Dynamics of a rocket with gravity $-GMm/r^2$

In[53]:=
```
Clear["Global`*"]
```

Numerical values used for graphs and comparisons

*** Data for Lunar Modules (LM) of Apollo missions, and the Moon surface,
https://en.wikipedia.org/wiki/Ascent_propulsion_system ,
https://en.wikipedia.org/wiki/Apollo_Lunar_Module ,
https://nssdc.gsfc.nasa.gov/nmc/spacecraft/display.action?id=1972-031C ,
https://www.hq.nasa.gov/office/pao/History/SP-4205/ch6-5.html ,

m_dry = 2150 kg LM dry mass od the ascent stage,
m_p = 2350 kg propellant mass, m0 = mf+mp,
t_burn = 465 s burn time,
I_sp = 311 s   specific impulse,

F_thrust = ca.15.5 kN   (to be verified below),
deltav = 2200 m/s (to be verified below),

*** Moon and astronomy data
gm = G M_moon = 6.674E-11 Nm2/kg2 * 7.346E22 kg,
y0= R_moon = 1.74E6 m,
g_moon = 1.623 m/s2

In[121]:=
```
params = {mp → 2350, mdry → 2150, tburn → 465, isp → 311, v0 → 0,
  gm → (6.674 × 10^-11 7.346 × 10^22), g0 → 9.81, rmoon → 1.74 × 10^6, gmoon → 1.623}
```

Out[121]=
```
{mp → 2350, mdry → 2150, tburn → 465, isp → 311, v0 → 0,
 gm → 4.90272 × 10^12, g0 → 9.81, rmoon → 1.74 × 10^6, gmoon → 1.623}
```

```
(* m0  = m(t=0)  = *) m0 = mdry + mp;
m0 /. params
```

Out[58]= 4500

In[69]:=
```
ve = isp g0 ;
ve /. params
```

Out[70]= 3050.91

In[73]:=
```
(* r = dm/dt = *) r = mp/tburn;
r /. params // N
```

Out[74]= 5.05376

```
(* fthrust should be equal about 15.5kN, fthrust=ve*R=  *) fthrust = ve r;
fthrust /. params
```

Out[94]= 15 418.6

```
(* Δv = ve Log[mf/m0]  should be about 2200 m/s *)
deltav = ve Log[m0/mdry]; deltav /. params
```

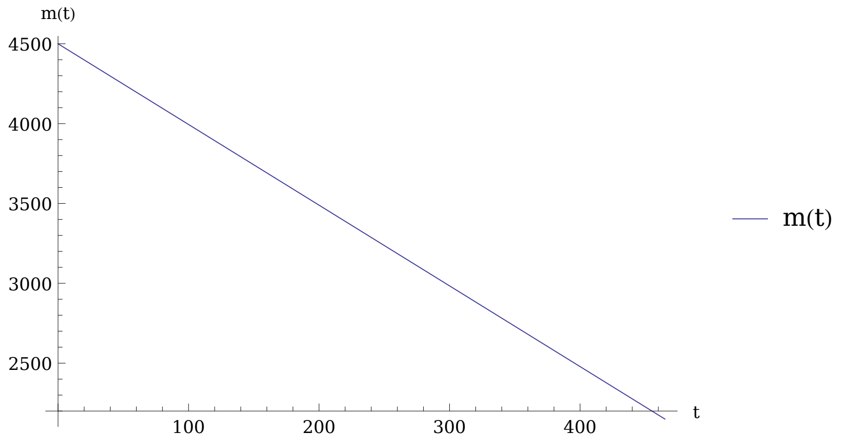
Out[100]= 2253.43

Mass vs time, m (t), with r=dm/dt=mass flow rate (exhausted mass per unit of time; r is assumed constant)

In[81]:=
```
m[t_] := m0 - r t
```

In[83]:=
```
Plot[m[t] /. params, {t, 0, tburn /. params},
 PlotLegends → {"m(t)"}, AxesLabel → {"t", "m(t)"}]
```

Out[83]=



Rocket moving along a straight line OY towards infinity, no other masses yet (only for testing DSolve)

In[95]:=
```
sol0 = DSolve[{y''[t] == fthrust/m[t], y[0] == 0, y'[0] == 0}, y, t]
```

Out[95]= $\left\{\left\{y \to \text{Function}\left[\{t\}, \frac{1}{mp}(g0\ isp\ mp\ t + g0\ isp\ mp\ t\ \text{Log}[mdry\ tburn + mp\ tburn] - g0\ isp\ mdry\ tburn\ \text{Log}[mdry\ tburn + mp\ tburn] - g0\ isp\ mp\ tburn\ \text{Log}[mdry\ tburn + mp\ tburn] - g0\ isp\ mp\ t\ \text{Log}[-mp\ t + mdry\ tburn + mp\ tburn] + g0\ isp\ mdry\ tburn\ \text{Log}[-mp\ t + mdry\ tburn + mp\ tburn] + g0\ isp\ mp\ tburn\ \text{Log}[-mp\ t + mdry\ tburn + mp\ tburn])\right]\right\}\right\}$

In[97]:=
```
y'[t] /. sol0 // FullSimplify
```

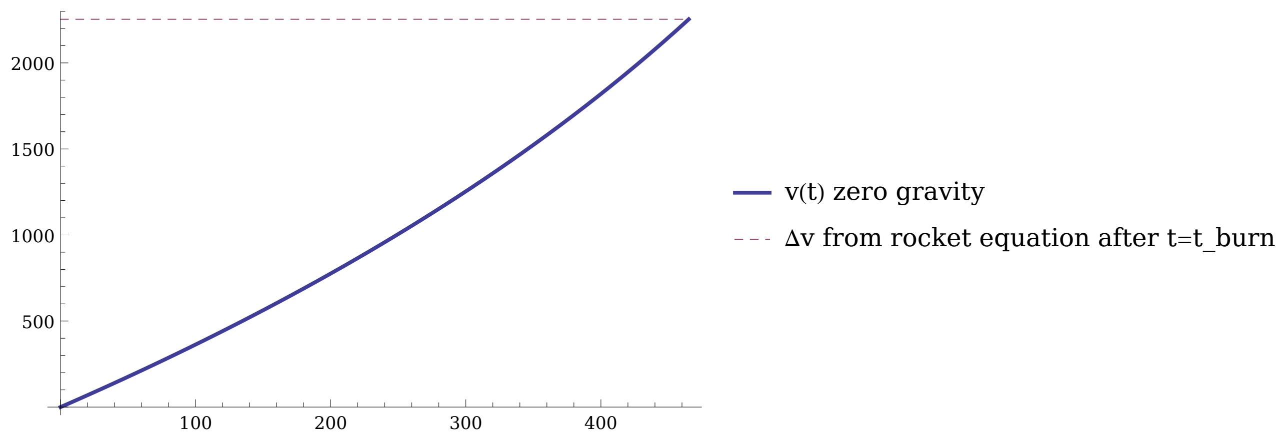
Out[97]= $\{g0\ isp\ (\text{Log}[(mdry + mp)\ tburn] - \text{Log}[-mp\ t + (mdry + mp)\ tburn])\}$

In[98]:=
```
y'[t] /. sol0 /. params
```

Out[98]= $\left\{\frac{1}{2350}\left(1.11516\times10^{8} - \frac{1.50025\times10^{13}}{2\,092\,500 - 2350\ t} + \frac{1.68487\times10^{10}\ t}{2\,092\,500 - 2350\ t} - 7.16964\times10^{6}\ \text{Log}[2\,092\,500 - 2350\ t]\right)\right\}$

In[103]:=
```
Plot[{y'[t] /. sol0 /. params, deltav /. params },
 {t, 0, tburn /. params}, PlotStyle -> {Thick, Dashed},
 PlotLegends → {"v(t) zero gravity", "Δv from rocket equation after t=t_burn"}]
```

Out[103]=

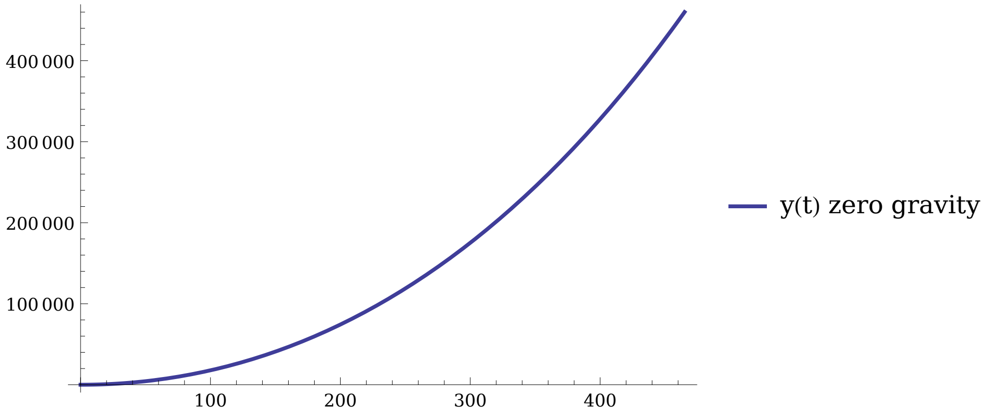


In[96]:=
```
y[t] /. sol0 // FullSimplify
```

Out[96]= $\left\{\frac{1}{\text{mp}} \text{g0 isp} \left(\text{mp t} + (\text{mp t} - (\text{mdry} + \text{mp})\ \text{tburn}) \left(\text{Log}[(\text{mdry} + \text{mp})\ \text{tburn}] - \text{Log}[-\text{mp t} + (\text{mdry} + \text{mp})\ \text{tburn}]\right)\right)\right\}$

In[105]:=
```
Plot[{y[t] /. sol0 /. params}, {t, 0, tburn /. params},
 PlotStyle -> {Thick}, PlotLegends → {"y(t)  zero gravity"}]
```

Out[105]=



Equations for g=const

In[122]:= $\text{yc[t\_]} := -\frac{\text{gmoon}\ t^2}{2} + \text{t v0} + \frac{\text{fthrust}\left(t + \left(-\frac{\text{m0}}{r} + t\right) \text{Log}\left[\frac{\text{m0}}{\text{m0} - r\ t}\right]\right)}{r}$

In[125]:=
```
yc[t]
```

Out[125]= $-\frac{\text{gmoon}\ t^2}{2} + \text{t v0} + \text{g0 isp}\left(t + \left(t - \frac{(\text{mdry} + \text{mp})\ \text{tburn}}{\text{mp}}\right) \text{Log}\left[\frac{\text{mdry} + \text{mp}}{\text{mdry} + \text{mp} - \frac{\text{mp t}}{\text{tburn}}}\right]\right)$

Rocket moving along a straight line OY from the surface of a planet/moon (y=0) towards infinity

(*
    $a_y(t)$ =
 $dv_y/dt\ a_y(t) = F_{net,y} / m(t)$ where $F_{net,y} = F_{thrust} - G\ M\ m(t) / y^2(t) = v_e R - G\ M\ m(t) / y^2(t)$
        and we assume that y=0 at the centre of a planet/moon of mass M

            and we need to solve

$$\frac{d^2y(t)}{dt^2} = \frac{R\ v_e}{m(t)} - \frac{G\ M}{y^2(t)}$$

             where $m(t) = m_0 - R\ t$, and $v_e = F_{thrust}/R$
*)

In[107]:= (* analytical solution of the differential eq. apparently not possible: *)

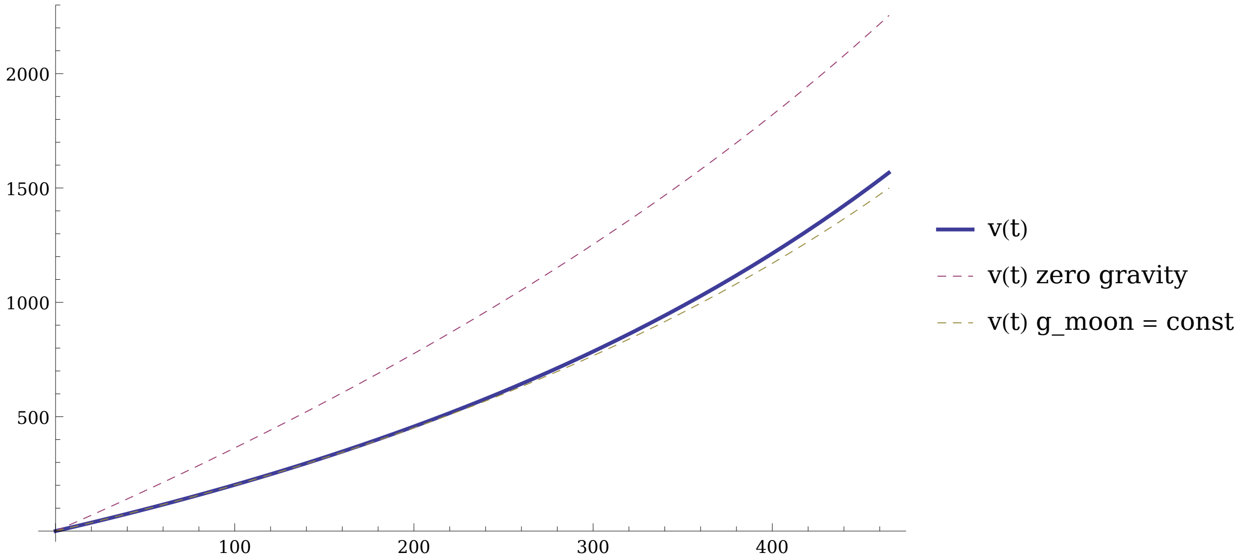
$\text{DSolve}\left[\left\{y''[t] == \frac{\text{fthrust}}{m[t]} - \frac{\text{gm}}{(y[t])^2}, y[0] == \text{y0}, y'[0] == 0\right\}, y, t\right]$

Out[107]= $\text{DSolve}\left[\left\{y''[t] == \frac{\text{g0 isp mp}}{\left(\text{mdry} + \text{mp} - \frac{\text{mp t}}{\text{tburn}}\right) \text{tburn}} - \frac{\text{gm}}{y[t]^2}, y[0] == \text{y0}, y'[0] == 0\right\}, y, t\right]$

In[111]:=
```
sol1 = NDSolve[{ (y''[t] == fthrust/m[t] - gm/(y[t])^2) /. params, y[0] == rmoon /. params, y'[0] == 0},
   y, {t, 0, tburn /. params}]
```
Out[111]=
```
{{y → InterpolatingFunction[{{0., 465.}}, <>]}}
```

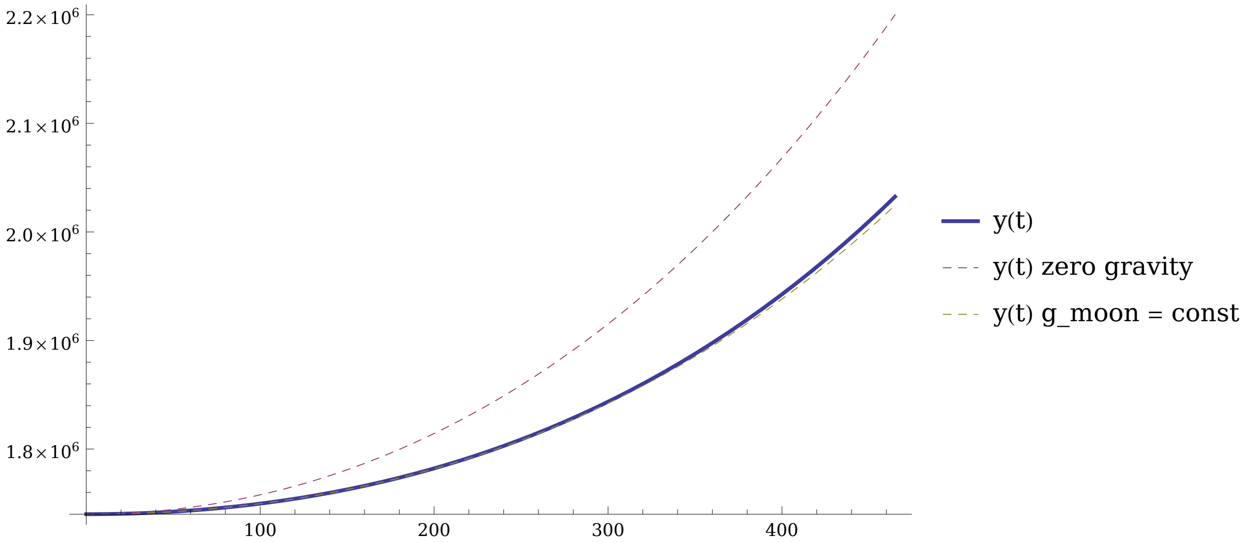
In[169]:=
```
Plot[{y'[t] /. sol1, (y'[t]) /. sol0 /. params, (yc'[t]) /. params },
  {t, 0, tburn /. params}, PlotStyle -> {Thick, Dashed, Dashed},
  PlotLegends → {"v(t)", "v(t)  zero gravity", "v(t)  g_moon  =  const"}]
```

Out[169]=



In[170]:=
```
Plot[{y[t] /. sol1, (y[t] + rmoon) /. sol0 /. params, (yc[t] + rmoon) /. params },
  {t, 0, tburn /. params}, PlotStyle -> {Thick, Dashed, Dashed},
  PlotLegends → {"y(t)", "y(t)  zero gravity", "y(t)  g_moon  =  const"}]
```

Out[170]=



In[161]:=
```
Table[{t, y[t]} /. sol1[[1]], {t, 0, 5}]
```
Out[161]= $\{\{0,\ 1.74\times10^{6}\},\ \{1,\ 1.74\times10^{6}\},\ \{2,\ 1.74\times10^{6}\},\ \{3,\ 1.74001\times10^{6}\},\ \{4,\ 1.74001\times10^{6}\},\ \{5,\ 1.74002\times10^{6}\}\}$

In[165]:=
```
Export["v.txt", Table[{t, y'[t] /. sol1[[1]]}, {t, 0, tburn /. params}], "Table"]
Export["y.txt", Table[{t, y[t] /. sol1[[1]]}, {t, 0, tburn /. params}], "Table"]
```
Out[165]= v.txt

Out[166]= y.txt

In[167]:=
```
Export["v-const-g.txt", Table[{t, (yc'[t]) /. params}, {t, 0, tburn /. params}], "Table"]
Export["y-const-g.txt",
 Table[{t, (yc[t] + rmoon) /. params}, {t, 0, tburn /. params}], "Table"]
```

Out[167]= v-const-g.txt

Out[168]= y-const-g.txt