G54 AAD

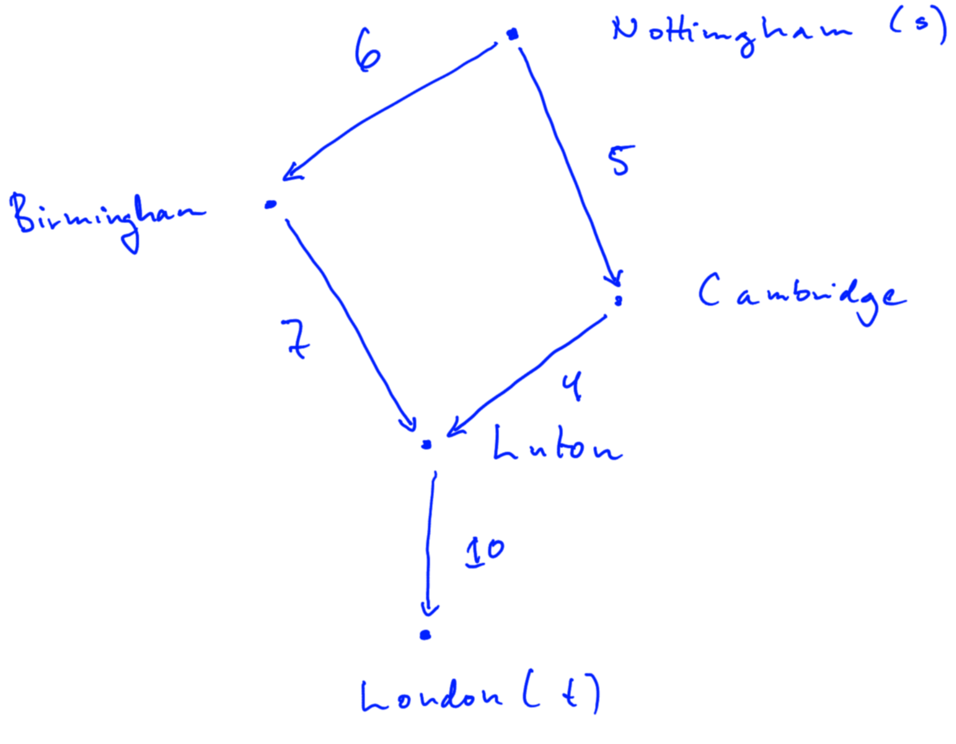
L11-12 Maximum Flow

Def: A flow network is a

- directed graph $G = (V, E)$,
- a source node $s \in V$ and a sink node $t \in V$.
- capacity function $c : V \times V \to \mathbb{R}$ where for all u, v

$$c(u,v) \geq 0 \quad \text{if } (u,v) \in E$$
$$c(u,v) = 0 \quad \text{if } (u,v) \notin E$$

Ex



Def Flow

A flow is a function $f: V \times V \to \mathbb{R}$ which satisfies the following constraints

1) Capacity Constraint

$\forall u, v \in V \qquad f(u,v) \leq c(u,v)$

2) Skew Symmetry

$\forall u, v \in V \qquad f(u,v) = -f(v,u)$

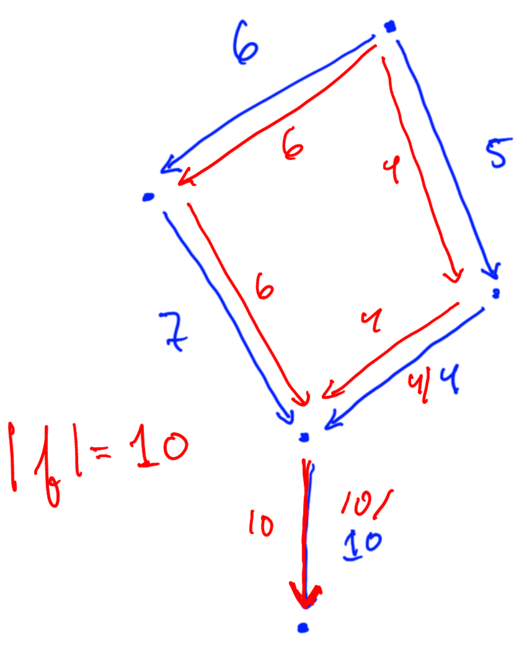
3) Flow Conservation

$\forall u \in V - \{s,t\} \quad \sum_{v \in V} f(u,v) = 0$

Def Value of Flow

The value of a flow is $|f| = \sum_{v \in V} f(s,v)$

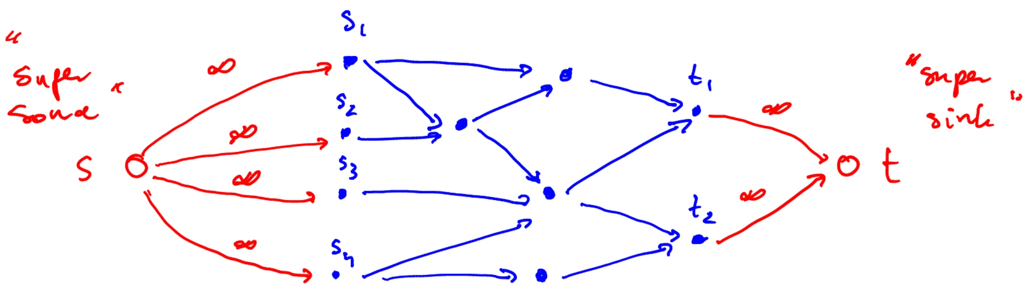
Ex



Def Maximum Flow Problem

Given a directed graph G and a capacity function c, find a flow f with maximal value.

Max Flow with Multiple Sources & Sinks



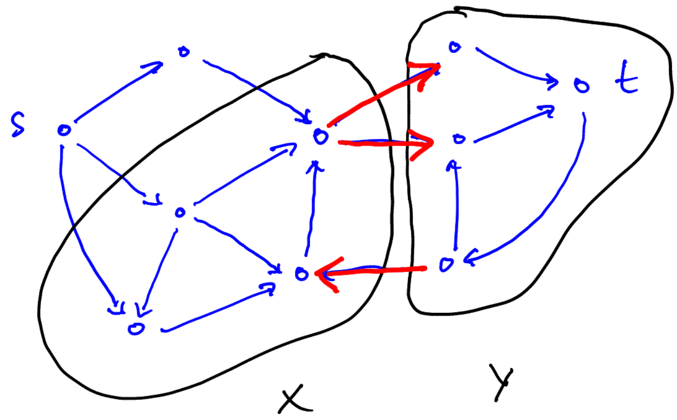
Max flow with multiple sources and sinks
can be transformed to single source single sink max flow
by adding

- "super source" node s with infinite edge capacities
- "super sink" node t with infinite edge capacities

Generalized Flow Function

Def Given $X, Y \subseteq V$, define

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$



Lemma

(1) For $X \subseteq V$ $\quad f(X,X) = 0$

(2) For $X, Y \subseteq V$ $\quad f(X,Y) = -f(Y,X)$

(3) For $X, Y, Z \subseteq V$ with $X \cap Y = \phi$

$$f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$$

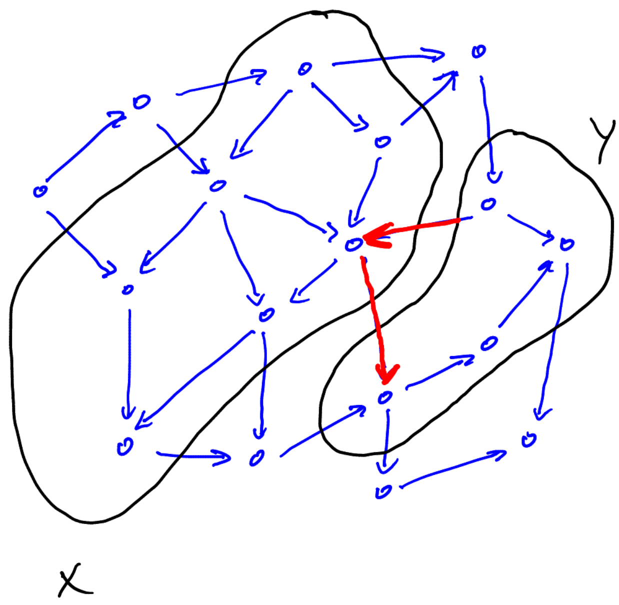
$$f(Z, X \cup Y) = f(Z,X) + f(Z,Y)$$

Proof (1)

$$
\begin{aligned}
f(x,x) &= \frac{1}{2}\left(f(x,x) + f(x,x)\right) \\
&= \frac{1}{2}\sum_{x\in X}\sum_{y\in X} f(x,y) + \sum_{x\in X}\sum_{y\in X} f(y,x) \\
&= \frac{1}{2}\left(\sum_{x\in X}\sum_{y\in X} f(x,y) + f(y,x)\right) \\
&= \frac{1}{2}\sum_{x\in X}\sum_{y\in X} 0 \qquad \text{(By Skew Symmetry)} \\
&= 0
\end{aligned}
$$

Proof (2)

$$
\begin{aligned}
f(X,Y) &= \sum_{x\in X}\sum_{y\in Y} f(x,y) \\
&= \sum_{x\in X}\sum_{y\in Y} - f(y,x) \\
&= - \sum_{y\in Y}\sum_{x\in X} f(y,x) \\
&= - f(Y,X)
\end{aligned}
$$

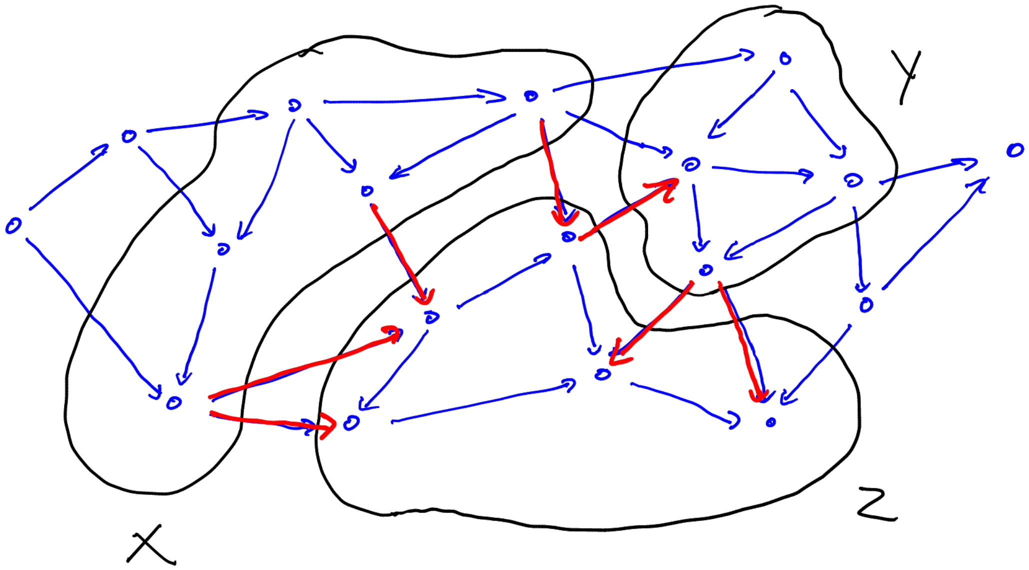


Proof (3)

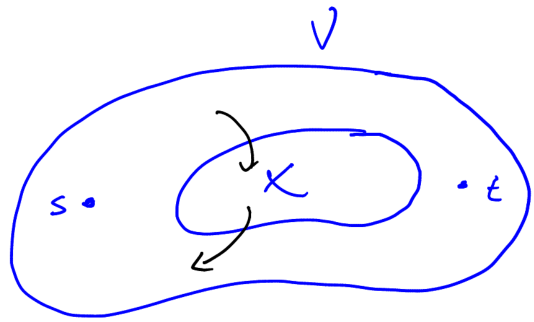
$$f(X \cup Y, Z) = \sum_{x \in X \cup Y} \sum_{z \in Z} f(x, z)$$

$$= \sum_{x \in X} \sum_{z \in Z} f(x, z) + \sum_{y \in Y} \sum_{z \in Z} f(y, z)$$

$$= f(X, Z) + f(Y, Z)$$



Flow Conservation



Lemma

For any $X \subseteq V - \{s, t\}$

$$f(X, V) = 0$$

Proof

$$f(X, V) = \sum_{x \in X} \sum_{y \in V} f(x, y)$$

$$= 0$$

Residual Network

Recall that $f(u,v) \le c(u,v)$.

Given a flow f on a network with capacity c

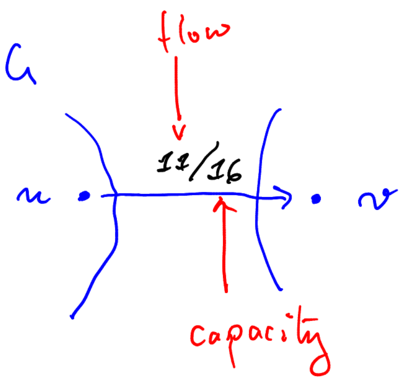
the residual capacity is

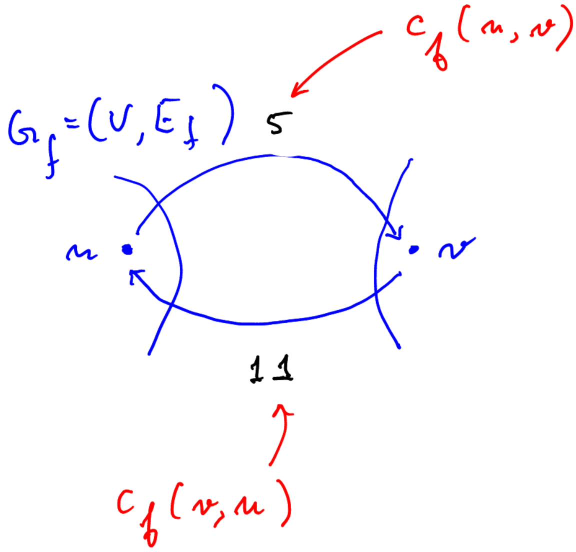
$$c_f(u,v) = c(u,v) - f(u,v), \text{ and}$$

the residual network is $G_f = (V, E_f)$ where

$$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$$

Ex





Def Addition of flows

$$(f_1 + f_2)(u,v) = f_1(u,v) + f_2(u,v)$$

Lemma

Given a flow network G with flow f.
Let f' be a flow in the residual network G_f.
Then $f + f'$ is a flow in G, and

$$|f + f'| = |f| + |f'|.$$

Proof

$$\begin{aligned}(f + f')(u,v) &= f(u,v) + f'(u,v)\\ &\leq f(u,v) + c_f(u,v)\\ &\leq f(u,v) + c(u,v) - f(u,v)\\ &= c(u,v)\end{aligned}$$

$\Rightarrow$ $f+f'$ satisfies the capacity constraint.

Proof (contd.)

$$(f+f')(u,v) = f(u,v) + f'(u,v)$$
$$= -f(v,u) - f'(v,u)$$
$$= -\left(f(v,u) + f'(v,u)\right)$$
$$= -(f+f')(v,u).$$

$\Rightarrow$ $f+f'$ satisfies Skew Symmetry

Given $u \in V - \{s,t\}$

$$\sum_{v \in V} (f+f')(u,v) = \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v)$$
$$= 0 + 0$$

$\Rightarrow$ $f+f'$ satisfies flow concentration.

$$|f+f'| = \sum_{v \in V} (f+f')(s,v)$$
$$= \sum_{v \in V} f(s,v) + \sum_{v \in V} f'(s,v)$$
$$= |f| + |f'|.$$

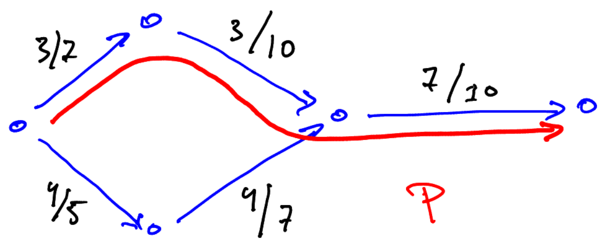
$\square$

Augmenting Path

A simple path in a residual network is called an augmenting path.

The residual capacity of a path is

$$c_f(p) = \min\{ c_f(u,v) \mid (u,v) \text{ is on } p \}$$



$c_f(p) = 3$

Lemma

Given a flow f and an augmenting path p, let

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ -c_f(p) & \text{if } (v,u) \text{ is on } p \\ 0 & \text{otherwise.} \end{cases}$$

Then f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corollary

Let f_p be as above, then

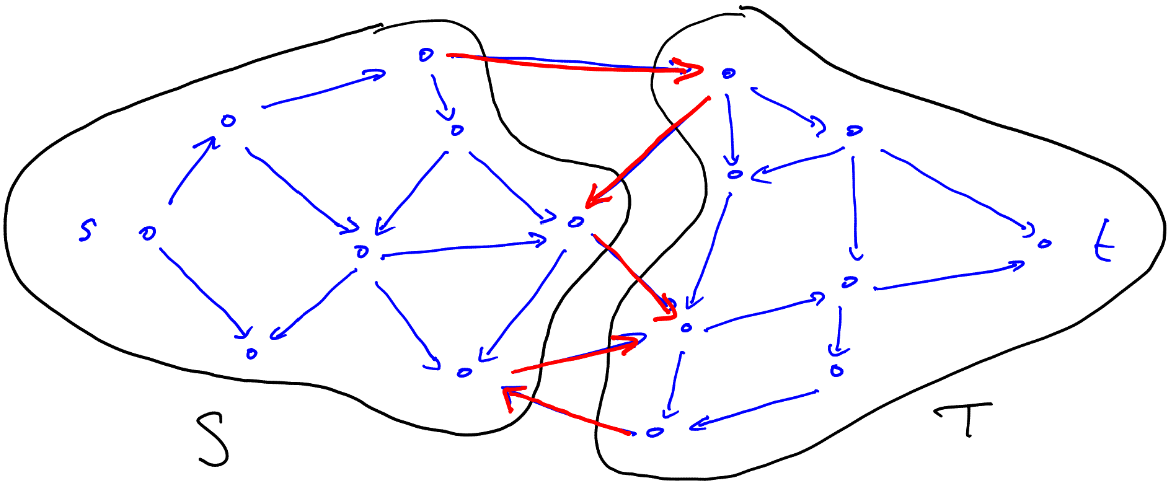
$$f' = f + f_p$$

is a flow in G with value $|f'| = |f| + |f_p|$.

Def Cuts

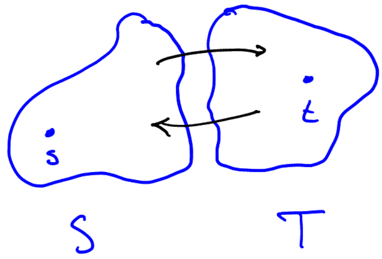
A partition of V into S and T such that $s \in S$ and $t \in T$ is called a cut.

The capacity of a cut is $c(S,T)$.



Lemma

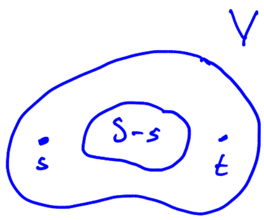
Let f be a flow in G, and (S,T) a cut of G. Then $f(S,T) = |f|$



$$f(S,T) = f(S,V) - f(S,S)$$

$$= f(S,V)$$

$$= f(s,V) + f(S-s,V)$$



$$= f(s,V)$$

$$= |f|.$$

Corollary

The value of any flow f in G is bounded from above by the capacity of any cut in G.

Proof

$$|f| = f(S,T)$$

$$\leq c(S,T)$$

Max Flow Min Cut Theorem

Theorem

If f is a flow in a network G, then the following are equivalent

(1) f is a max flow in G

(2) the residual network G_f contains no augmenting path

(3) $|f| = c(S,T)$ for some cut (S,T) of G.

Proof (1) => (2)

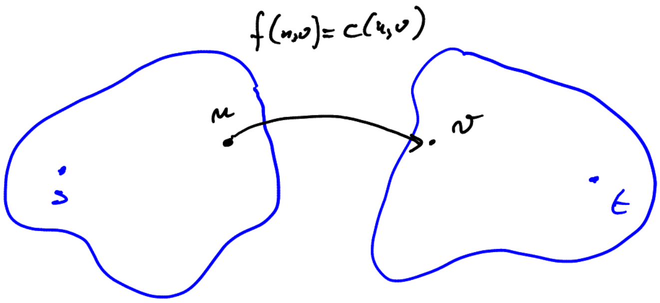
Assume that G_f contains an augmenting path p. Then $f + f_p$ is a flow in G with value $|f| + |f_p| > |f|$. This contradicts that f is a max flow.

(2) => (3)

Assume that G_f has no augmenting path.

Let

$$S = \{ v \mid \text{there exists a path from } s \text{ to } v \text{ in } G_f \}$$
$$T = V - S$$



For all pairs $u \in S$ and $v \in T$,

$$c_f(u, v) = 0$$

otherwise there would exist an augmenting path in G.

$$\Rightarrow \quad f(u,v) = c(u,v) - c_f(u,v)$$
$$= c(u,v)$$

$$\Rightarrow \quad |f| = c(S,T).$$

(3) $\Rightarrow$ (1)

For all flows f'

$$|f'| \leq c(S,T)$$

$\Rightarrow$ f is an optimal flow.

$\square$