Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018
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Lecture 8 – Routing and Location Problems

- Routing and Location Applications
  To identify applications of routing and location problems in logistics
- Minimum Spanning Tree Problem
  To apply Prim's greedy algorithm to solve MST problems
  To identify, model and solve minimum spanning tree problems
- Travelling Salesman Problem
  To identify, model and solve asymmetric and symmetric travelling salesman problems
- Facility Location Problems
  To identify, model and solve facility location problems

Additional Reading

Section 9.3 (about TSP) of (Taha, 2011).

Travelling Salesman Problem. Online at: http://www.math.uwaterloo.ca/tsp/, Accessed November 2017.

VRP Web. Online at: http://neo.lcc.uma.es/vrp/, Accessed November 2017.

TSP at the Movies: Yondu's Dart Problem. Martin J. Chlond. INFORMS Transactions on Education, Vol. 16, No. 3, pp. 110-111, 2016.

Look Here Comes the Library Van! Optimising the Timetable of the Library Service on the Isle of Wight . T. Rienthong, A. Walker, T. Bektas. OR Insight, Vol. 24, No. 1, pp. 49-62, 2011.

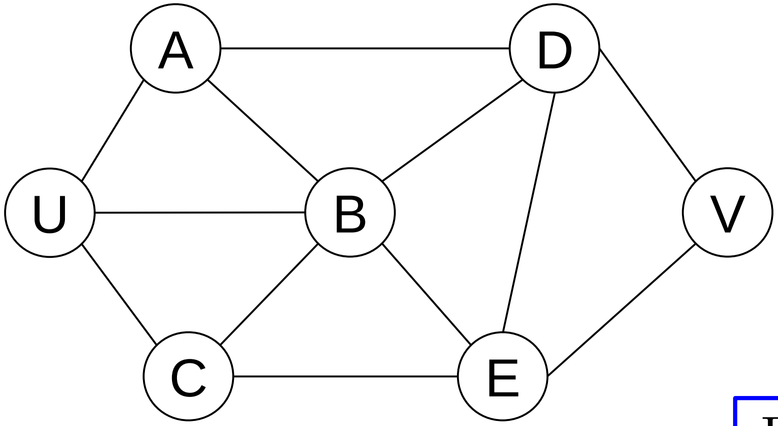
Integer Programming Formulation of Travelling Salesman Problems. C.E. Miller, A.W. Tucker, R.A. Zemlin. Journal of the ACM, Vol. 7, No. 4, pp. 326-329, 1960.

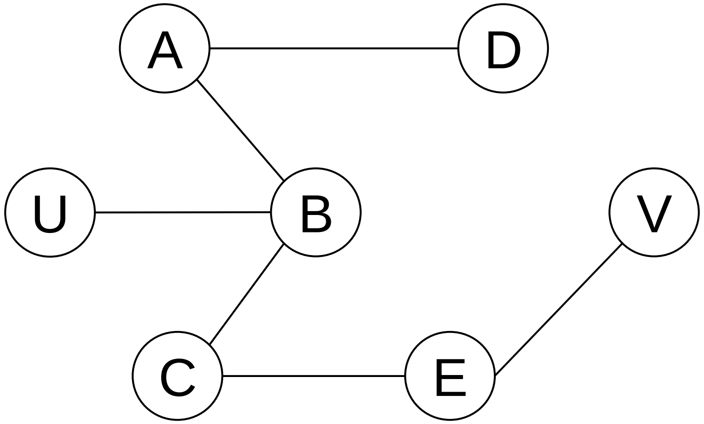
Routing and Location Applications

- Many practical applications in logistics and communication networks involve routing and location optimization problems where a set of points should be visited or served.
- In routing problems, common constraints include visiting all the points to be served and the elimination of sub-cycles or sub-tours. Examples are TSP and VRP.
- In location problems, common constraints include locating serving points in allowed positions and ensuring the points to be served are all considered. Examples are FLP and QAP.
- Routing and location problems are typically represented as networks, so they can also be seen as network optimization problems.

Minimum Spanning Tree Problem

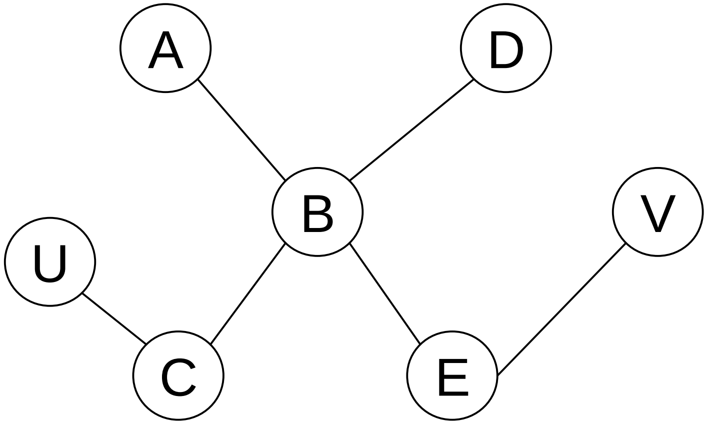
Given a connected graph G(V,E)





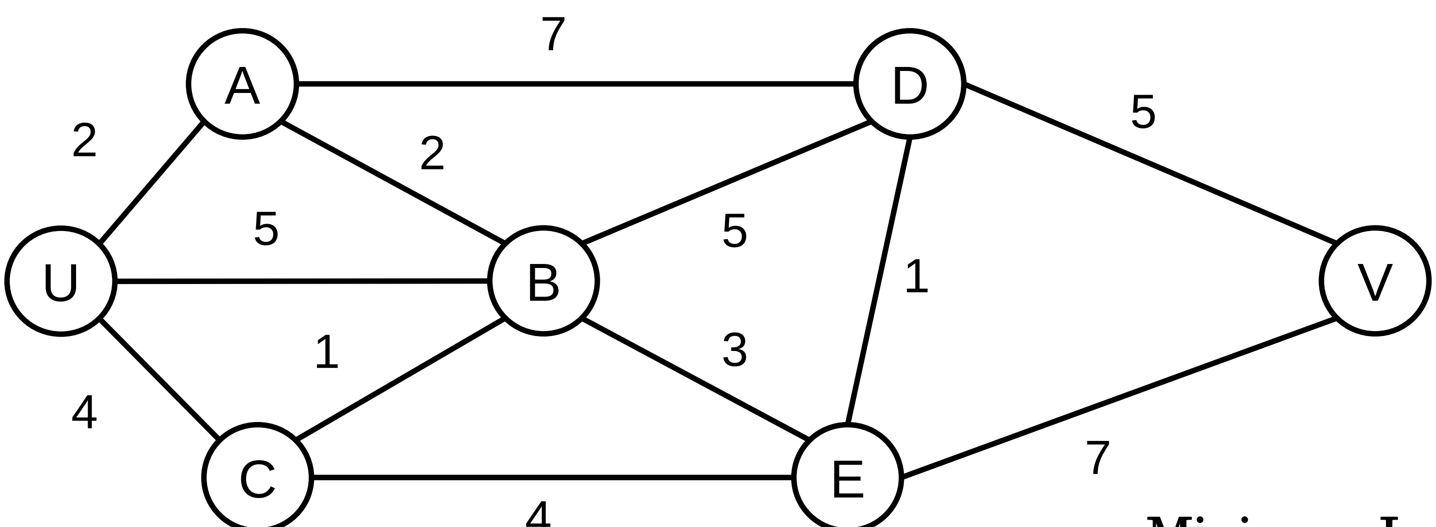
For N vertices, the spanning tree has N–1 edges.

A spanning tree is a sub-graph of G that has no cycles and contains all vertices of G. Two spanning trees for the above graph are shown on the right.



For a weighted connected graph G, a Minimum Spanning Tree (MST) is a spanning tree in with edges sum the minimum weight. Then, finding a MST is selecting the edges from G that form a spanning tree of the minimum weight.

Like for other optimization models (e.g. network flow problems), there are several algorithms for finding MST. One of those is the (greedy) Prim's algorithm.



Minimum Length: 14

Given:

Set of nodes V and a set of edges E for a graph G

N = |V| is the number of vertices or nodes

Wij is the weight of the edge between nodes i and j

The problem is to select N –1 edges from E whose sum of weights is the minimum and the selected edges do not form cycles.

Example. Sketch of a MST with N = 7 nodes and their edges.

$$\begin{bmatrix} W_{12} \\ W_{13} \\ W_{23} \\ \vdots \\ W_{57} \\ W_{67} \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{13} \\ X_{23} \\ \vdots \\ X_{57} \\ X_{67} \end{bmatrix} = Z$$

Select the number of edges for a spanning tree

$$X_{12} + X_{13} + \cdots + X_{57} + X_{67} = N - 1$$

Setting the constraints to avoid the formation of cycles is more complicated.

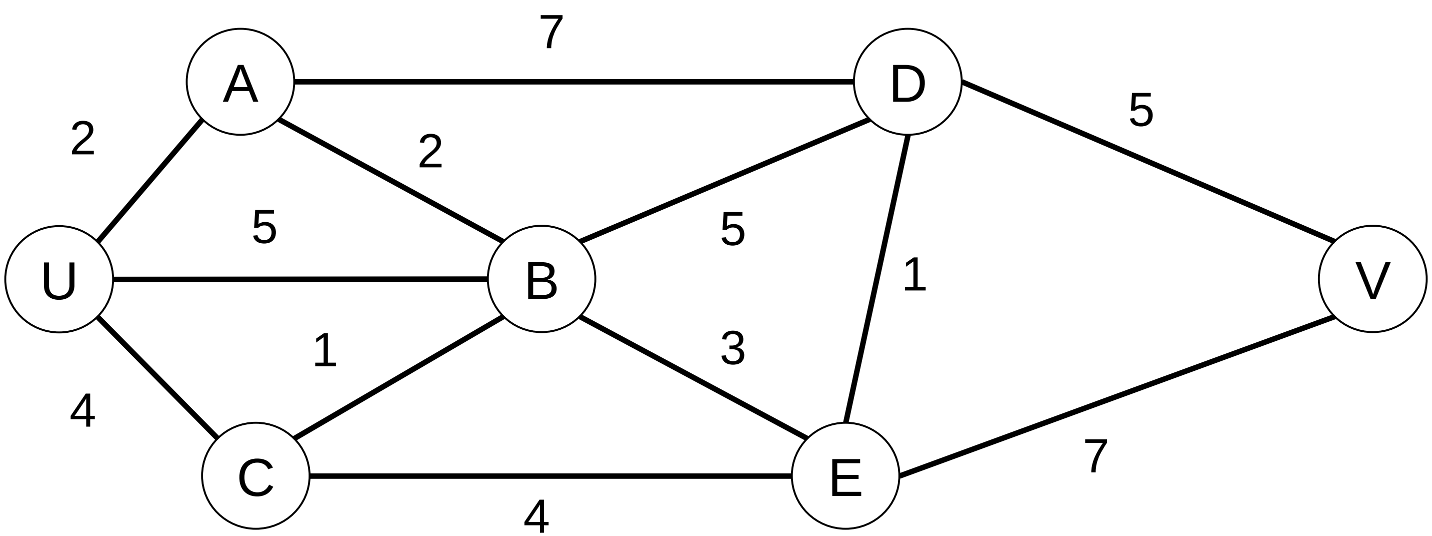
Example. Solve the MST Problem for the given graph.

Minimize: $Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} +$
$5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV}$

Subject to: $x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6$ (1)

All $x_{ij} \in \{0,1\}$ (2)

Solution: $X_{UA} = X_{UC} = X_{AB} = X_{BC} = X_{BE} = X_{DE} = 1$ and all other are 0.

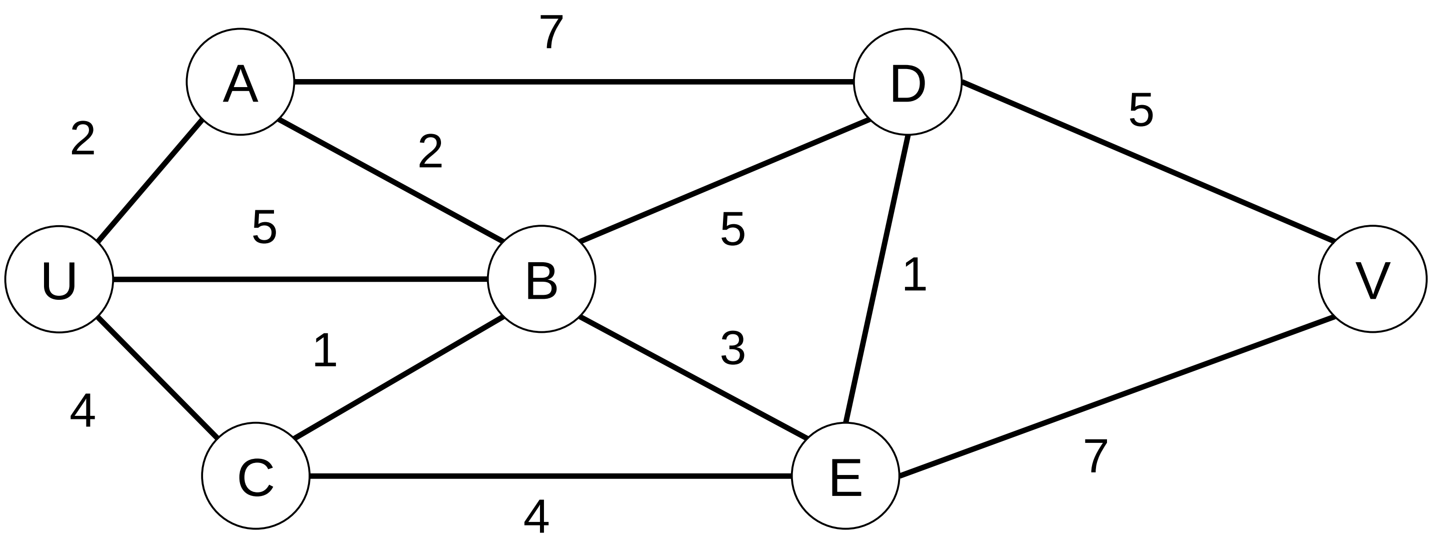


Z = 13

Example (cont). Solve the MST Problem for the given graph.

$$\begin{aligned}
\text{Minimize}: \quad & Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} + \\
& \quad 5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV} \\
\text{Subject to}: \quad & x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6 \qquad (1) \\
& x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{BC} \le 3 \qquad (2) \\
& \text{All } x_{ij} \in \{0,1\} \qquad (3)
\end{aligned}$$

Solution: $X_{UA} = X_{AB} = X_{BC} = X_{BE} = X_{CE} = X_{DE} = 1$ and all other are 0.



Z = 13

Example (cont). Solve the MST Problem for the given graph.

$$\text{Minimize: } Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} + 5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV}$$

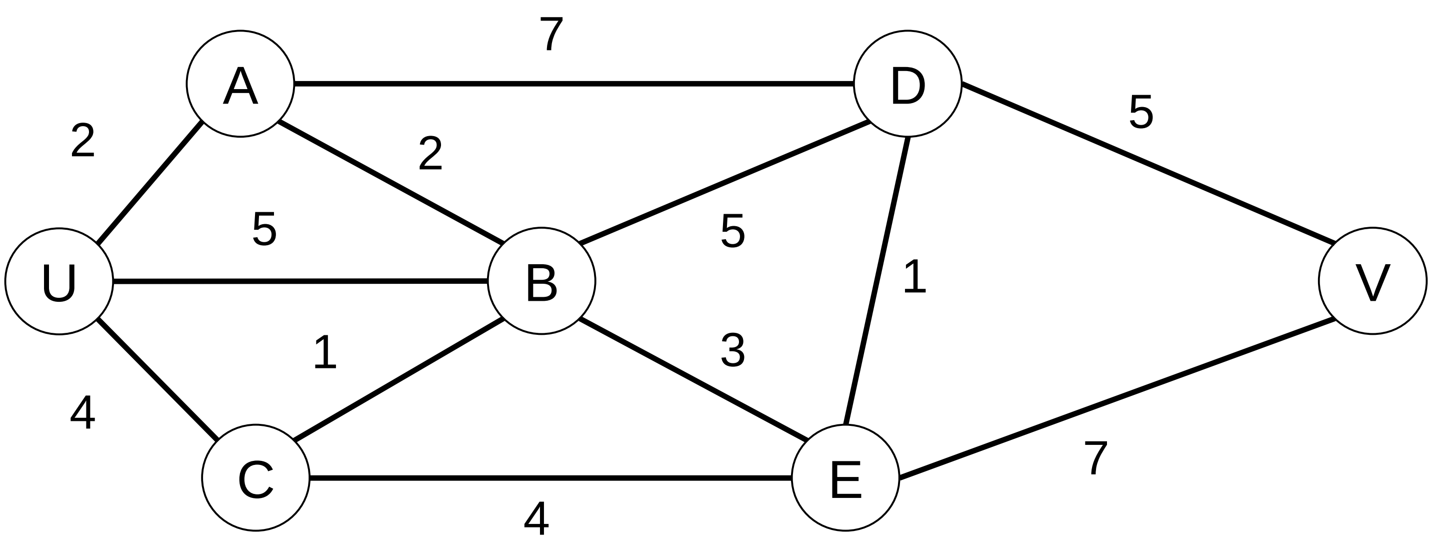
$$\text{Subject to: } x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6 \quad (1)$$

$$x_{UA} + x_{UC} + x_{AB} + x_{BC} \leq 3 \quad (2)$$

$$x_{BC} + x_{BE} + x_{CE} \leq 2 \quad (3)$$

$$\text{All } x_{ij} \in \{0,1\} \quad (4)$$

Solution: $X_{UA} = X_{UB} = X_{AB} = X_{BC} = X_{BE} = X_{DE} = 1$ and all other are 0.

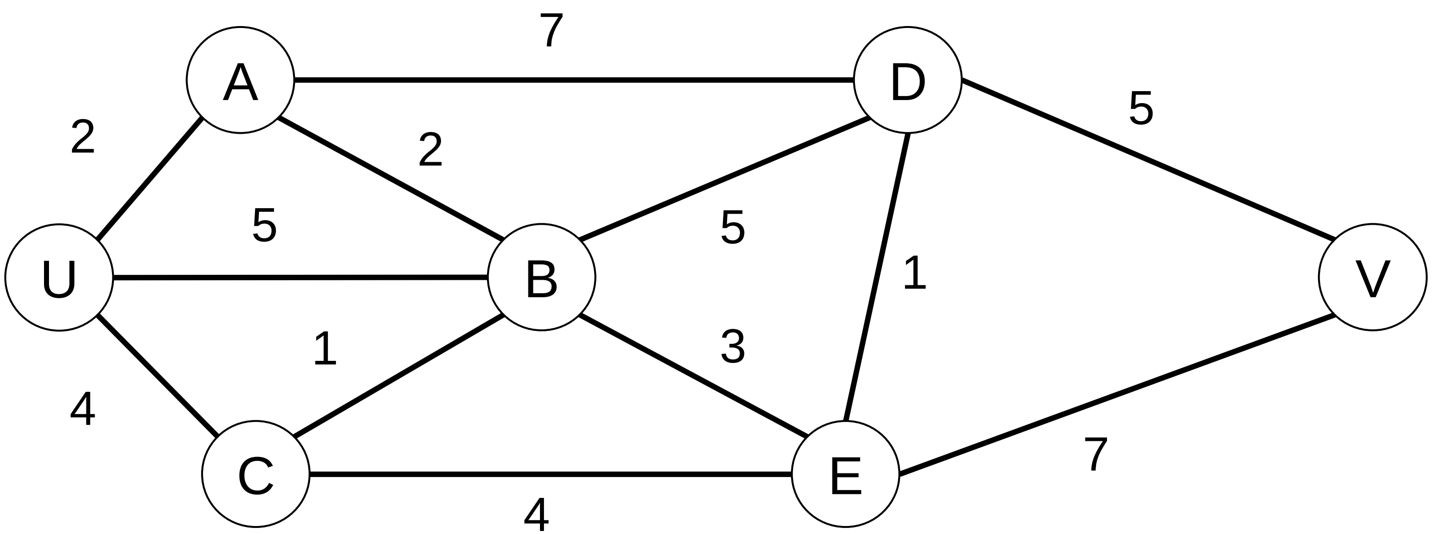


Z = 14

Example (cont). Solve the MST Problem for the given graph.

$$\begin{aligned}
\text{Minimize: } & Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} + \\
& \quad 5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV} \\
\text{Subject to: } & x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6 \qquad (1) \\
& x_{UA} + x_{UC} + x_{AB} + x_{BC} \le 3 \qquad (2) \\
& x_{BC} + x_{BE} + x_{CE} \le 2 \qquad (3) \\
& x_{UA} + x_{UB} + x_{AB} \le 2 \qquad (4) \\
& \text{All } x_{ij} \in \{0,1\} \qquad (5)
\end{aligned}$$

Solution: $X_{UA} = X_{AB} = X_{BC} = X_{BE} = X_{BD} = X_{DE} = 1$ and all other are 0.



Z = 14

Example (cont). Solve the MST Problem for the given graph.

$$\text{Minimize: } Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} + 5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV}$$

Subject to:

$$x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6 \quad (1)$$

$$x_{UA} + x_{UC} + x_{AB} + x_{BC} \leq 3 \quad (2)$$

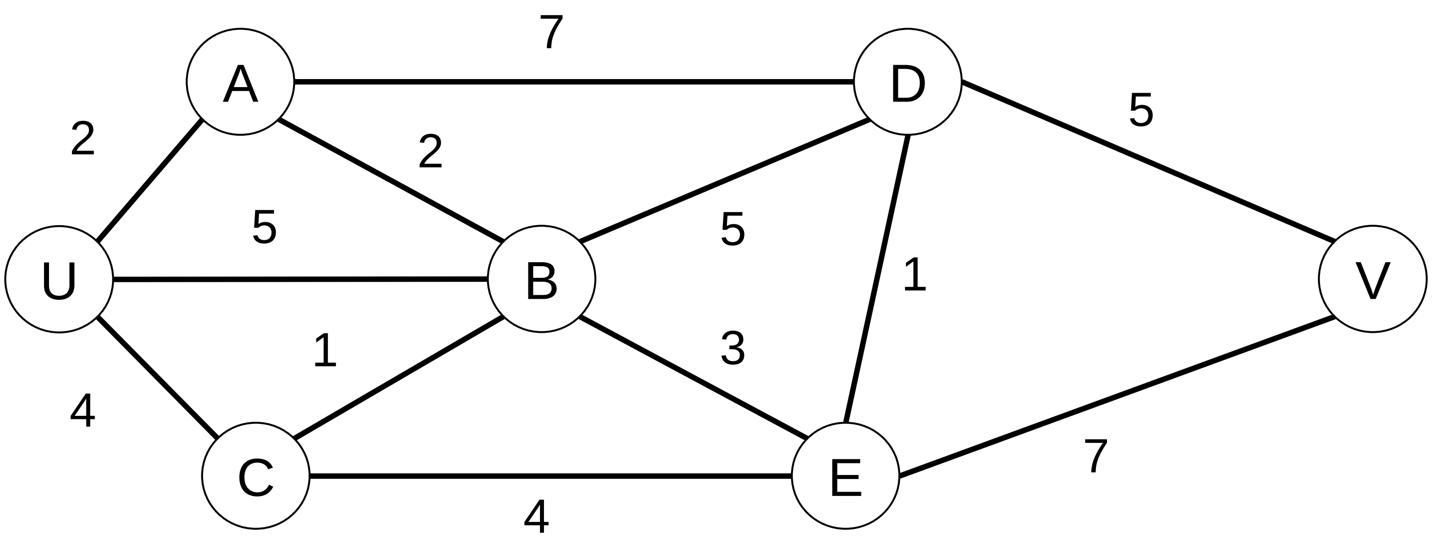
$$x_{BC} + x_{BE} + x_{CE} \leq 2 \quad (3)$$

$$x_{UA} + x_{UB} + x_{AB} \leq 2 \quad (4)$$

$$x_{BD} + x_{BE} + x_{DE} \leq 2 \quad (5)$$

$$\text{All } x_{ij} \in \{0,1\} \quad (6)$$

Solution:
$X_{UA} = X_{AB} = X_{BC} = X_{BE} = X_{DE} = X_{DV} = 1$ and all other are 0.



Z = 14

BIP Model for the MST Problem

$$\text{Minimize:} \quad Z = \sum_{i=1}^{N}\sum_{j=1}^{N} W_{ij} X_{ij}$$

$$\text{Subject to:} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} X_{ij} = N-1 \qquad (1)$$

$$\sum_{ij\in(S,S)} X_{ij} \le |S|-1 \quad \forall S \in V \qquad (2)$$

$$X_{ij} \in \{0,1\} \text{ for each edge between nodes } i \text{ and } j \qquad (3)$$

Where:

N:

V:

W_{ij}:

X_{ij}:

S:

$$\text{Minimise:} \quad Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2x_{AB} + 7x_{AD} + x_{BC} + 5x_{BD} + 3x_{BE} + 4x_{CE} + x_{DE} + 5x_{DV} + 7x_{EV}$$

$$\text{Subject to:} \quad x_{UA} + x_{UB} + x_{UC} + x_{AB} + x_{AD} + x_{BC} + x_{BD} + x_{BE} + x_{CE} + x_{DE} + x_{DV} + x_{EV} = 6 \qquad (1)$$

$$x_{UA} + x_{UC} + x_{AB} + x_{BC} \le 3 \qquad (2)$$

$$x_{BC} + x_{BE} + x_{CE} \le 2 \qquad (3)$$

$$x_{UA} + x_{UB} + x_{AB} \le 2 \qquad (4)$$

$$x_{BD} + x_{BE} + x_{DE} \le 2 \qquad (5)$$

$$\text{All } x_{ij} \in \{0,1\} \qquad (6)$$

Travelling Salesman Problem (TSP)

The Asymmetric TSP

Given:

A directed graph G with a set of nodes V and set of edges E

$N = |V|$ is the number of vertices or nodes

Each node represents a city

The distance between cities i and j is D_{ij}

The problem is to find a tour to visit all the cities so that the total travelled distance is minimized.

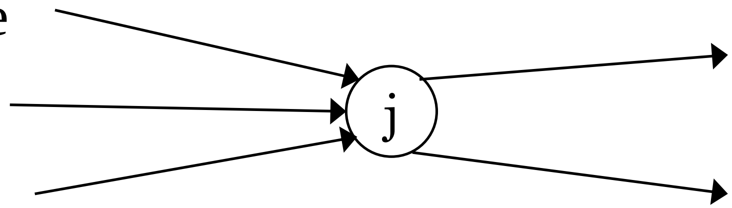
Two sets of constraints:

- Ensure that each node is visited once
- Ensure there are no sub-tours

Ensure each node is visited once.

One edge into the node

$$\sum_{i=1}^{N} X_{ij} = 1$$



One edge out of the node

$$\sum_{k=1}^{N} X_{jk} = 1$$

Setting the constraints to avoid the formation of sub-tours involves setting a constraint for each subset of edges that can potentially form a cycle.

Example. Sketch of an ATSP with N = 5 nodes and their edges.

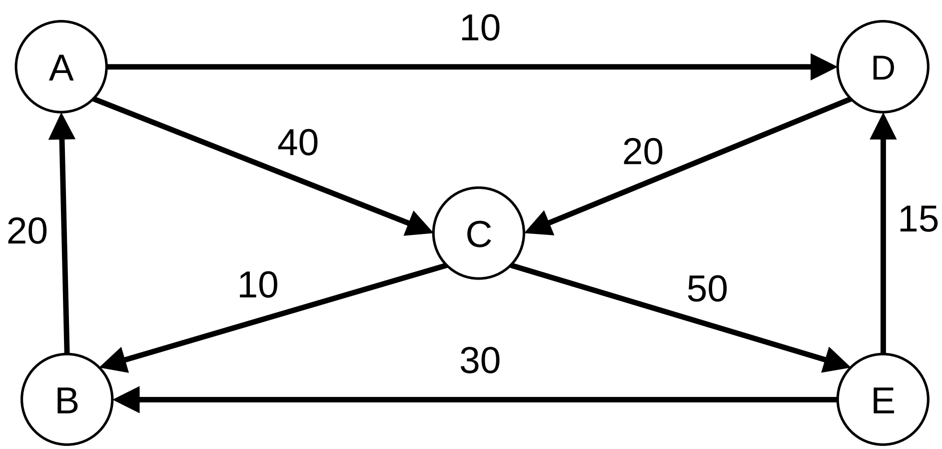
$$\begin{bmatrix} D_{13} \\ D_{14} \\ \vdots \\ D_{54} \end{bmatrix} \begin{bmatrix} X_{13} \\ X_{14} \\ \vdots \\ X_{54} \end{bmatrix} = Z$$

For each of the 5 nodes

$$\sum_{i=1}^{N} X_{ij} = 1 \quad \text{and} \quad \sum_{k=1}^{N} X_{jk} = 1$$

Plus the set of sub-tour elimination constraints.

Example. Solving the Asymmetric TSP, find shortest length tour.



Solution:
$X_{AD} = X_{DC} = X_{CE} = X_{EB} = X_{BA} = 1$ and all other are 0.

BIP Model for Solving the Above Asymmetric TSP

Minimize: $Z = 40x_{AC} + 10x_{AD} + 20x_{BA} + 10x_{CB} + 50x_{CE} + 20x_{DC} + 30x_{EB} + 15x_{ED}$

Subject to:

$x_{BA} = 1$ (1) $\quad$ $x_{AC} + x_{AD} = 1$ (6)

$x_{CB} + x_{EB} = 1$ (2) $\quad$ $x_{BA} = 1$ (7)

$x_{AC} + x_{DC} = 1$ (3) $\quad$ $x_{CB} + x_{CE} = 1$ (8)

$x_{AD} + x_{ED} = 1$ (4) $\quad$ $x_{DC} = 1$ (9)

$x_{CE} = 1$ (5) $\quad$ $x_{EB} + x_{ED} = 1$ (10)

All $x_{ij} \in \{0,1\}$

Minimum Length: 130

BIP Model for the Asymmetric TSP

Minimize: $\mathrm{Z} = \sum_{(i,j)\in E} D_{ij} X_{ij}$

Subject to: $\sum_{i=1}^{N} X_{ij} = 1$ for each node/city $j = 1 \ldots N$ (1)

$\sum_{k=1}^{N} X_{jk} = 1$ for each node/city $j = 1 \ldots N$ (2)

$\sum_{ij \in (S,S)} X_{ij} \leq |S| - 1 \quad \forall S \in V$ (3)

$X_{ij} \in \{0,1\}$ for each edge between nodes i and j (4)

Where:

N :

V :

D_{ij} :

X_{ij} :

S :

Adding all the sub-tour elimination constraints can make the model more difficult because potentially there is an exponential number of constraints required. So, in practice is often desirable to add these constraints as sub-tours arise during the optimization process.

The Symmetric TSP

Given:

A undirected graph G with a set of nodes V and set of edges E

$N = |V|$ is the number of vertices or nodes

Each node represents a city

Edges are undirected

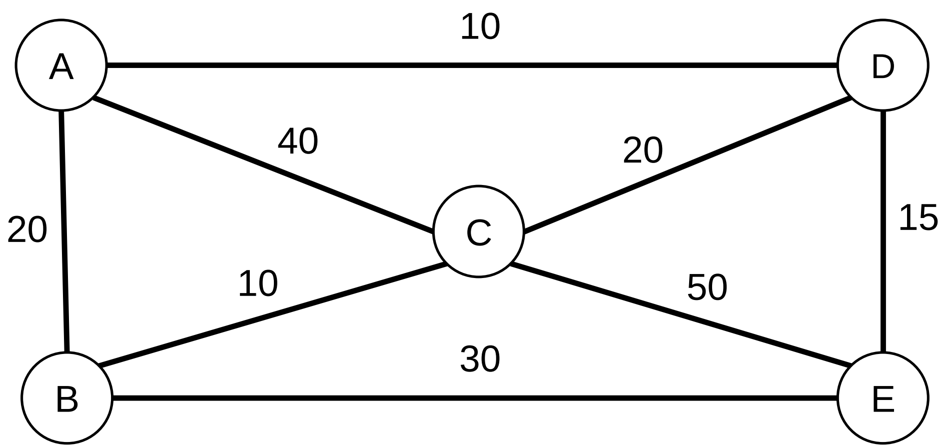
The distance between cities i and j is $D_{ij} = D_{ji}$

The problem is to find a tour to visit all the cities so that the total travelled distance is minimized.

Two sets of constraints:

- Ensure that each node is visited once
- Ensure there are no sub-tours

Example. Solving the Symmetric TSP, find shortest length tour.



Solution:
$X_{AD} = X_{DE} = X_{EB} = X_{BC} = X_{CA} = 1$ and all other are 0.

Minimize: $Z = \frac{1}{2}\begin{pmatrix}(20x_{AB} + 40x_{AC} + 10x_{AD}) + (20x_{AB} + 10x_{BC} + 30x_{BE}) + \\ (40x_{AC} + 10x_{BC} + 20x_{CD} + 50x_{CE}) + (10x_{AD} + 20x_{CD} + 15x_{DE}) + (30x_{BE} + 50x_{CE} + 15x_{DE})\end{pmatrix}$

Subject to: $x_{AB} + x_{AC} + x_{AD} = 2 \quad (1)$

$x_{AB} + x_{BC} + x_{BE} = 2 \quad (2)$

$x_{AC} + x_{BC} + x_{CD} + x_{CE} = 2 \quad (3)$

$x_{AD} + x_{CD} + x_{DE} = 2 \quad (4)$

$x_{BE} + x_{CE} + x_{DE} = 2 \quad (5)$

All $x_{ij} \in \{0,1\}$

Minimum Length: 105

BIP Model for the Symmetric TSP

$$\text{Minimize:} \quad Z = \frac{1}{2}\sum_{i=1}^{N}\sum_{e \in E_i} D_e X_e$$

$$\text{Subject to:} \quad \sum_{e \in E_i} X_e = 2 \quad \text{for each node/city } i = 1 \ldots N \qquad (1)$$

$$\sum_{e \in (S,S)} X_e \le |S| - 1 \quad \forall S \in V \qquad (2)$$

$$X_{ij} \in \{0,1\} \quad \text{for each edge between nodes } i \text{ and } j \qquad (3)$$

Where:

N :

V :

D_{ij} :

X_{ij} :

E_i :

S :

Like for the Asymmetric TSP, adding all the sub-tour elimination constraints in the Symmetric TSP can make the model more difficult because potentially there is an exponential number of constraints required. So, in practice is often desirable to add these constraints as sub-tours arise during the optimization process.

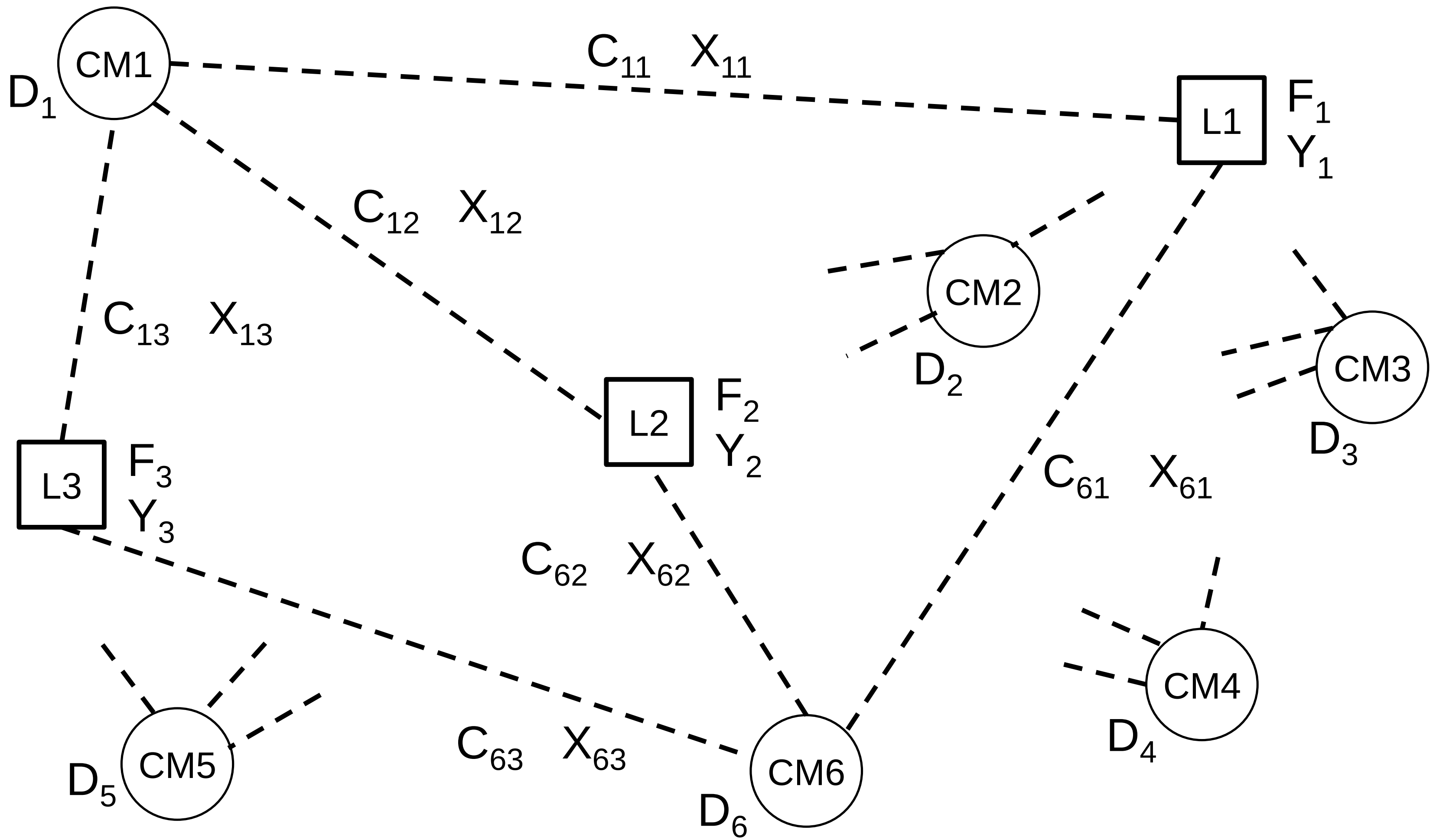
Facility Location Problem (FLP)

Given:

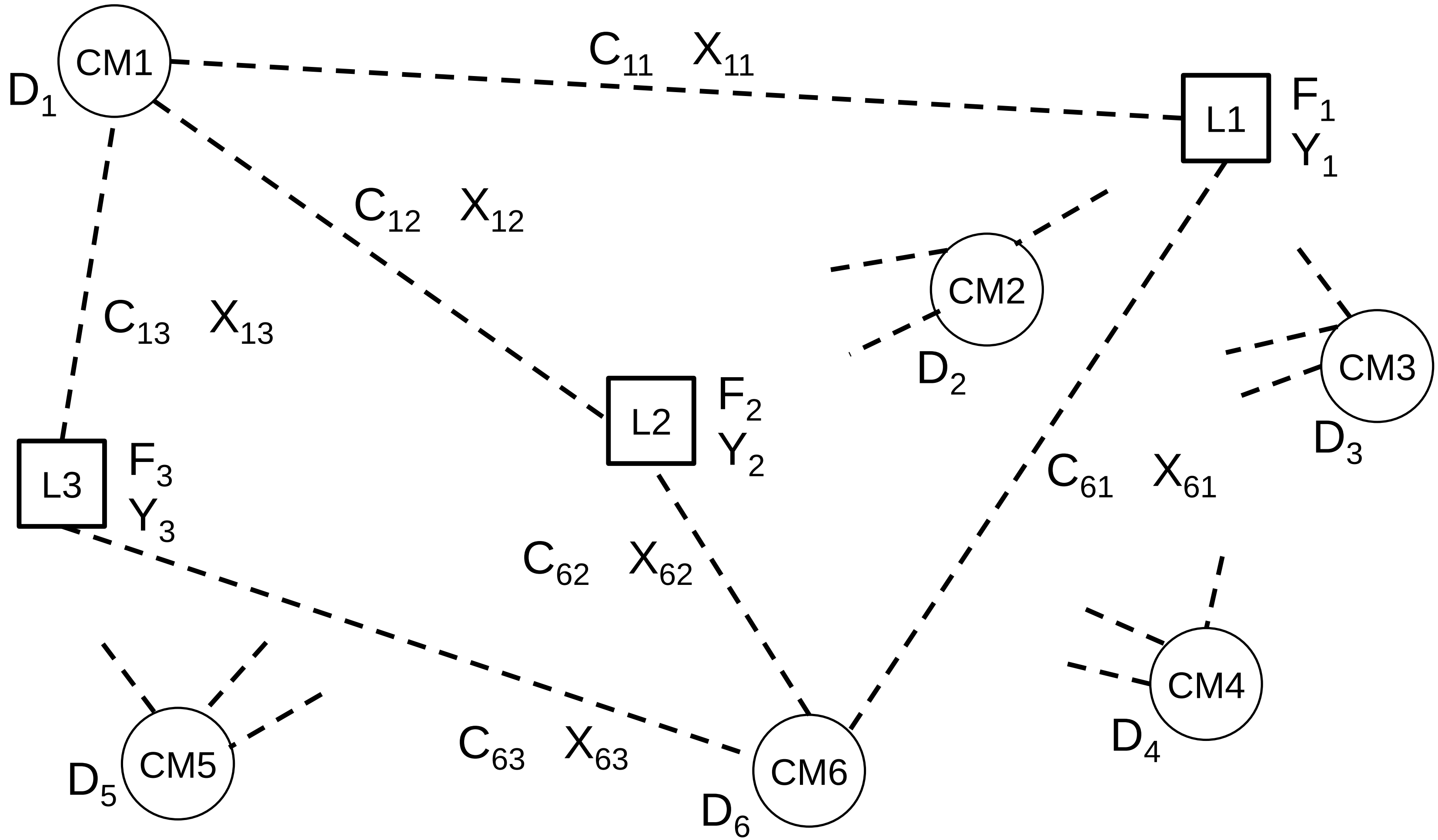
- A set of N customers each with given demand D_i
- A set of M potential locations for setting facilities to serve the customers demand
- The cost for satisfying the demand for customer i from facility located in location j is given by Cij
- The cost for setting a facility in location j is given by F_j

The problem is to choose the set of locations in which to set the facilities in order to satisfy the customers demand at the minimum overall cost.

Example. Sketch for a FLP with N=6 customers and M=3 locations.

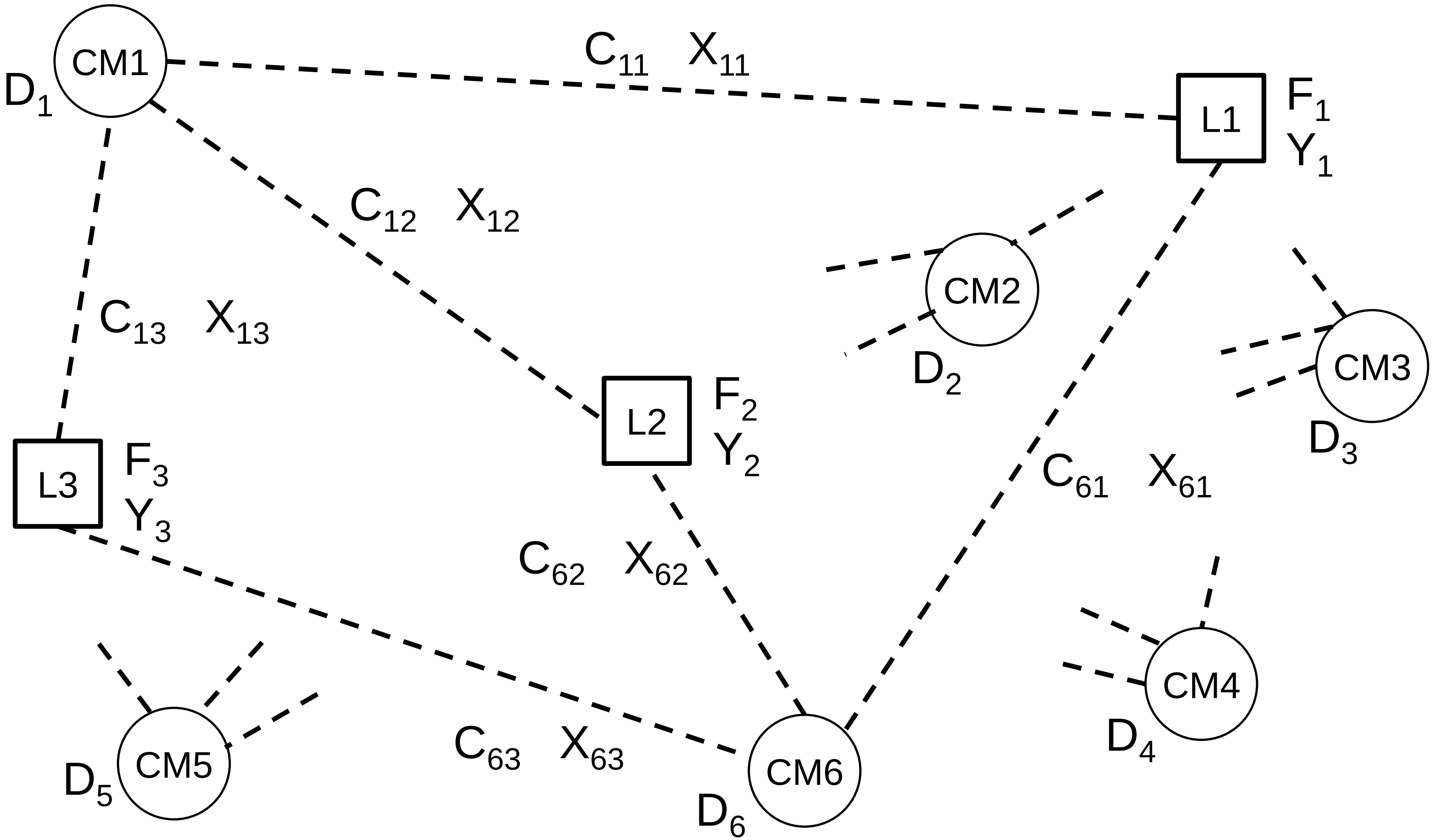


Example (cont). Sketch for a FLP with N=6 customers and M=3 locations.



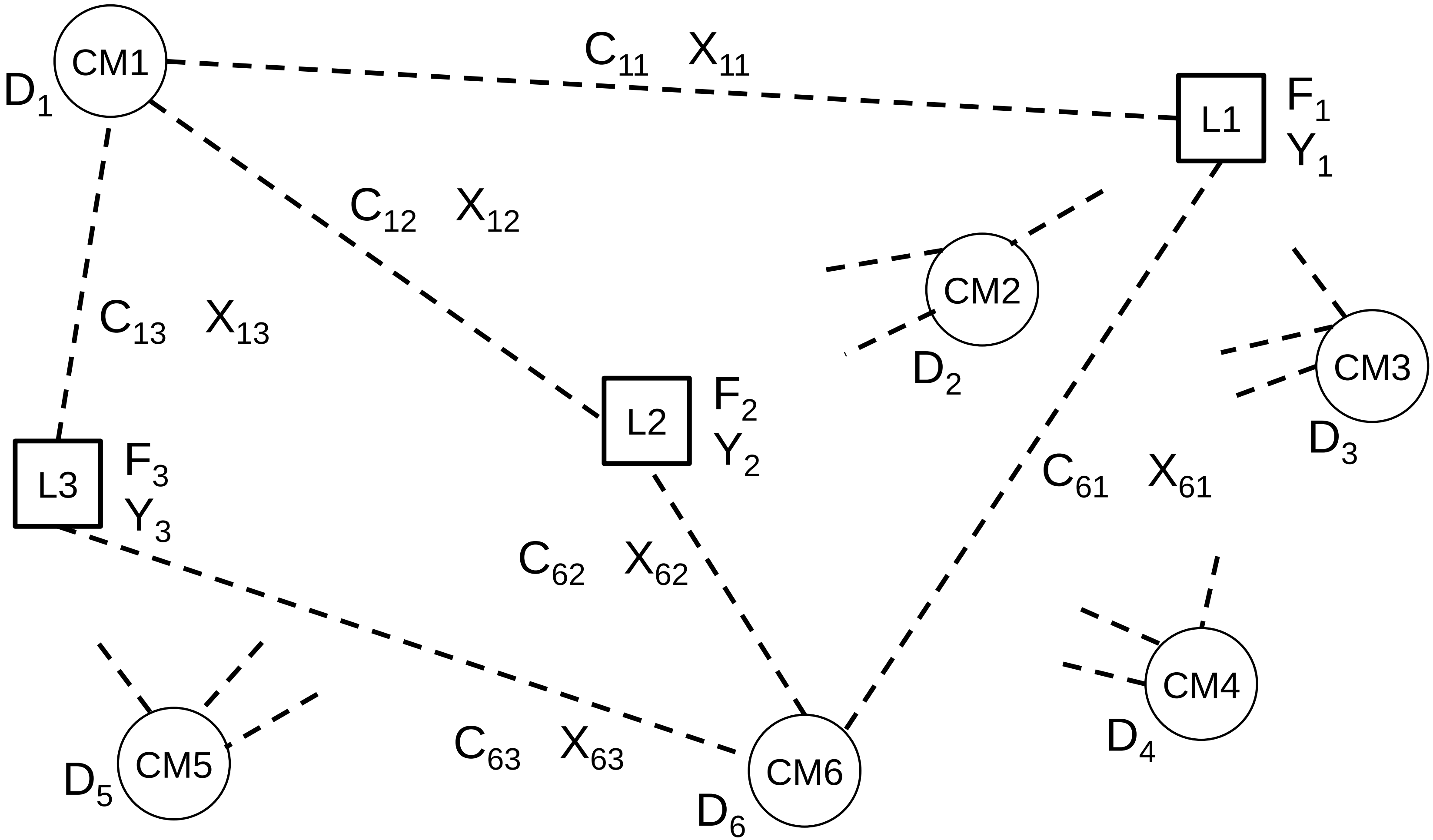
$$\text{Overall Cost:}\quad Z = (F_1Y_1 + F_2Y_2 + F_3Y_3) + (C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13}) + \ldots(C_{61}X_{61} + C_{62}X_{62} + C_{63}X_{63})$$

Example (cont). Sketch for a FLP with N=6 customers and M=3 locations.



$$\text{Demand for Customers}: \quad (X_{11} + X_{12} + X_{13}) = D_1 \quad \text{for each customer } i = 1 \ldots 6$$

Example (cont). Sketch for a FLP with N=6 customers and M=3 locations.



Use facility only if location is selected: $X_{11} \leq D_1 Y_1$ for $i = 1 \ldots 6$ and each $j = 1 \ldots 3$

BIP Model for the Uncapacitated Facility Location Problem

$$\text{Minimize:} \quad Z = \sum_{j=1}^{M} F_j Y_j + \sum_{i=1}^{N} \sum_{j=1}^{M} C_{ij} X_{ij}$$

$$\text{Subject to:} \quad \sum_{j=1}^{M} X_{ij} = D_i \qquad \text{for } i = 1,2,\ldots N \qquad (1)$$

$$X_{ij} - LY_j \leq 0 \quad \text{for } i = 1,2,\ldots N \text{ and } j = 1,2,\ldots M \qquad (2)$$

$$Y_j = 1 \text{ if a facility is located at location } j, 0 \text{ otherwise} \qquad (3)$$

$$X_{ij} = \text{partial demand of customer } i \text{ satisfied from facility in location } j \qquad (4)$$

L is a large enough number, for example $\geq D_i$

Where:

N :

M :

F_j :

D_i :

C_{ij} :

Y_j :

X_{ij} :

L :

Other variations of the FLP problem may specify a limited capacity for the facilities depending on the location and/or a required number of facilities to be installed.

Questions OR Comments