Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018
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Lecture 4 – Post-Optimality Analysis

- The Role of Post-Optimality Analysis
  To define the key concepts of post-optimality analysis
- Post-optimality Analysis Tools
  To interpret sensitivity analysis reports provided by Excel and LP-Solve
- Apply Post-optimality Analysis to Product-Mix Problems
  To model, solve and apply post-optimality analysis to product-mix optimization problems

Additional Reading

Sections 4.7 and 6.8 of the book (Hillier and Lieberman, 2015)

Sections 4.0 to 4.7 of the book (Ragsdle, 2015)

Section 3.6 of the book (Taha, 2017)

Teaching Supplement on Sensitivity Analysis from the article:

> A Teaching Supplement on Sensitivity Analysis for LP in Undergraduate Business Programs. Jomon Aliyas Paul, Leo MacDonald, INFORMS Transactions in Education, Vol. 16, No. 1, pp. 6–14, 2015.

The Role of Post-optimality Analysis

Post-optimality analysis can be conducted after finding the optimal solution to the given LP model. Basically, it consists of analysing what happens if some parts of the model change. Post-optimality analysis techniques include:

- Sensitivity Analysis refers to the interpretation of the optimal solution obtained and the evaluation of the impact that changes to problem parameters (such as objective function coefficients and constraint right-hand sides) could have on the given optimal solution.
- Limits Report (part of sensitivity analysis) is provided by some solvers and summarises for each variable, the optimal value, the lower limit and the upper limit (while still producing a feasible solution), together with the corresponding optimal objective value.
- Re-optimization (after post-optimality analysis and testing/revising the model) refers to solving the model again after having made some changes to the model, usually conducted to provide the decision-maker with alternative models and solutions.

Post-optimality Analysis Can Help to Answer…

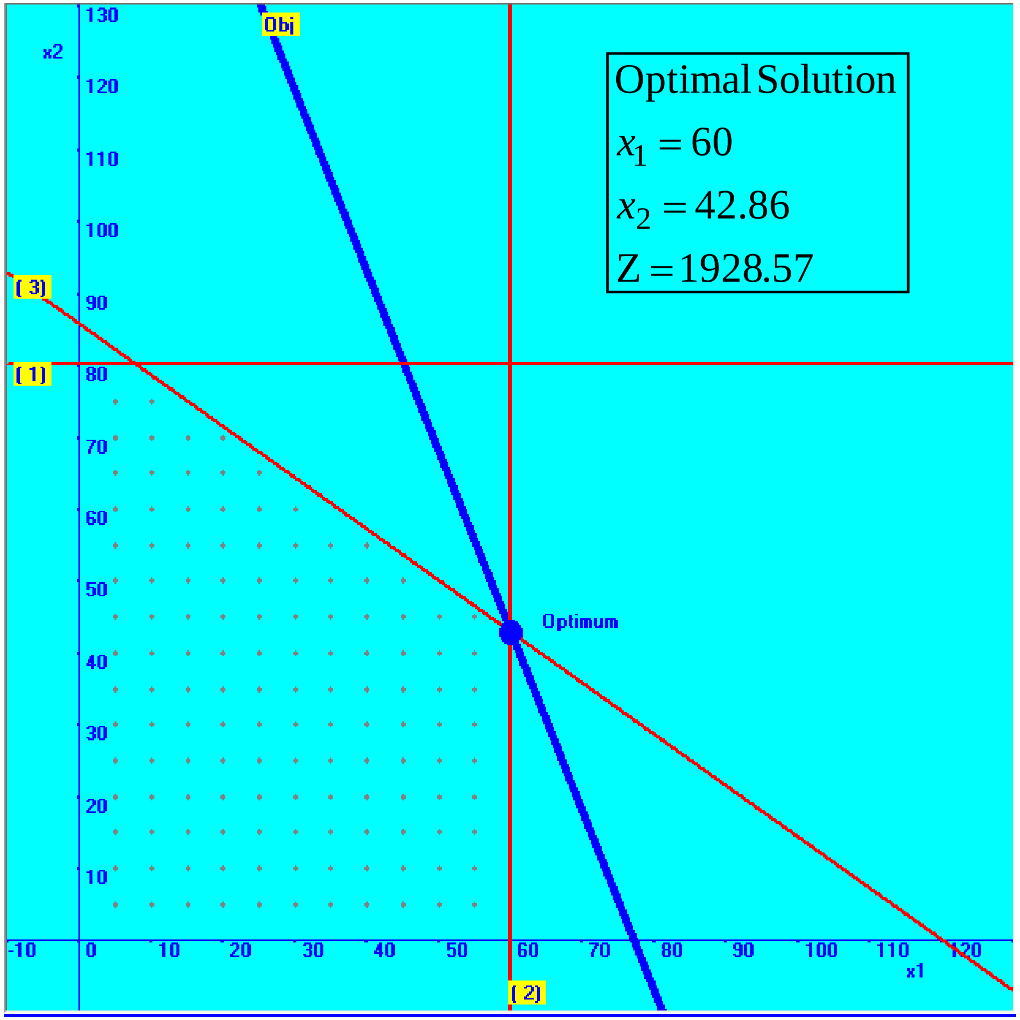
- How changes in the availability of constrained resources affect the optimal solution and the objective function?
- What is the slack, if any, of resources in the optimal solution?
- What resources are more critical in the optimization problem?
- Is there an alternative plan with the same profit or cost?
- How can the current optimal solution be improved?
- What cost is acceptable for increasing resources?

Example. Recall the WENBU product-mix optimization problem from the workshop 1.

WENBU is a food processing plant that produces hotdogs and buns. They grin their own flour for the buns at a maximum rate of 800 units/day. Each bun requires 10 units of flour. Each hotdog requires 25 units of pork. Their supplier can only provide 1500 units of pork per day. All other ingredients are in plentiful supply. The workforce consists of 2 part-time employees (5 hrs per day each). Each hotdog requires 5 minutes of labour and each bun requires 7 minutes of labour. Each hotdog gives a profit of 25 pence and each bun gives a profit of 10 pence. The company wants to know how many of each product to produce each day to achieve the highest possible profit.

Maximize:	$Z = 25x_1 + 10x_2$		maximize total profit
Subject to:	$0x_1 + 10x_2 \leq 800$	(1)	constraint on the flour
	$25x_1 + 0x_2 \leq 1500$	(2)	constraint on the pork
	$5x_1 + 7x_2 \leq 600$	(3)	constraint on the labour
	$x_1, x_2 \geq 0$	(4)	x_1 is hotdogs and x_2 is buns

Graphical post-optimality analysis for the WENBU problem.

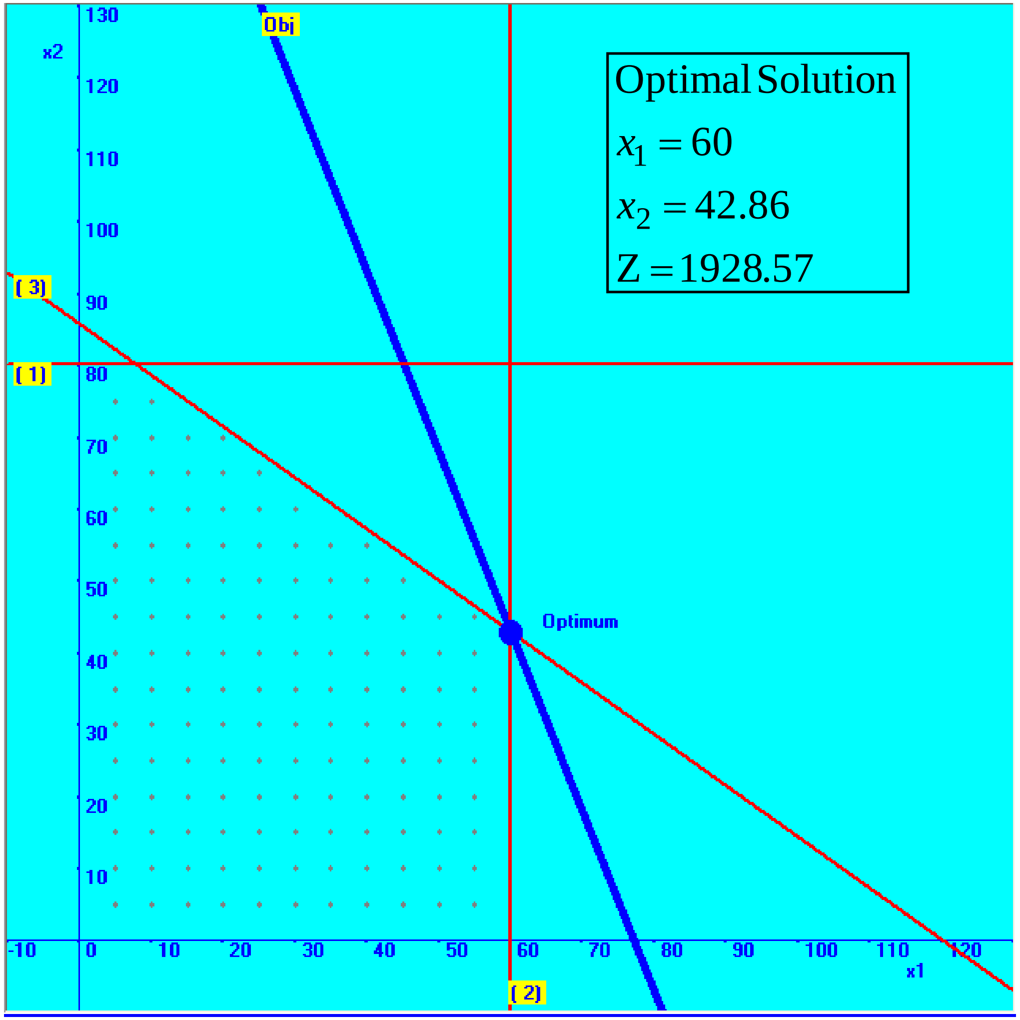


x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \le 800$	(1)	flour
	$25x_1 \le 1500$	(2)	pork
	$5x_1 + 7x_2 \le 600$	(3)	labour
	$x_1 \ge 0, x_2 \ge 0$	(4)	

- Is there any slack in the resources on the optimal solution?
- Which are the non-binding functional constraints on the optimal solution?

Graphical post-optimality analysis for the WENBU problem.

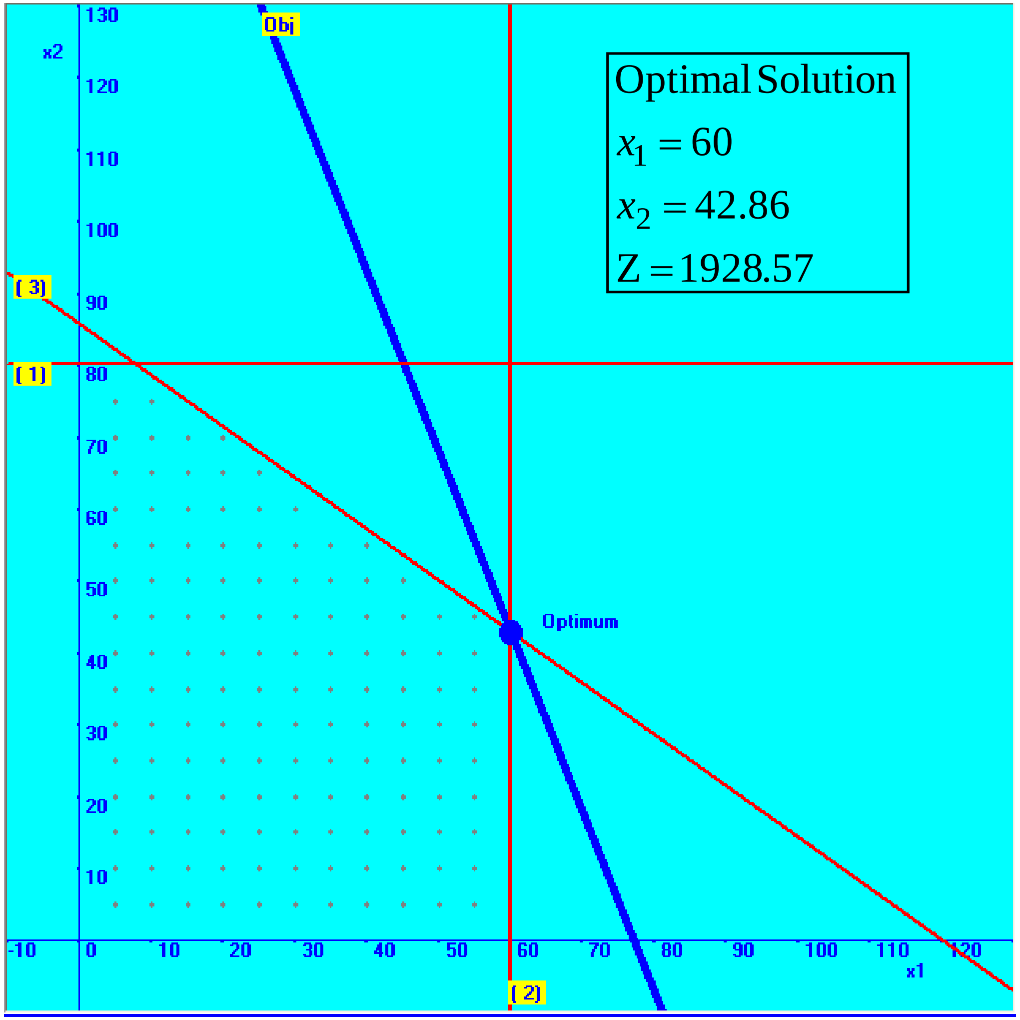


x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \leq 800$	(1)	flour
	$25x_1 \leq 1500$	(2)	pork
	$5x_1 + 7x_2 \leq 600$	(3)	labour
	$x_1 \geq 0, x_2 \geq 0$	(4)	

- What is the maximum price that should be paid to have more of the resources with no slack on the optimal solution?
- What is the value on the objective function of adding more of those resources?

Graphical post-optimality analysis for the WENBU problem.

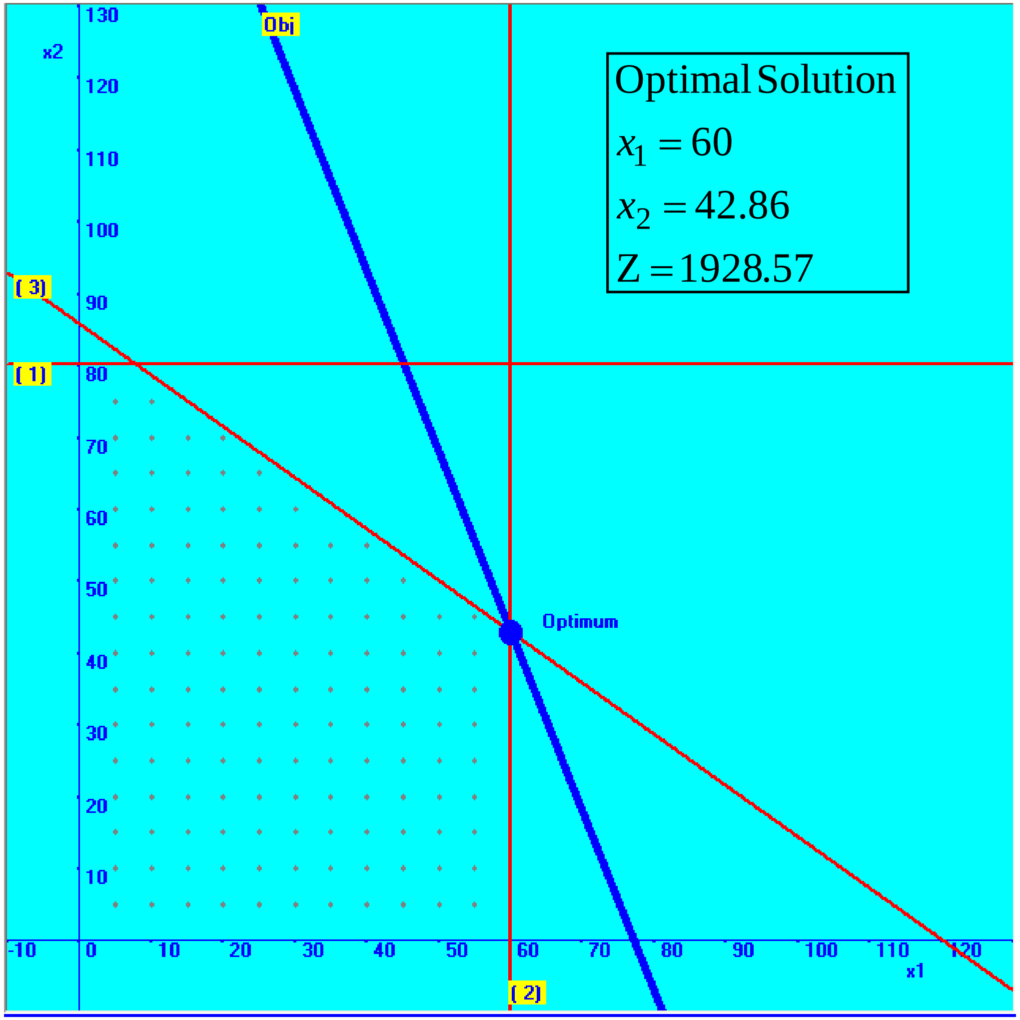


x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \leq 800$	(1)	flour
	$25x_1 \leq 1500$	(2)	pork
	$5x_1 + 7x_2 \leq 600$	(3)	labour
	$x_1 \geq 0, x_2 \geq 0$	(4)	

- How much can the profits per product change without changing the optimal solution?
- How much can the slope of the objective function change without changing the optimal solution?

Graphical post-optimality analysis for the WENBU problem.



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \leq 800$	(1)	flour
	$25x_1 \leq 1500$	(2)	pork
	$5x_1 + 7x_2 \leq 600$	(3)	labour
	$x_1 \geq 0, x_2 \geq 0$	(4)	

- Without changing the production of buns, how much can the production of hotdogs change while maintaining feasibility?
- With x2 = 42.86, how much can the value of decision variable x1 change without falling into infeasibility?

Post-optimality Analysis Tools

Most optimization solvers including Excel and LP-Solve provide post-optimality analysis reports.

Excel provides: answer report, sensitivity report, limits report

LP-Solve provides: constraints report, sensitivity report

Note: in order to produce the sensitivity report for all constraints, all the decision variables should be present in all the constraints in the LP-Solve model. For example, $0x_1 + 10x_2 \leq 800$.

Other powerful commercial solvers like CPLEX, LINGO, GUROBI, XPRESS, etc. all provide post-optimality analysis reports including tools for re-optimization.

Post-optimality Analysis Aids Decision Support

The Excel Answer Report

A1 — Microsoft Excel 15.0 Answer Report

Microsoft Excel 15.0 Answer Report
Worksheet: [ORModels1-PM.xlsx]Wenbu
Report Created: 24/10/2017 02:51:14
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time 100 sec, Iterations 100, Precision 0.000001
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 5%, Solve Without Integer Constraints, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$I$8	Used Profit/Product	0	1928.571429

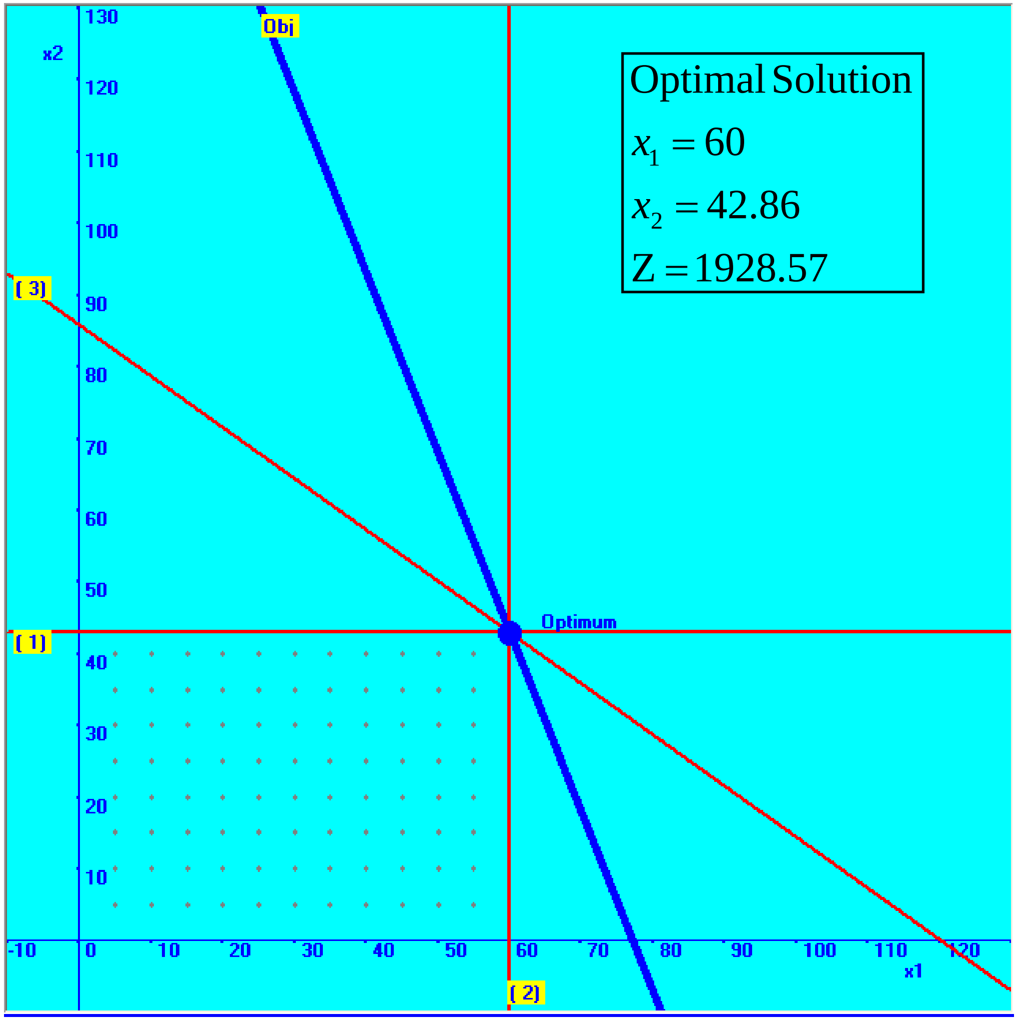
Variable Cells

Cell	Name	Original Value	Final Value	Integer
$H$5	Hotdogs Produced	0	60	Contin
$H$6	Buns Produced	0	42.85714286	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$8	Used Labour	600	$C$8<=$C$10	Binding	0
$D$8	Used Flour	428.5714286	$D$8<=$D$10	Not Binding	371.428571
$E$8	Used Pork	1500	$E$8<=$E$10	Binding	0

Changing right-hand side value of a constraint in WENBU.



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \le 428.57$	(1)	flour
	$25x_1 \le 1500$	(2)	pork
	$5x_1 + 7x_2 \le 600$	(3)	labour
	$x_1 \ge 0, x_2 \ge 0$	(4)	

- Solving for the minimum availability of resource corresponding to constraint (1) that still maintains the optimal solution and objective value.
- That is, with no more slack, all constraints are binding

The Excel Sensitivity Analysis Report

Microsoft Excel 15.0 Sensitivity Report
Worksheet: [ORModels1-PM.xlsx]Wenbu
Report Created: 24/10/2017 02:51:14

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$H$5	Hotdogs Produced	60	0	25	1E+30	17.85714286
$H$6	Buns Produced	42.85714286	0	10	25	10

Constraints

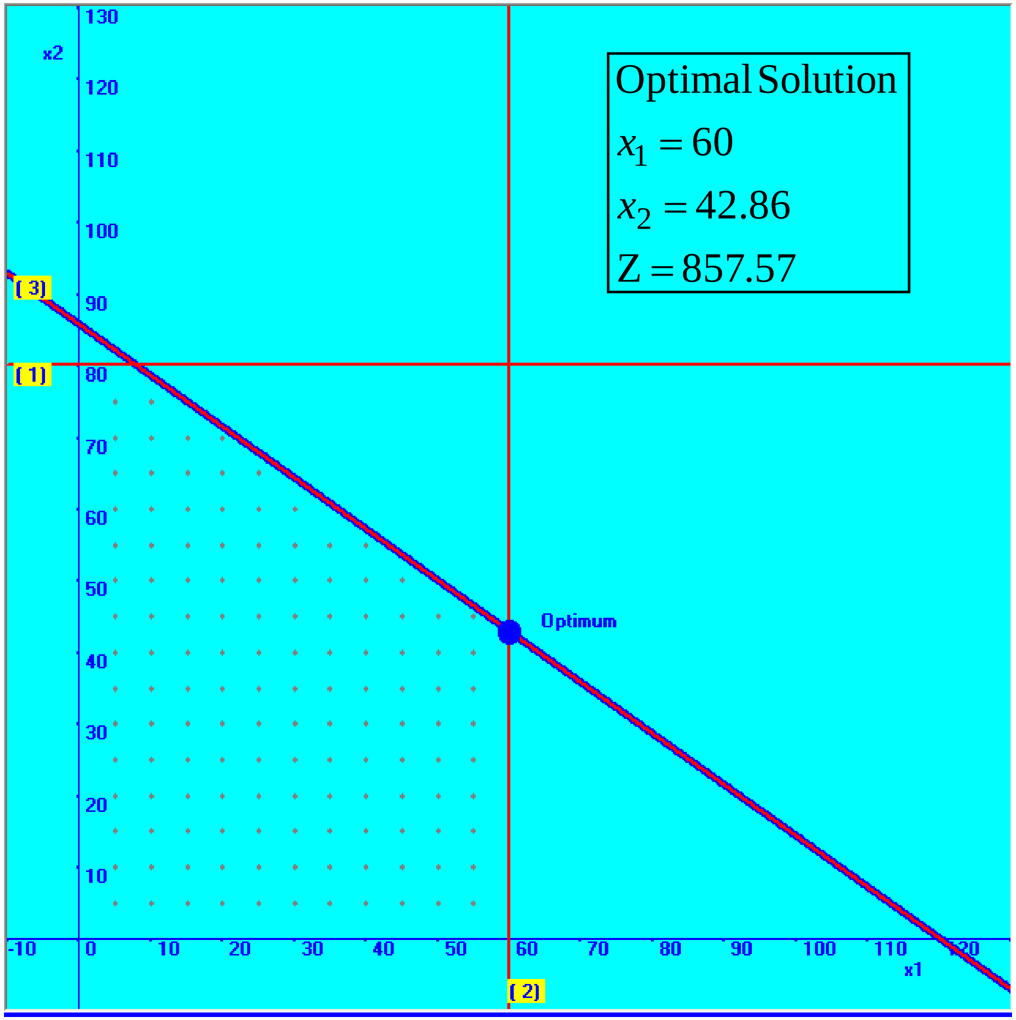
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$8	Used Labour	600	1.428571429	600	260	300
$D$8	Used Flour	428.5714286	0	800	1E+30	371.4285714
$E$8	Used Pork	1500	0.714285714	1500	1500	1300

Amount of resource used

Marginal value of resource

800 + inf. = inf. or 800 – 371.43 = 428.57
The shadow price remains the same

Changing coefficients in the objective function in WENBU.



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 7.15x_1 + 10x_2$		profit
Subject to:	$10x_2 \le 800$	(1)	flour
	$25x_1 \le 1500$	(2)	pork
	$5x_1 + 7x_2 \le 600$	(3)	labour
	$x_1 \ge 0, x_2 \ge 0$	(4)	

- From the original model, decreasing the coefficient of x1 in the objective function from 25 to 7.15, maintains the same optimal solution but different objective value.
- And this generates multiple optimal solutions.

The Excel Sensitivity Analysis Report

Microsoft Excel 15.0 Sensitivity Report
Worksheet: [ORModels1-PM.xlsx]Wenbu
Report Created: 24/10/2017 02:51:14

25 – 17.85 = 7.15
10 – 10 = 0
25 + 1E30 (infinity)
10 + 25 = 35

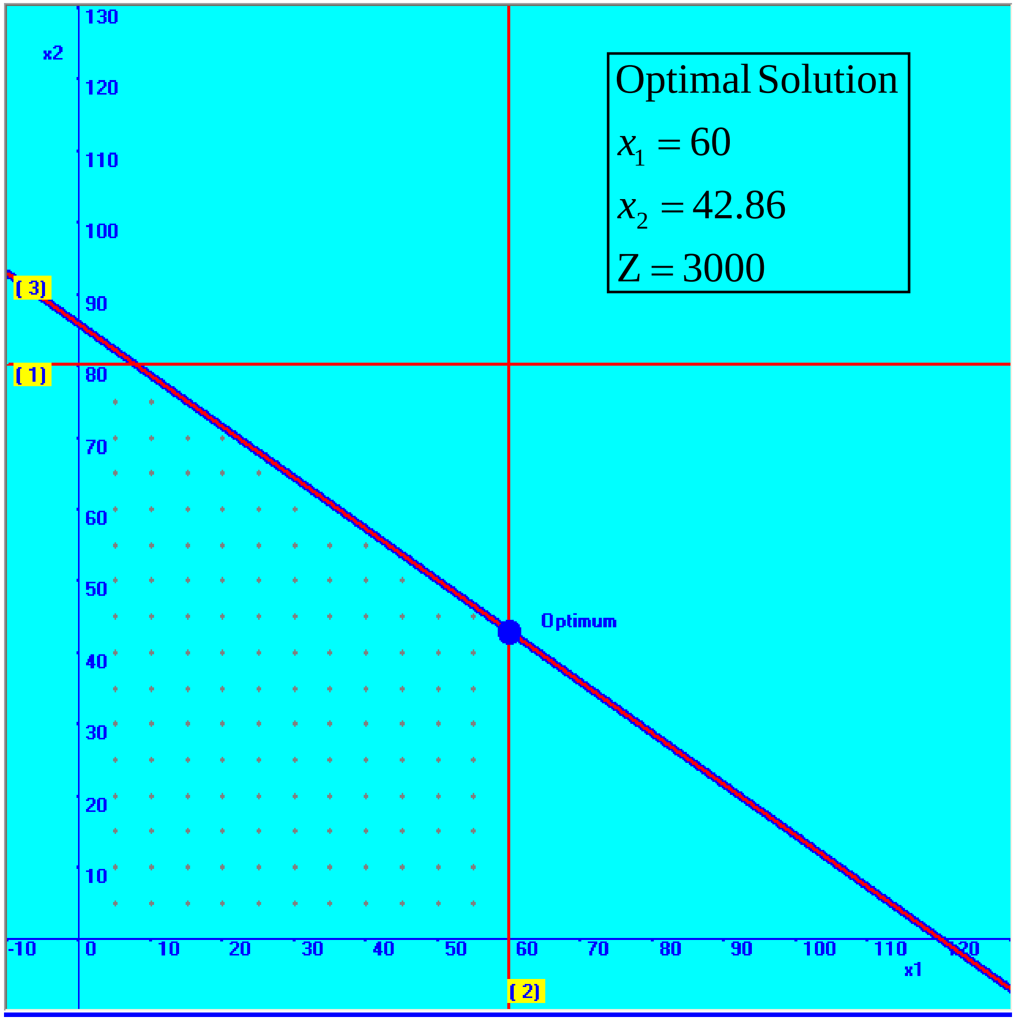
Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$H$5	Hotdogs Produced	60	0	25	1E+30	17.85714286
$H$6	Buns Produced	42.85714286	0	10	25	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$8	Used Labour	600	1.428571429	600	260	300
$D$8	Used Flour	428.5714286	0	800	1E+30	371.4285714
$E$8	Used Pork	1500	0.714285714	1500	1500	1300

Visual post-optimality analysis for the WENBU problem (cont.)



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 35x_2$		profit
Subject to:	$10x_2 \le 800$	(1)	flour
	$25x_1 \le 1500$	(2)	pork
	$5x_1 + 7x_2 \le 600$	(3)	labour
	$x_1 \ge 0, x_2 \ge 0$	(4)	

- From the original model, increasing the coefficient of x2 in the objective function from 10 to 35, maintains the same optimal solution but different objective value.
- And this generates multiple optimal solutions.

The Excel Sensitivity Analysis Report

Microsoft Excel 15.0 Sensitivity Report
Worksheet: [ORModels1-PM.xlsx]Wenbu
Report Created: 24/10/2017 02:51:14

25 – 17.85 = 7.15
10 – 10 = 0
25 + 1E30 (infinity)
10 + 25 = 35

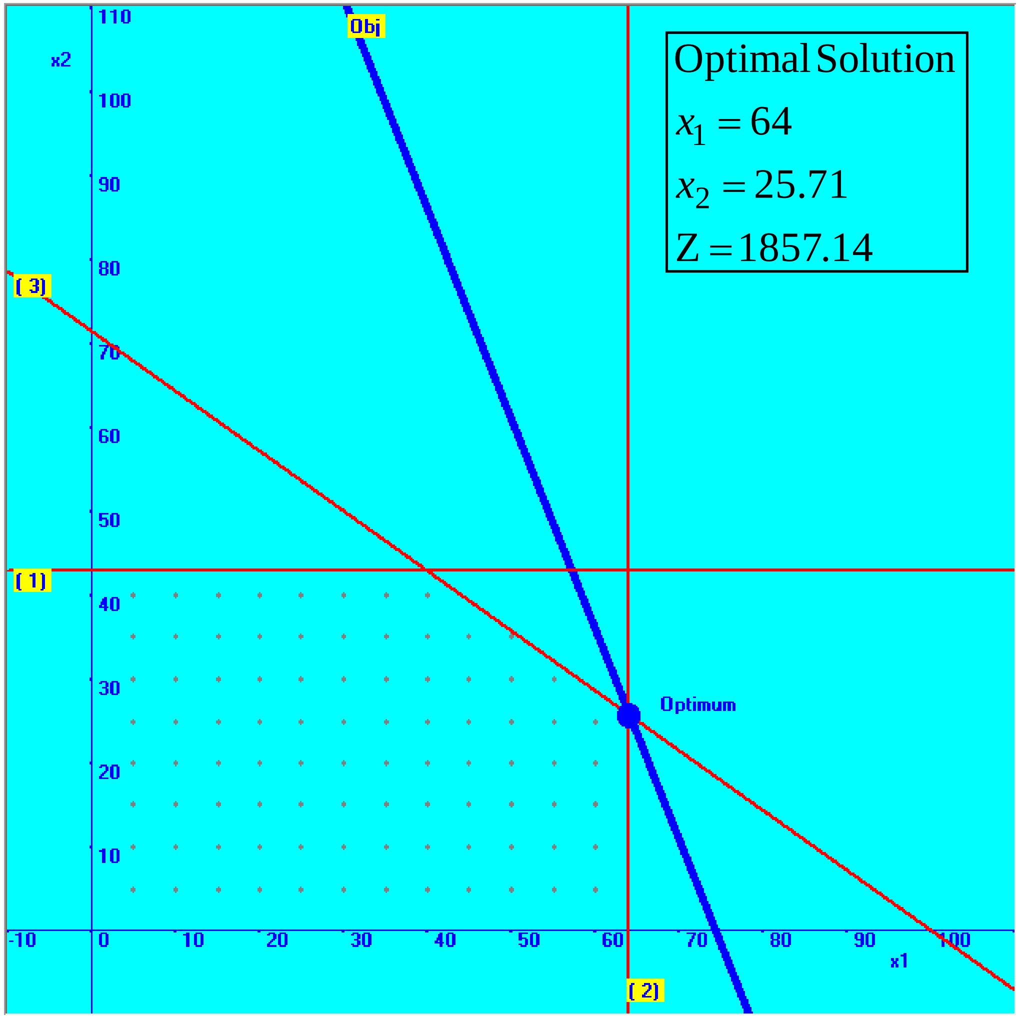
Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$H$5	Hotdogs Produced	60	0	25	1E+30	17.85714286
$H$6	Buns Produced	42.85714286	0	10	25	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$8	Used Labour	600	1.428571429	600	260	300
$D$8	Used Flour	428.5714286	0	800	1E+30	371.4285714
$E$8	Used Pork	1500	0.714285714	1500	1500	1300

Changing right-hand side value of a constraint in WENBU.



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 35x_2$		profit
Subject to:	$10x_2 \leq 428.57$	(1)	flour
	$25x_1 \leq 1600$	(2)	pork
	$5x_1 + 7x_2 \leq 500$	(3)	labour
	$x_1 \geq 0, x_2 \geq 0$	(4)	

- Increasing availability of resource (2) by 100 units adds some value to the objective function
- Decreasing availability of resource (3) by 100 units reduces some value to the objective function
- Constraint (1) now becomes non-binding

The Excel Sensitivity Analysis Report

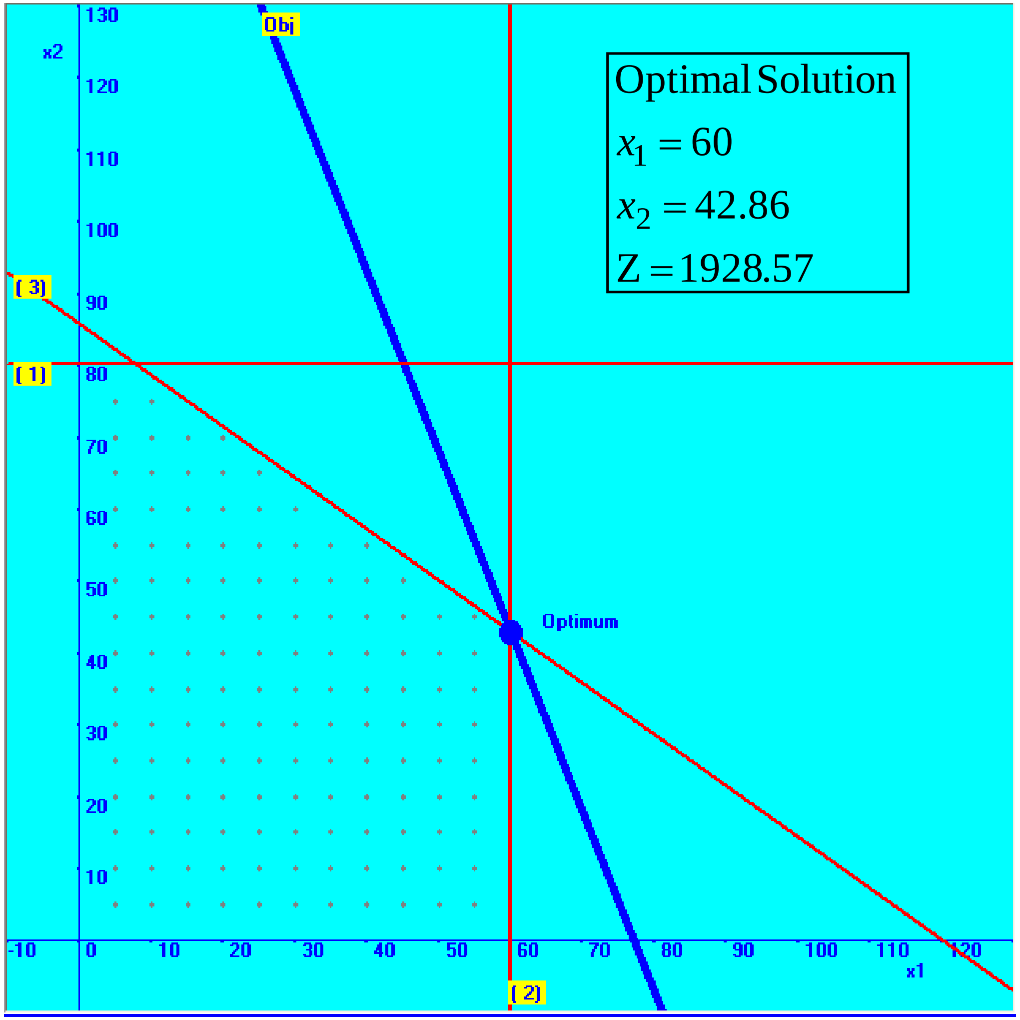
	A	B	C	D	E	F	G	H
1	**Microsoft Excel 15.0 Sensitivity Report**							
2	**Worksheet: [ORModels1-PM.xlsx]Wenbu**							
3	**Report Created: 24/10/2017 02:51:14**							
4								
5								
6	Variable Cells							
7				**Final**	**Reduced**	**Objective**	**Allowable**	**Allowable**
8		**Cell**	**Name**	**Value**	**Cost**	**Coefficient**	**Increase**	**Decrease**
9		$H$5	Hotdogs Produced	60	0	25	1E+30	17.85714286
10		$H$6	Buns Produced	42.85714286	0	10	25	10
11								
12	Constraints							
13				**Final**	**Shadow**	**Constraint**	**Allowable**	**Allowable**
14		**Cell**	**Name**	**Value**	**Price**	**R.H. Side**	**Increase**	**Decrease**
15		$C$8	Used Labour	600	1.428571429	600	260	300
16		$D$8	Used Flour	428.5714286	0	800	1E+30	371.4285714
17		$E$8	Used Pork	1500	0.714285714	1500	1500	1300
18								

Amount of resource used

Marginal value of resource

800 + 1500 = 3000 or 1500 – 1300 = 200
600 + 260 = 860 or 600 – 300 = 300
The shadow price remains the same

Changing the value of one decision variable only in WENBU.



x_1 is hotdogs and x_2 is buns

Maximize:	$Z = 25x_1 + 10x_2$		profit
Subject to:	$10x_2 \le 800$	(1)	flour
	$25x_1 \le 1500$	(2)	pork
	$5x_1 + 7x_2 \le 600$	(3)	labour
	$x_1 \ge 0, x_2 \ge 0$	(4)	

- Without changing the production of buns, how much can the production of hotdogs change while maintaining feasibility?
- With x2 = 42.86, how much can the value of decision variable x1 change without falling into infeasibility?

The Excel Limits Report

Microsoft Excel 15.0 Limits Report
Worksheet: [ORModels1-PM.xlsx]Wenbu
Report Created: 24/10/2017 02:51:15

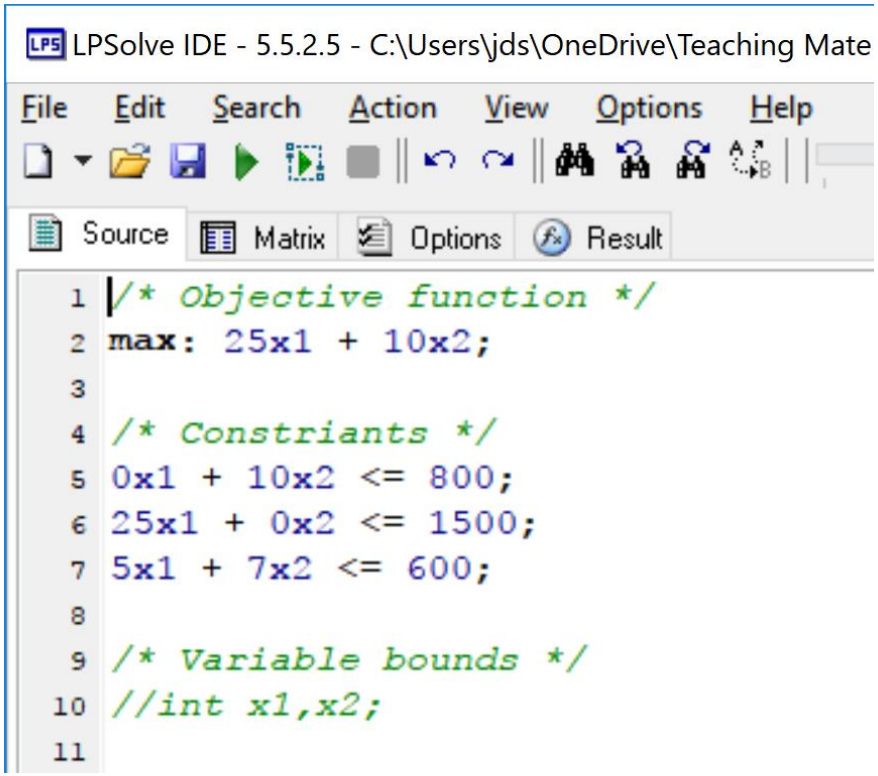
Cell	Objective Name	Value
$I$8	Used Profit/Product	1928.571429

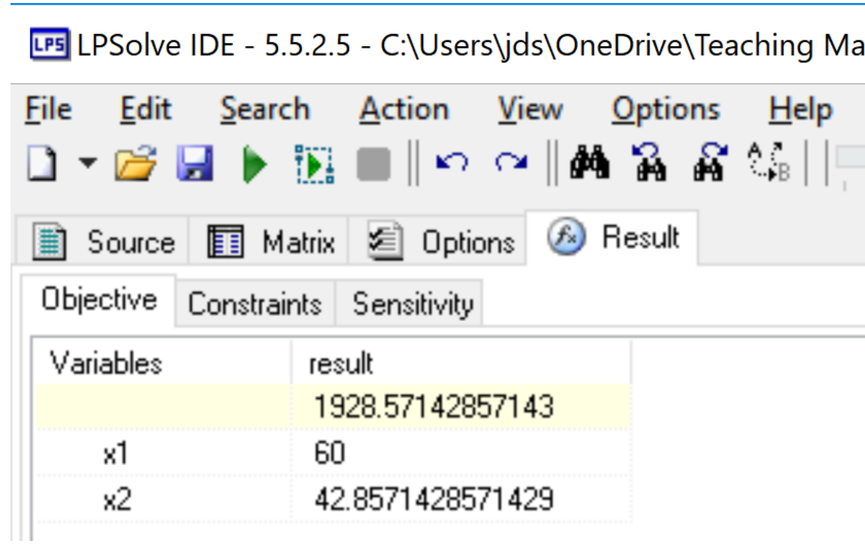
Cell	Variable Name	Value	Lower Limit	Objective Result	Upper Limit	Objective Result
$H$5	Hotdogs Produced	60	0	428.5714286	60	1928.571429
$H$6	Buns Produced	42.85714286	0	1500	42.85714286	1928.571429

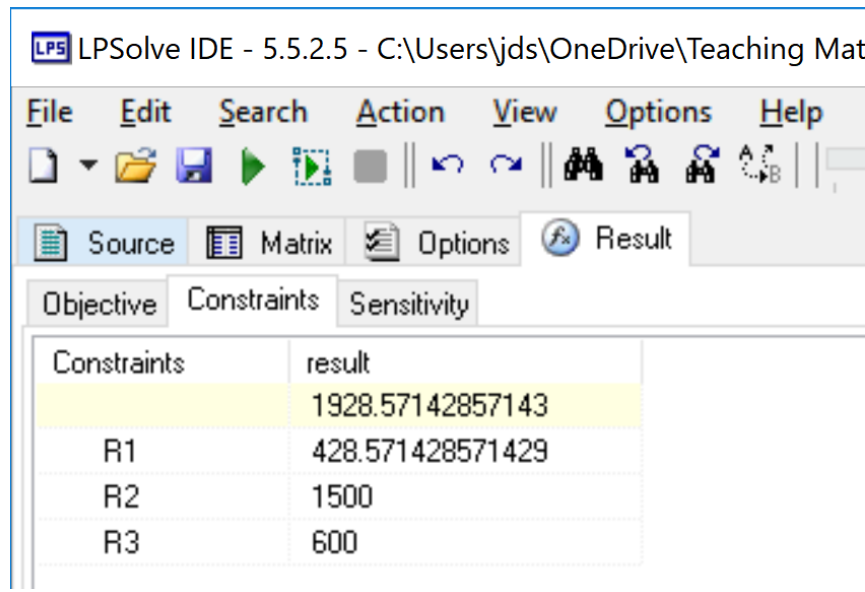
Lower and Upper Limits for the values of the decision variables that still produce a feasible solution; and the corresponding objective function values.

The LP-Solve Constraints Report

Produced fully if all decision variables appear in all constraints.

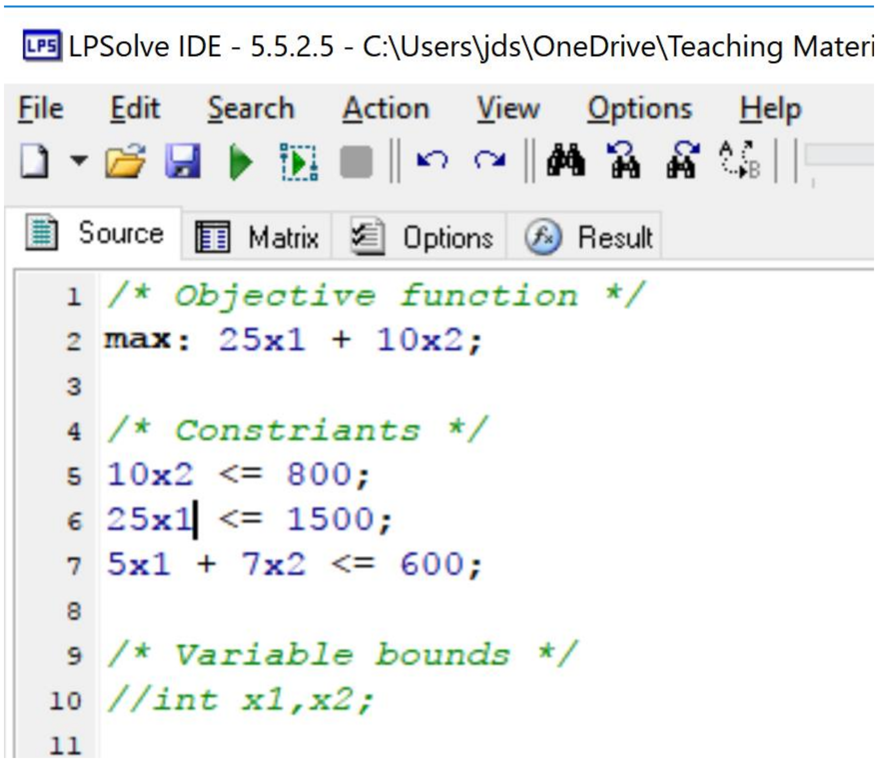


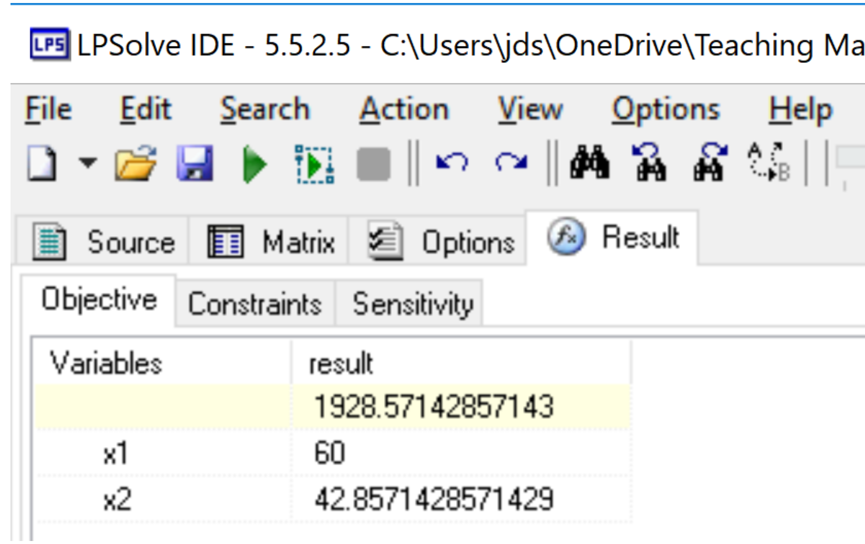


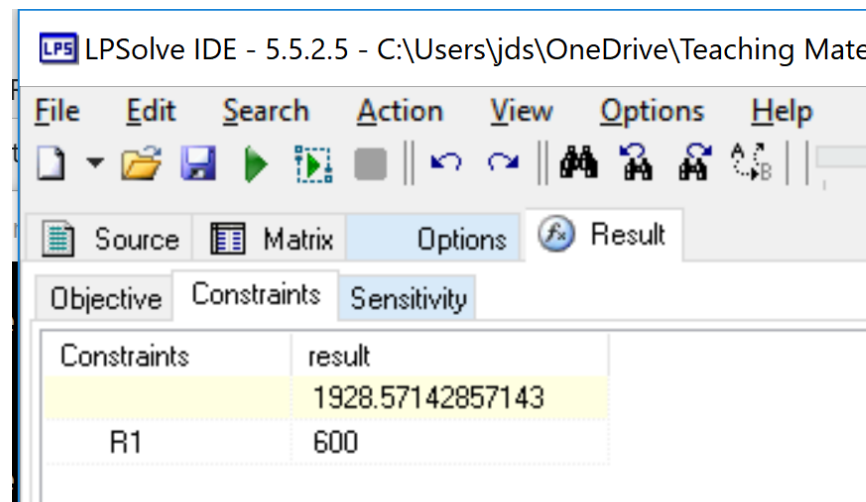


The LP-Solve Constraints Report

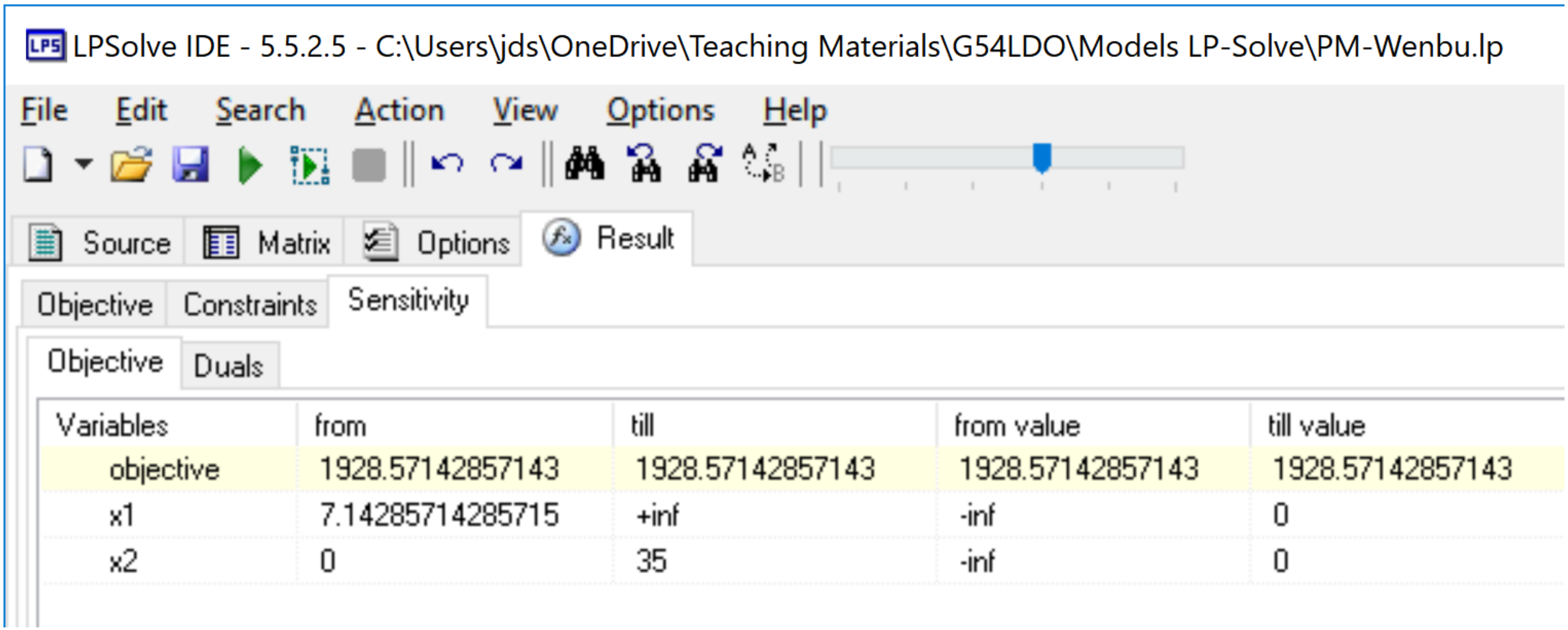
Not produced fully if some decision variables are missing in some constraints.





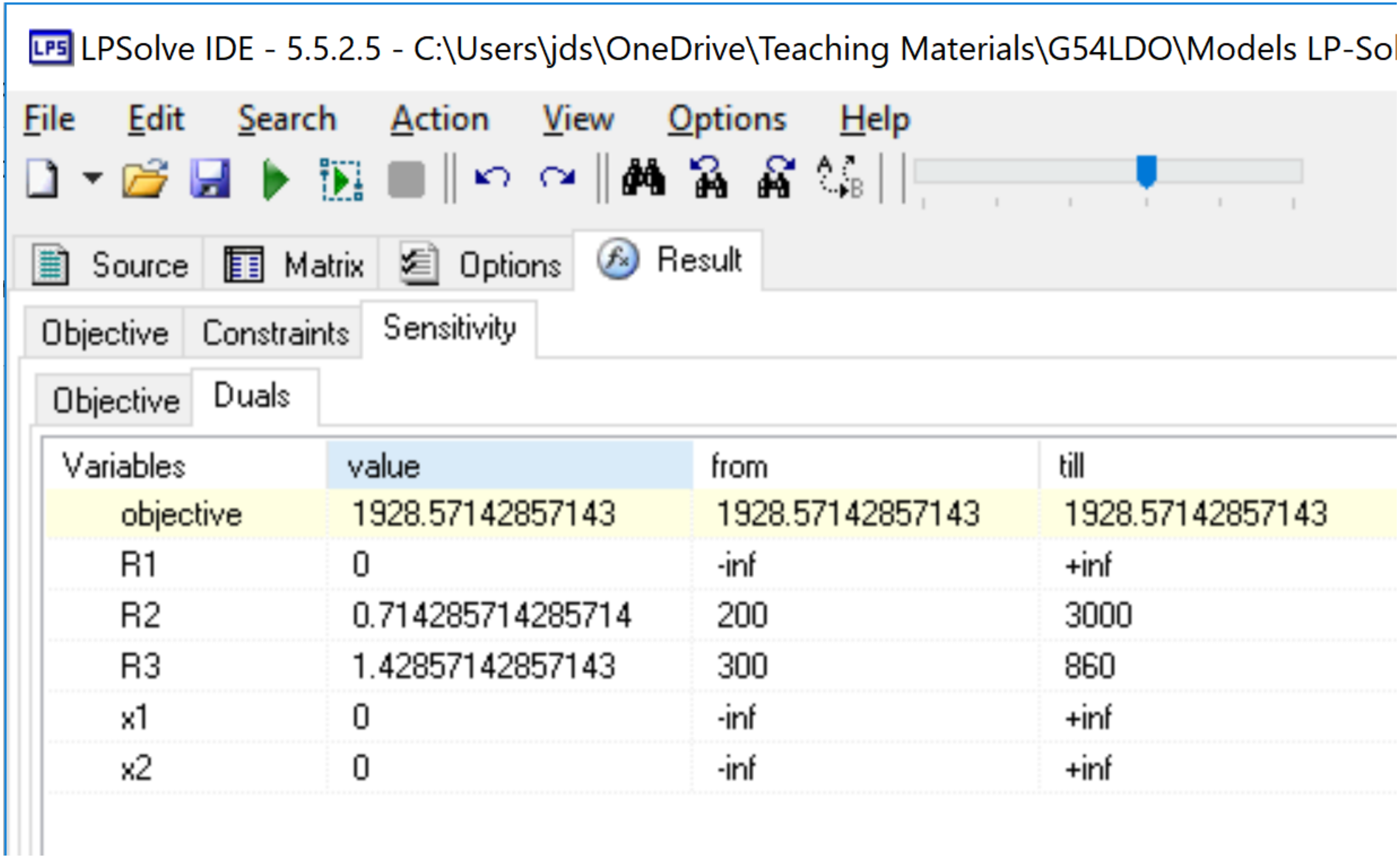


The LP-Solve Sensitivity Report – Objective Tab



Variables	from	till	from value	till value
objective	1928.57142857143	1928.57142857143	1928.57142857143	1928.57142857143
x1	7.14285714285715	+inf	-inf	0
x2	0	35	-inf	0

The LP-Solve Sensitivity Report – Duals Tab



Variables	value	from	till
objective	1928.57142857143	1928.57142857143	1928.57142857143
R1	0	-inf	+inf
R2	0.714285714285714	200	3000
R3	1.42857142857143	300	860
x1	0	-inf	+inf
x2	0	-inf	+inf

Apply Post-optimality Analysis to Product-Mix Problems

Interpret the post-optimality analysis reports in Excel and LP-Solve for the various product-mix optimization problems tackled so far in the module:

- Apex
- Apples
- Atlas
- Bank ABC
- Cargo Airplane
- Coffee Blend
- Furniture
- John Strong
- LP Model C
- Production P1P2
- Vegetables Distribution
- Wenbu
- and any other LP model that you encounter…

Questions OR Comments