Project 1 - Scientific Computing for Partial Differential Equations

Group 44
Alexander Cremer | Emil Kinman Maly

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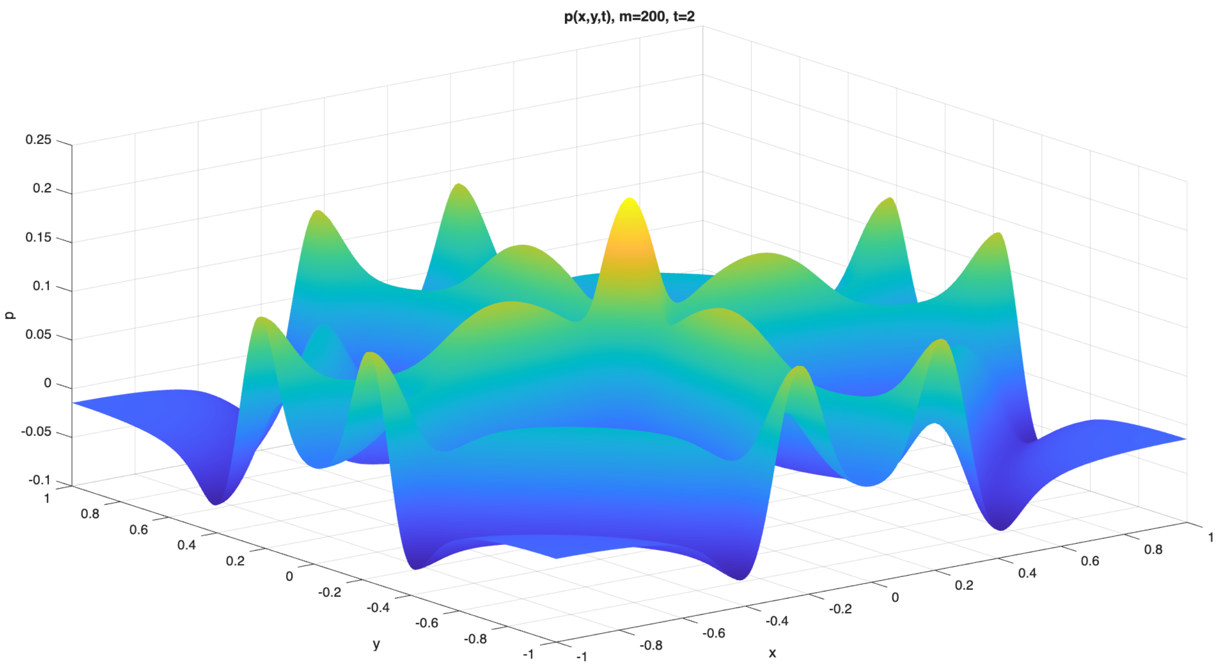

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1 Introduction

In this project we have aimed to solve the acoustic wave equation using a stable high-order accurate finite difference method, referred to as the SBP-Projection method. In the report we will show how we test various types of well-posed boundary conditions and compare a few different finite difference discretizations. Finally using the 4th order Runge-Kutta method to time-integrate the resulting ODE systems. We have used MATLAB to excecute the majority of the assignments.

2 Theory

$$\boldsymbol{C u_t} + \boldsymbol{A u_x} + \boldsymbol{B u_y} + \boldsymbol{D u} = 0 \tag{1}$$

2D acoustic wave equation with absorption, where:

$$\mathbf{u} = \begin{pmatrix} p \\ v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{\rho c^2} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad D = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, p is the pressure and v, w are the particle velocities in the x- and y-directions, respectively. The density is $\rho = \rho(x, y)$, and $c = c(x, y)$ is the speed of sound in the medium. The absorption parameter is $\beta = \beta(x, y)$, which is non-negative (zero gives undamped wave propagation).

For assignment 1 we analyze and solve the 1d acoustic wave equation on a bounded domain $x \in [x_l, x_r]$ given by:

$$\boldsymbol{C u_t} + \boldsymbol{A u_x} + \boldsymbol{D u} = 0, \qquad x \in [x_l, x_r], \tag{2}$$

where:

$$u = \begin{pmatrix} p \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \frac{1}{\rho c^2} & 0 \\ 0 & \rho \end{pmatrix}, \qquad D = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}.$$

We also have the general set of boundary conditions given by,

$$\begin{cases} p + a_l v = 0, & x = x_l, \\ p + a_r v = 0, & x = x_r. \end{cases} \tag{3}$$

where a_l and a_r are parameters.

In this project we use the SBP-projection method to impose our boundary conditions strongly. To do this we use the projection operator $\boldsymbol{P}$ to satisfy our discretized boundary conditions exactly:

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{\bar{H} L^T} (\boldsymbol{L \bar{H} L^T})^{-1} \boldsymbol{L} \tag{4}$$

Where $\bar{\boldsymbol{H}} = \boldsymbol{I_k} \otimes \boldsymbol{HC}$ is the norm of the quadrature matrix $\boldsymbol{H}$ and $\boldsymbol{L}$ is the boundary operator.
In the project we use the SBP central difference approximators to approximate $\frac{\partial u}{\partial x}$ defined as:

$$D_1 = H^{-1}(Q - \frac{1}{2}e_l e_l^T + \frac{1}{2}e_r e_r^T) \tag{5}$$

In assignment 3 the initial boundary–value problem is given by

$$\begin{aligned}
&\boldsymbol{Cu_t} + \boldsymbol{Au_x} + \boldsymbol{Bu_y} + \boldsymbol{Du} = 0, && x,\, y \in [-1, 1],\ t \geq 0,\\
&v = 0, && (x, y) \in \partial\Omega_{E,W},\ t \geq 0,\\
&w = 0, && (x, y) \in \partial\Omega_{S,N},\ t \geq 0,\\
&\boldsymbol{u} = \boldsymbol{g}(x, y), && x,\, y \in [-1, 1],\ t = 0,
\end{aligned} \tag{6}$$

where

$$u = \begin{pmatrix} p \\ v \\ w \end{pmatrix}$$

and with the initial data given by

$$\begin{aligned}
p(x, y, 0) &= e^{-100(x^2+y^2)},\\
v(x, y, 0) &= 0,\\
w(x, y, 0) &= 0.
\end{aligned} \tag{7}$$

3 Assignment 1

The first assignment focuses on analysing and solving the acoustic 1D wave equation on a bounded domain (2). Before beginning task 1 we show that the correct number of boundary conditions at each boundary is one for (2), which is shown by looking at the eigenvalues of the matrix $C^{-1}A$.

3.1 Task 1

In the first task we were asked to derive conditions for the parameters a_l and a_r for well-posedness. We were also supposed to present the resulting energy estimates with both *Dirichlet* and *Characteristic* boundary conditions.

We begin by applying the energy method on (2):

$$(u,\, Cu_t) + (u,\, Au_x) + (u,\, Du) = 0 \quad \Longleftrightarrow \quad (u,\, Cu_t) = -(u,\, Au_x) - (u,\, Du). \tag{8}$$

Introduce the complex conjugate to (8)

$$(u_t,\, C^*u) = -(u_x,\, A^*u) - (u,\, D^*u). \tag{9}$$

IBP on (9) yields:

$$(u_t,\, C^*u) = -\big[\, u^*A^*u \,\big]_{x_l}^{x_r} + (u,\, A^*u_x) - (u,\, D^*u). \tag{10}$$

(8) and (10) together gives us:

$$\frac{d}{dt}\|u\|_C^2 = -\big[\, u^*A^*u \,\big]_{x_l}^{x_r} - \underbrace{(u, (A - A^*)u_x)}_{=\,0} - (u, (D + D^*)u). \tag{11}$$

Since $A = A^*$ and $D = D^*$ (A and B are Hermitian), we get

$$\frac{d}{dt}\|u\|_C^2 = -\big[\, u^*A^*u \,\big]_{x_l}^{x_r} - (u, 2Du). \tag{12}$$

Because $D \geq 0$, the remaining condition for well-posedness is

$$\begin{aligned}
& \big[\, u^*A^*u \,\big]_{x_l}^{x_r} \geq 0 \\
\Leftrightarrow \quad & u^*A^*u\Big|_{x_r} - u^*A^*u\Big|_{x_l} \geq 0 \\
\Leftrightarrow \quad & \begin{pmatrix} p & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}\Bigg|_{x_r} - \begin{pmatrix} p & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}\Bigg|_{x_l} \geq 0 \\
\Leftrightarrow \quad & 2pv\Big|_{x_r} - 2pv\Big|_{x_l} \geq 0.
\end{aligned} \tag{13}$$

We now implement the general set of boundary conditions (3) by substitution of p and derive the desired conditions for a_l and a_r.

For the left boundary:

$$\begin{aligned}
& 2pv\big|_{x_l} = 2(-a_l v)v = -2a_l v^2 \geq 0 \\
\Rightarrow \quad & a_l \geq 0
\end{aligned} \tag{14}$$

And for the right boundary:

$$\begin{aligned}
& 2pv\big|_{x_r} = 2(-a_r v)v = -2a_r v^2 \geq 0 \\
\Rightarrow \quad & a_r \leq 0
\end{aligned} \tag{15}$$

For *Dirichlet* BC, it holds that

$$a_l = a_r = 0 \quad \Rightarrow \quad BT = -\left[\, u^* A^* u \,\right]_{x_l}^{x_r} = 0. \tag{16}$$

Thus, we get the energy rate

$$\begin{aligned} & \frac{d}{dt}\|u\|_C^2 = -(u, 2Du) \leq 0 \\ \Leftrightarrow \quad & \frac{d}{dt}E(t) = -(u, 2Du) \leq 0. \end{aligned} \tag{17}$$

Integrating in time leads to the energy estimate

$$\begin{aligned} & \int_0^t \frac{d}{dt}E(t)dt = \int_0^t -(u, 2Du)dt \leq 0 \\ \Leftrightarrow \quad & E(t) - E(0) = -\int_0^t (u, 2Du)dt \leq 0 \\ \Leftrightarrow \quad & E(t) = E(0) - \int_0^t (u, 2Du)dt \leq E(0) \end{aligned} \tag{18}$$

Or just simply

$$E(t) \leq E(0). \tag{19}$$

To determine the *Characteristic* boundary conditions, we start by diagonalizing (2): Get rid of C in front of u_t

$$u_t + C^{-1}Au_x + C^{-1}Du = 0. \tag{20}$$

Multiply by T^{-1} and inserting $TT^{-1} = I$

$$T^{-1}u_t + T^{-1}C^{-1}ATT^{-1}u_x + T^{-1}C^{-1}Du = 0. \tag{21}$$

We now let $\mathrm{w} = T^{-1}u$

$$\mathrm{w}_t + \underbrace{T^{-1}C^{-1}AT}_{=\Lambda}\mathrm{w}_x + \underbrace{T^{-1}C^{-1}Du}_{=\tilde{F}} = 0. \tag{22}$$

Here, we used that $T^{-1}(C^{-1}A)T = \Lambda$ and introduced the forcing function $\tilde{F} = T^{-1}C^{-1}Du$. We then obtain the system in diagonal form

$$\begin{aligned} & \mathrm{w}_t + \Lambda\mathrm{w}_x + \tilde{F} = 0 \\ \Leftrightarrow \quad & \mathrm{w}_t = -\Lambda\mathrm{w}_x - \tilde{F}. \end{aligned} \tag{23}$$

Since the matrix Λ is diagonal, every equation in the system (23) is a scalar advection equation.

The next step is to use the energy method on (23). Since the forcing function $\tilde{F}$ does not affect well-posedness, it is sufficient to study the homogenius version

$$\mathrm{w}_t = -\Lambda \mathrm{w}_x \tag{24}$$

$$(\mathrm{w},\ \mathrm{w}_t) = -(\mathrm{w},\ \Lambda \mathrm{w}_x). \tag{25}$$

The complex conjugate to (29) is

$$(\mathrm{w}_t,\ \mathrm{w}) = -(\mathrm{w}_x,\ \Lambda^* \mathrm{w}). \tag{26}$$

IBP on (26) gives

$$(\mathrm{w}_t,\ \mathrm{w}) = -\left[\, \mathrm{w}^* \Lambda^* \mathrm{w} \,\right]_{x_l}^{x_r} + (\mathrm{w},\ \Lambda^* \mathrm{w}_x). \tag{27}$$

Adding (29) and (27) together

$$\frac{d}{dt}\|\mathrm{w}\|^2 = -\left[\, \mathrm{w}^* \Lambda^* \mathrm{w} \,\right]_{x_l}^{x_r} - \underbrace{(\mathrm{w}, (\Lambda - \Lambda^*)\mathrm{w}_x)}_{=0}. \tag{28}$$

Since the eigenvalues of C^{-1} and A are real, $\Lambda = T^{-1}(C^{-1}A)T$ also has real eigenvalues and the interior term vanishes. Left is

$$\frac{d}{dt}\|\mathrm{w}\|^2 = -\left[\, \mathrm{w}^* \Lambda^* \mathrm{w} \,\right]_{x_l}^{x_r} = -\left[\sum_{i=1}^{m} \lambda_i \,|w_i|^2\right]_{x_l}^{x_r} \leq 0, \tag{29}$$

where λ_i are the eigenvales of $C^{-1}A$. We obtain

$$\begin{aligned} C = \begin{pmatrix} \frac{1}{\rho c^2} & 0 \\ 0 & \rho \end{pmatrix} \quad &\Rightarrow \quad C^{-1} = \begin{pmatrix} \rho c^2 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \\ \Leftrightarrow \quad C^{-1}A = \begin{pmatrix} \rho c^2 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} &\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \rho c^2 \\ \frac{1}{\rho} & 0 \end{pmatrix}. \end{aligned} \tag{30}$$

, with eigenvalues given by

$$\begin{aligned} det(C^{-1}A - \lambda I) = det \begin{pmatrix} -\lambda & \rho c^2 \\ \frac{1}{\rho} & -\lambda \end{pmatrix} = \lambda^2 - c^2 = 0 \\ \Leftrightarrow \quad \lambda = \pm c. \end{aligned} \tag{31}$$

Equation (29) then becomes

$$\begin{aligned}\frac{d}{dt}\|\mathrm{w}\|^2 &= -\Big[-c|w_1|^2 + c|w_2|^2 \Big]_{x_l}^{x_r} = \\ &= (c|w_1|^2 - c|w_2|^2)\Big|_{x_r} - (c|w_1|^2 - c|w_2|^2)\Big|_{x_l} = \\ &= (c|w_1(x_r)|^2 - c|w_2(x_r)|^2) - (c|w_1(x_l)|^2 - c|w_2(x_l)|^2) \leq 0.\end{aligned} \tag{32}$$

Note here that the sign of the eigenvalue determines the propagation direction,

$\lambda = -c < 0$ corresponds to a left-going wave,

$\lambda = c > 0$ corresponds to a right-going wave.

Since the characteristic variables w are the components of w, defined by

$$\mathrm{w} = T^{-1}u \tag{33}$$

and the general set of BC (3) are given in physical variables

$$u = \begin{pmatrix} p \\ v \end{pmatrix}, \tag{34}$$

we must express the boundary terms in the energy rate back in terms of u. Therefore we substitute $\mathrm{w} = T^{-1}u$ into the BT.

Using Matlab, we obtain

$$\begin{aligned} T^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & \rho c \\ 1 & -\rho c \end{pmatrix} \\ \Rightarrow \quad \mathrm{w} &= \frac{1}{2}\begin{pmatrix} 1 & \rho c \\ 1 & -\rho c \end{pmatrix}\begin{pmatrix} p \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p + \rho c v \\ p - \rho c v \end{pmatrix} \\ \Rightarrow \quad & \begin{cases} w_1 = \frac{1}{2}(p + \rho c v) \\ w_2 = \frac{1}{2}(p - \rho c v) \end{cases}. \end{aligned} \tag{35}$$

Implementing the general set of BC (3) by substitution of p gives

$$\begin{cases} w_1 = \frac{1}{2}(-a_l v + \rho c v), & x = x_l, \\ w_2 = \frac{1}{2}(-a_r v - \rho c v), & x = x_r, \end{cases}. \tag{36}$$

Imposing the $Characteristic$ BC amounts to specifying the incoming characteristic variables and leaving the outgoing ones free. Using equation (32) and taking the propagation directions mentioned before into account, we note that

For the right boundary:

The incoming wave corresponds to left-going propagation, i.e $w_2(x_r)$ is to be bound.

And for the left boundary:

The incoming wave corresponds to right-going propagation, i.e $w_1(x_l)$ is to be bound.

We therefore impose

$$\begin{aligned} w_2(x_r) &= 0 \\ w_1(x_l) &= 0. \end{aligned} \tag{37}$$

At the right boundary:

$$\begin{aligned} & w_2(x_r) = \frac{1}{2}(-a_r v - \rho c v) = 0 \\ \Leftrightarrow \quad & a_r = -\rho c. \end{aligned} \tag{38}$$

And at the left boundary:

$$\begin{aligned} & w_1(x_l) = \frac{1}{2}(-a_l v + \rho c v) = 0 \\ \Leftrightarrow \quad & a_l = \rho c. \end{aligned} \tag{39}$$

For *Characteristic* BC, it holds that

$$a_r = -\rho c, \quad a_l = \rho c \quad \Rightarrow \quad w_2(x_r) = w_1(x_l) = 0 \tag{40}$$

Thus, from (32) we get the energy rate

$$\begin{aligned} & \frac{d}{dt}\|w\|^2 = c(|w_1(x_r)|^2 + |w_2(x_l)|^2) \leq 0 \\ \Leftrightarrow \quad & \frac{d}{dt}E(t) = c(|w_1(x_r)|^2 + |w_2(x_l)|^2) \leq 0 \end{aligned} \tag{41}$$

Integrating in time leads to the energy estimate

$$\begin{aligned} & \int_0^t \frac{d}{dt}E(t)dt = c\int_0^t (|w_1(x_r)|^2 + |w_2(x_l)|^2)dt \leq 0 \\ \Leftrightarrow \quad & E(t) - E(0) = c\int_0^t (|w_1(x_r)|^2 + |w_2(x_l)|^2)dt \leq 0 \\ \Leftrightarrow \quad & E(t) = E(0) + c\int_0^t (|w_1(x_r)|^2 + |w_2(x_l)|^2)dt \leq E(0) \end{aligned} \tag{42}$$

Or simply, like for $Dirichlet$ BC,

$$E(t) \leq E(0). \tag{43}$$

3.2 Task 2

In this task we were asked to derive the SBP approximation of (2) where we impose 3 different sets of well posed boundary conditions derived in Task 1.

We start by rewriting (2) which we then discretisize into the semi-discrete equation below using the SBP difference approximator matrix D_x to "derive" discretely:

$$\boldsymbol{u_t = -C^{-1}(D_x + D)u} \tag{44}$$

Where the matrices C and D are given in the theory section and the matrix D_x is defined as (note that if we instead employ central difference operators we replace $D_\pm$ with D_1:

$$D_x = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}, \tag{45}$$

With the 3 discrete well posed boundary conditions:

$$\begin{cases} (e^{(2)} \otimes e_l^T)u = 0 \\ (e^{(2)} \otimes e_r^T)u = 0 \end{cases} \quad \begin{cases} (e^{(1)} \otimes e_l^T)u = 0 \\ (e^{(1)} \otimes e_r^T)u = 0 \end{cases} \quad \begin{cases} ((e^{(1)} + a_l e^{(2)}) \otimes e_l^T)u = 0 \\ ((e^{(1)} + a_r e^{(2)}) \otimes e_r^T)u = 0 \end{cases} \tag{46}$$

Which in turn give us the boundary operators:

$$L_1 = \begin{pmatrix} e^{(2)} \otimes e_l^T \\ e^{(2)} \otimes e_r^T \end{pmatrix}, \quad L_2 = \begin{pmatrix} e^{(1)} \otimes e_l^T \\ e^{(1)} \otimes e_r^T \end{pmatrix}, \quad L_3 = \begin{pmatrix} (e^{(1)} + a_l e^{(2)}) \otimes e_l^T \\ (e^{(1)} + a_r e^{(2)}) \otimes e_r^T \end{pmatrix} \tag{47}$$

Using the projection operator equation (4) from the theory section we now construct the projection matrix we will use to form our SBP-projection approximation:

$$\boldsymbol{u_t = -PC^{-1}(D_x + D)Pu} \tag{48}$$

We use the property $\bar{H}P = P^T\bar{H}$ and the energy method to prove stability using both central difference operators SBP operators D_1 and upwind SBP operators $D_\pm$. We start by dong the analysis and proving stability for the central difference operators by multiplying the equation (48) with $u^T\bar{H}$ (assuming that pressure and velocity can not be complex)

$$u^T\bar{H}u_t = -u^T\bar{H}PC^{-1}\left(A \otimes D_1 + D\right)Pu. \tag{49}$$

Using the transpose identity $(Pu)^T = u^T P^T$, we obtain:

$$u^T\bar{H}u_t = -u^T P^T\bar{H}C^{-1}\left(A \otimes D_1 + D\right)Pu. \tag{50}$$

We now introduce the variable

$$w = Pu.$$

Substituting gives:

$$u^T \bar{H} u_t = -w^T \bar{H} C^{-1} \left(A \otimes D_1 + D \right) w. \tag{51}$$

Using the identity

$$\bar{H} = (I_2 \otimes H)C,$$

we obtain:

$$u^T \bar{H} u_t = -w^T \left(I_2 \otimes H \right) \left(A \otimes D_1 + D \right) w. \tag{52}$$

Inserting the SBP decomposition

$$D_1 = H^{-1}(Q - \tfrac{1}{2} e_l e_l^T + \tfrac{1}{2} e_r e_r^T),$$

yields:

$$u^T \bar{H} u_t = -w^T (I_2 \otimes H)(A \otimes (H^{-1}(Q - \tfrac{1}{2} e_l e_l^T + \tfrac{1}{2} e_r e_r^T)) + D)w. \tag{53}$$

Writing out $\frac{1}{2} e_l e_l^T + \frac{1}{2} e_r e_r^T$ gives us tha matrix B:

$$= \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

Which allows us to rewrite equation (53):

$$u^T \bar{H} u_t = -w^T (I_2 \otimes H) \left(A \otimes \left(H^{-1} (Q + \frac{1}{2} B) \right) + D \right) w. \tag{54}$$

We now use the Kronecker rule $(A \otimes B)(M \otimes N) = (AM) \otimes (BN)$ to get the expression:

$$\begin{aligned} & u^T \bar{H} u_t = -w^T \left(I_2 A \otimes H H^{-1} (Q + \frac{1}{2} B) \right) w - w^T \left(I_2 \otimes H \right) D w \\ \Leftrightarrow \quad & u^T \bar{H} u_t = -w^T \left(A \otimes (Q + \frac{1}{2} B) \right) w - w^T \left(I_2 \otimes H \right) D w. \end{aligned} \tag{55}$$

We add the transpose and sum it all up using the property of the Q-matrix $Q + Q^T = 0$, which gives us:

$$\begin{aligned} & + u_t^T \bar{H} u = -w^T \left(A \otimes H H^{-1} (Q^T + \frac{1}{2} B) \right) w - w^T \left(I_2 \otimes H \right) D w \\ \Rightarrow \quad & \frac{d}{dt} \|u\|_{\bar{H}}^2 = -w^T (A \otimes B) w - w^T (I_2 \otimes H) D w. \end{aligned} \tag{56}$$

For the equation to be stable we need to confirm that $\frac{d}{dt} \|u\|_{\bar{H}}^2 \leq 0$. We see that the term $-w^T (I_2 \otimes H) D w$ satisfies $-w^T (I_2 \otimes H) D w \leq 0$. Thus we look at the term on the left:

$$\begin{aligned} & - w^T (A \otimes B) w \\ \Leftrightarrow \quad & 2 w_1^{(1)} w_1^{(2)} - 2 w_m^{(1)} w_m^{(2)} \end{aligned} \tag{57}$$

Which are the boundary terms. Our boundary operators (47) strongly impose the boundary conditions which means:

$$\frac{d}{dt}\|u\|_{\bar{H}}^2 \leq 0 \tag{58}$$

And thus verifies the expected stability. Let us now analyse the stability using the upwind SBP operators $D_{\pm}$. We start from the same equation as we did when analysing the central difference operator (48). We then proceed exactly like for the central difference SBP operator case by first introducing $Pu = w$, multiplying with $u^T\bar{H}$ adding the conjugate transpose and analysing the energy rate. Note, however, that we use the matrix D_x instead of $A \otimes D_1$ as we did in the previous analysis. Where:

$$\begin{aligned} D_x = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix} = H^{-1}\begin{pmatrix} 0 & Q_+ + \frac{1}{2}B \\ Q_- + \frac{1}{2}B & 0 \end{pmatrix} \\ \Rightarrow \quad HD_x = \begin{pmatrix} 0 & Q_+ + \frac{1}{2}B \\ Q_- + \frac{1}{2}B & 0 \end{pmatrix} \end{aligned} \tag{59}$$

So the energy method gives us:

$$u^T\bar{H}u_t = -u^T P^T \bar{H} C^{-1}\left(D_x + D\right) Pu \tag{60}$$

$$\Leftrightarrow u^T\bar{H}u_t = -w^T\bar{H}C^{-1}\left(D_x + D\right) w.$$

$$\Leftrightarrow [\bar{H} = (I_2 \otimes H)C] \Rightarrow u^T\bar{H}u_t = -w^T\left(I_2 \otimes H\right)\left(D_x + D\right) w$$

We use the fact that:

$$I_2 \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = HI_2$$

To derive the right hand side further:

$$u^T\bar{H}u_t = -w^T(HI_2D_x + HI_2D)w$$

$$\Leftrightarrow u^T\bar{H}u_t = -w^T(HD_x + HD)w$$

$$= -w^T\begin{pmatrix} 0 & Q_+ + \frac{1}{2}B \\ Q_- + \frac{1}{2}B & 0 \end{pmatrix} w - w^T HDw$$

We take the conjugate tranpose and add it all together:

$$+u^T\bar{H}u_t = -w^T\begin{pmatrix} 0 & Q_-^T + \frac{1}{2}B \\ Q_+^T + \frac{1}{2}B & 0 \end{pmatrix} w - w^T HDw$$

$$\Rightarrow \frac{d}{dt}||u||_{\bar{H}}^2 = -w^T\underbrace{\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}}_{Q_+ + Q_-^T = 0} w - 2w^T HDw$$

We once again not that the right hand term $-2w^T HDw$ is stable (meaning ≤ 0) and see that $-w^T \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} w$ gives the same expression as the one we derived for the central difference approximation. Meaning that:

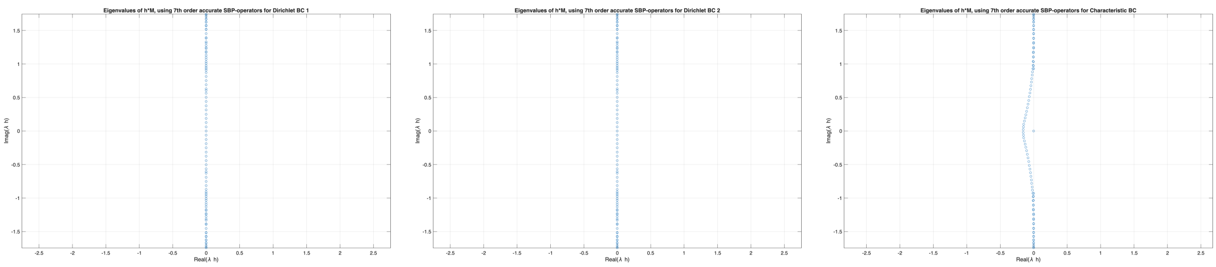
$$\frac{d}{dt}||u||^2_{\bar{H}} = \underbrace{2w_1^{(1)}w_1^{(2)} - 2w_m^{(1)}w_m^{(2)}}_{=0} \underbrace{-2w^T HDw}_{\leq 0} \tag{61}$$

$$\Leftrightarrow \frac{d}{dt}||u||^2_{\bar{H}} \leq 0$$

Which shows the expected result that both the upwind and central difference SBP-operators are stable for the given problem.

3.3 Task 3

In the third task we want to verify numerical stability. By rewriting the SBP-Projection approximations as $u_t = Mu$, we can analyse properties related to the numerical stability using the M-matrix. We start by defining all our parameters and matrices in Matlab, using given Matlab SBP-operators, which we combine to build our M-matrix. As noted in the problem statement we assume that $\rho = c = 1$, and $\beta = 0$. We compute the eigenvalues for the matrix M and note that eigenvalues are purely imaginary or have neagative Real components (which indicates damping as well as energy conservation as expected). We proceed by plotting the eigenvalues of the discretization matrix M multiplied by the grid-step h, when employing the 7th order accurate $D_\pm$ upwind SBP-operators. See figures below for both Dirichlet and characteristic boundary conditions with $m = 51$.



(a) Dirichlet BC 1 (b) Dirichlet BC 2 (c) Characteristic BC

Figure 1: Eigenvalue plots of hM for the different boundary conditions with m=51.

We now look at the CFL number for RK4 when employing 7th order order accurate upwind SBP-Projection approximations, for both Dirichlet and characteristic boundary conditions. The CFL is defined as the largest $\alpha = \frac{k}{h}$ where k is our time-step. For RK4 to be stable it needs to fulfill:

$$k|\lambda| \leq R \Leftrightarrow k \leq \frac{R}{|\lambda|} \tag{62}$$

Where $R \approx 2.78$ for RK4. Since the CFL is defined as the largest fraction alpha we et $k = \frac{R}{|\lambda|}$, which gives us the following expression for α :

$$\alpha = \frac{k}{h} \approx \frac{2.78}{|\lambda|h} \tag{63}$$

Which we compute in Matlab to get a CFL number $\alpha \approx 1.5930$ for both Dirichlet and characteristic boundary conditions with the maximum eigenvalue found to be $\lambda_{max} \approx 43.63$ for all boundary conditions.

3.4 Task 4

In the fourth task we implement the SBP-Projection method in MATLAB and verify the convergence rates of the numerical solutions. The solutions were computed under the same assumptions used before that $\rho = c = 1$, and $\beta = 0$. To verify the convergence of our numerical implementations we compare our numerical solution with the given analytical solution:

$$p(x,t) = \theta^{(2)}(x, L-t) - \theta^{(1)}(x, L-t), \qquad v(x,t) = \theta^{(1)}(x, L-t) + \theta^{(2)}(x, L-t) \tag{64}$$

Where the Gaussian profiles are described by the following given equations in the problem description:

$$\theta^{(1)}(x,t) = \exp\left(-\left(\frac{x-t}{r_*}\right)^2\right), \qquad \theta^{(2)}(x,t) = -\exp\left(-\left(\frac{x+t}{r_*}\right)^2\right).$$

To solve the differential equation system we use RK4 for time-inegration with CFL $= 0.05$ (to guarentee a small time-integration error). We copmute our numerical solutions for different m both using the $6th$ and $7th$ order accurate SBP-Projection approximations and compare them with the analytical solutions to derive both the error as well as the convergence rates 2 depending on m. Below is a table showing the error and convergence rates found.

m	$\|\|err_{(m)}\|\|_h^{(6th)}$	$q^{(6th)}$	$\|\|err_{(m)}\|\|_h^{(7th)}$	$q^{(7th)}$
101	$1.108_{10^{-2}}$	·	$1.269_{10^{-2}}$	·
201	$1.811_{10^{-4}}$	5.9353	$2.472_{10^{-5}}$	8.9934
401	$1.171_{10^{-5}}$	3.9502	$9.884_{10^{-8}}$	7.9667
601	$2.111_{10^{-6}}$	4.2264	$5.649_{10^{-9}}$	7.0584
801	$5.584_{10^{-7}}$	4.6222	$7.499_{10^{-10}}$	7.0192

Table 1: Discrete l^2 error and observed convergence rates for 6th-order central and 7th-order upwind SBP schemes.

As expected the $7th$ order accurate SPB-Projection approximations show a convergence rate approaching 7 and showcased a smaller error then the $6th$ order counterpart. The $6th$ order approximation, however, shows a discrepancy from the expected $6th$ order of accuracy, where the convergence rate unexpectedly is found to be closer to the expected order of accuracy for the smallest investigated m. We believe this to be caused by the step size still being too large and expect the convergence rate to approach 6 show a much clearer $6th$ order trend as h further decreases in size (m increases).

The convergence rate q was calculated using the given equation:

$$q = \frac{log(\frac{||err_{(m)}||_h}{||err_{(n)}||_h})}{log(\frac{n}{m})} \tag{65}$$

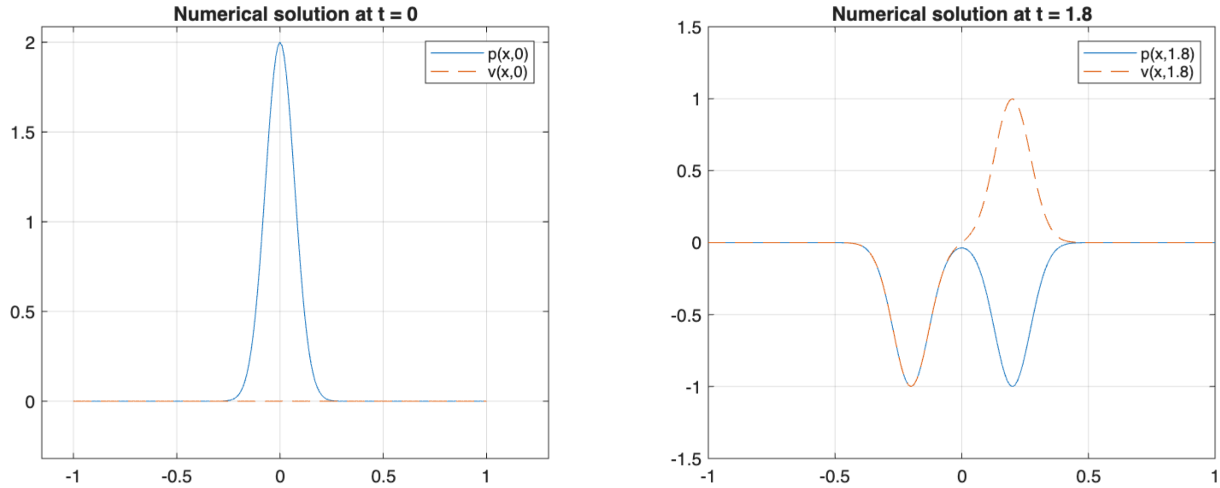


Figure 2: Plotted numerical solution at t = 0 (left) and at t = 1.8 (right).

4 Assignment 3

In this assignment we now study the 2D acoustic wave equation (1) on the square domain Ω, restricted by $x,\, y \in [-1,\, 1]$ with homogeneous boundary conditions corresponding to zero normal velocity on all four boundaries. The problem is discretized using the SBP-Projection method and time-integrated using RK4. Finally, 2D wave propagation in the presence of two materials is simulated and the solutions are plotted.

4.1 Well-posedness

The initial boundary-value problem is given by (6) and the matrices **C**, **A**, **B** and **D** are defined as in (1).

Applying the energy method to (6) yields:

$$\begin{aligned}(u,\, Cu_t) &= -(u,\, Au_x) - (u,\, Bu_y) - (u,\, Du) \underset{\text{IBP}}{=} \\ &\underset{\text{IBP}}{=} -u^*Au\Big|_{x=-1}^{x=1} + (u_x, Au) - u^*Bu\Big|_{y=-1}^{y=1} + (u_y, Bu) - (u, Du)\end{aligned} \tag{66}$$

Introducing and adding the complex conjugate to (66) yields the energy estimate

$$\frac{d}{dt}||u||_C^2 = u^*Au\Big|_{x=-1}^{x=1} + u^*Bu\Big|_{y=-1}^{y=1} - (u, 2Du). \tag{67}$$

Here, we used that $C = C^*$, $A = A^*$, $B = B^*$ and $D = D^*$.

Since C is symmetric positive definite and $D \geq 0$, the interior term is non-positive. The boundary conditions given by (6) prevent energy inflow through the boundary, implying that the boundary term is non–negative. Hence, the problem is well–posed.

4.2 Task 1

In this task we implement the SBP–Projection approximation of (6) in MATLAB, assuming $\rho = c = 1$ and $\beta = 0$. Furthermore, we time-integrate the resulting ODE system using RK4 and present solution plots obtained with seventh-order accurate SBP operators.

The SBP-Projection approximation of (6) can be written as:

$$u_t = -PC^{-1}\left(D_x + D_y + D\right)Pu, \tag{68}$$

where D_x, D_y, C^{-1}, D and u are:

$$\begin{aligned}
D_x &= \begin{bmatrix} 0 & D_+ \otimes I_m & 0 \\ D_- \otimes I_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_y = \begin{bmatrix} 0 & 0 & I_m \otimes D_+ \\ 0 & 0 & 0 \\ I_m \otimes D_- & 0 & 0 \end{bmatrix}, \\
C^{-1} &= \begin{bmatrix} \bar{\rho}c^2 & 0 & 0 \\ 0 & \bar{\rho}^{-1} & 0 \\ 0 & 0 & \bar{\rho}^{-1} \end{bmatrix}, \qquad D = \begin{bmatrix} \bar{\beta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} p \\ v \\ w \end{bmatrix}.
\end{aligned} \tag{69}$$

We begin by implementing the known matrices (69) in MATLAB.

To construct the projection operator

$$P = I - \tilde{H}^{-1}L^T\left(L\tilde{H}^{-1}L^T\right)^{-1}L, \tag{70}$$

we first construct the matrix $\tilde{H}$ starting from the 1D quadrature matrix H associated with the 7th order accurate upwind SBP operators provided. The 1D H is extended to 2D using the Kronecker product $H \otimes H$ yielding the 2D $\tilde{H}$ defined as

$$\tilde{H} = I_3 \otimes (H \otimes H), \tag{71}$$

The boundary operator L is then constructed to impose the discrete boundary conditions $v = 0$ on the east and west boundaries and $w = 0$ on the south and north boundaries. Giving us the following relations as well the boundary operator:

$$L = \begin{pmatrix} e^{(2)} \otimes e_W^T \\ e^{(2)} \otimes e_E^T \\ e^{(3)} \otimes e_S^T \\ e^{(3)} \otimes e_N^T \end{pmatrix}, \tag{72}$$

Where $e^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $e^{(3)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and e_W, e_S, e_N, e_E are defined as:

$$\begin{aligned}
e_W &= e_1 \otimes I_m \\
e_E &= e_m \otimes I_m \\
e_S &= I_m \otimes e_1 \\
e_N &= I_m \otimes e_m
\end{aligned} \tag{73}$$

With the above mentioned assumptions, where $\rho = c = 1$ and $\beta = 0$, we use the given initial conditions $p(x, y, 0) = e^{-100(x^2+y^2)}$, $v(x, y, 0) = w(x, y, 0) = 0$ as initial data and time integrate the ODE system with RK4 and CFL $= 0, 05$. Using the $7th$ order accurate SBP operators at $t = 0, t = 1, t = 2$ (with $m = 200$) gives us the following plots using MATLAB:

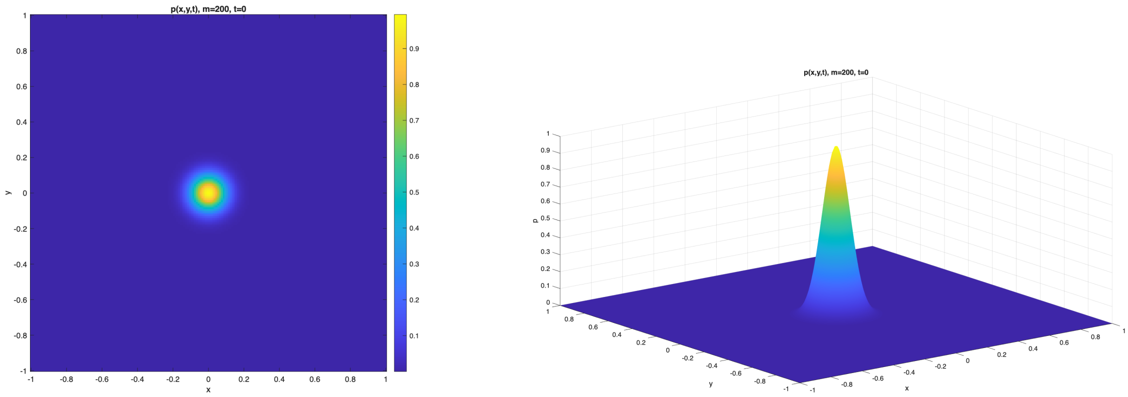


Figure 3: Pressure distribution at $t = 0$.

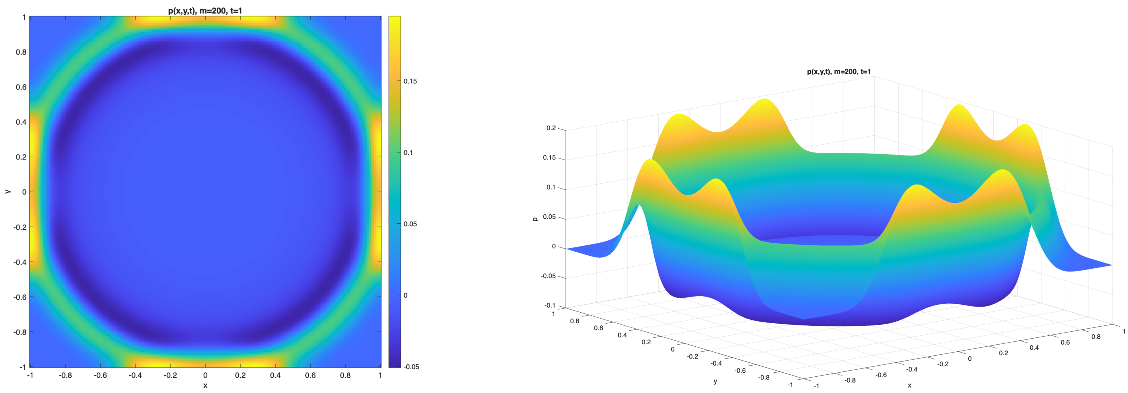


Figure 4: Pressure distribution at $t = 1$.

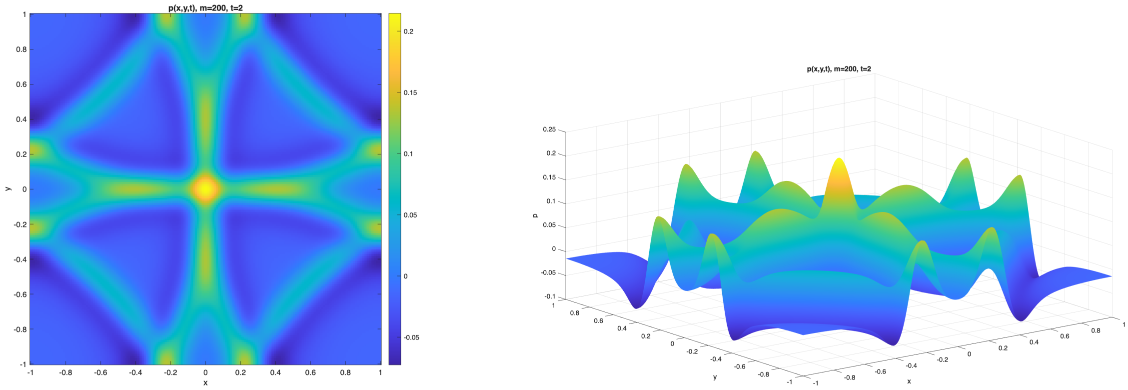


Figure 5: Pressure distribution at $t = 2$.

4.3 Task 2

In this task we simulate wave propagation in heterogeneous media by introducing spatially varying coefficients $\rho(x, y)$ (density), $c(x, y)$ (speed of sound) and $\beta(x, y)$ (absorption parameter).

We begin by fixing the density and speed of sound parameters to investigate the effect of the absorption coefficient. Setting $\beta(x) = 0$, for $x < 0$ and $\beta(x) = 4$ for $x > 0$ we see, as expected, a drastic dampening of the pressure for $x > 0$, visible in Fig 6.

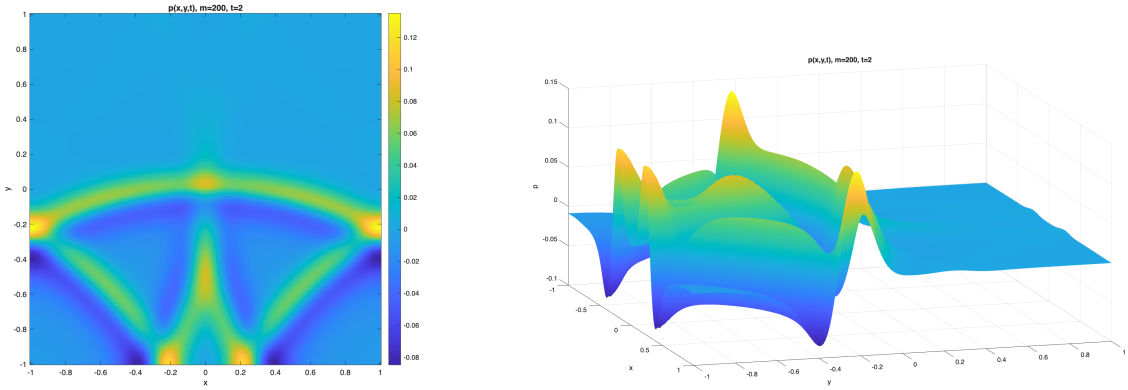


Figure 6: Pressure distribution at $t = 2$ when $\beta(x) = 0$ for $x < 0$ and $= 4$ for $x > 0$.

When instead changing the speed of sound in the heterogenous material we see the pressure reflecting in the cross-section between the two materials, in this case at $x = 0$.

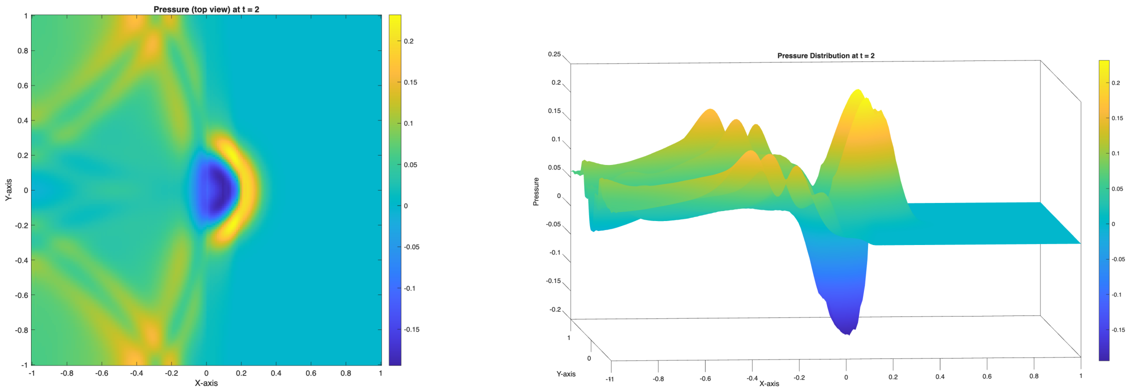


Figure 7: Pressure distribution at $t = 2$ when $\beta(x) = 0$, $\rho(x) = 1$ and $c = 1$ for $x < 0$ and $= 0.1$ for $x > 0$.

Finally looking at the velocity distribution in the material while varying the density $\rho(x)$, we see that the velocity is smaller in the region where the density is higher in accordance with Newtons second law.

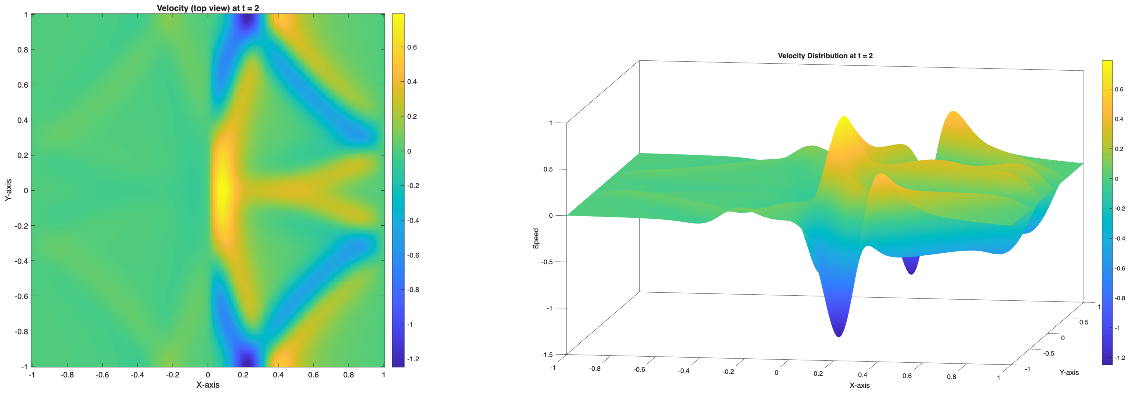


Figure 8: Velocity distribution at $t = 2$ when $\beta(x) = 0$, $c = 1$ and $\rho = 1$ for $x < 0$ and $= 0.1$ for $x > 0$.