G54 AAD 2015/2016

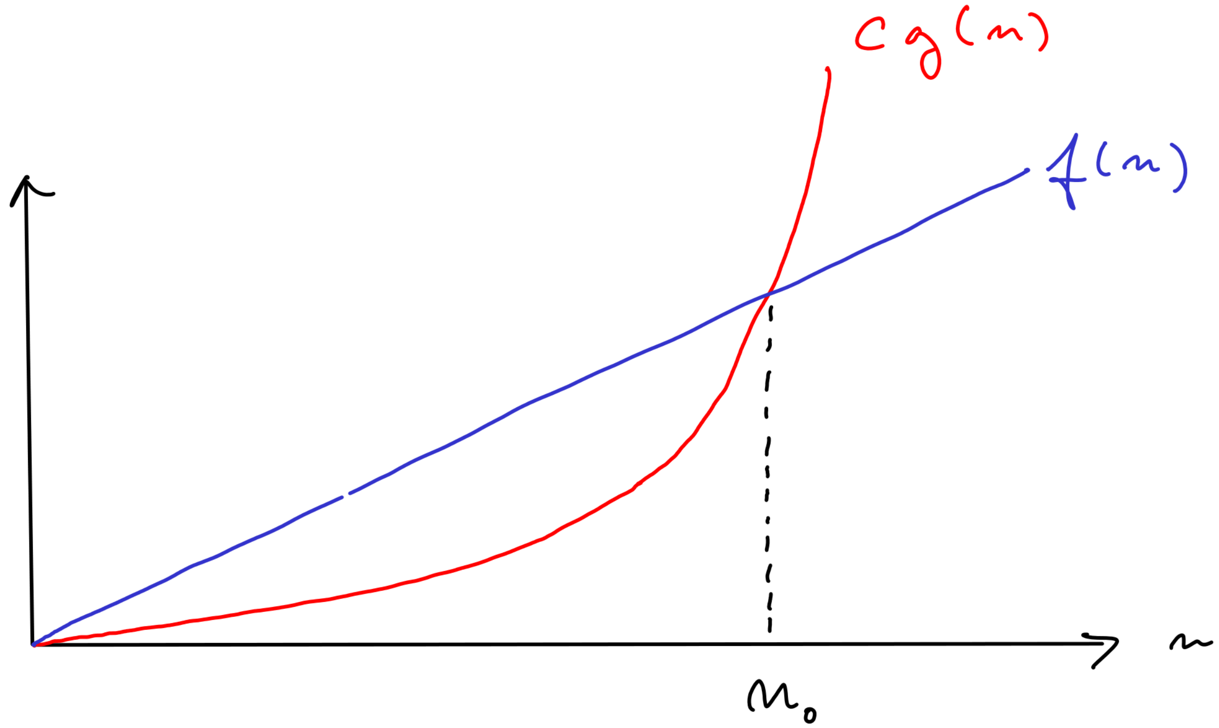
Lecture 01-03

Asymptotic Growth

Def:

- $f(n) \in O(g(n))$

  iff $\exists c, n_0 \geq 0$ s.t. $0 \leq f(n) \leq c\,g(n)$ $\forall n \geq n_0$



- $f(n) \in \Omega(g(n))$

  iff $g(n) \in O(f(n))$

- $f(n) \in \Theta(g(n))$

  iff $f(n) \in O(g(n))$ and

  $f(n) \in \Omega(g(n))$

Ex

$$n^3 + 2n \in O(n^3)$$

$$\in \Omega(n^3)$$

$$\in \Theta(n^3)$$

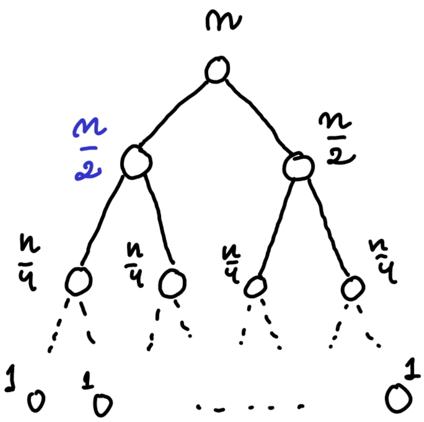
EXAMPLE OF DIVIDE & CONQUER

Divide problem in two equally sized sub-problems.
Solve the sub-problems, and combine their solutions in 1 time unit.
What is the running time?

$2^i = n$

def. $\Downarrow$

$i = \log n$



$1 = 2^0$

$2 = 2^1$

$4 = 2^2$

$\vdots$

$n = 2^{\log n}$

$$T(n) = \sum_{i=0}^{\log n} 2^i = \frac{2^{\log n + 1} - 1}{2 - 1} = 2n - 1.$$

$$\underbrace{= 2T(n/2) + 1.}_{\text{A recurrence}}$$

→ an equation (or inequality) that describes a function in terms of its value on smaller inputs.

Theorem (Master Theorem)

Let $T(n) = a\,T(n/b) + f(n)$

where $a \geq 1$, $b > 1$, $0 < c < 1$, and $\varepsilon > 0$ are constants

$$\begin{cases} \text{If } f(n) = O\left(n^{\log_b a - \varepsilon}\right) & \text{then } T(n) \in \Theta\left(n^{\log_b a}\right) \\ \quad\; f(n) = \Theta\left(n^{\log_b a}\right) & \text{then } T(n) \in \Theta\left(n^{\log_b a} \log n\right) \\ \quad\; f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) & \\ \quad\; \text{and } a f(n/b) \leq c f(n) & \text{then } T(n) \in \Theta(f(n)) \end{cases}$$

Lemma If $T(1) = \Theta(1)$ then

$$T(n) = \Theta\left(n^{\log_b a}\right) + \sum_{i=0}^{\log_b n - 1} a^i f(nb^{-i})$$

$\Rightarrow$ No longer a recurrence equation

Proof (by iteration method)

$$
\begin{aligned}
T(n) &= a\,T(nb^{-1}) + f(n) \\
&= a\left(a\,T(nb^{-2}) + f(nb^{-1})\right) + f(n) \\
&= a^2\,T(nb^{-2}) + f(n) + a\,f(nb^{-1}) \\
&\ \ \vdots \\
&= a^{\log_b n}\,T(1) + \sum_{i=0}^{\log_b n - 1} a^i f(nb^{-i}) \\
&= \Theta\left(n^{\log_b a}\right) + \underbrace{\sum_{i=0}^{\log_b n - 1} a^i f(nb^{-i})}_{=:\, g(n)}
\end{aligned}
$$

Motivating example (special case)

$f(n) = n^c, \quad c > 0$ const.

$$g(n) = \sum_{i=0}^{\log_b n - 1} a^i \cdot \left(\frac{n}{b^i}\right)^c = n^c \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^c}\right)^i$$

The recurrence depends on c, and a "critical point" occurs when $\frac{a}{b^c} = 1$, i.e. when $c = \log_b a$.

If $c = \log_b a$ then $g(n) = n^c \cdot \log_b(n)$

Proof of the Master Theorem

NB!

$$\log_b(x)\log_b(y) = \log_b(x)\log_b(y)$$

$$\log_b\left(x^{\log_b(y)}\right) = \log_b\left(y^{\log_b(x)}\right)$$

$$x^{\log_b(y)} = y^{\log_b(x)}$$

Case 1

$$f(n) \in \Theta(n^{\log_b a})$$

$$g(n) = \Theta\left(\sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b(a)} \sum_{i=0}^{\log_b(n-1)} a^i \cdot a^{\log_b\left(\frac{1}{b^i}\right)}\right)$$

$$= \Theta\left(n^{\log_b(a)} \sum_{i=0}^{\log_b(n-1)} a^i \cdot a^{-i}\right)$$

$$= \Theta\left(n^{\log_b(a)} \log n\right)$$

Case 2 $\quad f(n) = O(n^{\log_b a - \varepsilon})$

$$g(n) = O\left(\sum_{i=0}^{\log_b n - 1} a^i \cdot \left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\log_b n - 1} b^{\varepsilon i}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \cdot n^{\varepsilon}\right)$$

$\square$

Case 3

$$f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \quad \text{and}$$

$$a f(n/b) \leq c f(n) \quad \text{for some} \quad 0 < c < 1.$$

$$\Rightarrow a^i f(n/b^i) \leq c^i f(n)$$

$$g(n) = \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$$

$$\leq f(n) \sum_{i=0}^{\infty} c^i$$

$$= O(f(n))$$

Example 1

$$T(n) = 2T(n/2) + 1$$

$$\left.\begin{array}{l} a = 2 \\ b = 2 \end{array}\right\} \log_b a = 1$$

$$f(n) = 1 \in O\left(n^{\log_b a - \varepsilon}\right) \text{ for any } 0 < \varepsilon < 1.$$

$$T(n) = \Theta(n)$$

Example 2

$$T(n) = 9T(n/3) + n$$

$$\log_b a = \log_3 9 = 2$$

$$f(n) \in O(n^{\log_b a - \varepsilon})$$

$$T(n) \in \Theta(n^2)$$

$$\boxed{b^i = n \overset{\text{def.}}{\Longleftrightarrow} i = \log_b n}$$

Example 3

$$T(n) = T(2n/3) + 1$$

$$\log_b a = \log_{2/3} 1 = 0$$

$$f(n) \in \Theta\left(n^{\log_b a}\right)$$

$$T(n) = \Theta\left(n^{\log_b a} \log n\right) = \Theta(\log n)$$