Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018

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Lecture 3 – Product-Mix Optimization

- Some Special Cases in LP Models

  To interpret some special cases in LP models using the graphical method

- Product-Mix Problems

  To identify and develop models for product-mix optimization problems

  To formulate typical constraints arising in product-mix optimization problems

- Algebraic vs. Spreadsheet Models

  To appreciate the differences between developing algebraic and spreadsheet models to solve optimization problems

Additional Reading

Chapter on Linear Programming of any book in the reading list.

Sections 3.5, 3.6 and Chapter 21 of the book (Hillier and Lieberman, 2015) that cover the formulation of LP models on Spreadsheets.

Modelling Optimization Problems in the Unstructured World of Spreadsheets. D.G. Conway, C.T. Ragsdale. Omega: International Journal of Management Science, Vol. 25(3), pp. 313-322, 1997.

Designing Optimal Food Intake Patterns to Achieve Nutritional Goals for Japanese Adults Through the Use of LP Optimization Models. H. Okubo, S. Sasaki, K. Murakami, T. Yokoyama, N. Hirota, A. Notsu, M. Fukui, C. Date. Nutrition Journal, Vol. 14, pp. 57-66, doi: 10.1186/s12937-015-0047-7, 2015.

Summation Sign – An Interactive Learning Object. Sue Cobb, Matt Donaghy, Richard Field, Sandra Hill, John Horton. Available Online at: http://www.nottingham.ac.uk/toolkits/play_232

Some Special Cases in LP Models

Some special cases can occur in an LP formulation:

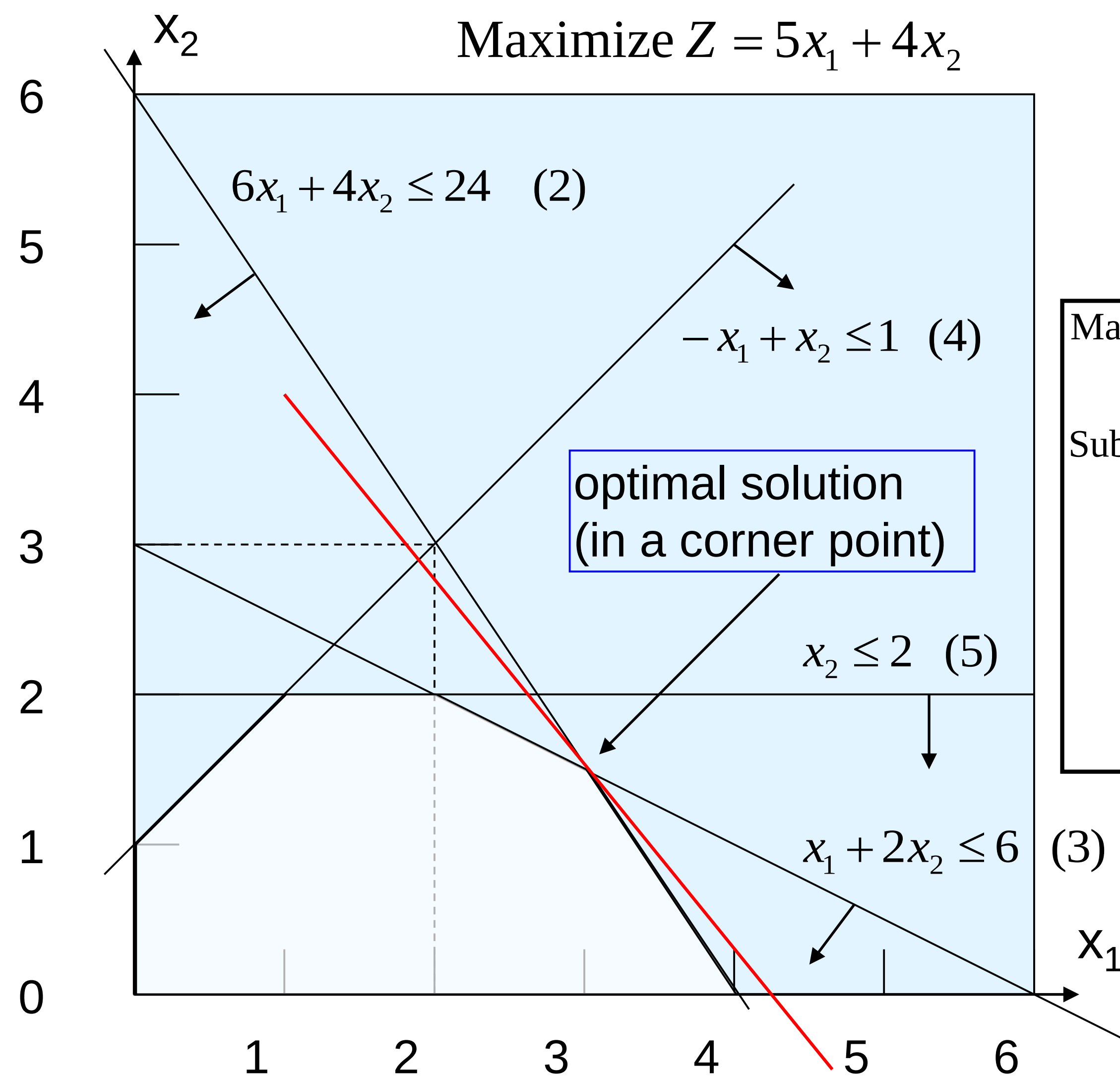
- There is no feasible solution (hence no optimal solution) because there is no feasible region
- The search space is unbounded (hence no defined optimal solution)
- There are multiple optimal solutions defined

The corner points define the feasible region in an LP formulation (with bounded feasible region). Then a corner-point feasible (CPF) solution is a solution that lies at a corner point of the feasible region.

The original ATLAS LP model

LP optimal solution:

$x_1 = 3.0$
$x_2 = 1.5$
$Z = 21.0$



Maximise:

$$Z = 5x_1 + 4x_2 \quad (1)$$

Subject to:

$$6x_1 + 4x_2 \le 24 \quad (2)$$
$$x_1 + 2x_2 \le 6 \quad (3)$$
$$-x_1 + x_2 \le 1 \quad (4)$$
$$x_2 \le 2 \quad (5)$$
$$x_1 \ge 0, x_2 \ge 0 \quad (6)$$

No feasible solution

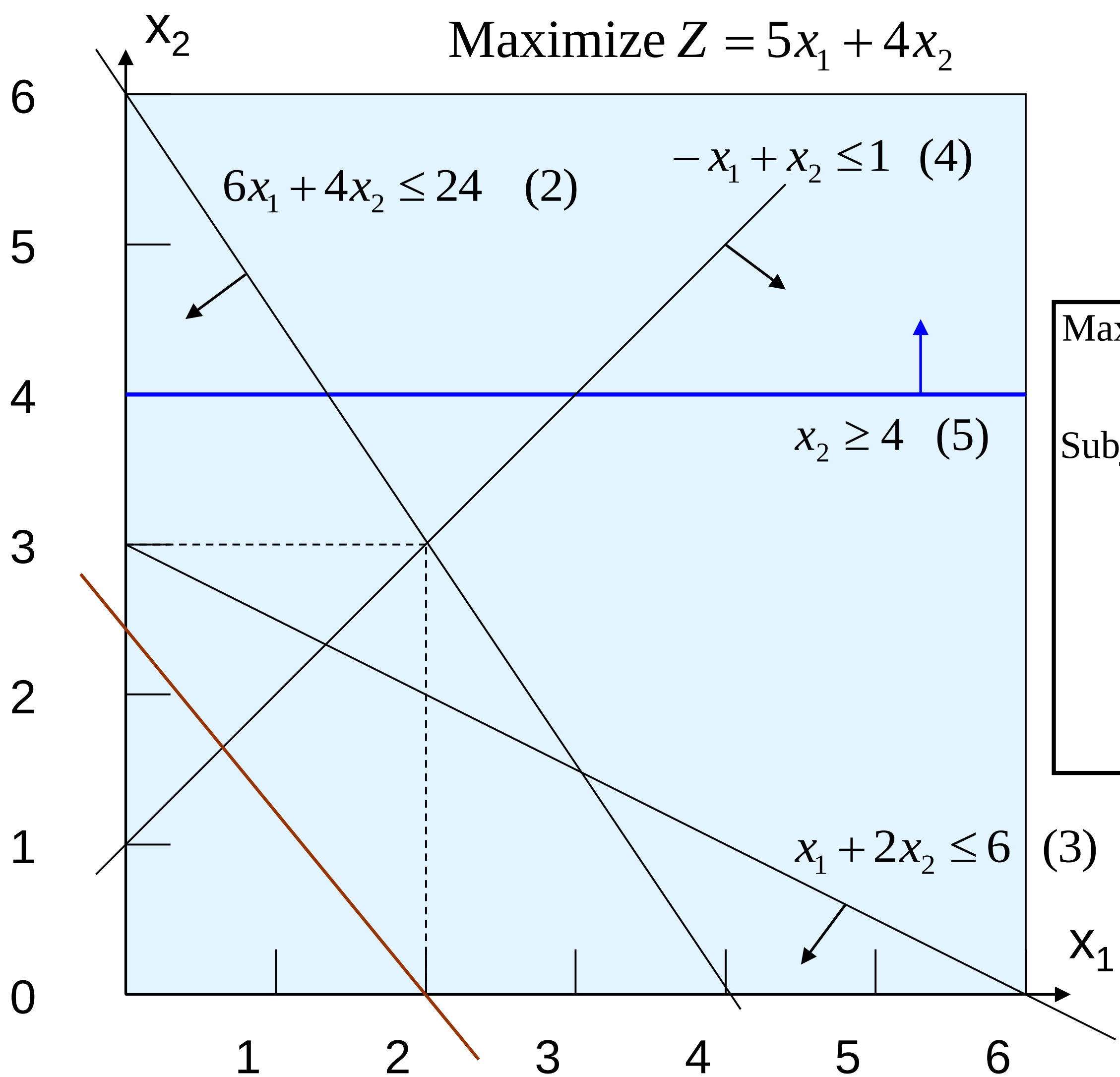
If we replace the constraint

$$x_2 \leq 2$$

By the constraint

$$x_2 \geq 4$$

Please pay attention, optimization solvers report when a model is infeasible.



Maximize $Z = 5x_1 + 4x_2$

Maximize:

$$Z = 5x_1 + 4x_2 \quad (1)$$

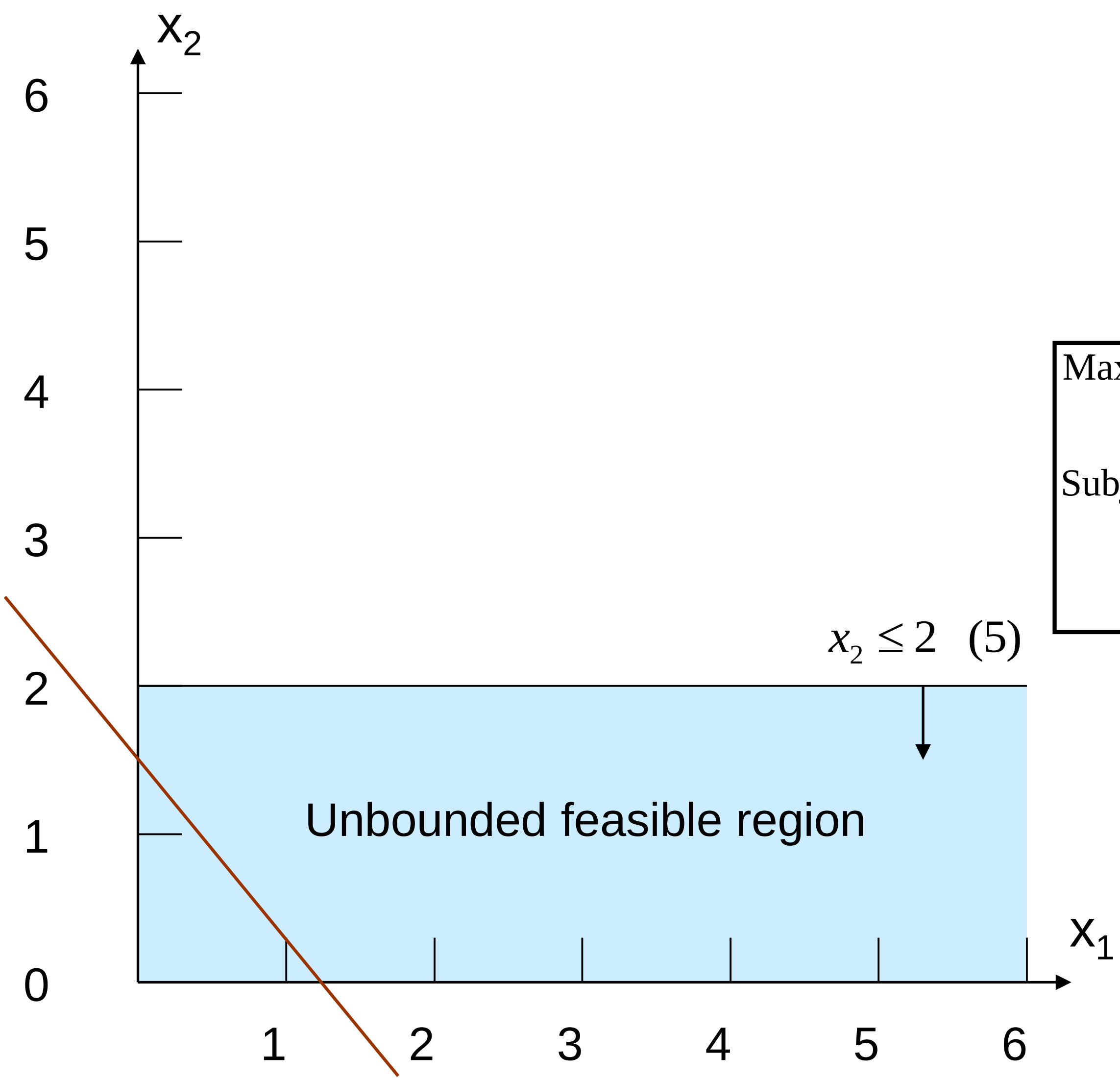
Subject to:

$$6x_1 + 4x_2 \leq 24 \quad (2)$$
$$x_1 + 2x_2 \leq 6 \quad (3)$$
$$-x_1 + x_2 \leq 1 \quad (4)$$
$$x_2 \geq 4 \quad (5)$$
$$x_1 \geq 0, x_2 \geq 0 \quad (6)$$

Unbounded search space

If we remove the constraints

$$6x_1 + 4x_2 \leq 24 \quad (2)$$
$$x_1 + 2x_2 \leq 6 \quad (3)$$
$$-x_1 + x_2 \leq 1 \quad (4)$$



Maximize:

$$Z = 5x_1 + 4x_2 \quad (1)$$

Subject to:

$$x_2 \leq 2 \quad (5)$$
$$x_1 \geq 0, x_2 \geq 0 \quad (6)$$

Please pay attention, optimization solvers report when a model is unbounded.

Multiple optimal solutions

If the problem has exactly one optimal solution, it must be a CPF solution. If the problem has multiple optimal solutions, at least two of these solutions must be CPF solutions.

If we replace the objective function

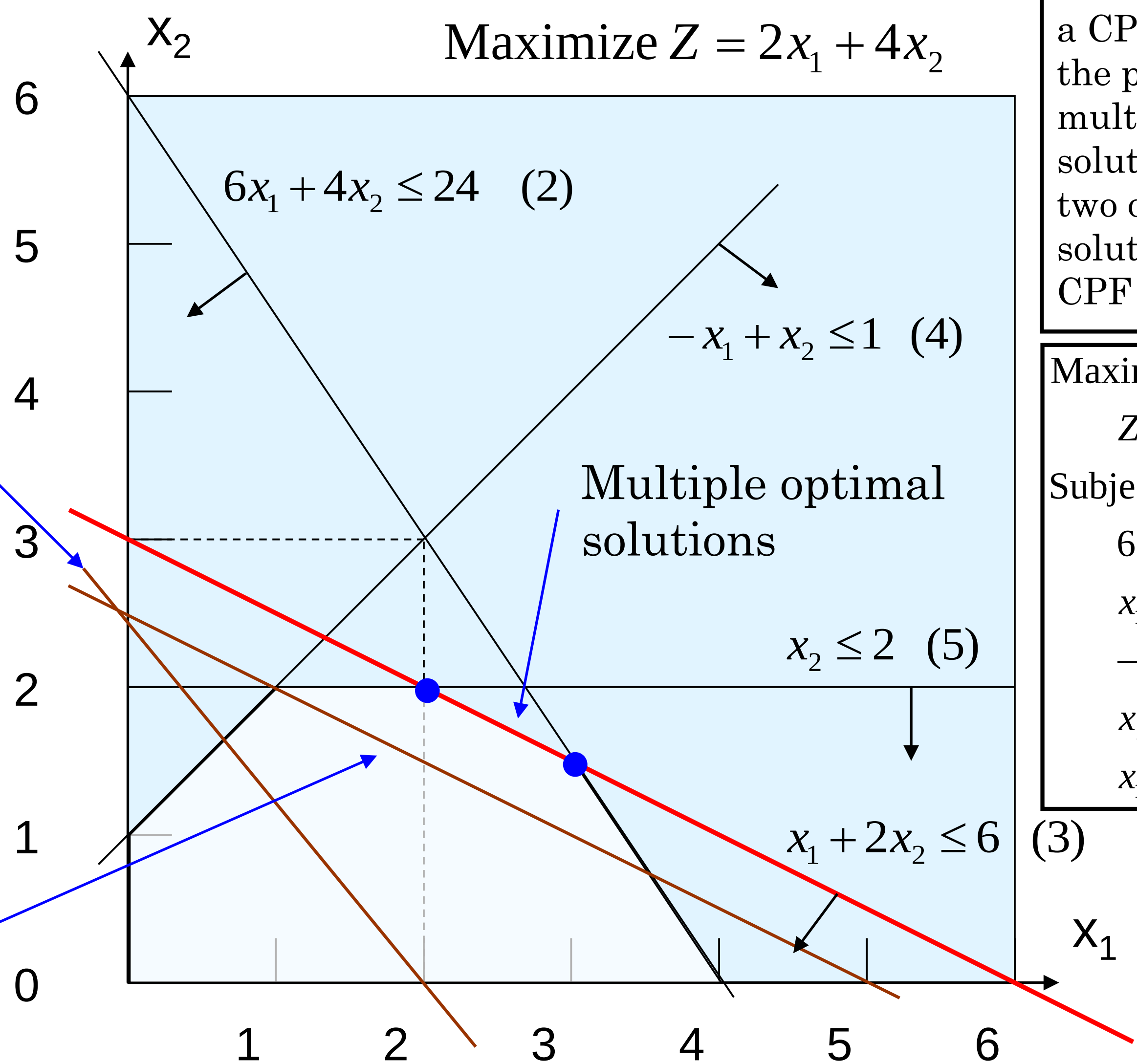
$$\text{Max}\, Z = 5x_1 + 4x_2$$

By the objective function

$$\text{Max}\, Z = 2x_1 + 4x_2$$

Draw line for the objective function for one solution

$$10 = 2x_1 + 4x_2$$



Maximize:

$$Z = 2x_1 + 4x_2 \quad (1)$$

Subject to:

$$6x_1 + 4x_2 \le 24 \quad (2)$$
$$x_1 + 2x_2 \le 6 \quad (3)$$
$$-x_1 + x_2 \le 1 \quad (4)$$
$$x_2 \le 2 \quad (5)$$
$$x_1 \ge 0, x_2 \ge 0 \quad (6)$$

An optimization solver would normally only report one optimal solution, the first one found.

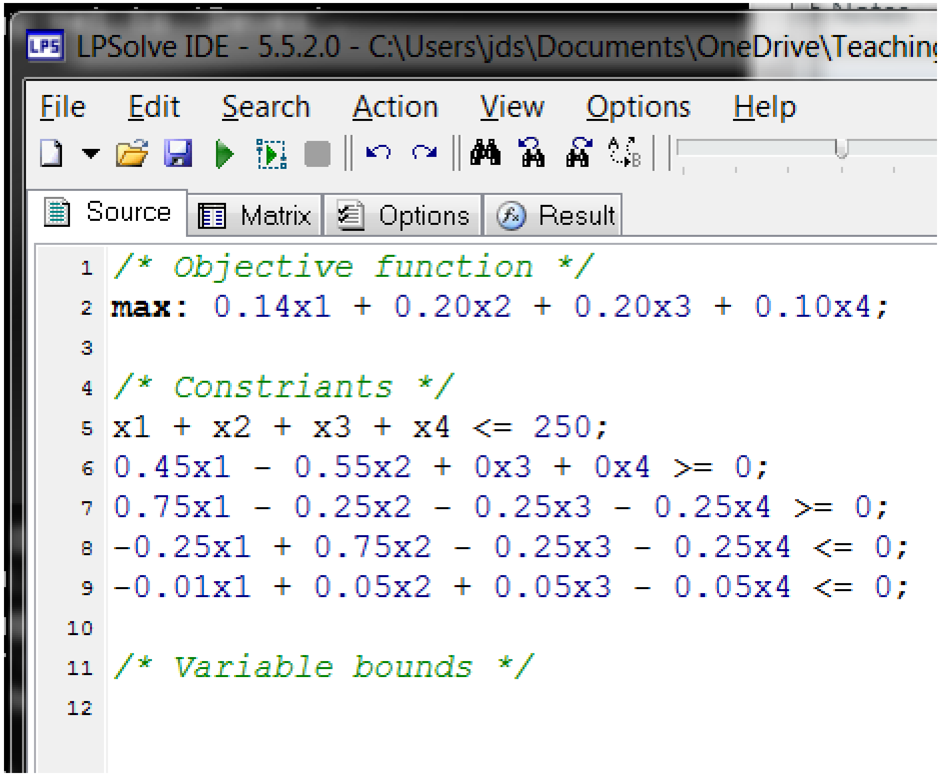
For a model with multiple optimal solutions, which solution a solver finds can depend on the 'layout' of the model as that influences in which order the solver tackles the constraints and explores the search space.

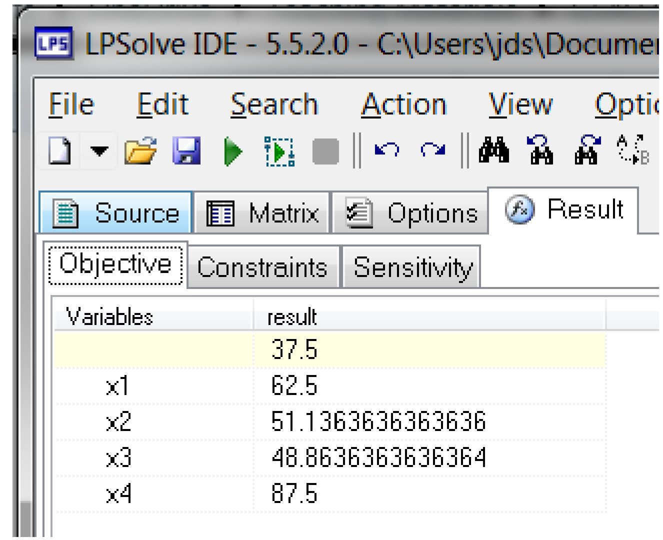
If for an LP model, it is known that there are multiple optimal solutions (e.g. the Bank ABC problem) then:

- How to find alternative optimal solutions in Excel and LP-Solve?

- How many alternative optimal solutions can

LP-Solve gives the following solution to the LP BANK ABC problem:

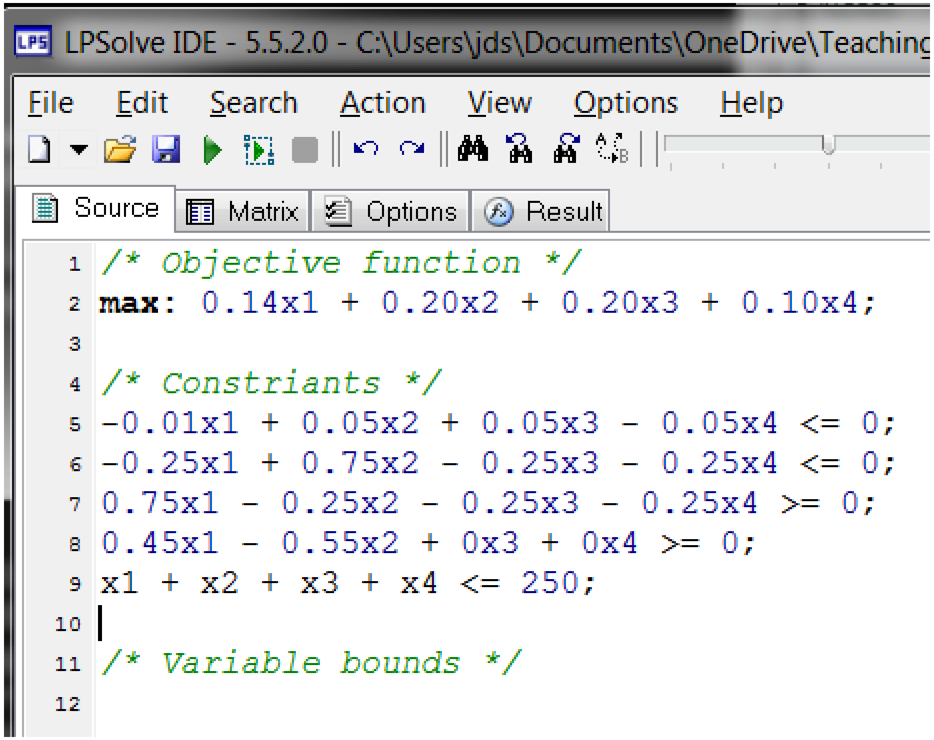


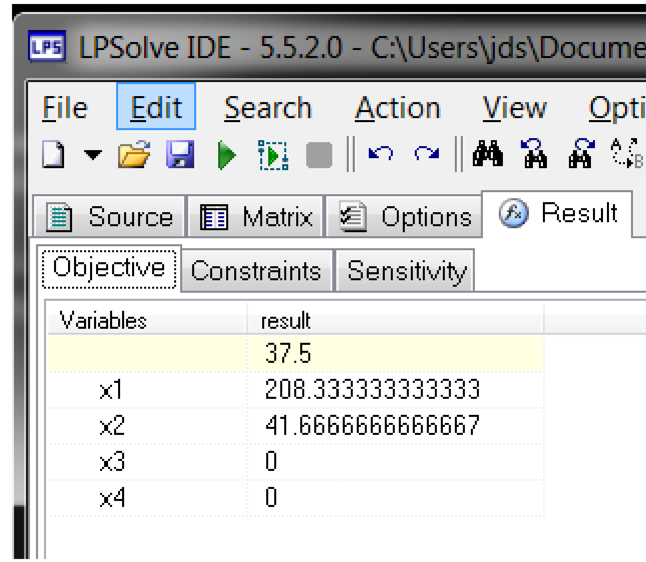


Constraints can be added to find another of the multiple optimal solutions, for example:

- Simply adding $x_1 \leq 62$, or
- Simply adding $x_3 \geq 50$, etc.

Simply shuffling the constraints and LP-Solve gives another of the multiple optimal solutions:





Constraints can be added to find another of the multiple optimal solutions, for example:

- Simply adding $x_3 + x_4 \geq 50$, or
- Simply adding $x_1 - x_2 \leq 100$, etc.

Feasible region split?

Are there really 2 feasible regions?

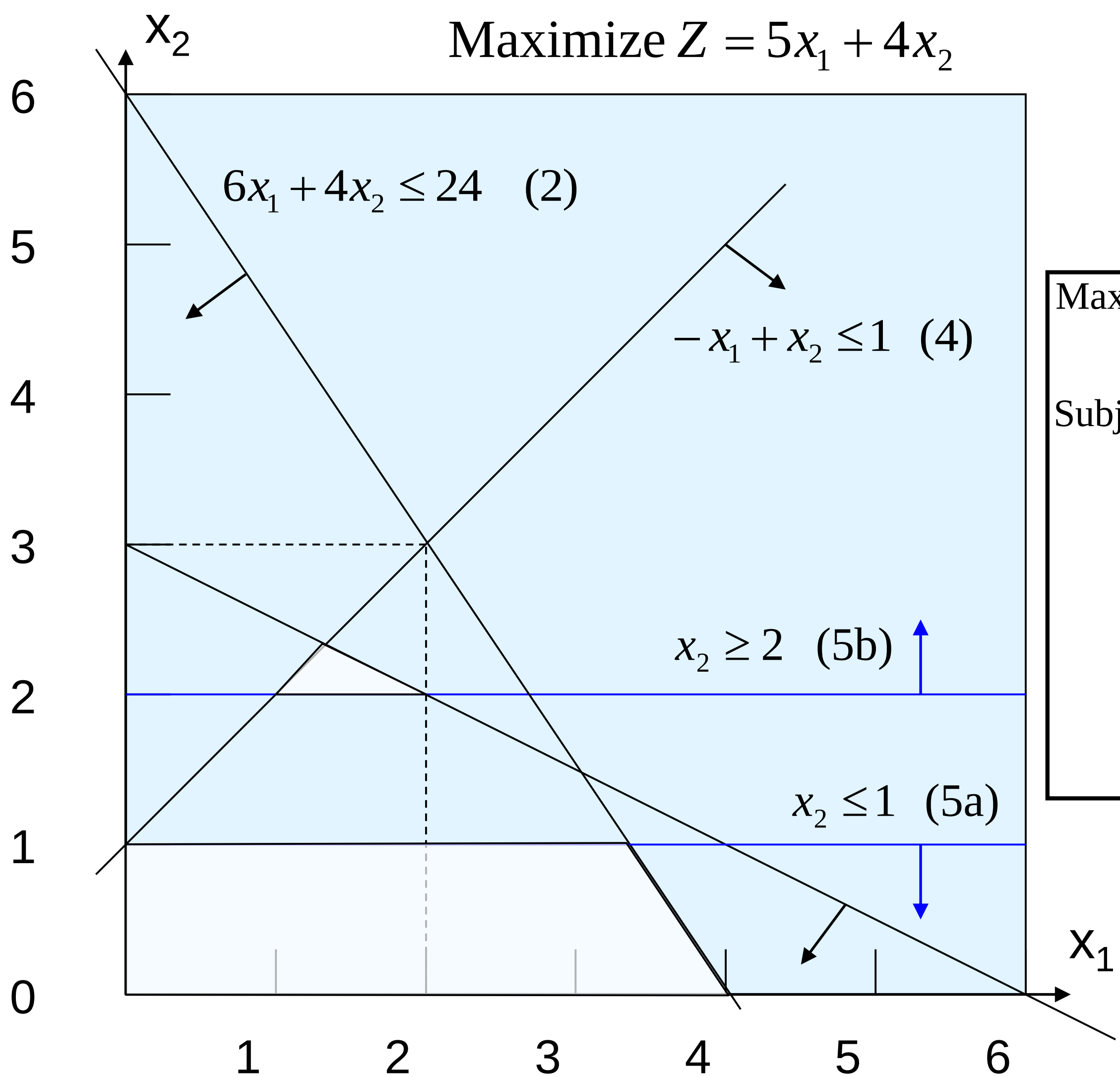
Replace constraint

$x_2 \le 2$

By the 2 following constraints (assume that they do not want the production of product B to be within a certain range.

$x_2 \le 1$

$x_2 \ge 2$



Maximize:

$Z = 5x_1 + 4x_2 \quad (1)$

Subject to:

$6x_1 + 4x_2 \le 24 \quad (2)$

$x_1 + 2x_2 \le 6 \quad (3)$

$-x_1 + x_2 \le 1 \quad (4)$

$x_2 \le 1 \quad (5a)$

$x_2 \ge 2 \quad (5b)$

$x_1 \ge 0, x_2 \ge 0 \quad (6)$

Product-Mix Problems

Product-mix problems are a type of LP problems where the objective is to find the optimal 'mix' for the products under consideration.

Examples include:

- optimize production planning activities
- product consumption to minimize costs
- optimize budget allocation to activities
- optimize the distribution of limited resources
- optimize the mix or blending of products

Typing Compact Algebraic Notation in the Moodle Tests

For typing algebraic expressions in 'compact notation' in the Moodle text editor for the tests, use simple text in the following format.

Maximize: Z = SUM[i=1 to 4] (a_i X_i)

Subject to: SUM[i=1 to 4] (b_ji x_i) <= c_j   for j=1 to 5
             x_i >= 0 for i=1 to 4

Equivalent to:

$$\text{Maximize:} \quad Z = \sum_{i=1}^{4} a_i X_i$$

$$\text{Subject to:} \quad \sum_{i=1}^{4} b_{ji} X_i \le c_j \quad \text{for } j = 1 \text{ to } 5$$

$$X_i \ge 0 \quad \text{for } i = 1 \text{ to } 4$$

The 'expanded notation is the one used in LP-Solve for example.

There is no need to type a multiplication sign but sometimes you might need to use parenthesis to establish precedence clearly, for example in this expression: a_i + (a_i + b_i)/2

Example. Type the LP model for the VEGETABLES DISTRIBUTION PROBLEM (see lecture 2 notes) in compact notation using simple text format.

$$\text{Minimize}: \quad Z = \sum_{i=1}^{15} (P_i - C_i) X_i \qquad (1)$$

$$\text{Subject to}: \quad X_i \geq \text{Min} \quad \text{for } i = 1 \ldots 15 \qquad (2)$$

$$X_i \leq \text{Max} \quad \text{for } i = 1 \ldots 15 \qquad (3)$$

$$\sum_{i=1}^{15} 1.25 X_i \leq 18000 \qquad (4)$$

$$\sum_{i=1}^{15} C_i X_i \leq 30000 \qquad (5)$$

$$X_i \geq 0 \quad \text{for } i = 1 \ldots 15 \qquad (6)$$

Note that using parenthesis for expression (5) in the simple text format might be redundant, just make sure it is very clear what the algebraic expression means.

Variables, Subscripts and Summations

Long algebraic expressions can be written in 'compact notation' or 'expanded notation'. Write the compact notation using simple text format.

$$\text{Max. } Z = \sum_{i=1}^{n} a_i x_i + b \sum_{i=n+1}^{m} x_i$$

$$\text{Max. } Z = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b\left(x_{n+1} + x_{n+2} + \cdots + x_m\right)$$

$$\text{Min. } Z = \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} x_{ij}$$

$$\begin{aligned}\text{Min. } Z = {} & g_{11}x_{11} + g_{12}x_{12} + \cdots + g_{1m}x_{1m} + g_{21}x_{21} + g_{22}x_{22} + \cdots + g_{2m}x_{2m} + \cdots \\ & + g_{n1}x_{n1} + g_{n2}x_{n2} + \cdots + g_{nm}x_{nm}\end{aligned}$$

$$\boxed{\sum_{i=1}^{n} x_{ij} \geq B_j \qquad j = 1 \ldots m}$$

$$
\begin{aligned}
&x_{11} + x_{21} + \cdots + x_{n1} \geq B_1 \\
&x_{12} + x_{22} + \cdots + x_{n2} \geq B_2 \\
&\vdots \\
&x_{1m} + x_{2m} + \cdots + x_{nm} \geq B_m
\end{aligned}
$$

Equalities and Inequalities

Constraints often involve these types of conditions. But usually they do not involve strict inequalities.

$$\boxed{\sum_{i=1}^{n} a_i x_i = b} \qquad \boxed{\sum_{i=1}^{n} a_i x_i \geq b} \qquad \boxed{\sum_{i=1}^{n} a_i x_i \leq b}$$

Example. Write the following in compact notation.

$$\boxed{\begin{aligned} Z = 25\left(0.85x_{1,1} + 0.60x_{2,1} + 0.40x_{3,1}\right) + 20\left(0.90x_{1,2} + 0.65x_{2,2} + 0.75x_{3,2}\right) + \\ 18\left(0.95x_{1,3} + 0.80x_{2,3} + 0.70x_{3,3}\right) + 12\left(0.35x_{1,4} + 0.25x_{2,4} + 0.30x_{3,4}\right) \end{aligned}}$$

$$\boxed{\begin{array}{ll} 0.85x_{1,1} + 0.60x_{2,1} + 0.40x_{3,1} \le 150 & (1) \\ 0.90x_{1,2} + 0.60x_{2,2} + 0.75x_{3,2} \le 750 & (2) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\ 0.95x_{1,15} + 0.80x_{2,15} + 0.60x_{3,15} \le 550 & (15) \end{array}}$$

Variables as Fractions or Ratios of Other Variables

Constraints often state that variables depend on other variables

$$\boxed{x_3 \geq 0.5x_1} \quad \boxed{x_2 \leq 0.5(x_4 + x_6)} \quad \boxed{x_2 \geq 12 + 0.25x_4} \quad \boxed{\frac{x_3}{2} = \frac{x_1}{3}}$$

These type of constraints are very common in Product-Mix LP models.

Constraints as Fractions of Other Constraints

Constraints that involve uniform distribution of resources

$$\boxed{100\frac{x_1 + x_2}{50} = 100\frac{x_3 + x_4}{30} = 100\frac{x_5 + x_6}{70}}$$

Slack and Surplus Variables in Constraints

$$\boxed{\begin{array}{l} x_1 + 3x_2 + 2x_3 \le 50 \\ 2x_1 - x_2 + x_3 \ge 15 \\ x_1, x_2, x_3 \ge 0 \end{array}} \quad \Longrightarrow \quad \boxed{\begin{array}{l} x_1 + 3x_2 + 2x_3 + u = 50 \\ 2x_1 - x_2 + x_3 - v = 15 \\ x_1, x_2, x_3, u, v \ge 0 \\ u \text{ is a slack variable} \\ v \text{ is a surplus variable} \end{array}}$$

Conflicting Constraints

$$\boxed{\begin{array}{l} x_1 + 3x_2 + 2x_3 = 10 \\ x_1 + 3x_2 + 2x_3 = 20 \\ x_1, x_2, x_3 \ge 0 \end{array}} \quad \Longrightarrow \quad \boxed{\begin{array}{l} x_1 + 3x_2 + 2x_3 + u_1 - v_1 = 10 \\ x_1 + 3x_2 + 2x_3 + u_2 - v_2 = 20 \\ x_1, x_2, x_3, u_1, v_1, u_2, v_2 \ge 0 \\ u_1, u_2 \text{ are slack variables} \\ v_1, v_2 \text{ are surplus variables} \\ \text{minimize the deviation becomes an objective} \end{array}}$$

Algebraic vs. Spreadsheet Models

Importance of Algebraic Models

- Models exhibit features and characteristics of the problem
- Modelling aids better understanding of the problem
- Models help to identify redundant constraints
- Models facilitate analysis of the problem
- Models facilitate (safe) experimentation
- Models facilitate treating multiple, conflicting objectives
- Given the model, algorithms can be used to solve the problem
- Developing a decision support system benefits from having a good model of the problem
- Many modelling languages are available for optimization

Developing Spreadsheet Optimization Models

Spreadsheets offer an intuitive way for developing and visualize models of optimization problems (instead of the algebraic form).

Like algebraic models, spreadsheet models also contain:

- data : cells with fixed given values
- decision variables: the changing cells
- constraints: output cells and solver parameters
- objective function: special output cell called the target cell

Parameters in the solver dialog box complete the model.

The overall process for modelling using spreadsheets:

1. Plan the model
2. Build the model
3. Test the model
4. Analyse model and results

Applying good planning and principles in spreadsheet modelling will help to produce models that are easy to understand, easy to debug and easy to modify.

It is not advisable to develop a spreadsheet model without proper planning as this might lead to a model that is poorly organised and difficult to interpret and expand.

Contrary to algebraic models, spreadsheets offer great flexibility for modelling but the risk is to develop less accurate, less robust and even non-linear models.

The Excel solver:

- developed by Frontline Systems Inc.
- uses the Generalized Reduced Gradient (GRG2) nonlinear optimization code.
- LP models use the simplex method with bounds on the variables and IP models use the branch-and-bound method.
- also referred to as 'What-if Analysis Tool

Follow these spreadsheet modelling principles if possible:

- first, enter all the available data
- the model structure should conform the data if possible
- a good layout can facilitate the modelling process
- organise and clearly identify the data, for clarity and also easier application of formulae
- enter each piece of data only once in the model
- separate data from formulae, avoid numbers directly in a formula
- avoid elaborate formulae, this helps to ensure the model is linear
- use formatting (labels, colours, shading, borders, etc.) to make the model easier to interpret
- show the entire model in the spreadsheet, including the equality and inequality signs of constraints
- use the various tools in the formulas menu for debugging spreadsheet models
- **be very careful, it is tedious to debug a spreadsheet model**

Questions OR Comments