CS 630, Fall 2024, Homework 7

Yifei Bao

November 18, 2024

Problem 1 ***Multi-path routing***

1. Since each edge succeeds independently with probability p, the total success probability is:
$P(p) = p \cdot p \cdot p = p^3$
When $p = 0.5$: $P(0.5) = (0.5)^3 = 0.125$
2. There are 4 non-overlapping paths from s to t, and the paths are independent.
The probability of one path failing is $1 - p^3$. The probability of all 4 paths failing is $(1 - p^3)^4$.
The probability of at least one path succeeding is:
$P(p) = 1 - (1 - p^3)^4$
When $p = 0.5$: $P(0.5) = 1 - (1 - 0.5^3)^4 = 1 - 0.875^4 = 0.41$
3. There are 4 overlapping paths from s to t:
$s \to v11 \to v12 \to t$
$s \to v11 \to v22 \to t$
$s \to v21 \to v12 \to t$
$s \to v21 \to v22 \to t$
$P(s \to ... \to v12) = 1 - (1 - p^2)^2$
$P(s \to ... \to v22) = 1 - (1 - p^2)^2$
$P(s \to ... \to t) = 1 - (1 - p(1 - (1 - p^2)^2))^2$
When $p = 0.5$: $P(0.5) = 0.39$

Problem 2 ***MAX Sketches***

1. Each sketch stores the maximum hash value of its inserted elements. The maximum hash value from the union of sets A and B is the maximum of the maximum hash values from each set.
Therefore, $S_{A \cup B} = \max(S_A, S_B)$
2. There are 2^{32} possible hash values. Exactly half of the hash values are $\geq 2^{31}$.
Therefore, $P(S \geq 2^{31}) = \frac{2^{31}}{2^{32}} = \frac{1}{2}$
3. $2^{32} - 2^{32-i} = 2^{32}\left(1 - \frac{1}{2^i}\right)$
Number of Hashes Above Threshold: 2^{32-i}.
$P\left(S \geq 2^{32} - 2^{32-i}\right) = \frac{2^{32-i}}{2^{32}} = \frac{1}{2^i}$
4. When inserting $n = 2^k$ items with uniformly random hash values between 0 and $2^{32} - 1$, the maximum value S has the CDF:
$P(S \leq s) = \left(\frac{s+1}{2^{32}}\right)^n$
To finding the Lower Bound, We seek s_0 such that:
$P(S \geq s_0) = 1 - P(S \leq s_0 - 1) \geq \frac{1}{2}$
$\left(\frac{s_0}{2^{32}}\right)^n \leq \frac{1}{2}$

$n \ln\left(\frac{s_0}{2^{32}}\right) \leq -\ln 2$
$\ln\left(\frac{s_0}{2^{32}}\right) \leq -\frac{\ln 2}{n}$
$s_0 \geq 2^{32} e^{-\frac{\ln 2}{n}} = 2^{32}\left(e^{-\frac{\ln 2}{2^k}}\right)$
And $e^{-x} \approx 1 - x$,
$s_0 \geq 2^{32}\left(1 - \frac{\ln 2}{2^k}\right)$
Therefore, the lower bound is: $S \geq 2^{32} - \frac{\ln 2 \times 2^{32}}{2^k}$