nil-Hecke algebra.

We consider the algebra $\mathbb{C}[x_1,\cdots,x_n]$. For each simple reflection s_i, $1\le i\le n-1$, we define Demazure operator ∂_i by

$$\partial_i : \mathbb{C}[x_1,\cdots,x_n] \to \mathbb{C}[x_1,\cdots,x_n]^{s_i} \subseteq \mathbb{C}[x_1,\cdots,x_n]$$
$$f \mapsto \frac{s_i(f)-f}{x_i-x_{i+1}} = \frac{s_i(f)-f}{\alpha_i}, \quad \alpha_i = x_i - x_{i+1}$$

Note that RHS is a polynomial.

We also consider the operators x_i defined by multiplication by x_i. Then let NH_n be the algebra generated by the operators ∂_i, x_i.

The defining relations of NH_n are

$$x_i x_j = x_j x_i$$
$$\partial_i^2 = 0$$
$$\partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i-j|>1$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$$
$$\partial_i x_j = x_j \partial_i \quad |i-j|>1$$
$$\partial_i x_i - x_{i+1}\partial_i = 1 = x_i \partial_i - \partial_i x_{i+1}$$

we also have $\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g)$

$\Rightarrow \partial_i$ is a Sym_n-module homomorphism

For example,

$$\partial_i \partial_{i+1} \partial_i(f) = \partial_i \partial_{i+1} \frac{s_i(f)-f}{\alpha_i}$$
$$= \partial_i\left(\frac{s_{i+1}\left(\frac{s_i(f)-f}{\alpha_i}\right) - \frac{s_i(f)-f}{\alpha_i}}{\alpha_{i+1}}\right)$$
$$= \partial_i \frac{\alpha_i s_{i+1}s_i(f) - \alpha_i s_{i+1}(f) - (\alpha_i+\alpha_{i+1})s_i(f) + (\alpha_i+\alpha_{i+1})f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})}$$
$$= \frac{1}{\alpha_i}\left(\frac{-\alpha_i s_i s_{i+1} s_i(f) + \alpha_i s_i s_{i+1}(f) - \alpha_{i+1} f + \alpha_{i+1} s_i(f)}{-\alpha_i(\alpha_i+\alpha_{i+1})\alpha_{i+1}} - \frac{\alpha_i s_{i+1}s_i(f) - \alpha_i s_{i+1}(f) - (\alpha_i+\alpha_{i+1})s_i(f) + (\alpha_i+\alpha_{i+1})f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})}\right)$$
$$= \frac{1}{\alpha_i}\left(\frac{\alpha_i s_i s_{i+1} s_i(f) - \alpha_i s_i s_{i+1}(f) - \alpha_{i+1} s_i(f) + \alpha_{i+1} f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})} - \frac{\alpha_i s_{i+1}s_i(f) - \alpha_i s_{i+1}(f) - (\alpha_i+\alpha_{i+1})s_i(f) + (\alpha_i+\alpha_{i+1})f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})}\right)$$
$$= \frac{1}{\alpha_i}\,\frac{\alpha_i s_i s_{i+1} s_i(f) - \alpha_i(s_i s_{i+1} + s_{i+1}s_i)(f) + \alpha_i s_i(f) + \alpha_i s_{i+1}(f) - \alpha_i f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})}$$
$$= \frac{s_i s_{i+1} s_i(f) - (s_i s_{i+1} + s_{i+1}s_i)(f) + s_i(f) + s_{i+1}(f) - f}{\alpha_i\alpha_{i+1}(\alpha_i+\alpha_{i+1})}$$

and similar for $\partial_{i+1}\partial_i\partial_{i+1}(f)$. so $\partial_{i+1}\partial_i\partial_{i+1} = \partial_i\partial_{i+1}\partial_i$.

Also, $\partial_i x_i(f) = \partial_i(x_i f) = \dfrac{s_i(x_i f) - x_i f}{\alpha_i} = \dfrac{x_{i+1} s_i(f) - x_i f}{\alpha_i}$

$$x_{i+1}\partial_i(f) = x_{i+1}\frac{s_i(f) - f}{\alpha_i} = \frac{x_{i+1} s_i(f) - x_{i+1} f}{\alpha_i}$$

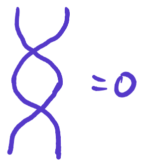
So $\partial_i x_i(f) - x_{i+1}\partial_i(f) = \dfrac{1}{\alpha_i}\left(x_{i+1} s_i(f) - x_i f - x_{i+1} s_i(f) + x_{i+1} f \right) = -f$

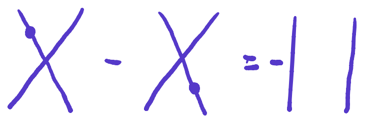
thus $\partial_i x_i - x_{i+1}\partial_i = -1$. Similar $x_i \partial_i - \partial_i x_{i+1} = -1$.

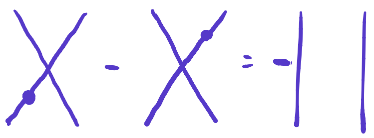
For each permutation $w \in S_n$, define $\partial_w = \partial_{i_1}\cdots\partial_{i_r}$, where $w = s_{i_1}\cdots s_{i_r}$ is a reduced expression.

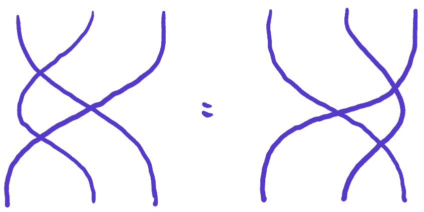
Remark. $\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0 & \text{otherwise} \end{cases}$

Let $NH = \bigoplus_n NH_n$. this can be defined diagramatically: objects are $n \in \mathbb{N}$, morphisms are generated by  : $2 \to 2$,  : $1 \to 1$, subject to relations

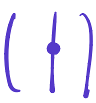
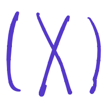
(1) 

(2) 

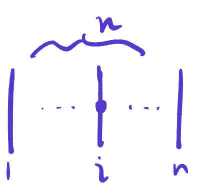


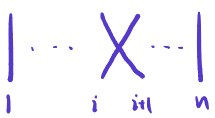
(3) 

Define a grading on NH by

deg() = 2, deg() = -2.

We will write

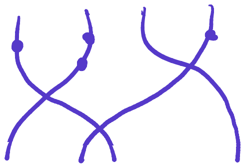
$x_i(n) =$  , element in NH_n.

$\partial_i(n) =$ 

Note that there are no morphisms $n\to m$ if $n\neq m$.

Let $Pol_n=\mathbb{Z}[x_1(n),\dots,x_n(n)]$, then we have an action of NH_n on Pol_n.

Thm. Let $B_n=\{x_1(n)^{u_1}\cdots x_n(n)^{u_n}\cdot\partial_w \mid w\in S_n,\ u_1,\dots,u_n\in\mathbb{N}\}$. Then NH_n is a free graded group with basis B_n.

$B_n =$ 

To prove linear independence we act on Pol_n: the actions of ∂_w on Pol_n is faithful.

We define $\mathfrak{S}_w(x)=\partial_{w^{-1}w_0}(x^{\rho_n})$ where $x^{\rho_n}=x_1^{n-1}\cdots x_{n-1}$, called Schubert polynomial.

Then
$$\partial_u(\mathfrak{S}_w)=\partial_u\partial_{w^{-1}w_0}(x^{\rho_n})=\begin{cases}\partial_{uw^{-1}w_0}(x^{\rho_n}) & \text{if } \ell(uw^{-1}w_0)=\ell(u)+\ell(w^{-1}w_0)\\ 0 & \end{cases}$$
$$=\begin{cases}\mathfrak{S}_{wu^{-1}} & \text{if } \ell(wu^{-1})=\ell(w)-\ell(u)\\ 0 & \end{cases}$$

Recall $\forall w\in S_n$, $\ell(ww_0)=\ell(w_0)-\ell(w)$

$w_0=\begin{pmatrix}1&2&\cdots&n\\ n&n-1&\cdots&1\end{pmatrix}$

If $a=\sum f_w\partial_w\neq 0$, $f_w\in Pol_n$, then we can find $h\in Pol_n$ such that $a(h)\neq 0$.

- Take w_1 minimal length such that $f_{w_1}\neq 0$. Then for w_0 = longest element in S_n, we have

$$a\,\partial_{w_1^{-1}w_0}=\left(\sum_w f_w\partial_w\right)\partial_{w_1^{-1}w_0}=f_{w_1}\partial_{w_0}$$

So $a\,\partial_{w_1^{-1}w_0}(x^{\rho_n})=f_{w_1}\partial_{w_0}(x^{\rho_n})=f_{w_1}\neq 0$

$\partial_w\partial_{w_1^{-1}w_0}=\partial_{ww_1^{-1}w_0}$

$\Leftrightarrow\ \ell(ww_1^{-1}w_0)=\ell(w)+\ell(w_1^{-1}w_0)$

$\Leftrightarrow\ -\ell(ww_1^{-1})=\ell(w)-\ell(w_1^{-1})$

$\Leftrightarrow\ \ell(w_1)-\ell(w)=\ell(ww_1^{-1})$

$\Leftrightarrow\ w_1\geq w$ under Bruhat order

$\Rightarrow\ w=w_1$

In fact, one can prove by induction that $\mathfrak{S}_e=1$.

$$\begin{aligned}\partial_{w_{0,n}}(x^{\rho_n})&=\partial_1\partial_2\cdots\partial_{n-1}\partial_{w_{0,n-1}}(x_1\cdots x_{n-1}x^{\rho_{n-1}})\\&=\partial_1\partial_2\cdots\partial_{n-1}(x_1\cdots x_{n-1}\partial_{w_{0,n-1}}(x^{\rho_{n-1}}))\\&=\partial_1\cdots\partial_{n-1}(x_1\cdots x_{n-1})\\&=\partial_1\cdots\partial_{n-2}(x_1\cdots x_{n-2})\\&=1\end{aligned}$$

$\deg(x_i(n))=2.$

Coro. Pol_n is a faithful graded module of NH_n, and NH_n is a free Pol_n-module of rank $n!$.

Thm. Pol_n is a free Sym_n-module of rank $n!$, with basis $(S_w)_{w\in S_n}$. Each S_w has degree $2\ell(w)$.

Proof.
$$\deg(S_w) = 2\ell(w_0) - 2\ell(w^{-1}w_0)$$
$$= 2\ell(w_0) - 2(\ell(w_0) - \ell(w))$$
$$= 2\ell(w)$$

Now we show that $(S_w)_{w\in S_n}$ is Sym_n-linearly independent. Suppose $\sum_w p_w S_w = 0$, $p_w \in Sym_n$, not all nonzero.
Let w' be of maximal length s.t. $p_{w'} \neq 0$. Then

$$\sum_w p_w \partial_{w^{-1}w_0}(x^{\rho_n}) = \sum_w p_w S_w = 0$$

but
$$\partial_{w'}\Big(\sum_w p_w \partial_{w^{-1}w_0}(x^{\rho_n})\Big) = \sum_w p_w \partial_{w'}\partial_{w^{-1}w_0}(x^{\rho_n})$$
$$= p_{w'}$$

$\partial_{w'}\partial_{w^{-1}w_0} = \partial_{w'w^{-1}w_0}$
$\Leftrightarrow \ell(w') - \ell(w) = -\ell(ww'^{-1})$
$\Leftrightarrow \ell(w'w^{-1}) = \ell(w) - \ell(w')$
$\Leftrightarrow w \geq w'$ in Bruhat order

so $p_{w'} = 0$, contradiction.

If $M = \bigoplus_{n\geq 0} M_n$, define
$$\mathrm{Dim}\, M = \sum_n \dim M_n \cdot q^n$$
Then $\mathrm{Dim}\, Sym_n = \frac{1}{\prod_{i=1}^n (1-q^{2i})}$, $\mathrm{Dim}\, Pol_n = \frac{1}{(1-q^2)^n}$

We also have $\sum_{w\in S_n} q^{2\ell(w)} = q^{\frac{n(n-1)}{2}}[n]!$, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$

Finally, compute graded dimension:
$$\mathrm{Dim}\Big(\bigoplus_w Sym_n S_w\Big) = \frac{\sum_{w\in S_n} q^{2\ell(w)}}{\prod_{i=1}^n (1-q^{2i})} = \frac{q^{\frac{n(n-1)}{2}}[n]!}{\prod_{i=1}^n (1-q^{2i})} = \frac{1}{(1-q^2)^n} = \mathrm{Dim}\, Pol_n$$

recall $\deg(x_i) = 2$.

For $M = \bigoplus M_n$, $N = \bigoplus N_n$, $\mathrm{HOM}(M,N) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(M, N\{d\})$.

Cor. $\mathrm{END}_{Sym_n}(Pol_n) \cong M_{n!}(Sym_n)$, matrix algebra. Its center is $\cong Sym_n$, a basis is given by $(E_{x,y})_{x,y\in S_n}$, with $E_{x,y}(S_w) = \delta_{y,w} S_x$, and $E_{x,y}$ is homogeneous of degree $2(\ell(x) - \ell(y))$

Thm. $NH_n \xrightarrow{\sim} \mathrm{END}_{Sym_n}(Pol_n)$

We have already seen this map is injective. It is surjective by graded dimension:
$$\mathrm{Dim}\, NH_n = \frac{1}{(1-q^2)^n} \cdot \sum_{w\in W} q^{-2\ell(w)} = \frac{1}{(1-q^2)^n} \cdot q^{-\frac{n(n-1)}{2}} \cdot [n]! = \frac{([n]!)^2}{\prod_{i=1}^n (1-q^{2i})} = ([n]!)^2 \cdot \mathrm{Dim}\, Sym_n$$
$$= \mathrm{Dim}\, Sym_n \cdot \sum_{x\in S_n, y\in S_n} q^{2\ell(x) - 2\ell(y)} = \mathrm{Dim}\, \mathrm{END}_{Sym_n}(Pol_n).$$

Cor. $Z(NH_n) = Sym_n$.

Now define $e_n := x^{\rho_n} \cdot \partial_{w_0} \in NH_n$. We have
$$e_n(\underbrace{S_{w_0}}_{=x^{\rho_n}}) = x^{\rho_n}\partial_{w_0}(x^{\rho_n}) = x^{\rho_n} = S_{w_0}, \quad \text{and} \quad e_n(S_w) = x^{\rho_n}\partial_{w_0}(\partial_{w^{-1}w_0}(x^{\rho_n})) = 0 \text{ for } w \neq w_0$$

so e_n corresponds to E_{w_0,w_0} under the isomorphism $NH_n \xrightarrow{\sim} M_{n!}(Sym_n)$. Then

$$P_n := q^{\frac{n(n-1)}{2}} NH_n \cdot e_n \qquad \text{i.e. } (q^m M)_n = M_{n-m}$$

$$= NH_n \cdot e_n \{-\tfrac{n(n-1)}{2}\}$$

is an indecomposable projective NH_n-module. Its head is denoted by L_n.

Prop. The left regular module NH_n is isomorphic to $[n]! P_n$ as a graded NH_n-module. Hence $\mathrm{Dim}\, L_n = [n]!$.

Proof. We have $NH_n \cong \bigoplus_{w \in S_n} NH_n \cdot E_{w,w}$. Right multiplication by E_{w,w_0} induces an isomorphism

$$NH_n \cdot E_{w,w} \cong q^{-2(\ell(w)-\ell(w_0))} NH_n \cdot E_{w_0,w_0}$$

$$f \cdot E_{w',w} \longmapsto f \cdot E_{w',w_0}, \quad \forall w' \in S_n$$

So
$$NH_n \cong \bigoplus_{w \in S_n} NH_n \cdot E_{w,w} \cong \bigoplus_{w \in S_n} q^{-2(\ell(w)-\ell(w_0))} \cdot NH_n \cdot E_{w_0,w_0} \cong \bigoplus_{w \in S_n} q^{-2(\ell(w)-\ell(w_0))} NH_n \cdot e_n$$

$$= q^{\frac{n(n-1)}{2}} [n]! \; NH_n \cdot e_n = [n]! \cdot P_n$$

Finally, $\mathrm{Dim}\, L_n = \mathrm{Dim}\, \mathrm{HOM}_{NH_n}(NH_n, L_n) = [n]!\, \mathrm{HOM}_{NH_n}(P_n, L_n) = [n]!$

Cor. $Pol_n \cong q^{-\frac{n(n-1)}{2}} P_n$.

Proof. We know that $NH_n \cong \mathrm{END}_{Sym_n}(Pol_n)$, so $Pol_n \cong P_n\{r\}$ for some $r \in \mathbb{Z}$. Now

$$\mathrm{Dim}\, Pol_n = \frac{1}{(1-q^2)^n}. \qquad \mathrm{Dim}\, P_n = \frac{1}{[n]!} \mathrm{Dim}\, NH_n = \frac{1}{[n]!} \cdot \frac{1}{(1-q^2)^n} \cdot \sum_{w \in W} q^{-2\ell(w)} = \frac{1}{(1-q^2)^n} \cdot q^{-\frac{n(n-1)}{2}}$$

so $Pol_n \cong q^{-\frac{n(n-1)}{2}} P_n$.

Categorification

We consider graded NH_n-modules:

NH_n-fmod , NH_n-pmod

these are graded categories with shift $\{r\}$, $r \in \mathbb{Z}$.

Let $T: NH_n \to NH_n$ be the anti-automorphism defined by

$$T(x_i) = x_i, \quad T(\partial_i) = \partial_i.$$

There are dualities on NH_n-pmod and NH_n-pmod by

$$M^* := \mathrm{HOM}_k(M,k), \qquad P^\# := \mathrm{HOM}_{NH_n}(P, NH_n).$$

where the module structures on the RHS are defined using T. That is,

For $m \in M$, $a \in NH_n$, $f \in \mathrm{HOM}_k(M,k)$,

$$(a\cdot f)(m) := f(T(a)m).$$

$$(ba\cdot f)(m) = f(T(a)T(b)m)$$
$$(b\cdot(a\cdot f))(m) = (a\cdot f)(T(b)m)$$
$$= f(T(a)T(b)m)$$

For $p \in P$, $a \in NH_n$, $f \in \mathrm{HOM}_{NH_n}(P, NH_n)$,

$$(a\cdot f)(p) := f(p)T(a)$$

$$(ba\cdot f)(p) = f(p)T(ba)$$
$$= f(p)T(a)T(b)$$
$$(b\cdot(a\cdot f))(p) = (a\cdot f)(p)T(b)$$
$$= f(p)T(a)T(b)$$

We define a bilinear pairing

$$\langle\, ,\rangle : K_0(NH_n\text{-pmod}) \times G_0(NH_n\text{-fmod}) \to \mathcal{A}$$

$$\langle [P], [M]\rangle := \mathrm{Dim}\, \mathrm{HOM}_{NH_n}(P, M).$$

Let $G_0(NH_n\text{-fmod})$ and $K_0(NH_n\text{-pmod})$ be Grothendieck groups. Then

$$-: G_0(NH_n\text{-fmod}) \to G_0(NH_n\text{-fmod}), \quad [M] \mapsto [M^*]$$

$$-: K_0(NH_n\text{-pmod}) \to K_0(NH_n\text{-pmod}), \quad [P] \mapsto [P^\#].$$

Note that they are anti-linear: set $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$.

$$(M\{d\})^* = \mathrm{HOM}(M\{d\}, k) = \mathrm{HOM}(M,k)\{-d\} = (M^*)\{-d\},$$

$$(P\{d\})^\# = \mathrm{HOM}_{NH_n}(P\{d\}, NH_n) = (P^\#)\{-d\}.$$

Recall $[P_n] \in NH_n\text{-pmod}$, $[L_n] \in NH_n\text{-fmod}$. We have

$$\mathrm{Dim}\, L_n^* = \overline{[n]!} = [n]!$$

So $L_n^* \cong L_n$. This implies $P_n^\# = P_n$. In fact, we have

$$\langle [P^\#], [M]\rangle = \mathrm{Dim}\, P^T \otimes_{NH_n} M = \overline{\langle [P], [M^*]\rangle}$$

We define an $\mathcal{A}$-bilinear pairing by

$$(\, ,\,) : K_0(NH_n\text{-pmod}) \times G_0(NH_n\text{-fmod}) \to \mathbb{Q}(q)$$
$$([P],[M]) := \langle [P^\#], [M]\rangle.$$

Then we have $(P_n, L_n) = 1$. Note that P_n and L_n are bar-invariant elements.

Now define $G_0(NH) = \bigoplus_{n\in\mathbb{N}} G_0(NH_n\text{-fmod})$, $K_0(NH) = \bigoplus_{n\in\mathbb{N}} K_0(NH_n\text{-pmod})$.

We have bar-involutions $\overline{}$ on $G_0(NH)$ and $K_0(NH)$, and an $\mathcal{A}$-bilinear pairing

$$(\ ,\ ): K_0(NH) \times G_0(NH) \to \mathbb{Q}(q)$$

Induction and Restriction:

For $m,n \in \mathbb{N}$ we consider the map

$$NH_m \otimes_k NH_n \to NH_{m+n}$$

$$x_i(m) \mapsto x_i(m) \qquad x_i(n) \mapsto x_{i+m}(m+n)$$

$$\partial_i(m) \mapsto \partial_i(m), \qquad \partial_i(n) \mapsto \partial_{i+m}(m+n)$$

This is an embedding of graded k-algebras.

Using the basis B_n of NH_n, we see that

- The graded left $(NH_m \otimes_k NH_n)$-module NH_{m+n} is graded free and has a homogeneous basis $\{\partial_w \mid w \in S_m \times S_n \backslash S_{m+n}\}$.
- The graded right $(NH_m \otimes_k NH_n)$-module NH_{m+n} is graded free and has a homogeneous basis $\{\partial_w \mid w \in S_{m+n}/S_m \times S_n\}$.

We define

$$\mathrm{Ind}_{m,n}^{m+n}: \mathrm{Rep}(NH_m \otimes_k NH_n) \to \mathrm{Rep}(NH_{m+n}), \quad M \mapsto NH_{m+n} \otimes_{NH_m \otimes_k NH_n} M$$

$$\mathrm{Res}_{m,n}^{m+n}: \mathrm{Rep}(NH_{m+n}) \to \mathrm{Rep}(NH_m \otimes_k NH_n), \quad N \mapsto 1_{NH_m \otimes_k NH_n} \cdot N.$$

Note that $G_0(NH_m\text{-fmod}) \otimes_{\mathcal{A}} G_0(NH_n\text{-fmod}) \xrightarrow{\sim} G_0((NH_m \otimes_k NH_n)\text{-fmod})$

$K_0(NH_m\text{-pmod}) \otimes_{\mathcal{A}} K_0(NH_n\text{-pmod}) \xrightarrow{\sim} K_0((NH_m \otimes_k NH_n)\text{-pmod})$.

We then obtain $\cdot$ and Δ on $G_0(NH)$ and $K_0(NH)$.

$$[M]\cdot[N] = [\mathrm{Ind}_{m,n}^{m+n} M \otimes_k N]$$

$$\Delta([M]) = \sum_{r=0}^{n} [\mathrm{Res}_{r,n-r}^{n} M]$$

By Frobenius reciprocity, we see that $G_0(NH)$ and $K_0(NH)$ are dual bialgebras.

Here we use twisted bialgebras:

$$(x_1 \otimes y_1)\cdot(x_2 \otimes y_2) = q^{-|y_1|\cdot|x_2|} x_1x_2 \otimes y_1y_2.$$

Recall Lusztig's integral form ${}_{\mathcal{A}}f = \mathcal{A}\cdot\theta$, with $r(\theta) = 1\otimes\theta + \theta\otimes 1$.

We have $\theta^{(m)}\cdot\theta^{(n)} = \begin{bmatrix} n+m \\ m \end{bmatrix}_q \theta^{(m+n)}$

$$r(\theta^{(n)}) = \sum_{r=0}^{n} q^{-r(n-r)}\theta^{(r)}\otimes\theta^{(n-r)}$$

There is also a Hopf pair $(,)$ on f defined by

$$(\theta^{(m)}, \theta^{(n)}) = \delta_{m,n}\prod_{i=1}^{m}\frac{1}{(1-q^{2i})}$$

It satisfies $(xy, z) = (x\otimes y, r(z))$

Thm. There is an isomorphism of twisted $\mathcal{A}$-bialgebras

$$\gamma : {}_{\mathcal{A}}f \to K_0(NH), \quad \theta^{(n)} \mapsto [P_n].$$

Moreover we have

(a) For $x \in {}_{\mathcal{A}}f$, $\gamma(\bar{x}) = \overline{\gamma(x)}$

(b) $(x, y) = (\gamma(x), \gamma(y))$.

Proof. We have to show

$$[\mathrm{Ind}_{m,n}^{m+n}(P_m\otimes_k P_n)] = \begin{bmatrix} m+n \\ m \end{bmatrix}_q [P_{m+n}]$$

It sufficies to consider graded dimensions:

$$\mathrm{Dim}\ \mathrm{Ind}_{m,n}^{m+n}(P_m\otimes_k P_n) = \mathrm{Dim}\ NH_{m+n}\otimes_{NH_m\otimes_k NH_n} P_m\otimes P_n$$

$$= \sum_{w\in S_{m+n}/S_m\times S_n} q^{-2l(w)}\cdot \mathrm{Dim}\ P_m\cdot\mathrm{Dim}\ P_n$$

$$= \frac{\sum_{w\in S_{m+n}} q^{-2l(w)}}{\sum_{x\in S_m} q^{-2l(x)}\cdot\sum_{y\in S_n} q^{-2l(y)}}\cdot\frac{1}{(1-q^2)^m}\cdot q^{-\frac{m(m-1)}{2}}\cdot\frac{1}{(1-q^2)^n}\cdot q^{-\frac{n(n-1)}{2}}$$

$$= q^{-\frac{(m+n)(m+n-1)}{2}}\cdot\frac{[m+n]!}{[m]!\,[n]!}\cdot\frac{1}{(1-q^2)^{m+n}}$$

$$= \begin{bmatrix} m+n \\ n \end{bmatrix}_q \mathrm{Dim}\ P_{m+n}.$$

Also, $(P_n, P_n) = [P_n, L_n]_q = \dfrac{\mathrm{Dim}\ P_n}{\mathrm{Dim}\ L_n} = \prod_{i=1}^{n}\dfrac{1}{1-q^{2i}}$.