Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018

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Lecture 9 – Multi-objective Optimization

- Problems With Multiple Objectives
  - To explain the need for considering multiple objectives in optimization
  - To explain the concept of multi-objective optimization
- Optimality in Multi-objective Problems
  - To illustrate the conflicting nature of multiple objectives
  - To describe the key concepts of multi-objective optimization
- Approaches for Multi-objective Optimization
  - To explain different techniques to tackle multiple-objectives
- Goal Programming
  - Formulate multi-objective problems using goal programming

Additional Reading

Supplement to Chapter 8 of (Hillier and Lieberman, 2015) on Linear Goal Programming and its Solution Procedures.

Chapter 8 of (Taha, 2007)

Chapter 8 of (Rardin, 1998)

A Review of Goal Programming and its Applications. M. Tamiz, D. Jones. Annals of Operations Research, Springer, 1995, 58, 39-53

An Integer Goal Programming Model to Allocate Offices to Staff in an Academic Institution. J. Giannikos, E. El-Darzi, P. Lees. Journal of the Operational Research Ssociety, Vol. 46, No.6, pp. 713-720, 1995.

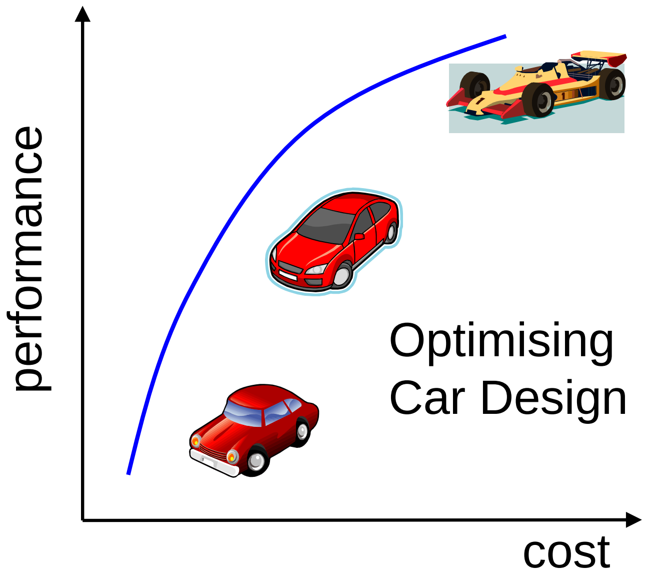
Problems with Multiple Objectives

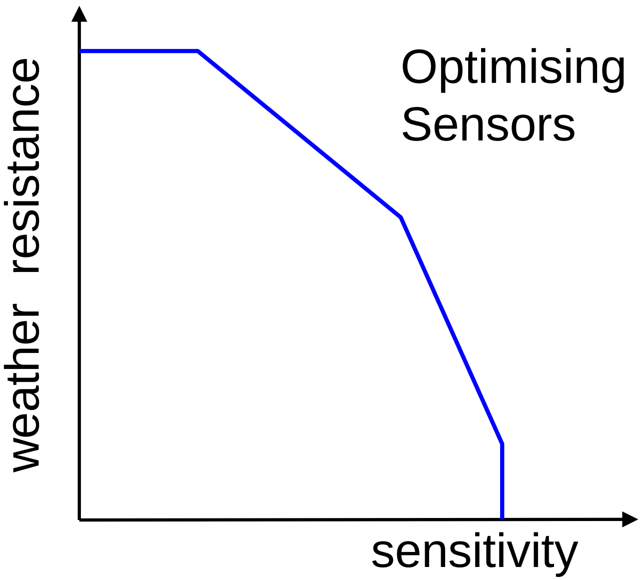
Many real-world problems have multiple objectives but it is acceptable to tackle them using a single-objective approach.

However, in many scenarios various objectives, perhaps some of them conflicting, should be optimized at the same time using a multi-objective approach.

Moreover, the same solution might be evaluated with different criteria expressed by different decision-makers.

Single-objective models are preferred to multi-objective ones because conflicting objectives make models less tractable.


performance
Optimising
Car Design
cost


weather resistance
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Sensors
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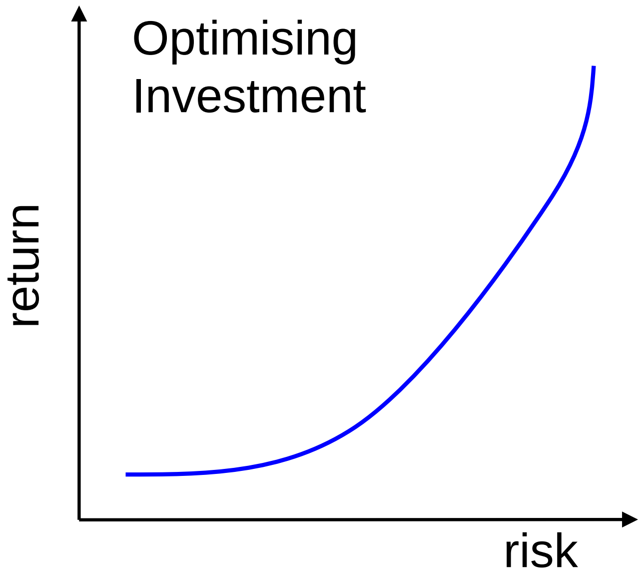
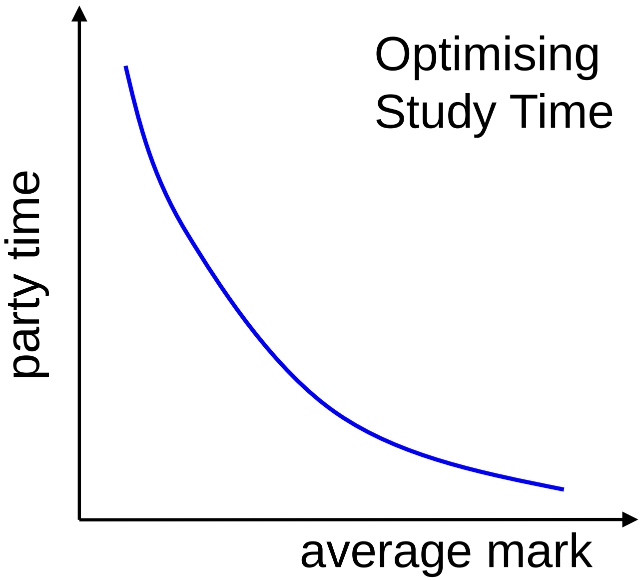

return
Optimising
Investment
risk


party time
Optimising
Study Time
average mark

Example. Consider the following 2-objective LP model and compare the optimum with respect to each objective using the graphical method.

$$\begin{aligned}
&\text{Maximize:} && Z_1 = 3x_1 + x_2 && \\
&\text{Minimize:} && Z_2 = x_1 - x_2 && \\
&\text{Subject to:} && x_1 + x_2 \leq 4 && (1)\\
& && x_1 \leq 3 && (2)\\
& && x_2 \leq 3 && (3)\\
& && x_1, x_2 \geq 0 && (4)
\end{aligned}$$

Objective Z_1 could be profit.

Objective Z_2 could be waste.

First, solve for Z_1 while ignoring Z_2 and then solve for Z_2 while ignoring Z_1.

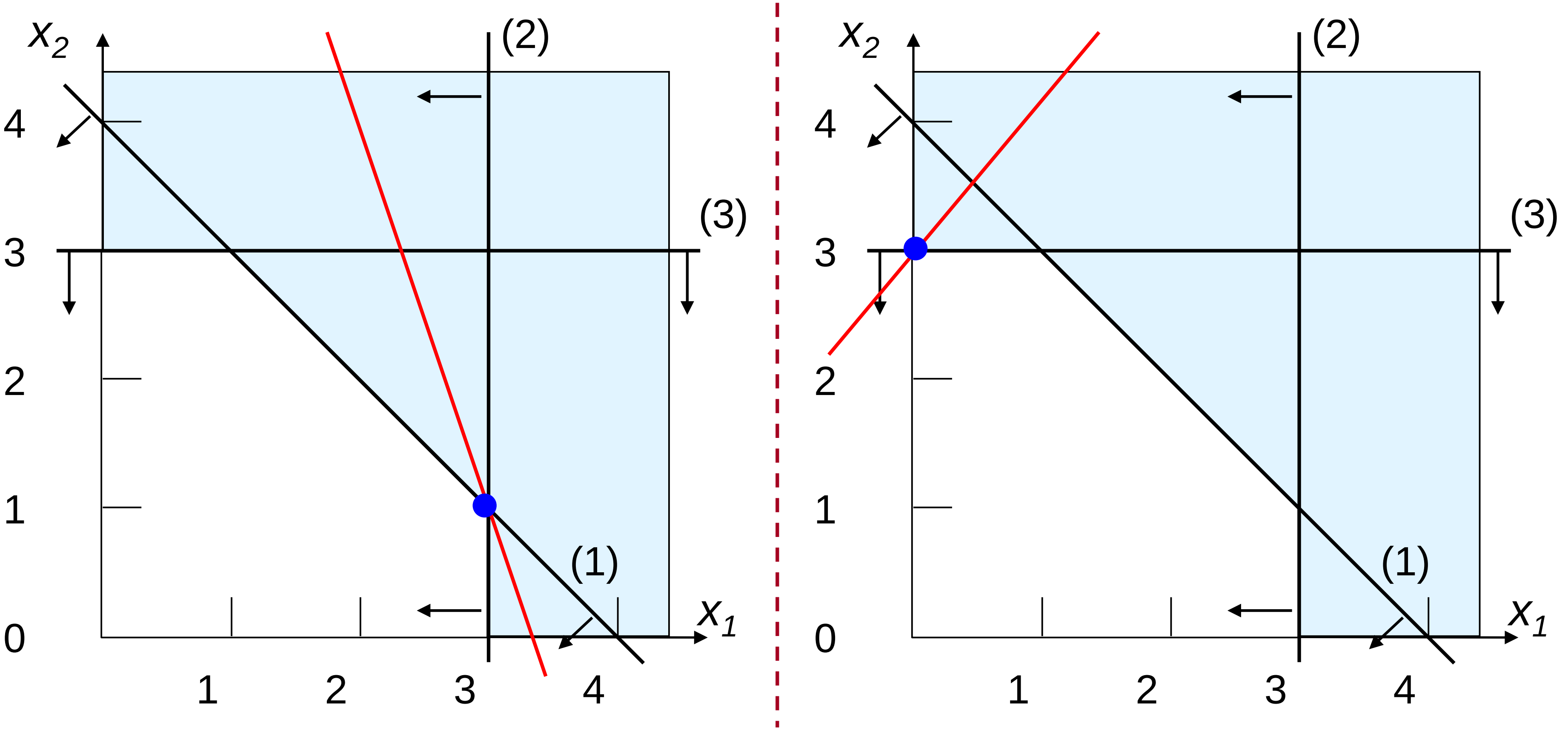
Example (cont.)

$\text{Maximize}: \quad Z_1 = 3x_1 + x_2$

Z_1=10 and Z_2=2

$\text{Minimize}: \quad Z_2 = x_1 - x_2$

Z_2= –3 and Z_1=3



Optimality in Multi-objective Problems

The objectives in multi-objective problems might be:

- Conflicting
- Incommensurable

As a result, the concept of optimal solution becomes less clear in multi-objective problems because each solution represents a trade-off or compromise between objectives.

Given a k-objective optimization problem:

Efficient solution. A solution S is efficient if there is no other solution that is at least as good as S is all k objectives and strictly better than S in one objective.

Efficient frontier. The collection of all efficient points corresponding to efficient solutions in a multi-objective optimisation problem.

An efficient solution is also called Pareto optimal solution or non-dominated solution.

The whole set of efficient solutions is also called the Pareto optimal set of the non-dominated set.

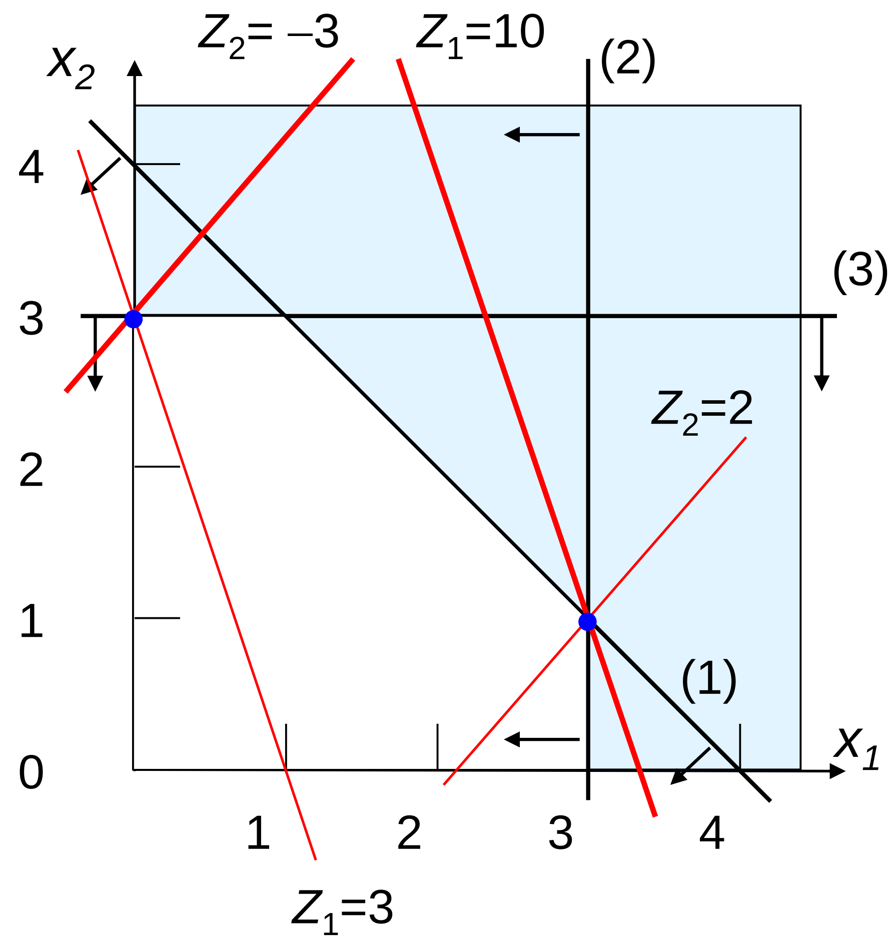
The efficient frontier is also called Pareto optimal front or the non-dominated front.

Other terms used in multi-objective optimization are trade-off and compromise.

Efficient solutions and efficient points

$$
\begin{aligned}
&\text{Maximize:} && Z_1 = 3x_1 + x_2 && \\
&\text{Minimize:} && Z_2 = x_1 - x_2 && \\
&\text{Subject to:} && x_1 + x_2 \leq 4 && (1)\\
& && x_1 \leq 3 && (2)\\
& && x_2 \leq 3 && (3)\\
& && x_1, x_2 \geq 0 && (4)
\end{aligned}
$$

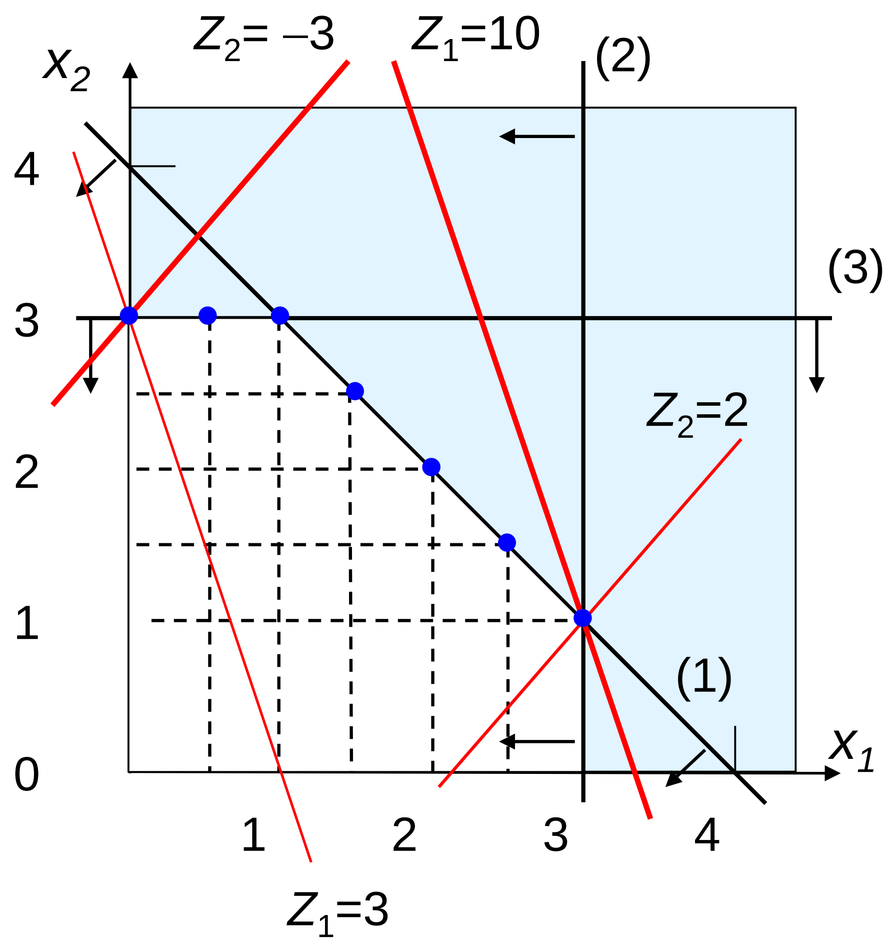
Efficient solutions	Efficient points
$x_1 = 0$	$Z_1 = 3$
$x_2 = 3$	$Z_2 = -3$
--------	----------
$x_1 = 3$	$Z_1 = 10$
$x_2 = 1$	$Z_2 = 2$



Efficient solutions and efficient points

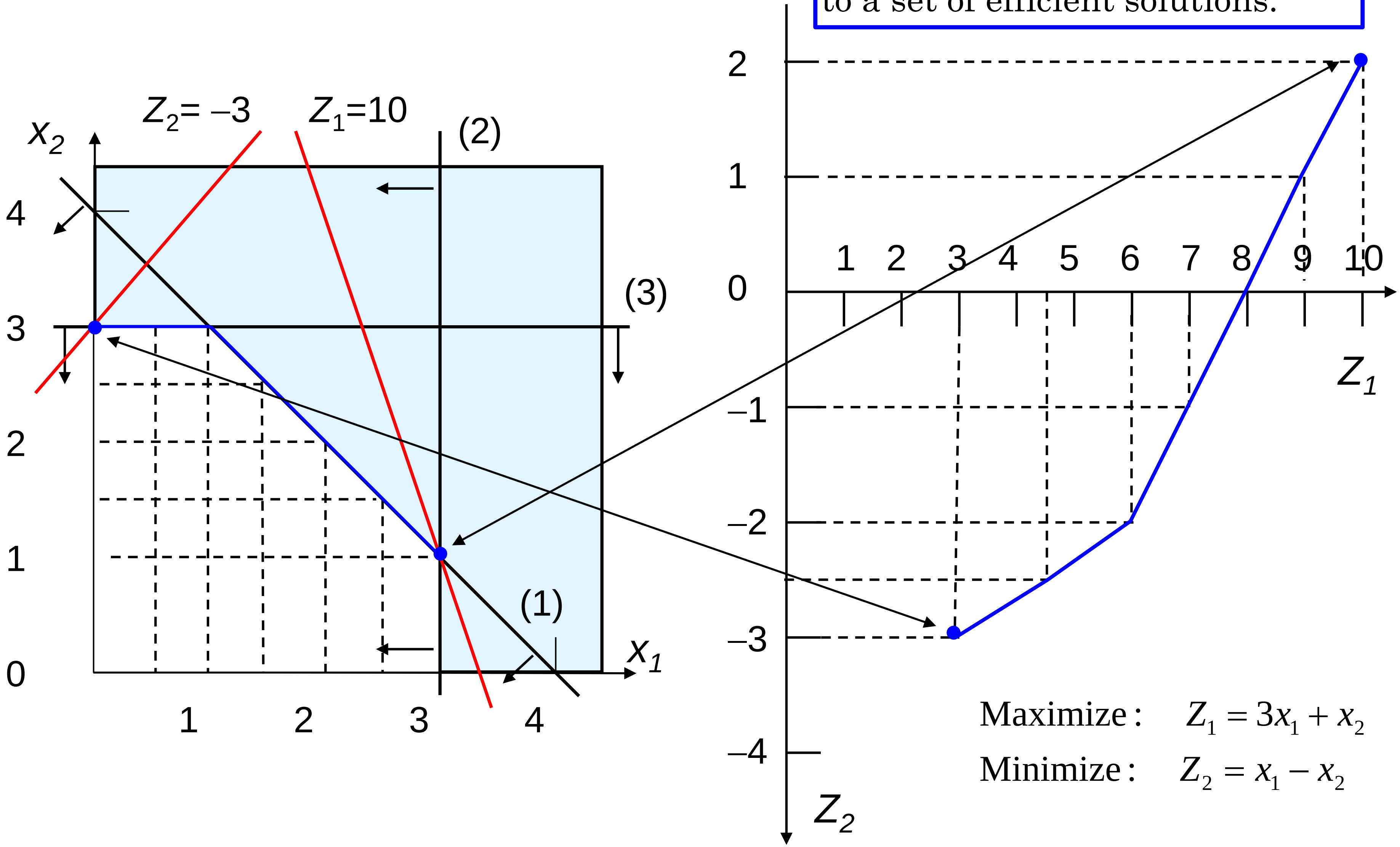
$$\text{Maximize}: \quad Z_1 = 3x_1 + x_2$$
$$\text{Minimize}: \quad Z_2 = x_1 - x_2$$
$$\text{Subject to}: \quad x_1 + x_2 \leq 4 \qquad (1)$$
$$x_1 \leq 3 \qquad (2)$$
$$x_2 \leq 3 \qquad (3)$$
$$x_1, x_2 \geq 0 \qquad (4)$$

ES (x_1,x_2)		EP (Z_1,Z_2)	
0	3	3	–3
0.5	3	4.5	–2.5
1	3	6	–2
1.5	2.5	7	–1
2	2	8	0
2.5	1.5	9	1
3	1	10	2



Decision Space and Objective Space

Efficient frontier formed by a set of efficient points corresponding to a set of efficient solutions.



$$\text{Maximize: } Z_1 = 3x_1 + x_2$$

$$\text{Minimize: } Z_2 = x_1 - x_2$$

Approaches for MOO

A multi-objective optimization (MOO) problem can be expressed as:

Optimize: $F(x) = \left(f_1(x), f_2(x), \ldots, f_k(x)\right)$

Subject to: $x \in X$

where $F(x)$ is the objective vector

$f_i(x)$ is the value of the i^{th} objective

x is the decison vector (set of decision variables)

X is the set of all feasible solutions

x^* is an efficient solution iff x^* is non-dominated with respect to X

x^* is also called Pareto-optimal solution

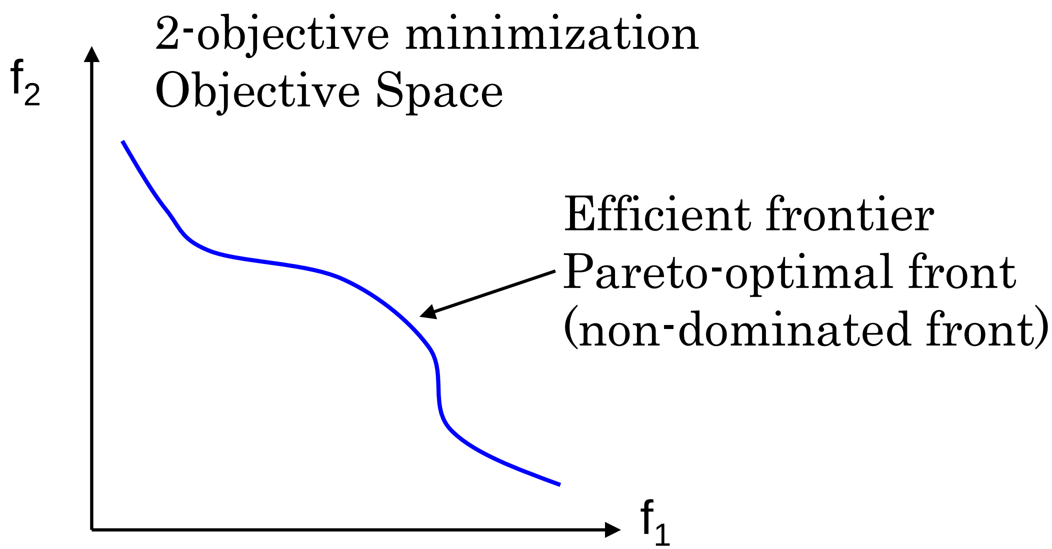
The Pareto optimal set or Efficient frontier is the set of all x^*

How to Combine Decision-Making and Search?

A priori – establish priority between objectives and then search for trade-off solution.

A posteriori – search for a set of efficient solutions and then select trade-off solution.

Interactive – decision-making and search are carried out simultaneously.

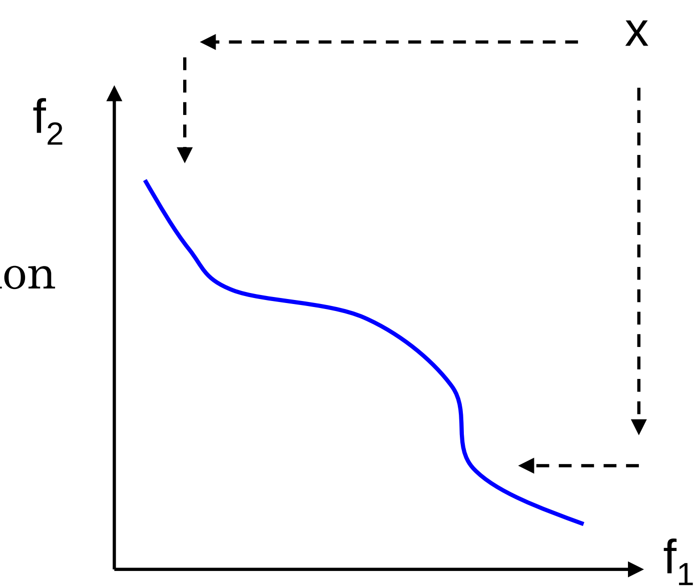


Lexicographic ordering

1. Order the objectives according to priority i=1,2,…,k
2. For each objective i in the priority order
   2.1 Solve the problem for the objective i
   2.2 Add constraint to restrict detriment in objective i

If each stage gives a single-objective optimum then the final solution is an efficient solution.

2-objective minimization
Objective Space



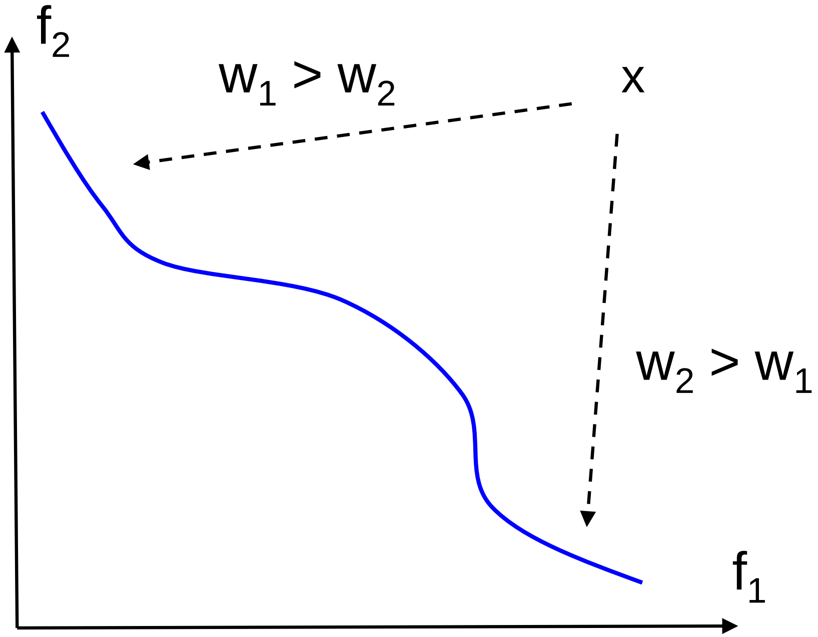
Weighted aggregation

Problem is treated as a single-optimisation problem

$$F(x) = w_1 f_1(x) + w_2 f_2(x) + \ldots w_k f_k(x)$$

If the weighted aggregation approach gives an optimum then this solution is an efficient solution.

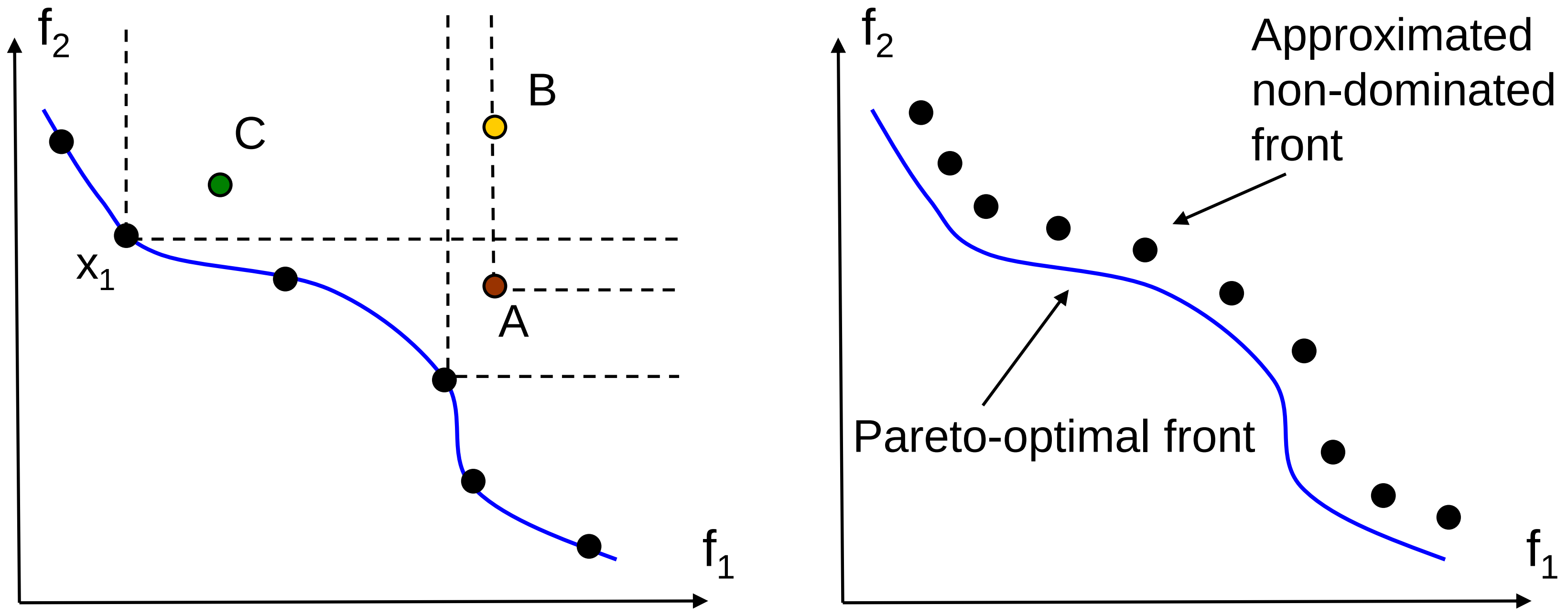
2-objective minimization
Objective Space



Pareto optimization

Search for a set of non-dominated solutions (hopefully Pareto optimal). A solution x_1 is non-dominated with respect to a set S if there is no other solution in S that dominates x_1. The aim is to obtain only efficient solutions.

2-objective minimization
Objective Space



Example. Use the lexicographic approach to solve the following 2-objective optimization problem.

$$\begin{aligned} &\text{Maximize:} \quad Z_1 = 3x_1 + x_2 \\ &\text{Minimize:} \quad Z_2 = x_1 - x_2 \\ &\text{Subject to:} \quad x_1 + x_2 \le 4 \qquad (1) \\ &\qquad\qquad\quad x_1 \le 3 \qquad (2) \\ &\qquad\qquad\quad x_2 \le 3 \qquad (3) \\ &\qquad\qquad\quad x_1, x_2 \ge 0 \qquad (4) \end{aligned}$$

Assuming priority order: Z_1 (higher priority) Z_2 (lower priority)

- First solve for Z_1 obtaining $x_1 = 3$, $x_2 = 1$, $Z_1 = 10$
- Then add the constraint $3x_1 + x_2 = 10$
- Now solve for Z_2 obtaining $x_1 = 3$, $x_2 = 1$, $Z_2 = 2$
- Then the solution vector (3,1) is an efficient solution and the point (10,2) is an efficient point
- Reversing the priority and following the same procedure gives the efficient solution (0,3) and efficient point (3,–3)

Example. Use the weighted approach to solve the following 2-objective optimization problem and obtain a Pareto optimal front with 3 solutions.

$$\begin{array}{lll} \text{Maximize:} & Z_1 = 3x_1 + x_2 & \\ \text{Minimize:} & Z_2 = x_1 - x_2 & \\ \text{Subject to:} & x_1 + x_2 \leq 4 & (1) \\ & x_1 \leq 3 & (2) \\ & x_2 \leq 3 & (3) \\ & x_1, x_2 \geq 0 & (4) \end{array}$$

Replace the two objective functions by:
$Z = w_1(3x_1+x_2) - w_2(x_1-x_2)$
Then solve for 3 combinations of weights, for example:
$(w_1=10,w_2=1),(w_1=4,w_2=10),(w_1=1,w_2=10)$
This gives the efficient solutions and efficient points:

Solution: (3,1)	Point: (10,2)
Solution: (1,3)	Point: (6,–2)
Solution: (0,3)	Point: (3,–3)

Goal Programming

Given a MOO problem, the goal programming approach consists in transforming the multi-objective formulation to a single-objective model by setting goals for each objective and minimizing the deviation from these goals.

1. Establish a goal or target value for each $f_i(x)$ of the k objectives in the problem
2. Convert the objective expressions to constraints using the goal for each objective
3. Introduce a deficiency variables d_i for each of the k objectives to model the deviation from the goal
4. Minimize the sum of deviation variables and solve as a single-objective problem

Example. Goal programming for a MOO problem.

$$\begin{aligned} &\text{Maximize:} \quad Z_1 = 3x_1 + x_2 \\ &\text{Minimize:} \quad Z_2 = x_1 - x_2 \\ &\text{Subject to:} \quad x_1 + x_2 \le 4 \quad (1) \\ &\qquad x_1 \le 3 \quad (2) \\ &\qquad x_2 \le 3 \quad (3) \\ &\qquad x_1, x_2 \ge 0 \quad (4) \end{aligned}$$

$\Rightarrow$

Set goals

$\Omega_1 = 8$

$\Omega_2 = 0$

$\Rightarrow$

$$\begin{aligned} &\text{Subject to:} \\ &\qquad 3x_1 + x_2 \ge 8 \quad (0a) \\ &\qquad x_1 - x_2 \le 0 \quad (0b) \\ &\qquad x_1 + x_2 \le 4 \quad (1) \\ &\qquad x_1 \le 3 \quad (2) \\ &\qquad x_2 \le 3 \quad (3) \\ &\qquad x_1, x_2 \ge 0 \quad (4) \end{aligned}$$

$\Downarrow$

Introduce deficiency variables d_1 and d_2

$\Leftarrow$

$$\begin{aligned} &\text{Minimize:} \quad Z = d_1 + d_2 \\ &\text{Subject to:} \\ &\qquad 3x_1 + x_2 + d_1 \ge 8 \quad (0a) \\ &\qquad x_1 - x_2 - d_2 \le 0 \quad (0b) \\ &\qquad x_1 + x_2 \le 4 \quad (1) \\ &\qquad x_1 \le 3 \quad (2) \\ &\qquad x_2 \le 3 \quad (3) \\ &\qquad x_1, x_2, d_1, d_2 \ge 0 \quad (4) \end{aligned}$$

Note: the deficiency variables are introduced with positive or negative sign (or both) depending on the inequality or equality in the constraint.

Example. Formulate the following MOO problem as a Goal Programming model. Assume that the decision maker has set these goals: at least Ω_1 for Z_1, at most Ω_2 for Z_2, at most Ω_3 for Z_3, Z_4 equal to Ω_4.

$$
\begin{array}{lll}
\text{Maximize:} & Z_1 = 4x_2 + 4.5x_3 + 5.5x_4 + 7x_5 & \\
\text{Minimize:} & Z_2 = 5x_1 + 4x_2 + 7.5x_3 + 5x_4 + x_5 & \\
\text{Minimize:} & Z_3 = (x_1 + x_3 + x_5)/10 & \\
\text{Maximize:} & Z_4 = 10x_1 + 2x_4 + 3x_5 & \\
\text{Subject to:} & x_1 + x_2 + x_3 = 140 & (1) \\
 & x_2 + 9x_3 + 6x_4 - x_5 = 0 & (2) \\
 & x_1 - 0.5x_4 + 0.7x_5 \leq 30 & (3) \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0 & (4)
\end{array}
$$

Example (cont.)

$$\text{Minimize: } d_1 + d_2 + d_3 + d_4 + d_5$$

$$\text{Subject to: } 4x_2 + 4.5x_3 + 5.5x_4 + 7x_5 + d_1 \geq \Omega_1 \qquad (0a)$$

$$5x_1 + 4x_2 + 7.5x_3 + 5x_4 + x_5 - d_2 \leq \Omega_2 \qquad (0b)$$

$$(x_1 + x_3 + x_5)/10 - d_3 \leq \Omega_3 \qquad (0c)$$

$$10x_1 + 2x_4 + 3x_5 + d_4 - d_5 = \Omega_4 \qquad (0d)$$

$$x_1 + x_2 + x_3 = 140 \qquad (1)$$

$$x_2 + 9x_3 + 6x_4 - x_5 = 0 \qquad (2)$$

$$x_1 - 0.5x_4 + 0.7x_5 \leq 30 \qquad (3)$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \qquad (4)$$

$$d_1, d_2, d_3, d_4, d_5 \geq 0 \qquad (5)$$