Before we head over to the Discrete Fourier Transform, let us take a look at the Fourier Transform itself: arguably one of the most important mathematical operations in signal processing.

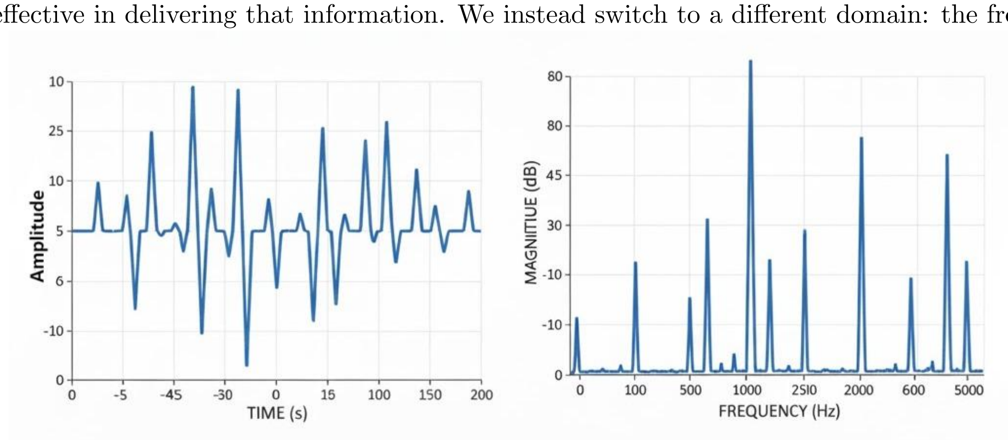
# Introduction

The Fourier Transform is one member of a broader family that includes the Fourier series, Discrete-Time Fourier Transform (DTFT), and the Discrete Fourier Transform (DFT) and others.

In signal processing, signals can be represented in multiple domains. The most common are the time domain and frequency domain. The time domain shows how the signal varies with respect to time and the information that can be obtained from this domain includes amplitude variations and timing information. However, the frequency content of the signal is not directly revealed.

For example, if I gave you an audio recording of your voice speaking, we just see the spikes (impulses) and the ups and downs (amplitude fluctuations) of the signal through time in the time-domain representation. While this gives us some information about the signal, for instance its amplitude at a particular time, it does not clearly indicate which frequencies are present.

If you would want to know the range of frequencies present, then a time domain representation would be ineffective in delivering that information. We instead switch to a different domain: the frequency domain.




# The Fourier Transform

The mathematical operation that makes the switch to the frequency domain possible is the Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Here, $x(t)$ is a continuous-time signal and $X(\omega)$ is its frequency-domain representation, where $\omega$ denotes angular frequency in radians per second.

The Fourier Transform works! There's no doubt about that. The question is why and how it works: how this expression is able to extract frequency information from a signal.

## The Complex Exponential

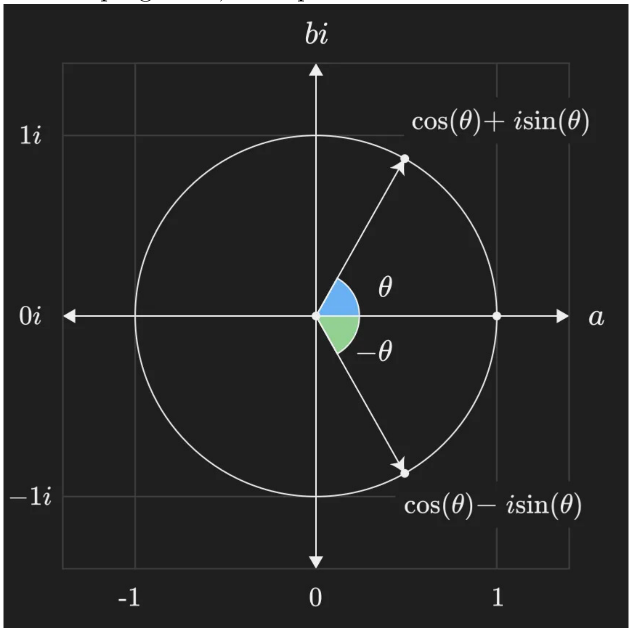
Most of the terms in the Fourier Transform are straightforward, but let's take a look at the exponential term: $e^{-j\omega t}$: quite a fascinating expression.

Oscillatory behaviour is pretty common and fundamental in both natural and engineered systems. In describing periodic behaviour, traditionally, trigonometric functions such as sine and cosine are used. However, dealing with sines and cosines can be mathematically lengthy and cumbersome: identities, phase relationships and others. What if we had a simple expression that captures all that.

That's where the complex exponential comes in. In its full form, using Euler's formula, its expressed as:

$$e^{-j\omega t} = cos(\omega t) - jsin(\omega t)$$

Another interesting and important property is that, we get to represent this geometrically on a complex plane. On the complex plane, $e^{-j\omega t}$ represents a vector of constant magnitude rotating at an angular speed $\omega$. As time progresses, the tip of this vector traces out a circle.




## Why is it in the Fourier Transform?

The complex exponential acts as a test signal, fishing out the frequencies in the signal: $x(t)$

Remember, the complex exponential results in a circle on the complex plane. Mathematically, the Fourier Transform computes an inner product between the signal $x(t)$ and the complex exponential of frequency $\omega$. This can be thought of as projecting the signal onto a set of rotating vectors, each corresponding to a different frequency.

If the signal contains a frequency component that matches that of the rotating vector, the product of the signal and the complex exponential produces a larger value. As the summation happens over time, this results in a large magnitude in the frequency domain at that particular frequency.

If the frequency of the signal and the rotating vector don't match, then a lower value is produced from their inner product: the positive and negative contributions cancel out over time.

One can relate this behaviour to the dot product in vector mathematics, which determines how closely vectors are to one another. If they are closely aligned, the value of the dot product is larger compared to when they are not. Example, a unit vector parallel to another unit vector have a dot product of 1 while two unit vectors at 90° have a dot product of zero.

The complex exponential, therefore, sort of acts like a frequency-filter, responding strongly when a matching frequency is present in the signal.

## The Discrete Fourier Transform

In digital systems, signals are discrete and finite in length. The Discrete Fourier Transform (DFT) is the discrete counterpart of the Fourier Transform. It is expressed mathematically as:

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

Here, $x[n]$ is a discrete-time signal of length $N$, and $k$ is the discrete frequency index. The DFT works on the same principle as the Fourier Transform, using complex exponentials to analyze the frequency content of discrete signals. I will not delve into the derivation of the formula, since I believe it is well covered by other resources.

The DFT can also be written as:

$$X(k) = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

where $W_N = e^{-j\frac{2\pi}{N}} = cos\left(\frac{2\pi}{N}\right) - jsin\left(\frac{2\pi}{N}\right)$.

The quantity $W_N^{kn}$ is commonly referred to as twiddle factor.

Like the Fourier Transform, the DFT produces complex-valued outputs. From these values, the magnitude spectrum and phase spectrum can be computed. The magnitude spectrum indicates the strength of each frequency component present in the signal, while the phase spectrum indicates the corresponding phase shifts. More of that would be covered in subsequent exercises.

It is also important to note that the DFT assumes the input signal is periodic with period $N$.