CS630 Graduate Algorithms

October 3, 2024

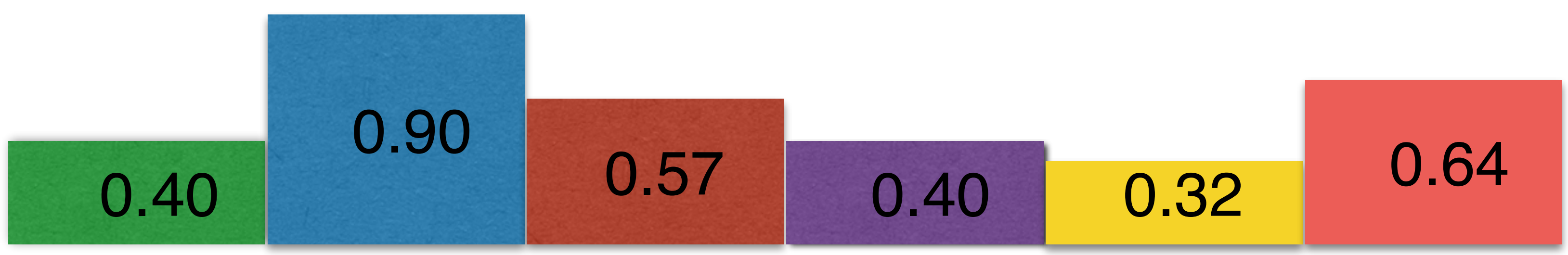
by Dora Erdos and Jeffrey Considine

- Bin packing and related problems

Bin Packing

Suppose you have a multiset of rational numbers $S = \{s_1, s_2, \ldots, s_n\}$ where $0 < s_i < = 1$ and you want to pack them into a minimal number of bins of size one. That is, you want to partition S such that the sum of each partition is at most one.

- What's the minimum bins you can pack all of S into?
- What partitioning of S will let you do that?



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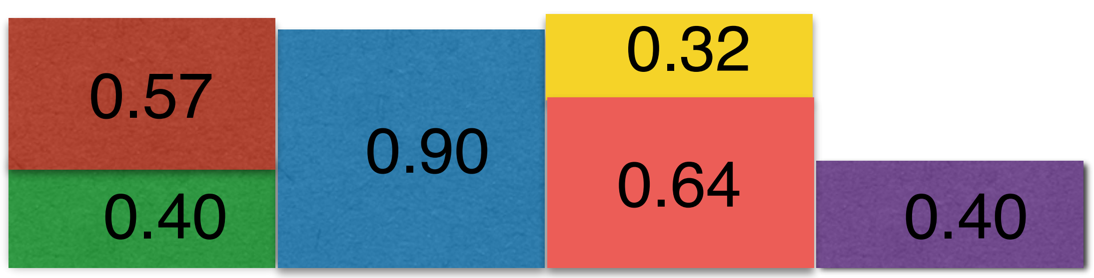

- Is this a hard problem in general?

Bin Packing is NP-Complete

$SAT \leq_P SUBSETSUM \leq_P PARTITION \leq_P BINPACKING$

SAT:

- Is this boolean formula (or circuit) satisfiable?
- Is there a set of inputs making it output true?
- OG NP-Complete problem

Subset-SUM:

- Given a multi-set of integers S, is there a subset of them adding up to T?
- One of Karp's list of 21 NP-Complete problems

Partition:

- Given a multiset of integers $S = \{s_1, \ldots, s_n\}$, is there a subset of them adding up to $\frac{1}{2}\sum s_i$?

Bin packing:

- Given a multiset of rational numbers $S = \{s_1, s_2, \ldots, s_n\}$, can you pack them into k bins of size one?

Bin Packing is NP-Complete

$$SAT \leq_P SUBSETSUM \leq_P PARTITION \leq_P BINPACKING$$

All of these problems are NP-Complete.

- No known polynomial time algorithms.
- No expectations of a general fast algorithm any time soon.

Next Fit Bin Packing - Approximation Algorithms?

How can we bound the approximation ratio?

- Lower bound for space required?

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- Lower bound for space required?

- Easy lower bound is $\left\lceil \sum s_i \right\rceil$

Next Fit Bin Packing

```
Start with one empty bin and call it “current”.
For each item:
    If there is room in the current bin,
        Put the item in the current bin.
    Otherwise,
        Start a new bin and make it current.
        Put the item in the new current bin.
```

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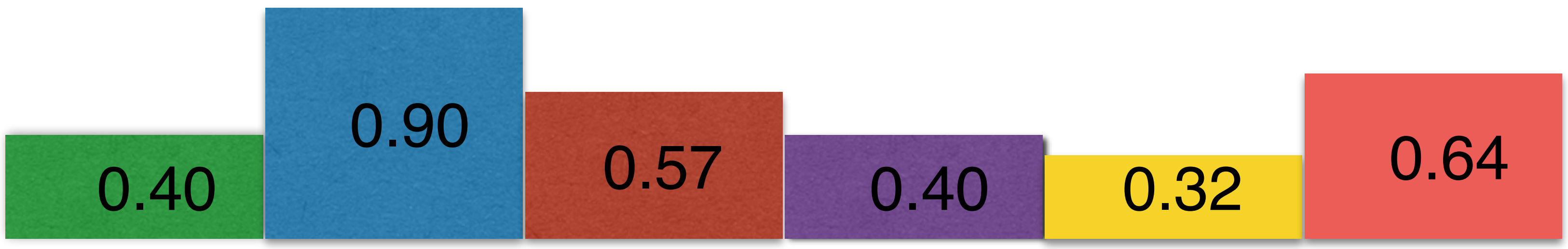

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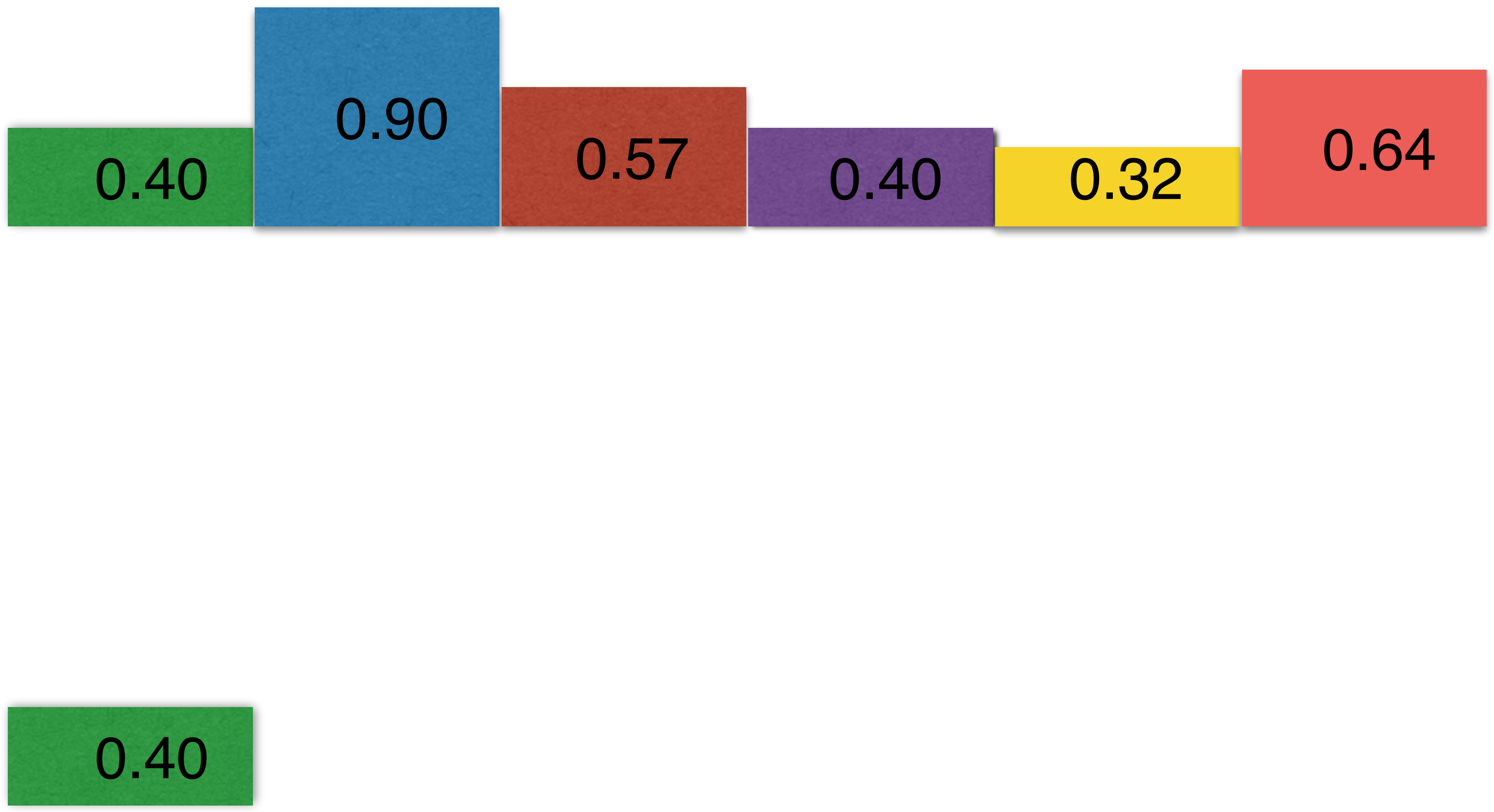

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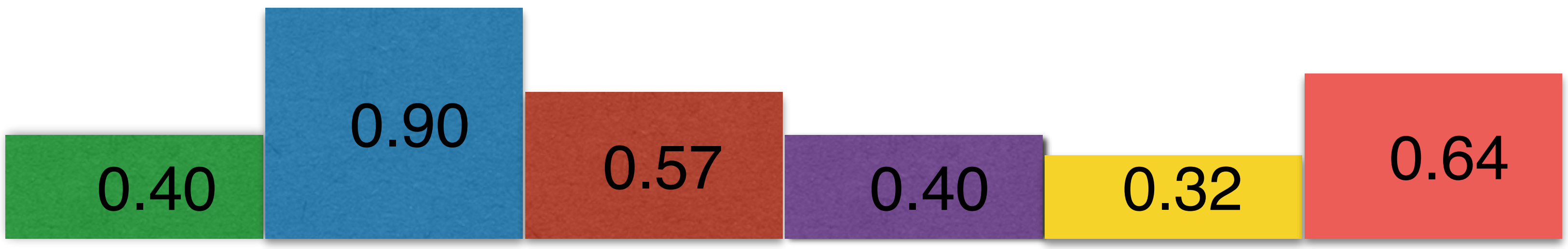

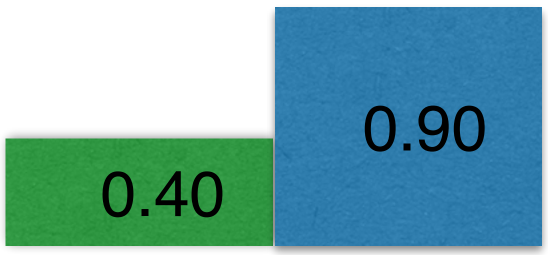

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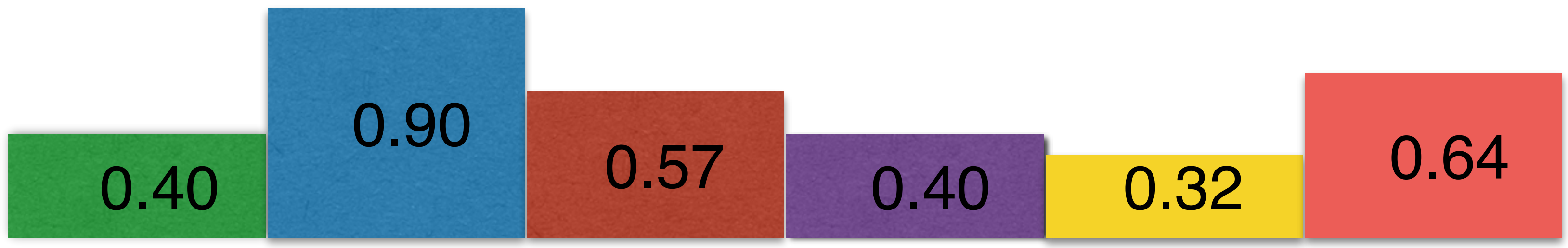

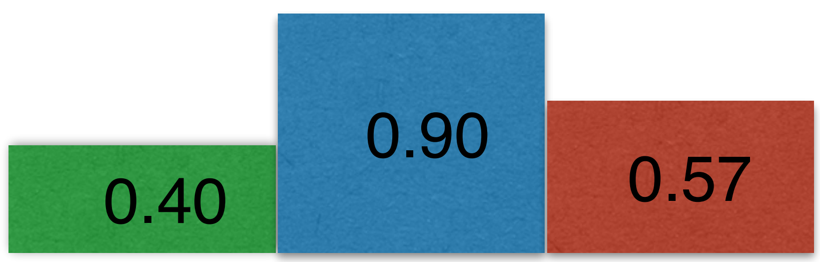

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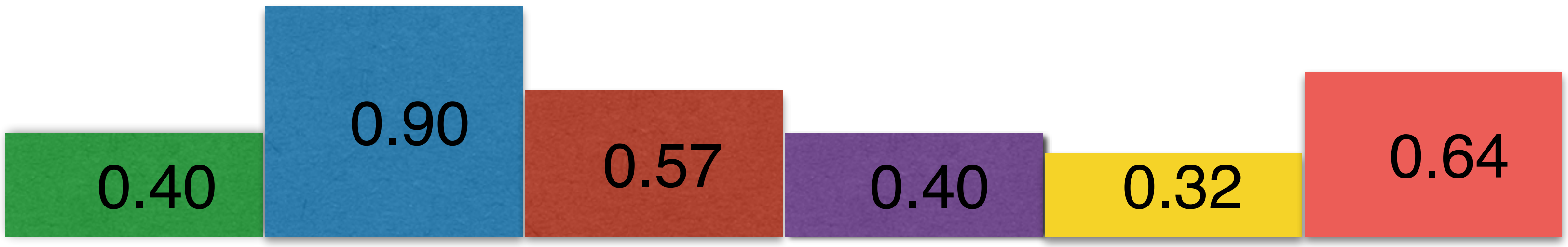

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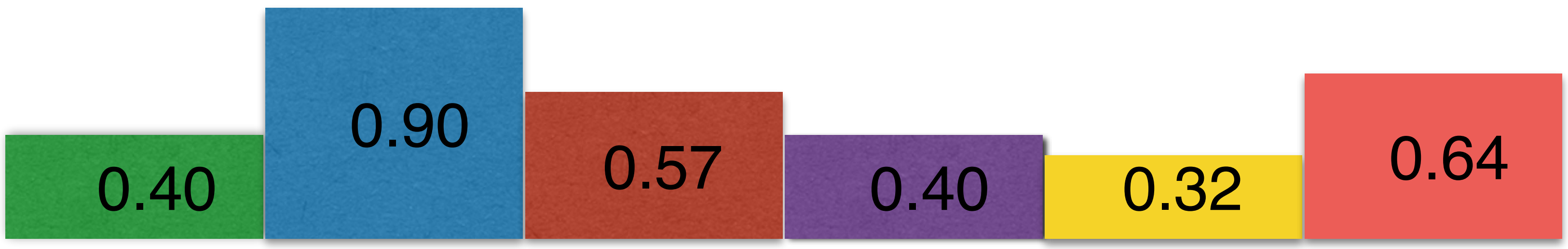

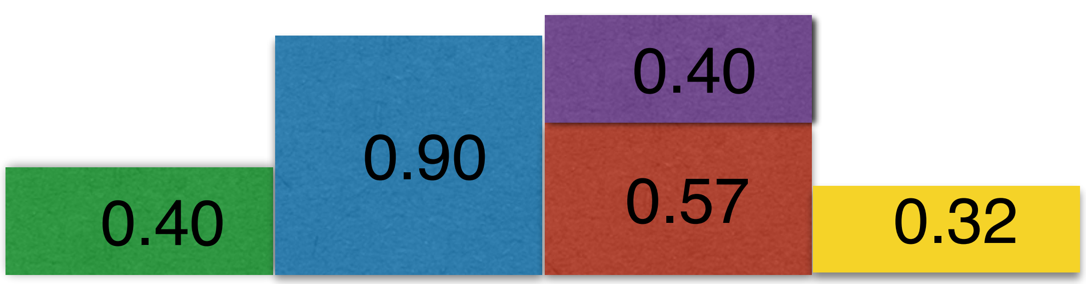

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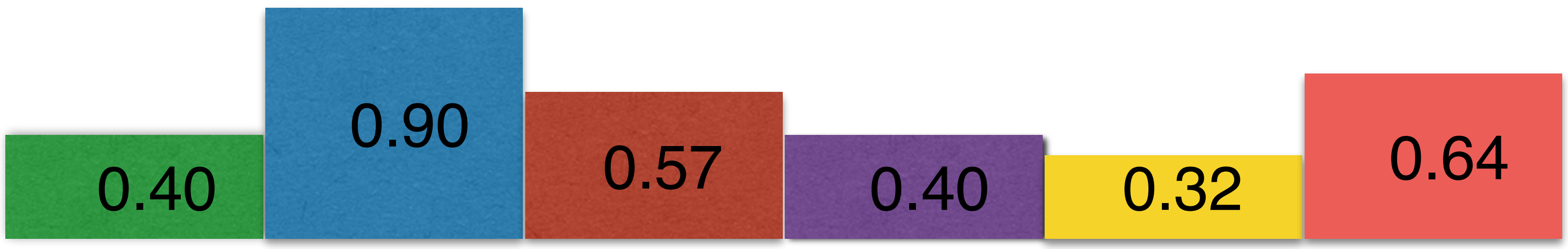

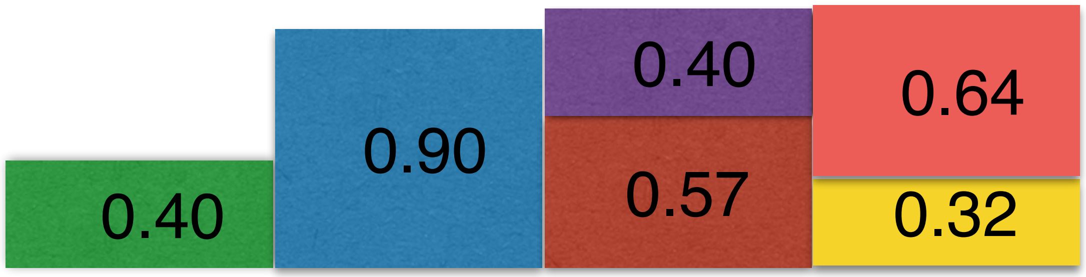

What do you think of this algorithm?

Top Hat

Which of the following are true about the Next Fit Bin Packing Algorithm?

- After starting a new current bin, the current and previous bin together contain items of at least size 1.
- Any two consecutive bins contain items of total size at least one.
- Next Fit Bin Packing returns optimal solutions to bin packing.
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Which of the following are true about the Next Fit Bin Packing Algorithm?

- After starting a new current bin, the current and previous bin together contain items of at least size 1.
  - True! If the total was less, the bin would not have bin split.
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- Next Fit Bin Packing has an approximation ratio of two.
  - True! We have most of the argument already.

Next Fit Bin Packing - Approximation Ratio

If $NF(L) = 1$, then $OPT(L) = 1$.

- Everything fit in the first bin.

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So, $NF(L) \leq 2\ OPT(L) - 1$

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Suppose we had a polynomial time bin packing algorithm with an approximation ratio $\alpha < 1.5$.

How fast could we solve partitioning?

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- Optimal bin packing answer is 2 if there is a partitioning and 3 otherwise.
- If $\alpha < 1.5$, then we can distinguish these cases.
- But we also said that partitioning is NP-Complete.
  - So that would mean $P = NP$.

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A polynomial time bin packing algorithm with an approximation ratio $\alpha < 1.5$ would be a major breakthrough.

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- Any chance for $1.5 \leq \alpha < 2$?

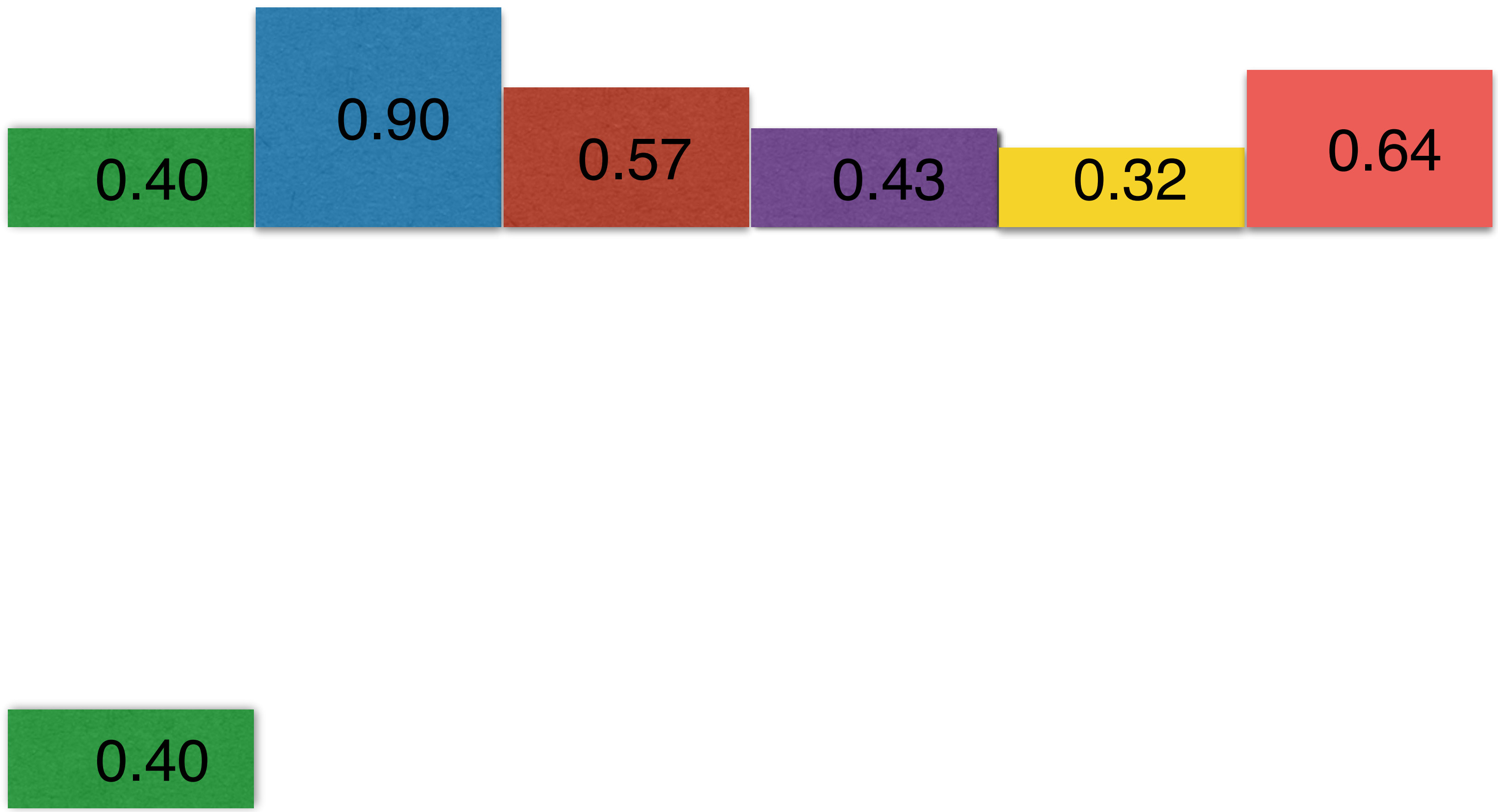
- Two simple algorithms First Fit and Best Fit have $\alpha = 1.7$.

First Fit Bin Packing

```
Initialize bin list to an empty list.
For each item:
    If any bin has room for the item,
       Put the item in the first bin with room.
    Otherwise,
        Start a new bin.
        Add the item to the new bin.
```

This can be implemented in $O(n \log n)$ time.

First Fit Bin Packing



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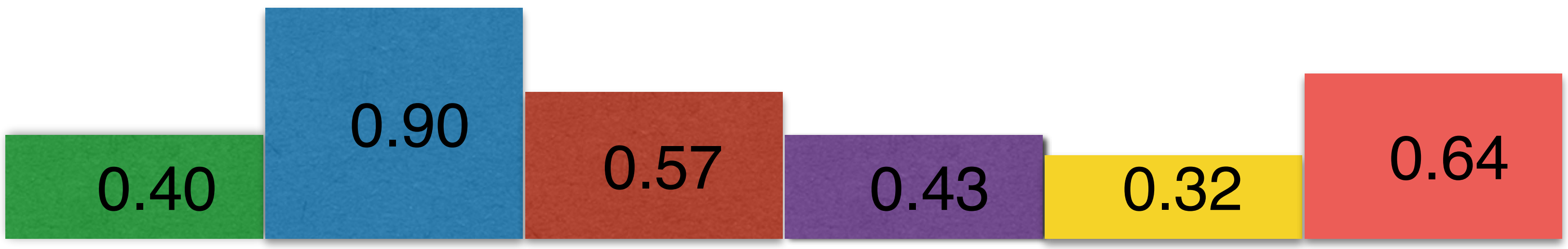

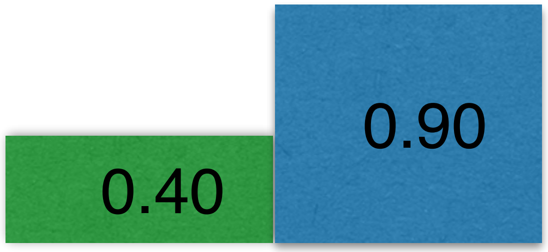

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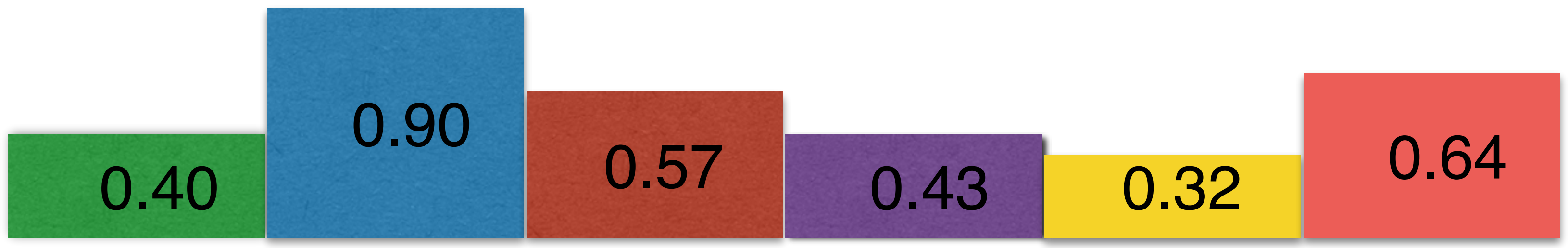

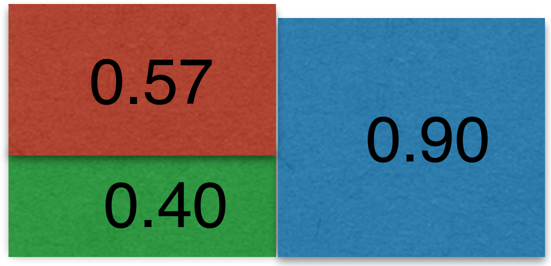

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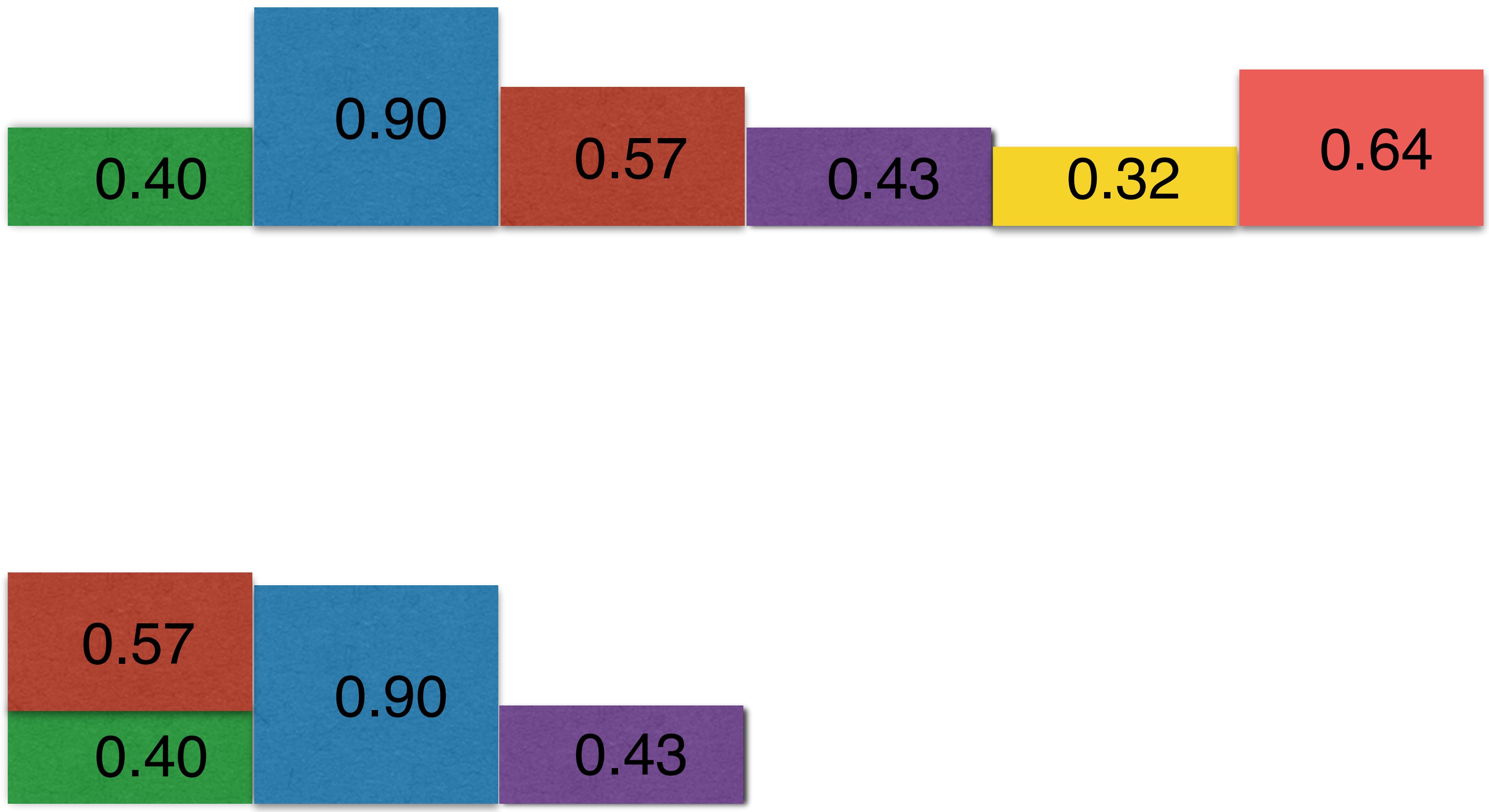

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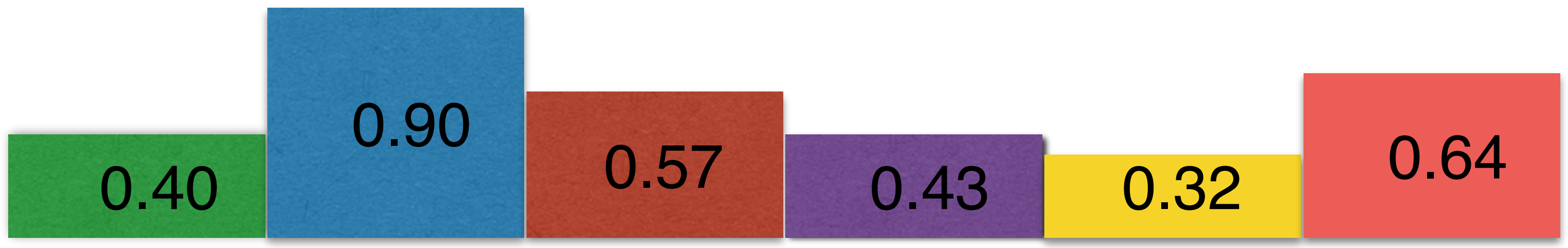

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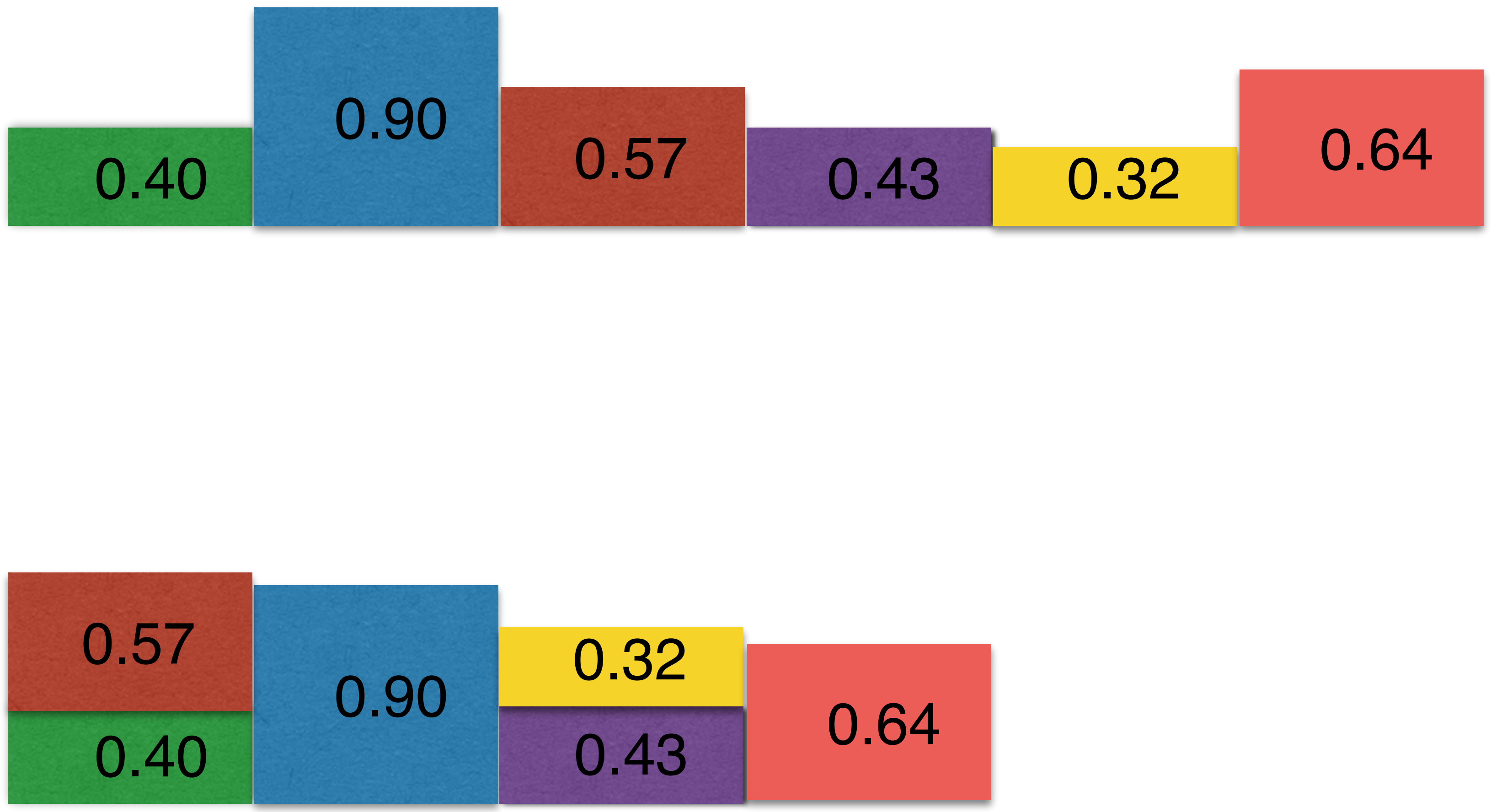


First Fit Bin Packing - Approximation Ratio

- 1971 - $FF(S) \leq 1.7OPT(S) + 3$
- 1972 - $FF(S) \leq 1.7OPT(S) + 2$
- 1976 - $FF(S) \leq 1.7OPT(S) + 0.9$
- 2010 - $FF(S) \leq 1.7OPT(S) + 0.7$
- 2013 - $FF(S) \leq \lfloor 1.7OPT(S) \rfloor$

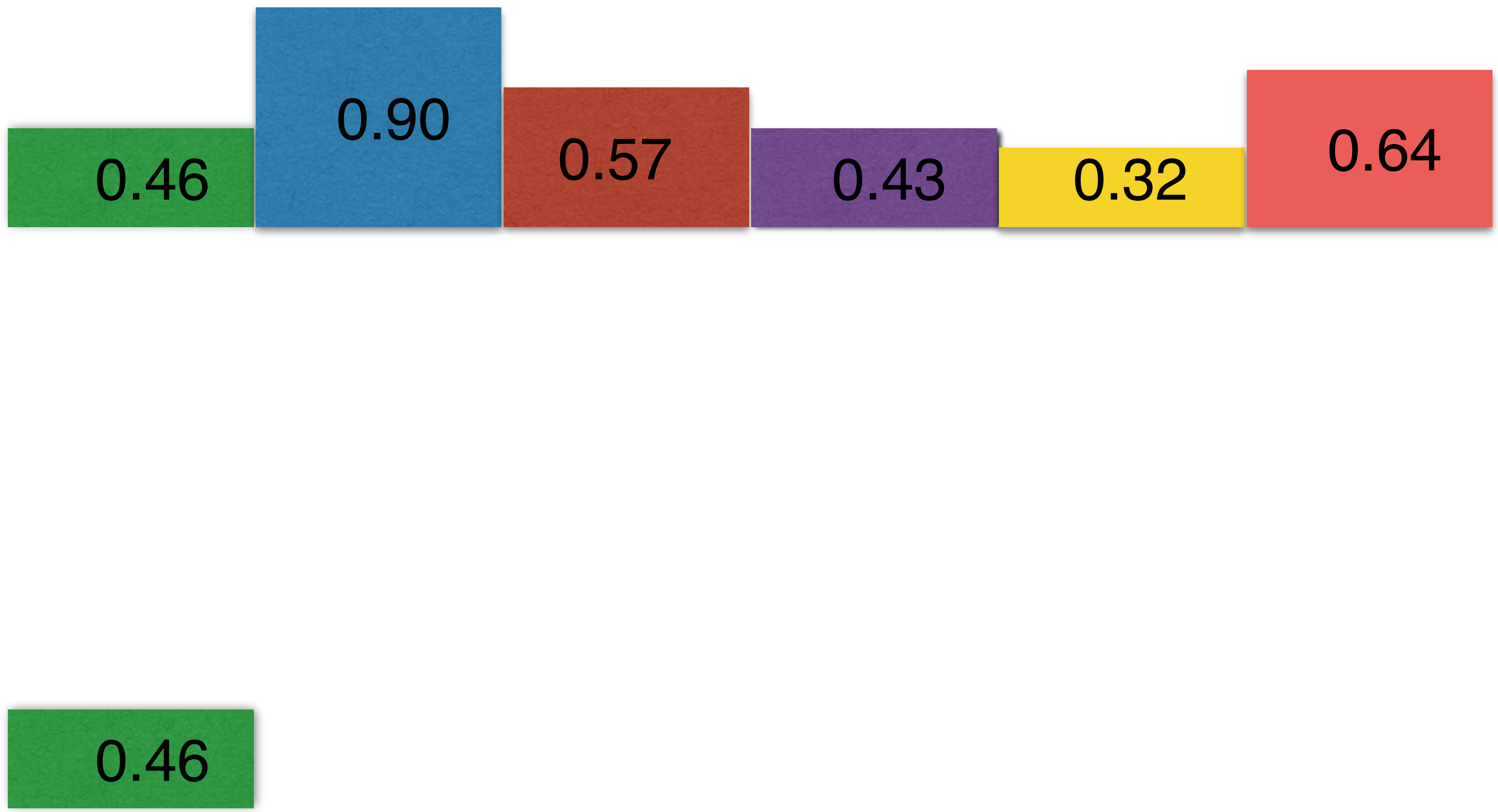
Last one is tight.

Best Fit Bin Packing

```
Initialize bin list to an empty list.
For each item:
    If any bin has room for the item,
       Put the item in the most heavily loaded bin with room.
    Otherwise,
        Start a new bin.
        Add the item to the new bin.
```

This can be implemented in $O(n \log n)$ time.

Best Fit Bin Packing



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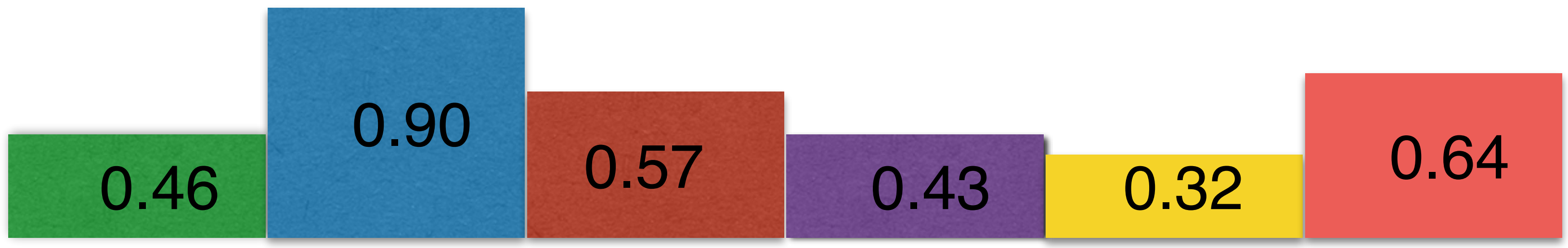

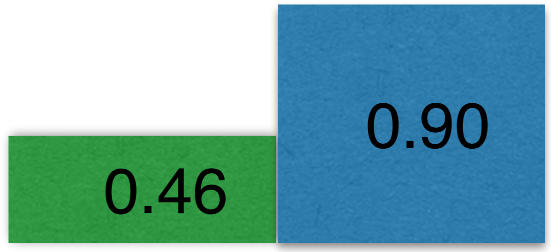

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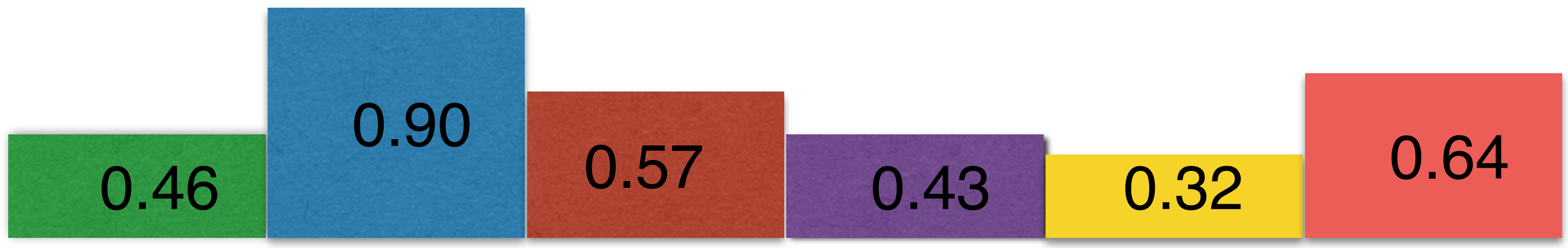

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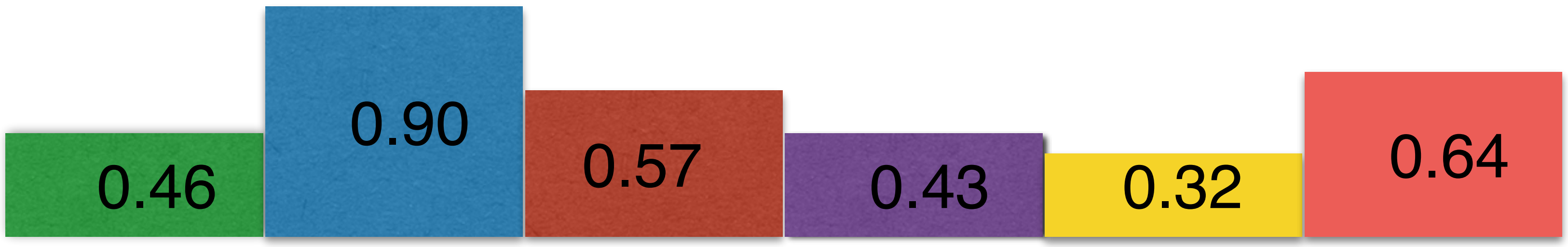

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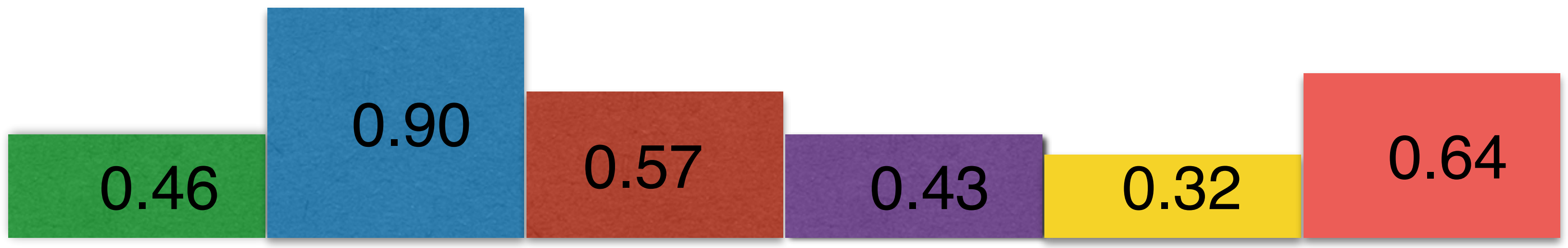

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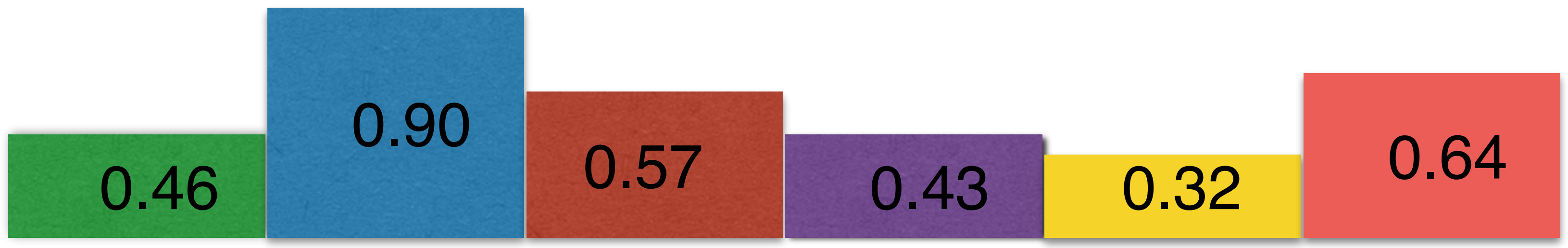

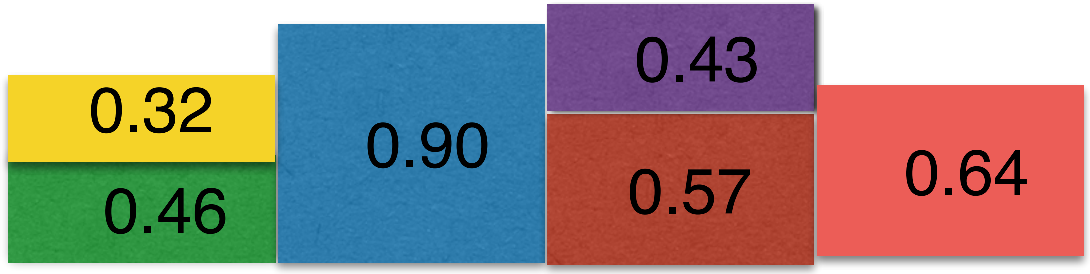

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This can be implemented in $O(n \log n)$ time.

Similar 1971 to 2013 history of approximation analysis to get same $\alpha = 1.7$.

First Fit Decreasing Bin Packing

```
Initialize bin list to an empty list.
Sort items in decreasing order by size.
For each item:
    If any bin has room for the item,
       Put the item in the first bin with room.
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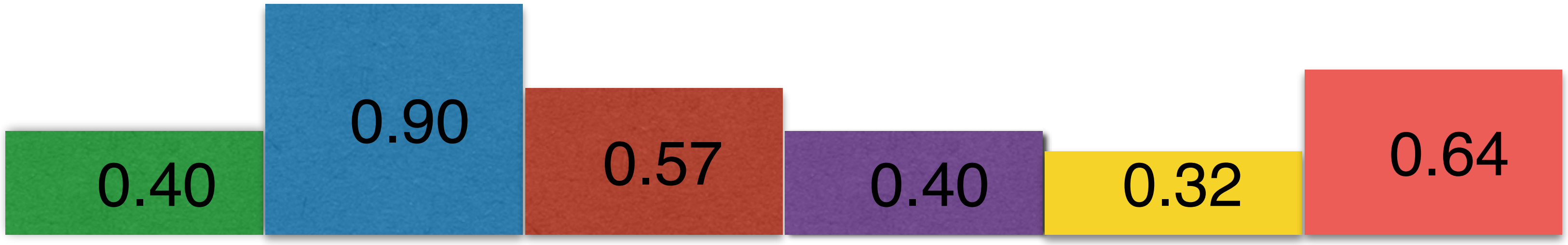

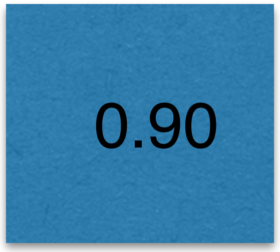

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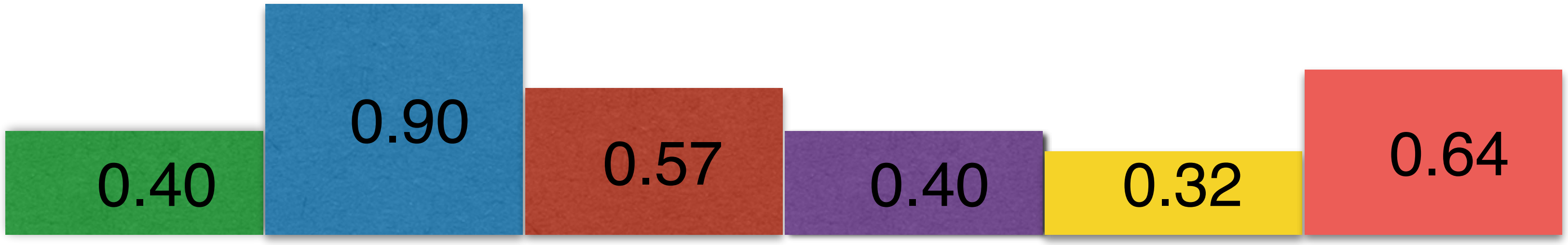

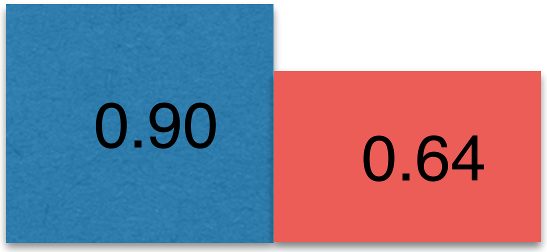

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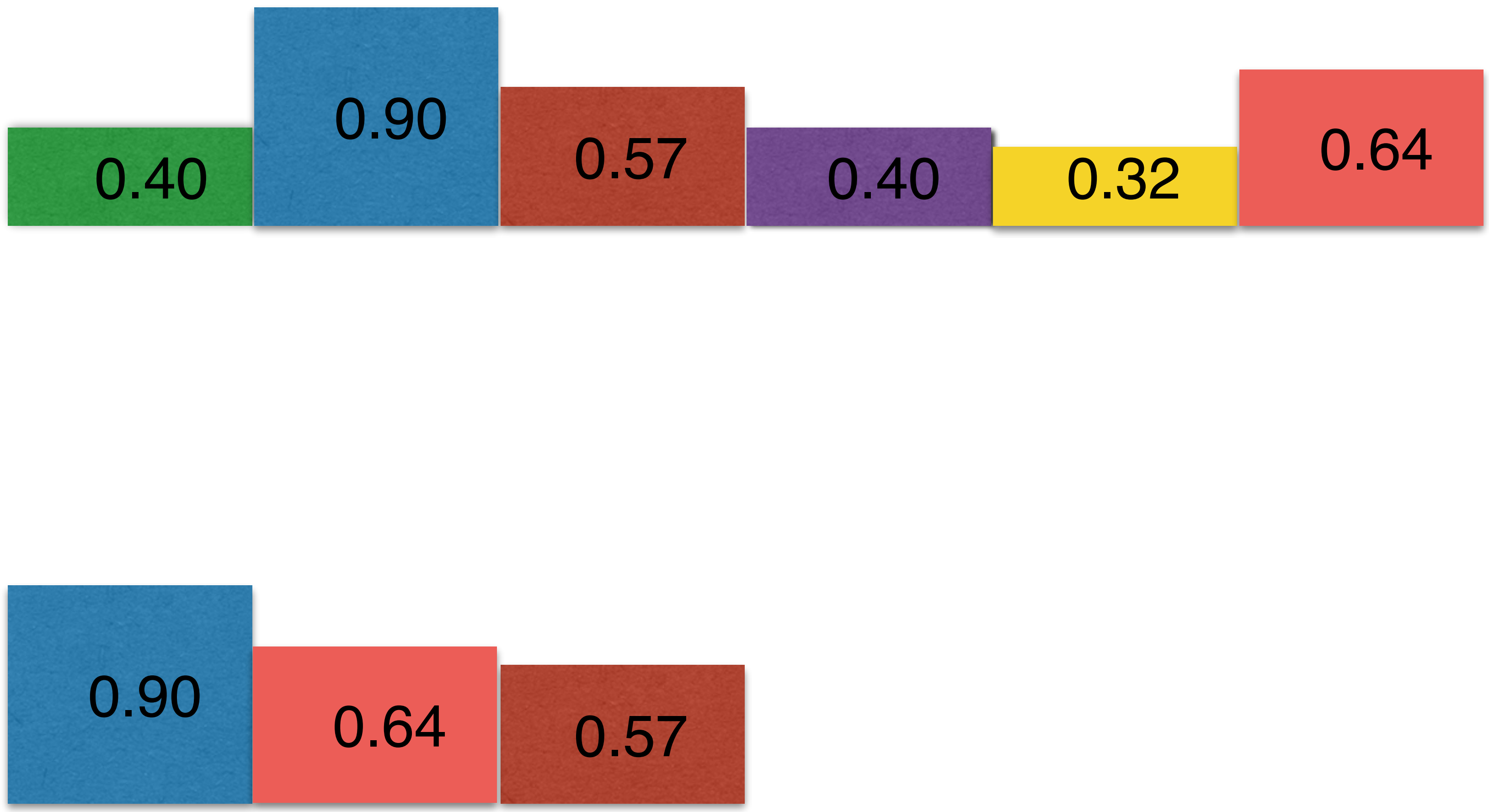

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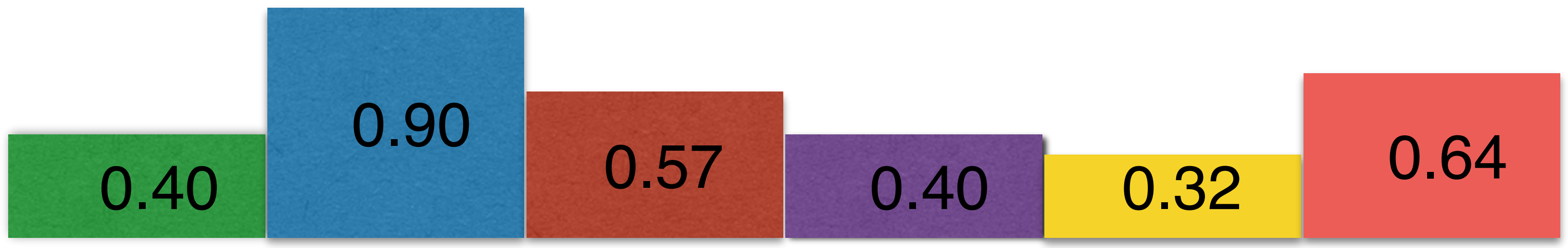
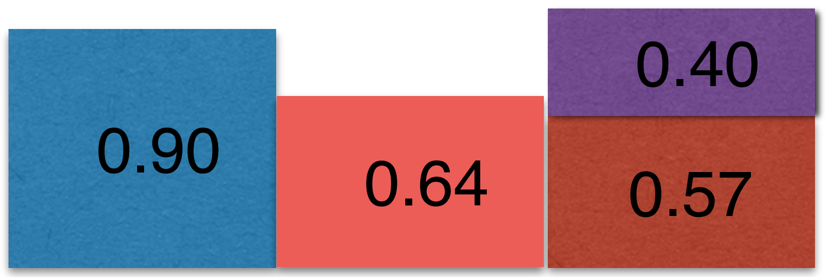
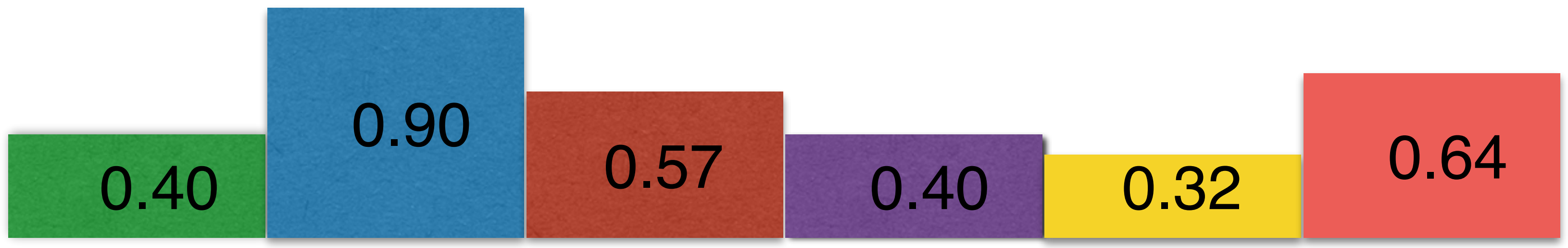




First Fit Decreasing Bin Packing



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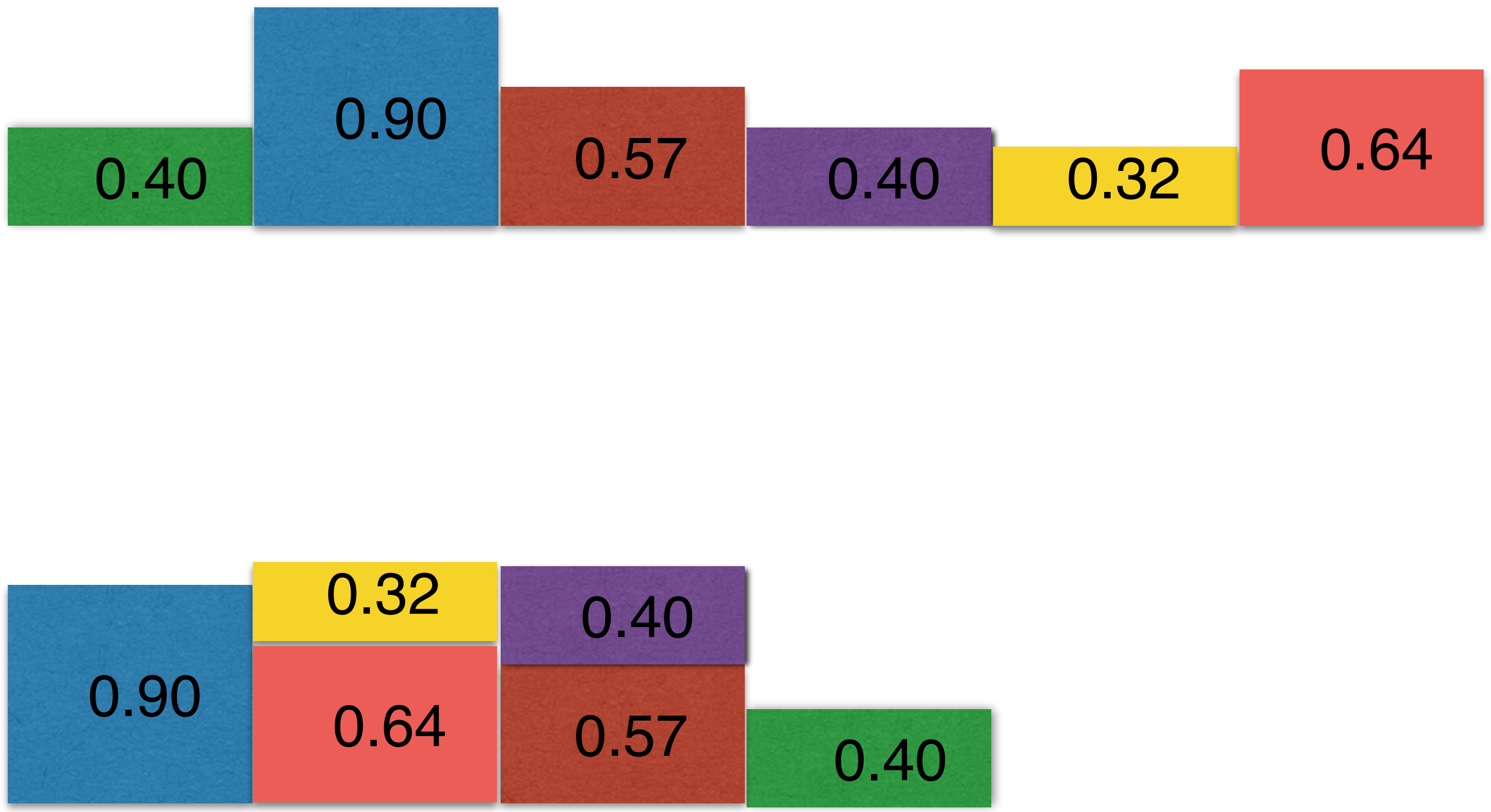

First Fit Decreasing Bin Packing

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Initialize bin list to an empty list.
Sort items in decreasing order by size.
For each item:
    If any bin has room for the item,
       Put the item in the first bin with room.
    Otherwise,
        Start a new bin.
        Add the item to the new bin.
```

This can be implemented in $O(n \log n)$ time.

$$FFD(L) \leq \frac{11}{9} OPT(L) + \frac{6}{9}$$

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If $OPT(L) = 2$, this only gives $FFD(L) \leq 3$, so no trivial $P = NP$.