G54 AAD

L12 FLOYD-WARSHALL & DYNAMIC PROGRAMMING

DYNAMIC PROGRAMMING

Problems amenable to dynamic programming have two key properties.

1) Optimal substructure

   An optimal solution to the problem contains optimal solutions to sub-problems.

2) Overlapping subproblems

   The subproblems are overlapping.
   A direct recursive solution to the problem would compute solutions to the same subproblems many times.

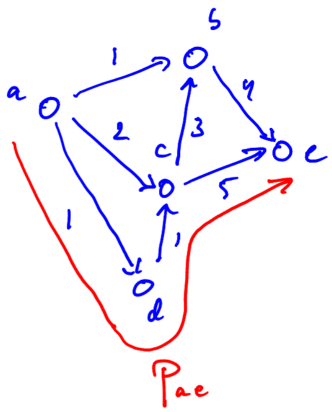
ALL PAIRS SHORTEST PATH

Def

A weighted, directed graph is a

- tuple $G = (V, E)$ where
  - V is a set called nodes, and
  - $E \subseteq V \times V$ is a set called edges, and
- a function $w : E \to \mathbb{R}$ called the weight function.

Ex



Def

A simple path is a sequence of nodes $(v_1, \dots, v_k)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in [k-1]$, and $v_i \neq v_j$ for all $i \neq j$.

The weight of a path is defined as

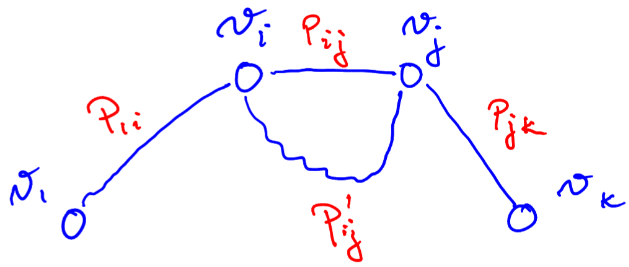
$$w(p) = \sum_{i=1}^{k-1} w((v_i, v_{i+1}))$$

ALL PAIRS SHORTEST PATH PROBLEM

Given a weighted directed graph $G = (V, E)$,
find the shortest path $P_{u,v}$ between all pairs of nodes $u, v \in V$.

LEMMA

If $P_{1k} = (v_1, \ldots, v_k)$ is the shortest path from v_1 to v_k,
then $P_{ij} = (v_i, \ldots, v_j)$ is the shortest path from v_i to v_j
for all $1 \leq i < j \leq k$.



Proof (By Contradiction)
Assume that there exists a path P'_{ij} from v_i to v_j which is shorter than P_{ij}. Then, the path P_{1i}, P'_{ij}, P_{jk} would be shorter than P_{1k} which contradicts that P_{1k} is the shortest path from v_1 to v_k.

DYNAMIC PROGRAMMING SOLUTION TO APSP.

Def: The intermediate nodes of a path $P = (v_1, \dots, v_k)$ are the nodes $\{v_2, \dots, v_{k-1}\}$.

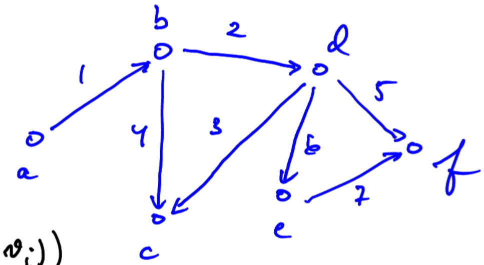
Idea: Consider the following more general problem

Given a weighted, directed graph $G = (V, E)$ and a subset $A \subseteq V$, find the shortest path P_{ij}^{A} between v_i and v_j among all paths that have intermediate nodes in A.

$\Rightarrow$ If $A = V$, then we have solved the original problem.

Special case $A = \phi$:

No intermediate nodes allowed.



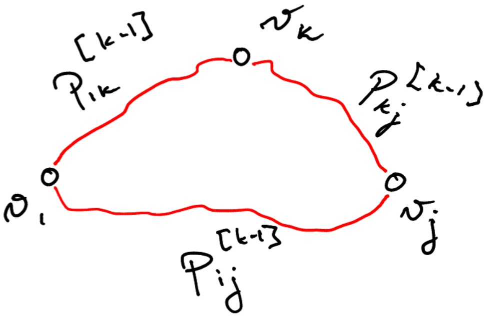
$\Rightarrow$ The solution is $w((v_i, v_j))$ for all pairs of nodes v_i and v_j.

General case

Assume solution for $A = [k-1]$

What is the solution for $A = [k]$?

Let $P_{ij}^{[k-1]}$ be the length of the shortest path from v_i to v_j with intermediate nodes $A = \{v_1, v_2, \dots, v_k\}$.



If $P_{ij}^{[k]}$ does not visit node k, then

$$P_{ij}^{[k]} = P_{ij}^{[k-1]}.$$

Otherwise, if $P_{ij}^{[k]}$ visits node k, then by Lemma 1

$$P_{ij}^{[k]} = P_{ik}^{[k-1]} + P_{kj}^{[k-1]}$$

Therefore

$$d_{ij}^{[k]} = \begin{cases} w_{ij} & \text{if } k = 0. \\ \min\{d_{ij}^{[k-1]}, d_{ik}^{[k-1]} + d_{kj}^{[k-1]}\} & \text{if } k > 0 \end{cases}$$

FLOYD-WARSHALL ALGORITHM

for $k=0$ to n

 for $i=1$ to n

  for $j=1$ to n

   if $k=0$ then {

    $d_{ij}^{[k]} = w_{ij}$

    $\Pi_{ij}^{[k]} = i$

   } else if $\left(d_{ij}^{[k-1]} > d_{ik}^{[k-1]} + d_{kj}^{[k-1]}\right)$ {

    $d_{ij}^{[k]} = d_{ik}^{[k-1]} + d_{kj}^{[k-1]}$

    $\Pi_{ij}^{[k]} = \Pi_{kj}^{[k-1]}$

   } else {

    $d_{ij}^{[k]} = d_{ij}^{[k-1]}$

    $\Pi_{ij}^{[k]} = \Pi_{ij}^{[k-1]}$

   }

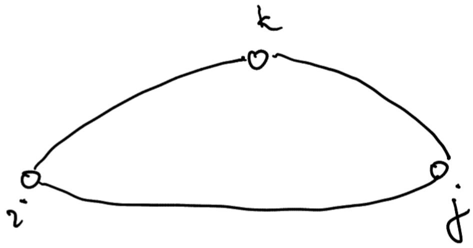
THM

Floyd-Warshall algorithm has running time $\Theta(n^3)$.

TRANSITIVE CLOSURE PROBLEM

Given a directed graph $G = (V, E)$,
the transitive closure is a graph $G' = (V, E')$
where $E' = \{ (u,v) \in V \times V \mid \text{there exists a path from } u \text{ to } v \text{ in } G \}$

Idea: Generalised problem where the path has intermediate vertices in $A \subseteq V$.



$$t_{ij}^{[k]} = \begin{cases} \text{FALSE} & \text{if } k=0 \text{ and } (i,j) \notin E \\ \text{TRUE} & \text{if } k=0 \text{ and } (i,j) \in E \\ t_{ij}^{[k-1]} \vee \left(t_{ij}^{[k-1]} \wedge t_{ij}^{[k-1]} \right) & \text{if } k>0. \end{cases}$$