CS 630 – Fall 2024 – Lab 11
Dec 4, 2024

Problem 1 *Stochastic Traffic Light*

Suppose we have a stochastic traffic light. At any moment, this traffic light may be red, green, or yellow. Each second, it randomly changes its color based on the following transition matrix.

before \ after	red	green	yellow
red	0.7	0.3	0.0
green	0.0	0.2	0.8
yellow	0.9	0.0	0.1

The matrix of these probabilities will be denoted A. That is,

$$A = \begin{bmatrix} 0.7 & 0.3 & 0.0 \\ 0.0 & 0.2 & 0.8 \\ 0.9 & 0.0 & 0.1 \end{bmatrix}$$

1. Suppose the traffic light is red now. Compute the probability distribution of colors in 2 seconds using the power method.

Solution: Let P_i denote the vector of the probability distribution after i seconds.

$$\begin{aligned} P_0 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ P_1 &= P_0 \times A = \begin{bmatrix} 0.7 & 0.3 & 0.0 \end{bmatrix} \\ P_2 &= P_1 \times A = \begin{bmatrix} 0.49 & 0.27 & 0.24 \end{bmatrix} \end{aligned}$$

P_2 is the answer.

2. Suppose the traffic light is green now. Explain how to compute the probability distribution in exactly 1 minute, while - instead of the Power Method - using matrix multiplication and repeated squaring.

   Can we say that after these 60 iterations this distribution has reached (or is very close to) the steady state distribution?

Solution: 1 minute is 60 seconds. $60 = 32 + 16 + 8 + 4 = 2^5 + 2^4 + 2^3 + 2^2$. Compute the following matrix multiplications.

$$\begin{aligned}
A^2 &= A \times A \\
A^4 &= A^2 \times A^2 \\
A^8 &= A^4 \times A^4 \\
A^{16} &= A^8 \times A^8 \\
A^{32} &= A^{16} \times A^{16} \\
A^{60} &= A^{32} \times A^{16} \times A^8 \times A^4 \\
&= \begin{bmatrix} 0.58536585 & 0.2195122 & 0.19512195 \\ 0.58536585 & 0.2195122 & 0.19512195 \\ 0.58536585 & 0.2195122 & 0.19512195 \end{bmatrix} \\
\text{probability distribution in 1 minute} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \times A^{60} \\
&= \begin{bmatrix} 0.58536585 & 0.2195122 & 0.19512195 \end{bmatrix}
\end{aligned}$$

Since all the rows of A^{60} are very similar, those rows must be close to the steady state probability distribution.

3. The simple structure of the transition matrix allows a simpler calculation of the steady state problems. Observe that the transition matrix only allows *changes* from red to green, green to yellow, and yellow to red. This means that the number of changes from a particular color, or to a particular color can only vary by one per color. That is, if there were 1000 transitions to red, then the number of transitions to green or yellow must be between 999 and 1001. Compute the average amount of time that the traffic light stays in each color (e.g. what is the average time it stays red). Then use those average times to calculate the average fraction of time in each color. (This is another way to calculate the steady-state distribution, but only for transition matrices of this structure.)

Solution: By inspection of the transition matrix and recalling the formula for the expectation of a geometric distribution $1/(1-p)$, the average time that the traffic light stays red is $1/(1-0.7)$, the average time that the traffic light stays green is $1/(1-0.2)$, and the average time that the traffic light stays yellow is $1/(1-0.1)$. So, the steady state fraction of time that the light is red is $\frac{1/(1-0.7)}{1/(1-0.7)+1/(1-0.2)+1/(1-0.1)} \approx 0.5853658536585366$. Similarly, the steady state fraction of time that the light is green is $\frac{1/(1-0.2)}{1/(1-0.7)+1/(1-0.2)+1/(1-0.1)} \approx$

0.21951219512195122 and the steady state fraction of time that the light is yellow is $\frac{1/(1-0.1)}{1/(1-0.7)+1/(1-0.2)+1/(1-0.1)} \approx 0.1951219512195122$.

Problem 2 *Power Law or Not?*

A power law is a relationship between two variables of the form $y = Cx^{-\alpha}$. The initial examples were for degrees in a graph, where for degree k, the probability of a particular vertex having degree k was $p(k) = Ck^{-\alpha}$. Similar relationships are abundant. Which of the following relationships are power laws?

1. The volume of a cube as a function of the length of a side of the cube.

   Solution: True. $V(l) = l^3$.

2. The distribution of vertex degrees in an Erdos-Renyi graph with parameters n and p (for fixed p).

   Solution: False. The degree distribution of such a graph follows a binomial distribution, not a power law distribution.

3. The distribution of bucket lengths in a hash table with chaining.

   Solution: False. The distribution of bucket lengths is closer to a geometric distribution - it drops off exponentially.

4. The vertex degree in the graph which is the union of vertices and edges of the paper collaboration graph (used for Erdös numbers) and the co-star graph (used for Bacon numbers).

   Solution: True. There are a few people (vertices) in common between these two graphs, but not many, so the probability density function is a weighted average of the two graphs. Adding two power laws is not exactly a power law unless the exponents are the same, but when assessing real world graphs, we allow some messiness.

5. Zipf's law observes that many sorted lists of measurements have the nth largest proportional to $1/n$. The best known example of Zipf's law is word frequencies for the English language where the most common word "the" has probability 7% and the next most common word "or" has probability roughly 3.5%. In that example, the frequency of the nth most common word in the English language is roughly $0.07n^{-1}$.

   Solution: True. With Zipf's law, $\alpha = 1$, and the scaling factor C depends on the data set.