CS630 Graduate Algorithms
November 21, 2024

Dora Erdos and Jeffrey Considine

graphs

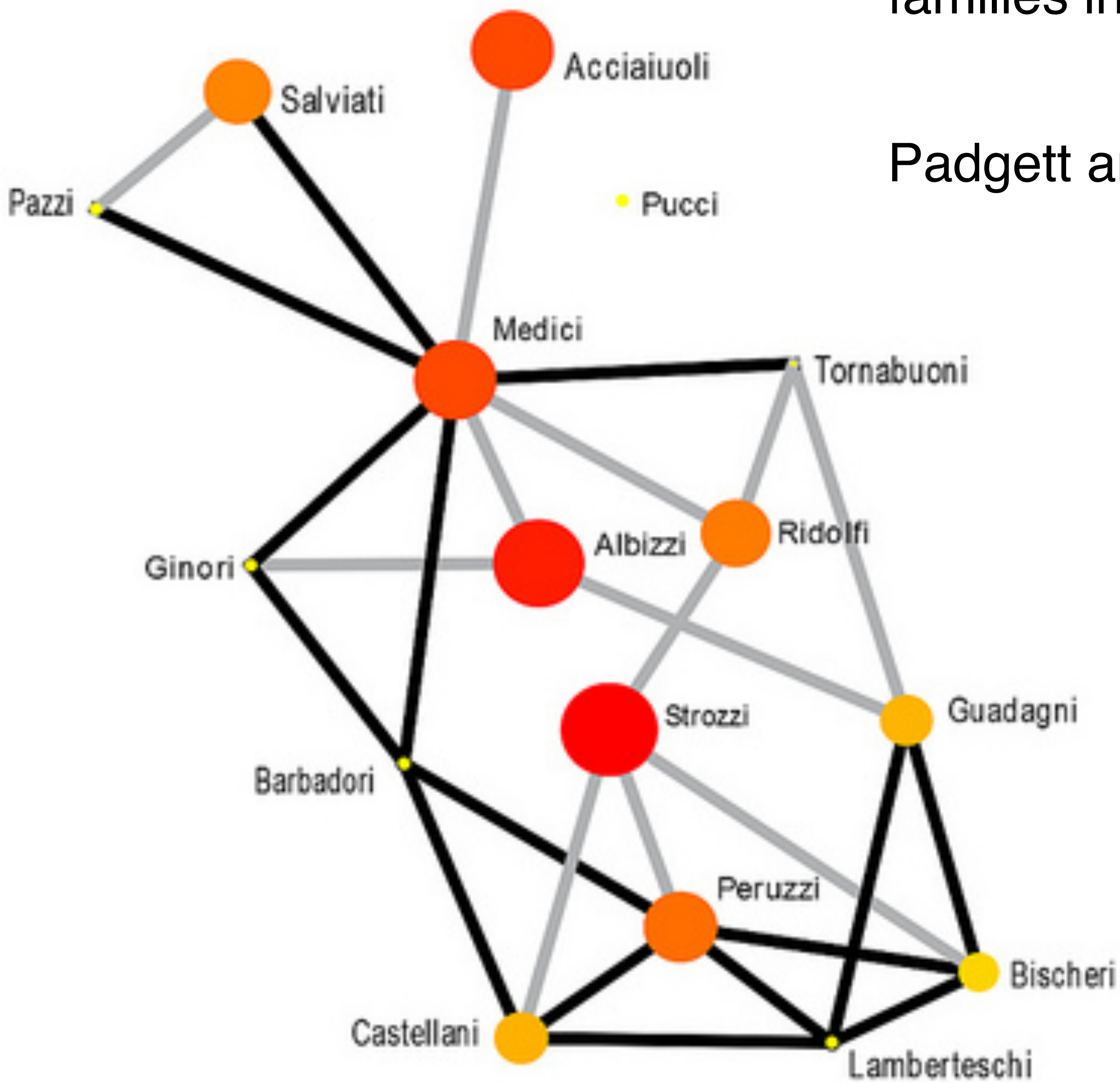
adjacency matrix & walks

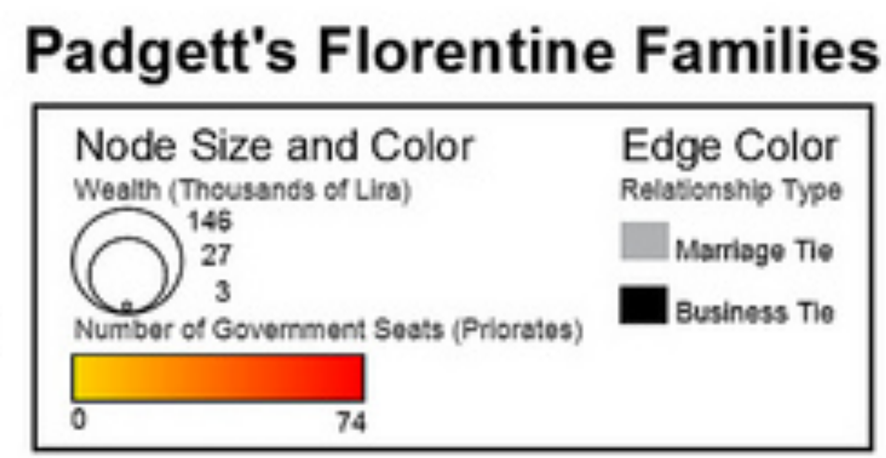
Slides by Dora Erdos

Florentine Families

Marriage and business ties between families in 15th century Florence.

Padgett and Ansell 1993





Centrality

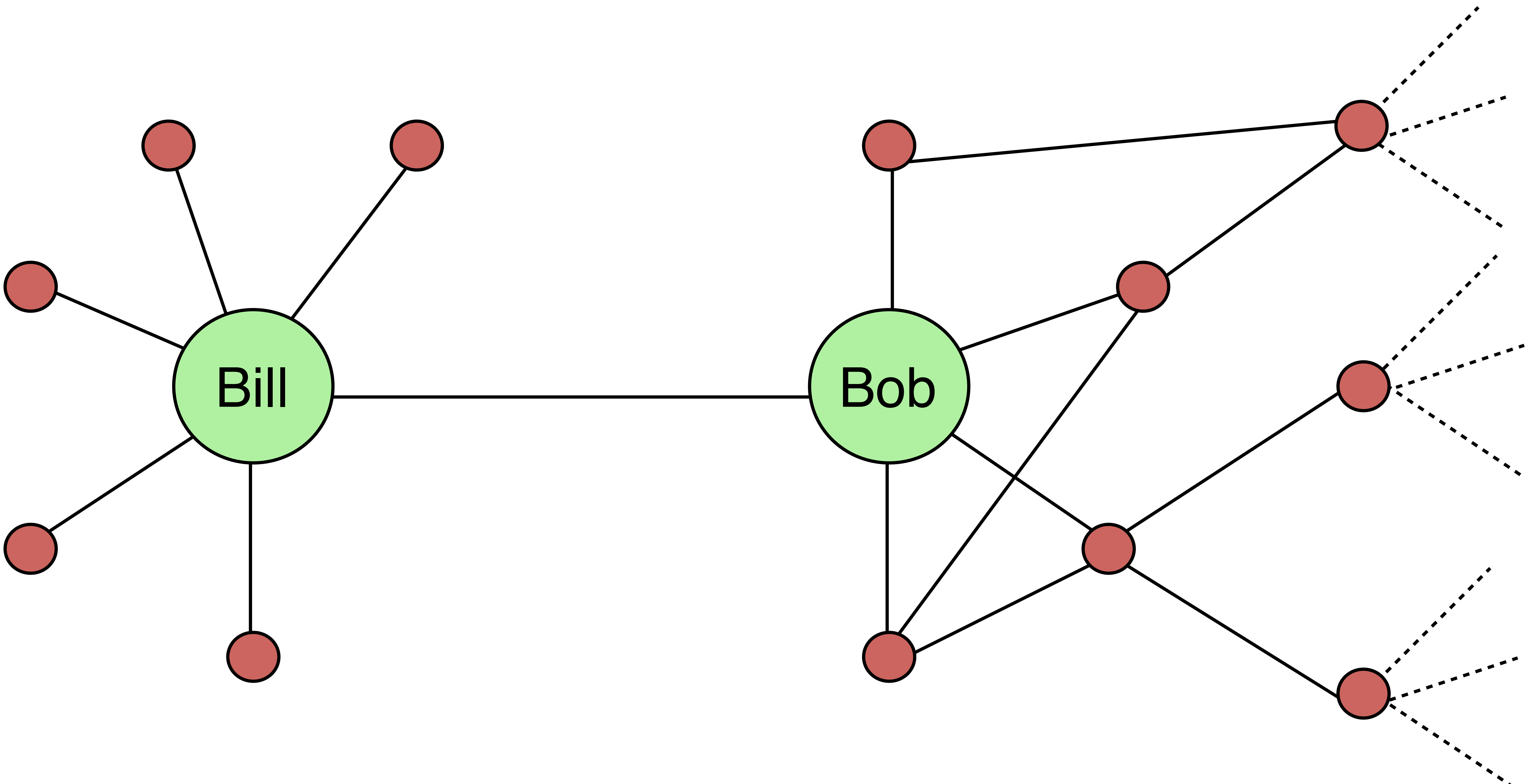
often we try to find the most important vertices in a graph

centrality:

- a measurement of importance
- there are many variants, most of them are related to the paths the vertex is on

Degree centrality

Which person is more “central” in this social network?



Vertices with higher degrees are more central.

Closeness Centrality

G(V,E) connected graph

characterize the average distance to nodes [Bavelas'53]

Lower distances to other vertices result in higher centrality C(v)

d(v,w) = length of shortest path

$$C(v) = \frac{1}{\sum_{w \ in \ V} d(v,w)}$$

Normalized closeness centrality

because of normalization can be used to compare nodes in networks of different size

$$C(v) = \frac{n-1}{\sum_{w \ in \ V} d(v,w)}$$

Closeness Centrality

G(V,E) connected graph

closeness centrality: $C(v) = \dfrac{n-1}{\sum_{w \text{ in } V} d(v,w)}$

Rank nodes based on closeness:

path graph:

star graph:

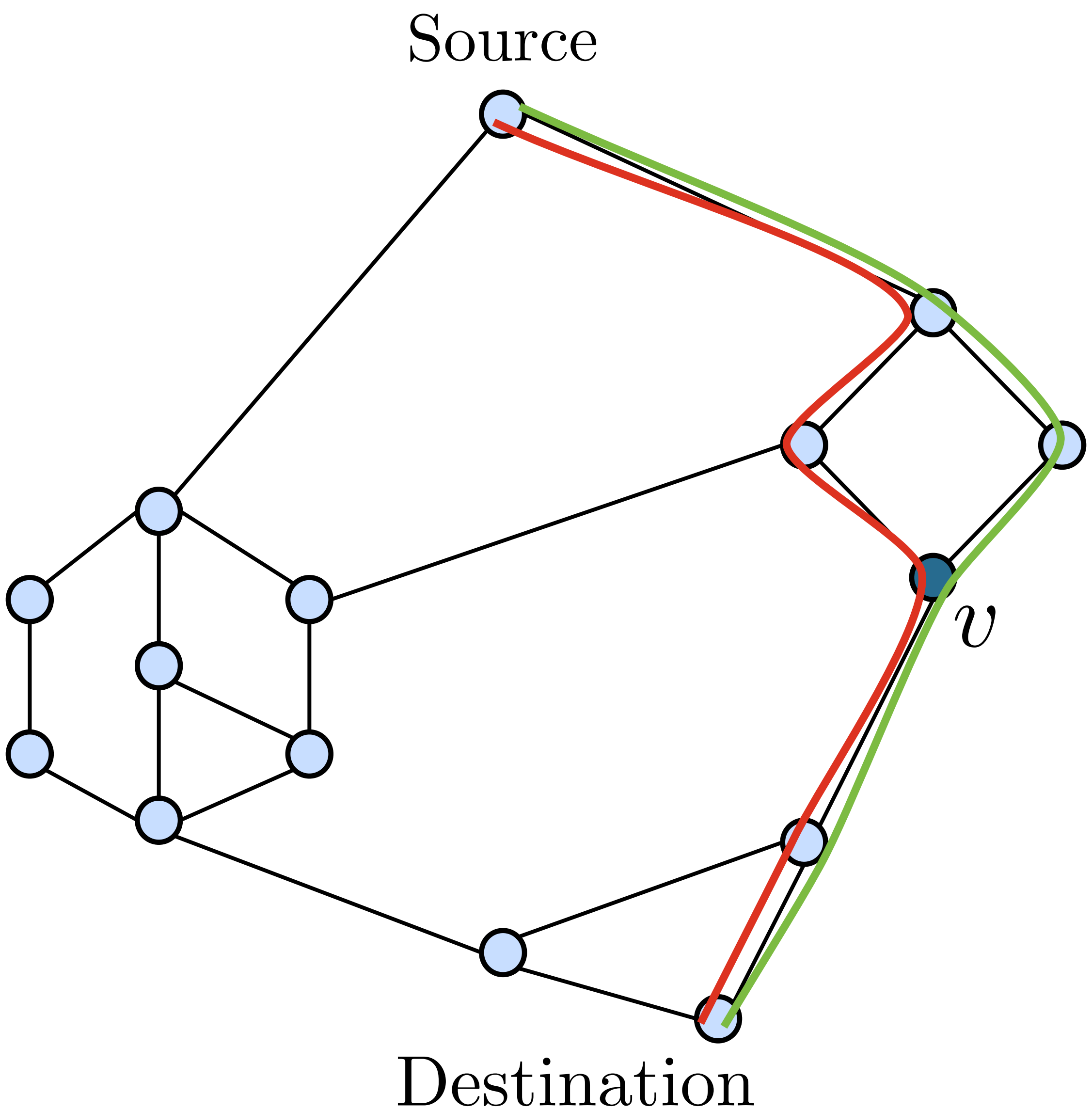
complete graph:

Path-based Centrality

- Network is a collection of paths, centrality corresponds to how many paths a vertex participates in

- characterize centrality variants based on type of paths
  - any paths (ex. propagation of gossip, radio broadcast)
  - shortest paths (ex. send message)
  - walks
    - vertices and edges can be visited multiple times

Path-based centrality



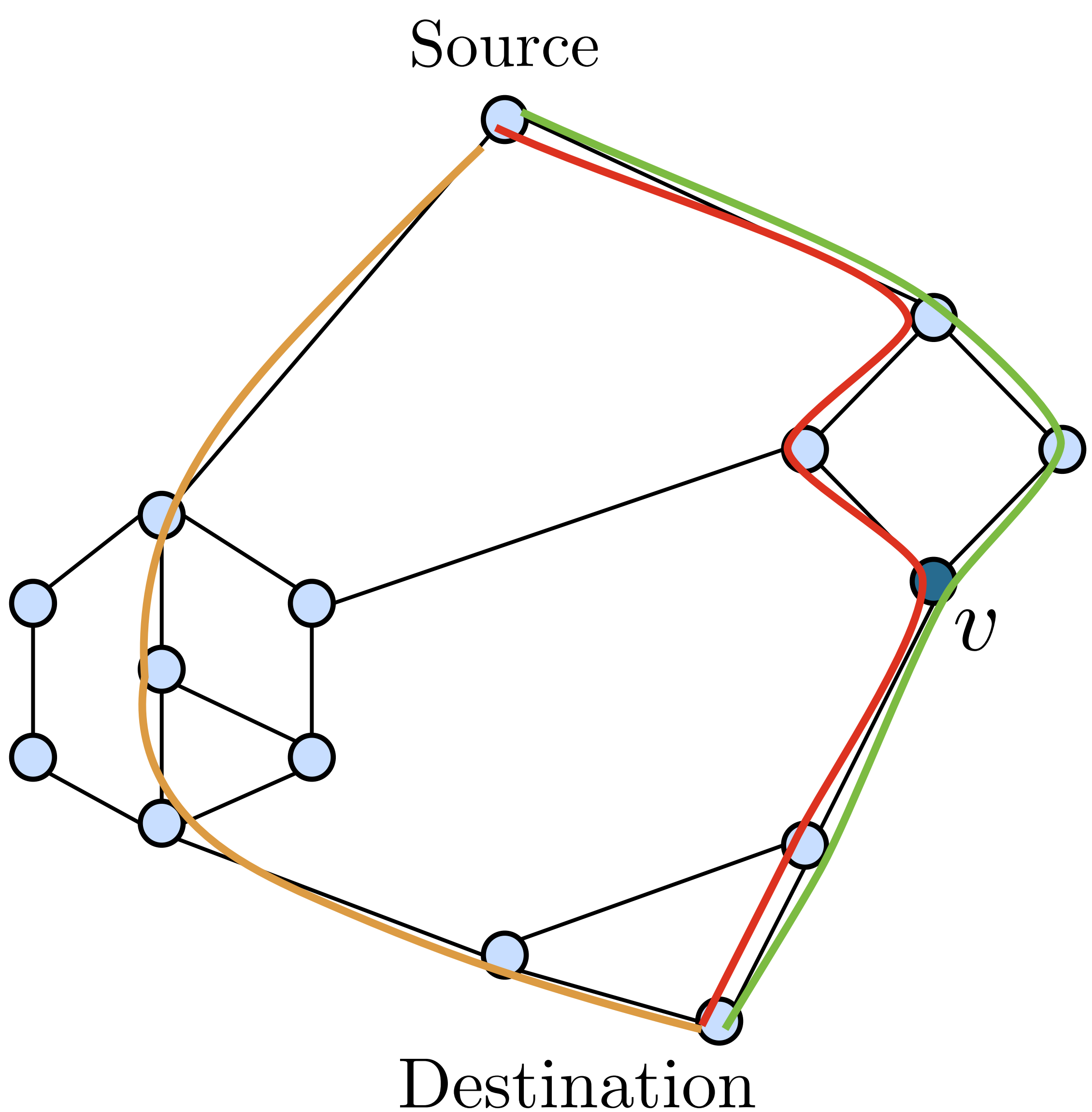
centrality(v) = number of *paths* it covers

Shortest paths centrality:

$$C_{st}(v) = 2$$

$$C(v) = \sum_{s,t \ in \ V} C_{st}(v)$$

Path-based centrality



centrality(v) = number of *paths* it covers

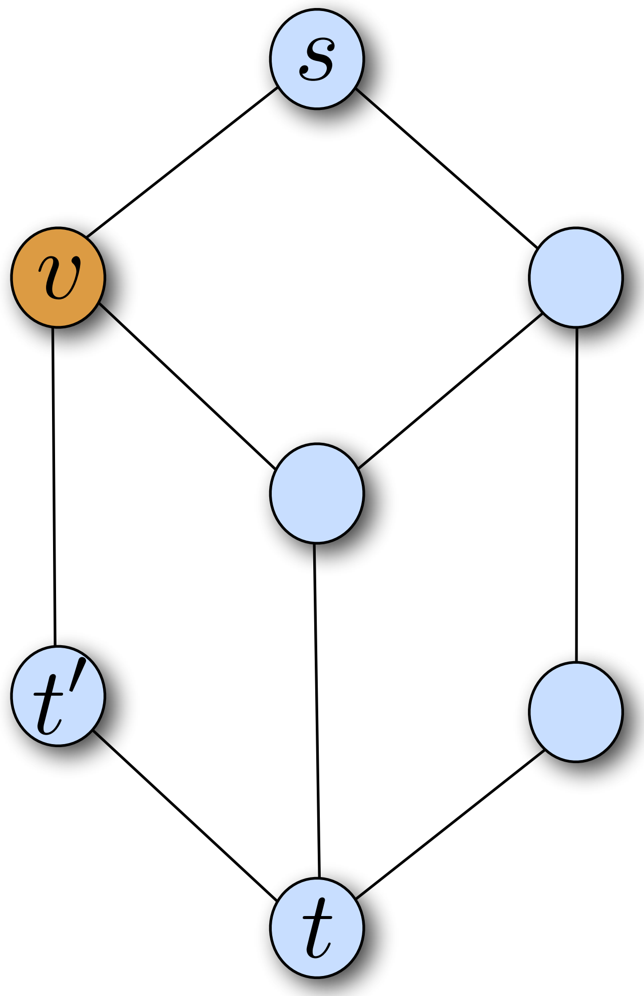
Shortest paths centrality:

$$C_{st}(v) = 2$$

$$C(v) = \sum_{s,t \ in \ V} C_{st}(v)$$

How would you change $C_{St}(v)$?

Dependency

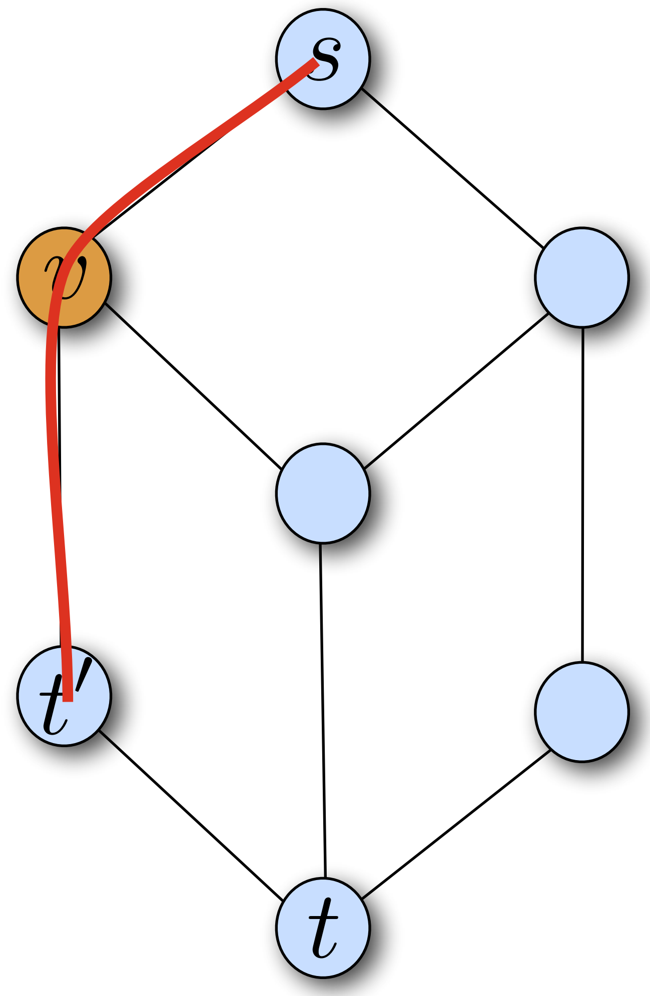


number of shortest paths from s to t'?

number of shortest paths from s to t?

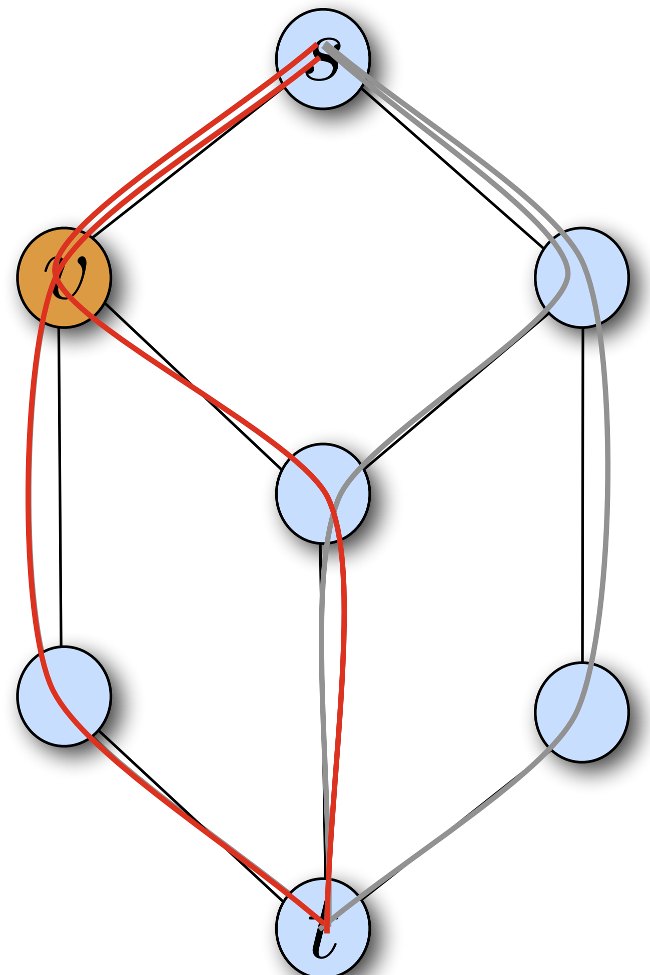
Dependency

$$\mathtt{dep}(\mathtt{s}, \mathtt{t}'|\mathtt{v}) = \frac{1}{1}$$



$$\mathtt{dep}(\mathtt{s}, \mathtt{t}|\mathtt{v}) = \frac{\#\mathtt{sh\_paths}(\mathtt{s}, \mathtt{t}|\mathtt{v})}{\#\mathtt{sh\_paths}(\mathtt{s}, \mathtt{t})}$$

Dependency



$$\mathtt{dep}(\mathtt{s},\mathtt{t}|\mathtt{v}) = \frac{2}{4}$$

$$\mathtt{dep}(\mathtt{s},\mathtt{t}|\mathtt{v}) = \frac{\#\mathtt{sh\_paths}(\mathtt{s},\mathtt{t}|\mathtt{v})}{\#\mathtt{sh\_paths}(\mathtt{s},\mathtt{t})}$$

Betweenness centrality

$$\mathtt{betweenness}(\mathtt{v}) = \sum_{\mathtt{s},\mathtt{t} \in \mathtt{V}} \mathtt{dep}(\mathtt{s}, \mathtt{t} | \mathtt{v})$$

Quantifies to what extent each source-destination pair depends on v.

$$\mathtt{dep}(\mathtt{s}, \mathtt{t} | \mathtt{v}) = \frac{\#\mathtt{sh\_paths}(\mathtt{s}, \mathtt{t} | \mathtt{v})}{\#\mathtt{sh\_paths}(\mathtt{s}, \mathtt{t})}$$

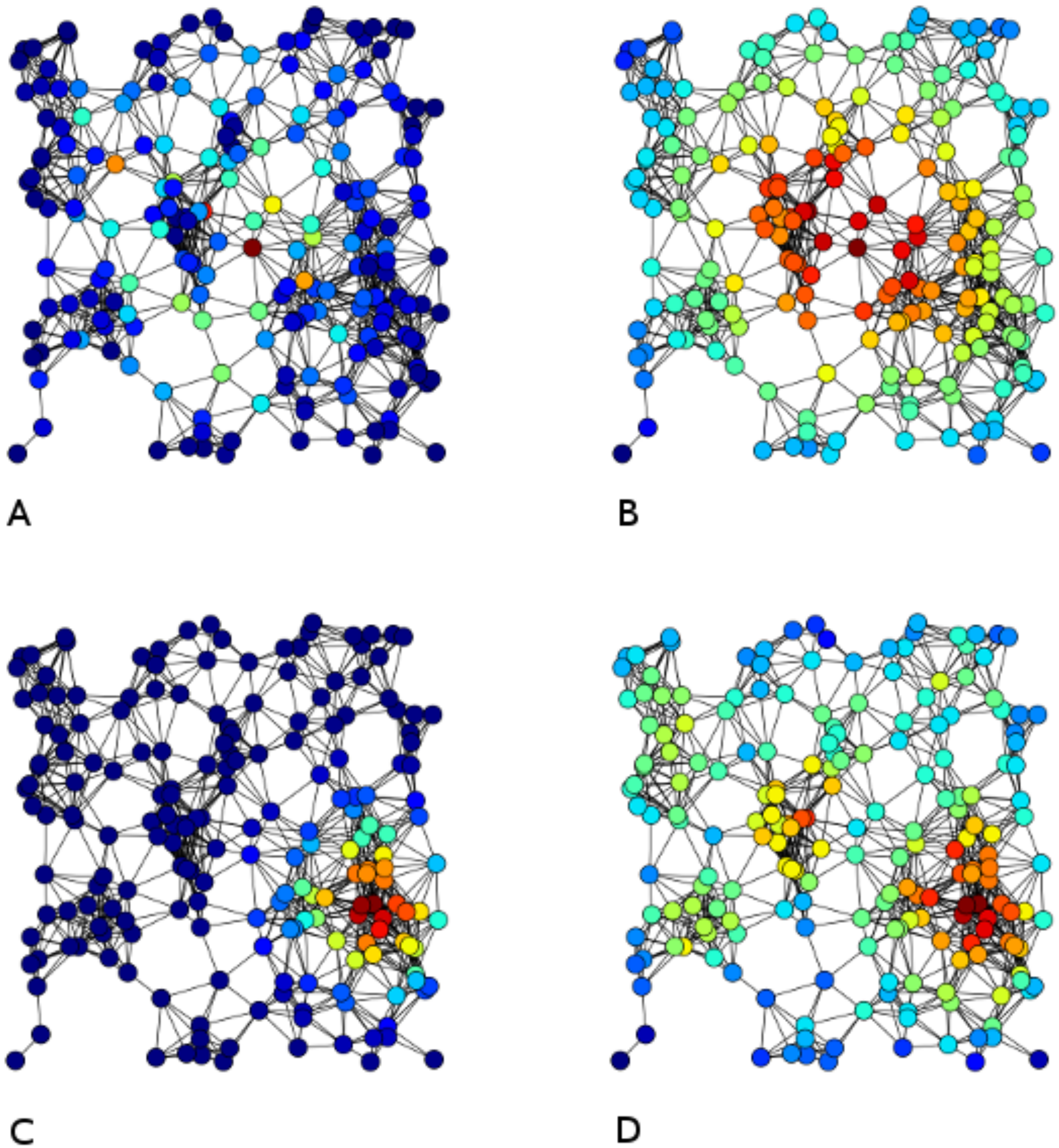
normalization

normalized closeness centrality: $C(v) = \dfrac{n-1}{\sum_{w\ in\ V} d(v,w)}$

betweenness: $\mathtt{dep}(\mathtt{s},\mathtt{t}|\mathtt{v}) = \dfrac{\#\mathtt{sh\_paths}(\mathtt{s},\mathtt{t}|\mathtt{v})}{\#\mathtt{sh\_paths}(\mathtt{s},\mathtt{t})}$

$$\mathtt{betweenness}(\mathtt{v}) = \sum_{\mathtt{s},\mathtt{t}\in\mathtt{V}} \mathtt{dep}(\mathtt{s},\mathtt{t}|\mathtt{v})$$

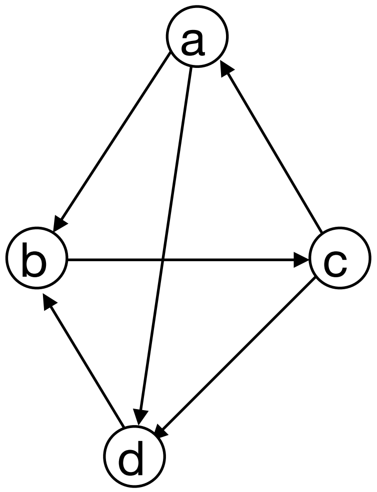
How would you normalize betweenness centrality?



A. betweenness
B. closeness
C. eigenvector
D. degree

dark red - most central
dark blue - least central

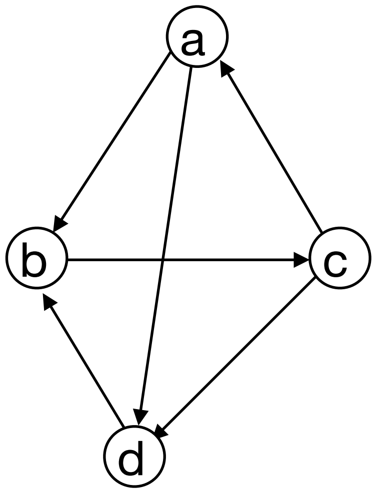
Graph adjacency matrix



D[i,j]=1 if there is a directed edge from i to j

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

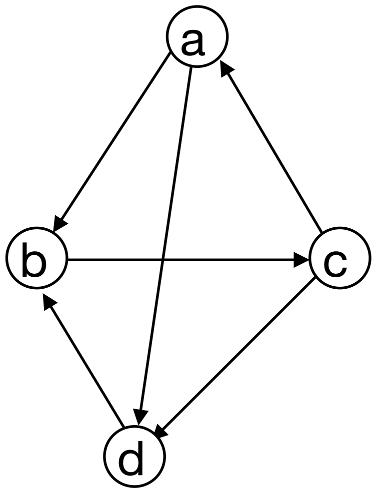
Graph adjacency matrix



D[i,j]=1 if there is a directed edge from i to j

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph adjacency matrix



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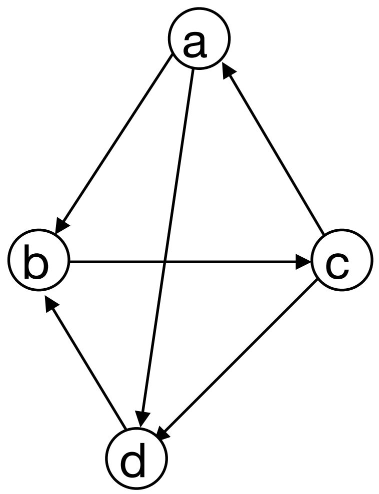
Graph adjacency matrix



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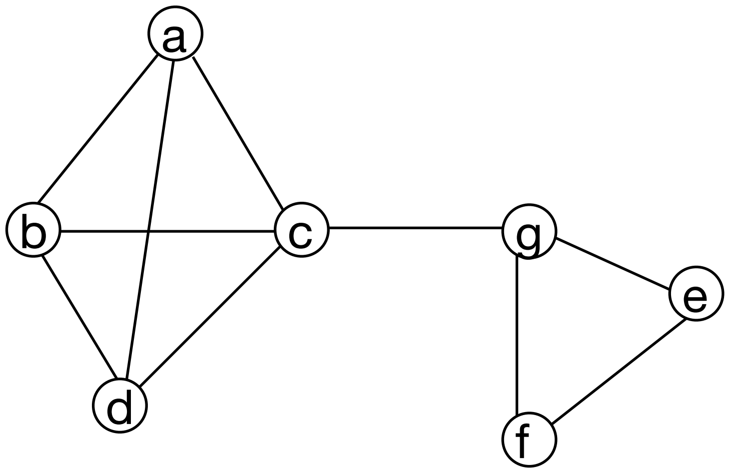
- what is the number in $D^2[i,j]$?
- what is the number in $D^3[i,j]$?
- what is the number in $D^k[i,j]$?
- where can we find the number of directed triangles that vertex a is contained in?
- what does it say about i and j if $\sum_{k=0}^{n-1} D^k[i,j] = 0$?

Graph adjacency matrix

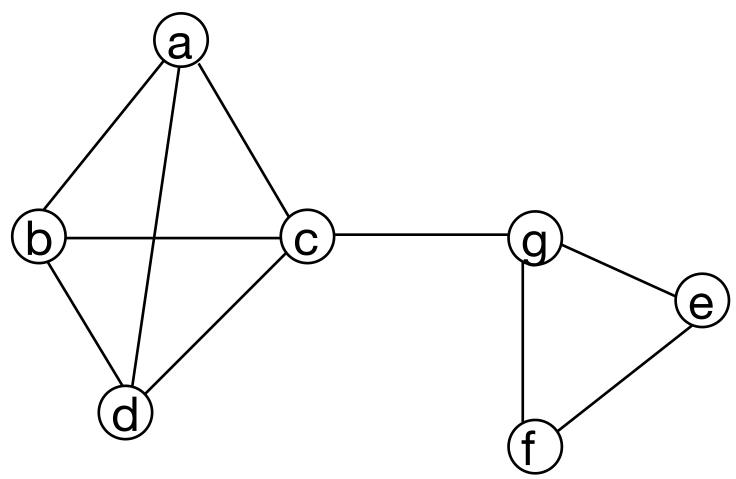
M[i,j]=1 if there is a directed edge from i to j



$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Graph adjacency matrix

M[i,j]=1 if there is a directed edge from i to j



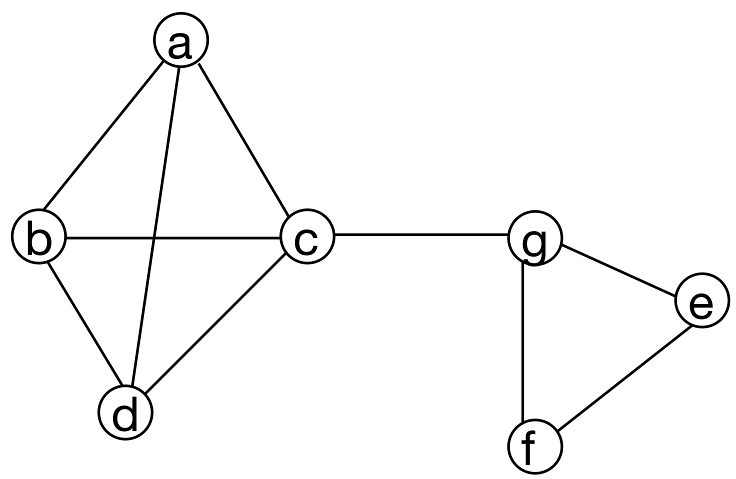
what can we infer from the fact that…

- M[a,c] = 0, M^2[a,c] = 0? M^3[a,c] = 1?
- M^2[a,a] = 3?
- Why is M^3[g,c] = 6 ?

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 2 & 0 & 0 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad M^3 = \begin{bmatrix} 6 & 7 & 8 & 7 & 1 & 1 & 2 \\ 7 & 6 & 8 & 7 & 1 & 1 & 2 \\ 8 & 8 & 6 & 8 & 1 & 1 & 6 \\ 7 & 7 & 8 & 6 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 3 & 2 & 4 \\ 2 & 2 & 6 & 2 & 4 & 4 & 2 \end{bmatrix}$$

Graph adjacency matrix

M[i,j]=1 if there is a directed edge from i to j



Let x = [1,0,0,0,0,0,0] represent vertex a

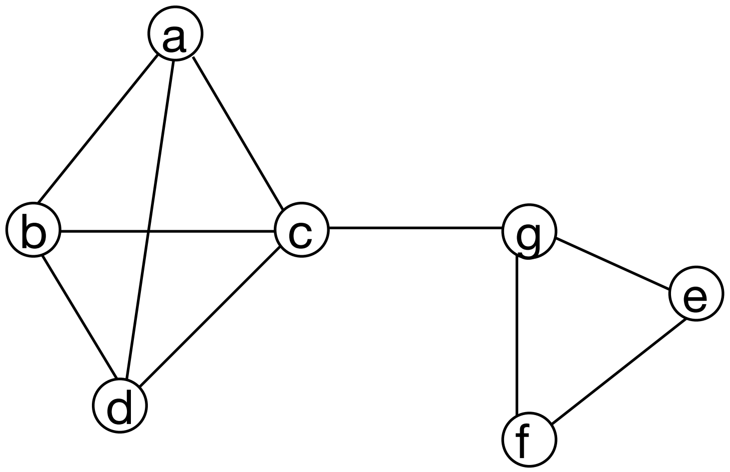
Then:
xM = [0,1,1,1,0,0,0]

xM^2 = [3,2,2,2,0,0,1]

xM^3 = [6,7,8,7,1,1,2]

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 2 & 0 & 0 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad M^3 = \begin{bmatrix} 6 & 7 & 8 & 7 & 1 & 1 & 2 \\ 7 & 6 & 8 & 7 & 1 & 1 & 2 \\ 8 & 8 & 6 & 8 & 1 & 1 & 6 \\ 7 & 7 & 8 & 6 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 3 & 2 & 4 \\ 2 & 2 & 6 & 2 & 4 & 4 & 2 \end{bmatrix}$$

Graph adjacency matrix



Let y = [1,0,1,0,0,0,0]

What is true about $yM^k[i,j]$?

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 2 & 0 & 0 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad M^3 = \begin{bmatrix} 6 & 7 & 8 & 7 & 1 & 1 & 2 \\ 7 & 6 & 8 & 7 & 1 & 1 & 2 \\ 8 & 8 & 6 & 8 & 1 & 1 & 6 \\ 7 & 7 & 8 & 6 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 3 & 2 & 4 \\ 2 & 2 & 6 & 2 & 4 & 4 & 2 \end{bmatrix}$$

Graph adjacency matrix

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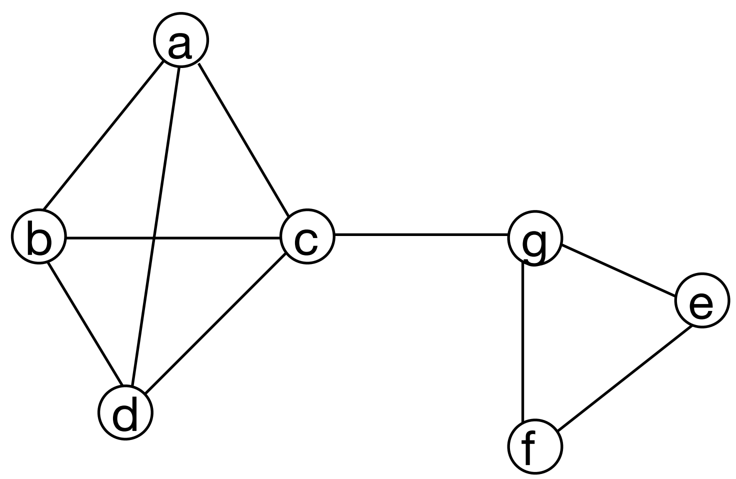


Let $x = [x_1, x_2, \ldots, x_n]$ represent the starting vertices of a walk.

xi can be 0/1

xi can be a probability

$xM^k[j]$ = the number of ways we can reach j in k steps if we start based on x.

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 2 & 0 & 0 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad M^3 = \begin{bmatrix} 6 & 7 & 8 & 7 & 1 & 1 & 2 \\ 7 & 6 & 8 & 7 & 1 & 1 & 2 \\ 8 & 8 & 6 & 8 & 1 & 1 & 6 \\ 7 & 7 & 8 & 6 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 3 & 2 & 4 \\ 2 & 2 & 6 & 2 & 4 & 4 & 2 \end{bmatrix}$$

Path-based Centrality

- Network is a collection of paths, centrality corresponds to how many paths a vertex participates in

- characterize centrality variants based on type of paths
  - any paths (ex. propagation of gossip, radio broadcast)
  - shortest paths (ex. send message)
  - walks
    - vertices and edges can be visited multiple times

Path lengths

- degree centrality - length 1 path
- adjacency matrix = # of length-1 paths A
- # of length-k paths A^k
- # of paths of length at most k $\sum_{i=1}^{k} A^i$

- infinite paths?
  - define centrality as the relative fraction of paths passing through a vertex
  - equivalent to asking: starting from a random vertex and taking some number of steps along the edges, how likely are we to end up in vertex v?

“status” as centrality measure

- vertices have a status (importance) score
- vertices have ‘high’ status if they are connected to vertices with ‘high’ status (this is a circular definition ;))

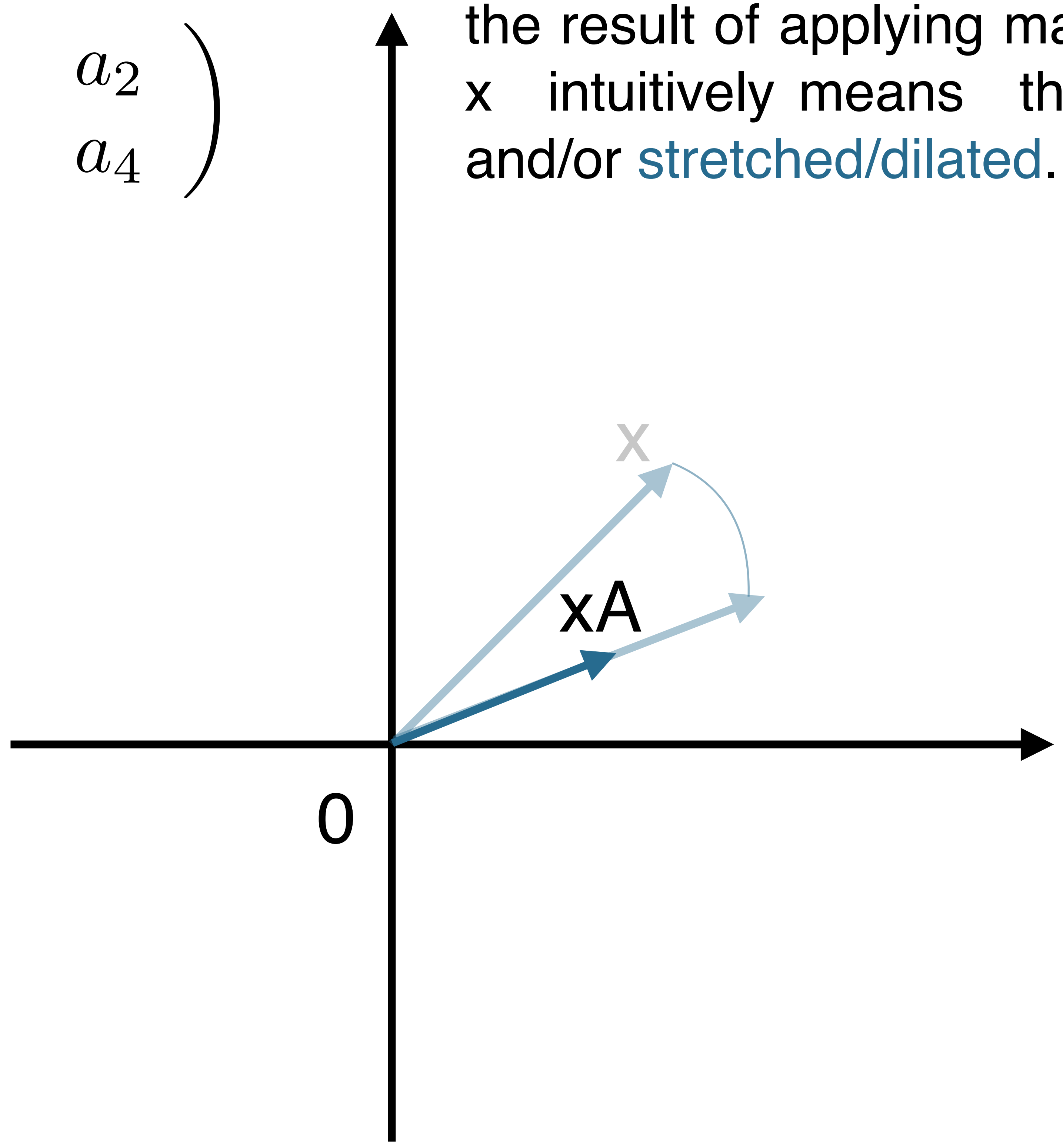
Centrality x_i of a node i is the sum of the centrality of its neighbors, scaled by some constant.

$$x_i = \frac{1}{\lambda} \sum_{(i,j) \in E} x_j$$

λ is a constant, that we’ll choose cleverly (later today)

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

the result of applying matrix A to a vector x intuitively means that x gets rotated and/or stretched/dilated.



Eigenvalue decomposition

Assume *A* is an *n x n* square matrix
u is an n-dim vector

$$v = uA = \sum_{j=1}^{n} u_j A[\,*\,, j]$$

You can think of *v* as a linear combination of the columns of *A*.

u is an eigenvector if applying *A* to it only changes its magnitude not its direction.

$$\lambda u = uA$$

If *A* has rank *n*, then there are *n* orthonormal eigenvectors and eigenvalues.

Eigenvalue decomposition of A

$$A = U \Lambda U^{-1}$$

U contains the eigenvectors as its columns
Λ is a diagonal matrix with the eigenvectors in its diagonal

Eigenvector centrality

Centrality x_i of a node *i* is the sum of the centrality of its neighbors, scaled by some constant.

$$x_i = \frac{1}{\lambda} \sum_{(i,j) \in E} x_j = \frac{1}{\lambda} \sum_{A[i,j]==1} x_j$$

Arrange the centrality values in a vector x = [x_1,x_2,….,x_n].

$$x = \frac{1}{\lambda} x A$$

Rearranging we get $x\lambda = xA$

x[i] is called the eigenvector centrality of node i.
if we choose λ to be the largest eigenvalue, than all x[i] are positive. (more on this later)

Path lengths

- degree centrality - length 1 path
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Eigenvalue decomposition and powers of a matrix

eigenvalue decomposition $A = V\Lambda V^T$

- V is orthonormal
- Λ is diagonal

compute A^k:

$$A = V\Lambda V^T$$

$$A^2 = V\Lambda V^T V\Lambda V^T = V\Lambda^2 V^T$$

$$A^k = V\lambda V^T V\lambda V^T \ldots V\Lambda V^T = V\Lambda^k V^T$$

Not sure how to use it yet, but there is definitely some kind of relationship between A^k and eigenvector centrality.

Random surfer model

Setting:

- on the Web there are sites and hyperlinks directing from one page to an other.
- remember the status score?
  - high status nodes (sites) are the ones pointed (linked) by high status nodes.
  - here links are endorsements by the author
- we are still on a quest to assign the “status” score to nodes

Random surfer:

- we start on a random page
- at each step we follow one of the outgoing links at random
- sometimes we get bored (e.g. quit browsing), and start over at some random page
- if we were to walk for a looooong time, then our starting page doesn’t matter anymore
- we should find the probability that at any given time we are in any given page
- it turns out that after many steps this probability is independent of the walk itself
- this probability distribution (called stationary distribution) is Pagerank

Walking on the graph — power method

Let $x^{(0)} = [x_1, x_2, \ldots, x_n]$ correspond to some initial state of the vertices.

- The $^{(0)}$ in the notation corresponds to the 0th iteration, not an exponent.

Now take a step

$$x^{(1)} = x^{(0)} A$$

$x_i^{(1)}$ is the weight-scaled notion of being in node i.

Another step, then many more

$$x^{(2)} = x^{(1)} A = x^{(0)} A^2$$

$$x^{(k)} = x^{(k-1)} A = x^{(0)} A^k$$

Walking on the graph — power method

Let $x^{(0)} = [x_1, x_2, \ldots, x_n]$ correspond to some initial state of the vertices.

- The $^{(0)}$ in the notation corresponds to the 0th iteration, not an exponent.

In summary we have

$$x^{(1)} = x^{(0)} A$$

$$x^{(k)} = x^{(0)} A^k$$

If x can be any kind of weight, then these would be really large numbers. So let's normalize in each iteration.

$$\text{Assume } \|x^{(0)}\|_2 == 1 \text{ (or take } x^{(0)} = \frac{x^{(0)}}{\|x^{(0)}\|_2}\text{)}$$

$$x^{(1)} = x^{(0)} A$$

$$x^{(1)} = \frac{x^{(1)}}{\|x^{(1)}\|_2}$$

After k iterations

$$x^{(k)} = x^{(k-1)} A = x^{(0)} A^k$$

$$x^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|_2}$$