\}$ is a subset of $\{1, \dots, d\}$. Let $\operatorname{vol} \in \Lambda^d V$ be the orientation of V of norm 1 given by $\operatorname{vol} = e_1 \wedge \dots \wedge e_d$. Then the Hodge $*$ -operator is defined by

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \cdot \operatorname{vol}$$

for $\alpha, \beta \in \Lambda^k V$. This determines $*$ since the exterior product defines a nondegenerate pairing $\Lambda^k V \times \Lambda^{d-k} V = \operatorname{vol} \cdot \mathbb{R}$. One easily sees that $*$: $\Lambda^k V \rightarrow \Lambda^{d-k} V$. The most important properties

of the Hodge $*$ -operator are collected in the following proposition.

operator prop

Proposition 2.1.6. *Let $(V, \langle \cdot, \cdot \rangle)$ be an oriented euclidian vector space of dimension d . Let e_1, \dots, e_d be an orthonormal basis of V and let $\text{vol} \in \bigwedge^d V$ be the orientation of norm one given by $e_1 \wedge \dots \wedge e_d$. The Hodge $*$ -operator associated to $(V, \langle \cdot, \cdot \rangle, \text{vol})$ satisfies the following conditions:*

(a) *If $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{d-k}\}$ are complementary, then*

$$*(e_{i_1} \wedge \dots \wedge e_{i_k}) = \varepsilon \cdot e_{j_1} \wedge \dots \wedge e_{j_{d-k}}$$

*where $\varepsilon = \text{sgn}(i_1, \dots, i_k, j_1, \dots, j_{d-k})$. In particular, $*1 = \text{vol}$.*

(b) *The $*$ -operator is self-adjoint up to a sign: for $\alpha \in \bigwedge^k V$ we have*

$$\langle \alpha, *\beta \rangle = (-1)^{k(d-k)} \langle *\alpha, \beta \rangle.$$

(c) *The $*$ -operator is involutive up to a sign:*

$$(*|_{\bigwedge^k V})^2 = (-1)^{k(d-k)}.$$

(d) *The $*$ -operator is an isometry on $(\bigwedge^* V, \langle \cdot, \cdot \rangle)$.*

In our situation we will usually have $d = 2n$ and $*$ and $\langle \cdot, \cdot \rangle$ will be considered on the dual space $\bigwedge^* V^*$. Let us now come back to the situation considered before. Associated to $(V, \langle \cdot, \cdot \rangle, J)$ we had introduced the Lefschetz operator $L : \bigwedge^k V^* \rightarrow \bigwedge^{k+2} V^*$. The **dual Lefschetz operator** Λ is the operator $\Lambda : \bigwedge^* V^* \rightarrow \bigwedge^* V^*$ that is adjoint to L with respect to $\langle \cdot, \cdot \rangle$, i.e. $\Lambda\alpha$ is uniquely determined by the condition

$$\langle \Lambda\alpha, \beta \rangle = \langle \alpha, L\beta \rangle$$

for all $\beta \in \bigwedge^* V^*$. The \mathbb{C} -linear extension of the dual Lefschetz operator will also be denoted by Λ .

Remark 2.1.1. Recall that J induces a natural orientation on V . Thus, the Hodge $*$ -operator is well-defined. Using an orthonormal basis $x_i, y_i = Jx_i$ as above, a straightforward calculation yields

$$\omega^n = n! \cdot \text{vol}$$

where ω is the associated fundamental form.

or expression

Proposition 2.1.7. *The dual Lefschetz operator Λ is of degree -2 and we have $\Lambda = *^{-1} \circ L \circ *$.*

Proof. The first assertion follows from the fact that L is of degree two and that the decomposition $\bigwedge^* V^* = \bigoplus_k \bigwedge^k V^*$ is orthogonal. Now by the definition of the Hodge $*$ -operator we have

$$\langle L\beta, \alpha \rangle \cdot \text{vol} = L\beta \wedge *\alpha = \omega \wedge \beta \wedge *\alpha = \beta \wedge (\omega \wedge *\alpha) = \langle \beta, *^{-1}(L(*\alpha)) \rangle \cdot \text{vol}.$$

Since α and β are arbitrary, this shows $\Lambda = *^{-1} \circ L \circ *$, as desired. \square

Recall that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ had been defined as the Hermitian extension to $V_{\mathbb{C}}^*$ of the scalar product $\langle \cdot, \cdot \rangle$ on V^* . It can further be extended to a positive definite hermitian form on $\bigwedge^* V_{\mathbb{C}}^*$. Equivalently, one could consider the extension of (\cdot, \cdot) on $\bigwedge^* V^*$ to an Hermitian form on $\bigwedge^* V_{\mathbb{C}}^*$. In any case, there is a natural positive Hermitian product on $\bigwedge^* V_{\mathbb{C}}^*$ which will also

be called $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. The Hodge $*$ -operator associated to $(V, \langle \cdot, \cdot \rangle, \text{vol})$ is extended \mathbb{C} -linearly to $*$: $\Lambda^k V_{\mathbb{C}}^* \rightarrow \Lambda^{2n-k} V_{\mathbb{C}}^*$. On $\Lambda^* V_{\mathbb{C}}^*$ these two operators are now related by

$$\alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \text{vol}.$$

Clearly, the Lefschetz operator L and its dual Λ on $\Lambda^* V_{\mathbb{C}}^*$ are also formally adjoint to each other with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, and the equality $\Lambda = *^{-1} \circ L \circ *$ is still valid.

operator prop

Proposition 2.1.8. *Let $(V, \langle \cdot, \cdot \rangle, \text{vol})$ be as above and $n = \dim_{\mathbb{C}}(V, J)$.*

- (a) *The decomposition $\Lambda^k V_{\mathbb{C}}^* = \bigoplus \Lambda^{p,q} V^*$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$.*
- (b) *The Hodge $*$ -operator maps $\Lambda^{p,q} V^*$ to $\Lambda^{n-q, n-p} V^*$.*
- (c) *The dual Lefschetz operator Λ is of bidegree $(-1, -1)$.*

We continue to assume that $\dim_{\mathbb{R}} V = 2n$, and let $H : \Lambda^* V \rightarrow \Lambda^* V$ be the counting operator defined by

$$H = \sum_{k=0}^{2n} (n-k) \cdot \Pi^k.$$

It is clear that each $\Lambda^k V^*$ is an eigenspace of H . With H, L, Λ, Π , etc., we dispose of a large number of linear operators on $\Lambda^* V^*$ and one might wonder whether they commute. In fact, they do not, but their commutators can be computed.

representation

Theorem 2.1.9. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space endowed with a compatible almost complex structure J . Consider the linear operators H, L, Λ , and Π .*

- (a) $[L, J] = [\Lambda, J] = 0$.
- (b) $[H, L] = 2L$, $[H, \Lambda] = -2\Lambda$, and $[\Lambda, L] = H$.

To prove Theorem ^{almost complex space $\mathfrak{sl}(2, \mathbb{C})$ representation} 2.1.9, it is necessary to introduce some notation which will allow us to effectively work with the convectors in $\Lambda^* V^*$. We consider multi-indices $I = (i_1, \dots, i_k)$, where $i_1 < \dots < i_k$ are distinct elements of $\{1, \dots, n\}$, and set $|I| = k$. If $I = \{i_1, \dots, i_k\}$, then we write

$$z^I = z^{i_1} \wedge \dots \wedge z^{i_k}, \quad x^I = x^{i_1} \wedge \dots \wedge x^{i_k}$$

and so on. If M is a multiindex, we let

$$\alpha^M = \prod_{i \in M} z^i \wedge \bar{z}^i = (-2i)^{|M|} \prod_{i \in M} x^i \wedge y^i.$$

In this last product it is clear that the ordering of the factors is irrelevant, since the terms commute with one another, and we shall use the same symbol M to denote the ordered k -tuple and its underlying set of elements, provided that this leads to no confusion. Any element of $\Lambda^* V^*$ can be written in the form

$$\sum'_{A, B, M} c_{A, B, M} z^A \wedge \bar{z}^B \wedge \alpha^M$$

where $c_{A, B, M} \in \mathbb{C}$, and A, B , and M are (for a given term) mutually disjoint multiindices, and, as before, the prime on the summation sign indicates that the sum is taken over multiindices whose elements are strictly increasing sequences (what we shall call an increasing multiindex).

We have the following fundamental and elementary lemma which shows the interaction between the $*$ -operator (defined in terms of the real structure) and the bigrading on $\wedge^* V^*$ (defined in terms of the almost complex structure).

ed form lemma

Lemma 2.1.10. *Suppose that A , B , and M are mutually disjoint increasing multiindices. Then*

$$*(z^A \wedge \bar{z}^B \wedge \alpha^M) = \gamma(a, b, m) z^A \wedge \bar{z}^B \wedge \alpha^N$$

for a nonvanishing constant $\gamma(a, b, m)$, where $a = |A|$, $b = |B|$, $m = |M|$, and N is the complement of $A \cup B \cup M$. Moreover,

$$\gamma(a, b, m) = i^{a-b} (-1)^{\frac{p(p+1)}{2} + m} (-2i)^{p-n}$$

where $p = a + b + 2m$ is the total degree of $z^A \wedge \bar{z}^B \wedge \alpha^M$.

Proof. Let $\beta = z^A \wedge \bar{z}^B \wedge \alpha^M$. If $A = A_1 \cup A_2$ for some multiindex A , let

$$\epsilon_A^{A_1 A_2} = \begin{cases} 0 & \text{if } A_1 \cap A_2 \neq \emptyset \\ 1 & \text{if } A_1 A_2 \text{ is an even permutation of } A \\ -1 & \text{if } A_1 A_2 \text{ is an odd permutation of } A \end{cases}$$

Using this notation it is easy to see that

$$z^A = \sum_{A=A_1 \cup A_2} \epsilon_A^{A_1 A_2} \cdot i^{a_2} x^{A_1} \wedge y^{A_2},$$

where the sum runs over all decompositions of A into increasing multiindices, and we set $a_1 = |A_1|$, $a_2 = |A_2|$, etc. We then obtain

$$\beta = (-2i)^m \sum_{\substack{A=A_1 \cup A_2 \\ B=B_1 \cup B_2}} \epsilon_A^{A_1 A_2} \epsilon_B^{B_1 B_2} \cdot i^{a_2 - b_2} x^{A_1} \wedge y^{A_2} \wedge x^{B_1} \wedge y^{B_2} \wedge \prod_{\mu \in M} x^\mu \wedge y^\mu.$$

We want to compute $*\beta$, having expressed β in terms of a real basis, and we shall do this term by term and then sum the result. To simplify the notation, consider the case where $B = \emptyset$. In this case,

$$*(z^A \wedge \alpha^M) = (-2i)^m \sum_{A=A_1 \cup A_2} \epsilon_A^{A_1 A_2} i^{a_2} * \left\{ x^{A_1} \wedge y^{A_2} \wedge \prod_{\mu \in M} x^\mu \wedge y^\mu \right\}. \quad (1.4) \quad \text{almost complex}$$

It is clear that the result of $*$ acting on the bracketed expression is of the form

$$\pm x^{A_2} \wedge y^{A_1} \wedge \prod_{\mu \in N} x^\mu \wedge y^\mu \quad (1.5) \quad \text{almost complex}$$

where $N = (A \cup M)^c$. The only problem is to determine the sign. To do this it suffices (because of the commutativity of $\prod_{\mu \in M} x^\mu \wedge y^\mu$) to consider the product

$$x^{A_1} \wedge y^{A_2} \wedge x^{A_2} \wedge y^{A_1} = (-1)^{a_2^2} x^{A_1} \wedge y^{A_1} \wedge x^{A_2} \wedge y^{A_2}.$$

In general, we have

$$x^C \wedge y^C = (-1)^{\frac{|C|(|C|-1)}{2}} x^{\mu_1} \wedge y^{\mu_2} \wedge \dots \wedge x^{\mu_{|C|}} \wedge y^{\mu_{|C|}}$$

and applying this to our problem above, we see immediately that the sign in (1.5) is of the almost complex space Hodge
form

$$(-1)^{a_2^2 + \frac{a_1(a_1-1)}{2} + \frac{a_2(a_2-1)}{2}} =: (-1)^r$$

From this and [\(1.4\)](#), we conclude that [almost complex space Hodge star on standard form lemma-1](#)

$$*(z^A \wedge \alpha^M) = (-2i)^m \sum_{A=A_1 \cup A_2} \epsilon_A^{A_1 A_2; a_2} (-1)^r x^{A_2} \wedge y^{A_1} \wedge \prod_{\mu \in N} x^\mu \wedge y^\mu. \quad (1.6) \quad \text{almost complex}$$

The idea now is to change variables in the summation. We write

$$\epsilon_A^{A_1 A_2} = (-1)^{a_1 a_2} \epsilon_A^{A_2 A_1}, \quad i^{a_2} = i^a (-1)^{a_1} i^{a_1},$$

and by substituting this in [\(1.6\)](#), we obtain that [almost complex space Hodge star on standard form lemma-3](#)

$$*(z^A \wedge \alpha^M) = i^a (-2i)^m \sum_{A=A_1 \cup A_2} \epsilon_A^{A_2 A_1; a_1} \cdot \{(-1)^{r+a_1+a_1 a_2}\} \cdot x^{A_2} \wedge y^{A_1} \wedge \prod_{\mu \in N} x^\mu \wedge y^\mu,$$

which is, modulo the bracketed term, of the right form to be $(z^A \wedge \alpha^M)$. A priori, this term depends on the decompositions $A = A_1 \cup A_2$; however, one can verify that in fact

$$(-1)^{r+a_1+a_1 a_2} = (-1)^{\frac{a(a+1)}{2}} = (-1)^{\frac{p(p+1)}{2}+m}$$

and the bracketed constant pulls out in front the summation, so

$$*(z^A \wedge \alpha^M) = i^a (-1)^{\frac{p(p+1)}{2}+m} (-2i)^{p-n} z^A \wedge \alpha^N.$$

The general case can be treated similarly. □

Proof of Theorem [2.1.9](#). [almost complex space \$\mathfrak{sl}\(2, \mathbb{C}\)\$ representation](#) The first assertion follows from the fact that L and Λ are homogeneous operators and are real. We now prove the second one. It is immediate that, for $\alpha \in \wedge^k V^*$, we have

$$[H, L](\alpha) = (n - k - 2)(\omega \wedge \alpha) - \omega \wedge ((n - k)\alpha) = -2\omega \wedge \alpha,$$

$$[H, \Lambda](\alpha) = (n - k + 2)(\Lambda \alpha) - \Lambda((n - k)\alpha) = -2\Lambda \alpha.$$

This proves the first two equalities. Now using the notation in Lemma [2.1.10](#), we observe that [almost complex space Hodge star on standard form lemma](#)

$$L(z^A \wedge \bar{z}^B \wedge \alpha^M) = \frac{i}{2} \left(\sum_{i=1}^n z^i \wedge \bar{z}^i \right) \wedge z^A \wedge \bar{z}^B \wedge \alpha^M = \frac{i}{2} z^A \wedge \bar{z}^B \wedge \left(\sum_{i \notin A \cup B \cup M} \alpha^{M+\{i\}} \right)$$

On the other hand, we see that, using Lemma [2.1.10](#) and the definition of Λ , [almost complex space Hodge star on standard form lemma](#)

$$\Lambda(z^A \wedge \bar{z}^B \wedge \alpha^M) = \frac{2}{i} z^A \wedge \bar{z}^B \wedge \left(\sum_{i \in M} \alpha^{M-\{i\}} \right).$$

Using these formulas, one obtains easily, assuming that $z^A \wedge \bar{z}^B \wedge \alpha^M$ has total degree k ,

$$(\Lambda L - L \Lambda)(z^A \wedge \bar{z}^B \wedge \alpha^M) = (n - k) z^A \wedge \bar{z}^B \wedge \alpha^M,$$

and part (c) of Theorem [2.1.9](#) follows immediately. [almost complex space \$\mathfrak{sl}\(2, \mathbb{C}\)\$ representation](#) □

Corollary 2.1.11. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space endowed with a compatible almost complex structure J . The triple (Λ, H, L) defines a natural $\mathfrak{sl}(2, \mathbb{C})$ -representation on $\wedge^* V^*$.*

Corollary 2.1.12. *For any $\alpha \in \wedge^k V^*$, we have*

$$[L^i, \Lambda] \alpha = i(k - n + i - 1) L^{i-1}(\alpha).$$

Proof. This can be proved by induction on i , using the relation $[\Lambda, L] = H$. □

Following the terminology of representations of $\mathfrak{sl}(2, \mathbb{C})$, an element $\alpha \in \Lambda^k V^*$ is called **primitive** if $\Lambda \alpha = 0$. The linear subspace of all primitive elements $\alpha \in \Lambda^k V^*$ is denoted by $P^k \subseteq \Lambda^k V^*$. Accordingly, an element $\alpha \in \Lambda^k V_{\mathbb{C}}^*$ is called **primitive** if $\Lambda \alpha = 0$. Clearly, the subspace of those is just the complexification of P^k .

decomposition

Proposition 2.1.13. *Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidian vector space of dimension $2n$ with a compatible almost complex structure J and let L and Λ be the associated Lefschetz operators.*

(a) *There is a **Lefschetz decomposition** of the form*

$$\Lambda^k V^* = \bigoplus_{i \geq 0} L^i (P^{k-2i})$$

which is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

(b) *If $k > n$, then $P^k = 0$, and $P^k = \{\alpha \in \Lambda^k V^* : L^{n-k+1} \alpha = 0\}$ for $k \leq n$.*

(c) *The map $L^{n-k} : P^k \rightarrow \Lambda^{2n-k} V^*$ is injective for $k \leq n$.*

(d) *The map $L^{n-k} : \Lambda^k V^* \rightarrow \Lambda^{2n-k} V^*$ is bijective for $k \leq n$.*

Remark 2.1.2. Since L , Λ , and H are of pure type $(1, 1)$, $(-1, -1)$ and $(0, 0)$, respectively, the Lefschetz decomposition is compatible with the bidegree decomposition. Therefore we conclude

$$P^k = \bigoplus_{p+q=k} P^{p,q}$$

where $P^{p,q} = P_{\mathbb{C}}^k \cap \Lambda^{p,q} V^*$. Since Λ and L are real, we also have $\overline{P^{p,q}} = P^{q,p}$.

In particular, we have

$$\Lambda^0 V_{\mathbb{C}}^* = P^{0,0} = P_{\mathbb{C}}^0 = \mathbb{C}, \quad \Lambda^1 V_{\mathbb{C}}^* = P^{1,0} \oplus P^{0,1},$$

and

$$\Lambda^2 V_{\mathbb{C}}^* = \Lambda^{2,0} V^* \oplus \Lambda^{1,1} V^* \oplus \Lambda^{0,2} V^* = P^{2,0} \oplus (P^{1,1} \oplus \omega \mathbb{C}) \oplus P^{0,2}.$$

Roughly, the Lefschetz operators and its dual Λ induce a reflection of $\Lambda^* V^*$ in the middle exterior product $\Lambda^n V^*$. But there is another operator with this property, namely the Hodge $*$ -operator. We now want to prove some fundamental results concerning the relationship between the operators $*$, L , and Λ which are important in the theory of Kähler manifolds. The development we give here is due to Hecht and differs from the more traditional viewpoint of Weil in that a global representation of both $\mathrm{SL}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$ on the Hermitian exterior algebra is utilized, leading to some simple ordinary differential equations which simplifies some of the combinatorial arguments found by Weil.

Let $\{z_i = \frac{1}{2}(x_i - Jx_i)\}$ be an orthonormal basis for $V^{1,0}$, so that $\{x_i, y_i = Jx_i\}$ is an orthonormal real basis for V . With this, the fundamental form ω has the form

$$\omega = \frac{i}{2} \sum_{i=1}^n z^i \wedge \bar{z}^i = \sum_{i=1}^n x^i \wedge y^i.$$

Now if η is any k -form in $\Lambda^* V_{\mathbb{C}}^*$, we let

$$e(\eta)(\alpha) := \eta \wedge \alpha$$

be the operator acting on $\Lambda^* V_{\mathbb{C}}^*$ given by wedging with η . We note that

$$L = e(\omega) = \frac{i}{2} \sum_{i=1}^n e(z^i) e(\bar{z}^i), \quad \Lambda = e^*(\omega) = -\frac{i}{2} \sum_{i=1}^n e^*(\bar{z}^i) e^*(z^i). \quad (1.7)$$

It is clear that $[L, e(\eta)] = 0$ since ω is a 2-form. As for $[\Lambda, e(\eta)]$, we have the following proposition.

Proposition 2.1.14. *Let ω be the fundamental form of $(V, \langle \cdot, \cdot \rangle)$ and η be a real 1-form. Then*

$$[\Lambda, e(\eta)] = -J e^*(\eta) J^{-1}. \quad (1.8)$$

In particular, if η is a $(1, 0)$ -form, we have

$$[\Lambda, e(\eta)] = -i e^*(\bar{\eta}), \quad [\Lambda, e(\bar{\eta})] = i e^*(\eta). \quad (1.9)$$

Proof. If η is a real 1-form, then we claim that

$$e(\eta)^* = *e(\eta) *.$$

To see this, we note that for any $\alpha, \beta \in \Lambda^* V_{\mathbb{C}}^*$ with degrees $k-1$ and k , we have (note that $\eta \wedge * \beta$ is of degree $2n - k + 1$)

$$\langle e(\eta) \alpha, \beta \rangle \cdot \text{vol} = \eta \wedge \alpha \wedge * \beta = (-1)^{k-1} \alpha \wedge (\eta \wedge * \beta) = \alpha \wedge * (* (\eta \wedge * \beta)) = \langle \alpha, *e(\eta) * \beta \rangle \cdot \text{vol}.$$

Now by computing the right side $*e(\eta)*$, we see

$$e^*(z^j)(z^I \wedge \bar{z}^K) = \begin{cases} 0, & j \notin I, \\ 2\varepsilon(j, I) z^{I-\{j\}} \wedge \bar{z}^K, & j \in I, \end{cases} \quad (1.10)$$

$$e^*(\bar{z}^k)(z^I \wedge \bar{z}^K) = \begin{cases} 0, & k \notin K, \\ 2(-1)^{|I|+n(k, K)} z^I \wedge \bar{z}^{K-\{k\}}, & k \in K. \end{cases} \quad (1.11)$$

where $\eta(j, I)$ is the number of indices in I that is strictly smaller than j , and similar for $\eta(k, K)$. Now by (1.7), we then have

$$\begin{aligned} [\Lambda, e(z^j)] &= -\frac{i}{2} \left[\sum_{i=1}^n e^*(\bar{z}^i) e^*(z^i) e(z^j) - \sum_{i=1}^n e(z^j) e^*(\bar{z}^i) e^*(z^i) \right] \\ &= -\frac{i}{2} [e^*(\bar{z}^j) e^*(z^j) e(z^j) - e(z^j) e^*(\bar{z}^j) e^*(z^j)] \end{aligned}$$

since $e(z^j)$ commutes with $e^*(z^i)$ and $e^*(\bar{z}^i)$ for $i \neq j$, which follows from (1.10) and (1.11).

We now consider the action of $[\Lambda, e(z^j)]$ on $z^I \wedge \bar{z}^K$, which is distinguished into two cases: if $j \in I$, then $e(z^j)(z^I \wedge \bar{z}^K) = 0$, so

$$[\Lambda, e(z^j)](z^I \wedge \bar{z}^K) = \frac{i}{2} (-1)^{n(j, I)} e(z^j) e^*(\bar{z}^j) (z^{I-\{j\}} \wedge \bar{z}^K).$$

Again, if $j \notin K$ then the right side is identically zero, and otherwise, we have

$$\begin{aligned} [\Lambda, e(z^j)](z^I \wedge \bar{z}^K) &= i \cdot (-1)^{n(j, I) + \eta(j, K) + |I| - 1} e(z^j) (z^{I-\{j\}} \wedge \bar{z}_{K-\{j\}}) \\ &= -i \cdot (-1)^{|I| + \eta(j, K)} z^I \wedge \bar{z}_{K-\{j\}}. \end{aligned}$$

On the other hand, if $j \notin I$, then $e^*(z^j)(z^I \wedge \bar{z}^K) = 0$, so

$$[\Lambda, e(z^j)](z^I \wedge \bar{z}^K) = -\frac{i}{2} e^*(\bar{z}^j) e^*(z^j) (z^j \wedge z^I \wedge \bar{z}^K).$$

Again, this is zero if $j \notin K$, and if $j \in K$ we have

$$[\Lambda, e(z^j)](z^I \wedge \bar{z}^K) = -i \cdot (-1)^{|I|+n(j,K)} z^I \wedge \bar{z}^{K-\{j\}}.$$

In both cases, we have $[\Lambda, e(z^j)](z^I \wedge \bar{z}^K) = -ie^*(\bar{z}^j)(z^I \wedge \bar{z}^K)$, so the first equality of (1.9) follows. The second one can be proved similarly. Finally, to see (1.8), we simply note that any real 1-form can be written in the form $\eta = \alpha + \beta$, where α is of type $(1, 0)$ and β is of type $(0, 1)$. Then we can check that

$$-ie^*(\alpha) = -Je^*(\bar{\alpha})J^{-1}, \quad ie^*(\beta) = -Je^*(\beta)J^{-1}.$$

Combine this with (1.9), we see (1.8) follows. \square

With these preparations made, we now want to prove two basic lemmas due to Hecht. We introduce the following operator on $\Lambda^* V_{\mathbb{C}}^*$ induced by the action of $\mathrm{SL}(2, \mathbb{C})$ on $\Lambda^* V_{\mathbb{C}}^*$ by the representation of $\mathfrak{sl}(2, \mathbb{C})$. Recall that the Weyl element θ in $\mathrm{SL}(2, \mathbb{C})$ is given by

$$\theta = \exp\left(\frac{i\pi}{2}(e + f)\right) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The point is that conjugation by θ in $\mathfrak{sl}(2, \mathbb{C})$ gives rise to a reflection with respect to basis (e, h, f) (the Weyl group reflection). Namely,

$$\theta h \theta^{-1} = -h, \quad \theta e \theta^{-1} = f, \quad \theta f \theta^{-1} = e.$$

Now we let $\#$ be the action of θ on $\Lambda^* V_{\mathbb{C}}^*$, that is,

$$\# = \theta \Lambda^* V_{\mathbb{C}}^* = \exp\left(\frac{\pi i}{2}(e + f)\right) \Lambda^* V_{\mathbb{C}}^* = \exp\left(\frac{\pi i}{2}(\Lambda + L)\right).$$

Lemma 2.1.15. *Let (V, ρ) be a representation of $\mathfrak{sl}(2, \mathbb{C})$ and $v \in V$ be a primitive element of weight n . Let π be the induced representation of $\mathrm{SL}(2, \mathbb{C})$ on V , then for $0 \leq k \leq n - 1$,*

$$\pi(\theta)\rho(f)^k(v) = i^n \frac{k!}{(n-k)!} \rho(f)^{n-k} v. \quad (1.12)$$

Proof. Let $\mathrm{SL}(2, \mathbb{C})$ act on \mathbb{C}^2 by left matrix multiplication, and we consider $\mathrm{Sym}^n(\mathbb{C}^2)$, the n -fold symmetric tensor product of \mathbb{C}^2 with itself. Then V is isomorphic to this representation, so we consider $V = \mathrm{Sym}^n(\mathbb{C}^2)$. If $n = 1$, we let

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be a basis of \mathbb{C}^2 . Then we have $h \cdot v = v$, $e \cdot v = 0$, and $f \cdot v = w$, so v is a primitive vector. We also note that $\theta \cdot v = iw = ifv$, so (1.12) holds in this case.

Now the general case is obtained by taking symmetric power on \mathbb{C}^2 . That is, if we set

$$\omega_i = \binom{n}{i} v^{n-i} w^i$$

then $\omega_0 = v^n$ is a primitive vector of V , and we have

$$\rho(f)^k \omega_0 = \frac{n!}{(n-k)!} v^{n-k} w^k = \frac{1}{k!} \omega_k.$$

Therefore, the action of θ on $\rho(f)^k \omega_0$ is given by

$$\pi(\theta)\rho(f)^k \omega_0 = \frac{n!}{(n-k)!} \pi(\theta)(v^{n-k} w^k) = i^n \frac{k!}{(n-k)!} \frac{n!}{k!} v^k w^{n-k} = i^n \frac{k!}{(n-k)!} \rho(f)^{n-k} \omega_0.$$

Since every irreducible representation of V is isomorphic to some $\text{Sym}^n(\mathbb{C}^2)$, we see the claim follows. \square

Lemma 2.1.16. *Let η be a real 1-form. Then*

$$\#e(\eta)\#^{-1} = -iJe^*(\eta)J^{-1}. \quad (1.13)$$

Proof. For simplicity we write $e = e(\eta)$. Now for $t \in \mathbb{C}$, we consider the operator

$$e_t = \exp(it(\Lambda + L)) \cdot e \cdot \exp(-it(\Lambda + L)).$$

We note that $e_{\pi/2} = \#e\#^{-1}$. We will see that e_t satisfies a simple differential equation with initial condition $e_0 = e$, which can be easily solved, and evaluating the solution at $t = 1/2\pi$ will give the desired result. We first note that by the equality $\exp(\text{ad}(A)) = \text{Ad}(\exp(A))$, we have

$$e_t = \exp(\text{ad}(it(\Lambda + L)))(e) = \sum_{k=0}^{\infty} \frac{\text{ad}^k(it(\Lambda + L))}{k!}(e). \quad (1.14)$$

Now $\text{ad}^k(\Lambda + L)$ is a sum of monomials in $\text{ad}(\Lambda)$ and $\text{ad}(L)$. Since $\Lambda L = L\Lambda + H$, $\text{ad}(L)(e) = 0$, and $\text{ad}(H)(e) = -e$ (since η is of degree 1), we see that e_t can be expressed in the form

$$e_t = \sum_{k=0}^{\infty} a_k(t) \text{ad}^k(\Lambda)(e),$$

where $a_k(t)$ are real-analytic functions in t . Now (1.8) implies that $\text{ad}^k(\Lambda)(e) = 0$ for $k \geq 2$, since Λ commutes with J and e^* . Thus

$$e_t = a_0(t)e + a_1(t)\text{ad}(\Lambda)(e). \quad (1.15)$$

We now consider differentiating with respect to t . From (1.14), we see e_t satisfies the differential equation

$$\begin{cases} e'_t = i(\text{ad}(\Lambda) + \text{ad}(L))(e_t), \\ e_0 = e. \end{cases}$$

We can solve this using (1.15). Namely, we have

$$e_t = a'_0(t)e + a'_1(t)\text{ad}(\Lambda)(e), \quad (1.16)$$

which must equal to the product

$$i(\text{ad}(\Lambda + L))[a_0(t)e + a_1(t)\text{ad}(\Lambda)(e)] = ia_0(t)\text{ad}(\Lambda)(e) + ia_1(t)\text{ad}(L)\text{ad}(\Lambda)(e),$$

using the fact that $\text{ad}^2(\Lambda)(e) = 0$ and $\text{ad}(L)(e) = 0$. But

$$\text{ad}(L)\text{ad}(\Lambda)(e) = \text{ad}([L, \Lambda])(e) + \text{ad}(\Lambda)\text{ad}(L)(e) = \text{ad}(-H)(e) = e,$$

so the right side of (1.16) must equal to $ia_0(t)\text{ad}(\Lambda)(e) + ia_1(t)e$. Comparing the coefficients, we conclude that

$$\begin{cases} a'_0(t) = ia_1(t), \\ a'_1(t) = ia_0(t). \end{cases}$$

Then by letting $a_0(t) = \cos t$ and $a_1(t) = \sin t$, we find that

$$e_t = \cos t \cdot e + \sin t \cdot \text{ad}(\Lambda)(e).$$

Now set $t = \pi/2$, we then find that $e_{\pi/2} = i[\Lambda, e]$, which proves the lemma. \square

and Hodge star

Lemma 2.1.17. *For any $\alpha \in \Lambda^k V^*$, we have*

$$*\alpha = i^{k^2-n} J^{-1} \# \alpha. \quad (1.17) \quad \text{almost complex}$$

Proof. We first note that the $*$ -operator is characterized by

$$*1 = \text{vol} = \frac{1}{n!} L^n(1), \quad *e(\eta) = (-1)^k e^*(\eta) *, \quad (1.18) \quad \text{almost complex}$$

as an operator on $\Lambda^k V_{\mathbb{C}}^*$ for any real 1-form η , as the forms obtained from 1 by repeated application of $e(\eta)$ span $\Lambda^* V^*$. The first equation is clear, and the second one follows from Proposition 2.1.6(c): for a k -form α , we have

$$*e(\eta)\alpha = *e(\eta) * *^{-1} \alpha = (-1)^k e^*(\eta) * \alpha.$$

Now consider the operator

$$\tilde{*} = i^{k^2-n} J \#$$

defined on $\Lambda^k V_{\mathbb{C}}^*$. We recall that by (1.12),

$$\# L^k \alpha = i^n \frac{k!}{(n-k)!} L^{n-k} \alpha$$

if α is primitive of weight n . But 1 is just a primitive form of weight n , so apply this equality we have

$$\# 1 = \frac{i^n}{n!} L^n(1).$$

We then conclude that

$$\tilde{*}1 = i^{-n} \cdot \frac{i^n}{n!} L^n(1) = \frac{L^n(1)}{n!} = \text{vol}.$$

Similarly, if η is a real 1-form and α is a k -form, using (1.13) we see

$$\begin{aligned} \tilde{*}e(\eta)\alpha &= i^{(k+1)^2-n} J^{-1} \# e(\eta)\alpha = i^{k^2-n} (-1)^k i J^{-1} \# e(\eta) \#^{-1} \alpha \\ &= i^{k^2-n} (-1)^k e^*(\eta) J^{-1} \# \alpha = (-1)^k e^*(\eta) \tilde{*} \alpha. \end{aligned}$$

This verifies (1.18) for $*$, so $* = \tilde{*}$. □

We now prove the following mysterious but extremely useful result about the interplay between L and $*$.

and L power**Proposition 2.1.18.** *Let α be a primitive k -form in $\Lambda^* V^*$. Then for $0 \leq j \leq n-k$, we have*

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} J \alpha \quad (1.19)$$

Proof. Let V_{α}^* be the subspace of $\Lambda^* V^*$ generated by $\{L^j \alpha : 0 \leq j \leq n-k\}$. Then V_{α}^* is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$, and by (1.12) we have

$$\# L^j \alpha = i^{n-k} \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha.$$

Hence, by Lemma 2.1.17, if $\alpha \in P^k$ we have (recall that $J^2 = \sum_k (-1)^k \Pi^k$)

$$\begin{aligned} *L^j \alpha &= i^{(k+2j)^2-n} J^{-1} \# L^j \alpha = i^{k^2-n} J^{-1} i^{n-k} \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha \\ &= i^{k^2-k} (J^{-1})^2 \frac{j!}{(n-k-j)!} L^{n-k-j} J \alpha \end{aligned}$$

$$\begin{aligned}
&= i^{k^2-k} (-1)^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha \\
&= i^{k^2-k} (-1)^k \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha \\
&= (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha.
\end{aligned}$$

□

Example 2.1.19. Here are a few instructive special cases. Let $j = k = 0$ and $\alpha = 1$, then we obtain

$$*1 = \frac{1}{n!} L^n 1 = \frac{\omega^n}{n!}.$$

Thus, $\text{vol} = \omega^n/n!$ as was claimed before. Also, for $k = 0$, $j = 1$, and $\alpha = 1$, the proposition yields

$$*\omega = \frac{\omega^{n-1}}{(n-1)!}$$

Finally, if α is a primitive $(1, 1)$ -form, then

$$*\alpha = \frac{-1}{(n-2)!} \omega^{n-2} \wedge \alpha.$$

Let $(V, \langle \cdot, \cdot \rangle, J)$ be as before and let ω be the associated fundamental form. The **Hodge-Riemann pairing** of V is the bilinear form

$$Q : \Lambda^k V^* \times \Lambda^k V^* \rightarrow \mathbb{R}, \quad (\alpha, \beta) \mapsto (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}$$

where $\Lambda^{2n} V^*$ is identified with \mathbb{R} via the volume form vol . By definition $Q = 0$ on $\Lambda^k V^*$ for $k > n$. We will also denote by Q the \mathbb{C} -linear extension of the Hodge-Riemann pairing to $\Lambda^* V_{\mathbb{C}}^*$. Our final result will be an orthogonal condition on Q :

near relation

Proposition 2.1.20 (Hodge-Riemann bilinear relation). *Let $(V, \langle \cdot, \cdot \rangle, J)$ be an Euclidian vector space endowed with a compatible almost complex structure. Then the associated Hodge-Riemann pairing Q satisfies*

$$Q(\Lambda^{p,q} V^*, \Lambda^{p',q'} V^*) = 0$$

for $(p, q) \neq (q', p')$ and

$$i^{p-q} Q(\alpha, \bar{\alpha}) = (n - (p + q))! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$$

for nonzero $\alpha \in P^{p,q}$ with $p + q \leq n$.

Proof. From the definition of Q and the fact that ω is of type $(1, 1)$, the first assertion follows immediately. We only need to verify the second assertion. By definition,

$$\begin{aligned}
Q(\alpha, \bar{\alpha}) \cdot \text{vol} &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k} \bar{\alpha} \\
&= (-1)^{\frac{k(k-1)}{2}} \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \text{vol},
\end{aligned}$$

where $k = p + q$ and β is the p -form given by $*\bar{\beta} = L^{n-k} \bar{\alpha}$. Hence $*^2 \bar{\beta} = (-1)^k \bar{\beta}$ and, on the other hand, by Proposition [2.1.18](#) we have

$$*^2 \bar{\beta} = *L^{n-k} \bar{\alpha} = (-1)^{\frac{k(k+1)}{2}} (n-k)! i^{p-q} \bar{\alpha}.$$

Thus $\beta = (-1)^{\frac{k(k+1)}{2} + k} (n-k)! i^{p-q} \alpha$ and we then see

$$Q(\alpha, \bar{\alpha}) = (-1)^{\frac{k(k+1)}{2} + \frac{k(k-1)}{2} + k} (n-k)! i^{p-q} \langle \alpha, \alpha \rangle_{\mathbb{C}}.$$

This yields $i^{p-q}Q(\alpha, \bar{\alpha}) = (n-k)! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$ for nonzero $\alpha \in P^{p,q}$. \square

Example 2.1.21. Suppose that $n \geq 2$ and consider the decomposition $(\wedge^{1,1}V^*)_{\mathbb{R}} = \omega\mathbb{R} \oplus P_{\mathbb{R}}^{1,1}$, where $(-)\mathbb{R}$ denotes the intersection with \wedge^2V^* . This decomposition is Q -orthogonal, because $(\alpha \wedge \omega) \wedge \omega^{n-2} = \alpha \wedge \omega^{n-1} = L^{n-1}\alpha = 0$ for any $\alpha \in P^2$ (Proposition 2.1.13(b)). Moreover, Q is a positive definite symmetric bilinear form on $\omega\mathbb{R}$ and a negative definite symmetric bilinear form on $P_{\mathbb{R}}^{1,1}$.

2.1.2 Tangent space and differential forms on \mathbb{C}^n

A real manifold M is studied by means of its tangent bundle TM , the collection of all tangent spaces T_xM for $x \in M$, and its k -form bundle $\wedge^k T^*M$. In this subsection we will apply the linear algebra developed previously to the form bundles of an open subset $M = U \subseteq \mathbb{C}^n$. The bidegree decomposition induces a decomposition of the exterior differential d which is well suited for the study of holomorphic functions on U . We conclude by a local characterization of so called Kähler metrics which will be of central interest in the global setting.

Let $U \subseteq \mathbb{C}^n$ be an open subset. Then U can in particular be considered as a $2n$ -dimensional real manifold. For $x \in U$ we have its real tangent space T_xU at the point x which is of real dimension $2n$. A canonical basis of T_xU is given by the tangent vectors

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}.$$

where $z^i = x^i + iy^i$ are the standard coordinates on \mathbb{C}^n . Moreover, the vectors $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial y^n}$ are global trivializing sections of TU .

Each tangent space T_xU admits a natural almost complex structure defined by

$$J : T_xU \rightarrow T_xU, \quad \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}.$$

which is compatible with the global trivialization. We shall regard J as a vector bundle endomorphism of the real vector bundle TU over U . The dual basis of $(T_xU)^*$ is denoted by $dx^1, \dots, dx^n, dy^1, \dots, dy^n$. Recall that the induced almost complex structure on T_xU in terms of this dual basis is described by $J(dx^i) = -dy^i$, $J(dy^i) = dx^i$.

The general theory developed in the previous subsections applies to this almost complex structure and yields the following result.

Proposition 2.1.22. *The complexified tangent bundle $TU_{\mathbb{C}} := TU \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles*

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U,$$

such that the complex linear extension of J satisfies

$$J|_{T^{1,0}U} = i \cdot \text{id}, \quad J|_{T^{0,1}U} = -i \cdot \text{id}.$$

The vector bundles $T^{1,0}U$ and $T^{0,1}U$ are trivialized by the sections

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} - i \frac{\partial}{\partial y^i} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + i \frac{\partial}{\partial y^i} \right)$$

for $i = 1, \dots, n$, respectively.

The complexified cotangent bundle $T_{\mathbb{C}}^*U := T^*U \otimes_{\mathbb{R}} \mathbb{C}$ admits an analogous decomposition $T^*U_{\mathbb{C}} = (T^*U)^{1,0} \oplus (T^*U)^{0,1}$, where $(T^*U)^{1,0}$ and $(T^*U)^{0,1}$ are trivialized by the dual basis $dz^i :=$

$dx^i + idy^i$ and $d\bar{z}^i := dx^i - idy^i$, for $i = 1, \dots, n$, respectively. Note that these decompositions are compatible with restriction to smaller open subsets $U' \subseteq U$.

Proposition 2.1.23. *Let $f : U \rightarrow V$ be a holomorphic map between open subsets $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$. Then the \mathbb{C} -linear extension of the differential $df : T_x U \rightarrow T_{f(x)} V$ respects the above decomposition, i.e., $df(T_x^{0,1} U) \subseteq T_{f(x)}^{0,1} V$ and $df(T_x^{1,0} U) \subseteq T_{f(x)}^{1,0} V$.*

Proof. Since f is holomorphic, the extension of df is given by the Jacobian matrix

$$\begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$$

It then follows that df maps $T^{1,0} U$ into $T^{1,0} V$ and $T^{0,1} U$ into $T^{0,1} V$. \square

In a similar fashion, we can use the previous results in order to decompose the bundles of k -forms. Let $U \subseteq \mathbb{C}^n$ be an open subset. Over U one defines the complex vector bundles

$$\Lambda^{p,q} T^* U = \Lambda^p (T^* U)^{1,0} \otimes \Lambda^q (T^* U)^{0,1}.$$

By $\mathcal{A}_{\mathbb{C}}^k(U)$ and $\mathcal{A}^{p,q}(U)$ we denote the spaces of sections of $\Lambda^k T_{\mathbb{C}}^* U$ and $\Lambda^{p,q} T^* U$, respectively. Now by Proposition ??, we have the following decomposition for $\Lambda^k T_{\mathbb{C}}^* U$:

Proposition 2.1.24. *There are natural decompositions*

$$\Lambda_{\mathbb{C}}^k T^* U = \bigoplus_{p+q=k} \Lambda^{p,q} T^* U, \quad \mathcal{A}_{\mathbb{C}}^k(U) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(U).$$

We note that the restriction map $\mathcal{A}^k(U) \rightarrow \mathcal{A}^k(U')$ for an open subset $U' \subseteq U$ respects this decomposition. As before, the projection operators $\Lambda_{\mathbb{C}}^k T^* U \rightarrow \Lambda^{p,q} T^* U$ and $\mathcal{A}_{\mathbb{C}}^k(U) \rightarrow \mathcal{A}^{p,q}(U)$ will be denoted by $\Pi^{p,q}$.

We now consider the complex extension $d : \mathcal{A}_{\mathbb{C}}^k(U) \rightarrow \mathcal{A}_{\mathbb{C}}^{k+1}(U)$ of the usual exterior differential d on k -forms. Then

$$\partial : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p+1,q}(U), \quad \bar{\partial} : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p,q+1}(U)$$

are defined by $\partial := \Pi^{p+1,q} \circ d$ and $\bar{\partial} := \Pi^{p,q+1} \circ d$. For any local function f we have

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i + \sum_i \frac{\partial f}{\partial y^i} dy^i = \sum_i \frac{\partial f}{\partial z^i} dz^i + \sum_i \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i. \quad (1.20) \quad \boxed{\text{manifold complex differential expression on}}$$

Thus, f is holomorphic if and only if $\bar{\partial} = 0$. Moreover, using (1.20), the operators ∂ and $\bar{\partial}$ can be explicitly expressed as

$$\begin{aligned} \partial(f dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}) &= \sum_{k=1}^n \frac{\partial f}{\partial z^k} dz^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}, \\ \bar{\partial}(f dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}) &= \sum_{k=1}^n \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}, \end{aligned}$$

on \mathbb{C}^n prop

Proposition 2.1.25. *For the differentials ∂ and $\bar{\partial}$, we have*

$$(a) \quad d = \partial + \bar{\partial}.$$

$$(b) \quad \partial^2 = \bar{\partial}^2 = 0 \text{ and } \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

(c) The operators ∂ and $\bar{\partial}$ satisfy the Leibniz rule, i.e.,

$$\partial(\alpha \wedge \beta) = \partial(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \partial(\beta),$$

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}(\beta),$$

for $\alpha \in \mathcal{A}^{p,q}(U)$ and $\beta \in \mathcal{A}^{r,s}(U)$.

Proof. The first assertion follows from the description of ∂ and $\bar{\partial}$ given above, and the second one from the expansion $0 = d^2 = (\partial + \bar{\partial})^2$. For the third one, we recall that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d\beta.$$

Taking the $(p+r+1, q+s)$ -parts on both sides one obtains the Leibniz rule for ∂ . Similarly, taking $(p+r, q+s+1)$ -parts proves the assertion for $\bar{\partial}$. \square

Since ∂ and $\bar{\partial}$ share the usual properties of the exterior differential d and reflect the holomorphicity of functions, it seems natural to build up a holomorphic analogue of the de Rham complex. As we work here exclusively in the local context, only the local aspects will be discussed. Of course, locally the de Rham complex is exact due to the standard Poincaré lemma. We will show that this still holds true for $\bar{\partial}$ (and ∂).

Proposition 2.1.26 ($\bar{\partial}$ -Poincaré Lemma on \mathbb{C}). Consider an open neighbourhood of the closure of a bounded one-dimensional disc $B_\varepsilon \subseteq \bar{B}_\varepsilon \subseteq U \subseteq \mathbb{C}$. For $\alpha = f d\bar{z} \in \mathcal{A}^{0,1}(U)$ the function

$$g(z) = \frac{1}{2\pi i} \int_{B_\varepsilon} \frac{f(w)}{w-z} dw \wedge d\bar{w}$$

on B_ε satisfies $\alpha = \bar{\partial}g$.

The following proposition and its corollary are known as the Grothendieck-Poincaré lemma. The first proof of it is due to Grothendieck and was presented by Serre in the Séminaire Cartan in 1958.

Proposition 2.1.27 ($\bar{\partial}$ -Poincaré Lemma on \mathbb{C}^n). Let U be an open neighbourhood of the closure of a bounded polydisc $B_\varepsilon \subseteq \bar{B}_\varepsilon$ in \mathbb{C}^n . If $\alpha \in \mathcal{A}^{p,q}(U)$ is $\bar{\partial}$ -closed and $q > 0$, then there exists a form $\beta \in \mathcal{A}^{p,q-1}(B_\varepsilon)$ with $\alpha = \bar{\partial}\beta$ on B_ε .

Corollary 2.1.28 ($\bar{\partial}$ -Poincaré Lemma on Polydisc). Let B be a polydisc in \mathbb{C}^n which can be unbounded. If $\alpha \in \mathcal{A}^{p,q}(B)$ is $\bar{\partial}$ -closed and $q > 0$, then there exists $\beta \in \mathcal{A}^{p,q-1}(B)$ with $\alpha = \bar{\partial}\beta$.

Remark 2.1.3. It is easy to see that $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$ for any $\alpha \in \mathcal{A}^{p,q}(U)$, so by conjugation we can also form a ∂ -Poincaré lemma for open disk U in \mathbb{C}^n .

Corollary 2.1.29 ($\bar{\partial}\partial$ -Poincaré Lemma on Polydisc). Let B be a polydisc in \mathbb{C}^n . If $\alpha \in \mathcal{A}^{p,q}(B)$ is d -closed, then for any point $x \in B$ there is a neighbourhood U of x and $\beta \in \mathcal{A}^{p-1,q-1}(U)$ such that $\partial\bar{\partial}\beta = \alpha$ in U .

Proof. The proof consists of an application of the Poincaré lemmas for the operators d , ∂ , and $\bar{\partial}$. Namely, since $d\alpha = 0$, for each $x \in B$ there is a neighbourhood U of x and $\eta \in \mathcal{A}^{k-1}(U)$ such that $d\eta = \alpha$, where $k = p+q$ is the total degree of α . Thus we see that if we write $\eta = \sum_{i,j} \eta^{i,j}$, we have

$$d\eta = \bar{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q}, \quad \bar{\partial}\eta^{p-1,q} = \partial\eta^{p,q-1} = 0.$$

and by shrinking U there exists (by the $\bar{\partial}$ and ∂ Poincaré lemmas) forms $\beta_1 \in \mathcal{A}^{p-1, q-1}(U)$ and $\beta_2 \in \mathcal{A}^{p-1, q-1}(U)$ so that

$$\partial\beta_1 = \eta^{p, q-1}, \quad \bar{\partial}\beta_2 = \eta^{p-1, q}.$$

Now we then see

$$\alpha = d\beta = \bar{\partial}\beta_1 + \partial\bar{\partial}\beta_2 = \partial\bar{\partial}(\beta_2 - \beta_1).$$

This completes the proof of the corollary. \square

So far, only the consequences of the existence of a natural (almost) complex structure on each $T_x U$ have been discussed. Following the presentation of the last part, we shall conclude by combining this with certain metric aspects of the manifold U . Let $U \subseteq \mathbb{C}^n$ be an open subset and consider a Riemannian metric g on U . For what follows we may always assume that U is a polydisc. The metric g is compatible with the natural (almost) complex structure on U if for any $x \in U$ the induced scalar product g_x on $T_x U$ is compatible with the induced almost complex structure J , i.e. $g_x(v, w) = g_x(Jv, Jw)$ for all $v, w \in T_x U$. Recall that in this situation a natural $(1, 1)$ -form $\omega \in \mathcal{A}^{1,1}(U) \cap \mathcal{A}^2(U)$ defined by

$$\omega = g(J(-), (-)),$$

which is called the fundamental form of g . Moreover, $h := g - i\omega$ defines a positive Hermitian form on the complex vector spaces $(T_x U, g_x)$ for any $x \in U$.

Example 2.1.30. Let g be the constant standard metric such that

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$$

is an orthonormal basis for any $T_x U$. Clearly, complex structure and g are compatible. The form ω in this case is

$$\omega = \frac{i}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i.$$

An arbitrary metric g on U , if compatible with the almost complex structure, is uniquely determined by the matrix $h_{ij}(z) := h(\partial/\partial x^i, \partial/\partial x^j)$. The fundamental form can then be written as (Proposition [2.1.4](#))

$$\omega = \frac{i}{2} \sum_{i,j=1}^n h_{ij} dz^i \wedge d\bar{z}^j. \quad (1.21) \quad \text{complex manif}$$

Even if g is not the standard metric, one might try to change the complex coordinates such that it becomes the standard metric with respect to the new coordinates. Of course, this cannot always be achieved, but a reasonable class of metrics can be then defined: we say the metric g **osculates in the origin to order two** to the standard metric if $(h_{ij}) = \delta_{ij} + O(|z|^2)$. Explicitly, the condition means

$$\frac{\partial h_{ij}}{\partial z^k}(0) = \frac{\partial h_{ij}}{\partial \bar{z}^k}(0) = 0$$

for all i, j, k . In other words, the power series expansion of (h_{ij}) differs from the constant matrix (δ_{ij}) by terms of order at least two, thus terms of the form $a_{ij}z^k + b_{ij}\bar{z}^k$ do not occur.

Osculating metrics will provide the local models of Kähler metrics which will be extensively studied in the later chapters. Here is the crucial fact:

iff two order

Proposition 2.1.31. *Let g be a compatible metric on U and let ω be the associated fundamental form. Then $d\omega = 0$ if and only if for any point $x \in U$ there exist a neighbourhood V of $0 \in \mathbb{C}^n$ and a local biholomorphic map $f : V \cong f(V) \subseteq U$ with $f(0) = x$ and such that f^*g osculates in the origin to order two to the standard metric.*

Proof. First note that for any local biholomorphic map f the pull-back $f^*\omega$ is the associated fundamental form to f^*g . In particular, ω is closed on $f(V)$ if and only if $f^*\omega$ is closed. Thus, in order to show that $d\omega = 0$ one can assume that the metric g osculates to order two to the standard metric and then one verifies that $d\omega$ vanishes in the origin. But the latter follows immediately from

$$\frac{\partial h_{ij}}{\partial z^k}(0) = \frac{\partial h_{ij}}{\partial \bar{z}^k}(0) = 0.$$

For the other direction let us assume that $d\omega = 0$. We fix a point $x \in U$. After translating we may assume that $x = 0$. By a linear coordinate change we may furthermore assume that $(h_{ij})(0) = (\delta_{ij})$. Thus

$$h_{ij} = \delta_{ij} + \sum_k a_{ijk} z^k + \sum_k b_{ijk} \bar{z}^k + O(|z|^2).$$

Thus, we have $a_{ijk} = \frac{\partial h_{ij}}{\partial z^k}(0)$ and $b_{ijk} = \frac{\partial h_{ij}}{\partial \bar{z}^k}(0)$. The assumption $d\omega(0) = 0$ together with (1.21) implies $a_{ijk} = a_{kji}$ and $b_{ijk} = b_{ikj}$. Furthermore, since ω is real, $h_{ij} = \bar{h}_{ji}$ and thus $b_{ijk} = \bar{a}_{jik}$. New holomorphic coordinates in a neighbourhood of the origin can now be defined by

$$w^j = z^j + \frac{1}{2} \sum_{i,k=1}^n a_{ijk} z^i z^k.$$

Then we have

$$dw^j = dz^j + \frac{1}{2} \sum_{i,k=1}^n a_{ijk} (dz^i) z^k + \frac{1}{2} \sum_{i,k=1}^n a_{ijk} z^i (dz^k) = dz^j + \sum_{i,k=1}^n a_{ijk} z^k dz^i$$

and similarly

$$d\bar{w}^j = d\bar{z}^j + \sum_{i,k=1}^n b_{ijk} \bar{z}^k d\bar{z}^i.$$

Therefore, up to terms of order at least two, we find that

$$\begin{aligned} \frac{i}{2} \sum_{j=1}^n dw^j \wedge d\bar{w}^j &= \frac{i}{2} \sum_{j=1}^n \left[dz^j \wedge d\bar{z}^j + \left(\sum_{i,k=1}^n a_{ijk} z^k dz^i \right) \wedge d\bar{z}^j + dz^j \wedge \left(\sum_{i,k=1}^n b_{jik} \bar{z}^k d\bar{z}^i \right) \right] \\ &= \frac{i}{2} \left[\sum_{j=1}^n dz^j \wedge d\bar{z}^j + \sum_{i,j=1}^n \left(\sum_{k=1}^n a_{ijk} z^k \right) dz^i \wedge d\bar{z}^j + \sum_{i,j=1}^n \left(\sum_{k=1}^n b_{jik} \bar{z}^k \right) dz^j \wedge d\bar{z}^i \right] \\ &= \omega. \end{aligned}$$

This completes the proof of the proposition. \square

Example 2.1.32. Any compatible metric on $U \subseteq \mathbb{C}$ satisfies the above condition. Clearly, the three-form $d\omega$ vanishes for dimension reasons.

2.2 Complex manifolds