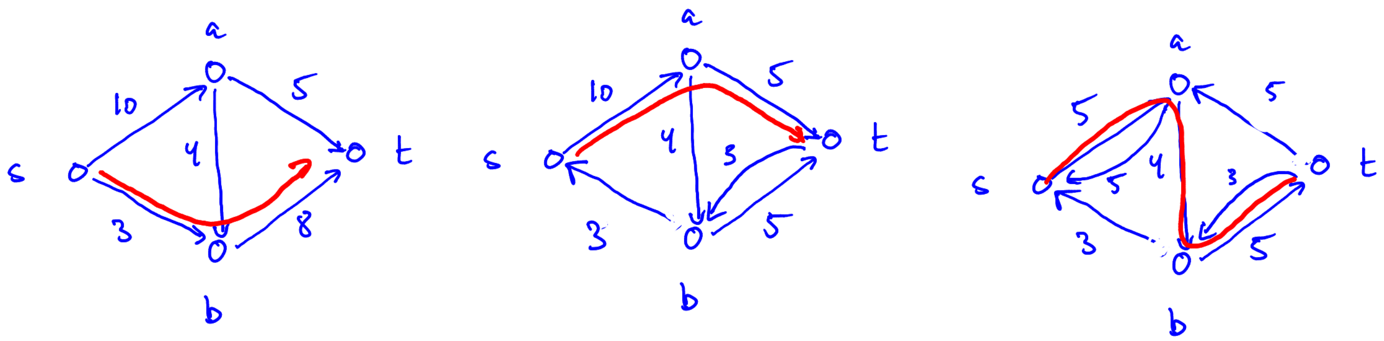
C54 AAD

Runtime Analysis of Edmonds-Karp Alg.

FORD - FULKERSON ALGORITHM

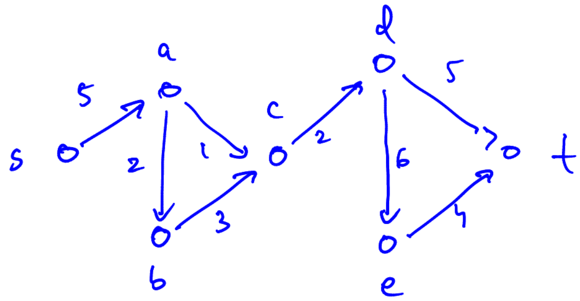
```
1  for each edge (u,v) ∈ E
2        f[u,v] = 0
3        f[v,u] = 0
4  while there exists an augmenting path p in G_f
5        c_f(p) = min_{(u,v)∈p} c_f(u,v)
6        for each edge (u,v) on p
7              f[u,v] ← f[u,v] + c_f(p)
8              f[v,u] ← -f[u,v]
```

Step 4 of the algorithm can be implemented in different ways. When the augmenting path p is found by searching for a shortest path, the resulting algorithm is called Edmonds - Karp algorithm.



Def. Given a residual network G_f, let $\delta_f(u,v)$ be the length of the shortest path from u to v in G_f.

Ex



G_f

$\delta_f(s,a) = 1$

$\delta_f(s,c) = 2$

$\delta_f(s,d) = 3$

Lemma

The shortest path $\delta_f(u,v)$ in a residual network G_f increases monotonically with each augmentation in the Edmonds-Karp algorithm.

Proof (by contradiction)

Let f' be the flow obtained by augmenting f with a shortest path p from s to t in G_f.
Assume by contradiction that there exists a node v s.t.

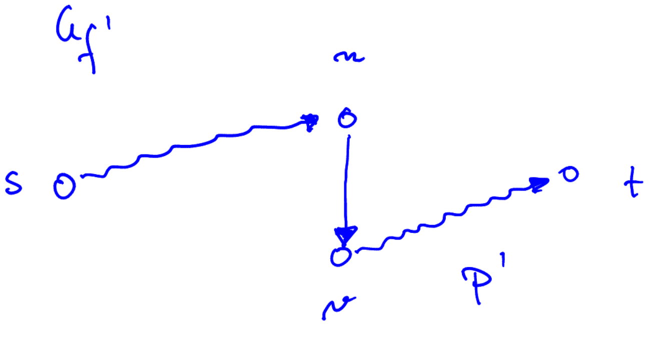
$$\delta_{f'}(s,v) < \delta_f(s,v) \qquad (1)$$

Without loss of generality, assume that v is the closest node to s in $G_{f'}$ where the distance to s decreased.
Assume the shortest path from s to v has last edge (u,v).

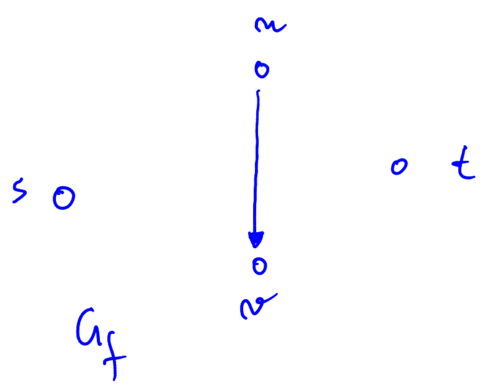
$$\delta_{f'}(s,u) \geq \delta_f(s,u) \qquad (2)$$

Since (u,v) is on the shortest path p

$$\delta_{f'}(s,v) = \delta_f(s,u) + 1. \qquad (3)$$



Case 1 $(u,v) \in E_f$



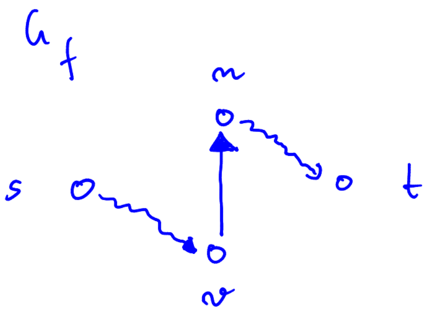
Then

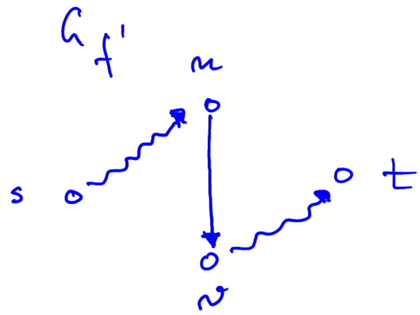
$$\begin{aligned}
\delta_f(s,v) &\leq \delta_f(s,u) + 1 \\
&\leq \delta_{f'}(s,u) + 1 \quad \text{by (2)} \\
&= \delta_{f'}(s,v) \quad \text{by (3)}
\end{aligned}$$

$\Rightarrow$ contradicts (1).

Case 2 $(u,v) \notin E_f$.

Note that we assume that $(u,v) \in E_{f'}$. Hence, $(v,u) \in E_f$ and f' is obtained by augmenting the flow along edge (v,u). Thus, the edge (v,u) must be on the shortest path from s to t in G_f.





$$\delta_{f'}(s,v) < \delta_f(s,v) \qquad (1)$$

$$\delta_{f'}(s,u) \geq \delta_f(s,u) \qquad (2)$$

$$\delta_{f'}(s,v) = \delta_{f'}(s,u) + 1 \qquad (3)$$

$$\begin{aligned}\delta_f(s,v) &= \delta_f(s,u) - 1 \\ &\leq \delta_{f'}(s,u) - 1 \\ &= \delta_f(s,v) - 2\end{aligned}$$

$\Rightarrow$ Contradicts (3).

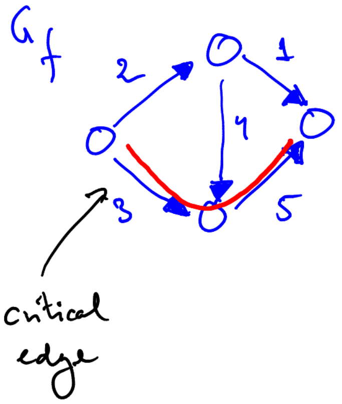
Def

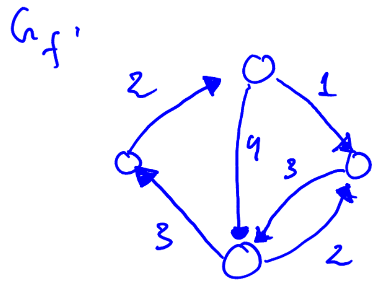
An edge (u,v) on an augmenting path f in G_f is called critical if

$$c_f(u,v) = c_f(p).$$

Note that any critical edge (u,v) disappears when the flow is augmented by p.

Ex





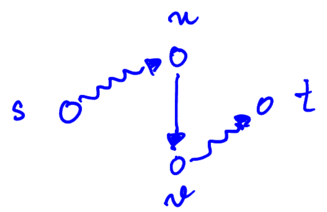
Thm

The Edmonds-Karp algorithm augments the flow at most $O(VE)$ times.

Proof

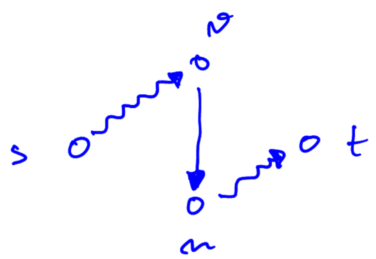
Every time the flow is augmented, at least one critical edge is removed. Hence, the number of times the flow is augmented is no larger than the number of times a critical edge can disappear.

Assume that (u,v) is a critical edge in G_f that is removed.



$$\delta_f(s,v) = \delta_f(s,u) + 1.$$

In order for the edge (u,v) to reappear, there must exists a late flow f' such that $(v,u) \in G_{f'}$, and (v,u) is on the shortest path p' from s to t in $G_{f'}$.



$$\delta_{f'}(s,u) = \delta_{f'}(s,v) + 1.$$

Hence

$$\begin{aligned} \delta_{f'}(s,u) &= \delta_{f'}(s,v) + 1 \\ &\geq \delta_f(s,v) + 1 \qquad \text{(by Lemma)} \\ &= \delta_f(s,u) + 2. \end{aligned}$$

Therefore, every time (u,v) reappears, the distance between s and u increases by at least 2.

The maximal distance is $|V|-2$.

Hence each edge (u,v) is critical at most $O(V)$ times.

There are E edges, and each edge is critical at most $O(V)$ times.
The number of path augmentations is less than the number of times edges become critical.
Hence, there are at most $O(VE)$ path augmentations. $\square$

Corollary

The Edmonds-Karp alg. has running time $O(VE^2)$, where V is the number of vertices, and E is the number of edges.

Proof

A shortest path in G_f can be found in $O(E)$ time using breadth first search.
The flow must be augmented at most $O(VE)$ times.