CS630 Graduate Algorithms

September 17, 2024

by Dora Erdos and Jeffrey Considine

- Polynomial reductions

slides by Dora Erdos and Jeffrey Considine, based on Kevin Wayne, Adam Smith

Polynomial Reduction to Problems (informal)

Think of the max-flow solving algorithm (in our case FF - there are others) as a call to an API.

In the applications we used this API to solve other problems.

- max bipartite matching
- disjoint paths
- baseball elimination
- blood donation, dance competition, spies, etc.

Can we do this API-style trick with other algorithms/solvers too?

Polynomial-time reductions

Suppose there is an algorithm to solve problem Y.

- the algorithm is unknown to us, we treat it as a *black-box* or *API*

Polynomial Reduction. Problem X is polynomial-time reducible to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to an oracle/black box that solves problem Y.

Notation. $X \leq_P Y$.

Bipartite Matching $\leq_p$ Max Flow

- define flow graph corresponding to the bipartite graph
- find max flow (using FF)
- iterate over middle edges and assign pairs with non-zero flow to the matching

Longest Path and Hamiltonian Path problems

Longest Path: Given a weighted graph *G* and an integer *k*, is there a simple path of lengths at least *k*?

- if G does have a path of length k, it is easy to “prove” by returning the path itself
  - Proof by example - the path is an easy certificate for Longest Path

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  - Proof by example - the path is another easy certificate

Hamiltonian Path - TopHat

Hamiltonian path problem: Given an *un*weighted graph G, is there a simple path in G that contains *all* vertices?

Suppose that there is an algorithm Long( ) that solves the Longest path problem.

Question: we can use Long( ) to solve the Ham-path problem. Which of these best describes this approach?

A. Pick a random source s. Run DFS on G from s to find the “deepest” path. If this contains every node, then we found the Hamiltonian path.

B. Assign weight 1 to each edge in G then call Long(G, n-1). If it returns n-1 as the longest simple path length, then G contains a Hamiltonian path.

C. Find whether there is a Hamiltonian path in G. If yes, then return that Long(G) is n-1.

Longest Path and Hamiltonian Path problems

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- we don't know of any polynomial algorithm for it
  - No super-polynomial lower bound either
  - Belongs to a class called NP-Complete → next lecture's topic
- if G does have a path of length k, it is easy to "prove" by returning the path itself

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Hamiltonian-path problem: Given an unweighted graph G(V,E) is there a simple path that contains every node exactly once?

There is a polynomial-time reduction from Hamiltonian-path to Longest Path. Which means that any algorithm solving LP can also be used to solve Ham-Path with polynomial number of extra steps.

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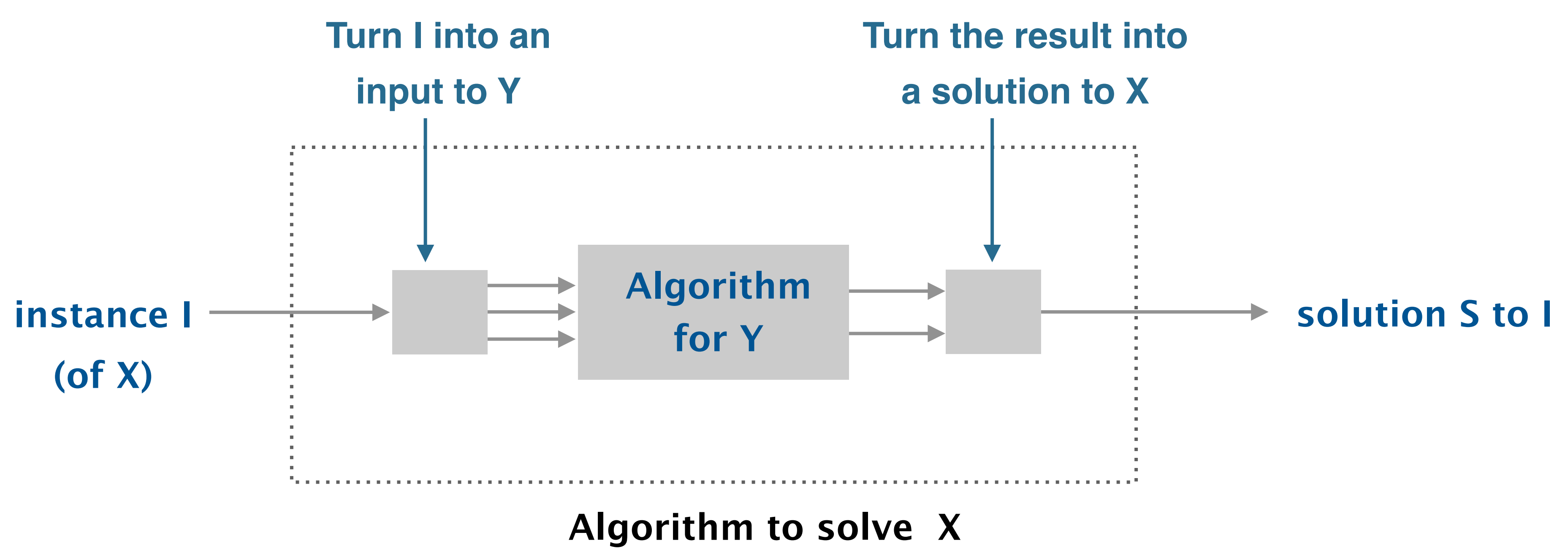


Hamiltonian path $\leq_P$ Longest Path

Same result, new notation.

Circuit Value Problem

Given a description of a Boolean circuit and its input values, calculate the output value of the circuit.

A Boolean circuit is usually describe with a directed acyclic graph with vertices labeled as inputs, outputs, or internal gates (of type NOT/AND/OR…)

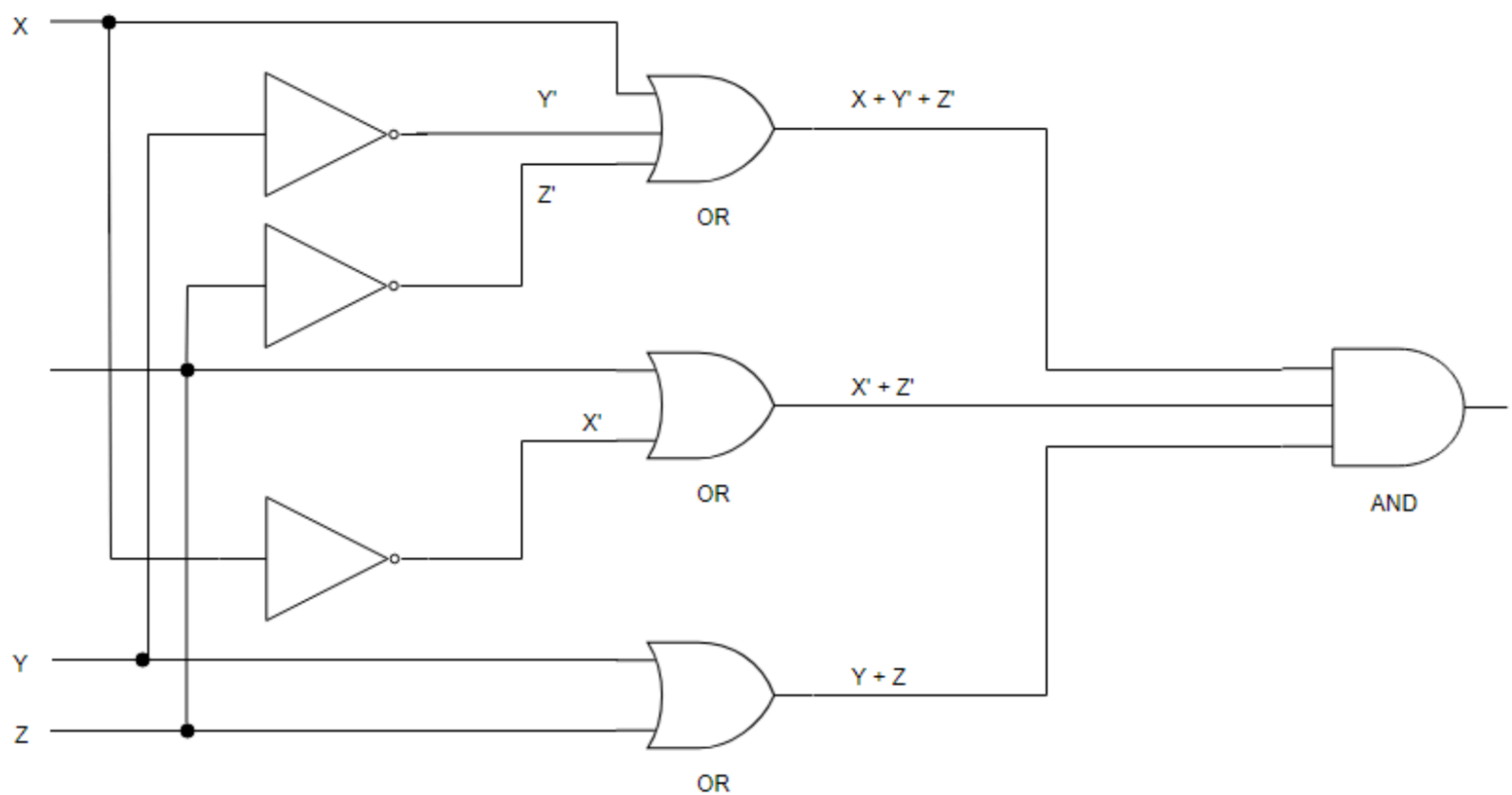
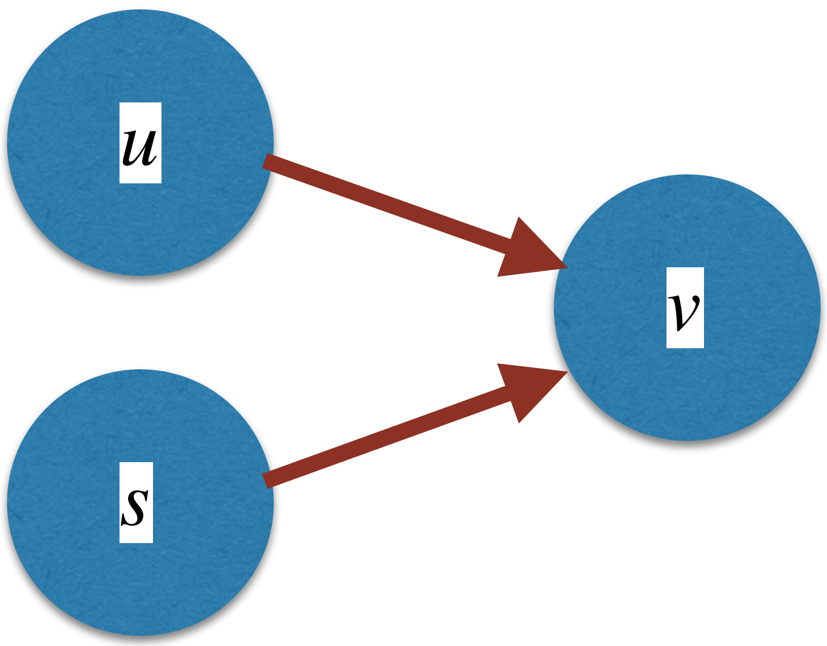


Image source: https://online.visual-paradigm.com/

Topological Sort

Given a directed acyclic graph, generate an order of the vertices such that if there is an edge $u \to v$, then u comes before v in the new order.



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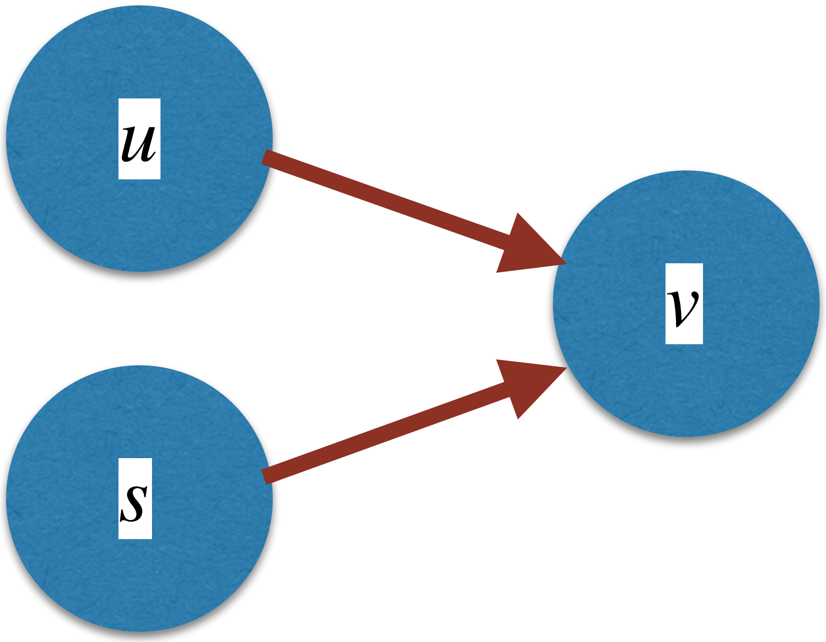

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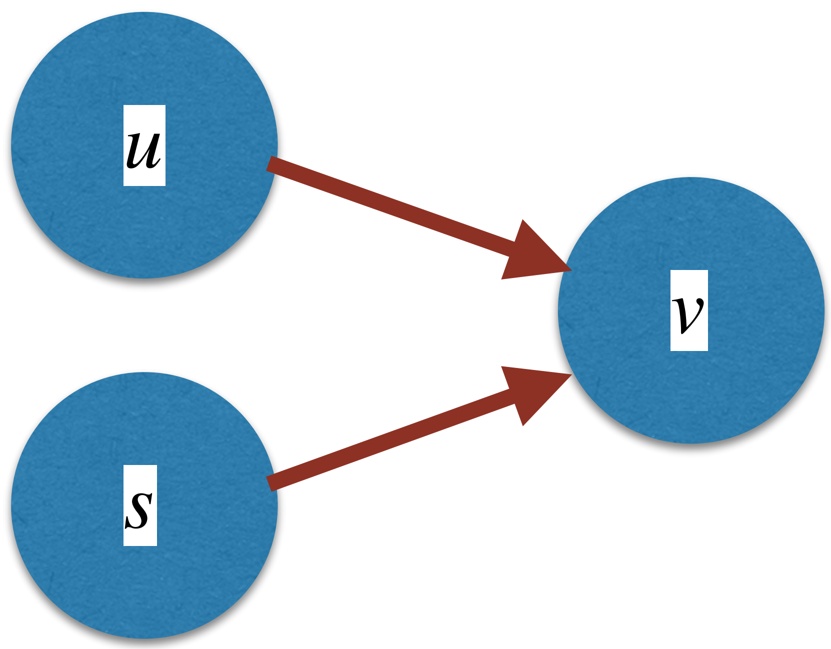

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How does this relate to the previous problem?

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- Perform a topological sort of the circuit DAG.
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  - When we get to a node, all its inputs are already calculated.

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No recursion after topological sort.

(Topological sort usually uses modified DFS.)

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Old measure of vector similarity.

$$u \cdot v = u^T v = \cos\ \theta\ \|u\|\ \|v\|$$

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- Map documents of interest to vectors of features.
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Popular again with new machine learning developments

- Turns out many large language model make really good vectors.
- Revival of cosine similarity to retrieve related documents, and drive Retrieval Augmented Generation (RAG)

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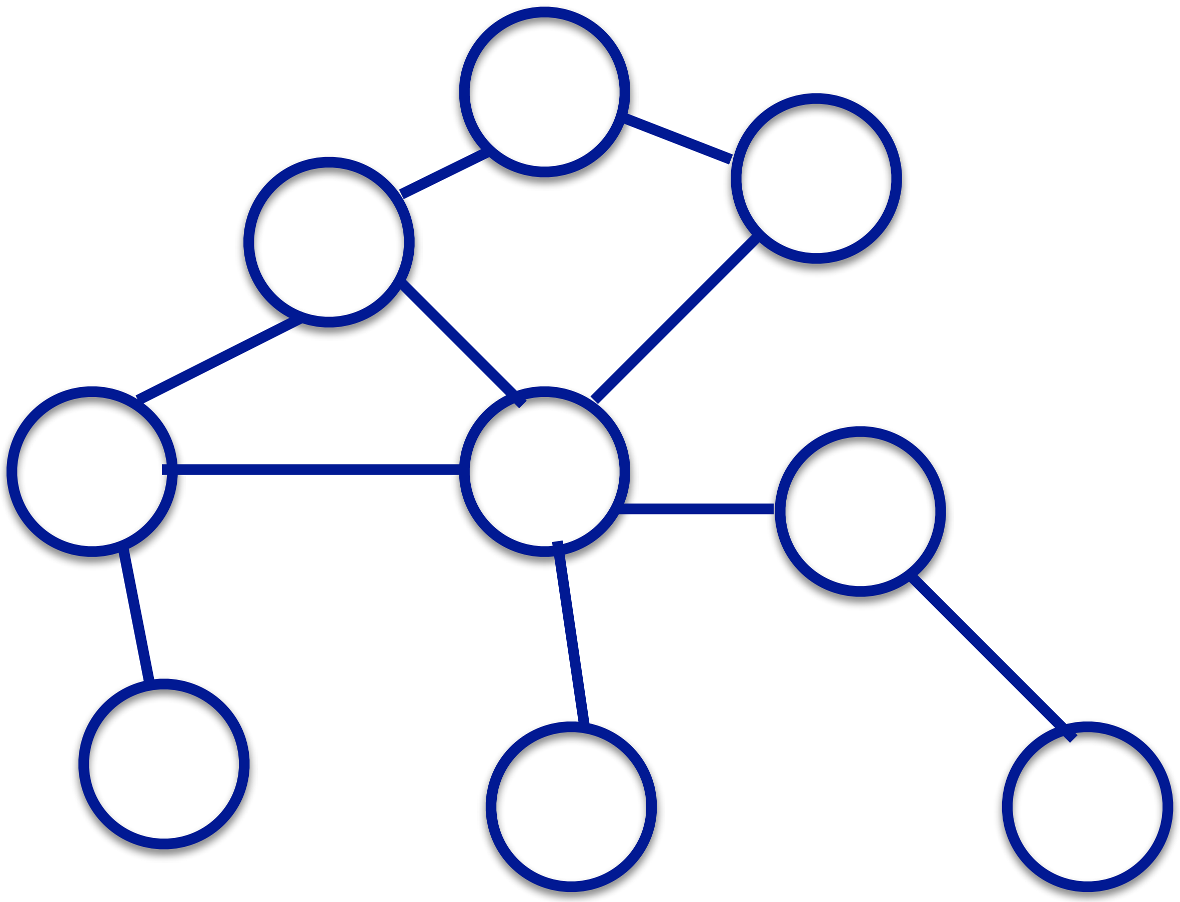
TLDR:

- Reduce cosine similarity to nearest neighbors by normalizing lengths.
- Then highest cosine similarity is the same as smallest distance.

Vertex Cover

Given a graph $G = (V, E)$, find a minimum subset of V such that each edge is covered. That is, find minimum $V' \subseteq V$ such that $\forall (u, v) \in E \; (u \in V' \lor v \in V')$.



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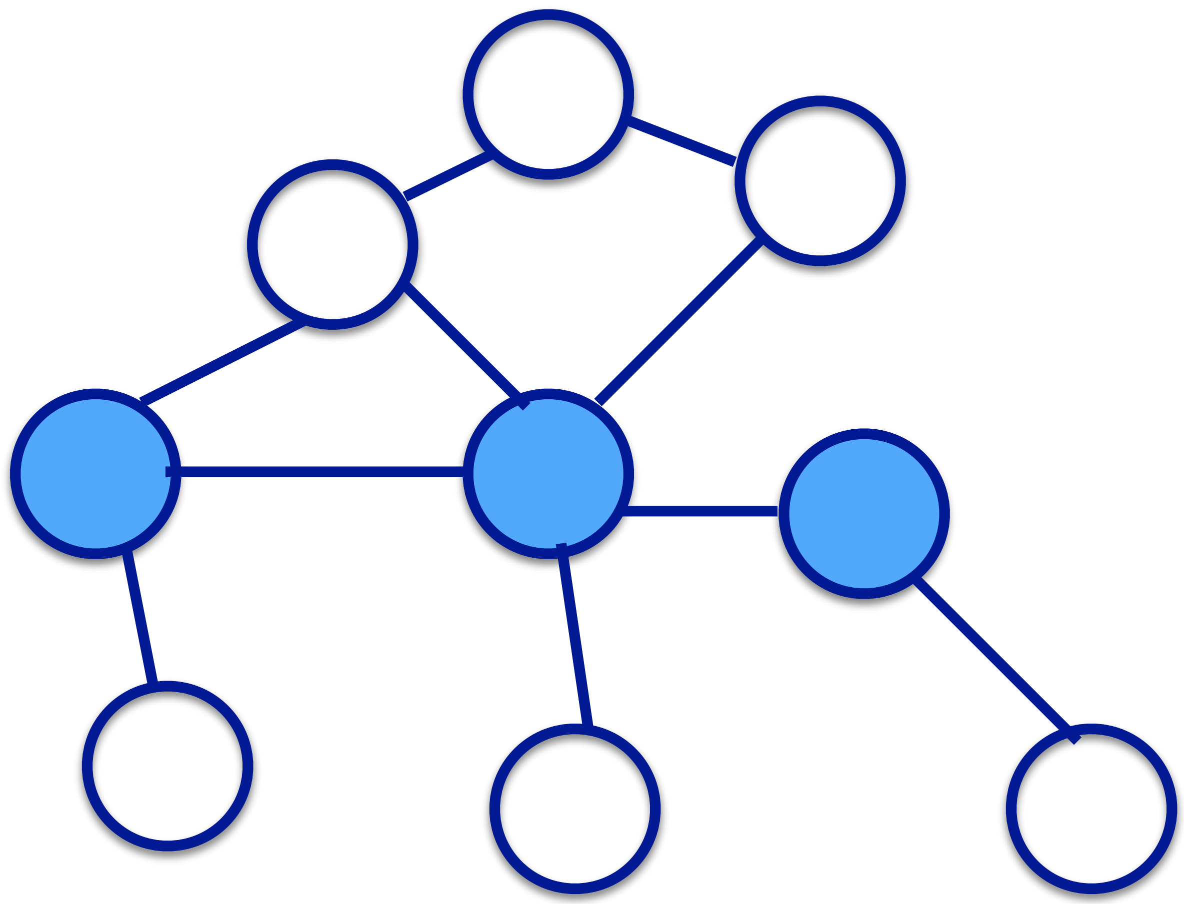

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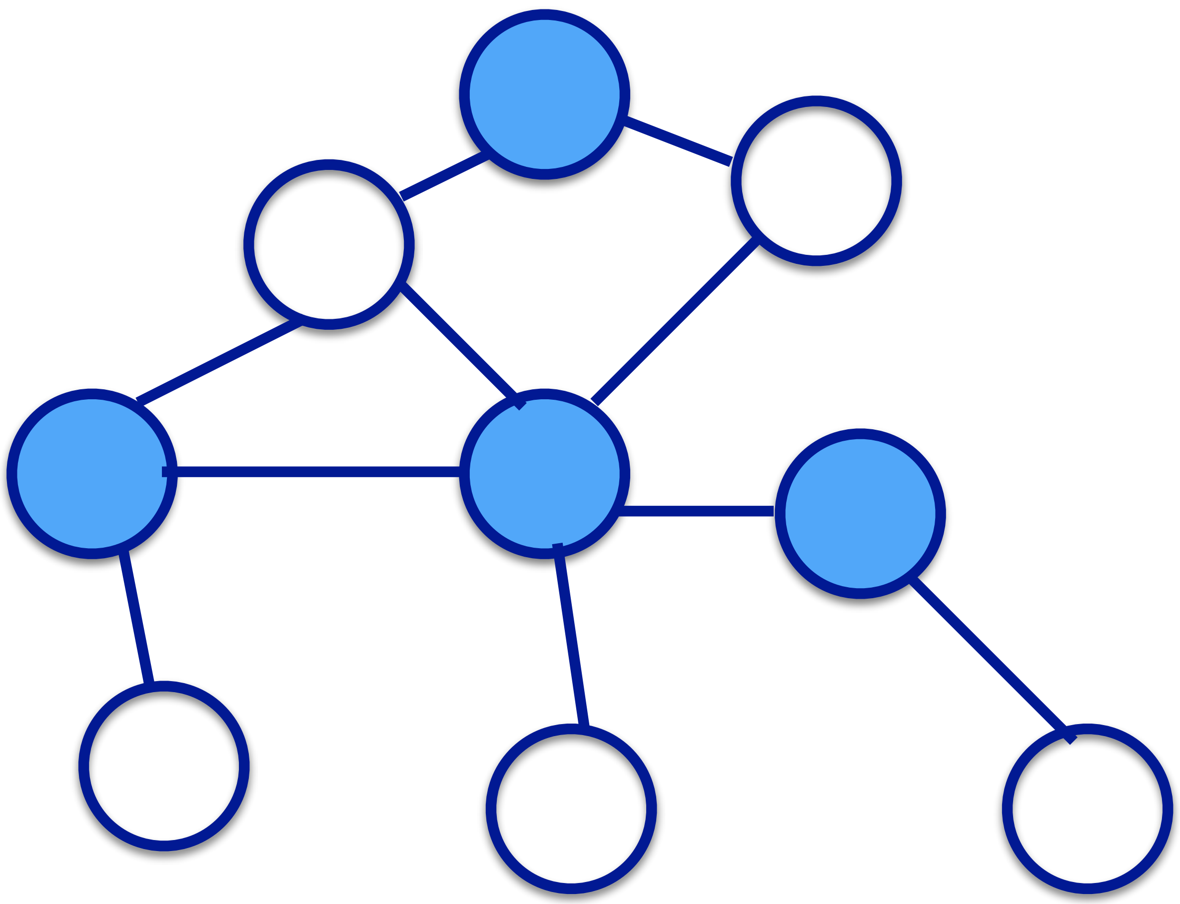

Set Cover

Finding the minimum set of courses to fulfill your HUB requirements

Algorithm: always pick the course with the largest number of additional HUB units

	CS111	CS112	CS132	AN101	AN103	EE150	BI306	AR307
Quant Reas	X	X	X					X
Crit. think	X	X	X		X			X
Creat/Inno	X	X						
Digi/multi			X					
res and inf				X		X	X	X
Social Inq				X	X			
Indiv in Com					X			
Sci Inq						X	X	X
History						X		

AN101 Intro to Cult Anthropology AN103 Anthropology through Ethnography AS105 EE150 Sustainable Energy BI306 Bio of global change AR307 Archeological Science

Set Cover

Given a set of n items $U = \{u_1, u_2, \ldots, u_n\}$ and m subsets of these items $S_1, S_2, \ldots S_m$ find a minimum number of subsets, such that their union contains every element in U.

Vertex Cover $\leq_P$ Set Cover

Reduction:

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Reduction:

1. VC edges $\rightarrow$ SC items

Vertex Cover $\leq_P$ Set Cover

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2. VC vertices → SC sets of the VC edge SC items

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So,

- Picking an SC set $\leftrightarrow$ picking a VC vertex
- Items covered by picked SC set $\leftrightarrow$ VC edges covered by picked VC vertex

Vertex Cover to Set Cover

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1. VC edges → SC items
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So,

- Picking an SC set ↔ picking a VC vertex
- Items covered by picked SC set ↔ VC edges covered by picked VC vertex

Spoiler for next time: Set Cover $\leq_P$ Vertex Cover too!