Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018

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Lecture 2 – Linear Programming vs. Integer Programming

- Basics of Linear Programming

  To identify the elements of LP models

- Formulating LP Models

  To interpret and write LP formulations

- Graphical Method for LP Models

  To solve small LP models with the graphical method

- Assumptions of LP Models

  To identify conditions and limitations of LP models

- Basics of Integer Programming

  To distinguish LP and IP models and explain the LP Relaxation technique

Additional Reading

Chapter Introduction to Linear Programming of (Hillier and Lieberman, 2015).

Chapter Introduction to Optimization and Linear Programming of (Ragsdale, 2015).

Chapter on Linear Programming of any other book in the reading list.

Supplement Appendix 6 of (Hillier and Lieberman, 2015) gives a review of simultaneous linear equations.

LP Graphic Tutorial, available at:
http://www.msubillings.edu/BusinessFaculty/Harris/LP_Problem_intro.htm

LP Graphers, available at:
https://www.zweigmedia.com/utilities/lpg/index.html
http://www.wolframalpha.com/widgets/view.jsp?id=7fa77b668578a893653c674b2be3865c

Basics of Linear Programming

- LP is one of the most important tools in optimization technology

- The typical LP problem involves:
    - limited resources
    - competing activities
    - measure of solution quality
    - constraints on the use of resources (functional)
    - constraints on the decision variables bounds (e.g. non-negativity)
    - the optimal solution(s)

- LP is a mathematical model with:
    - parameters and decision variables
    - linear algebraic expressions (objective function and constraints)
    - the aim of planning activities

Example. Identifying elements in the Bank ABC LP model:

$$
\begin{array}{lll}
\text{Maximize:} & Z = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4 & (1) \\
\text{Subject to:} & x_1 + x_2 + x_3 + x_4 \le 250 & (2) \\
 & 0.45x_1 - 0.55x_2 \ge 0 & (3) \\
 & 0.75x_1 - 0.25x_2 - 0.25x_3 - 0.25x_4 \ge 0 & (4) \\
 & -0.25x_1 + 0.75x_2 - 0.25x_3 - 0.25x_4 \le 0 & (5) \\
 & -0.01x_1 + 0.05x_2 + 0.05x_3 - 0.05x_4 \le 0 & (6) \\
 & x_1, x_2, x_3, x_4 \ge 0 & (7)
\end{array}
$$

Concepts in LP Models

- Decision variables
- Parameters
- Objective function
- Constraints
  - Functional
  - Non-negativity
- Feasible solutions
- Feasible region
- Infeasible solutions
- Infeasible region
- Search space
- Optimal solutions
- Binding constraints
- Solution vs. Objective

Example of LP Model for APEX Problem

Maximize :	$Z = 120x_1 + 80x_2$	(1)
Subject to :	$x_1 \leq 40$	(2)
	$x_2 \leq 10$	(3)
	$20x_1 + 10x_2 \leq 500$	(4)
	$x_1 - x_2 \leq 5$	(5)
	$x_2 - x_1 \leq 5$	(6)
	$x_1 \geq 0, x_2 \geq 0$	(7)

Optimal solution:	Binding Constraints: Non-binding Constraints:

The size of an LP model can be estimated by:

- The number of decision variables
- The number of constraints
- The size of the search space

Example. Estimate the size of the APEX problem LP model.

$$
\begin{aligned}
\text{Maximize:}\quad & Z = 120x_1 + 80x_2 && (1)\\
\text{Subject to:}\quad & x_1 \le 40 && (2)\\
& x_2 \le 10 && (3)\\
& 20x_1 + 10x_2 \le 500 && (4)\\
& x_1 - x_2 \le 5 && (5)\\
& x_2 - x_1 \le 5 && (6)\\
& x_1 \ge 0, x_2 \ge 0 && (7)
\end{aligned}
$$

Number of decision variables:

Number of Constraints:

Size of the search space:

Example. Estimate the size of the VEGETABLES DISTRIBUTION LP model.

$$\text{Minimize: } Z = \sum_{i=1}^{15} (P_i - C_i) X_i \quad (1)$$

$$\text{Subject to: } X_i \geq \text{Min} \quad \text{for } i = 1 \ldots 15 \quad (2)$$

$$X_i \leq \text{Max} \quad \text{for } i = 1 \ldots 15 \quad (3)$$

$$\sum_{i=1}^{15} 1.25 X_i \leq 18000 \quad (4)$$

$$\sum_{i=1}^{15} C_i X_i \leq 30000 \quad (5)$$

$$X_i \geq 0 \quad \text{for } i = 1 \ldots 15 \quad (6)$$

Number of decision variables:

Number of Constraints:

Size of the search space:

Example. Estimate the size of the following optimization model.

$$\text{Maximize: } Z = \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} x_{ij} \qquad (1)$$

$$\text{Subject to: } \sum_{j=1}^{m} x_{ij} = 1 \quad i = 1 \ldots n \qquad (2)$$

$$\sum_{i=1}^{n} x_{ij} \geq 2 \quad j = 1 \ldots m \qquad (3)$$

$$x_{ij} \in \{0,1\} \quad i = 1 \ldots n \text{ and } j = 1 \ldots m \qquad (4)$$

Number of decision variables:

Number of Constraints:

Size of the search space:

Formulating LP Models

A General LP Formulation

An LP formulation is associated to the problem of finding the optimal allocation of limited resources to competing activities, i.e. planning the activities. Constraints can involve inequalities ($\geq$ or $\leq$), strict inequalities (> or <), or equalities. The objective function and all constraints are linear algebraic expressions.

Recommended Steps

- Identify parameters (numerical data)
- Identify decision variables (x_1, x_2, etc.)
- Formulate objective function (maximize or minimize Z)
- Formulate functional constraints (linear algebra)
- Specify non-negativity / integrality constraints (for all x_i)

Format of General LP Formulation

$$\text{Maximize}: \quad Z = c_1 x_1 + c_2 x_2 + \ldots c_n x_n$$

$$\text{Subject to}:$$

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n \le b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \le b_2 \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n \le b_m \\
&x_1 \ge 0, x_2 \ge 0 \ldots x_n \ge 0
\end{aligned}$$

Example. Formulating an LP model for a simple optimization problem.

The company ATLAS produces two products A and B from two raw materials M1 and M2. The problem is to plan the production, i.e. determine the amount to produce of A and B, in order to maximise the profit. The quantity produced of B cannot larger than the quantity produced of A by more than 1 unit. Also, the demand (required production) of product B is known to be at most 2 units. The availability of raw materials and their requirement for the production of products A and B are shown in the table.

Identify elements of the optimization problem:

	units of material required to produce one unit of product		
	product A	product B	max availability
Material M1	6	4	24
Material M2	1	2	6
Profit per unit	5	4	

Construct Formal LP Model

Define algebraic linear expressions

	units of material required to produce one unit of product		
	product A	product B	max availability
Material M1	6	4	24
Material M2	1	2	6
Profit per unit	5	4	

Decision variables

$$\boxed{\begin{aligned} x_1 &= \text{units produced of product A} \\ x_2 &= \text{units produced of product B} \end{aligned}}$$

Objective function

$$\boxed{\text{Maximize} \quad Z = 5x_1 + 4x_2}$$

	product A	product B	max availability
Material M1	6	4	24
Material M2	1	2	6
Profit per unit	5	4	

Functional Constraints

$6x_1 + 4x_2$ use of material M1 for product A and B

$6x_1 + 4x_2 \leq 24$ availability of material M1

$x_1 + 2x_2$ use of material M2 for products A and B

$x_1 + 2x_2 \leq 6$ availability of material M2

$x_2 \leq x_1 + 1$ difference in demand between products

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$ maximum demand of product B

Non-negativity constraints

$x_1 \geq 0$ and $x_2 \geq 0$ production cannot be negative

Formal LP Model for the ATLAS Optimization Problem

	product A	product B	max availability
Material M1	6	4	24
Material M2	1	2	6
Profit per unit	5	4	

$$\text{Maximize: } Z = 5x_1 + 4x_2 \qquad (1)$$

$$\text{Subject to: } 6x_1 + 4x_2 \leq 24 \qquad (2)$$

$$x_1 + 2x_2 \leq 6 \qquad (3)$$

$$-x_1 + x_2 \leq 1 \qquad (4)$$

$$x_2 \leq 2 \qquad (5)$$

$$x_1, x_2 \geq 0 \qquad (6)$$

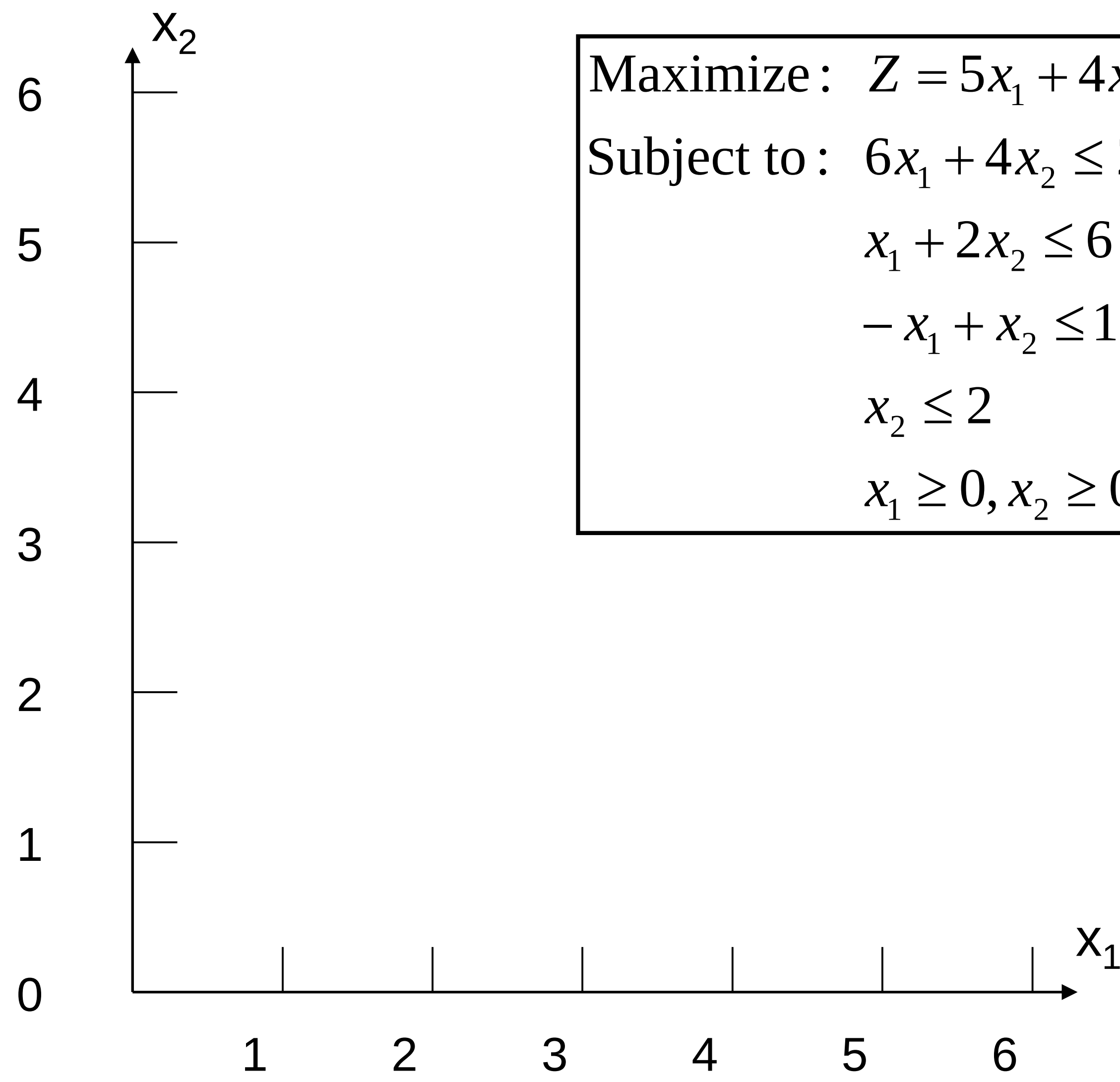
Graphical Method for LP Models

Develop a Computer (or other) Solving Procedure

Derive solutions to the problem according to the model.

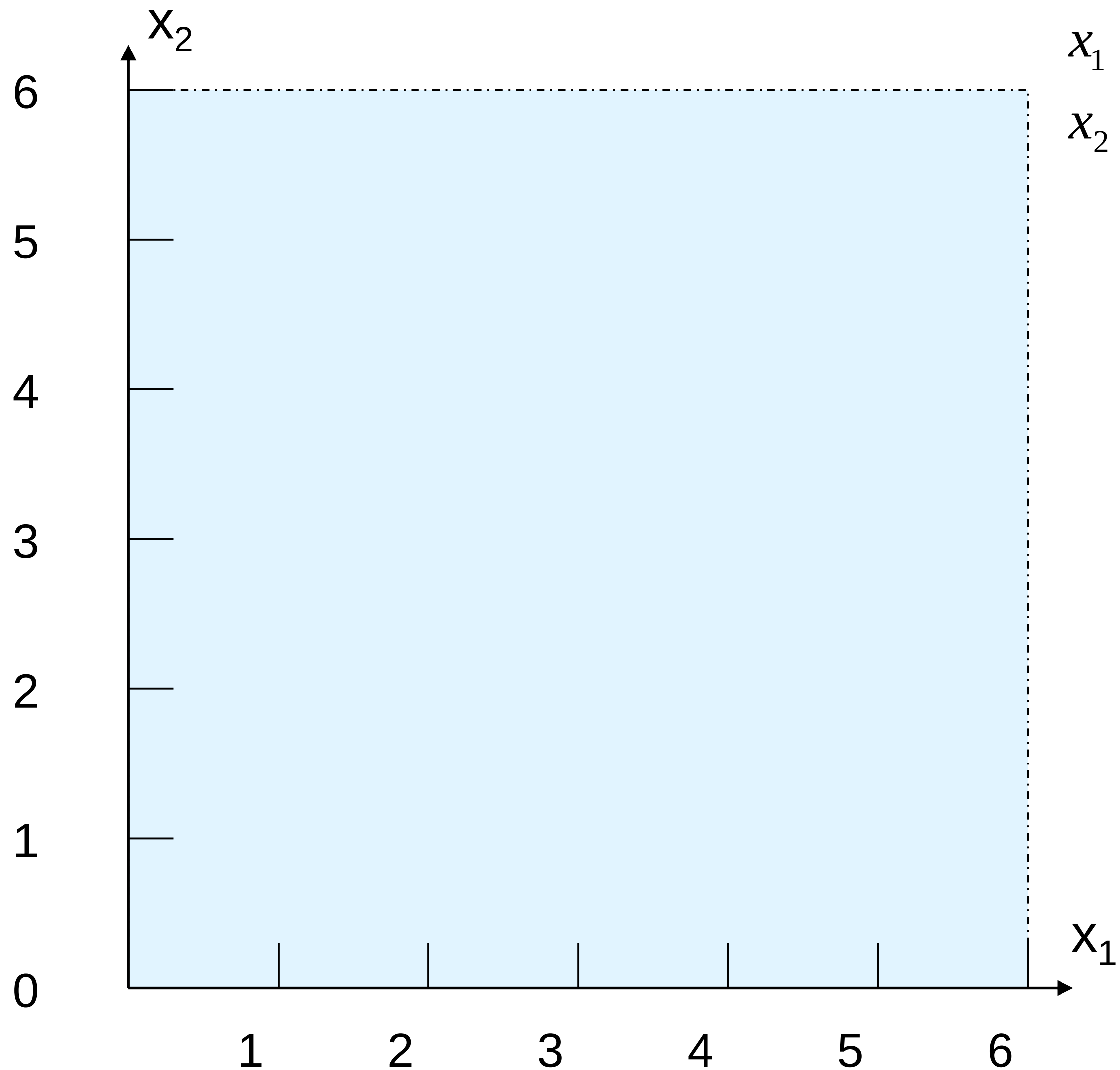
- Graphical Method
  - For LP models with 2 decision variables (maybe 3)
  - A two-dimensional graph to visualise decision variables, objective function, constraints, feasible region, infeasible region, search space, optimal solutions
  - Find solutions by ‘trial and error’ (with some logic behind it)
- Simplex Method Variants
  - For larger LP models with 3 or more decision variables
- Exact Solver or Heuristic Approach
  - Very large models might not be solvable to optimality in practical computation time

Example Graphical method to solve the ATLAS LP model



$$\begin{array}{lll} \text{Maximize:} & Z = 5x_1 + 4x_2 & (1) \\ \text{Subject to:} & 6x_1 + 4x_2 \le 24 & (2) \\ & x_1 + 2x_2 \le 6 & (3) \\ & -x_1 + x_2 \le 1 & (4) \\ & x_2 \le 2 & (5) \\ & x_1 \ge 0, x_2 \ge 0 & (6) \end{array}$$

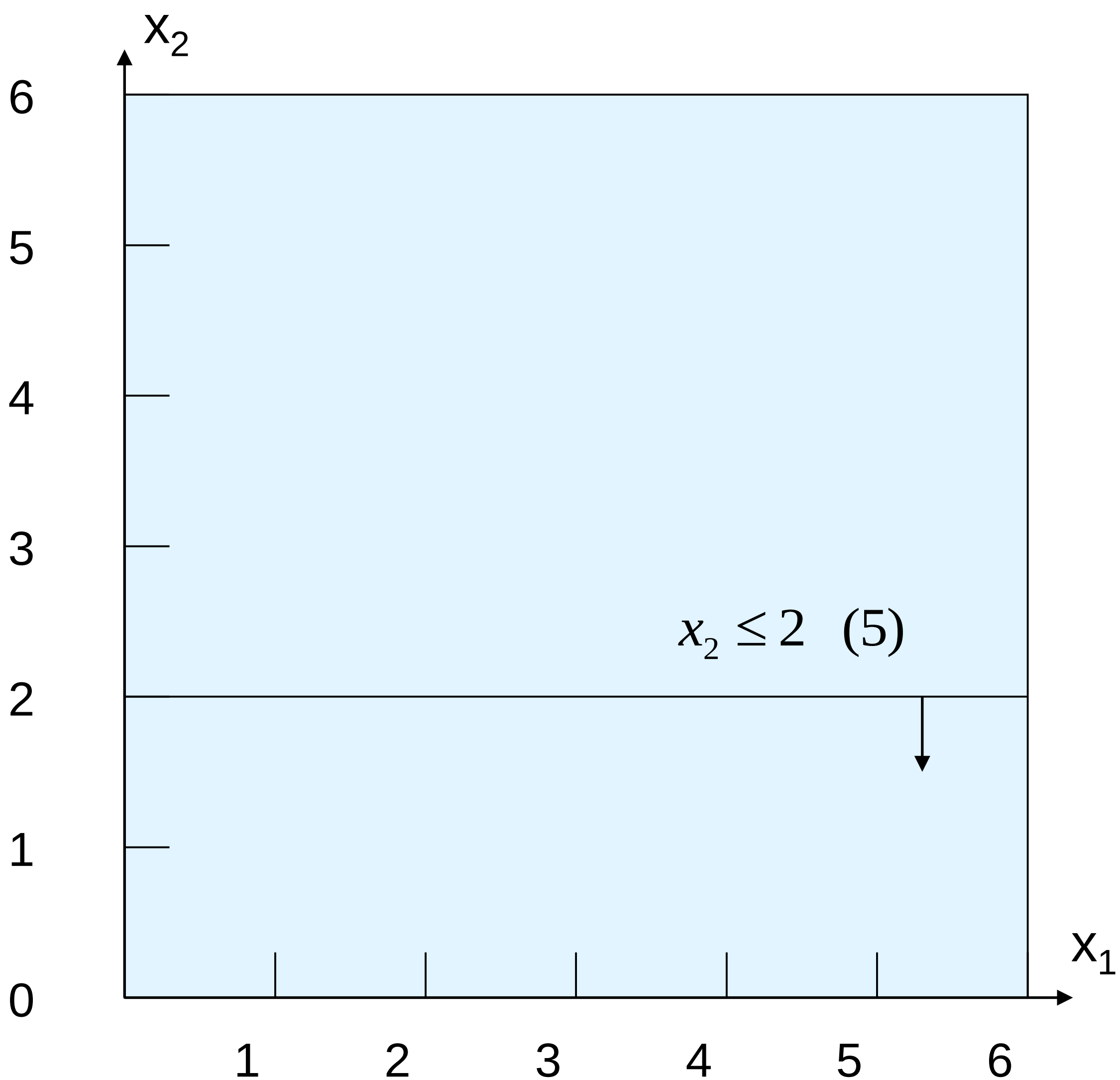
Example (cont.) Graphical method to solve ATLAS LP model



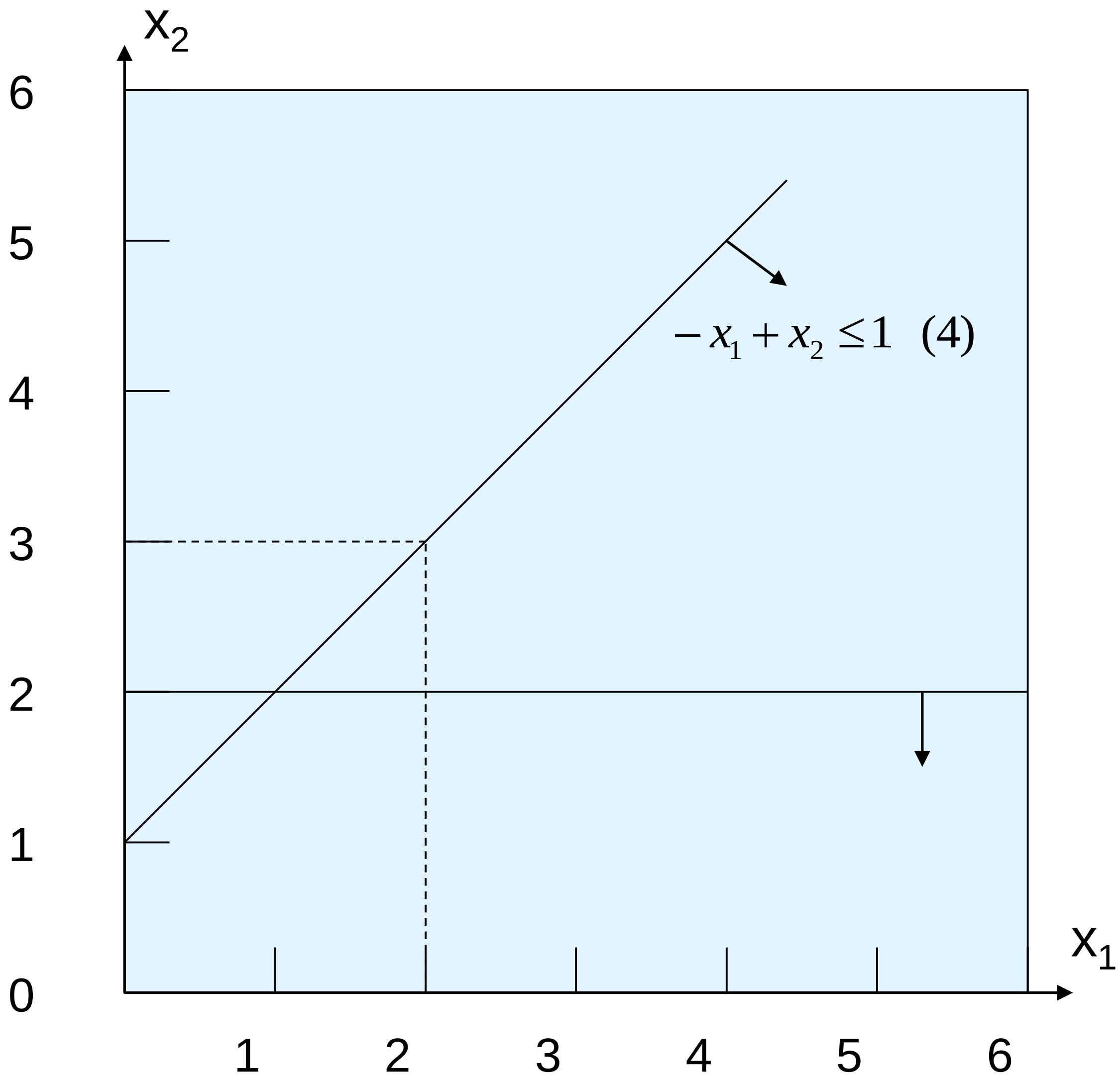
$x_1 \geq 0$ and

$x_2 \geq 0$ (6)

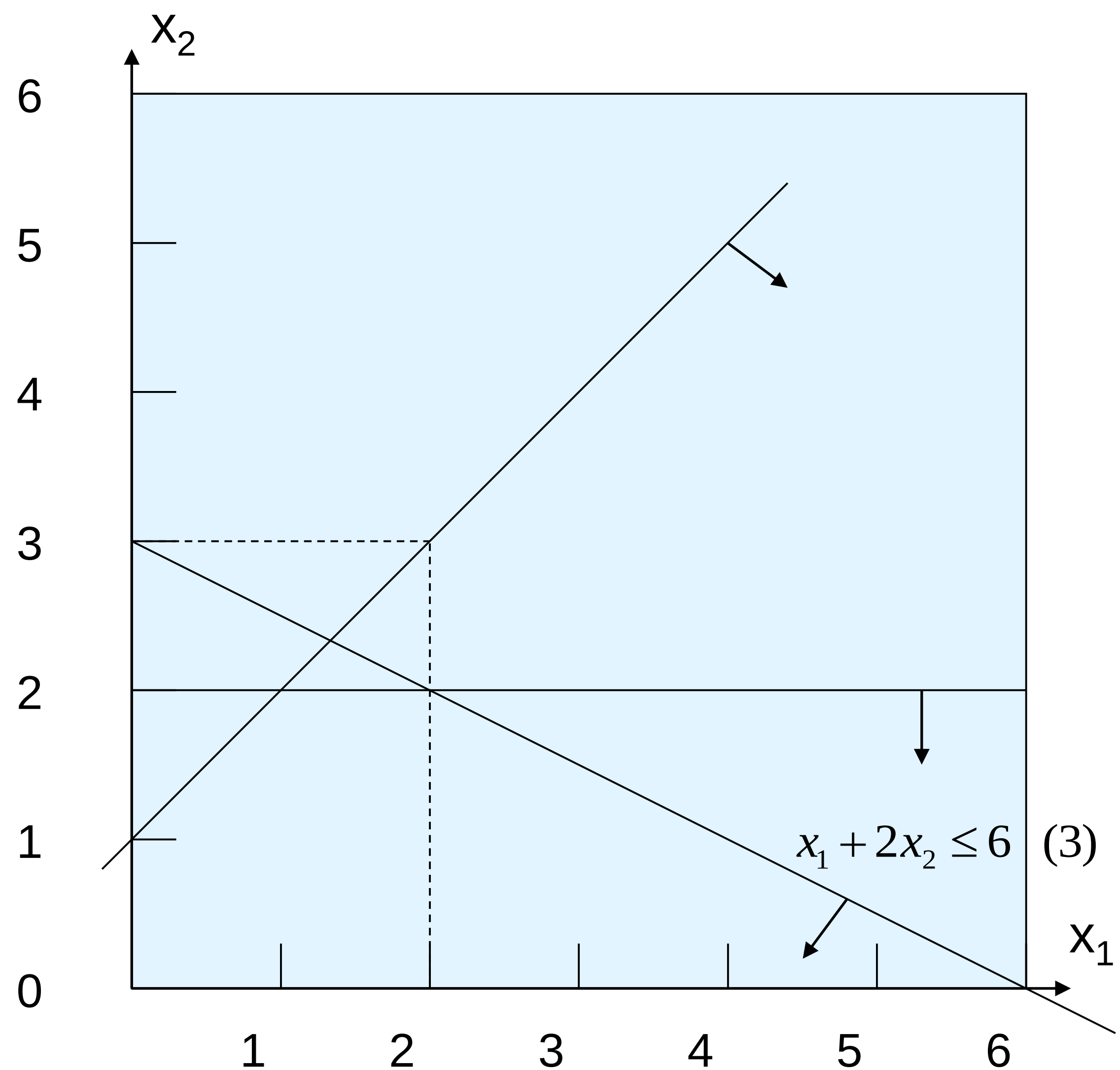
Example (cont.) Graphical method to solve ATLAS LP model



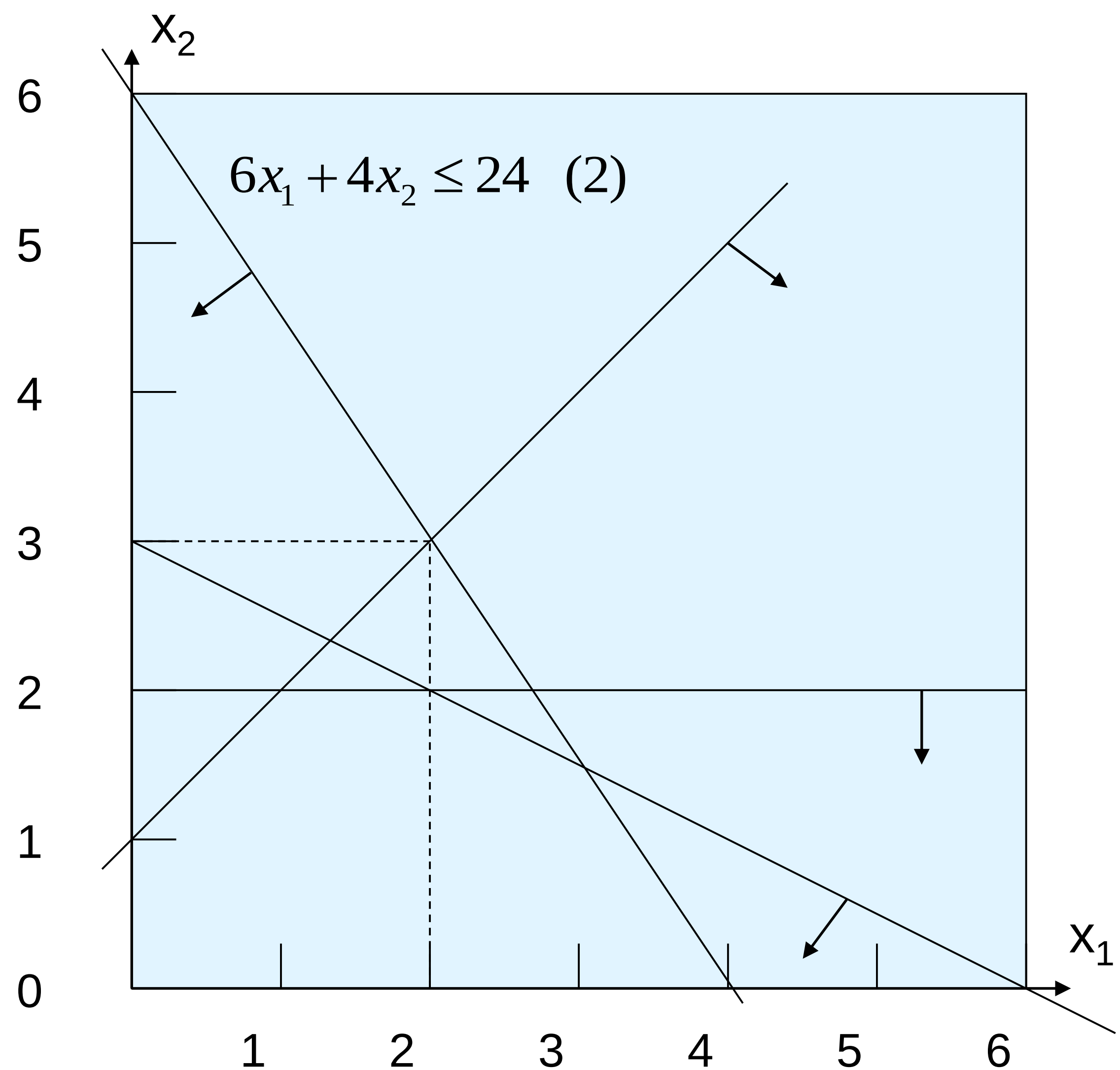
Example (cont.) Graphical method to solve ATLAS LP model



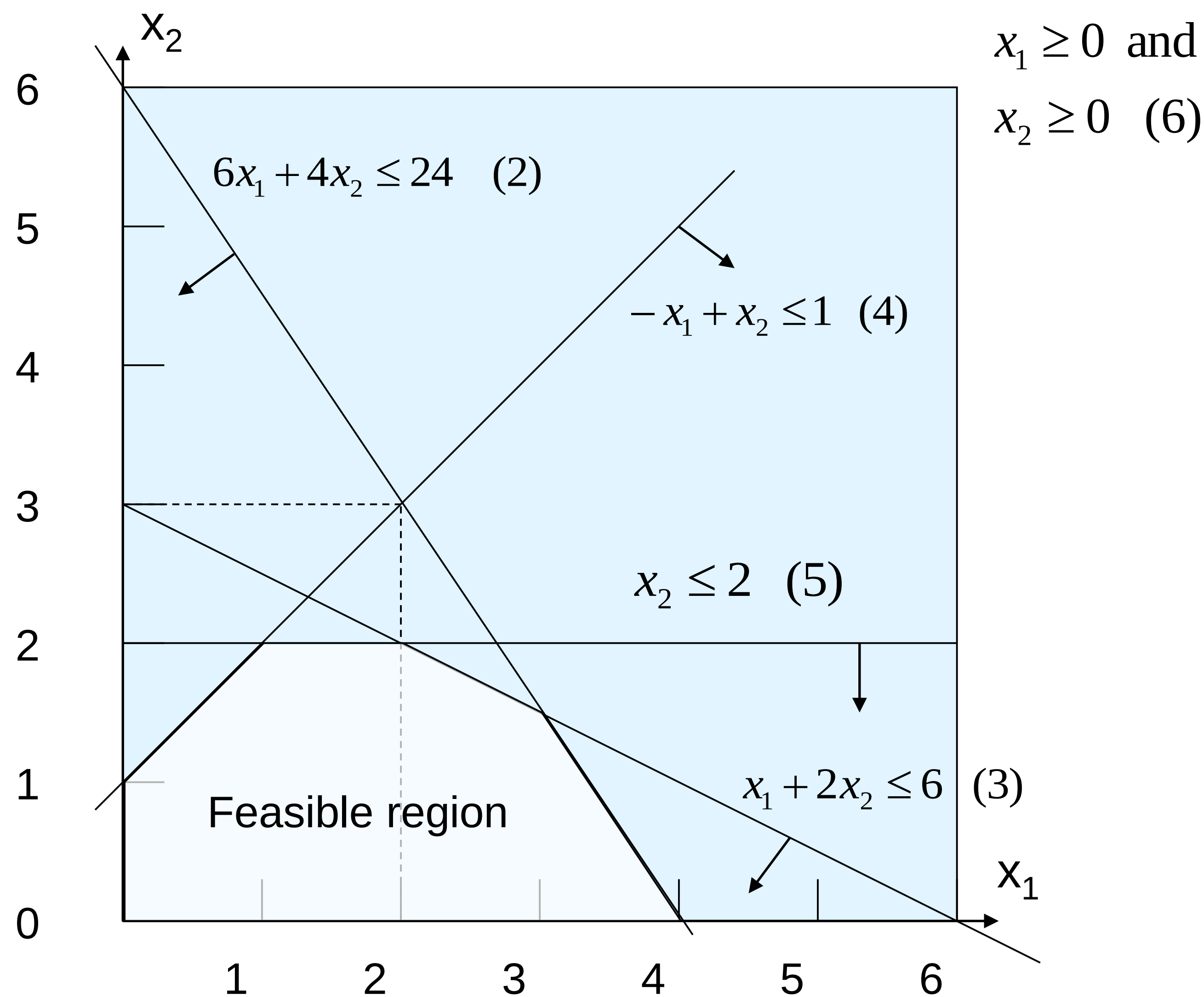
Example (cont.) Graphical method to solve ATLAS LP model



Example (cont.) Graphical method to solve ATLAS LP model



Example (cont.) Graphical method to solve ATLAS LP model



Example (cont.) Graphical method to solve ATLAS LP model

Start with a solution in the feasible region

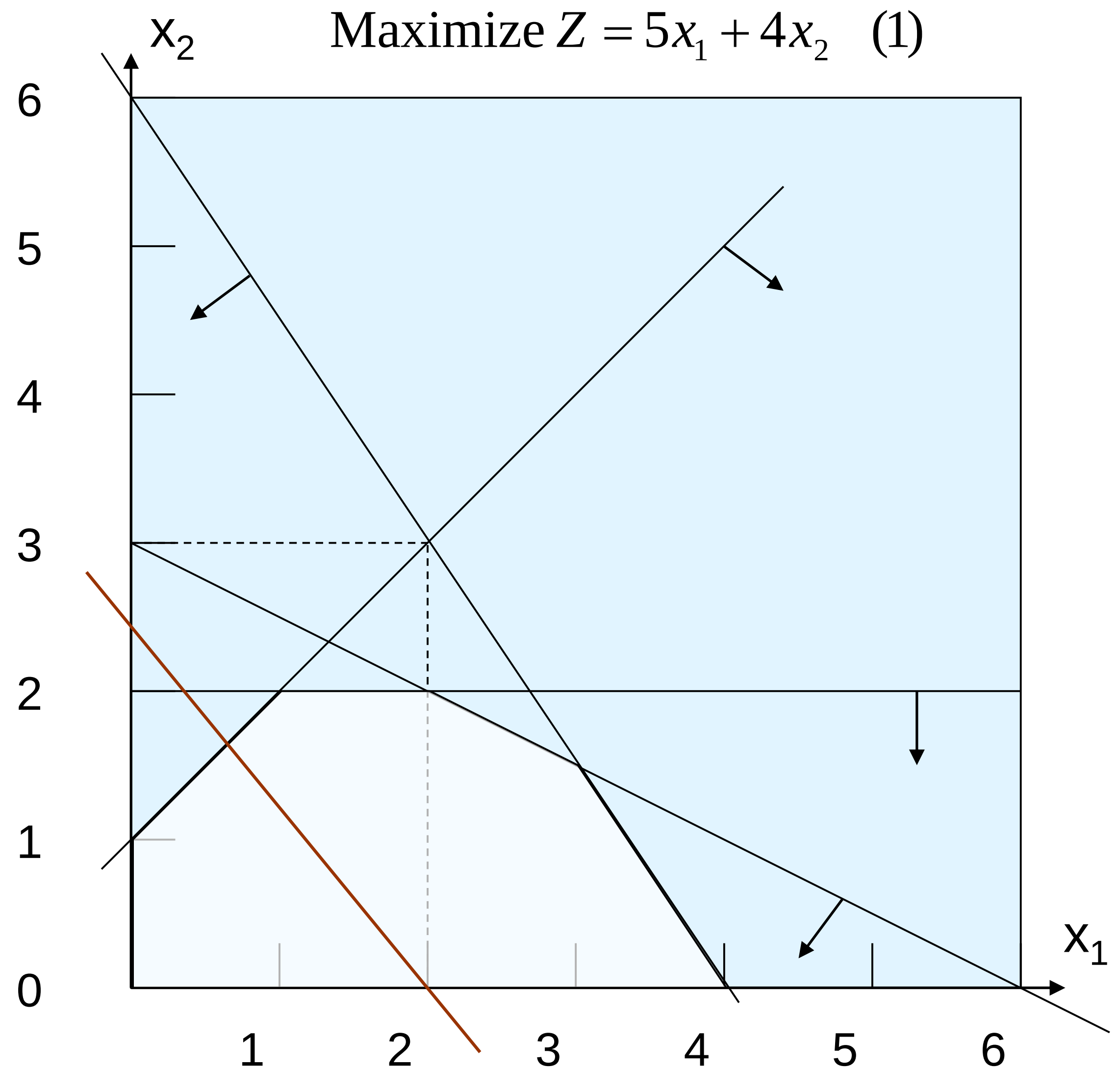
Assume profit $Z=10$

Then $10=5x_1+4x_2$

Draw line for the objective function

If $x_1 = 0$
Then $x_2 = 2.5$

If $x_1 = 0$
Then $x_2 = 2$



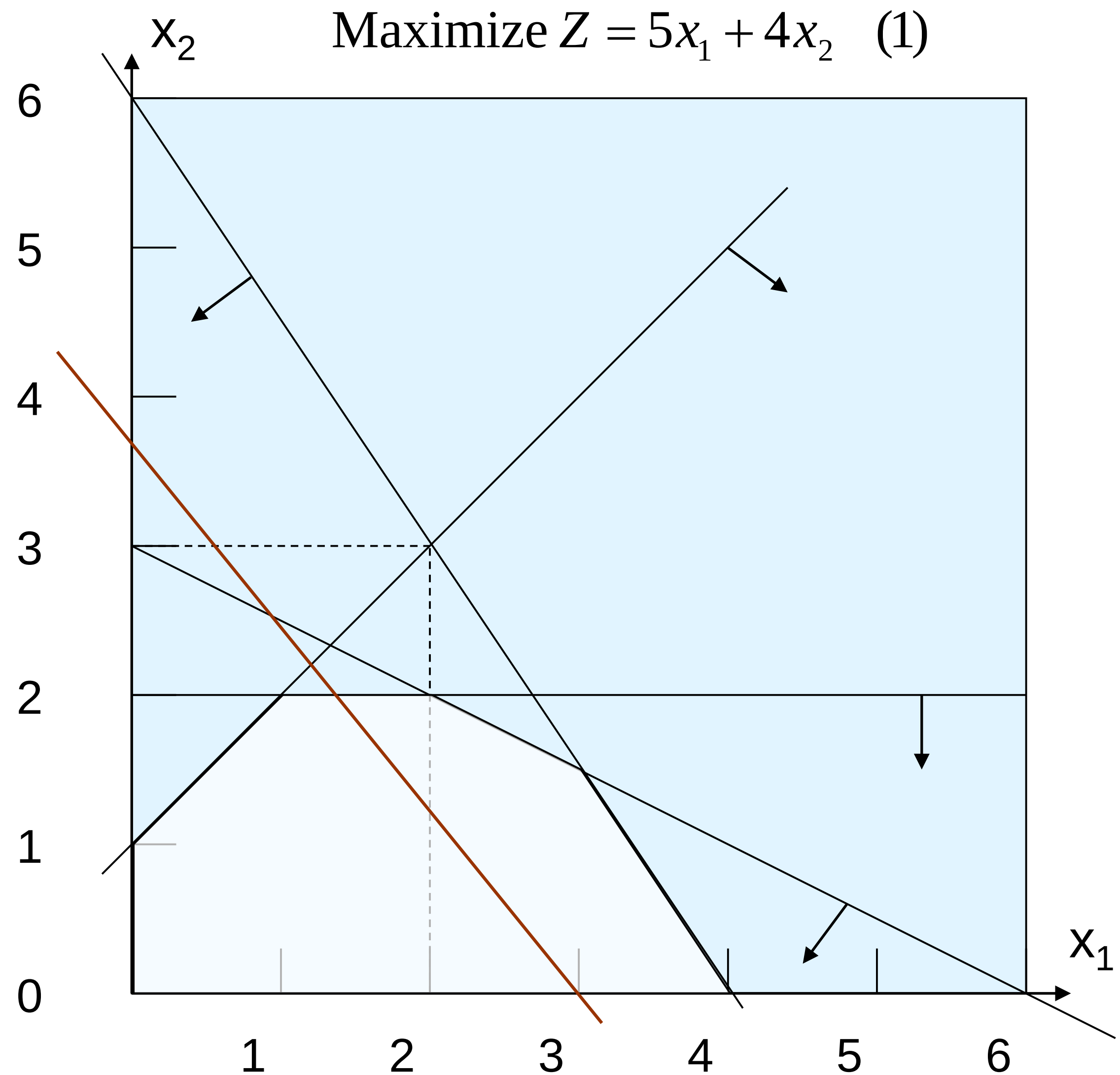
Example (cont.) Graphical method to solve ATLAS LP model

Find better solutions in the feasible region

Assume profit Z=15

Then $15=5x_1+ 4x_2$

Identify direction of improvement



Example (cont.) Graphical method to solve ATLAS model

Find point at the intersection of (2) and (3)

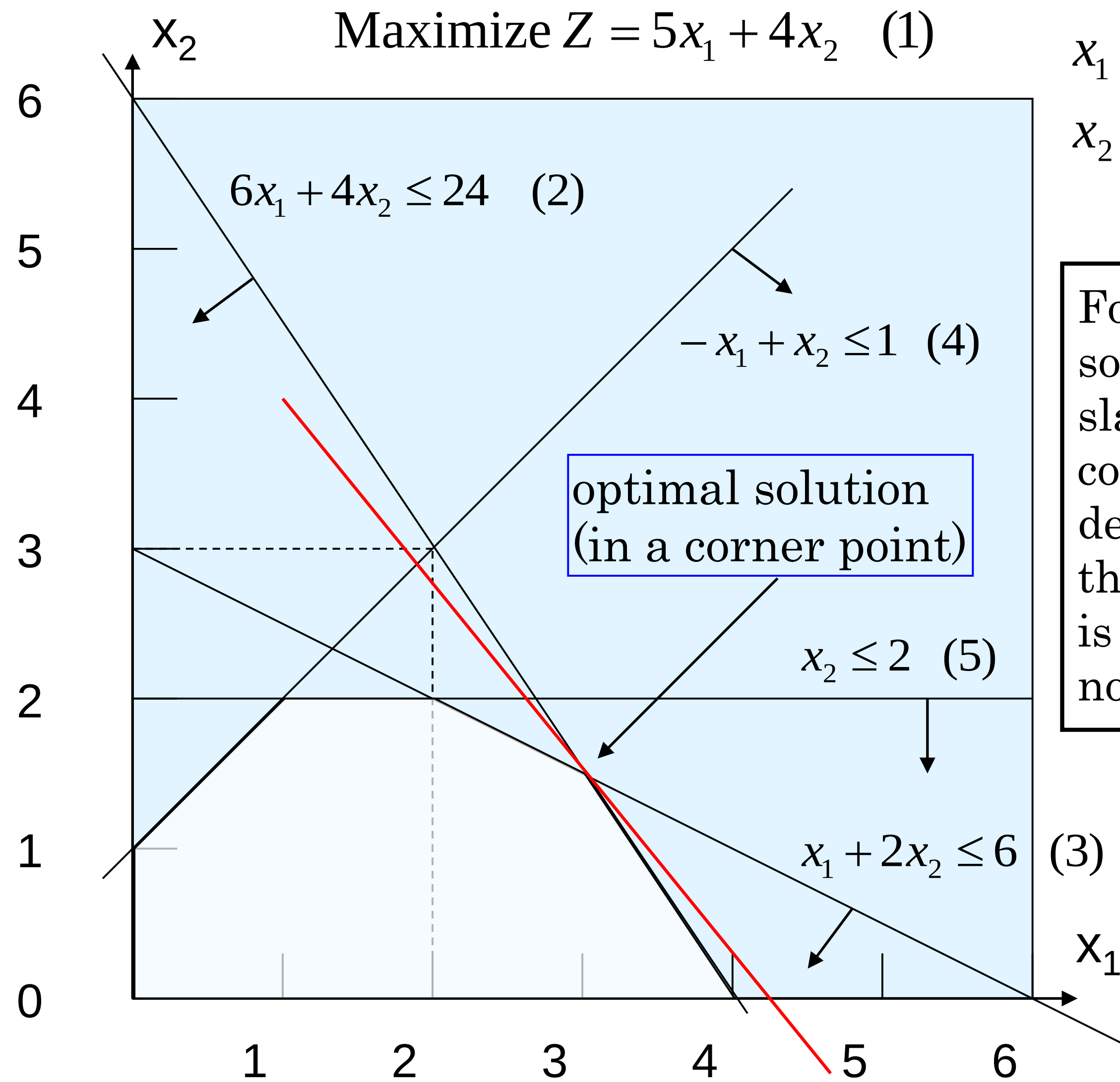
$6x_1 + 4x_2 = 24$
$x_1 + 2x_2 = 6$

$6x_1 + 4x_2 = 24$
$-2x_1 - 4x_2 = -12$

$4x_1 = 12$
$x_1 = 3$

$x_1 + 2x_2 = 6$
$3 + 2x_2 = 6$
$x_2 = 1.5$

$Z = 5x_1 + 4x_2$
$Z = 21$

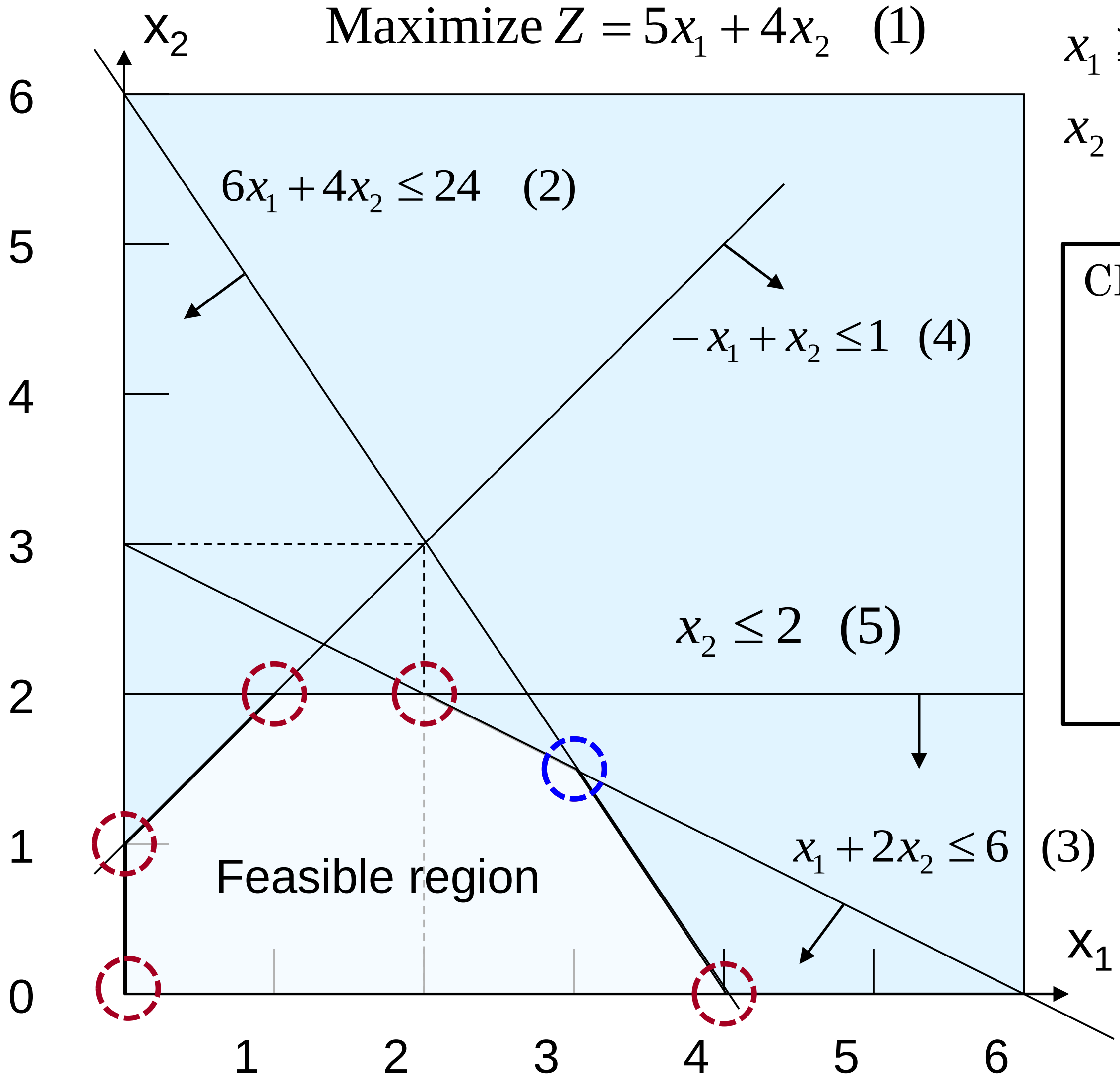


For the optimal solution, the slack in each constraint determines if the constraint is binding or non-binding.

Example (cont.) Graphical method to solve ATLAS LP model

An LP model with feasible solutions and a bounded feasible region must have CPF (corner-point feasible) solutions and at least one optimal solution.

The best CPF solution must be an optimal solution.



$x_1 \ge 0$ and

$x_2 \ge 0$ (6)

CPF solutions?

Summary of LP Graphical Method

Alternatively, after step 3, evaluate all the CPFs and determine which ones represent an optimal solution.

1. Draw a two-dimensional graph
2. Draw lines representing the constraints
3. Identify the feasible and infeasible regions
4. Start with some solution in the feasible region and draw the line representing the objective function
5. Find better values for the objective function (and corresponding values for the decision variables) within the feasible region
6. Identify the direction that improves on the objective function
7. Identify the point(s) that contains the optimal solution(s)

Assumptions of LP Models

General LP formulations are limited to problems where the objective function and all constraints are given by linear functions. There are four assumptions in general LP formulations: proportionality, additivity, divisibility and certainty.

$$
\left.
\begin{array}{ll}
\text{Maximize:} & Z = c_1 x_1 + c_2 x_2 + \ldots c_n x_n \\
\text{Subject to:} & \\
& a_{11} x_1 + a_{12} x_2 + \ldots a_{1n} x_n \le b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots a_{2n} x_n \le b_2 \\
& \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots a_{mn} x_n \le b_m \\
& x_1 \ge 0, x_2 \ge 0 \ldots x_n \ge 0
\end{array}
\right\}
$$

objective function

constraints

parameters

competing activities

limited resources

linear functions

non-negativity

Proportionality. The contribution of each activity to the value of the objective function and to the left-hand side of each functional constraint is proportional to the level of the activity.

Examples Where Proportionality Holds	Examples Where Proportionality Does No Hold
Maximise $Z = 5x_1 + 4x_2$	Maximise $Z = 5x_1 + 4x_2$ if $x_2 > 2$ $Z = 5x_1 + x_2$ otherwise
Minimise $Z = 0.5x_1 + 0.25x_2$	Minimise $Z = 0.5x_1^{3/2} + 0.25x_2$
Inequality constraint $x_2 - x_1 \leq 1$	Inequality constraint $x_2^2 - x_1 \leq 1$
Equality constraint $2x_1 + 3x_2 = 50$	Equality constraint $x_1^{\log x} + 2x_2 = 50$

Additivity. Every linear function is the sum (not cross-product terms) of the individual contributions of the respective activities.

Examples Where Additivity Holds	Examples Where Additivity Does No Hold
Maximise $Z = 5x_1 + 4x_2$	Maximise $Z = 5x_1 + 4x_2 + x_1 x_2$
Minimise $Z = 0.5x_1 + 0.25x_2$	Minimise $Z = 0.5x_1 - x_1 x_2$
Inequality constraint $x_2 - x_1 \leq 1$	Inequality constraint $x_1 x_2{}^2 - x_1 \leq 1$
Equality constraint $2x_1 + 3x_2 = 50$	Equality constraint $2x_1 x_2 = 50$

Divisibility. Every decision variable can have any non-negative value including integers and non-integers that satisfies the constraints.

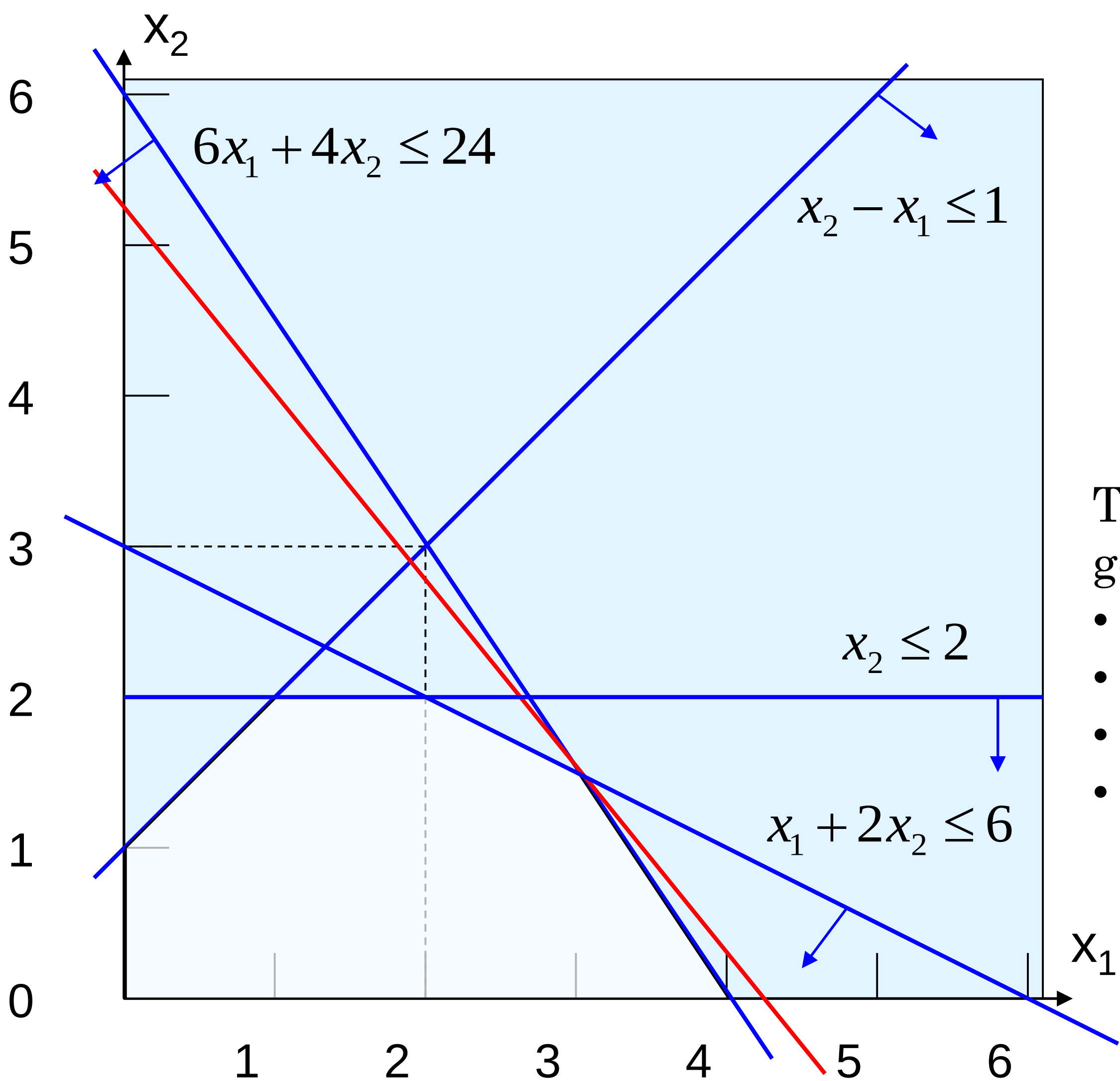
Examples Where Divisibility Holds	Examples Where Divisibility Does No Hold
x_1 = liters produced of product A	x_1 = salesmen assigned to branch A
x_2 = liters produced of product B	x_2 = salesmen assigned to branch B
x_1 = watts of electricity from A to B	x_1 = number of trucks from A to B
x_2 = watts of electricity from B to C	x_2 = number of trucks from B to C

Certainty. Parameter values are known and constant.

Examples Where Certainty Holds	Examples Where Certainty Does No Hold
Maximise $Z = 5x_1 + 4x_2$ profits due to x_1, x_2 are constant Subject to $Ax_1 + Bx_2 = 50$ where $A = 2$ and $B = 3$ the required quantities of x_1 and x_2 in this constraint are constant	Maximise $Z = 5x_1 + 4x_2$ profits due to x_1, x_2 change a lot Subject to $Ax_1 + Bx_2 = 50$ where $2 \leq A, B \leq 20$ the required quantities of x_1 and x_2 can change a lot during production

Consider the ATLAS problem LP model:

$$\begin{array}{llr} \text{Maximize:} & Z = 5x_1 + 4x_2 & (1) \\ \text{Subject to:} & 6x_1 + 4x_2 \le 24 & (2) \\ & x_1 + 2x_2 \le 6 & (3) \\ & -x_1 + x_2 \le 1 & (4) \\ & x_2 \le 2 & (5) \\ & x_1 \ge 0, x_2 \ge 0 & (6) \end{array}$$



The 4 assumptions of general LP formulations hold

- Proportionality
- Additivity
- Divisibility
- Certainty

Basics of Integer Programming

Difficulty of LP, IP, BIP and MIP problems

- LP problems are in general easier to solve than IP and BIP problems.
- BIP problems are considered easier to solve than IP problems.
- The size of IP and BIP problems grows exponentially with the number of decision variables.
- The LP-relaxation is often used to solve IP problems.
- The non-integer (i.e. fractional) decision variables in MIP problems do not have much effect of the difficulty of the problem.

Many Applications of BIP Models

- Invest of not on that company?
- Assign or not a person to that timeslot?
- Build or not a road between two cities?
- Include or not that site in the tour?
- Deliver or not this product with that truck?
- Schedule or not this match in premium TV time?
- Assign or not this airport gate to that aircraft?
- Produce or not that new article in this factory?
- Include or not this item in my luggage?
- Should activity A follow or precede activity B?

YES or NO Decisions

LP Relaxation for IP and BIP Models

The LP Relaxation of an IP or BIP problem formulation is the relaxed version of the model in which all variables are allowed to take fractional values.

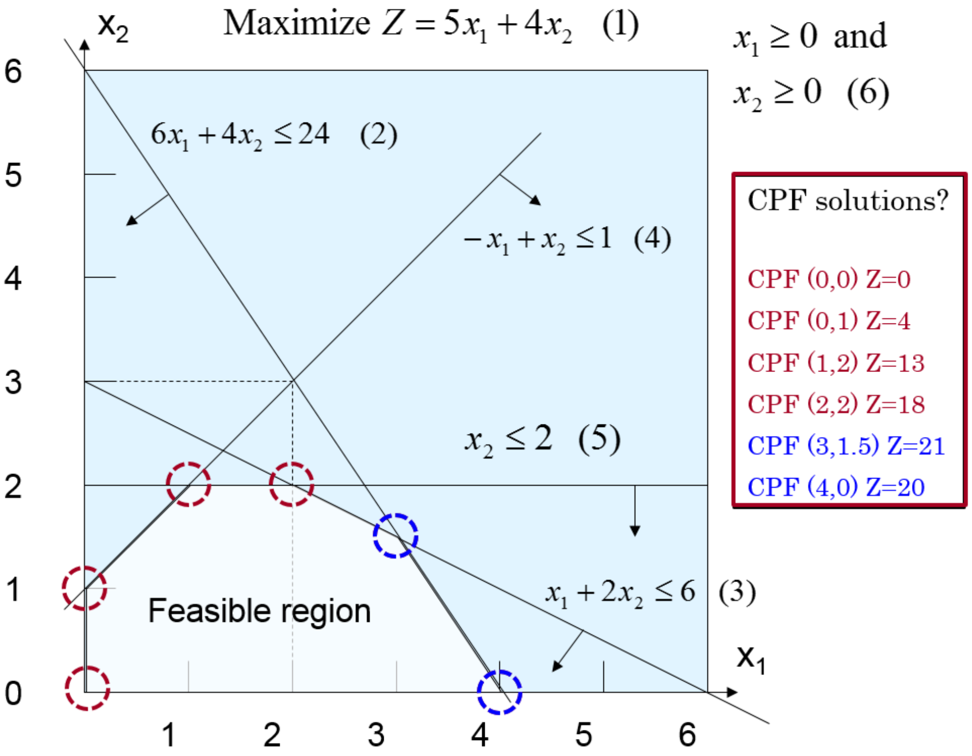
There are some special problems (e.g. Minimum Cost Flow with integer parameters) for which solving the LP relaxation gives also the optimal solution to the original integer problem.

Very often, for problems of considerable size, the optimal solution to the LP relaxation does not satisfy the integer restrictions of the original problem.

The optimal LP relaxation solution (after rounding) might be infeasible for the original problem.

Example of LP Relaxation Consider the ATLAS company problem and assume that the decision variables should now be integer. The optimal solution to the LP relaxation is shown below.

$$
\begin{aligned}
\text{Maximize:} \quad & Z = 5x_1 + 4x_2 && (1)\\
\text{Subject to:} \quad & 6x_1 + 4x_2 \le 24 && (2)\\
& x_1 + 2x_2 \le 6 && (3)\\
& -x_1 + x_2 \le 1 && (4)\\
& x_2 \le 2 && (5)\\
& x_1, x_2 \ge 0 \text{ are integer} && (6)
\end{aligned}
$$



LP relaxation solution:

$$x_1 = 3.0 \quad x_2 = 1.5 \quad Z = 21.0$$

IP solution:

$$x_1 = 4 \quad x_2 = 0 \quad Z = 20$$

Example

JOHN STRONG needs to decide what daily combination of corn and soybean is better for his diet while minimizing the cost. He should eat at least 750 units of these two ingredients combined each day and he has been advised to take at least 30% of protein and at most 5% of fiber. The nutrients composition of corn and soybean is as shown in the table.

	units of nutrients per each unit of ingredients		
	protein	fiber	cost per unit
Corn	0.09	0.02	0.30
Soybean	0.60	0.06	0.90

Formulate the LP model and find the optimal solution using the graphical method.

Example (cont.)

Define mathematical linear expressions

- Decision variables

$$\boxed{\begin{array}{l} x_1 = \text{units of corn} \\ x_2 = \text{units of soybean} \end{array}}$$

- Objective function

$$\boxed{\text{Minimise } Z = 0.3x_1 + 0.9x_2}$$

- Constraints

$$\boxed{x_1 + x_2 \geq 750 \quad \text{minimum amount required}}$$

$$\boxed{\begin{array}{l} 0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2) \quad \text{protein requirement} \\ 0 \geq 0.21x_1 - 0.3x_2 \end{array}}$$

$$\boxed{\begin{array}{l} 0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2) \quad \text{fiber requirement} \\ 0 \leq 0.03x_1 - 0.01x_2 \end{array}}$$

$$\boxed{x_1 \geq 0 \text{ and } x_2 \geq 0 \quad \text{Eating cannot be negative}}$$

Example (cont.) Solve with the graphical method

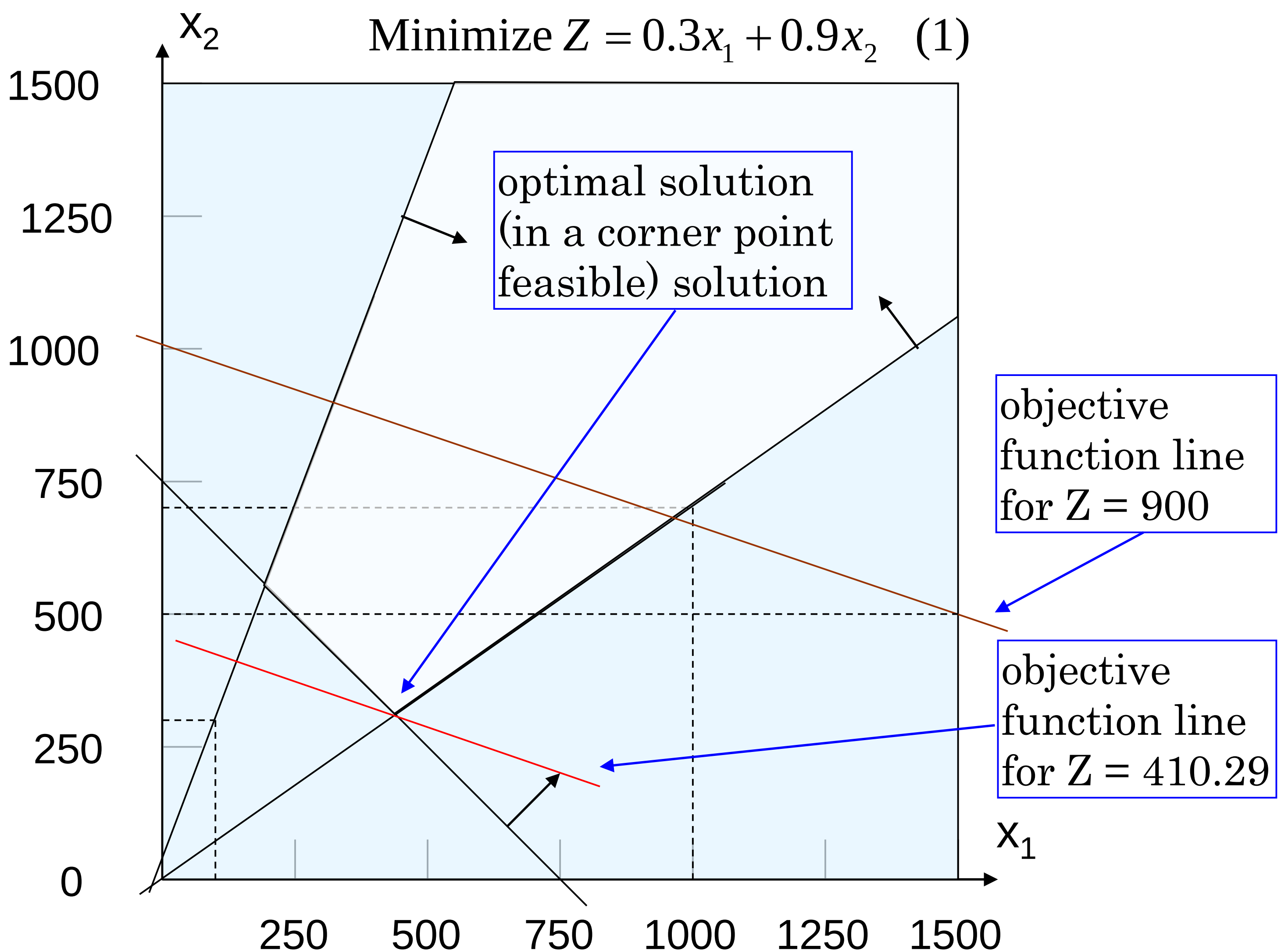
Find the point containing the optimal feasible solution

$x_1 = ?$

$x_2 = ?$

$Z = 0.3x_1 + 0.9x_2$

$Z = ?$



Example (cont.) Solve with the graphical method

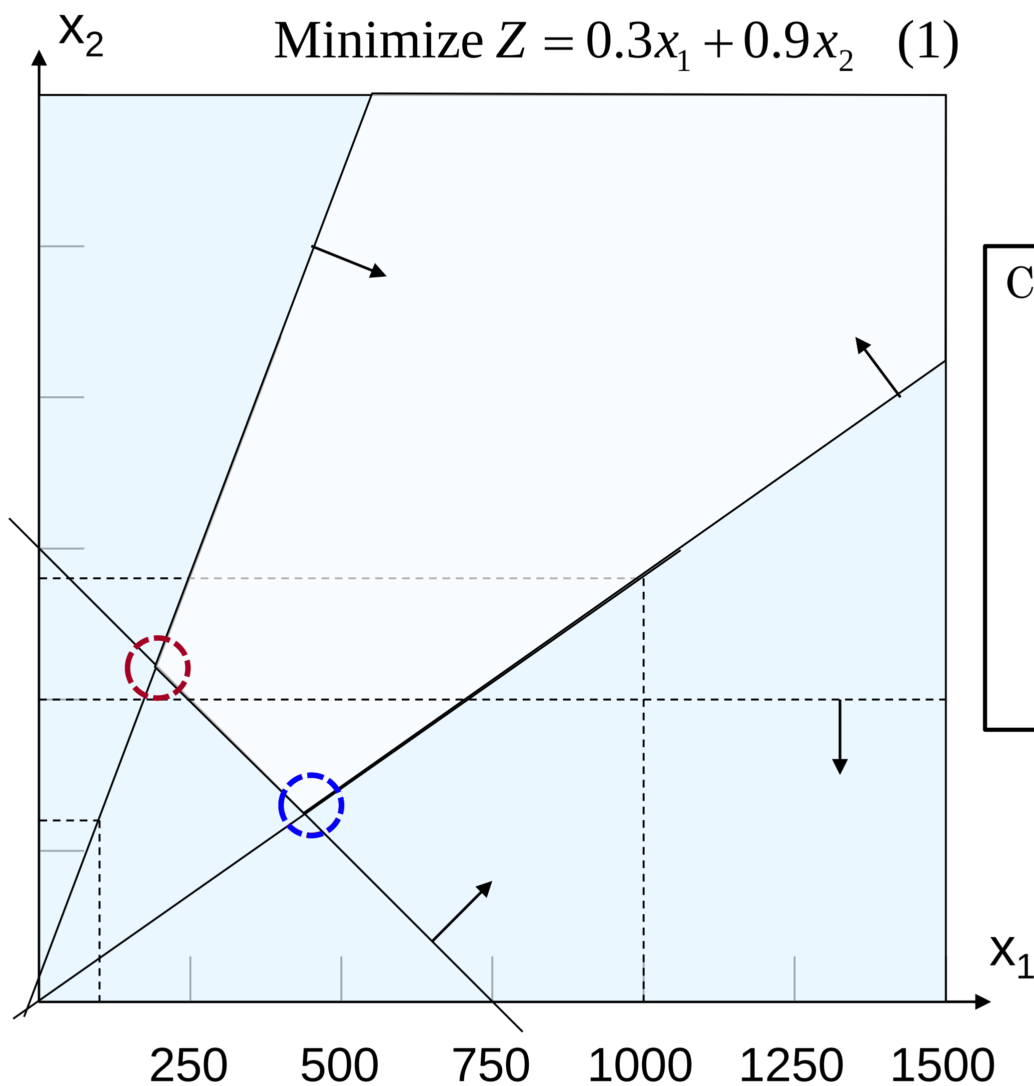
Find the point containing the optimal feasible solution

$x_1 = ?$

$x_2 = ?$

$Z = 0.3x_1 + 0.9x_2$

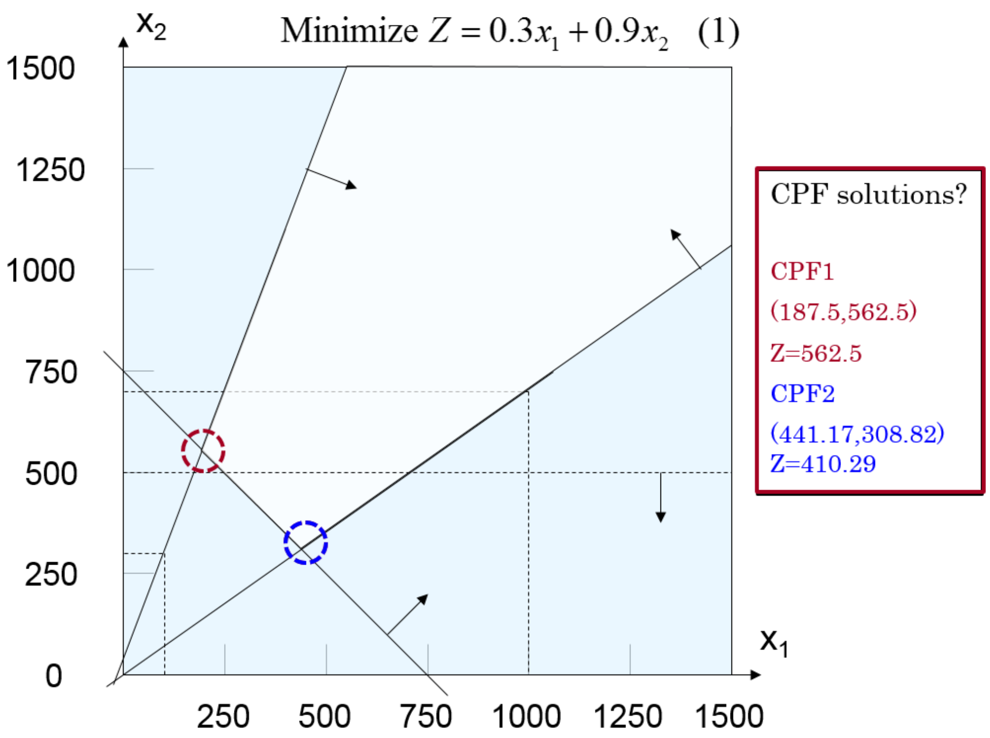
$Z = ?$



CPF solutions?

Example of LP Relaxation Consider the JOHN STRONG problem and assume that the decision variables should now be integer. The optimal solution to the LP relaxation is shown below.

$$
\begin{aligned}
\text{Minimize}: \quad & Z = 0.3x_1 + 0.9x_2 & (1)\\
\text{Subject to}: \quad & x_1 + x_2 \geq 750 & (2)\\
& 0.21x_1 - 0.3x_2 \leq 0 & (3)\\
& 0.03x_1 - 0.01x_2 \geq 0 & (4)\\
& x_1, x_2 \geq 0 \text{ are integer} & (5)
\end{aligned}
$$



LP relaxation solution:

$$
\boxed{\begin{aligned} x_1 &= 441.17\\ x_2 &= 308.82\\ Z &= 410.29 \end{aligned}}
$$

IP solution:

$$
\boxed{\begin{aligned} x_1 &= 441\\ x_2 &= 309\\ Z &= 410.4 \end{aligned}}
$$

Example

A FURNITURE maker has:

- 6 units of wood in stock
- 28 hours of workshop time available

He can make two types of furniture:

- A table takes 2 units of wood and 7 hours
- A chair takes 1 unit of wood and 8 hours

He estimates that he will male a profit of:

- £120 per table
- £80 per chair

The problem is to determine the optimal production plan to maximize the profit.

Formulate the optimization model and illustrate with the graphical method to contrast the integer and lp-relaxation optimal solutions.

	table (x_1)	chair (x_2)	max availability
Wood	2	1	6
Time	7	8	28
Profit per unit	120	80	

$$
\begin{aligned}
&\text{Maximize:} && Z = 120x_1 + 80x_2 && (1)\\
&\text{Subject to:} && 2x_1 + x_2 \leq 6 && (2)\\
& && 7x_1 + 8x_2 \leq 28 && (3)\\
& && x_1, x_2 \geq 0 \text{ and integer} && (4)
\end{aligned}
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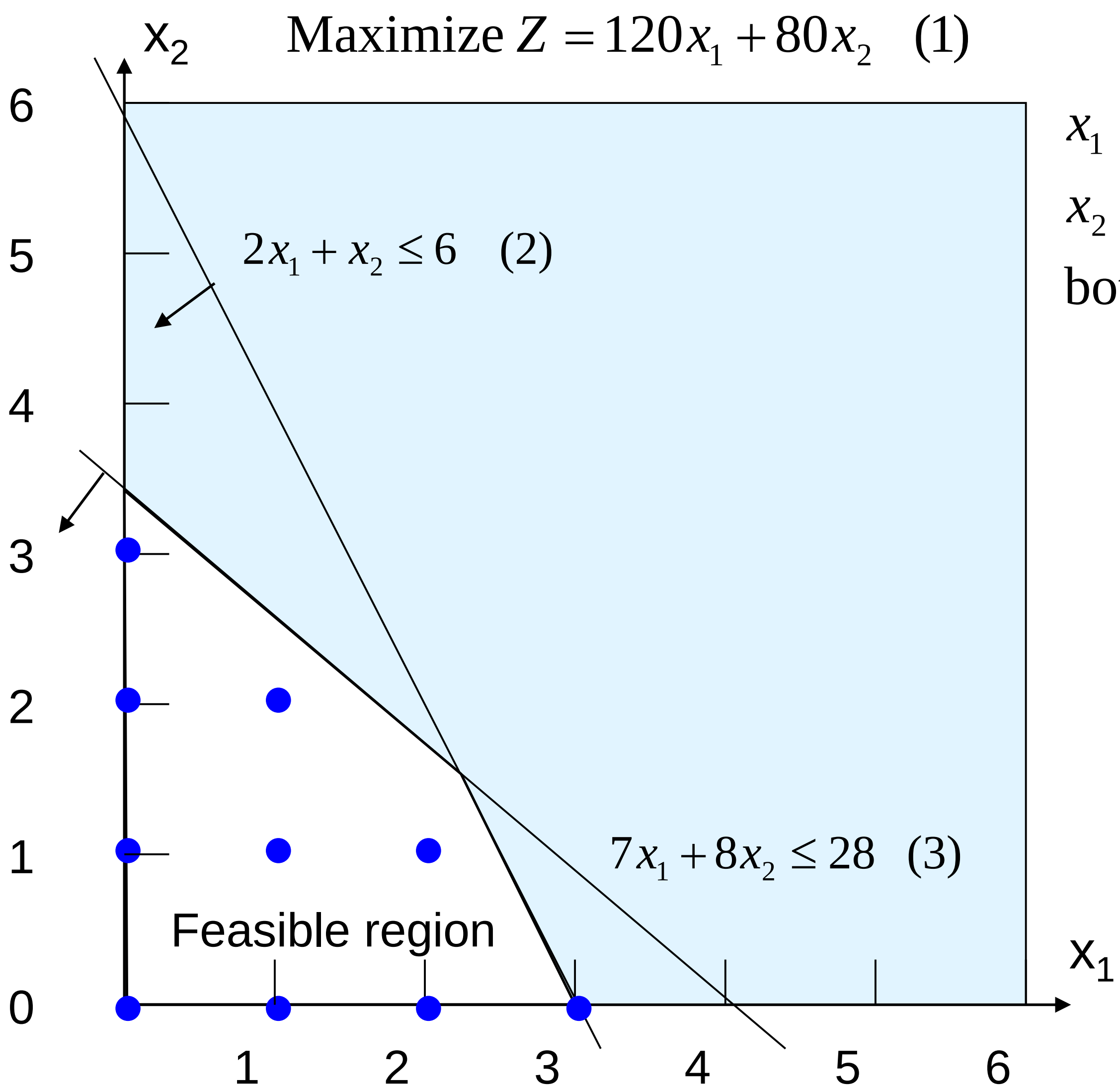
$$

Example (cont.) Graphical method to solve FURNITURE IP model

Identify the feasible solutions for the IP model.

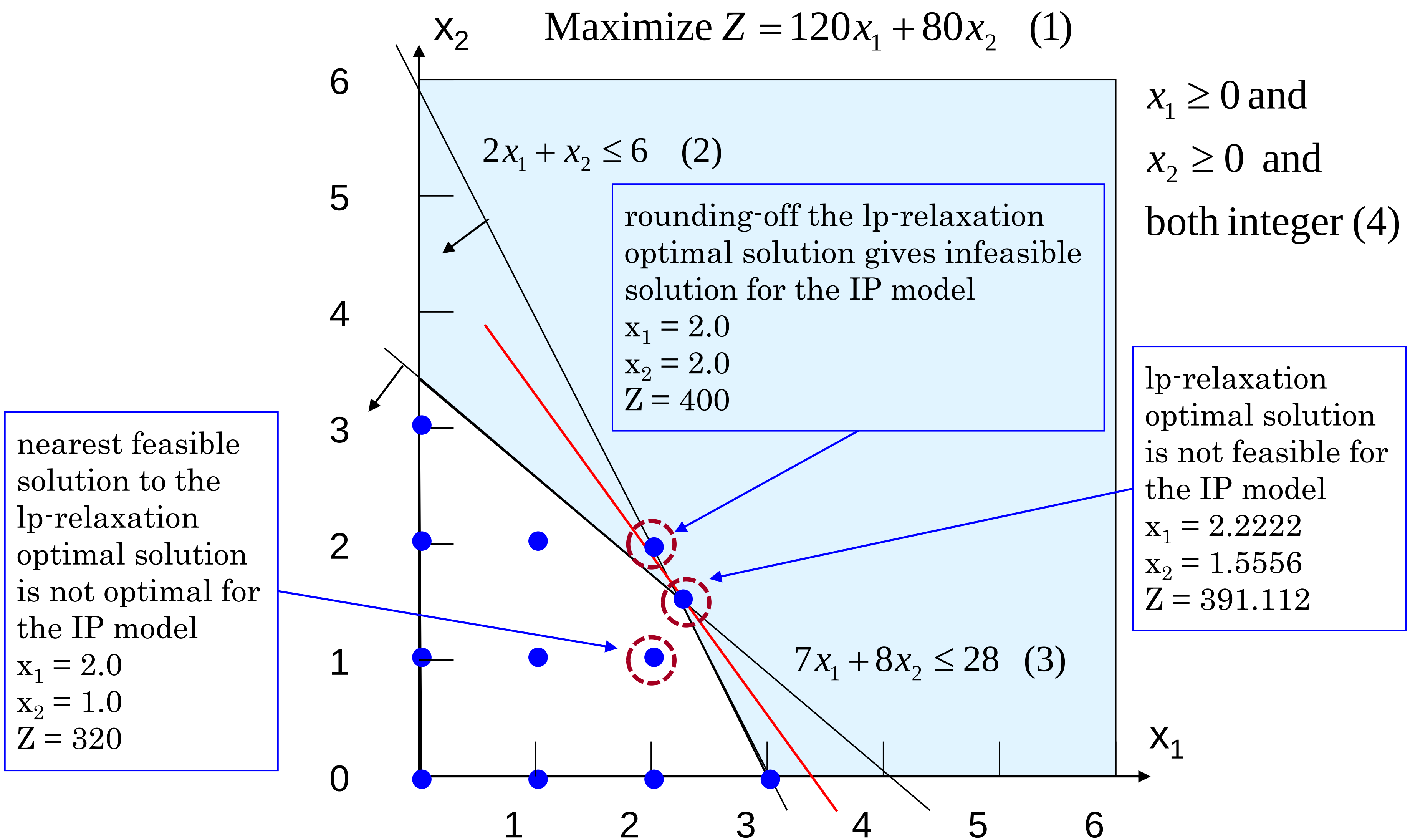


$x_1 \geq 0$ and

$x_2 \geq 0$ and

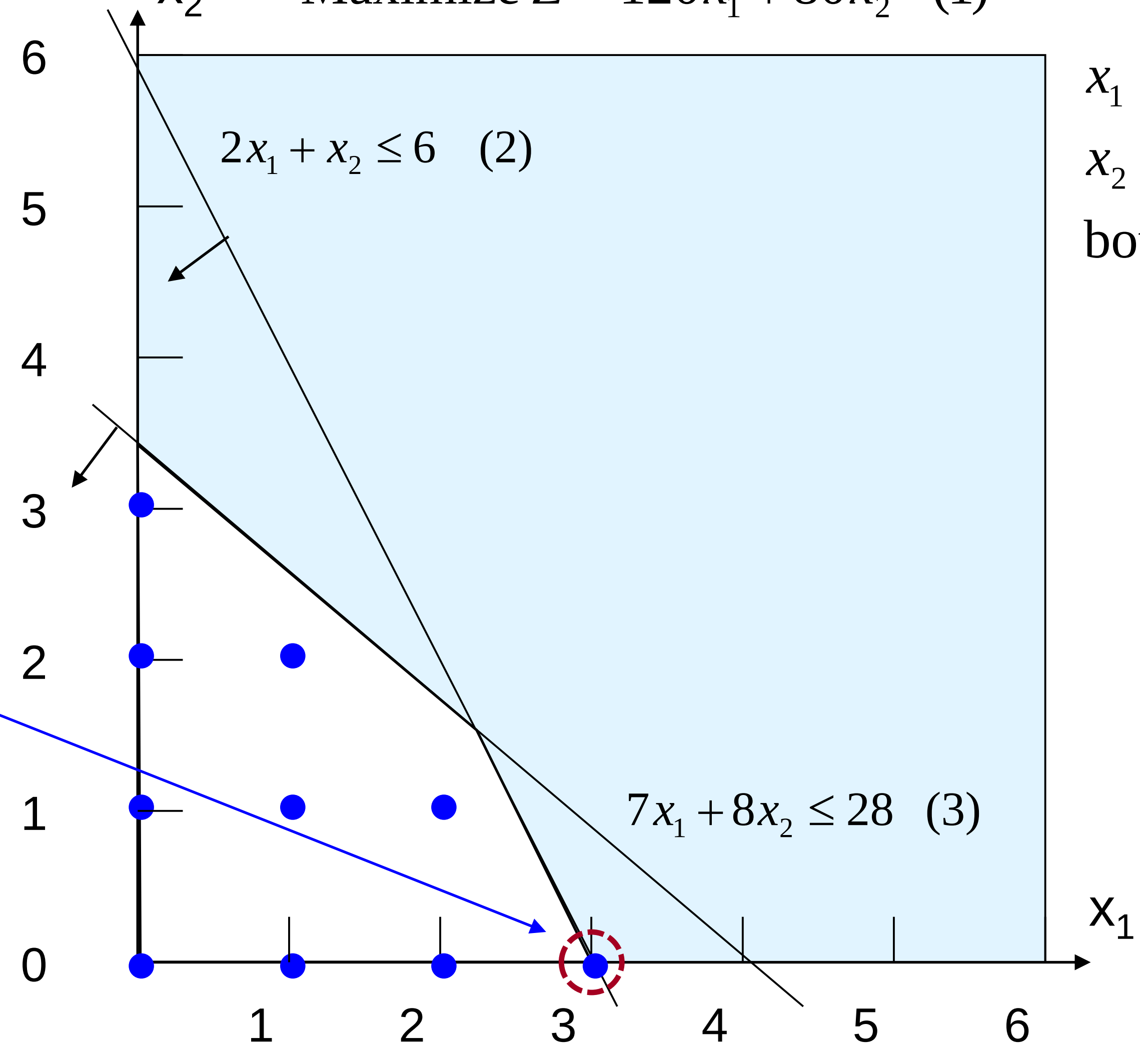
both integer (4)

Example (cont.) Contrast integer and lp-relaxation optimal solutions



Example (cont.) Contrast integer and lp-relaxation optimal solutions

$$\text{Maximize } Z = 120x_1 + 80x_2 \quad (1)$$



$x_1 \ge 0$ and

$x_2 \ge 0$ and

both integer (4)

integer optimal solution for the IP model is a CPF feasible solution
$x_1 = 3.0$
$x_2 = 0$
$Z = 360$

Questions OR Comments