G53FUZ
Fuzzy Sets and Systems

Fuzzy Concepts

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Overview

- Basic concepts of fuzzy sets
  - definitions; notation
  - membership functions
- Essential properties of fuzzy sets
  - α-cuts, support, normality, convexity
- Basic operations
  - complement, intersection, union
- Advanced operations
  - axioms, operators families

Classical (Crisp) Sets

Characteristic Functions

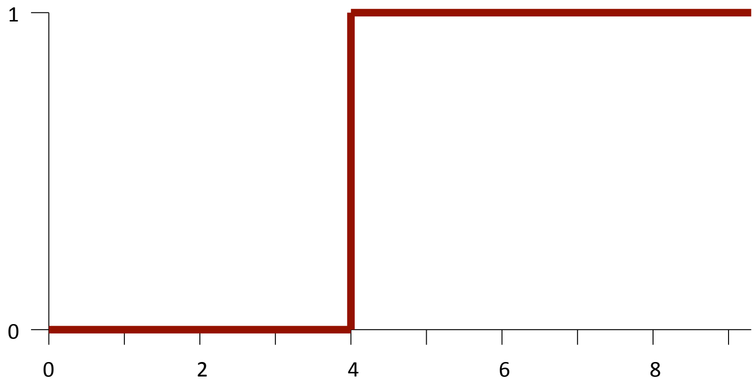
- Elements of the universal set X are defined to be either members or not of a set A by a *characteristic function*
  – for a given set A, this function assigns a value $\mu_A(x)$ to every $x \in X$, such that

$$\mu_A(x) = \begin{cases} 1 & \text{if and only if (iff) } x \in A \\ 0 & \text{if and only if (iff) } x \notin A \end{cases}$$

- Thus, the function maps elements of the universal set to the set containing 0 and 1
  – this can be denoted by $\mu_A(x): X \rightarrow \{ 0, 1 \}$

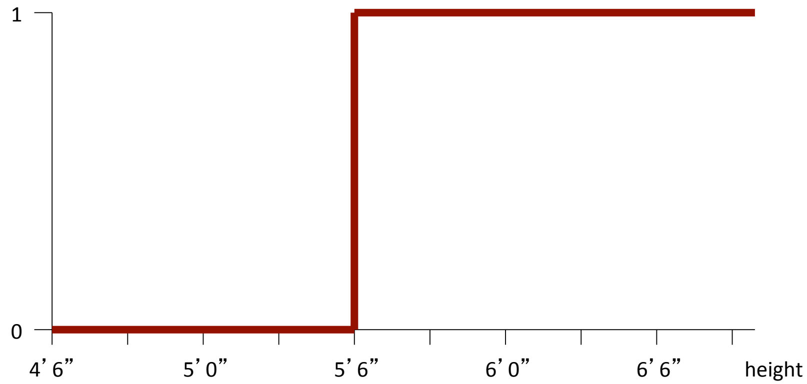
Diagramatically

- The set of real numbers greater than 4, $\mu_A(x) = 1$ iff $x \geq 4$, may be illustrated as



Tall People

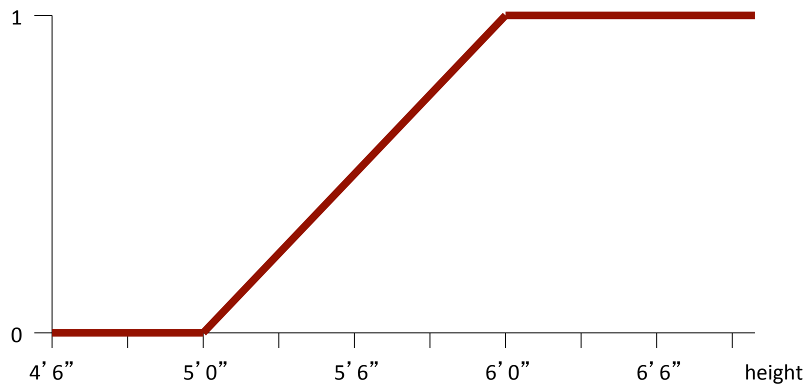
- But what about the set of tall people?
  Say, $\mu_{tall}(x) = 1$ iff $x \geq 66$ (inches, i.e. 5' 6")???



Fuzzy Sets

Fuzzy Tall People

- Let's modify the sharp (crisp) cut-off for the set of tall people into a smooth transition, e.g.



Formal Definition

- The Boolean characteristic function of conventional sets is replaced by a *membership function* that returns a real value in [0, 1]
  – this can be denoted by $\mu_A(x): X \rightarrow [\,0, 1\,]$
- So, for the tall example previously

$$\mu_{tall}(x) = \begin{cases} 1 & x \geq 72'' \\ \dfrac{x-60}{12} & 60'' < x < 72'' \\ 0 & x \leq 60'' \end{cases}$$

- Note that membership values can also be listed

  tall = { 0/*Danny DeVito*, 0.9/*Bob J*, 1/*Michael Jordan* }

Notation

- Fuzzy sets (as with crisp sets) can be either discrete or continuous
  – fuzzy set notation can (initially) be confusing
- Discrete sets

$$A = \mu_1/x_1 + \mu_2/x_2 + \mu_3/x_3 + \ldots + \mu_n/x_n$$

or

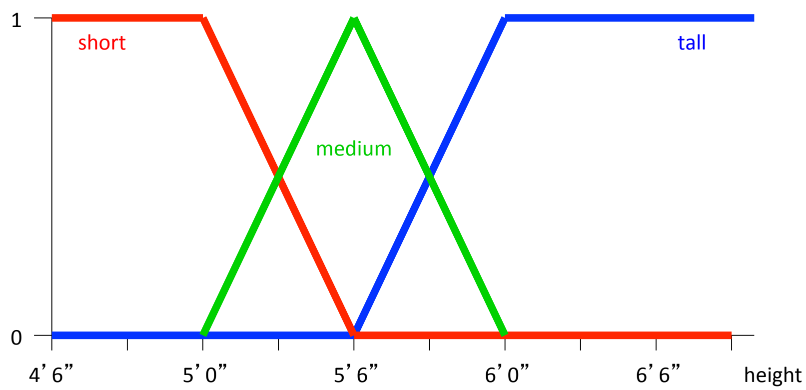
$$A = \Sigma_{i=1,n}\, \mu_i \,/\, x_i$$

- Continuous sets

$$A = \int_X \mu(x) \,/\, x$$

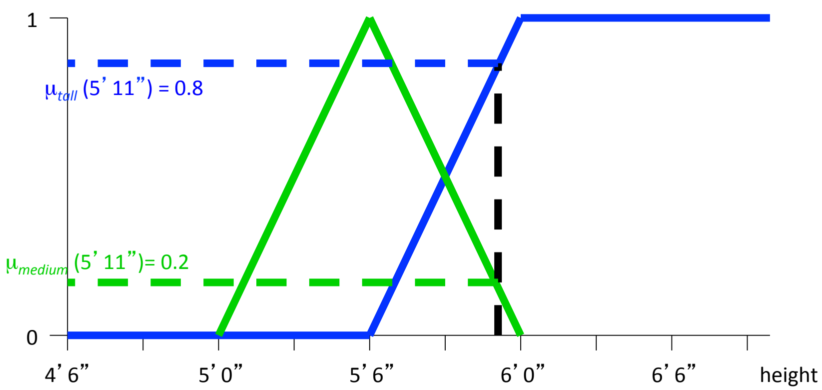
Examples

- Examples of further fuzzy sets for height



Multiple Memberships

- A given element or value can belong to multiple fuzzy sets with differing memberships



Exercises

- Write down the fuzzy set (memberships/ elements) of your three closest neighbours for the fuzzy set *tall*
- On the universe of real numbers, draw a fuzzy set of the concept *about five*
- Draw a fuzzy set for the concept *middle-aged* (people)
  - write down the lower and upper age limit of the set of middle-aged people

Meaning of Fuzzy Grades

Meaning of Grades?

- Consider
  - membership of *Bob* in set of *tall people* is 0.7
- Likelihood view
  - 70% of the population would describe *Bob* as *tall*
- Random set view
  - 70% of the population described '*tall*' as an interval containing *Bob's* height
- Typicality view
  - *Bob's* height is 70% along the scale of tallness

Fuzzy Sets and Probabilities

- Fuzzy memberships are *not* probabilities
  - there is no probability involved in a person's height
  - memberships are better interpreted as *compatibilities*
- Consider you are given two bottles of liquid
  - bottle A
    - the liquid is drinkable with probability 0.9
  - bottle B
    - the liquid is drinkable with fuzzy membership 0.9
- Which do you drink, and why?

Fuzzy Sets and Probabilities

- Bottle A:
  - there is 0.9 chance bottle is drinkable
  - 0.1 chance bottle is filled with undrinkable (poison?)
  - potentially risky!
- Bottle B:
  - 0 is fuzzy totally undrinkable (poison?)
  - 1 is fuzzy totally drinkable (beer? red wine?)
  - 0.9 is close to fully drinkable (water?)

Properties

α-Cuts

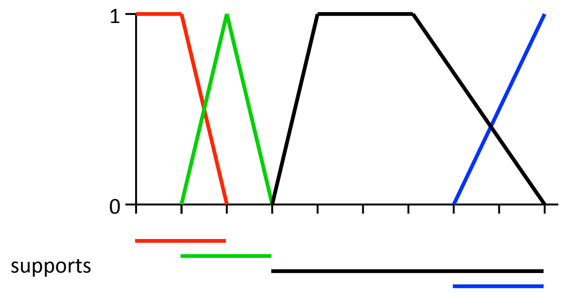
- An important concept which establishes a relationship between crisp sets and fuzzy sets is the concept of an α-*cut*
  - an α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of A with membership greater than or equal to the specified value of α
  - this can be written as

    $A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \}$
- The *strong* α-*cut*, $A_{\alpha+}$, can also be defined

  $A_{\alpha+} = \{ x \in X \mid \mu_A(x) > \alpha \}$

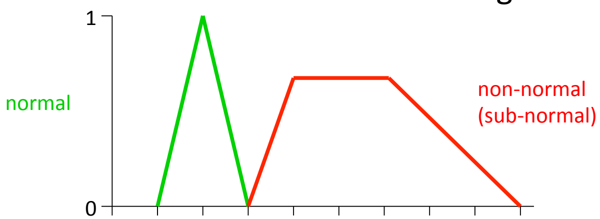
Support

- The *support* of a fuzzy set, A, is the stong α-cut of A for $\alpha = 0$ (A_{0+})
  - i.e. the crisp set of elements where the membership is greater than zero



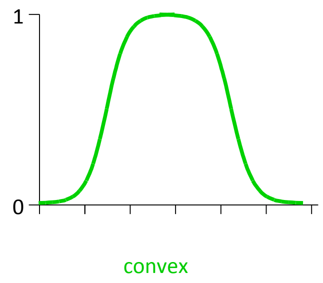
Normality

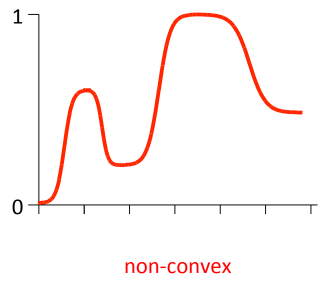
- The *height* of a fuzzy set is the largest membership grade attained by any element of that set
- A fuzzy set is *normalised* if at least one of its elements attains the maximum possible grade
  - if membership grades are in [0,1], it is normalised when at least one element has height 1



Convexity

- A fuzzy set is convex if and only if each of its α-cuts is a convex set
  - iff $\mu_A(\lambda\mathbf{r} + (1 - \lambda)\mathbf{s}) > \min[\mu_A(\mathbf{r}), \mu_A(\mathbf{s})]$ $\forall$ $\mathbf{r},\mathbf{s} \in \Re^n$ and $\forall$ $\lambda \in [0, 1]$





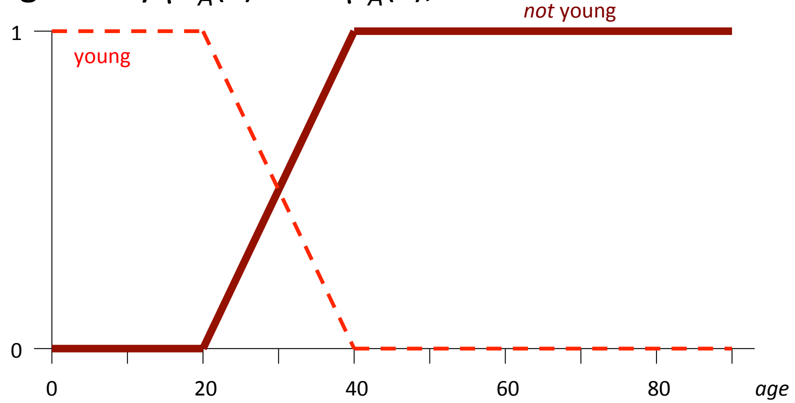
Exercises

- Given the following fuzzy sets
  - A = 0.0/1 + 0.1/2 + 0.7/3 + 0.6/4 + 0.7/5 + 0.4/6 + 0.1/7
  - B = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.5/6 + 0.1/7
  - C = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7
- What are the alpha cuts
  - $A_{0.2}$, $B_{0.5}$, $C_{0.9}$
- What is the support of each?
- Which of the sets are normal? / convex?

Basic Operations

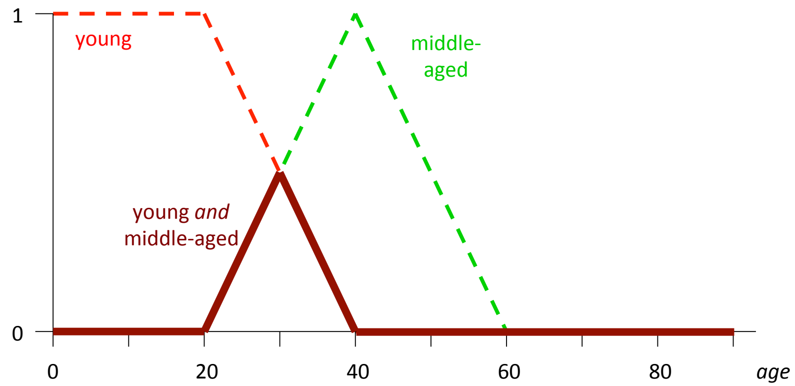
Complement

- The fuzzy complement, $\bar{A}$, of fuzzy set A is given by $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$, $\forall x \in U$



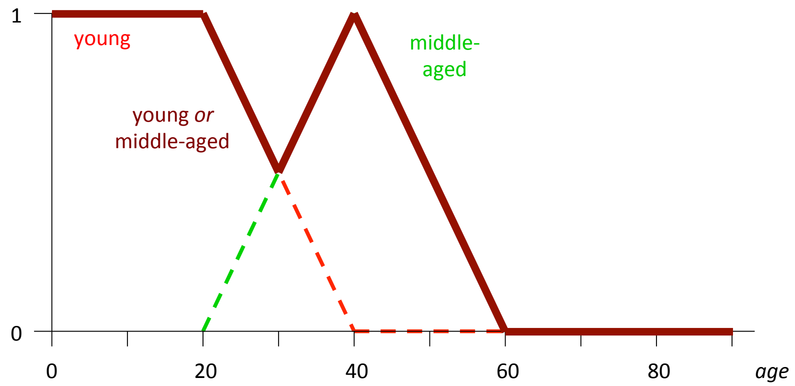
Intersection

- The fuzzy intersection, $A \cap B$, of two fuzzy sets A and B, is given by $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$



Union

- The fuzzy union, $A \cup B$, of two fuzzy sets A and B, is given by $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$



Exercises

- Given the following two fuzzy sets
  - A = 0.0/1 + 0.1/2 + 0.5/3 + 1.0/4 + 0.7/5 + 0.4/6 + 0.1/7
  - B = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7
  - write down the fuzzy sets
    - NOT B
    - A AND B
    - A OR B
    - A AND $\bar{A}$

Parameterised Operations

Complement

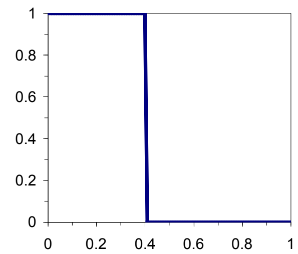
- A complement is a function which converts a fuzzy set, A, to another set, $\bar{A}$
  - c: [0,1] → [0,1]
- The function *must* satisfy the following axioms
  - **axiom *c*1**: $c(0) = 1$ and $c(1) = 0$
    - must behave like crisp sets (*boundary conditions*)
  - **axiom *c*2**: for all $a, b \in [0,1]$:
    if $a < b$, then $c(a) \geq c(b)$
    - c is *monotonic non-increasing*
  - where $a = \mu_A(x)$ and $b = \mu_B(x)$

Complement

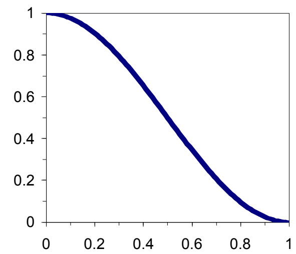
- Very often (in most cases), further requirements may be placed
  - **axiom *c*3**: c should be a *continuous* function
  - **axiom *c*4**: c should be *involutive*
    - that is $c(c(a)) = a$ for all $a \in [0,1]$
- Functions satisfying axiom 3 form a special sub-class of general fuzzy complements
  - all functions satisfying axiom 4 are necessarily continuous, and so form a further nested sub-class

Example Complements

(a)



(b)

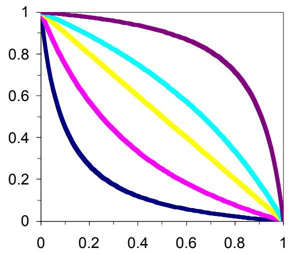


$$(a) \quad \overline{\mu(x)} = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{if } a > t \end{cases}$$

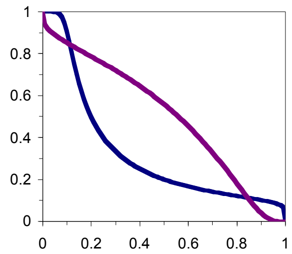
$$(b) \quad \overline{\mu(x)} = \tfrac{1}{2}(1 + \cos(\pi a))$$

Example Complements

(c)



(d)



$$(c) \quad \overline{\mu(x)} = \frac{1-a}{1+\lambda a}$$

$$(d) \quad \overline{\mu(x)} = \left(\frac{1-a^w}{1+\lambda a^w}\right)^{1/w}$$

Intersection

- Fuzzy intersections are represented by an established class of functions called *triangular norms* or *t-norms*
- A t-norm is a function which takes two arguments in [0,1] and returns a value in [0,1]
  - i: [0,1] x [0,1] → [0,1]
- Thus
  - $\mu_{A \cap B}(x) = i(\mu_A(x), \mu_B(x))$

Intersection Axioms

- The function *must* satisfy the following axioms
  - **axiom *i*1**: $i(1,1)=1$, $i(0,1)=i(1,0)=i(0,0)=0$
    - must behave like crisp sets (*boundary conditions*)
    - sometimes written like $1 \otimes a = a$
  - **axiom *i*2**: $i(a, b) = i(b, a)$ (*commutative*)
  - **axiom *i*3**: if $a \le a'$ and $b \le b'$, then $i(a,b) \le i(a',b')$ (*monotonic*)
  - **axiom *i*4**: $i(i(a,b), c) = i(a, i(b,c))$ (*associative*)

Optional Intersection Axioms

- The function *may* satisfy the following axioms
  - **axiom *i*5**: *i* is *continuous*
  - **axiom *i*6**: $i(a, a) = a$ (*idempotent*)
- The minimum **min(*a*, *b*)** is the *only* t-norm which satisfies axioms *i*1 to *i*6

Common T-Norms

- Schweizer and Sklar (1963)

  $a \otimes b = \max(0, a^p + b^p - 1)^{1/p}$

- This gives

$a \otimes_{-\infty} b$	=	$\min(a, b)$	(*standard*)
$a \otimes_0 b$	=	ab	(*product*)
$a \otimes_1 b$	=	$\max(0, a + b - 1)$	(*bounded diff.*)
$a \otimes_\infty b$	=	(a if b = 1; b if a = 1; else 0)	(*drastic sum*)

Union

- Fuzzy unions are represented by an established class of functions called *triangular conorms* or *t-conorms*
- A t-conorm is a function which takes two arguments in [0,1] and returns a value in [0,1]
  - u: [0,1] x [0,1] → [0,1]
- Thus
  - $\mu_{A \cup B}(x) = u(\mu_A(x), \mu_B(x))$

Union Axioms

- The function *must* satisfy the following axioms
  - **axiom *u*1**: $u(0,0)=0$, $u(0,1)=u(1,0)=u(1,1)=1$
    - must behave like crisp sets (*boundary conditions*)
    - sometimes written like $0 \oplus a = a$
  - **axiom *u*2**: $u(a, b) = u(b, a)$ (*commutative*)
  - **axiom *u*3**: if $a \le a'$ and $b \le b'$, then $u(a,b) \le u(a',b')$ (*monotonic*)
  - **axiom *u*4**: $u(u(a,b), c) = u(a, u(b,c))$ (*associative*)

Optional Union Axioms

- The function *may* satisfy the following axioms
  - **axiom *u*5**: u is *continuous*
  - **axiom *u*6**: $u(a, a) = a$ (*idempotent*)
- The maximum **max(*a*, *b*)** is the *only* t-conorm which satisfies axiom u6
  - as well as all the others!

Common T-Conorms

- Schweizer and Sklar (1963)

  $a \oplus b = 1 - \max(0, (1-a)^{-p} + (1-b)^{-p} - 1)^{1/p}$

- This gives

  $a \oplus_{-\infty} b = \max(a, b)$ (*standard*)

  $a \oplus_0 b = a + b - ab$ (*probabilistic sum*)

  $a \oplus_1 b = \min(1, a + b)$ (*bounded sum*)

  $a \oplus_\infty b =$ (a if $b = 0$; b if $a = 0$; else 1) (*drastic*)

Other Operations

Exercises

- Given the following two fuzzy sets
  - $A = 0.0/1 + 0.1/2 + 0.5/3 + 1.0/4 + 0.7/5 + 0.4/6 + 0.1/7$
  - $B = 0.1/1 + 0.3/2 + 0.6/3 + 0.7/4 + 0.8/5 + 0.9/6 + 1.0/7$
  - write down (A AND B) and (A OR B), using
    - min, max
    - product, probabilistic sum
    - bounded difference, bounded sum
- Calculate ANDs using min and product for
  - Jack is 0.9/tall and 0.9/old
  - Fred is 0.9/tall and 0.2/old
  - which do you think makes more sense, and why?

Summary

- Lecture summary
  - fuzzy sets are extensions of conventional (crisp) sets that allow everyday notions to be represented
  - fuzzy memberships are not probabilities
    - but their precise meaning is open to interpretation
  - min and max are used for basic AND and OR
  - there are many alternative operator families
- Next lecture
  - linguistic variables