CEE216 AXIAL FEM NOTES[1]

I. Matrix Notation Review

- Matrix $A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$
  - General Case: A_{ij}, where *i* describes the row and *j* describes the column
  - Dimensions written as *m x n*, or *rows x columns*
    - If m=1, then A = $[A_{11}, A_{12}\ldots, A_{1n}]$ (row vector)
    - If n=1, then A = $[A_{11}, A_{21}\ldots, A_{m1}]^T$ (column vector)
  - Transpose of a matrix is denotated by A^T (as seen above) and it is obtained by interchanging the row and columns
    - If $A^T = A$ → A is symmetric
- Matrix Product: $\underset{(m\,x\,p)}{C} = \underset{(m\,x\,n)}{A} \cdot \underset{(n\,x\,p)}{B}$
  - The inner dimensions need to match (see the n)
  - The column number of matrix A matches row number of matrix B
- Unit Matrix I: $\delta_{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$, δ_{ij} is the Kronecker delta
- Finite Element Method (FEM) is the most common method of stress analysis
  - It is a matrix-based method to solve structural problems (biomedical, mechanical, civil, aircraft, etc.)
  - Problem Set Up:
    1) Discretize body into elements and define nodes @ each end of the element
       - Typically, each element has the same geometry area (A) and material Young's modulus (E)
    2) Define geometry and loads
    3) Set up governing equations

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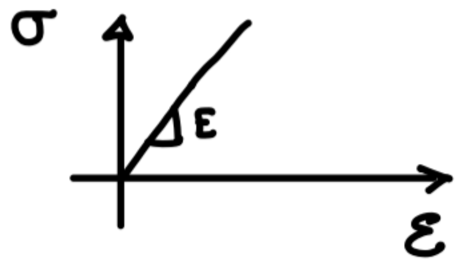
II. Assumptions in Linear Analysis

1. *Small displacements (deformations)*
   This means the structure does not change appreciably after applying loads

2. *Linear elastic material response*



3. *The response of the structure is static*
   No dynamic effects → frequencies of load are small compared with frequency of the entire structure

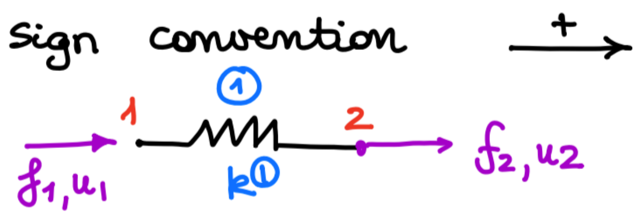
III. Governing Equations and Conditions

1. *Equilibrium*: $\sum F = 0$ (consequence of Newton's 2nd Law)

2. *Compatibility*: requires that the displacements are continuous everywhere in body

3. Constitutive Law (stress-strain law):
   $\sigma = E\varepsilon$ (Hooke's Law)
   $f = k\delta$ (for a spring, f= force, k= spring constant from geometry [N/m], δ= elongation [m])

IV. Element Stiffness for 1D Springs



- Quantities measured in the positive direction:
  - $f_i \equiv$ nodal forces
  - $u_i \equiv$ nodal displacements (also called degrees of freedom (DOF))

- A spring itself is defined with 2 nodes (shown in red), 1 element (blue)
- Constitutive Law (see 3rd governing equation and condition above)

  - Spring Law: $f = k\delta$
  - At node 2: $f_2 = k^{(1)}(u_2 - u_1)$
  - At node 1: $f_1 = -f_2$ due to equilibrium, so $f_1 = -k^{(1)}(u_2 - u_1)$

- Writing these results in matrix form:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k^{(1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

** STIFFNESS MATRIX OF THE ELEMENT e: $k^{(e)} = k^{(1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

  - Symmetric because $k^{(e)} = k^{(e)T}$
  - Its diagonal > 0
  - Size = # of DOF (in this case 2x2)
  - Physical meaning of the values in matrix k_{ij}:
    - The jth column represents the forces applied to the spring to maintain the deformed shape if there's a unit displacement at node j and zero at all other nodes

V. Stiffness Matrix for a Bar – 1D Rod Element

- Quantities measured in the positive direction:
  - $f_i \equiv$ nodal forces
  - $u_i \equiv$ nodal displacements (also called degrees of freedom)

- Internal forces = External forces
  - $f_{int} = kd = f_{ext}$
    $where$ k = stiffness matrix (size is the number of DOF)
- Equations
  - Linear elastic: $\sigma = E\varepsilon$
  - Definition of strain: $\varepsilon = \frac{\Delta u}{L}$
  - Stress = $\frac{Force}{Area}$

(1) $f_2 = A\sigma = AE\varepsilon = AE\frac{u_2 - u_1}{L}$

(2) $f_1 = -f_2 = -AE\frac{u_2 - u_1}{L}$

- These results in matrix form:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$f^{(e)} = \quad k^{(e)} \quad d^{(e)}$$

(e) = element

VI. Example of a System of 1-D Elements

- The theory from I-V can be extrapolated for a system of 1-d elements
- The example below is a statically indeterminate problem, and the problem consists of 3 elements in blue and 4 nodes in red
- Given the geometry (L_i), material properties for each element (E_i, A_i) and the external loads (f_i) the goal is to
    - (1) define the stiffness matrix of each element
    - (2) assemble into the global stiffness matrix (its size is the # of nodes or DOF)
    - (3) solve for the nodal displacements (u_i) using $ku = f$

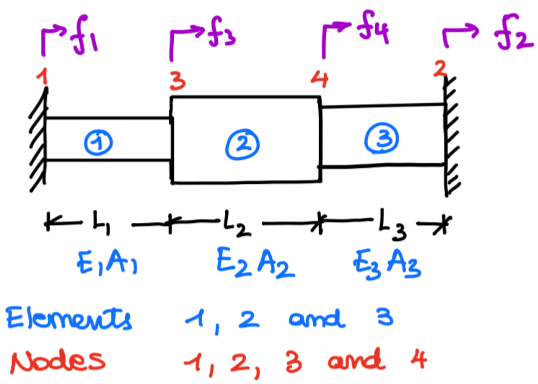


Table 1. Connectivity table relates nodes and elements.

ELEMENT	NODE i	NODE j
(1)	1	3
(2)	3	4
(3)	4	2

- f_i = forces at each node
    - $f_{1,2}$ = unknown reactions at walls
    - $f_{3,4}$ = unknown reactions at walls

1) The stiffness matrix for each element is $k_i = \frac{E_i A_i}{L_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

ELEMENT (1) the values in red are the nodes corresponding to the element

$$k_1 = \frac{E_1 A_1}{L_1} \begin{array}{c} \begin{array}{cc} 1 & \quad 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 3 \end{array} \rightarrow \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = k_1 \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

ELEMENT (2)

$$k_2 = \frac{E_2 A_2}{L_2} \begin{array}{c} \begin{array}{cc} 3 & \quad 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \end{array} \rightarrow \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = k_2 \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

ELEMENT (3)

$$k_3 = \frac{E_3 A_3}{L_3} \begin{array}{cc} \begin{array}{cc} 4 & 2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 4 \\ 2 \end{array} \end{array} \rightarrow \begin{bmatrix} f_4 \\ f_2 \end{bmatrix} = k_3 \begin{bmatrix} u_4 \\ u_2 \end{bmatrix}$$

The global stiffness matrix is a 4x4 since there are 4 DOF (one per node). Each element is part of the global stiffness matrix, and we call assembling the global matrix the placing of each component.

2) The global stiffness matrix can be obtained by placing each component at each node:

$$\boldsymbol{K} = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

This result could have also been obtained by augmenting each of the element matrices:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Adding the augmented matrices leads to the same result as before:

$$\boldsymbol{K} = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Boundary Conditions

Because of the fixed walls at nodes 1 & 2, we know that: $u_1 = u_2 = 0$. If there were a gap Δ between the bar and the wall, that is the displacement of the node.

3) The system that we need to solve is:

$$\boldsymbol{Ku} = \boldsymbol{f}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

We can eliminate the first two rows and columns since u_1 & u_2 are known.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

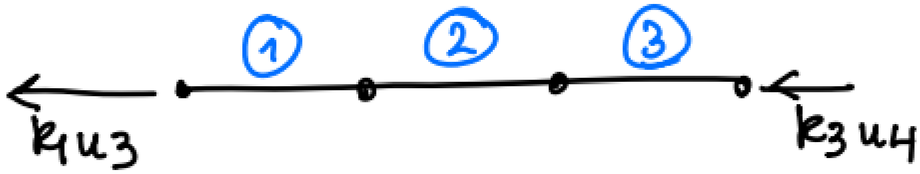
The new system is:

$$\underset{\text{(Known)}}{\begin{bmatrix} f_3 \\ f_4 \end{bmatrix}} = \underset{\text{(Known)}}{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}} \underset{\text{(Unknown)}}{\begin{bmatrix} u_3 \\ u_4 \end{bmatrix}}$$

After solving for displacements u_3 & u_4, we can find the reactions at the walls: f_1 & f_2:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{cases} f_1 = k_1(0) + 0(0) - k_1(u_3) + 0(u_4) = -k_1(u_3) \\ f_2 = -k_2(u_4) \end{cases}$$



4) Finding the stresses:

ELEMENT (1)

$$\sigma_1 = E_1 \varepsilon_1 = E_1 \frac{u_3 - u_1}{L_1} = E_1 \frac{u_3}{L_1}$$

ELEMENT (2)

$$\sigma_2 = E_2 \varepsilon_2 = E_2 \frac{u_4 - u_3}{L_2}$$

ELEMENT (3)

$$\sigma_3 = E_3 \varepsilon_3 = E_3 \frac{u_2 - u_4}{L_3}$$