Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018

Dr Dario Landa-Silva

http://www.cs.nott.ac.uk/~pszjds/

Lecture 5 – Network Flow Optimization

- Applications of Network Flow Optimization
  To describe the scope of network optimization problems and their applications
- Transportation Problem
  To identify and develop models for transportation problems
- Minimum Cost Flow Problem
  To identify and develop models for minimum cost flow problems
- Maximum Flow Problem
  To identify and develop models for maximum flow problems
- Shortest Path as a Flow Problem
  To apply the minimum cost flow optimization model to solve shortest path problems

Additional Reading

Sections 9.1, 10.5 and 10.6 of (Hillier and Lieberman, 2015).

Chapter 5 on Network Modelling of (Ragsdale, 2015).

Section 5.3 of (Williams, 2013).

A Comprehensive Empirical Analysis of 16 Heuristics for the Transportation Problem. N. Storozhyshina, F. Pargar, F.J. Vasko. OR Insight, Vol. 24, Issue 1, pp. 63-76, 2011.

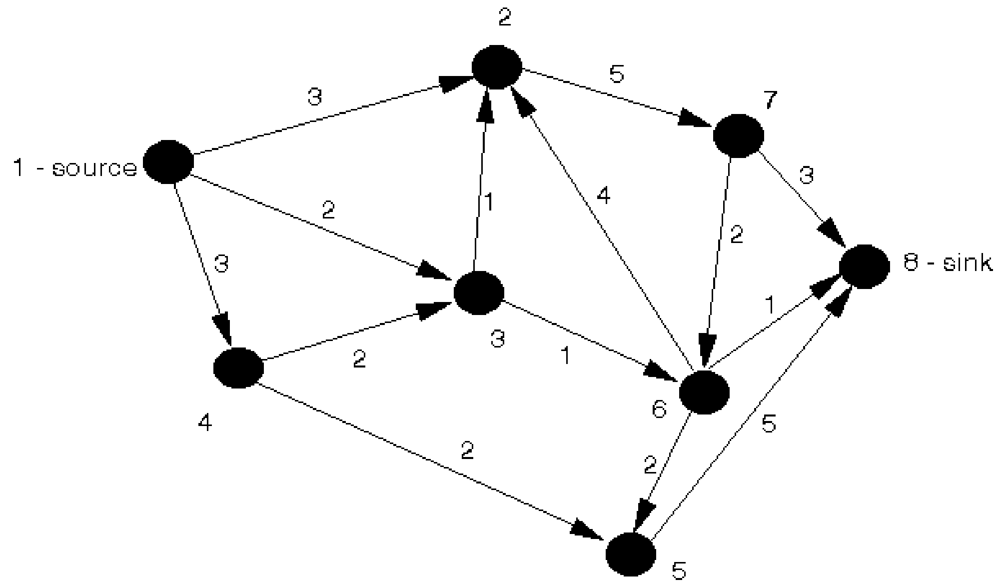
Balancing a Transportation Problem – Is It Really That simple? F.J Vasko, N. Storozhyshinaa. OR Insight, Vol. 24, Issue 3, pp. 205-214, 2011.

Efficient Maximum Flow Algorithms. A.V. Goldberg, R.E. Tarjan. Communications of the ACM, Vol. 57, Issue 8, pp. 82-89, 2014.

Applications of Network Optimization

Network flow optimization problems arise in many real-world applications including:

- Communications
- Transportation
- Distribution
- Project planning
- Circuit design
- Production
- Financial planning



http://people.brunel.ac.uk/~mastjjb/jeb/or/netflow.html

Many network flow optimization problems (although not all) can be formulated as Linear Programming (LP) models.

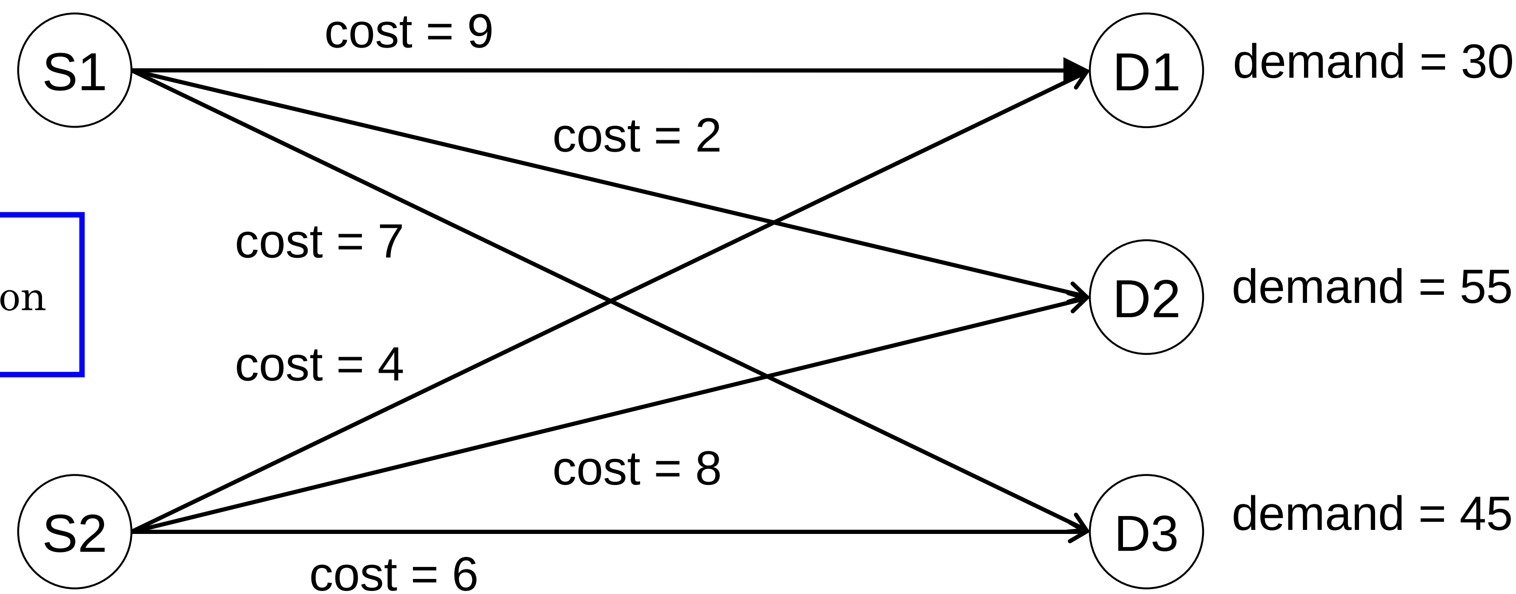
Examples of network flow optimization problems:

- Find the shortest route between two points in a city
- Find the critical path of activities in a large project
- Find the fastest route to send data through a network
- Design the best network for mobile communications
- Design the best layout for an electronic circuit
- Design the best route map for an airline
- Design the best evacuation plan for a city
- Find the maximum cash flow in a financial network
- Find the maximum flow in a water distribution network
- Find the minimum distribution cost in a supply chain

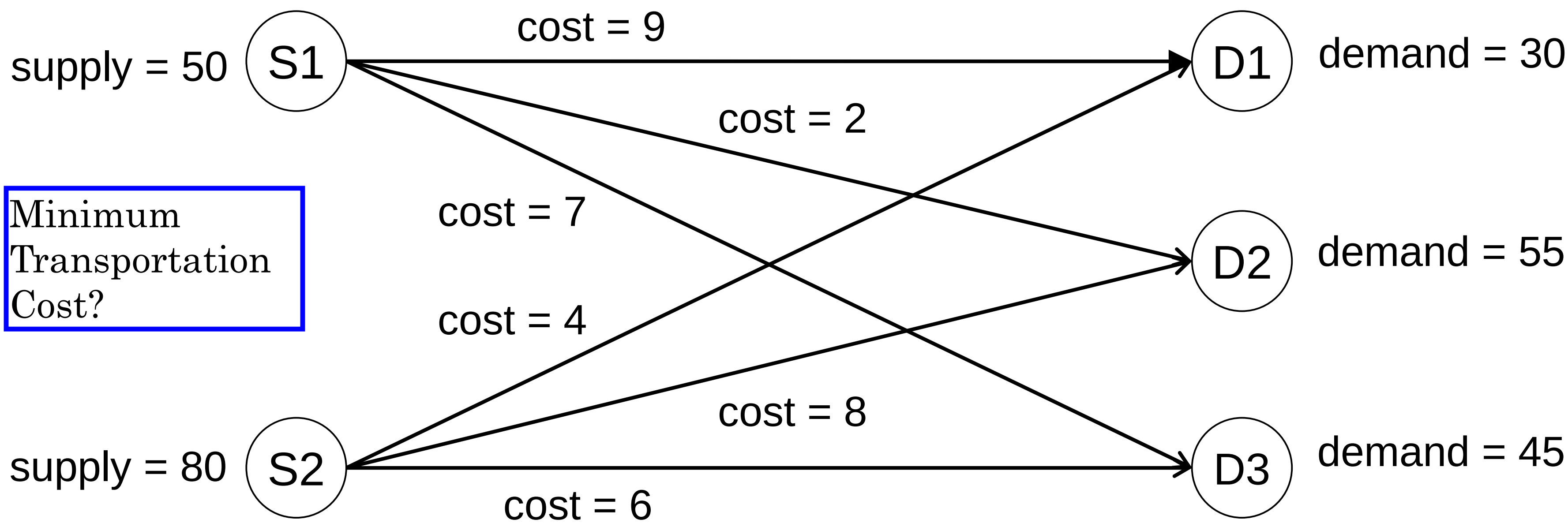
The focus in this type of problems is to optimize the flow of one more commodities.

Transportation Problem

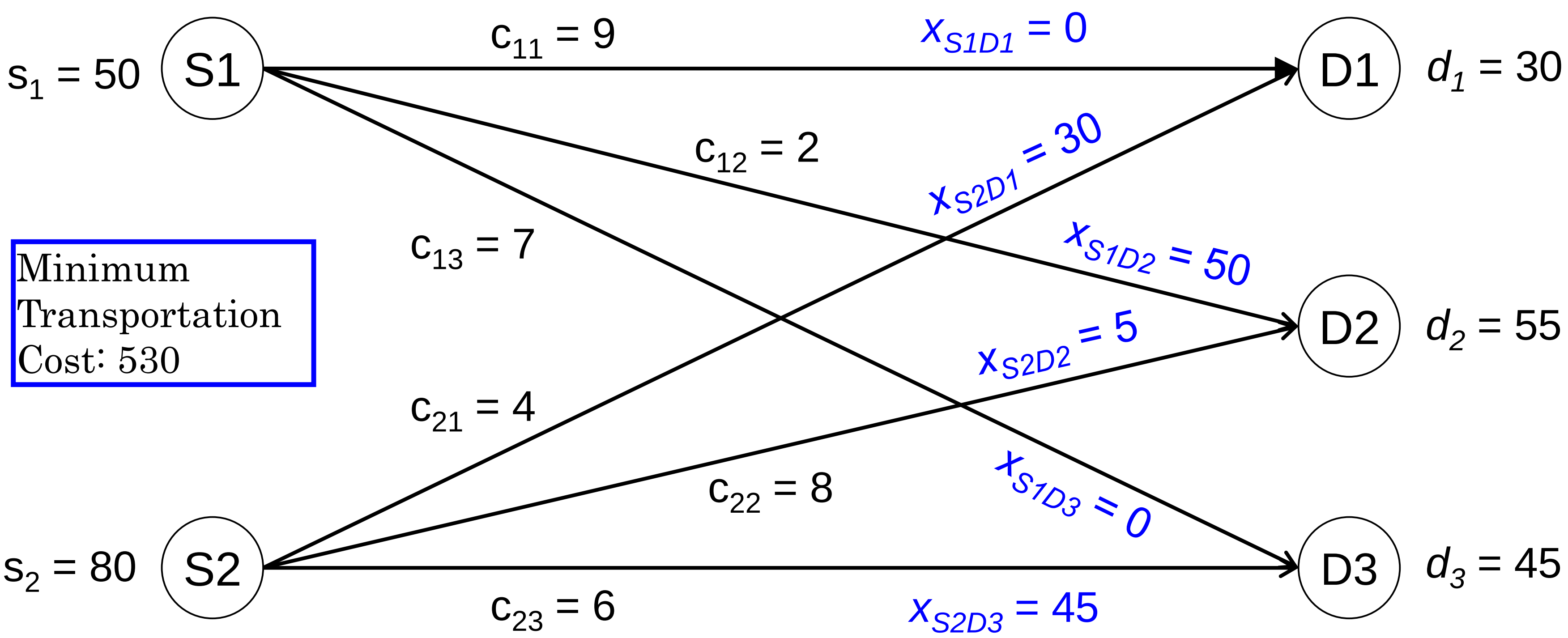
Consider a directed network with a set of supply nodes (sources) and a set of demand nodes (destinations). A given transportation cost per unit of flow and unlimited flow capacity are associated to each arc. The problem is to distribute the entire commodity supply from the sources to the destinations such as to minimize the total transportation cost. Linear Programming can solve transportation problems very efficiently.



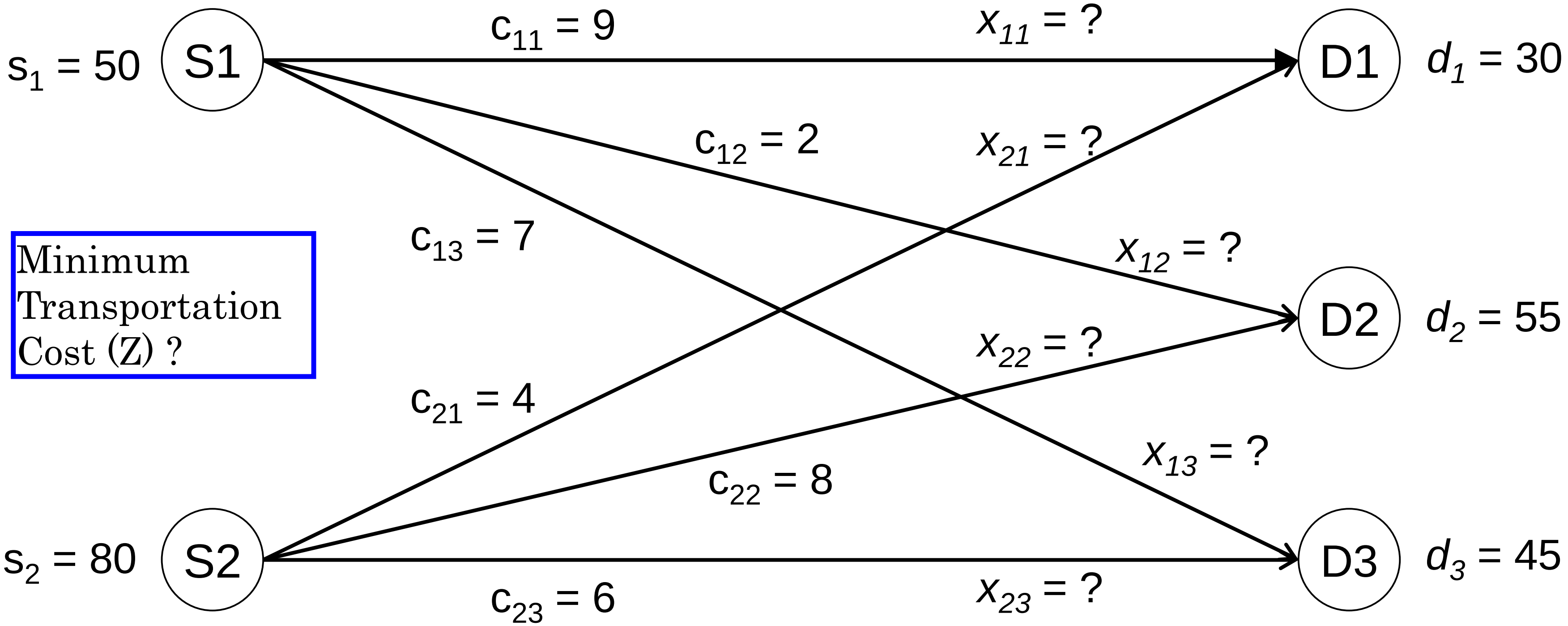
Minimum Transportation Cost?


cost = 9
supply = 50
S1
D1
demand = 30
cost = 2
Minimum
Transportation
Cost?
cost = 7
D2
demand = 55
cost = 4
cost = 8
supply = 80
S2
D3
demand = 45
cost = 6

Optimal solution to this small transportation problem:



Minimum Transportation Cost: 530



$$
\begin{aligned}
&\text{Minimize:} \quad Z = 9x_{11} + 2x_{12} + 7x_{13} + 4x_{21} + 8x_{22} + 6x_{23} \\
&\text{Subject to:} \quad x_{11} + x_{12} + x_{13} = 50 && (1) \\
&\qquad x_{21} + x_{22} + x_{23} = 80 && (2) \\
&\qquad x_{11} + x_{21} = 30 && (3) \\
&\qquad x_{12} + x_{22} = 55 && (4) \\
&\qquad x_{13} + x_{23} = 45 && (5) \\
&\qquad x_{ij} \ge 0 \quad \text{for all } i \text{ and } j && (6)
\end{aligned}
$$

LP Model for Transportation Problem

$$\text{Minimize:} \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} X_{ij}$$

$$\text{Subject to:} \quad \sum_{j=1}^{m} X_{ij} = S_i \quad \text{for } i = 1,2\ldots n \qquad (1)$$

$$\sum_{i=1}^{n} X_{ij} = D_j \quad \text{for } j = 1,2\ldots m \qquad (2)$$

$$X_{ij} \geq 0 \quad \text{for all } i \text{ and } j \qquad (3)$$

Where:

n :

m :

C_{ij} :

X_{ij} :

S_i :

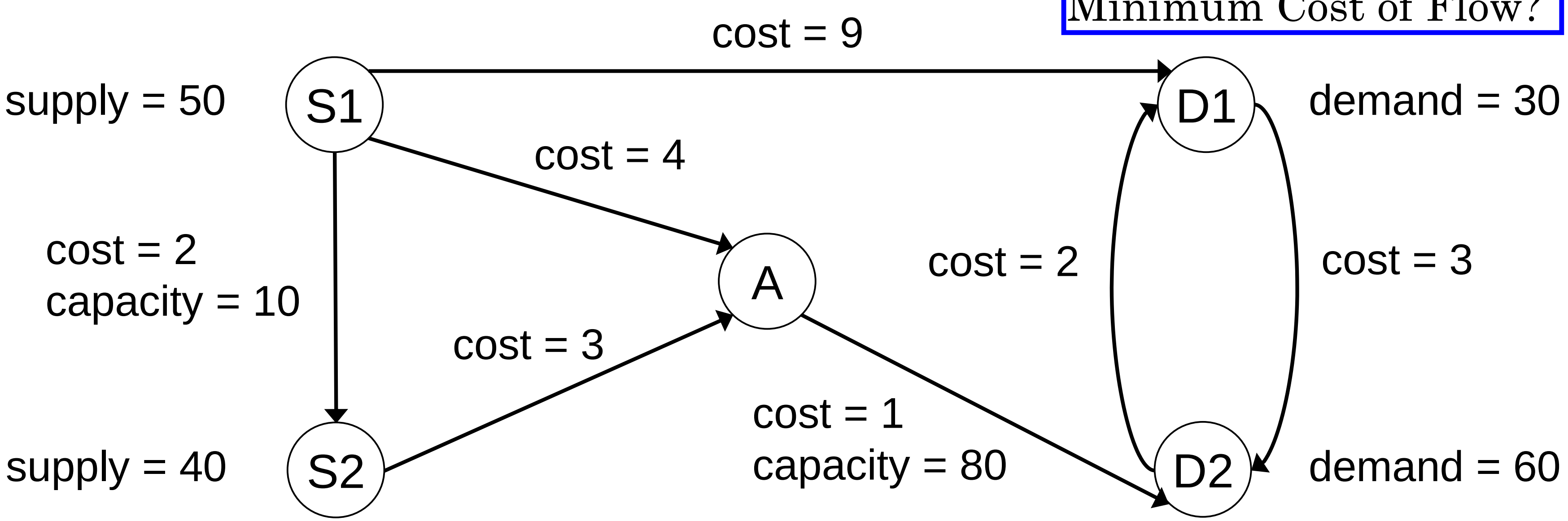
D_j :

This LP model of the transportation problem assumes that the problem is balanced.

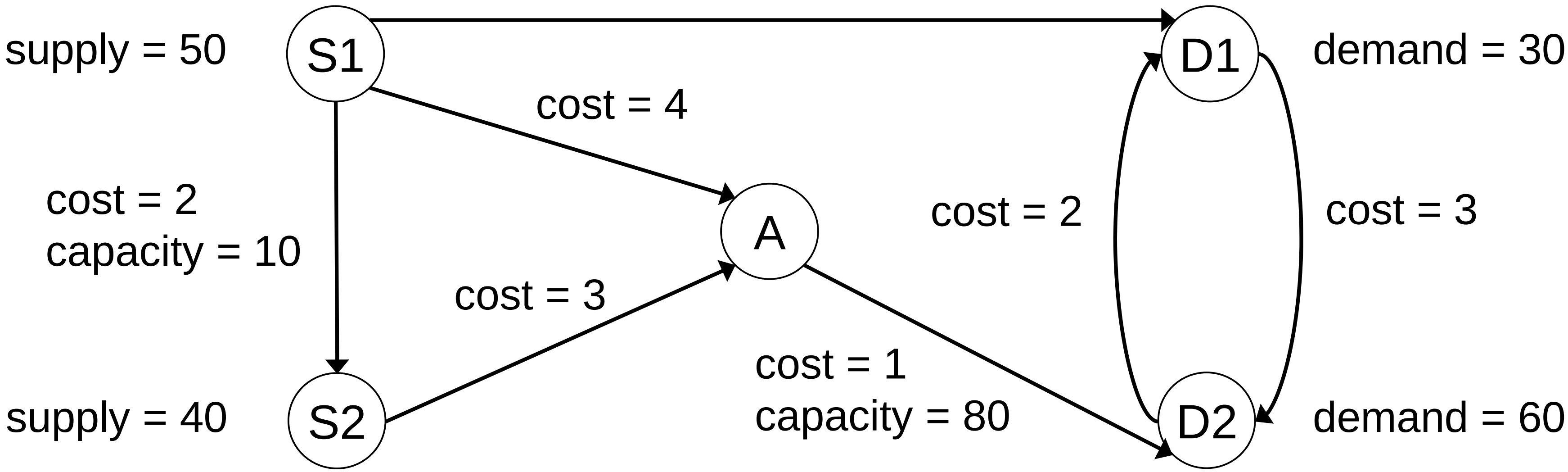
Minimum Cost Flow Problem

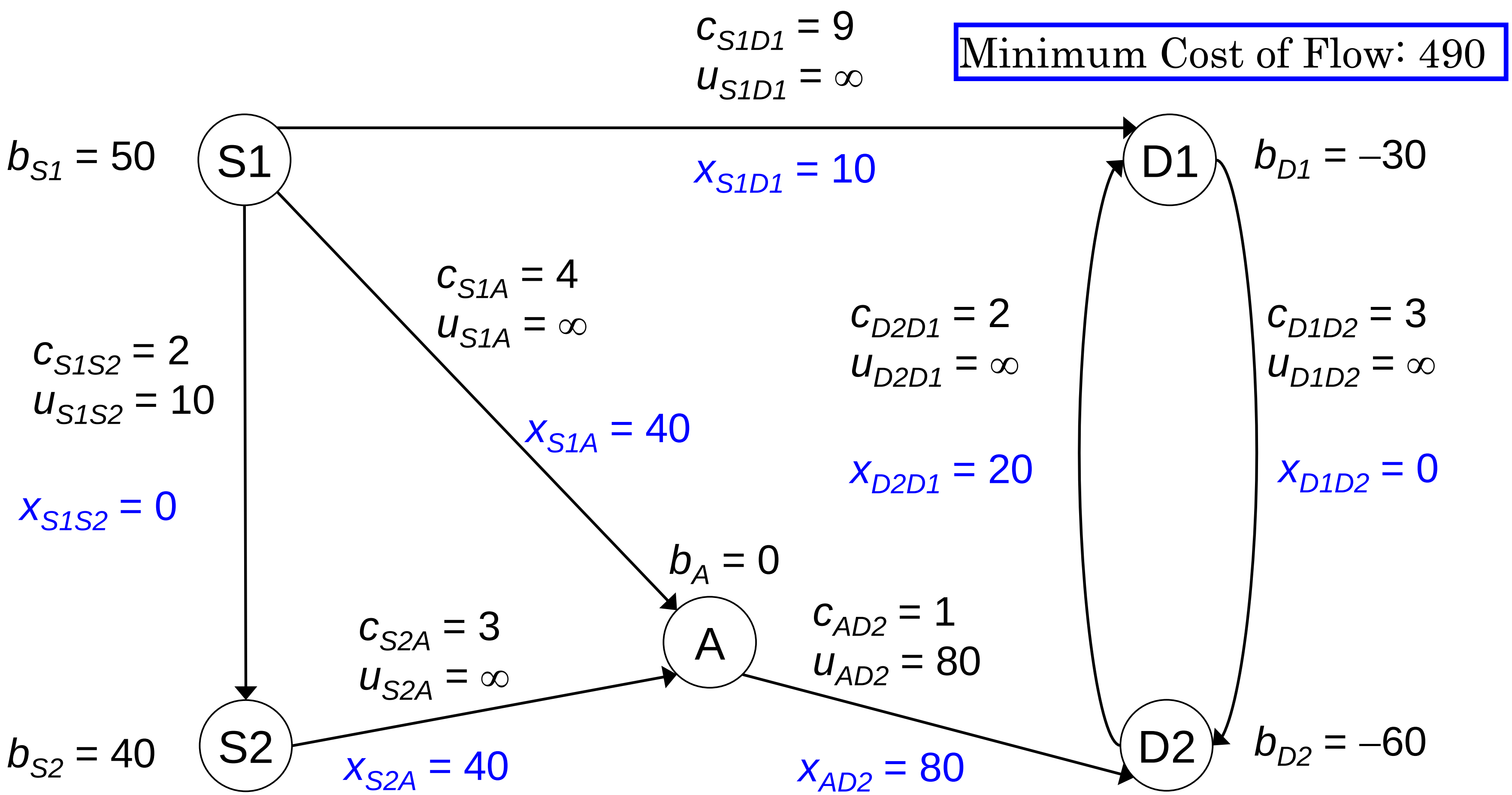
Consider a directed network with a set of nodes, at least one node supplies flow, at least one node demands flow and there is a set of transhipment nodes. A given cost per unit of flow and non-negative maximum flow capacity are associated to each arc. The problem is to distribute the entire flow supply through the network from the sources to the destinations such as to minimize the total flow cost. Linear Programming can solve minimum cost flow problems very efficiently.

Minimum Cost of Flow?



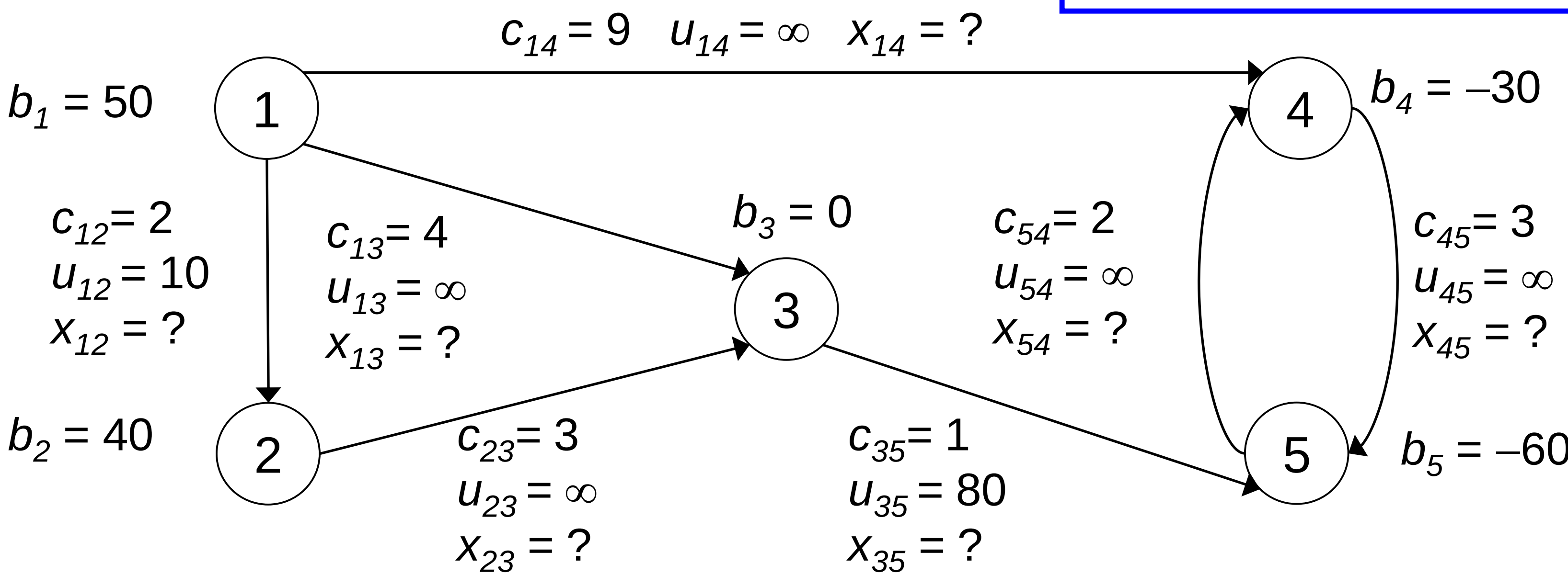
Minimum Cost of Flow?





$c_{S1D1} = 9$
$u_{S1D1} = \infty$
Minimum Cost of Flow: 490
$b_{S1} = 50$
S1
$x_{S1D1} = 10$
D1
$b_{D1} = -30$
$c_{S1A} = 4$
$u_{S1A} = \infty$
$c_{D2D1} = 2$
$u_{D2D1} = \infty$
$c_{D1D2} = 3$
$u_{D1D2} = \infty$
$c_{S1S2} = 2$
$u_{S1S2} = 10$
$x_{S1A} = 40$
$x_{D2D1} = 20$
$x_{D1D2} = 0$
$x_{S1S2} = 0$
$b_A = 0$
$c_{AD2} = 1$
$u_{AD2} = 80$
$c_{S2A} = 3$
$u_{S2A} = \infty$
A
$b_{S2} = 40$
S2
$x_{S2A} = 40$
$x_{AD2} = 80$
D2
$b_{D2} = -60$

Minimum Cost of Flow (Z) ?



$$\text{Minimize: } Z = 2x_{12} + 4x_{13} + 9x_{14} + 3x_{23} + x_{35} + 3x_{45} + 2x_{54}$$

$$\text{Subject to: } x_{12} + x_{13} + x_{14} = 50 \qquad (1)$$

$$x_{23} - x_{12} = 40 \qquad (2)$$

$$x_{35} - x_{13} - x_{23} = 0 \qquad (3)$$

$$x_{45} - x_{14} - x_{54} = -30 \qquad (4)$$

$$x_{54} - x_{35} - x_{45} = -60 \qquad (5)$$

$$0 \le x_{12} \le 10 \text{ , } 0 \le x_{35} \le 80 \text{ and } x_{ij} \ge 0 \qquad (6)$$

LP Model for Minimum Cost Flow Problem

$$\text{Minimize}: \quad Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij}$$

$$\text{Subject to}: \quad \sum_{j=1}^{n} X_{ij} - \sum_{j=1}^{n} X_{ji} = B_i \quad \text{for } i = 1,2 \ldots n \qquad (1)$$

$$0 \le X_{ij} \le U_{ij} \quad \text{for each arc } i \rightarrow j \qquad (2)$$

Where:

n :

C_{ij} :

X_{ij} :

B_i :

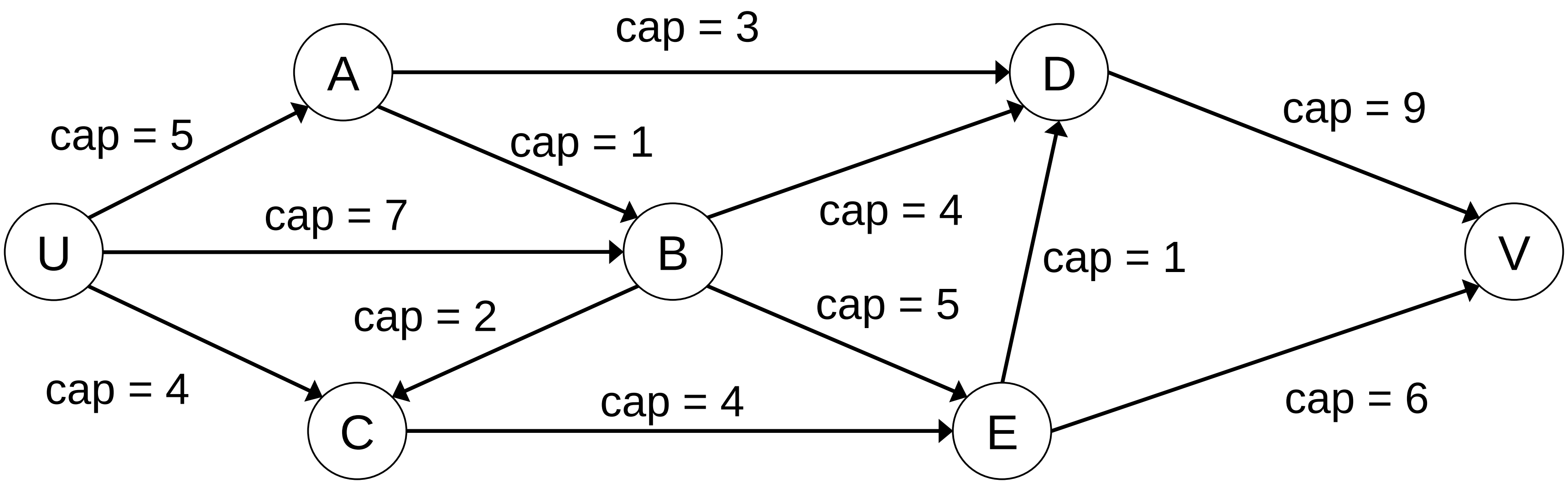
U_{ij} :

This LP model of the minimum cost flow problem is a more general case of the transportation problem.

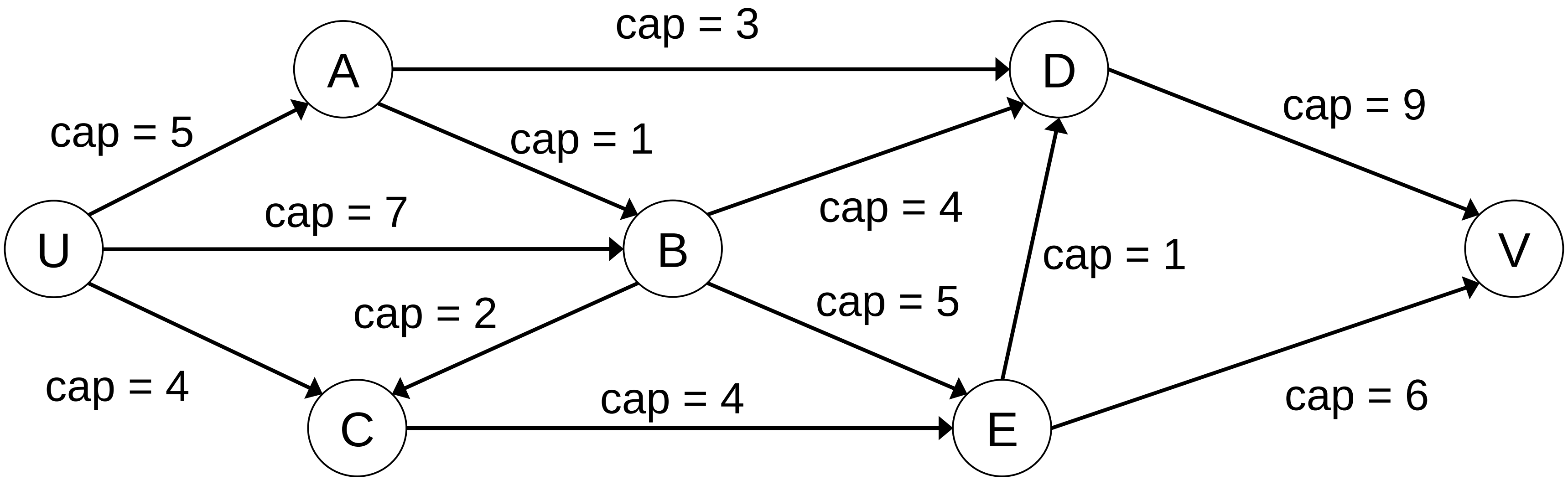
Maximum Flow Problem

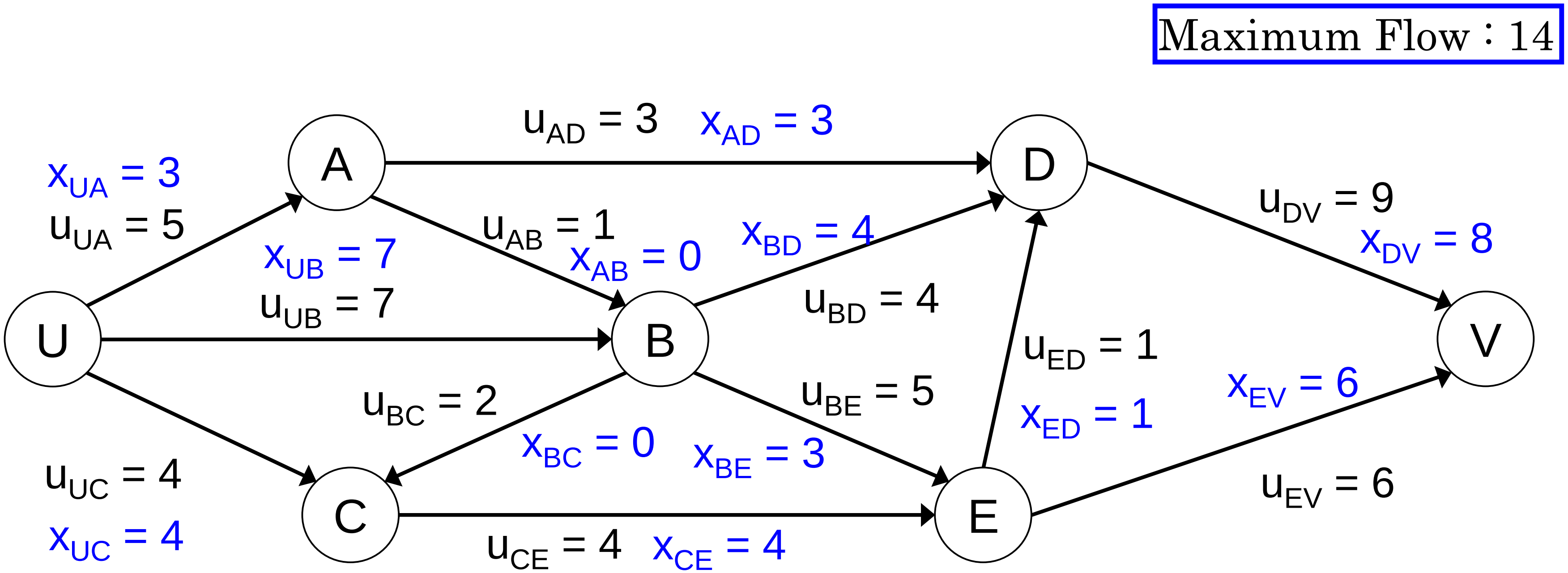
Consider a directed network with a set of nodes, source nodes, sink nodes and transhipment nodes. A non-negative maximum flow capacity is associated to each arc. The problem is to maximize the total flow from the sources to the sinks. Linear Programming can solve maximum flow problems very efficiently.

Maximum Flow?

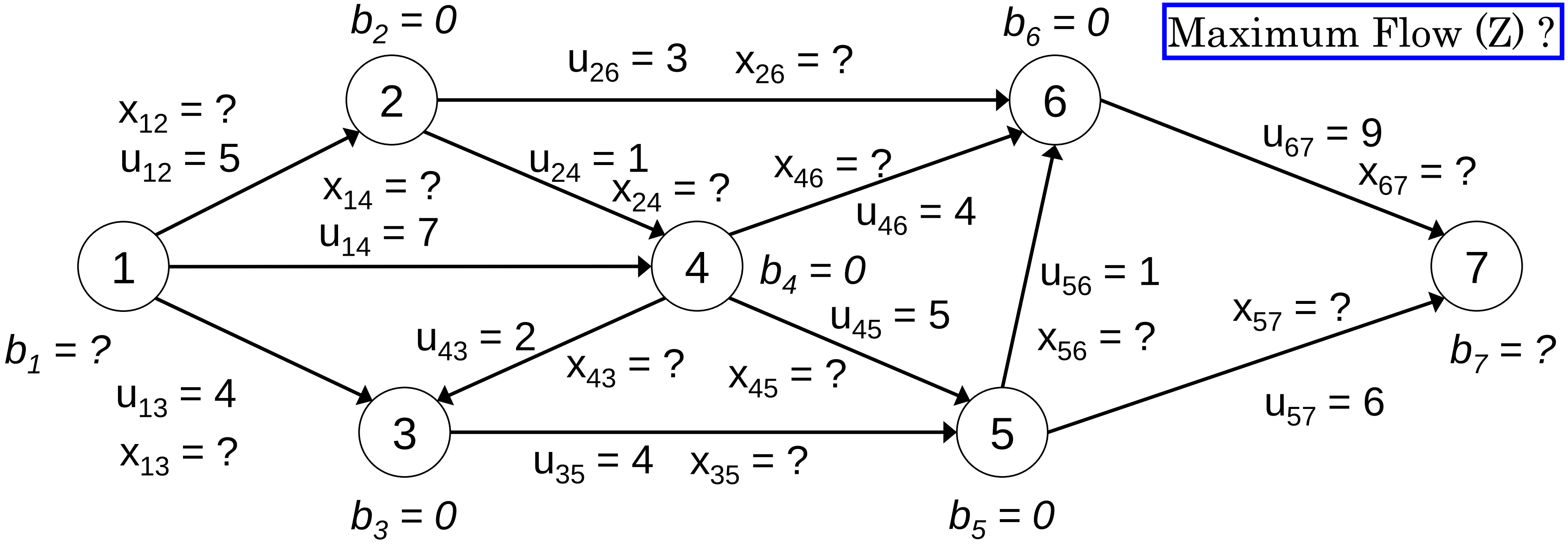


Maximum Flow?





Maximum Flow : 14
$x_{UA} = 3$
$u_{UA} = 5$
$u_{AD} = 3$ $x_{AD} = 3$
$u_{AB} = 1$
$x_{AB} = 0$
$x_{UB} = 7$
$u_{UB} = 7$
$x_{BD} = 4$
$u_{BD} = 4$
$u_{DV} = 9$
$x_{DV} = 8$
$u_{ED} = 1$
$x_{ED} = 1$
$u_{BE} = 5$
$x_{BE} = 3$
$u_{BC} = 2$
$x_{BC} = 0$
$x_{EV} = 6$
$u_{EV} = 6$
$u_{UC} = 4$
$x_{UC} = 4$
$u_{CE} = 4$ $x_{CE} = 4$
U
A
B
C
D
E
V



$$
\begin{aligned}
&\text{Maximize: } Z = x_{12} + x_{13} + x_{14} \quad \text{or} \quad Z = x_{57} + x_{67} \\
&\text{Subject to: } -x_{12} + x_{24} + x_{26} = 0 && (1) \\
&-x_{13} - x_{43} + x_{35} = 0 && (2) \\
&-x_{14} - x_{24} + x_{43} + x_{45} + x_{46} = 0 && (3) \\
&-x_{35} - x_{45} + x_{56} + x_{57} = 0 && (4) \\
&-x_{26} - x_{46} - x_{56} + x_{67} = 0 && (5) \\
&0 \le x_{ij} \le u_{ij} \quad \text{for each arc } i \to j && (6)
\end{aligned}
$$

LP Model for Maximum Flow

$$\text{Maximize}: \quad Z = \sum_{j=2}^{n} X_{1j} \quad \text{involving each arc } 1 \to j$$

$$\text{alternatively}: \quad Z = \sum_{i=1}^{n-1} X_{in} \quad \text{involving each arc } i \to n$$

$$\text{Subject to}: \quad \sum_{j=1}^{n} X_{ij} - \sum_{j=1}^{n} X_{ji} = B_i \ \text{ for } \ i = 2 \ldots (n-1) \qquad (1)$$

$$0 \le X_{ij} \le U_{ij} \ \text{ for each arc } \ i \to j \qquad (2)$$

Where:

n :

X_{ij} :

B_i :

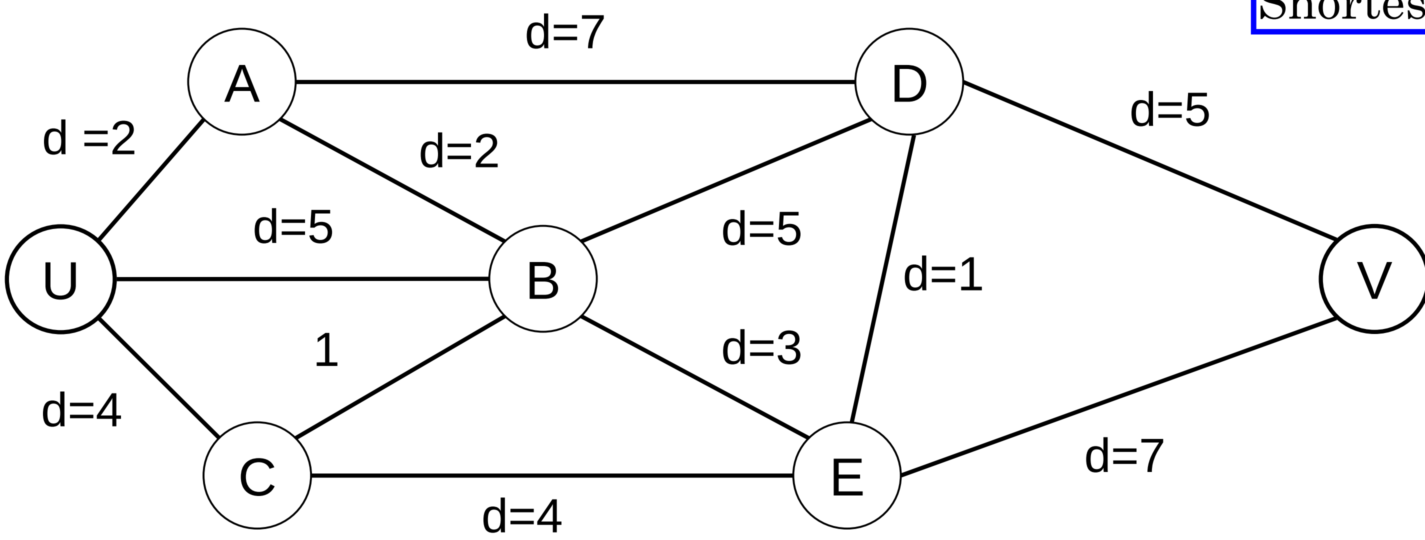
U_{ij} :

This general LP model of the maximum flow problem assumes that there is one source and one sink node.

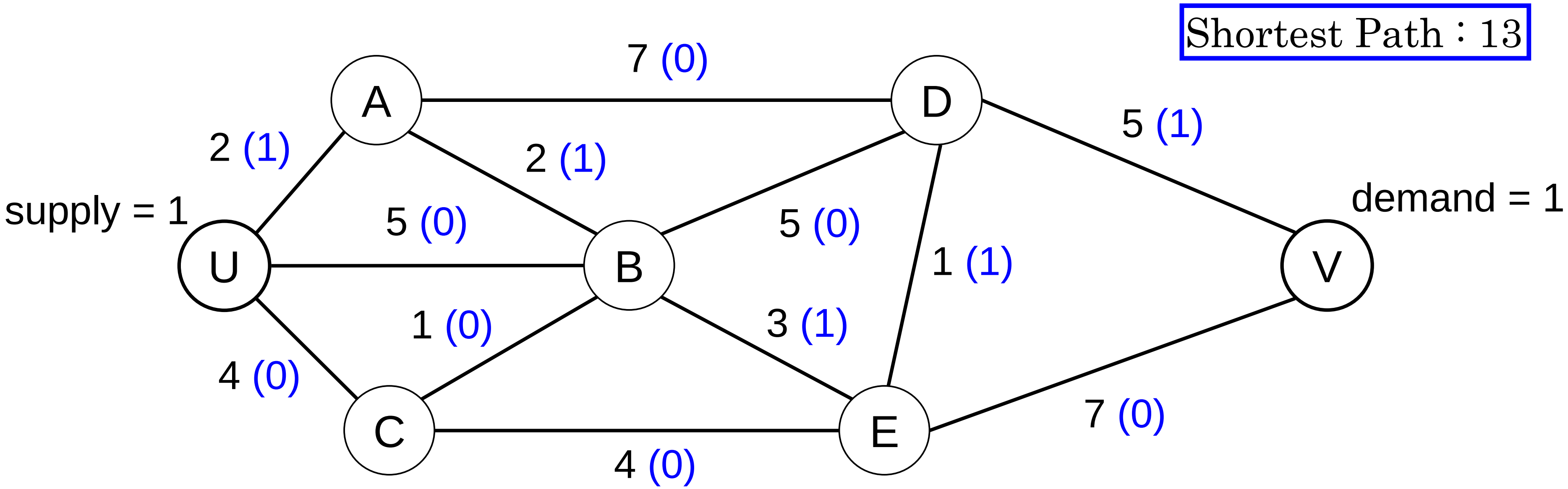
Shortest Path as a Flow Problem

Consider an undirected network with non-negative distance associated to each link. The objective is to find the shortest path between the origin node and the destination node. Several algorithms exist for the shortest path problem but Linear Programming can also solve shortest path very efficiently.

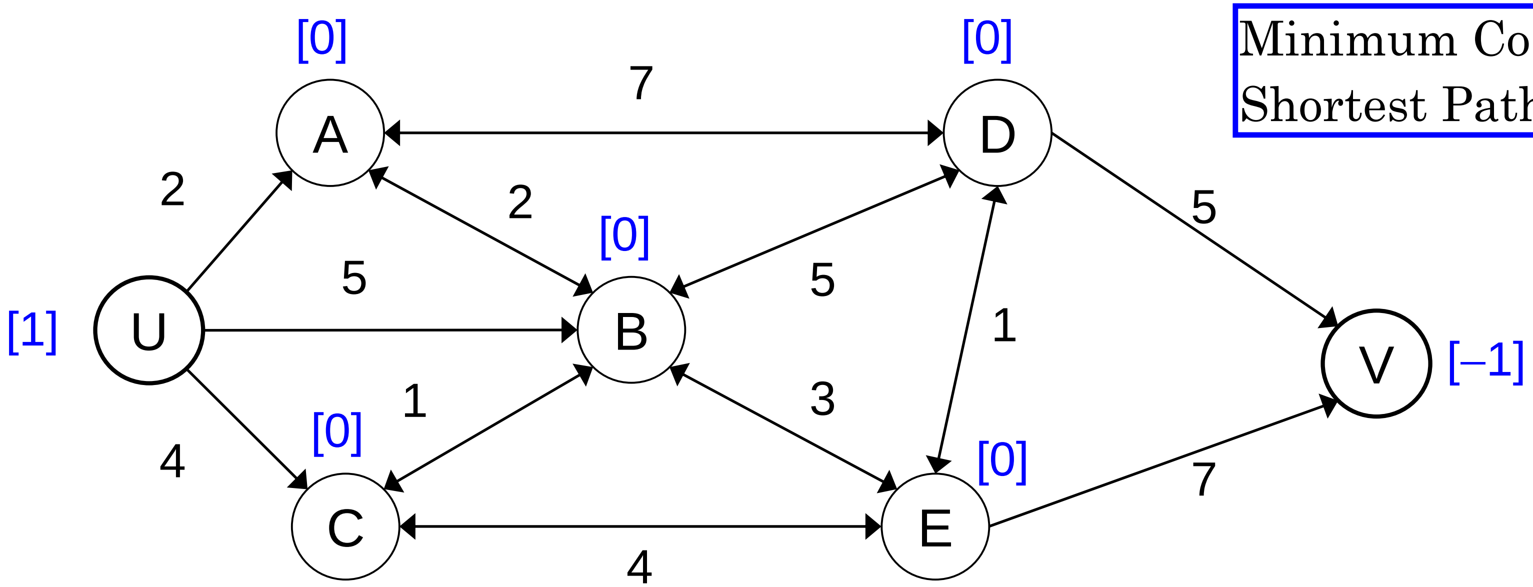
Shortest Path?



Modelling Shortest Path as Minimum Cost Flow



- distance d_{ij} becomes cost of flow c_{ij}
- maximum flow capacity is set as ∞ for all arcs
- flow in each arc in the shortest path will be 1, other flows will be 0
- origin node becomes supply node with supply $b_u = 1$
- destination node becomes demand node with demand $b_v = -1$
- all the other nodes are transhipment nodes with $b_i = 0$
- undirected links are replaced by directed arcs



Minimum Cost of Flow: 13
Shortest Path: 13

Minimize: $Z = 2x_{UA} + 5x_{UB} + 4x_{UC} + 2(x_{AB} + x_{BA}) + (x_{BC} + x_{CB}) + 7(x_{AD} + x_{DA}) +$
$4(x_{CE} + x_{EC}) + 5(x_{BD} + x_{DB}) + 3(x_{BE} + x_{EB}) + (x_{DE} + x_{ED}) + 5x_{DV} + 7x_{EV}$ (1)

Subject to: $x_{UA} + x_{UB} + x_{UC} = 1$ (2)

$x_{AB} + x_{AD} - x_{UA} - x_{BA} - x_{DA} = 0$ (3)

$x_{BA} + x_{BC} + x_{BD} + x_{BE} - x_{UB} - x_{AB} - x_{CB} - x_{DB} - x_{EB} = 0$ (4)

$x_{CB} + x_{CE} - x_{UC} - x_{BC} - x_{EC} = 0$ (5)

$x_{DA} + x_{DB} + x_{DE} + x_{DV} - x_{AD} - x_{BD} - x_{ED} = 0$ (6)

$x_{EB} + x_{EC} + x_{ED} + x_{EV} - x_{BE} - x_{CE} - x_{DE} = 0$ (7)

$-x_{DV} - x_{EV} = -1$ (8)

$0 \le x_{ij} \le 1$ or more precisely $x_{ij} \in \{0,1\}$ for $i, j \in \{U, V, A, B, C, D, E\}$ (9)

Write the LP Model for This Problem

MEDEQUIP is a company that has two production factories and three customers. The table gives the output produced by each factory, amount ordered by each customer, and cost of shipment per unit of product for each combination of factory and customer. The company wants to decide how much to send from each factory to each customer so as to meet the customers' demand and minimise the total shipment cost.

To / From	Unit Shipping Cost			Output
	Customer 1	Customer 2	Customer 3	
Factory 1	£600	£800	£700	400
Factory 2	£400	£900	£600	500
Order Size	300	200	400	

Data? Decision Variables?

Objective Function? Constraints?

Questions OR Comments