Linear and Discrete Optimization (G54LDO)

Semester 1 of Academic Session 2017-2018
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Lecture 10 – Algebraic Simplex and B&B Methods

- Principles of Simplex
  To explain the geometric and algebraic concepts and mechanisms of the Simplex method
- The Algebraic Simplex Method
  To describe the various elements of the Simplex method: slack variables, augmented solution, basic solution, basic variables, non-basic variables, optimality test
  To apply the algebraic Simplex method to small LP problems
- Principles of Branch and Bound
  To explain the principles of B&B search
- Steps of Branch and Bound
  To describe the three main steps of B&B
  To apply B&B to solve examples of BIP models

Additional Reading

Sections 4.1 to 4.4 of (Hillier&Lieberman, 2010) for Simplex. Available at: http://faculty.ksu.edu.sa/72966/Documents/chap04.pdf

Chapters on the Simplex (algebraic) method in any book in the reading list.

Demo of the Algebraic Simplex Method: http://faculty.mercer.edu/schultz_sr/courses/ise302/lp/algebraic/page1.html

Hybrid Simplex Improved GA for Global Numerical Optimization. Z.W. Ren, Y. San, J.F. Chen. Acta Automatic Sinica, Vol. 33, No. 1, pp. 91-95, 2007.

Sections 11.5 to 11.7 of (Hillier&Lieberman, 2010) for B&B.

Chapters on the B&B method of any book in the reading list.

On the Use of Integer Programming Versus Evolutionary Solver in Spreadsheet Optimization. Kenneth R. Baker, Jeffrey D. Camm. INFORMS Transactions on Education, Vol. 5, No. 3, pp. 1-7, 2005.

Principles of Simplex

- Developed by G. Dantzig in 1947, this is a very efficient algebraic procedure to solve very large LP problems.

- The underlying principles of the Simplex method are based on the geometric concepts:
  - constraint boundaries
  - corner-point solutions
  - corner-point feasible (CPF) solutions
  - corner-point infeasible solutions
  - adjacent CPF solutions
  - edges of the feasible region
  - a CPF solution is optimal if it does not have adjacent CPF solutions that are better

Geometric Concepts of Simplex

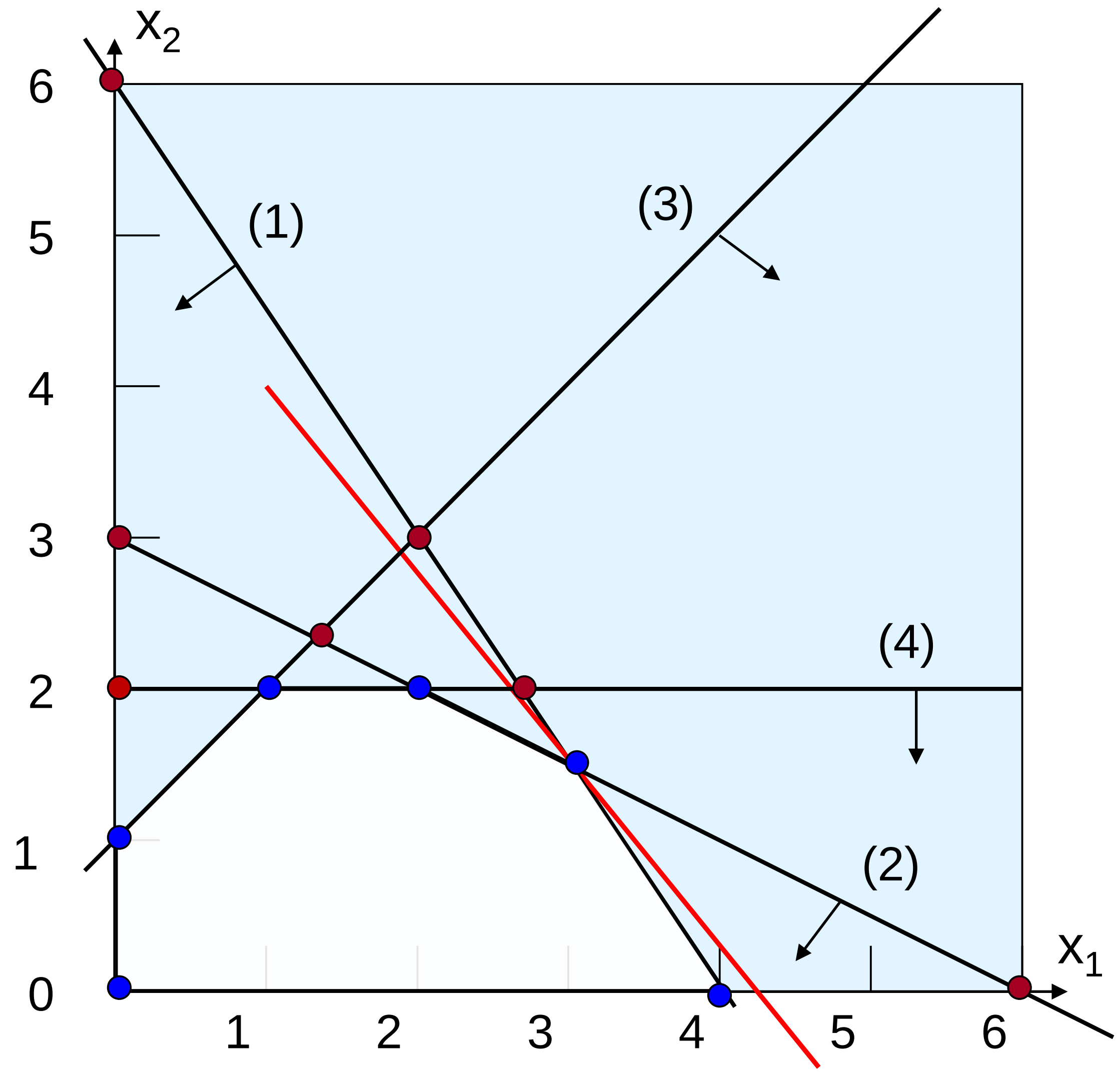
$$\text{Maximize}: \ Z = 5x_1 + 4x_2$$

Subject to:

$$6x_1 + 4x_2 \le 24 \quad (1)$$
$$x_1 + 2x_2 \le 6 \quad (2)$$
$$-x_1 + x_2 \le 1 \quad (3)$$
$$x_2 \le 2 \quad (4)$$
$$x_1 \ge 0, x_2 \ge 0 \quad (5)$$

ATLAS problem example

Identify:
- constraint boundaries
- CP solutions
- CPF solutions
- CP infeasible solutions
- adjacent CPF solutions
- edges of feasible region



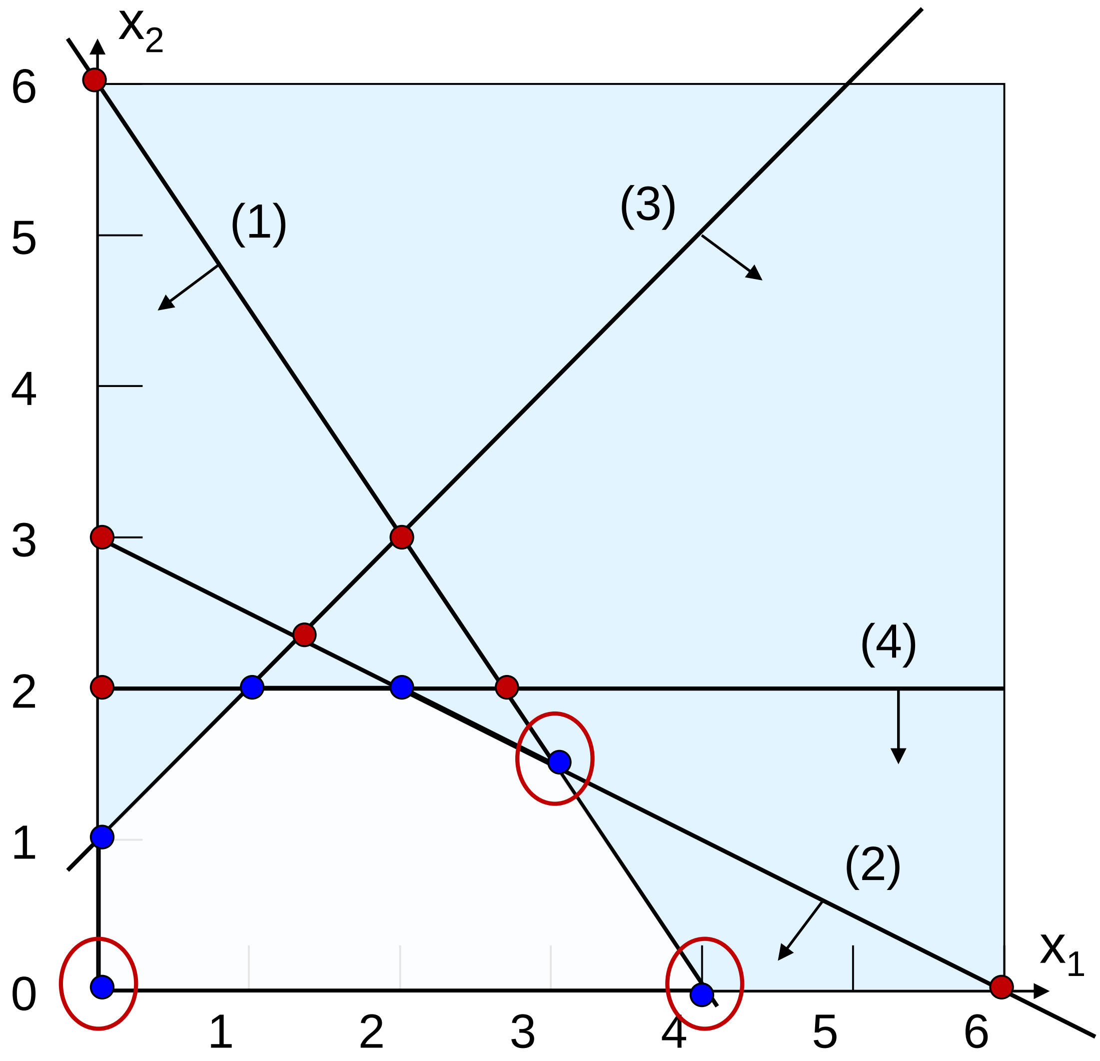
The Simplex Procedure

1. Initialisation
2. Optimality Test
3. If Optimal, Stop
4. If not Optimal, perform iteration: move to better adjacent CPF solution and then go to step 2.

Maximize: $Z = 5x_1 + 4x_2$

Subject to:

$$6x_1 + 4x_2 \leq 24 \quad (1)$$
$$x_1 + 2x_2 \leq 6 \quad (2)$$
$$-x_1 + x_2 \leq 1 \quad (3)$$
$$x_2 \leq 2 \quad (4)$$
$$x_1 \geq 0, x_2 \geq 0 \quad (5)$$



Key Concepts of the Simplex Procedure

- It focuses only on CPF solutions (assuming there is at least one).
- In each iteration, another CPF solution is explored.
- If possible, the origin ($x_i = 0\ \forall i$) being a CPF solution, is selected as the initial point (for maximization).
- The procedure moves to adjacent CPF solutions by moving along the edges of the feasible region.
- Better adjacent CPF solutions are identified by positive rate of improvement on the objective value Z.
- If the rate of improvement along the edges emanating from the current CPF solution are all negative, then the current CPF solution is optimal.

Importance of the Simplex Method

- Simplex is the standard method for solving LP problems.
- Even modern algorithms to solve large IP problems are based on successive LP relaxations solved using Simplex.
- Variants of the Simplex method are tailored for specific types of problems.
- The Primal Simplex method is simply known as Simplex.
- The Dual Simplex method is efficient for re-optimization after the model has changed (change of coefficients, change of right-hand side values, addition/deletion of constraints).
- The Revised Simplex method is useful as part of a technique called Column Generation for solving very large LP problems.

The Algebraic Simplex Method

$$\begin{array}{llr} \text{Maximise:} & Z = 5x_1 + 4x_2 & (0) \\ \text{Subject to:} & 6x_1 + 4x_2 \le 24 & (1) \\ & x_1 + 2x_2 \le 6 & (2) \\ & -x_1 + x_2 \le 1 & (3) \\ & x_2 \le 2 & (4) \\ & x_1 \ge 0, x_2 \ge 0 & (5) \end{array}$$

Example 1 Apply the algebraic Simplex method to solve the ATLAS maximization problem.

Convert constraints to equations

Introduce 1 slack variable in each functional constraint to obtain equalities:

$$6x_1 + 4x_2 \le 24 \quad \Rightarrow \quad 6x_1 + 4x_2 + x_3 = 24 \quad \text{and} \quad x_3 \ge 0$$

$$x_1 + 2x_2 \le 6 \quad \Rightarrow \quad x_1 + 2x_2 + x_4 = 6 \quad \text{and} \quad x_4 \ge 0$$

$$-x_1 + x_2 \le 1 \quad \Rightarrow \quad -x_1 + x_2 + x_5 = 1 \quad \text{and} \quad x_5 \ge 0$$

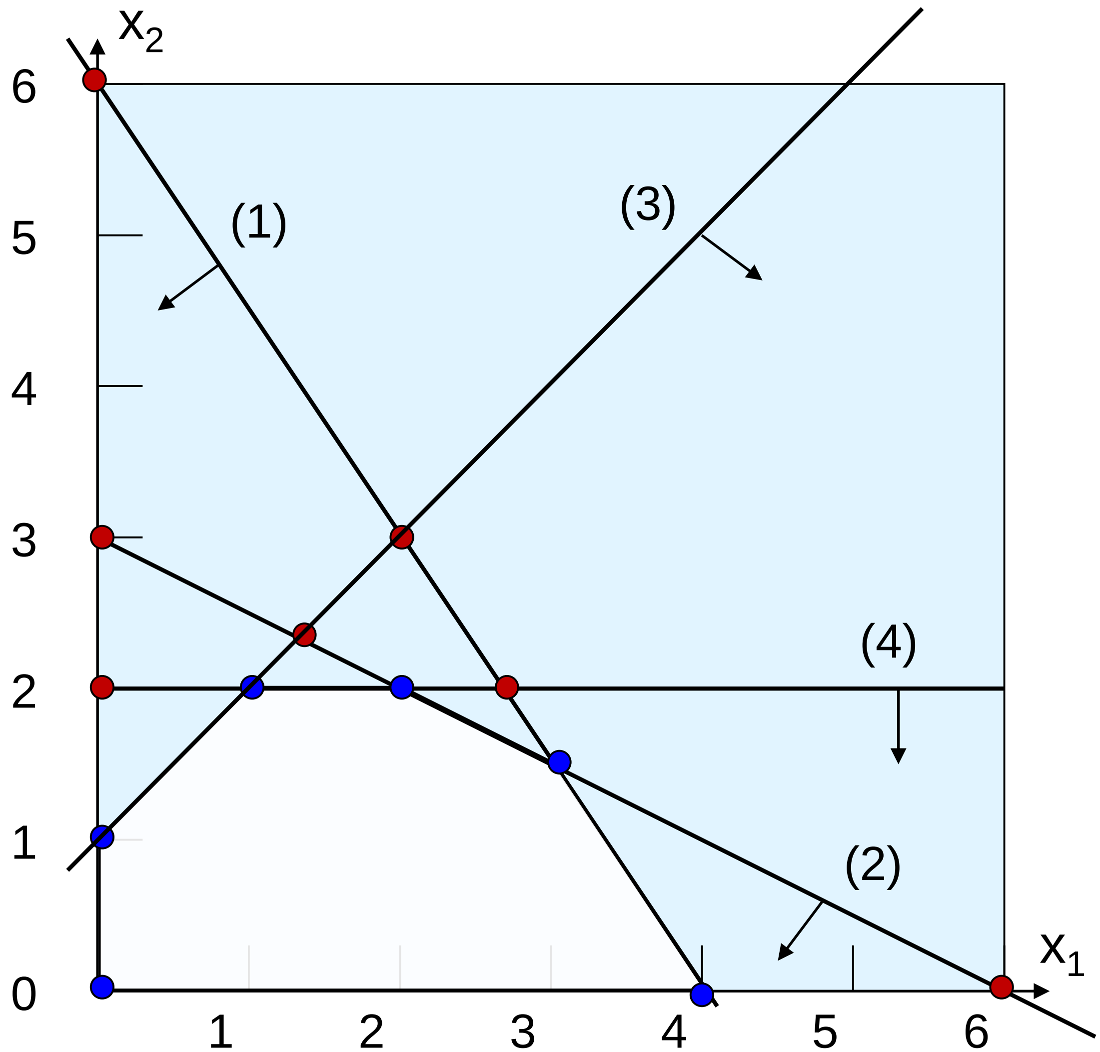
$$x_2 \le 2 \quad \Rightarrow \quad x_2 + x_6 = 2 \quad \text{and} \quad x_6 \ge 0$$

The result is an augmented model

$$\begin{aligned}
&\text{Maximise:} && Z - 5x_1 - 4x_2 = 0 && (0)\\
&\text{Subject to:} && 6x_1 + 4x_2 + x_3 = 24 && (1)\\
& && x_1 + 2x_2 + x_4 = 6 && (2)\\
& && -x_1 + x_2 + x_5 = 1 && (3)\\
& && x_2 + x_6 = 2 && (4)\\
& && x_i \geq 0 \ \text{ for } i = 1 \ldots 6 && (5)
\end{aligned}$$

For a given solution, the value of the slack variable indicates if the solution is:

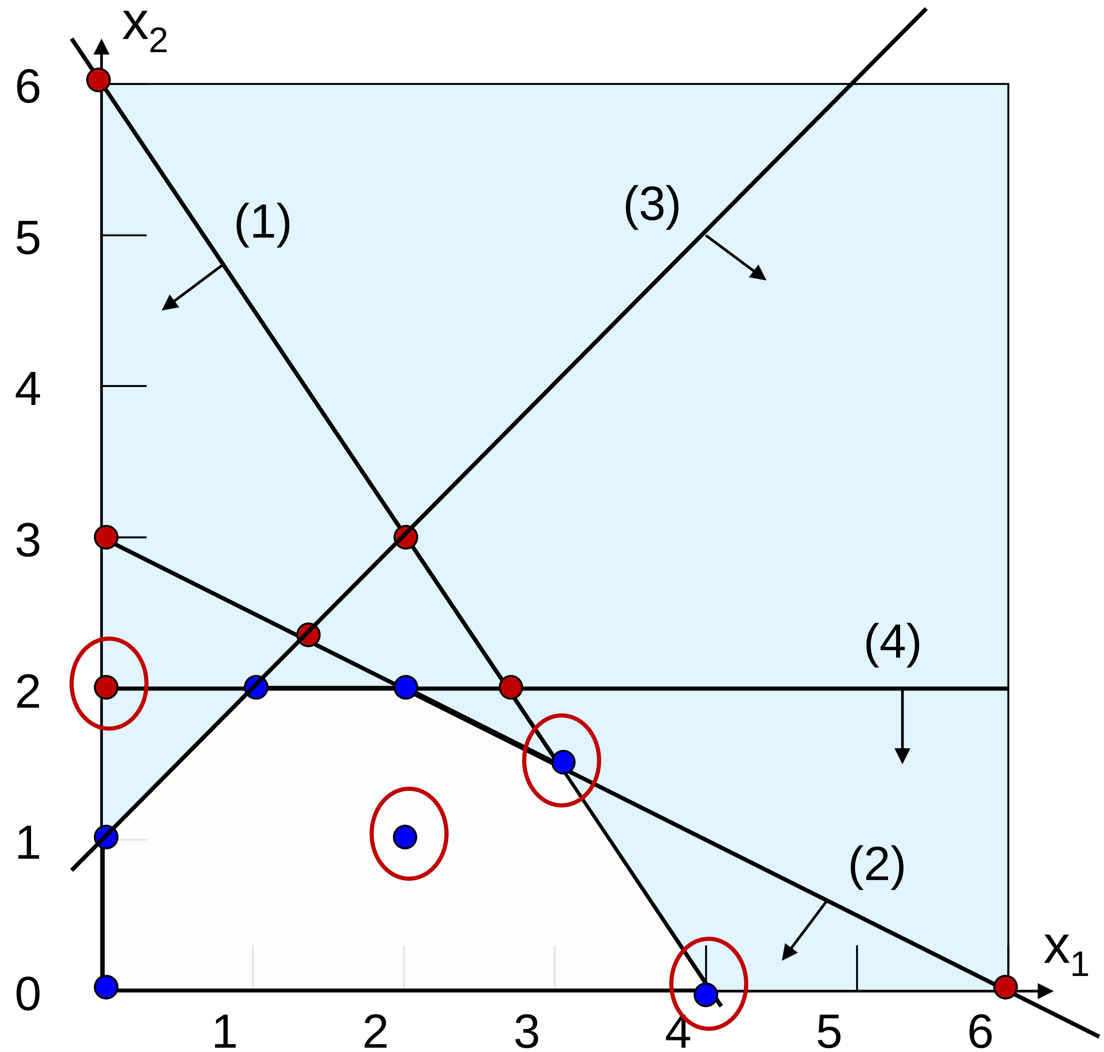
- On the constraint boundary
- On the feasible region
- On the infeasible region



Augmented solutions and basic solutions

$$
\begin{array}{lll}
\text{Maximize:} & Z - 5x_1 - 4x_2 = 0 & (0) \\
\text{Subject to:} & 6x_1 + 4x_2 + x_3 = 24 & (1) \\
& x_1 + 2x_2 + x_4 = 6 & (2) \\
& -x_1 + x_2 + x_5 = 1 & (3) \\
& x_2 + x_6 = 2 & (4) \\
& x_i \geq 0 \ \text{ for } i = 1 \ldots 6 & (5)
\end{array}
$$

For a given solution, the augmented solution is obtained by calculating the value of the slack variables. A basic solution is an augmented corner-point solution.

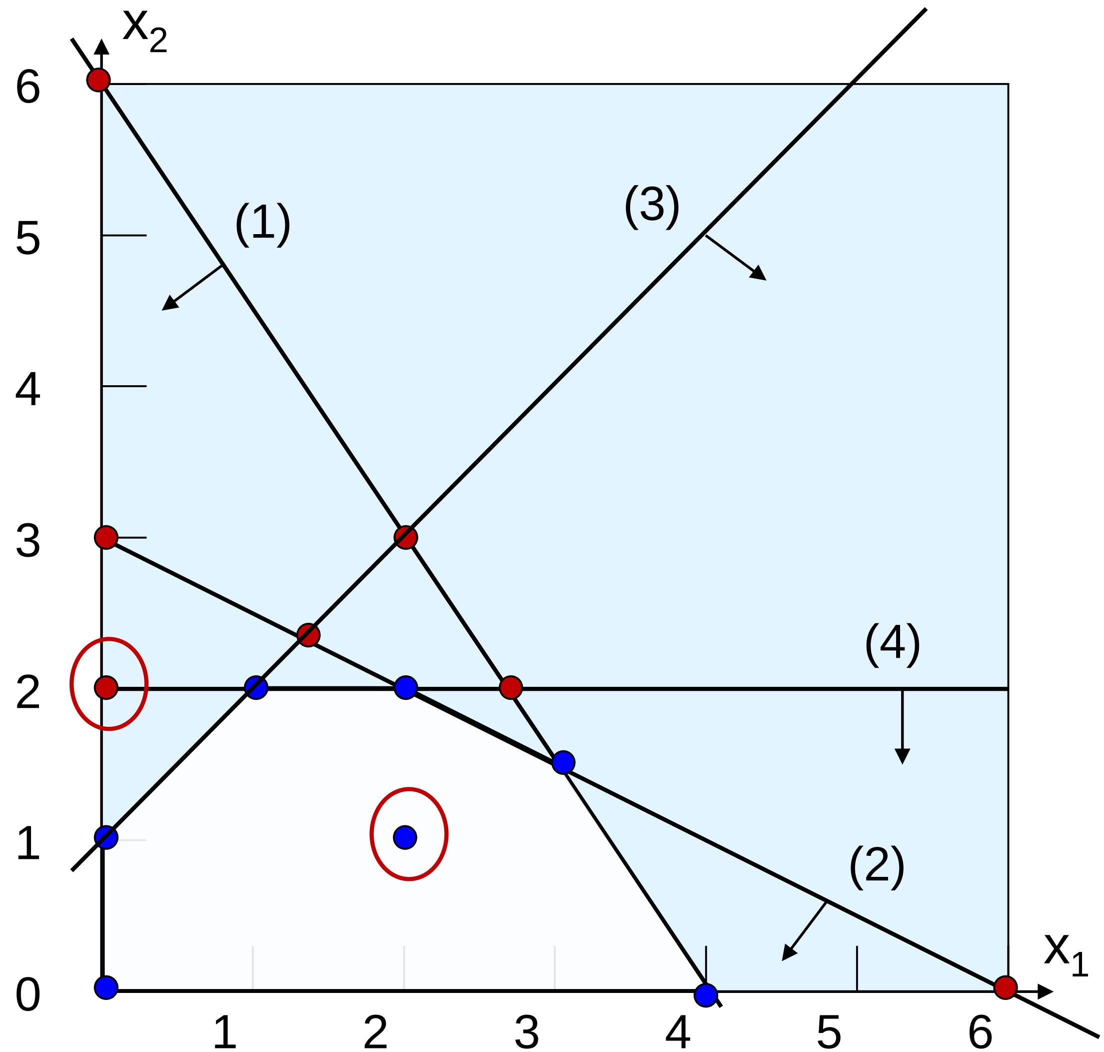


Augmented solutions and basic solutions

$$
\begin{aligned}
&\text{Maximize: } Z - 5x_1 - 4x_2 = 0 && (0)\\
&\text{Subject to: } 6x_1 + 4x_2 + x_3 = 24 && (1)\\
&x_1 + 2x_2 + x_4 = 6 && (2)\\
&-x_1 + x_2 + x_5 = 1 && (3)\\
&x_2 + x_6 = 2 && (4)\\
&x_i \geq 0 \text{ for } i = 1 \ldots 6 && (5)
\end{aligned}
$$

Feasible Solution
(2,1)
Augmented Solution
(2,1,8,2,2,1)

CP Infeasible Solution
(0,3)
Augmented Solution
(0,3,12,0,–2,–1)

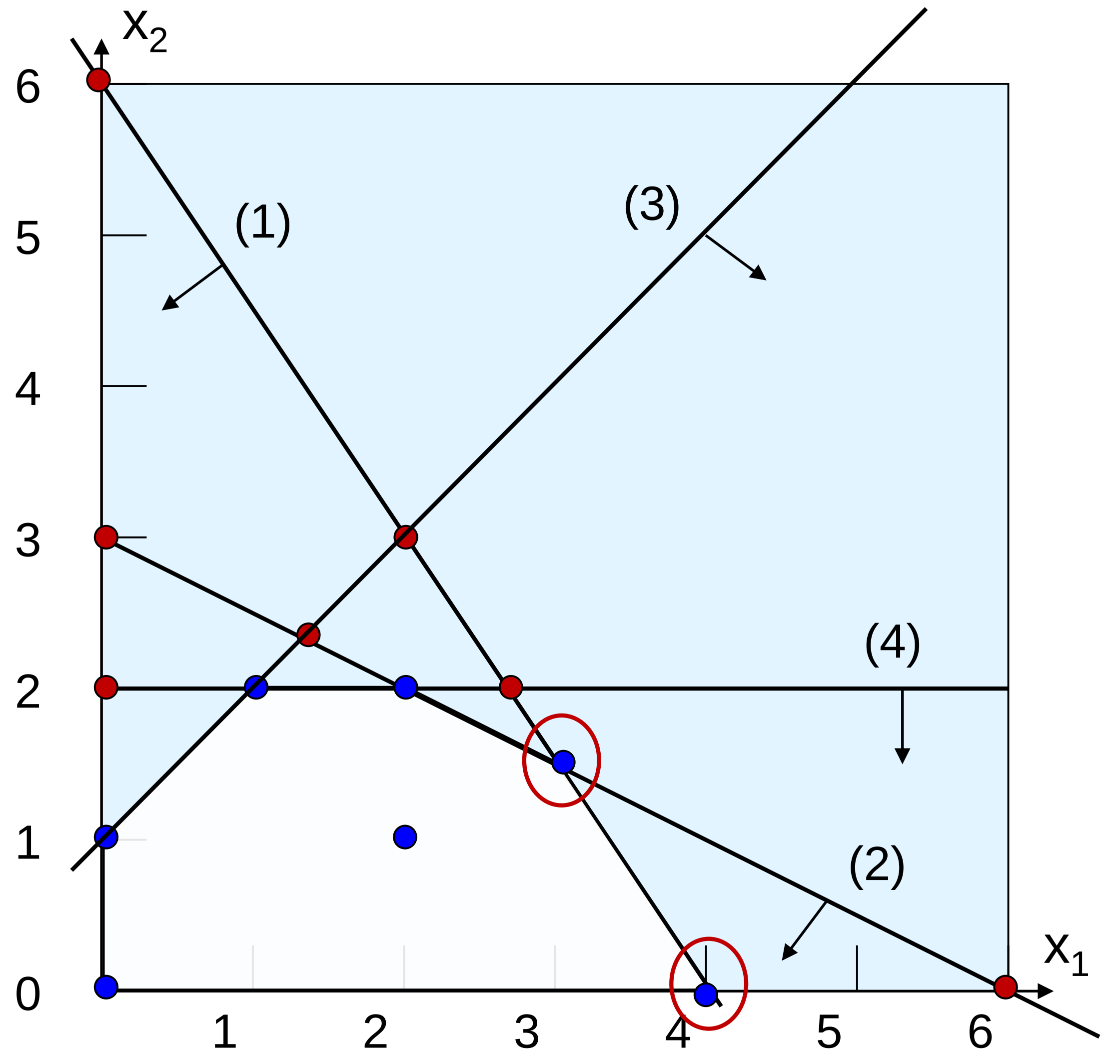


Augmented solutions and basic solutions

$$\begin{aligned}
&\text{Maximize: } && Z - 5x_1 - 4x_2 = 0 && (0)\\
&\text{Subject to: } && 6x_1 + 4x_2 + x_3 = 24 && (1)\\
& && x_1 + 2x_2 + x_4 = 6 && (2)\\
& && -x_1 + x_2 + x_5 = 1 && (3)\\
& && x_2 + x_6 = 2 && (4)\\
& && x_i \geq 0 \ \text{ for } i = 1\ldots6 && (5)
\end{aligned}$$

CPF Solution
(4,0)
Augmented Basic Solution
(4,0,0,2,5,2)

CPF Solution
(3,1.5)
Augmented Basic Solution
(3,1.5,0,0,2.5,0.5)



Algebraic procedure from the augmented model

From the set of variables in the augmented model, some are designated as basic variables and the others are designated as non-basic variables during the solution procedure.

The non-basic variables are set to zero and the basic variables are calculated by solving the equations (augmented constraints).

Initialisation

x_1 and x_2 are designated as the non-basic variables, so

$x_1 = x_2 = 0$

non-basic: $x_1 = 0$, $x_2 = 0$
basic: x_3, x_4, x_5, x_6

Solve for the basic variables from the augmented model:

$$\text{Maximize}: Z - 5x_1 - 4x_2 = 0 \qquad (0)$$

non-basic: $x_1 = 0, x_2 = 0$

basic: x_3, x_4, x_5, x_6

Subject to:

$$6x_1 + 4x_2 + x_3 = 24 \quad (1) \quad \Rightarrow \quad x_3 = 24$$

$$x_1 + 2x_2 + x_4 = 6 \quad (2) \quad \Rightarrow \quad x_4 = 6$$

$$-x_1 + x_2 + x_5 = 1 \quad (3) \quad \Rightarrow \quad x_5 = 1$$

$$x_2 + x_6 = 2 \quad (4) \quad \Rightarrow \quad x_6 = 2$$

$$x_i \geq 0 \text{ for } i = 1 \ldots 6 \quad (5)$$

The initial basic feasible (BF) solution: (0,0,24,6,1,2)

Optimality Test

The initial BF solution (0,0,24,6,1,2) is not optimal

$$\boxed{Z = 5x_1 + 4x_2 \qquad (0)}$$

Iteration 1

$$Z = 5x_1 + 4x_2 \qquad (0)$$

1. Determine direction of movement.

Rate of improvement 5 (corresponding to x_1) is better.

The non-basic variable x_1 is now called the entering basic variable.

2. Determine when to stop the movement.

swap x_1 with basic variable

Increase x_1 as much as possible but maintain the other non-basic variables on zero ($x_2 = 0$), maintain the satisfaction of the equations, and maintain variables non-negative.

$$\begin{array}{llcll}
6x_1 + 4x_2 + x_3 = 24 & (1) & \Rightarrow & x_3 = 24 - 6x_1 & \text{so} \quad x_1 \le 4 \Leftarrow \\
x_1 + 2x_2 + x_4 = 6 & (2) & \Rightarrow & x_4 = 6 - x_1 & \text{so} \quad x_1 \le 6 \\
-x_1 + x_2 + x_5 = 1 & (3) & \Rightarrow & x_5 = 1 + x_1 & \text{so} \quad x_1 \text{ no bounds} \\
x_2 + x_6 = 2 & (4) & \Rightarrow & x_6 = 2 &
\end{array}$$

(cont. Step 2)

Then, x_1 can be increased up to 4 as indicated by the augmented constraint (1), restricted by x_3.

The basic variable x_3 becomes the leaving basic variable.

3. Solving for the new BF solution

non - basic: $x_2 = 0$, $x_3 = 0$

basic: $x_1 = 4$, x_4, x_5, x_6

Initial BF solution: (0,0,24,6,1,2)

New BF solution: (4,0,0,?,?,?)

$$Z - 5x_1 - 4x_2 = 0 \qquad (0)$$

$$6x_1 + 4x_2 + x_3 = 24 \qquad (1)$$

$$x_1 + 2x_2 + x_4 = 6 \qquad (2)$$

$$-x_1 + x_2 + x_5 = 1 \qquad (3)$$

$$x_2 + x_6 = 2 \qquad (4)$$

(cont. Step 3)

The coefficient pattern of the leaving basic variable x_3 (0,1,0,0,0) is reproduced for the entering variable x_1 by means of algebraic manipulation.

$$Z - 5x_1 - 4x_2 = 0 \qquad (0)$$
$$6x_1 + 4x_2 + x_3 = 24 \qquad (1)$$
$$x_1 + 2x_2 + x_4 = 6 \qquad (2)$$
$$-x_1 + x_2 + x_5 = 1 \qquad (3)$$
$$x_2 + x_6 = 2 \qquad (4)$$

$$Z - \frac{2}{3}x_2 + \frac{5}{6}x_3 = 20 \qquad (0)$$
$$x_1 + \frac{2}{3}x_2 + \frac{1}{6}x_3 = 4 \qquad (1)$$
$$\frac{4}{3}x_2 - \frac{1}{6}x_3 + x_4 = 2 \qquad (2)$$
$$\frac{5}{3}x_2 + \frac{1}{6}x_3 + x_5 = 5 \qquad (3)$$
$$x_2 + x_6 = 2 \qquad (4)$$

Given the new non-basic variables $x_2=0$ and $x_3=0$ then:
$Z=20$, $x_1=4$, $x_4=2$, $x_5=5$, $x_6=2$

non-basic: $x_2 = 0$, $x_3 = 0$

basic: $x_1 = 4$, $x_4 = 2$, $x_5 = 5$, $x_6 = 2$

The new BF solution: (4,0,0,2,5,2)

Optimality Test

The new BF solution (4,0,0,2,5,2) is not optimal.

$$Z = 20 + \frac{2}{3}x_2 - \frac{5}{6}x_3 \qquad (0)$$

non - basic : $x_2 = 0$, $x_3 = 0$

basic : $x_1 = 4$, $x_4 = 2$, $x_5 = 5$, $x_6 = 2$

Iteration 2

1. Determine direction of movement.

Rate of improvement 2/3 (corresponding to x_2) is better.

The non-basic variable x_2 is now the entering basic variable.

2. Determine when to stop the movement.

swap x_2 with basic variable

Increase x_2 as much as possible but maintain the other non-basic variables on zero ($x_3 = 0$), maintain the satisfaction of the equations, and maintain variables non-negative.

(cont. Step 2)

$$x_1 + \frac{2}{3}x_2 + \frac{1}{6}x_3 = 4 \quad (1) \quad \Rightarrow \quad x_1 = 4 - \frac{2}{3}x_2 \text{ so } x_2 \le 6$$

$$\frac{4}{3}x_2 - \frac{1}{6}x_3 + x_4 = 2 \quad (2) \quad \Rightarrow \quad x_4 = 2 - \frac{4}{3}x_2 \text{ so } x_2 \le \frac{6}{4}$$

$$\frac{5}{3}x_2 + \frac{1}{6}x_3 + x_5 = 5 \quad (3) \quad \Rightarrow \quad x_5 = 5 - \frac{5}{3}x_2 \text{ so } x_2 \le 3$$

$$x_2 + x_6 = 2 \quad (4) \quad \Rightarrow \quad x_6 = 2 - x_2 \text{ so } x_2 \le 2$$

Then, x_2 can be increased up to 6/4=1.5 as indicated by the augmented constraint (2), restricted by x_4 .

The basic variable x_4 becomes the leaving basic variable.

non - basic : $x_3 = 0, x_4 = 0$

basic : $x_1, x_2 = 1.5, x_5, x_6$

3. Solving for the new BF solution

$$\text{non-basic}: x_3 = 0, x_4 = 0$$
$$\text{basic}: x_1, x_2 = 1.5, x_5, x_6$$

Previous BF solution: (4,0,0,2,5,2)

New BF solution: (?,1.5,0,0,?,?)

The coefficient pattern of the leaving basic variable x_4 (0,0,1,0,0) is reproduced for the entering variable x_2 by means of algebraic manipulation.

$$Z - \frac{2}{3}x_2 + \frac{5}{6}x_3 = 20 \quad (0)$$
$$x_1 + \frac{2}{3}x_2 + \frac{1}{6}x_3 = 4 \quad (1)$$
$$\frac{4}{3}x_2 - \frac{1}{6}x_3 + x_4 = 2 \quad (2)$$
$$\frac{5}{3}x_2 + \frac{1}{6}x_3 + x_5 = 5 \quad (3)$$
$$x_2 + x_6 = 2 \quad (4)$$

$$\Longrightarrow$$

$$Z + \frac{3}{4}x_3 + \frac{1}{2}x_4 = 21 \quad (0)$$
$$x_1 + \frac{1}{4}x_3 - \frac{1}{2}x_4 = 3 \quad (1)$$
$$x_2 - \frac{1}{8}x_3 + \frac{3}{4}x_4 = \frac{3}{2} \quad (2)$$
$$\frac{3}{8}x_3 - \frac{5}{4}x_4 + x_5 = \frac{5}{2} \quad (3)$$
$$\frac{1}{8}x_3 - \frac{3}{4}x_4 + x_6 = \frac{1}{2} \quad (4)$$

(cont. Step 3)

Since the new non-basic variables $x_3=0$ and $x_4=0$ then:
$Z=21$, $x_1=3$, $x_2=1.5$, $x_5=5/2$, $x_6=1/2$

The new BF solution: (3,1.5,0,0,2.5,0.5)

$$\boxed{\begin{array}{l}\text{non-basic: } x_3 = 0, x_4 = 0 \\ \text{basic: } x_1 = 3, x_2 = 1.5, x_5 = 2.5, x_6 = 0.5\end{array}}$$

Optimality Test

The new BF solution (3,1.5,0,0,2.5,0.5) is optimal because in the new objective function both x_3 and x_4 have negative coefficient, so increasing any of them would lead to a worse adjacent BF solution.

$$Z = 21 - \frac{3}{4}x_3 - \frac{1}{2}x_4 \qquad (0)$$

Special Cases in the Simplex Method

The form of the Algebraic Simplex method explained in this lecture works for a given format of the LP model:

- Maximization objective (starting from solution $x_i=0\ \forall i$ is straightforward)
- Inequalities in the constraints are of type $\leq$ (adding slack variables makes sense)
- The right-hand side values b_i in the constraints are positive
- Non-negativity constraints in all decision variables.

To deal with variations of the above LP model formal, some algebraic manipulation is required to prepare the augmented model.

Unbounded Search Space

No leaving basic variable can be identified in Step 2 of a given iteration. This will happen if there is no bound on the increase of the entering basic variable.

Multiple Optimal Solutions

The Simplex method stops immediately after the first optimal solution is found. Multiple optimal solutions exist if at least one of the non-basic variables has a coefficient of zero in the objective function equation. Additional optimal solutions can be obtained by performing additional iterations of the Simplex method, each time choosing a non-basic variable with zero coefficient as the entering basic variable.

Example 2 Apply the algebraic Simplex method to solve the WENBU maximization problem.

$$
\begin{array}{llr}
\text{Maximise:} & Z = 25x_1 + 10x_2 & (0) \\
\text{Subject to:} & 10x_2 \leq 800 & (1) \\
& 25x_1 \leq 1500 & (2) \\
& 5x_1 + 7x_2 \leq 600 & (3) \\
& x_1 \geq 0, x_2 \geq 0 & (4)
\end{array}
$$

Augmented Model

$$
\begin{array}{ll}
(0) & Z - 25x_1 - 10x_2 = 0 \\
(1) & 10x_2 + x_3 = 800 \\
(2) & 25x_1 + x_4 = 1500 \\
(3) & 5x_1 + 7x_2 + x_5 = 600 \\
(4) & x_1 \geq 0, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Example 2 (cont.)

Initialisation

non-basic: $x_1 = 0, x_2 = 0$

basic: x_3, x_4, x_5

$$
\begin{array}{lllll}
(0) & Z - 25x_1 - 10x_2 = 0 & & & \\
(1) & 10x_2 + x_3 = 800 & \Rightarrow & x_3 = 800 \\
(2) & 25x_1 + x_4 = 1500 & \Rightarrow & x_4 = 1500 \\
(3) & 5x_1 + 7x_2 + x_5 = 600 & \Rightarrow & x_5 = 600
\end{array}
$$

Optimality Test

New BF solution $(0,0,800,1500,600)$ with $Z = 0$ is not optimal.

$$Z = 24x_1 + 10x_2 \qquad (0)$$

Example 2 (cont.)

Iteration 1

Entering basic variable: x_1

$$
\begin{array}{lll}
(0)\ Z - 25x_1 - 10x_2 = 0 & & \\
(1)\ 10x_2 + x_3 = 800 & \Rightarrow\ x_3 = 800 & (\text{no bound on } x_1) \\
(2)\ 25x_1 + x_4 = 1500 & \Rightarrow\ x_4 = 1500 - 25x_1 & (x_1 \le 60) \\
(3)\ 5x_1 + 7x_2 + x_5 = 600 & \Rightarrow\ x_5 = 600 - 5x_1 & (x_1 \le 120)
\end{array}
$$

x_1 can increase up to 60 as given by (2)

Leaving basic variable: x_4

New BF solution: (60,0,?,0,?)

non - basic : $x_2 = 0$, $x_4 = 0$

basic : $x_1 = 60$, x_3, x_5

$$
\begin{array}{ll}
(0)\ Z - 25x_1 - 10x_2 = 0 \ \Rightarrow & Z - 10x_2 + x_4 = 1500 \\
(1)\ 10x_2 + x_3 = 800 \ \Rightarrow & 10x_2 + x_3 = 800 \\
(2)\ 25x_1 + x_4 = 1500 \ \Rightarrow & x_1 + \tfrac{1}{25}x_4 = 60 \\
(3)\ 5x_1 + 7x_2 + x_5 = 600 \ \Rightarrow & 7x_2 - \tfrac{1}{5}x_4 + x_5 = 300
\end{array}
$$

New BF solution (60,0,800,0,300) $\quad Z = 1500$

Example 2 (cont.)

$$\text{non-basic: } x_2 = 0\,,\, x_4 = 0$$
$$\text{basic: } x_1 = 60,\, x_3 = 800,\, x_5 = 300$$

Optimality Test

New BF solution (60,0,800,0,300) with Z = 1500 is not optimal.

$$Z = 1500 + 10x_2 - x_4 \qquad (0)$$

Iteration 2

Entering basic variable: x_2

(0) $Z - 10x_2 + x_4 = 1500$

(1) $10x_2 + x_3 = 800 \quad \Rightarrow \quad x_3 = 800 - 10x_2 \quad (x_2 \leq 80)$

(2) $x_1 + \frac{1}{25}x_4 = 60 \quad \Rightarrow \quad x_1 = 60 \quad$ (no bound on x_2)

(3) $7x_2 - \frac{1}{5}x_4 + x_5 = 300 \Rightarrow \quad x_5 = 300 - 7x_2 \quad \left(x_2 \leq \frac{300}{7}\right)$ $\Leftarrow$

x_2 can increase up to 300/7 ≈ 42.86 as given by (3)

Leaving basic variable: x_5

New BF solution: (?,300/7,?,0,0)

$$\text{non-basic: } x_4 = 0,\, x_5 = 0$$
$$\text{basic: } x_1,\, x_2 = 300/7,\, x_3$$

Example 2 (cont.)

non - basic: $x_4 = 0, x_5 = 0$

basic: $x_1, x_2 = 300/7, x_3$

(0) $Z - 10x_2 + x_4 = 1500 \quad \Rightarrow$

(1) $10x_2 + x_3 = 800 \quad \Rightarrow$

(2) $x_1 + \frac{1}{25} x_4 = 60 \quad \Rightarrow$

(3) $7x_2 - \frac{1}{5} x_5 + x_5 = 300 \quad \Rightarrow$

$Z + \frac{25}{35} x_4 + \frac{10}{7} x_5 = \frac{13500}{7}$

$x_3 + \frac{10}{35} x_4 - \frac{10}{7} x_5 = \frac{2600}{7}$

$x_1 + \frac{1}{25} x_4 = 60$

$x_2 - \frac{1}{35} x_4 + \frac{1}{7} x_5 = \frac{300}{7}$

New BF solution (60,300/7,2600/7,0,0) Z = 13500/7

non - basic: $x_4 = 0, x_5 = 0$

basic: $x_1 = 60, x_2 = 300/7, x_3 = 2600/7$

Optimality Test

New BF solution (60,42.86,371.42,0,0) with Z = 1928.57 is optimal

$$Z = \frac{13500}{7} - \frac{25}{35} x_4 - \frac{10}{7} x_5 \qquad (0)$$

Principles of Branch and Bound

Given a problem with general integer or binary decision variables, finding the optimal solution involves exploring the search space represented as a search tree.

The search tree represents all feasible and infeasible solutions to the problem. The size of the tree grows exponentially with the number of general integer and binary decision variables.

Exploring the whole search tree is equivalent to carrying out an exhaustive enumeration procedure.

Solution techniques based on tree search attempt to explore only the promising areas of the search space.

Example. The search space for the IP version of the ATLAS problem can be represented as a search tree.

$$\begin{aligned}
\text{Maximize:}\quad & Z = 5x_1 + 4x_2 & \\
\text{Subject to:}\quad & 6x_1 + 4x_2 \le 24 & (1)\\
& x_1 + 2x_2 \le 6 & (2)\\
& -x_1 + x_2 \le 1 & (3)\\
& x_2 \le 2 & (4)\\
& x_1, x_2 \ge 0 \text{ are integer} & (5)
\end{aligned}$$

Possible values:

$0 \le x_1 \le 4$

$0 \le x_2 \le 2$

Optimal values:

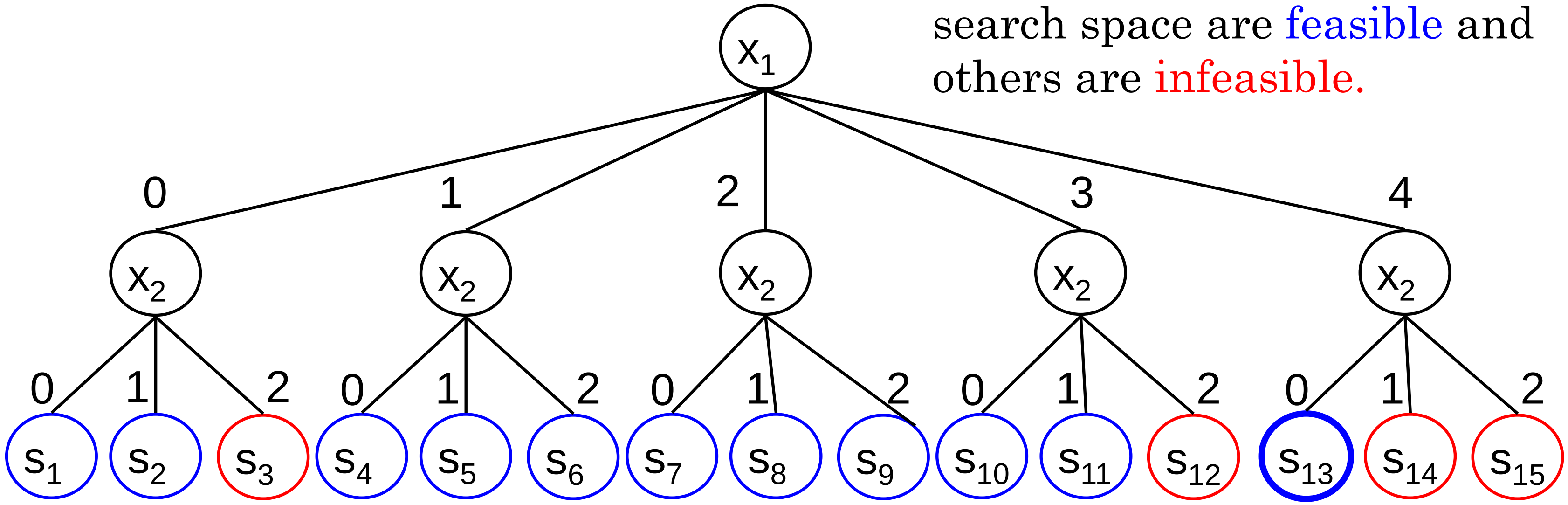
$x_1 = 4$

$x_2 = 0$

Considering all constraints, we have $0 \le x_2 \le 2$

Then, considering all constraints and $0 \le x_2 \le 2$ then we have $0 \le x_1 \le 4$

Some of the solutions in the search space are feasible and others are infeasible.



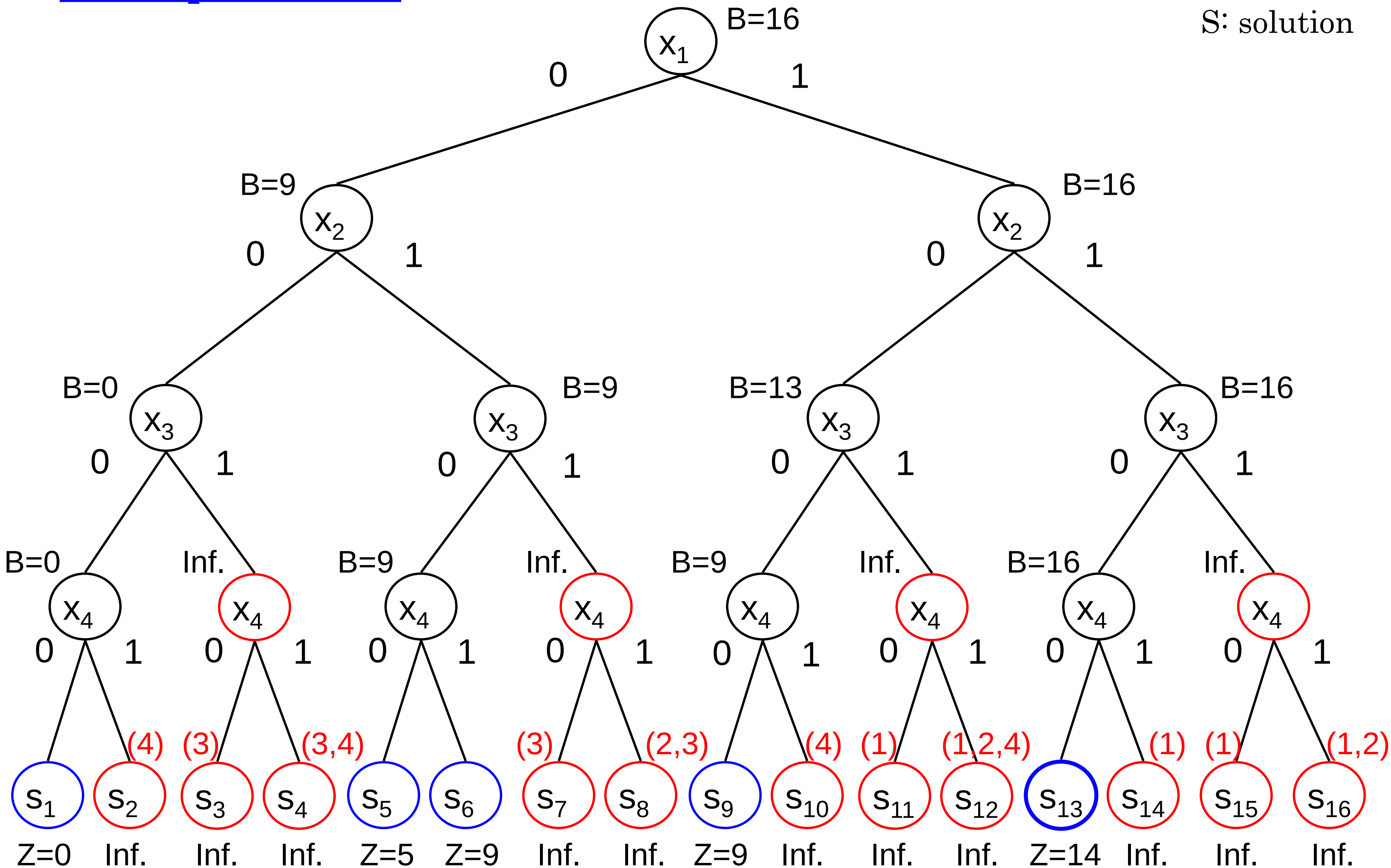
Example. Draw the search tree that represents the search space of the following BIP problem.

$$
\begin{aligned}
\text{Maximize:}\quad & Z = 9x_1 + 5x_2 + 6x_3 + 4x_4 \\
\text{Subject to:}\quad & 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 && (1) \\
& x_3 + x_4 \le 1 && (2) \\
& -x_1 + x_3 \le 0 && (3) \\
& -x_2 + x_4 \le 0 && (4) \\
& x_1, x_2, x_3, x_4 \in \{0,1\} && (5)
\end{aligned}
$$

This search space can be represented with a binary search tree of depth equal to the number of decision variables.

The number of possible solutions (branches in the tree) is given by $2^4 = 16$.

Example (cont.)



Example. Sketch the search tree that represents the search space of the multiple knapsack problem.

$$\text{Maximize}: \quad Z = \sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} X_{ij}$$

$$\text{Subject to}: \quad \sum_{i=1}^{N} S_i X_{ij} \le B_j \quad \text{for } j = 1,2,\ldots M \qquad (1)$$

$$\sum_{j=1}^{M} X_{ij} \le 1 \qquad \text{for } i = 1,2,\ldots N \qquad (2)$$

$$X_{ij} = 1 \text{ if item } i \text{ is packed into knapsack } j, 0 \text{ otherwise} \qquad (3)$$

The depth of the binary search tree is equal to N×M

The Branch and Bound Method

The principle underlying the Branch and Bound (B&B) solution technique is divide and conquer.

If the search tree is to large to explore, the problem is divided into smaller problems which then are solved in a similar way.

The B&B technique uses:

- Branching: the problem is divided by partitioning the entire set of feasible solutions into smaller and smaller subsets.
- Bounding: The bound (best possible solution) of a subset of solutions is computed (via relaxing the IP) to determine the location of incumbent (potentially optimal) solutions.
- Conquering: When a sub-problem is conquered (fathomed), there is no need to further branch that sub-tree.

Considerations in B&B

Branching

- The ordering of the branching variables has an effect on the search speed.

Bounding

- A relaxation of the sub-problem is used to compute the bounds. One relaxation commonly used is the LP relaxation.

Conquering (Fathoming)

- Conquering corresponds to eliminating parts of the search tree in order to search more efficiently. There are 3 ways to conquer a sub-problem (node):
  1. Unique optimal integer solution is found by solving the relaxed sub-problem.
  2. The computed bound of the sub-problem is no better that the current incumbent solution.
  3. Solving the relaxed problem does not find feasible solutions.

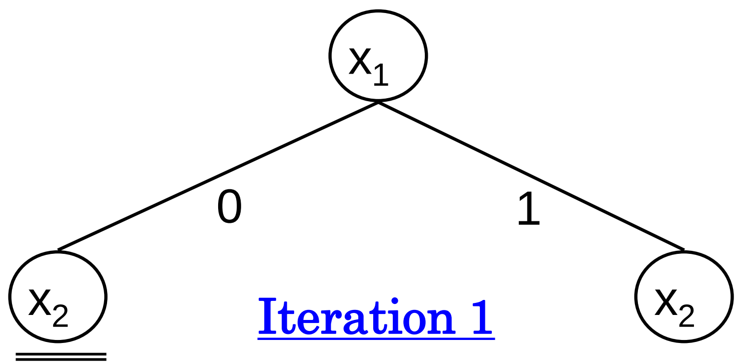
Example. Apply the B&B algorithm to solve the following BIP problem.

$$
\begin{array}{ll}
\text{Maximise:} & Z = 9x_1 + 5x_2 + 6x_3 + 4x_4 \\
\text{Subject to:} & 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \quad (1) \\
& x_3 + x_4 \le 1 \quad (2) \\
& -x_1 + x_3 \le 0 \quad (3) \\
& -x_2 + x_4 \le 0 \quad (4) \\
& x_1, x_2, x_3, x_4 \in \{0,1\}
\end{array}
$$

Initialise: Solution to LP relaxation of the whole problem (note that below, Sub stands for sub-problem)

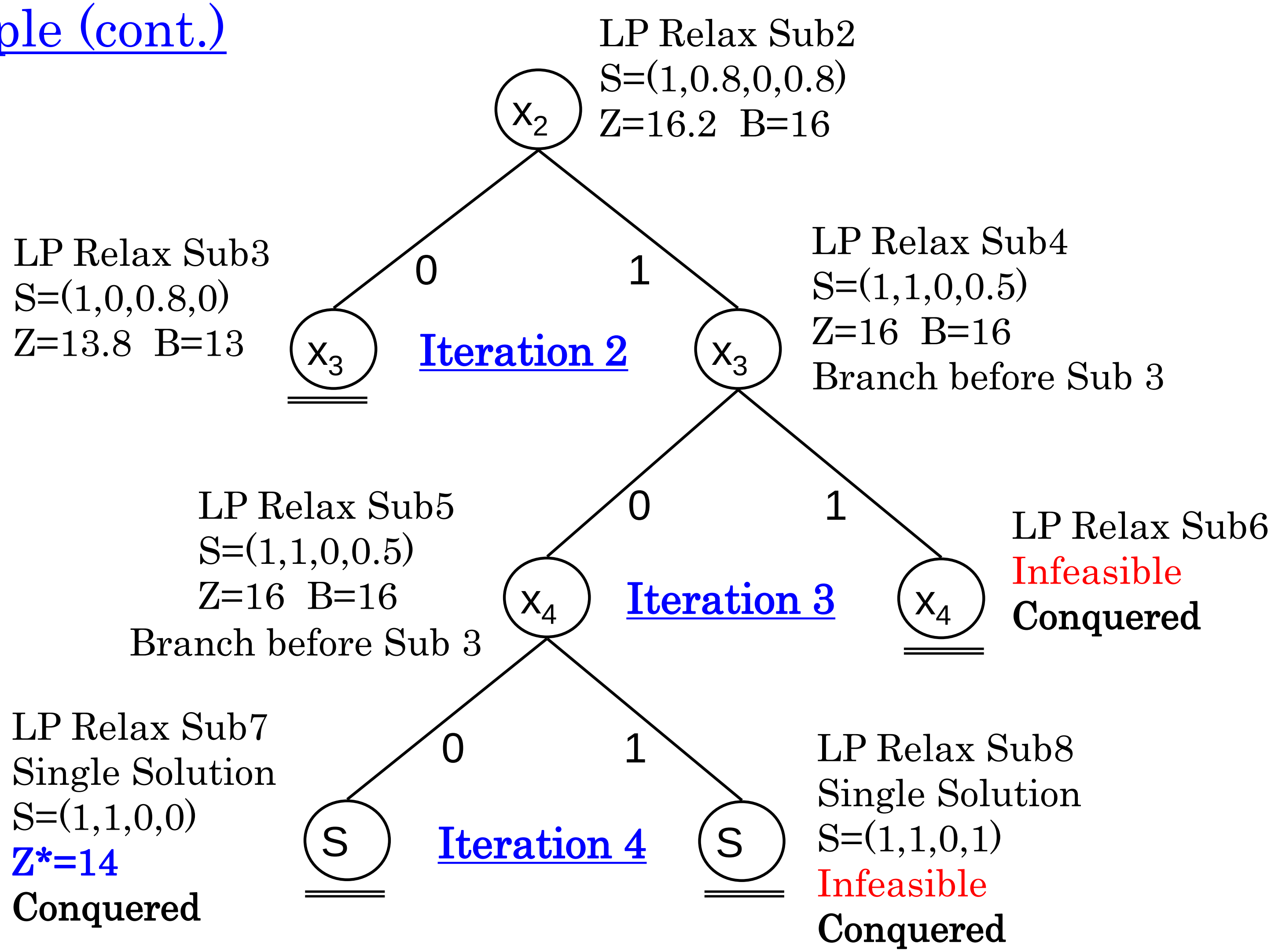
S=(0.8333,1,0,1) Z= 16.5

LP Relax Sub1
S=(0,1,0,1)
Z*=9
Conquered



LP Relax Sub2
S=(1,0.8,0,0.8)
Z=16.2 B=16

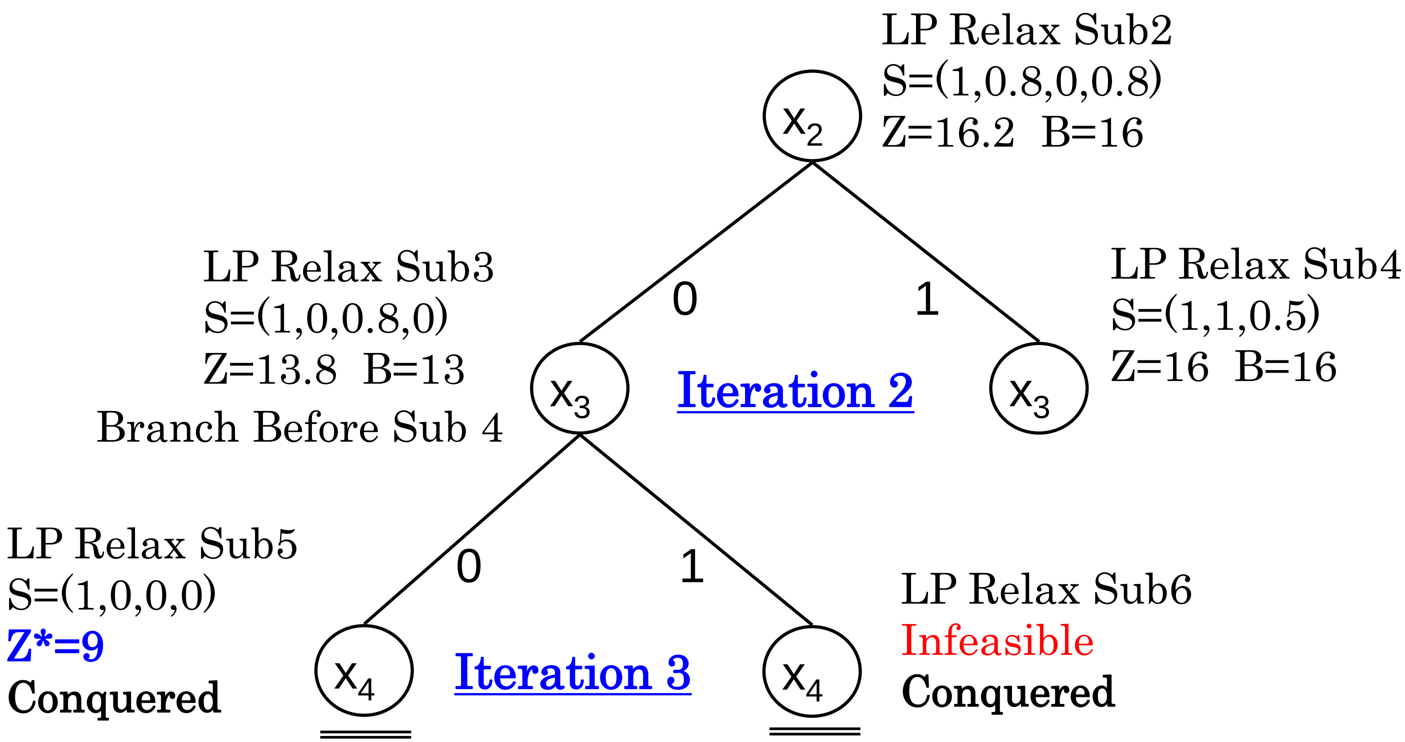
Example (cont.)



Conquer-Test to Subproblem 3

Optimality-Test Passed

Example (cont.) If branching was to occur in sub-problem 3 before sub-problem 4:



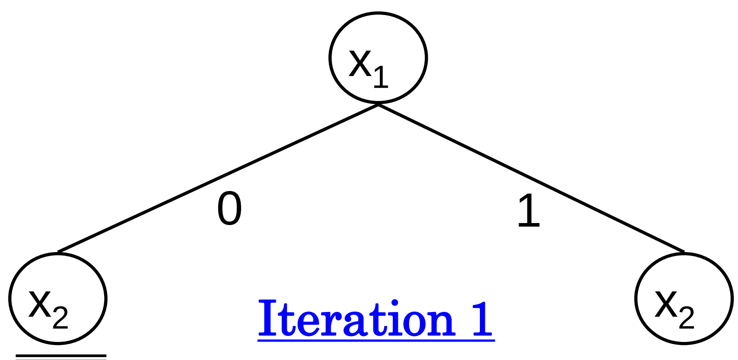
Next branching would be in sub-problem 4 giving as result the same optimal solution

Example. Apply the B&B algorithm to solve the following BIP problem.

$$
\begin{aligned}
\text{Minimise:}\quad & Z = 5x_1 + 6x_2 + 7x_3 + 8x_4 + 9x_5 \\
\text{Subject to:}\quad & 3x_1 - x_2 + x_3 + x_4 - 2x_5 \geq 2 \quad (1) \\
& x_1 + 3x_2 - x_3 - 2x_4 + x_5 \geq 0 \quad (2) \\
& -x_1 - x_2 + 3x_3 + x_4 + x_5 \geq 1 \quad (3) \\
& x_1, x_2, x_3, x_4, x_5 \in \{0,1\}
\end{aligned}
$$

Initialise: Solution to LP relaxation of the whole problem (note that below, Sub stands for sub-problem)
S=(0.5,0,0.5,0,0) Z= 6

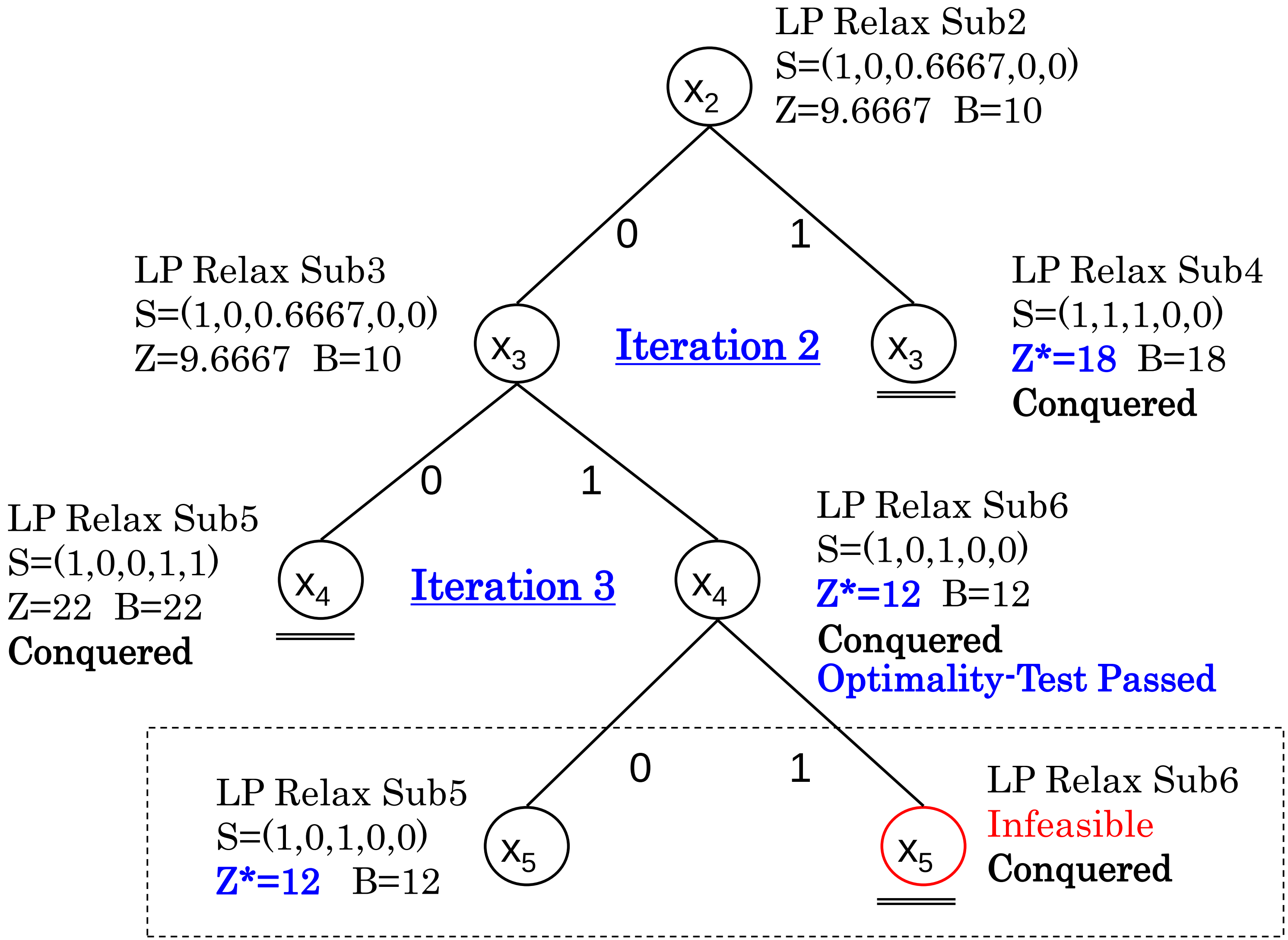
LP Relax Sub1
Infeasible
Conquered



Iteration 1

LP Relax Sub2
S=(1,0,0.6667,0,0)
Z=9.6667 B=10

Example (cont.)



After the optimality test passed, no need to solve (branch) for these last two nodes.

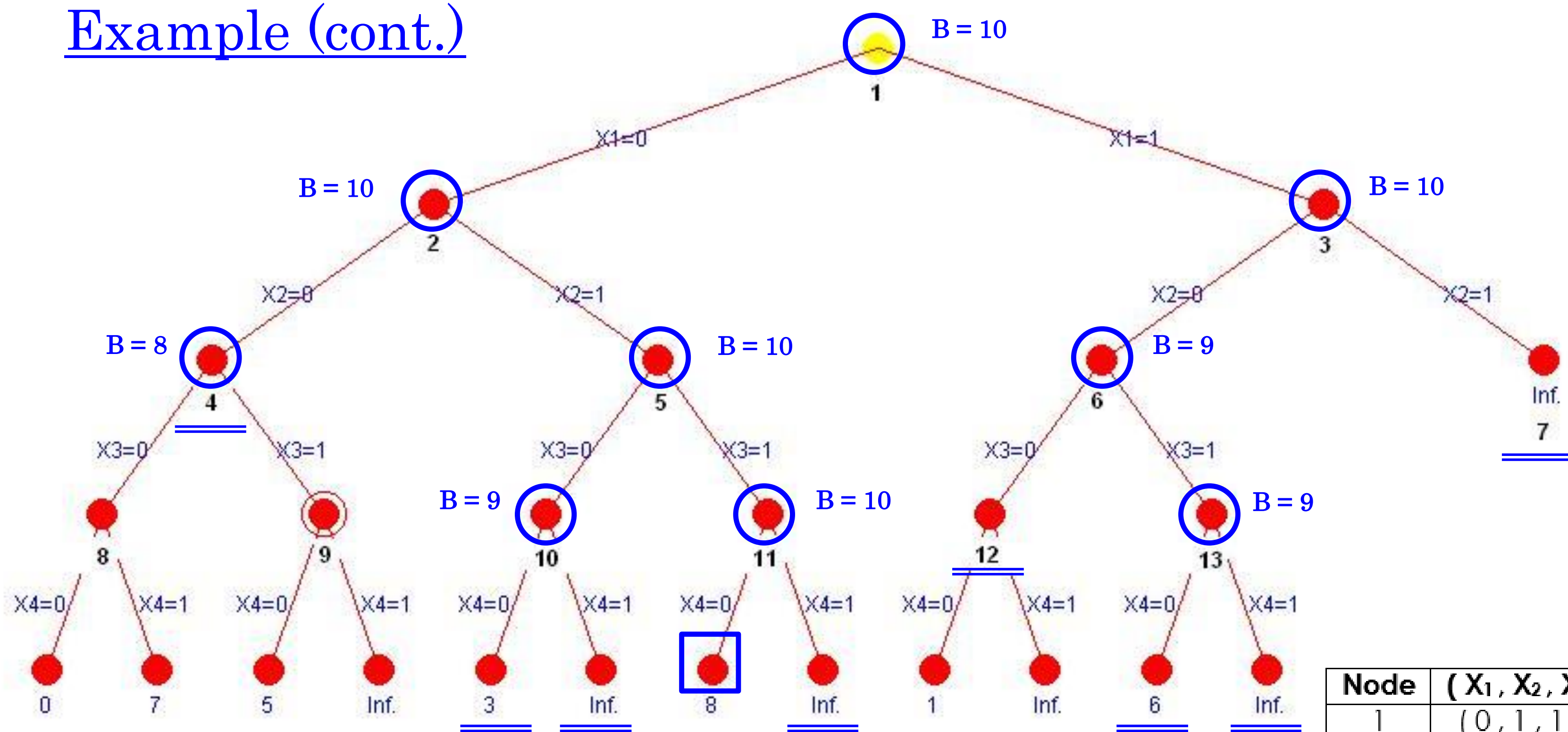
Example. Consider the following BIP model:

$$
\begin{array}{llr}
\text{Maximise: } & Z = x_1 + 3x_2 + 5x_3 + 7x_4 & \\
\text{Subject to: } & x_1 + 2x_2 + 10x_3 + 20x_4 \le 20 & (1) \\
& x_1 + 2x_2 \le 2 & (2) \\
& x_1, x_2, x_3, x_4 \in \{0,1\} & (3)
\end{array}
$$

The binary tree for the above problem is shown next. Each leaf node shows whether the corresponding solution is infeasible ('Inf.' means infeasible) or feasible with a given value for Z.

The table shown below the tree gives the optimal solution to the LP Relaxation corresponding to each node in the tree (the label for each node is indicated in bold below the node).

Example (cont.)



Node	(X_1, X_2, X_3, X_4)	Z
1	(0, 1, 1, 0.4)	10.8
2	(0, 1, 1, 0.4)	10.8
3	(1, 0.5, 1, 0.4)	10.3
4	(0, 0, 1, 0.5)	8.5
5	(0, 1, 1, 0.4)	10.8
6	(1, 0, 1, 0.45)	9.15
7	(1, 1, –, –)	Inf.
8	(0, 0, 0, 1)	7
9	(0, 0, 1, 0.5)	8.5
10	(0, 1, 0, 0.9)	9.3
11	(0, 1, 1, 0.4)	10.8
12	(1, 0, 0, 0.95)	7.65
13	(1, 0, 1, 0.45)	9.15

Apply the Branch and Bound method to solve this BIP problem. Step by step, explain clearly the steps indicating the sequence in which the nodes are explored until the optimal solution is found. Specify and justify the branch ordering criteria used.

B&B Algorithm for BIP
(assuming maximization)

Initialise
Optimal Z = $-\infty$ (really poor value), Bound, Conquer-Test, Optimality-Test

$\Downarrow$

Branch
Select most recently created unconquered sub-problem and larger bound
Divide sub-problem by fixing the branching variable to 0 or 1

$\Downarrow$

Bound
Solve each relaxed sub-problem and obtain a (usually round-down) value for Z

$\Downarrow$

Conquer-Test
Check if sub-problem can be conquered in one of the 3 ways
Discard conquered sub-problems

$\Downarrow$

Optimality-Test
If no more unconquered sub-problems remain, then Stop. If stopped and incumbent solution exists then is optimal. If unconquered sub-problems remain, then go to Branch step

Example. Consider the following BIP model:

$$
\begin{array}{lll}
\text{Minimize:} & Z = y & \\
\text{Subject to:} & \sum_{i=1}^{20} 2x_i + y = 21 & (1) \\
 & x_i \in \{0,1\} \text{ for } i = 1,\ldots,20 & (2) \\
 & y \in \{0,1\} & (3)
\end{array}
$$

The above problem is quite simple and trivial to solve.

However, the binary tree for this problem is sill quite large and the Branch and Bound algorithm needs to explore a large number of nodes before confirming that the optimal solution has been found.

Tailoring the B&B Technique

The B&B framework is flexible and can be tailored to exploit the specific structure of an optimization problem.

Different rules to select the branching variables, i.e. to divide the search tree, can be used.

Different relaxation techniques to solve the sub-problems, i.e. to compute the bounds, can be used.

The conquering (fathoming) is generally the same:

- no better bound found
- no feasible solution found
- optimal solution found

Other algorithms for solving IP and BIP models are Branch and Cut, Branch and Price, etc.

Notes About Excel Solver's B&B

- Excel solver's Branch & Bound uses breadth first search on 'unconquered' nodes with largest bounds (for maximization).
- The Precision value should be set to an appropriate value for linear and non-linear models while Integer Optimality (%) (zero tolerance is better but slower) is used only in the Branch & Bound method.
- The Excel solver incorporates Simplex, Generalised Reduced Gradient, Branch & Bound and Evolutionary Algorithm.
- Other more powerful solvers: Open Solver and LP_Solve also use some form of B&B.
- VBA can be used to control the Excel solver and manipulate the spreadsheets.
- Spreadsheet formula language is targeted to general computations and not only optimization. Other solvers for optimization include GUROBI, GAMS, CPLEX, LINGO, etc.