Linear and Discrete Optimization (G54LDO)

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Lecture 7 – Packing Optimization Problems

- Capacity and Indicator Variables

  To recognise typical capacity constraints

  To explain the use of indicator binary variables

- Knapsack Type Problems

  To recognise knapsack type problems and apply the corresponding BIP formulation

- Bin Packing Type Problems

  To recognise bin-packing type problems and apply the corresponding BIP formulation

- Example Applications

  To apply packing optimization models to some optimization scenarios

Additional Reading

Sections 11.1 to 11.5 of (Hillier and Lieberman, 2010).

Section 9.1 of (Taha, 2011).

The Knapsack Problem. John J. Bartholdi. Building Intuition – Insights From Basic Operations Management Models and Principles, Chapter 2, pp. 19-31, 2008.

List of Knapsack Problems. Wikipedia, http://en.wikipedia.org/wiki/List_of_knapsack_problems, Accessed November 2017.

Bin Packing Problem. Wikipedia, https://en.wikipedia.org/wiki/Bin_packing_problem, Accessed November 2017.

Designing Difficult Office Space Allocation Problem Instances with Mathematical Programming. Özgür Ülker, Dario Landa-Silva. Experimental Algorithms, Lecture Notes in Computer Science, Vol. 6630, pp. 280-291, Springer-Verlag, 2011. Available here.

Heuristic Approach for Automated Shelf Space Allocation. Dario Landa-Silva, Fathima Marikar, Khoi Le. Proceedings of the 24th ACM Symposium on Applied Computing (SAC 2009), Vol. 2, pp. 922-928, ACM Press, Hawaii USA, March 2009. Available here.

Capacity and Indicator Variables

- Some problems involve selecting items for packing them into containers of limited capacity.
- The most common constraint is the limited capacity of the containers available.
- In some packing problems, besides binary variables to decide whether at item is selected or not for packing, indicator binary variables are also needed to indicate the use or not of some containers.

Indicator Variables

This technique helps to 'link' variables for dependency:

$Y_1 - Y_2 \leq 0$ — Assuming Y_1, Y_2 are binary, it means Y_1 may be one only if Y_2 is one.

$X - MY \leq 0$ — Assuming Y binary, X not binary, M positive number, it means X may be above zero (but not greater than M), only if Y is one.

Examples of Using Indicator Variables

$$X_1 \le MY_1$$

$$Y_1 \le X_1$$

$$M = 3000$$

Y_1 is binary

X_1 is integer

INTERNET CONNECTION Problem

Number of minutes from company A (X_1) can be more than zero (but no greater than 3000) only when the corresponding connection fee (Y_1) is paid and the connection fee is paid only when the number of minutes from company A is not zero.

Any other example seen so far that uses indicator variables?

Knapsack Type Problems

The Knapsack Problem

Given:

A knapsack (container) of given space capacity B

A set of N items

Each item has a given size S_i and generates profit P_i if it is included in the knapsack.

The problem is to select which items to pack in the knapsack so that the total profit is maximized without exceeding the capacity of the knapsack.

Example. Given a limited budget of 15, select the subset of investments among 8 alternatives so as to maximize the return.

Example (cont.) Sketch of a knapsack problem with N = 8, B = 15.

Assume the optimal solution is to select investments 2, 4, 6 and 7

$$\text{Knapsack Capacity} = \text{B} = 15$$

$$\begin{array}{ll}
\text{Items Sizes} & [S_1 \quad \boxed{S_2} \quad S_3 \quad \boxed{S_4} \quad S_5 \quad \boxed{S_6} \quad \boxed{S_7} \quad S_8] \\
\text{Item Profits} & [P_1 \quad \boxed{P_2} \quad P_3 \quad \boxed{P_4} \quad P_5 \quad \boxed{P_6} \quad \boxed{P_7} \quad P_8] \\
\text{Items Selected} & [X_1 \quad \boxed{X_2} \quad X_3 \quad \boxed{X_4} \quad X_5 \quad \boxed{X_6} \quad \boxed{X_7} \quad X_8]
\end{array}$$

Then :

$X_2 = X_4 = X_6 = X_7 = 1$

$X_1 = X_3 = X_5 = X_8 = 0$

$S_2 X_2 + S_4 X_4 + S_6 X_6 + S_7 X_7 \leq B$

$\text{Overall Profit} = P_2 X_2 + P_4 X_4 + P_6 X_6 + P_7 X_7$

BIP Model for the Knapsack Problem

$$\text{Maximize:} \quad Z = \sum_{i=1}^{N} P_i X_i$$

$$\text{Subject to:} \quad \sum_{i=1}^{N} S_i X_i \le B \qquad (1)$$

$$X_i = 1 \text{ if item } i \text{ is packed, 0 otherwise} \qquad (2)$$

Where:

N:

B:

S_i:

P_i:

X_i:

BIP model for problem instance with $N = 8$ and $B = 15$

$$\text{Maximize:} \quad Z = P_1X_1 + P_2X_2 + P_3X_3 + P_4X_4 + P_5X_5 + P_6X_6 + P_7X_7 + P_8X_8$$

$$\text{Subject to:} \quad S_1X_1 + S_2X_2 + S_3X_3 + S_4X_4 + S_5X_5 + S_6X_6 + S_7X_7 + S_8X_8 \le 15 \qquad (1)$$

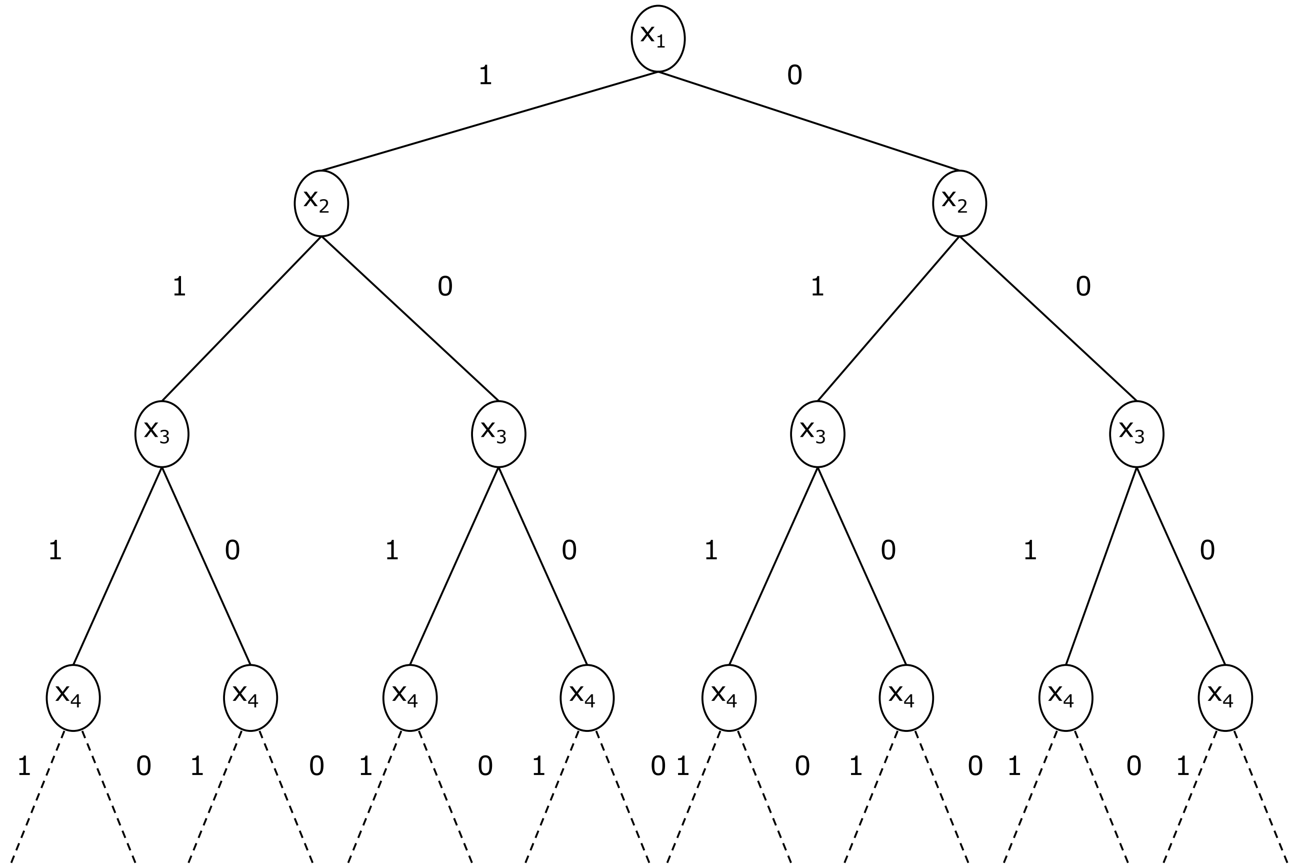
$$X_i \in \{0,1\} \text{ for } i = 1\ldots8 \qquad (2)$$

Combinatorial Nature of the Knapsack Problem (N=8)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_{N-1} \\ P_N \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-1} \\ X_N \end{bmatrix} = Z$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{N-1} \\ S_N \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-1} \\ X_N \end{bmatrix} \leq B$$

There are 2^N ways of selecting a subset from the set of N items. This includes the empty set.
Sketch for the first 4 decisions variables shown below.



The Multiple Knapsack Problem

Given:

A set of M knapsacks each of given capacity B_j

A set of N items

Each item has a given size S_i

Each item generates profit P_{ij} if it is included in knapsack j.

The problem is to select which items to pack in each of the knapsacks so that the total profit is maximized without exceeding the capacity of any knapsack.

Example. Given a limited budget in each of 2 years, select the subset of investments among 3 alternatives so as to maximize the overall return. Each investment can only be made in at most one year and its profit depends on the year.

Example. Sketch of the multiple knapsack problem.

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1M} \\ P_{21} & P_{22} & \cdots & P_{2M} \\ P_{31} & P_{32} & \cdots & P_{3M} \\ \vdots & \vdots & \vdots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NM} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1M} \\ X_{21} & X_{22} & \cdots & X_{2M} \\ X_{31} & X_{32} & \cdots & X_{3M} \\ \vdots & \vdots & \vdots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NM} \end{bmatrix} = Z$$

Objective function is the sum-product of decision variables and profits.

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \\ \vdots \\ X_{N1} \end{bmatrix} \le B_1 \quad \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \\ \vdots \\ X_{N2} \end{bmatrix} \le B_2 \quad \cdots \quad \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{1M} \\ X_{2M} \\ X_{3M} \\ \vdots \\ X_{NM} \end{bmatrix} \le B_M$$

The capacity constraint for each knapsack is the sum-product of the knapsack's decision variables and the item sizes.

$$\begin{aligned} &[X_{11} + X_{12} + X_{13} + \cdots + X_{1M}] \le 1 \\ &[X_{21} + X_{22} + X_{23} + \cdots + X_{2M}] \le 1 \\ &[X_{31} + X_{32} + X_{33} + \cdots + X_{3M}] \le 1 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ &[X_{N1} + X_{N2} + X_{N3} + \cdots + X_{NM}] \le 1 \end{aligned}$$

The constraint for each item being packed exactly in one knapsack is the sum of the item's decision variables on all knapsacks.

BIP Model for the Multiple Knapsack Problem

$$\text{Maximize:} \quad Z = \sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} X_{ij}$$

$$\text{Subject to:} \quad \sum_{i=1}^{N} S_i X_{ij} \le B_j \quad \text{for } j = 1,2,\ldots M \qquad (1)$$

$$\sum_{j=1}^{M} X_{ij} \le 1 \qquad \text{for } i = 1,2,\ldots N \qquad (2)$$

$$X_{ij} = 1 \text{ if item } i \text{ is packed into knapsack } j, 0 \text{ otherwise} \qquad (3)$$

BIP model for problem instance with $N = 3$ and $M = 2$

$$\text{Maximize:} \quad Z = P_{11}X_{11} + P_{12}X_{12} + P_{21}X_{21} + P_{22}X_{22} + P_{31}X_{31} + P_{32}X_{32}$$

$$\text{Subject to:} \quad S_1 X_{11} + S_2 X_{21} + S_3 X_{31} \le B_1 \qquad (1)$$

$$S_1 X_{12} + S_2 X_{22} + S_3 X_{33} \le B_2 \qquad (2)$$

$$X_{11} + X_{12} + X_{13} \le 1 \qquad (3)$$

$$X_{21} + X_{22} + X_{23} \le 1 \qquad (4)$$

$$X_{31} + X_{32} + X_{33} \le 1 \qquad (5)$$

$$X_{ij} \in \{0,1\} \text{ for } i = 1\ldots8 \text{ and } j = 1\ldots2 \qquad (6)$$

Where:

N :

M :

B_j :

S_i :

P_{ij} :

X_{ij} :

The Generalised Assignment Problem

Given:

A set of N tasks

A set of M workers

Each worker j has available time T_j

Assigning task i to worker j has a cost C_{ij}

The time that worker j takes to do task i is t_{ij}

The problem is to assign all the tasks to the workers so that the total cost is minimized without exceeding the time available of any worker (a worker can take more than one task).

Example. Given a set of 10 computer processors each with limited computation time, assign the set of 75 processing jobs so that the cost is minimized.

Example. Sketch of the generalised assignment problem.

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1M} \\ c_{21} & c_{22} & \cdots & c_{2M} \\ c_{31} & c_{32} & \cdots & c_{3M} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NM} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ x_{31} & x_{32} & \cdots & x_{3M} \\ \vdots & \vdots & \vdots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix} = Z$$

Objective function is the sum-product of decision variables and profits.

$$\begin{aligned} &[x_{11} + x_{12} + x_{13} + \cdots + x_{1M}] = 1 \\ &[x_{21} + x_{22} + x_{23} + \cdots + x_{2M}] = 1 \\ &[x_{31} + x_{32} + x_{33} + \cdots + x_{3M}] = 1 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ &[x_{N1} + x_{N2} + x_{N3} + \cdots + x_{NM}] = 1 \end{aligned}$$

The constraint for each task being assigned exactly to one worker is the sum of the task's decision variables on all workers.

$$\begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \\ \vdots \\ t_{N1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ \vdots \\ x_{N1} \end{bmatrix} \le T_1 \quad \begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \\ \vdots \\ t_{N2} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ \vdots \\ x_{N2} \end{bmatrix} \le T_2 \quad \cdots \quad \begin{bmatrix} t_{1M} \\ t_{2M} \\ t_{3M} \\ \vdots \\ t_{NM} \end{bmatrix} \begin{bmatrix} x_{1M} \\ x_{2M} \\ x_{3M} \\ \vdots \\ x_{NM} \end{bmatrix} \le T_M$$

The time capacity constraint for each worker is the sum-product of the worker's decision variables and the tasks' times.

BIP Model for the Generalised Assignment Problem

$$\text{Minimize:} \quad Z = \sum_{i=1}^{N}\sum_{j=1}^{M} C_{ij} X_{ij}$$

$$\text{Subject to:} \quad \sum_{j=1}^{M} X_{ij} = 1 \qquad \text{for } i = 1,2,\ldots N \qquad (1)$$

$$\sum_{i=1}^{N} t_{ij} X_{ij} \le T_j \qquad \text{for } j = 1,2,\ldots M \qquad (2)$$

$$X_{ij} = 1 \text{ if task } i \text{ is assigned to worker } j, \text{ 0 otherwise} \qquad (3)$$

Where:

N :

M :

T_j :

t_{ij} :

C_{ij} :

X_{ij} :

Bin Packing Type Problems

The Bin Packing Problem

Given:

A set of M bins all of the same capacity B

A set of N items

Each item has a given size S_i

The problem is pack all the items into the least number of bins without exceeding the capacity of any bin (it is assumed that there are enough bins for packing all items).

Example. Given a set of 150 items of different sizes and unlimited number of 'boxes' of capacity B, pack all of the items using the minimum number of boxes.

Example. Sketch of the bin-packing problem.

$$\begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_M \end{bmatrix}$$ Large enouhg number of bins

Objective function is to use the least number of bins.

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \\ \vdots \\ X_{N1} \end{bmatrix} \le Y_1 B \qquad \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{22} \\ X_{32} \\ \vdots \\ X_{N2} \end{bmatrix} \le Y_2 B \quad \cdots \quad \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} \begin{bmatrix} X_{1M} \\ X_{2M} \\ X_{3M} \\ \vdots \\ X_{NM} \end{bmatrix} \le Y_M B$$

Linking constraints for the capacity of each bin not to be exceeded use indicator variables.

$$\begin{aligned}
&[X_{11} + X_{12} + X_{13} + \cdots + X_{1M}] = 1 \\
&[X_{21} + X_{22} + X_{23} + \cdots + X_{2M}] = 1 \\
&[X_{31} + X_{32} + X_{33} + \cdots + X_{3M}] = 1 \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&[X_{N1} + X_{N2} + X_{N3} + \cdots + X_{NM}] = 1
\end{aligned}$$

Constraints to ensure each item is packed in exactly one bin.

BIP Model for the Bin Packing Problem

Minimize: $Z = \sum_{j=1}^{M} Y_j$

Subject to: $\sum_{i=1}^{N} S_i X_{ij} \leq Y_j B \quad \text{for } j = 1,2,\ldots M$ (1)

$\sum_{j=1}^{M} X_{ij} = 1 \quad \text{for } i = 1,2,\ldots N$ (2)

$X_{ij} = 1$ if item i is packed into bin j, 0 otherwise (3)

$Y_j = 1$ if bin j is used, 0 otherwise (4)

Where:

N:

M:

B:

S_i:

Y_j:

X_{ij}:

Example. Capital Investments Problem

An investor is considering 7 capital investments. Each investment can be made only once. The profit P_i and required capital C_i for each investment are given. The total amount of capital available for these investments is denoted by B. Investments 1 and 2 are mutually exclusive. Investment 5 can only be chosen if investment 7 is chosen. Neither 3 and 4 can be taken unless one of the two first investments is chosen. The objective is to select the combination of investments that maximizes the total profit.

Questions OR Comments