CS630 Graduate Algorithms

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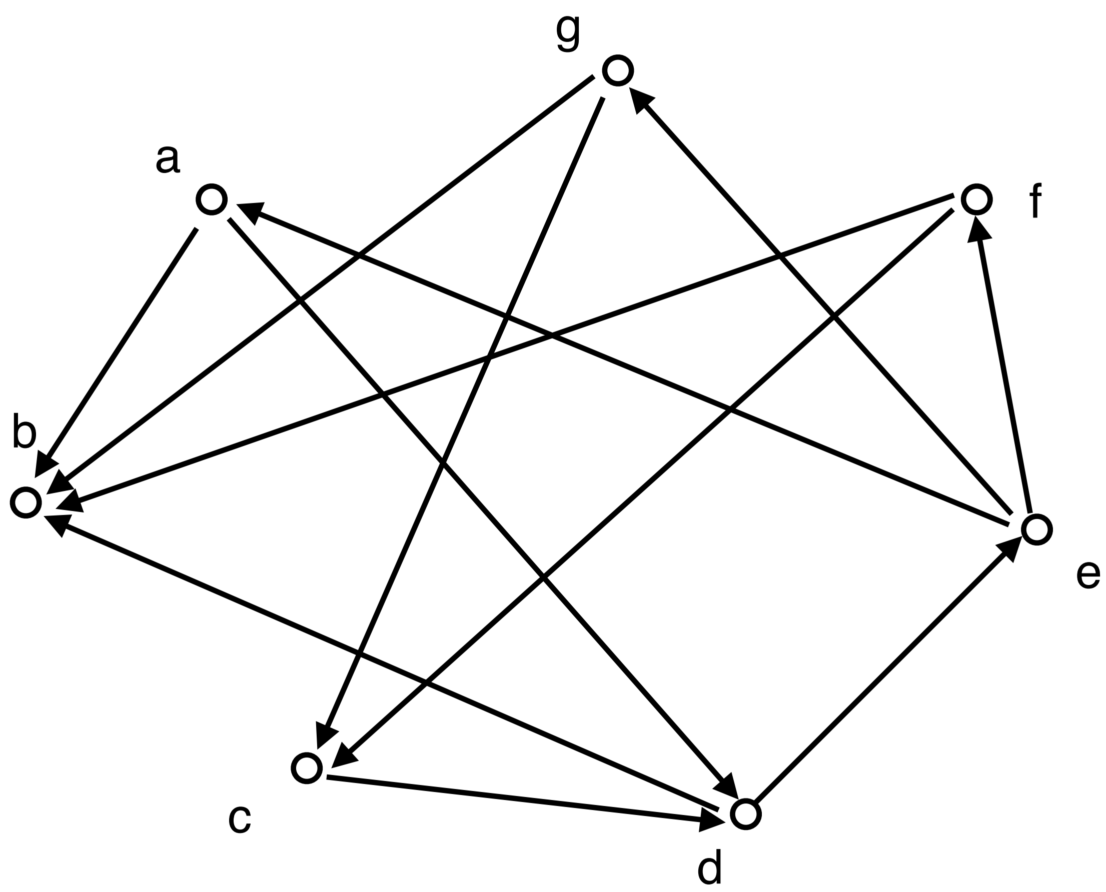
- approximation algorithms
  - vertex cover 2-approximation
  - set cover
  - vertex cover ln n - approx
  - dominating set
  - independent set

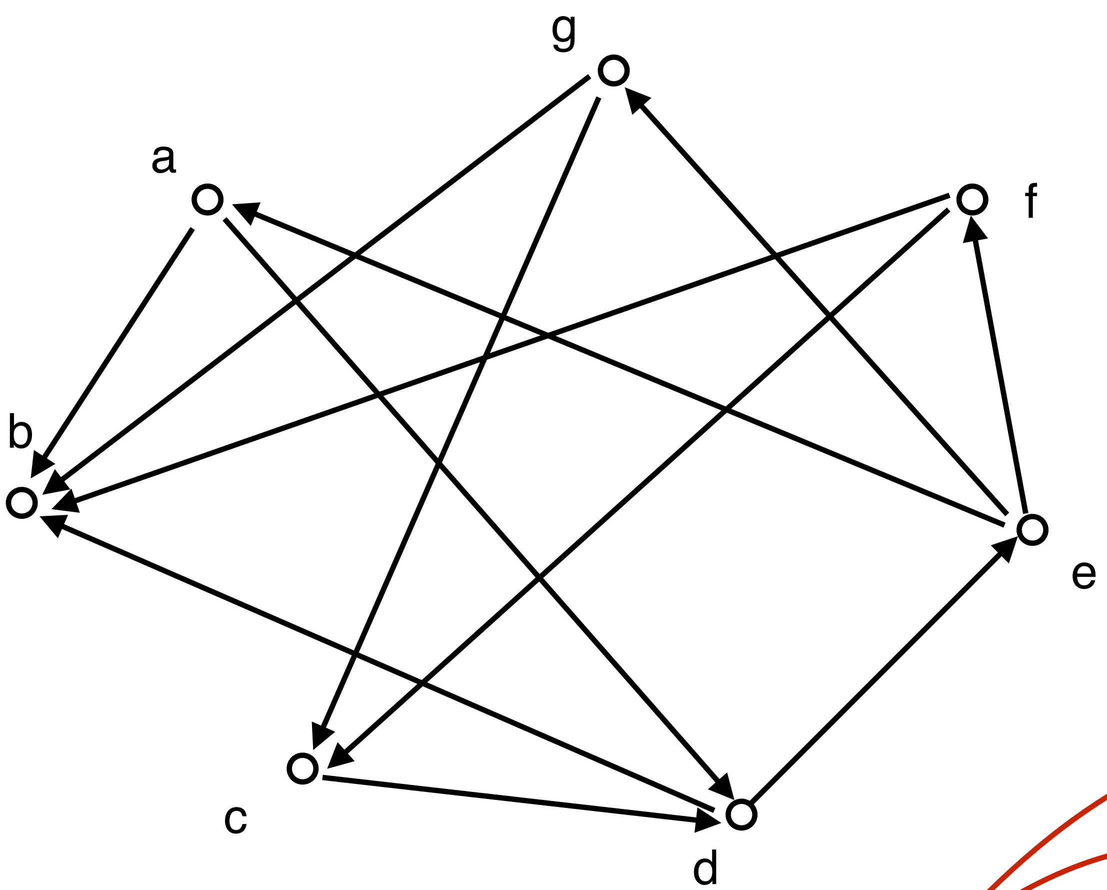
Acyclic Subgraph

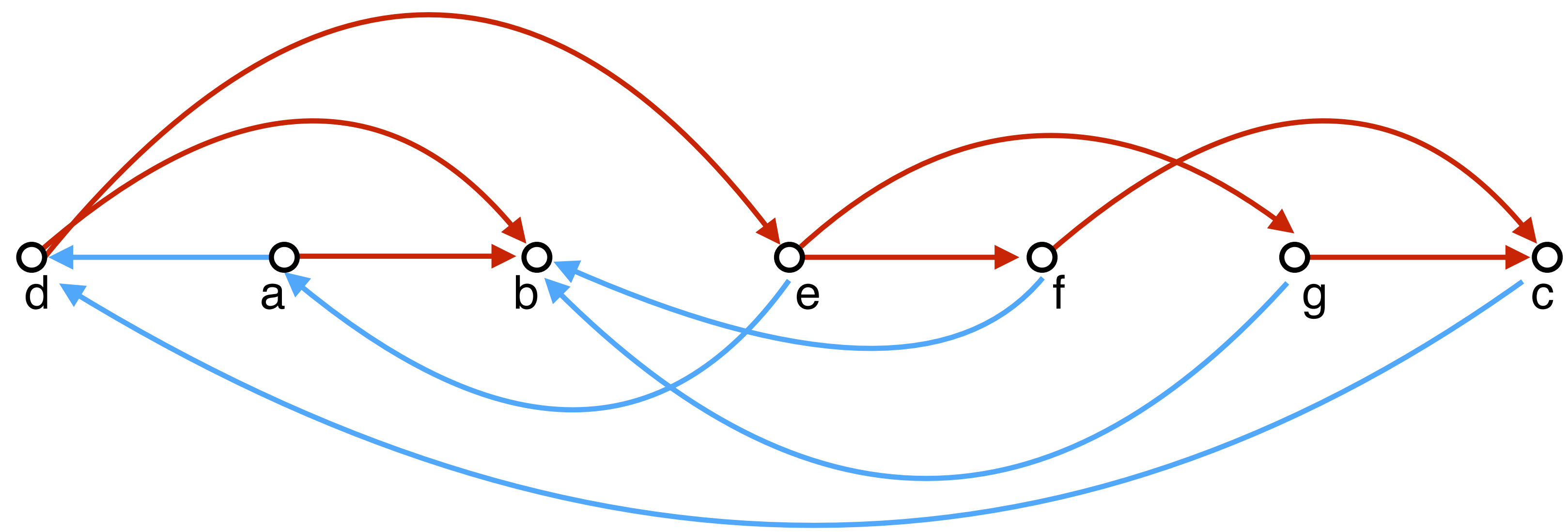


Given graph G, find it's largest acyclic subgraph:

delete some of its edges, such that the remaining graph doesn't contain any directed cycles and has the maximum number of edges.

Acyclic Subgraph





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Vertex Cover 2x-optimal greedy algorithm

Vertex Cover: Given a graph G(V,E) find the smallest subset of vertices S, such that it forms a vertex cover. That is, every edge (u,v) has at least one of its nodes in S.

Algorithm 1: GreedyVC($G(V, E)$)

$S \leftarrow$ empty set of vertices;
for (u, v) *is an edge* **do**
    if $u \notin S$ *AND* $v \notin S$ **then**
        $S \leftarrow S \cup \{u, v\}$;
return S;

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Claim: The GreedyVC( ) algorithm returns a set S that is *at most twice* as large as the smallest vertex cover.

Vertex Cover 2x-optimal greedy algorithm

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proof:

- Consider the set A of edges that this algorithm chooses.
- None of these edges share a vertex, hence any vertex cover must include *at least* |A| vertices
- Set S contains $2 \cdot |A|$ vertices

Approximation algorithms

Suppose that the optimal solution to an optimization problem P has value m*, and algorithm A returns a solution with value m. We say that A is an approximation algorithm with approximation factor c (also called approx. ratio) if on *any* input

if P is a *minimization* problem then $m^* \leq m \leq c \cdot m^*$

- sometimes we use the notation $\frac{1}{c} \cdot m \leq m^*$

or if P is a *maximization* problem then $m \leq m^* \leq c \cdot m$

We say that A is a c-approximation algorithm

Approximation algorithms

A is a c-approximation algorithm if it returns the value m and we have $m \leq c \cdot m^*$ or $m^* \leq c \cdot m$ for the min/max problem.

- c is always $c \geq 1$
- if c = 1, then A always yields the optimal solution

Goal:

- find A for which we can prove that it is a c-approximation for all inputs
- the smaller the c the better
- sometimes for efficiency we may use an approximation algorithm even if there exist a (slow) polynomial optimal algorithm

Greedy approximation algorithm for Independent Set

Independent Set: Given a graph G(V,E), an independent set is a subset of its vertices S*, such that for each edge (u,v) at most one of u or v is in S*

Greedy algorithm to find the max independent set:

GreedyIS is a $(D+1)$-approximation

Let be the maximum degree in G and let S be the set returned by GreedyIS

GreedyIS is a $(D+1)$-approximation

Let D be the maximum degree in G and let S be the set returned by GreedyIS

Goal: find a lower bound on |S|

- a node u is in V—S (thus not in S) because it has a neighbor v in s
- Each v in S has at most D neighbors
- we get $|V-S| \leq D \cdot |S|$
- Adding up the two we get $|V| = |V-S| + |S| \leq D \cdot |S| + |S| = (D+1)|S|$
- in conclusion $|OPT| \leq |V| \leq (D+1)|S|$

Set Cover greedy algorithm

Set Cover: Given a universe U of items $i_1, i_2, \ldots, i_n$ and subsets of items $S_1, S_2, \ldots, S_m$, select a minimum number of the subsets so that their union contains every item in U.

Greedy algorithm:

Set Cover greedy algorithm

Set Cover: Given a universe U of items $i_1, i_2, \ldots, i_n$ and subsets of items $S_1, S_2, \ldots, S_m$, select a minimum number of the subsets so that their union contains every item in U.

Greedy algorithm:

In each iteration select the set that covers the most additional items.

Algorithm 1: GreedySC$(U, S_1, \ldots S_m)$

$X \leftarrow U$ `/* uncovered elements in U */`
$C \leftarrow$ empty set of subsets;
while X *is not empty* **do**
    Select S_i that covers the most items in X;
    $C \leftarrow C \cup S_i$;
    $X \leftarrow X \setminus S_i$;
return C;

SC greedy approximation

Theorem: if the optimal solution to SC uses k sets, than the greedy solution uses at most $k \ln n$ sets

reminder from calculus: for any t > 0 we have $\left(1 - \frac{1}{t}\right)^t < \frac{1}{e}$

Conclusion: GreedySC is an $\ln n$ -approximation

SC greedy approximation

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proof:

- since the optimal solution uses k sets, there is at least one set in the opt that covers 1/k fraction of all items
- since GreedySC selects the largest set, it also covers at least $\frac{n}{k}$ items
- after the first iteration at most $n\left(1-\frac{1}{k}\right)$ remain uncovered
- again, there must be a set in the cover that contains at least 1/k of the remaining
- thus after two iterations $n\left(1-\frac{1}{k}\right)^2$ are uncovered
- after $k \ln n$ rounds there are at most $n\left(1-\frac{1}{k}\right)^{k \ln n}$ uncovered items left
- $n\left(1-\frac{1}{k}\right)^{k \ln n} < \left(\frac{1}{e}\right)^{\ln n} = 1$
- there are at most $k \ln n$ sets returned by the greedy algorithm

SC greedy approximation — how to get k ln n

Reminder from calculus for any t > 0

$$\left(1 - \frac{1}{t}\right)^t < \frac{1}{e}$$

After r iterations the number of uncovered elements is

$$n\left(1 - \frac{1}{k}\right)^r$$

Use trick to get $1 \leq n\left(\left(1 - \frac{1}{k}\right)^k\right)^{\frac{r}{k}} < n\left(\frac{1}{e}\right)^{\frac{r}{k}}$

Some manipulations:

$$e^{\frac{r}{k}} < n \Rightarrow \frac{r}{k} < \ln n \Rightarrow r < k \ln n$$

After r iterations there are no uncovered vertices left.

GreedySC for Vertex Cover

Can we use the approximate solution for Set Cover to solve Vertex Cover?

Dominating Set

Dominating Set: Given a graph G(V,E) a dominating set is a subset of its vertices S, such that for each node v either v is in S or it has a neighbor in S.

DS problem: Given G, find a minimum size dominating set.

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Independent Set: Given a graph G(V,E), an independent set is a subset of its vertices S, such that for each edge (u,v) at most one of u or v is in S

claim: The *maximum* independent set is also a dominating set.

What is the relationship between DS and IS?

Dominating Set and Set Cover

Design an $\ln n$ -approximation algorithm for Dominating Set.