Summary - Circuits and Systems

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1 Sinusoids

1.1 Sinusoidal Signals

The most general mathematical formula for a cosine signal is

$$x(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi)$$

where A is called the **amplitude** (>0), ω_0 the **radian frequency**,f_0 the **cyclic frequency** and ϕ the **phase**.

Here are some useful formula with the sinusoids:

$$\sin(\theta) = \cos(\theta - \pi/2)$$
$$\cos(\theta + 2\pi k) = \cos(\theta) \text{ for } k \in \mathbb{Z}$$
$$\cos(-\theta) = \cos(\theta) \text{ and } \sin(-\theta) = -\sin(\theta)$$
$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\cos(\theta)\cos(\beta) = \frac{1}{2}(\cos(\theta + \beta) + \cos(\theta - \beta))$$

The **period** of the sinusoid, denoted by T_0, is the length of one cycle of the sinusoid. We have the following relations:

$$\omega_0 T_0 = 2\pi \Rightarrow T_0 = \frac{2\pi}{\omega_0} \Rightarrow T_0 = \frac{1}{f_0}$$

1.2 Phase Shift and Time Shift

The phase shift parameter ϕ determines the time locations of the maxima and minima of a cosine wave. Time shift is essentially a redefinition of the time origin of the signal:

$$x_0(t - t_1) = A\cos(\omega_0(t - t_1)) = A\cos(\omega_0 t + \phi)$$
$$\Rightarrow t_1 = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0} \Rightarrow \phi = -2\pi f_0 t_i = -2\pi\frac{t_1}{T_0}$$

1.3 Complex Exponentials and Phasors

1.3.1 Complex Number

Complex numbers may be represented by the notation $z = (x, y)$, where $x = \Re(z)$ the real part and $y = \Im(z)$ the imaginary part. In the Cartesian form, this complex number is represented as $z = x + yj$ where $j = \sqrt{-1}$. We can also represent a complex number in the polar form:

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan(\frac{x}{y})$$
$$z = re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

And here some basic rules:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$
$$z_1 \cdot z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

The complex exponential signal is defined as

$$z(t) = Ae^{j(\omega_0 t + \theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$$

Then our cosine signals can be written as

$$x(t) = \Re(Ae^{\omega_0 t + \theta}) = A\cos(\omega_0 t + \theta)$$

1.3.2 Complex Amplitude

$$z(t) = Ae^{j(\omega_0 t+\theta)} = Ae^{j\theta}e^{j(\omega_0 t)} = Xe^{j\omega_0 t}$$

The complex number X, called the complex amplitude, is a polar representation created from the amplitude and the phase shift of the complex exponential signal.

1.3.3 Inverse Euler Formulas

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

These equations can be used to express $\cos(\omega_0 t + \phi)$ in terms of a positive and a negative frequency complex exponential:

$$A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}}{2}\right) = \frac{1}{2}Xe^{j\omega_0 t} + \frac{1}{2}X^* e^{-j\omega_0 t}$$

1.3.4 Phasor Addition Rule

The sum of two or more cosine signals each having the same frequency but having different amplitudes and phase shifts, can be expressed as a single equivalent cosine signal (with the same frequency).

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \Re((\sum_{k=1}^{N} X_k)e^{j\omega_0 t})$$

Note also that:

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

2 Spectrum Representation

2.1 The Spectrum of a Sum of Sinusoids

A spectrum is a compact representation of the frequency content of a signal that is composed of sinusoids. In other words, it's simply the collection of amplitude, phase, and frequency information that allows us to express the signal in the form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) = X_0 + \Re(\sum_{k=1}^{N} X_k e^{j2\pi f_k t})$$

The inverse Euler formula gives a way to represent $x(t)$ in the alternative form:

$$x(t) = X_0 + \sum_{k=1}^{n} \left\{ \frac{1}{2}X_k e^{j\omega_k t} + \frac{1}{2}X_k^* e^{-j\omega_k t} \right\}$$

It is common to refer to the spectrum as the **frequency-domain representation** of the signal. Note that the constant component X_0 is often called **DC component**.

2.2 Beat Notes

When we multiply two sinusoids having different frequencies, we can create an interesting audio effect called a **beat note**. But beat notes are also produced by adding two sinusoids with nearly identical frequencies:

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$$

The two frequencies can be expressed as $f_1 = f_c - f_\Delta$ and $f_2 = f_c + f_\Delta$ where we have defined a center frequency $f_c = \frac{1}{2}(f_1 + f_2)$ and a deviation frequency $f_\Delta = \frac{1}{2}(f_1 - f_2)$, which is much smaller that f_c. In this case we can conclude that:

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2\cos(2\pi f_\Delta t)\cos(2\pi f_c t)$$

2.3 Frequency modulation

We can use different mathematical formula to create signals whose frequency is time-varying. We can adopt the following general notation for the class of signals with time-varying angle function:

$$x(t) = A\cos(\psi(t))$$

For example, we can create a signal with quadratic angle function by defining:

$$\psi(t) = 2\pi\mu t^2 + 2\pi f_0 t + \phi$$

Now we can define the instantaneous frequency for these signals as the slope of the angle function

$$\omega(t) = \frac{d}{dt}\psi(t) \Rightarrow f(t) = 2\mu t + f_0$$

The linear FM signals are also called chirp signals, or simply chirps.

2.4 Periodic Waveforms

A periodic signal satisfies the condition that $x(t + T_0) = x(t)$ for all t. T_0 is called the period of $x(t)$ and if it is the smallest such repetition interval, it is called the **fundamental period**.
The harmonically related frequencies are all frequencies that are multiples of a frequency f_0.
In other words, the fundamental frequency is:

$$f_0 = gcd(f_1, ..., f_k)$$

and we say that a signal is periodic if it has a rational fundamental frequency.

2.5 Fourier Series

This is a general theory that shows how any periodic signal can be synthesized with a sum of harmonically related sinusoids.
How do we derive the coefficients for the harmonic sum? The answer is that we use the Fourier series integral :

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

where T_0 is the fundamental period of $x(t)$.

A special case is that the DC component is obtained by

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

This analysis formula goes hand in hand with the synthesis formula for periodic signals, which is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

2.5.1 The Square Wave

It is defined for on cycle by :

$$s(t) = \begin{cases} 1 & \text{for } 0 \leq t < 0.5T_0 \\ 0 & \text{for } 0.5T_0 \leq t \leq T_0 \end{cases}$$

and the Fourier series coefficients

$$a_k \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \pm5, ... \\ 0 & k = \pm2, \pm4, \pm6, ... \\ 0.5 & k = 0 \end{cases}$$

2.5.2 The Triangle Wave

It is defined for on cycle by :

$$s(t) = \begin{cases} 2t/T_0 & \text{for } 0 \leq t < 0.5T_0 \\ 2(T_0 - t)/T_0 & \text{for } 0.5T_0 \leq t \leq T_0 \end{cases}$$

and the Fourier series coefficients

$$a_k \begin{cases} \frac{2}{\pi^2 k^2} & k = \pm1, \pm3, \pm5, ... \\ 0 & k = \pm2, \pm4, \pm6, ... \\ 0.5 & k = 0 \end{cases}$$

3 Sampling and Aliasing

In this section, we will speak about the conversion of signals between the analogue (continuous-time) and digital (discrete-time) domains. The **sampling theorem** states that when the sampling rate is greater than twice the highest frequency contained in the spectrum of the analogue signal, the original signal can be reconstructed. The process of converting from digital back to analogue is called **reconstruction**.

Sinusoidal waveforms of the form $x(t) = A\cos(\omega t + \phi)$ are examples of continuous-time signals. A **discrete-time signal** is represented mathematically by an indexed sequence of numbers. We can sample a continuous-time signal at equally spaced time instants

$$x[n] = x(nT_s) \qquad -\infty < n < \infty$$

The individual values of $x[n]$ are called samples of the continuous-time signal. The fixed time interval between samples, T_s, can also be expressed as a fixed **sampling rate**, f_s, in samples per second:

$$f_s = \frac{1}{T_s} \quad \text{samples/sec}$$

3.1 Sampling Sinusoidal Signals

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \phi) = A\cos(\hat{\omega} n + \phi)$$

where we have defined $\hat{\omega}$, the **discrete-time frequency**, to be

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s} \quad radians$$

3.2 Aliasing

The phenomenon that we are calling **aliasing** is due to the fact that trigonometric functions are periodic with period 2π.

For example, we take the signal $x_1(t) = \cos(200\pi t)$, which we convert in discrete-time signal $x_1[n] = \cos(0.4\pi n)$. Then, we have another signal $x_2(t) = \cos(1200\pi t)$ and the correspondent discrete-time signal $x_2[n] = \cos(2.4\pi n) = \cos(2.4\pi n - 2\pi n) = \cos(0.4\pi n)$.

So, with two different origin signals, we arrive to the same discrete-time signal, and this phenomenon is called aliasing. The **principal alias** is the alias with the smallest $\hat{\omega}$ of all the aliases. In summary, we can write the following general formulas for all aliases of a sinusoid with frequency $\hat{\omega}$:

$$\hat{\omega}_0, \hat{\omega}_0 + 2\pi l, 2\pi l - \hat{\omega}_0 \qquad (l \in \mathbb{Z})$$

3.3 The Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = \frac{1}{T_s}$ that is greater than $2f_{max}$. The minimum sampling rate of $2f_{max}$ is called the **Nyquist rate**.

In a ideal reconstruction, the selection rule is arbitrary, but the ideal C-to-D converter always selects the lowest possible frequency components (the principal aliases). These frequencies are guaranteed to be found in the range $-\pi < \hat{\omega} \leq \pi$, so when converting from $\hat{\omega}$ to analogue frequency, the output frequency always be found between $-\frac{1}{2}f_s$ and $\frac{1}{2}f_s$

The **over-sampling** is action to obey the constraint of the sampling theorem so that we will avoid the problems of aliasing and folding. In this case, the original waveform will be reconstructed exactly.

When $f_s < 2f_0$, the signal is **under-sampled**. In the case where the sampling rate and the frequency of the sinusoid are the same, the samples are always taken at the same place on the waveform, so we get the equivalent of sampling a constant (DC).

A **folding** is a alias of a negative frequency. i.e. $\hat{\omega} = -1.6\pi + 2\pi l \rightarrow 0.4\pi$ is a folding of $\hat{\omega} = 1.6\pi$. It's when the negative frequency become the positive one. When a folding appear, the sign of the phase of the signal will be changed.

3.4 Discrete-to-Continuous Conversion

The purpose of the ideal discrete-to-continuous converter is to interpolate a smooth continuous-time function through the input samples $y[n]$. A general formula that describes a broad class of D-to-C converters is given by the equation

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

where $p(t)$ is the characteristic pulse shape of the converter.

Here is the simplest pulse shape that is a **symmetric square** of the form

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & otherwise \end{cases}$$

Here is a pulse consisting of the **first-order polynomial segment**

$$p(t) = \begin{cases} 1 - |t|/T_s & -T_s < t \leq T_s \\ 0 & otherwise \end{cases}$$

Here is the pulse that gives "ideal D-t-C conversion"

$$p(t) = \frac{\sin(\frac{\pi}{T_s}t)}{\frac{\pi}{T_s}t} \text{ for } -\infty < t < \infty$$

In summary, for al signals composed of sinusoids, we have the following step in the conversion chain:

$$x(t) = \sum_{k=0}^{N} x_k(t) \Rightarrow x[n] = x(nT_s) = \sum_{k=0}^{N} x_k[n] \Rightarrow y(t) = \sum_{n=-\infty}^{\infty} x[n]p(t-nT_s) = \sum_{k=0}^{N} \left(\sum_{n=-\infty}^{\infty} x_k[n]p(t-nT_s) \right) = x(t)$$

3.5 FIR Filters

A **filter** is a system that is designed to remove some component or modify some characteristic of a signal. The **FIR filters** are systems for which each output sample is the sum of a finite number of weighted samples of the input sequence.

3.6 The running-Average Filter

This type of filter compute a moving or running average of two or more consecutive numbers of the sequence. For example, we have each value of the output sequence that is the sum of three consecutive input sequence values divided by three (noncausal filter):

$$y[n] = \frac{1}{3}(x[n] + x[n+1] + x[n+2])$$

This equation is called a **difference equation**. Averaging is commonly used whenever data fluctuate and must be smoothed prior to interpretation.

In the computation, present, past or future values can be used. We call the **sliding window** the range of value used for the computation. A filter that uses only the present and past values of the input is called a **causal filter**. Therefore, a filter that uses future values of the input is called **noncausal**.

3.7 The general FIR Filter

The general difference equation for a causal FIR filter is:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

If the coefficients b_k are not all the same, then we might say that it is a weighted running average filter. The parameter M is the order of the FIR filter. The number of filter coefficients is also called the filter length: $L = M + 1$.

3.8 The Unit Impulse Response

3.8.1 Unit Impulse Sequence

The unit impulse is the simplest sequence because it has ony one nonzero value:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

It is used to do compact representation of signals and any sequence can be represented in this way:

$$x[n] = \sum_{k} x[k]\delta[n-k]$$

3.8.2 Unit Impulse Response Sequence

When the input to the FIR filter is a unit impulse sequence, $x[n] = \delta[n]$, the output is, by definition, the uni impulse response, which we will denote by $h[n]$:

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & n = 0, 1, 2, ..., M \\ 0 & otherwise \end{cases}$$

In other words, $h[n]$ is simply the sequence of difference equation coefficients. The length of the impulse response sequence is finite and this is why the system is called a finite impulse response (FIR).

3.8.3 The Unit-Delay System

One of the important system is the operator that performs a delay or shift by an amount n_0

$$y[n] = x[n-n_0] \Rightarrow h[n] = \delta[n-n_0]$$

The delay system is actually the simplest of FIR filters, it has only one nonzero coefficient.

3.9 Convolution and FIR filters

A general expression for the FIR filter's output can be derived in terms of the impulse response:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

When the relation between the input and the output of the FIR filter is expressed in terms of the input and the impulse response, it is called a finite convolution sum, and we say that the output is obtained by convolving the sequences $x[n]$ and $h[n]$. This operation called convolution is equivalent to polynomial multiplication.

3.10 Linear Time-Invariant (LTI) Systems

Linearity and time invariance are two general properties, lead to simplifications of mathematical analysis and greater insight and understanding of system behavior.

A discrete-time system is said to be **time-invariant** if, when an input is delayed by n_0, the output is delayed by the same amount:

$$x[n-n_0] \longmapsto y[n-n_0]$$

Linear systems have the property that if $x_1[n] \longmapsto y_1[n]$ and $x_2[n] \longmapsto y_2[n]$, then

$$x[n] = \alpha x_1[n] + \beta x_2[n] \longmapsto \alpha y_1[n] + \beta y_2[n]$$

A system that satisfies both properties is called a **linear time-invariant** system, or simply **LTI**. FIR filter satisfy both the linearity and time invariance conditions so it is an example of an LTI system. All LTI systems can be represented by a convolution sum:

$$y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

Here are some interesting aspect of convolution:

- Notation: $y[n] = x[n] \star h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$
- Convolution with an Impulse: $x[n] \star \delta[n-n_0] = x[n-n_0]$
- Commutative Property: $x[n] \star h[n] = h[n] \star x[n]$
- Associative Property: $(x_1[n] \star x_2[n]) \star x_3[n] = x_1[n] \star (x_2[n] \star x_3[n])$

3.10.1 Cascade Filter

The coefficient for a cascade of two LTI systems are:

$$h[n] = h_1[n] \star h_2[n]$$

4 Frequency Response of FIR Filters

4.1 Sinusoidal Response of FIR Systems

$$x(t) = Ae^{j\phi}e^{j\omega t} \Rightarrow x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k A e^{j\phi} e^{j\hat{\omega}(n-k)} = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) Ae^{j\phi}e^{j\hat{\omega}n} = H(e^{j\hat{\omega}})Ae^{j\phi}e^{j\hat{\omega}n}$$

The quantity $H(e^{j\hat{\omega}})$ is called the **frequency-response function** for the system.

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

The principle of superposition makes it very easy to find the output of a linear time-invariant system if the input is a sum of complex exponential signals:

$$\begin{aligned} y[n] &= H(e^{j0})X_0 + \sum_{k=1}^{N} \left(H(e^{j\hat{\omega_k}}) \frac{X_k}{2} e^{j\hat{\omega_k}n} + H(e^{-j\hat{\omega_k}}) \frac{X_k^*}{2} e^{-j\hat{\omega_k}n} \right) \\ &= H(e^{j0})X_0 + \sum_{k=1}^{N} |H(e^{j\hat{\omega_k}})||X_k| \cos(\hat{\omega_k}n + \angle X_k + \angle H(e^{j\hat{\omega_k}})) \end{aligned}$$

When the input is a complex exponential signal, we do not need to deal with the time-domain description of the system but we can solve the problem with a **frequency-domain** approach. In other words, we think about how the spectrum of the signal is modified by the system rather than considering what happens to the individual samples of the input signal.

4.2 Steady-State and Transient Response

Taking a complex exponential signal that starts at $n = 0$ and is nonzero only for $0 \leq n$:

$$x[n] = Xe^{j\hat{\omega}n}u[n] = \begin{cases} Xe^{j\hat{\omega}n} & 0 \leq n \\ 0 & n < 0 \end{cases}$$

The output of an LTI FIR system for this input is

$$y[n] = \sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)} u[n-k]$$

By considering different values of n and the fact that $u[n-k]=0$ for $k>n$, it follows that the sum can be expressed as:

$$y[n]=\begin{cases} 0 & n<0 \\ \left(\sum_{k=0}^{n} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n} & 0\leq n<M \\ \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n} & M\leq n \end{cases}$$

This output can be considered to be defined over three distinct regions:

- $n<0$, the input is zero, therefore the output is zero too
- **Transient** part of the output: In this region, the complex multiplier of $e^{j\hat{\omega}n}$ depends upon n.
- **Steady-state** part: in this region, the output is identical to the output that would be obtained if the input were defined over the doubly infinite interval. If, at some time $n>M$, the input changes frequency or goes to zero, another transient region will occur.

4.3 Properties of the Frequency Response

- **Relation to impulse Response and Difference Equation**:

$$\begin{array}{ccc} \text{Time Domain} & \longleftrightarrow & \text{Frequency Domain} \\ h[n]=\sum_{k=0}^{M} h[k]\delta[n-k] & \longleftrightarrow & H(e^{j\hat{\omega}})=\sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} \end{array}$$

- **Periodicity**: It is always a periodic function with period 2π
- **Conjugate Symmetry**: It has a symmetry in its magnitude and phase that allows us to concentrate on just half of the period when plotting whenever $b_k=b_k^*$:

$$b_k=b_k^*\rightarrow H(e^{-j\hat{\omega}})=H^*(e^{j\hat{\omega}})\rightarrow |H(e^{-j\hat{\omega}})|=|H^*(e^{j\hat{\omega}})|\rightarrow \angle H(e^{-j\hat{\omega}})=-\angle H^*(e^{j\hat{\omega}})$$

4.3.1 Cascaded LTI Systems

The frequency response of a cascade connection of two LTI systems is simply the product of the individual frequency responses:

$$\begin{array}{ccc} \text{Convolution} & \longleftrightarrow & \text{Multiplication} \\ h_1[n]\star h_2[n] & \longleftrightarrow & H_1(e^{j\hat{\omega}})H_2(ej\hat{\omega}) \end{array}$$

4.4 Running-Average Filtering

A simple linear time-invariant system is defined by the equation:

$$y[n]=\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

This system is called an L-point running averager, because the output at time n is computed as the average of x[n] and the L-1 previous samples of the input. The frequency response of the L-point running averager is

$$H(e^{j\hat{\omega}})=\frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}=D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}$$

where $D_L(e^{j\hat{\omega}})$ (often called the **Dirichlet function**) is

$$D_L(e^{j\hat{\omega}})=\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

4.5 Filtering Sampled Continuous-Time Signals

If the frequency of the continuous-time signal satisfies the condition of the sampling theorem, then the ideal D-to-C converter will reconstruct the original frequency without any aliasing:

$$x(t) \Rightarrow x[n] = x(nT_s) = Xe^{j\hat{\omega}n} \Rightarrow y[n] = H(e^{j\hat{\omega}})Xe^{j\hat{\omega}n} \Rightarrow y(t) = H(e^{j\omega T_s})Xe^{j\omega t}$$

where $H(e^{j\omega T_s})$ is called the **analog frequency response**.

4.6 Some common filters

- **Low-pass filter (LPF)**:

$$b_k = \{1, 2, 1\} \Rightarrow h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \Rightarrow H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

- **High-pass filter (HPF)**:

$$b_k = \{1, -1\} \Rightarrow h[n] = \delta[n] - \delta[n-1] \Rightarrow H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$$

- **Band-pass filter**: For example, you can cascade an LPF and a HPF