Unit 2 Tutorials: Limits and Continuity

INSIDE UNIT 2

Slope of a Tangent Line Visually

by Sophia

WHAT'S COVERED

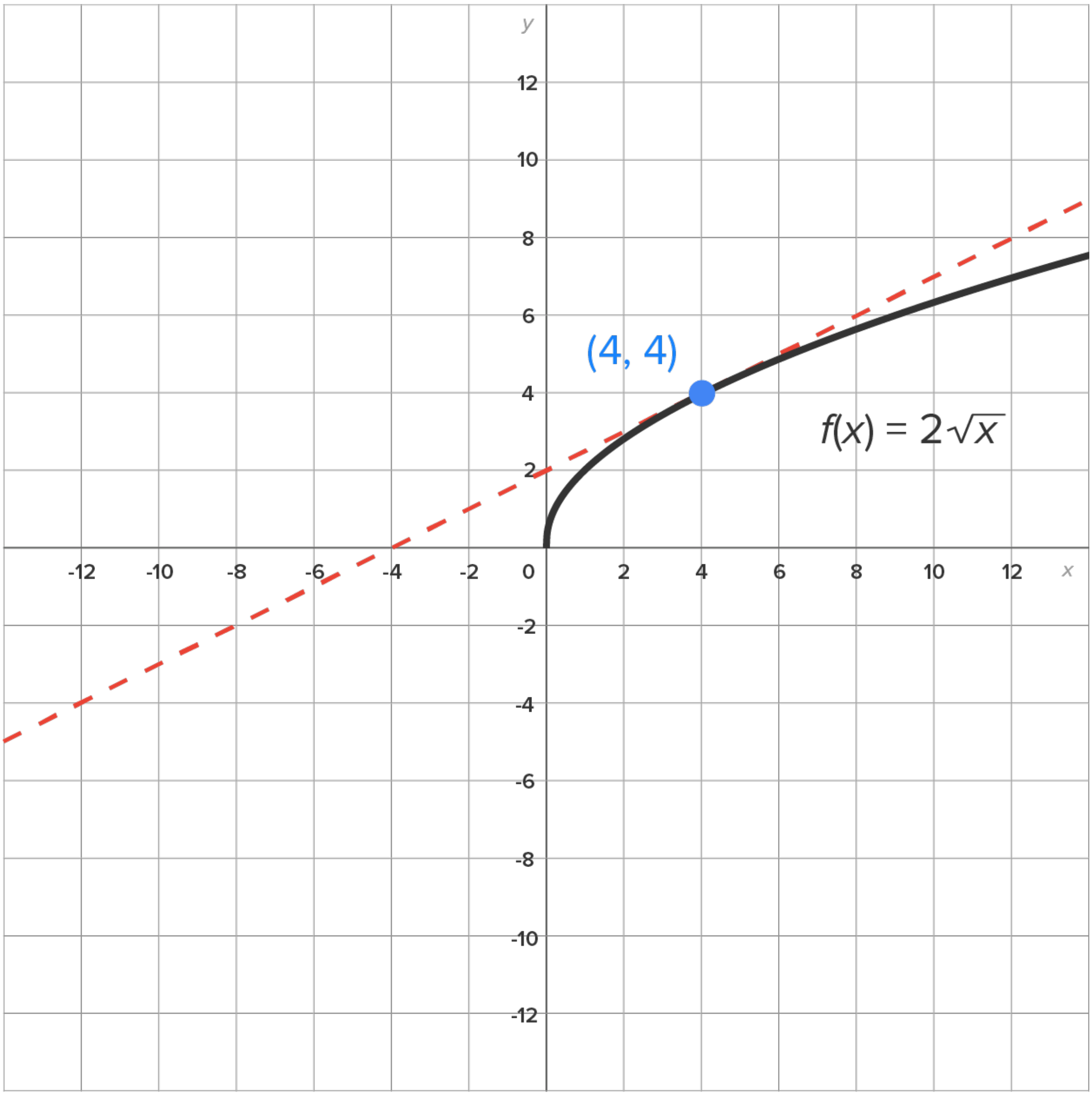
In this lesson, you will learn what a tangent line is and how to estimate its slope graphically. Specifically, this lesson will cover:

1. Estimating the Slope of a Tangent Line Graphically

1. Estimating the Slope of a Tangent Line Graphically

A **tangent line** is a line that touches a graph at one specific point (but does not cross it).

→ EXAMPLE The graph of $f(x)=2\sqrt{x}$ and its tangent line at (4, 4) are shown below. Use this picture to estimate the slope of the tangent line.



In order to estimate the slope of a line, two points are needed. Thus, we need another point on the line besides (4, 4) to estimate the slope of this line. Inspecting closely, it looks like the point (8, 6) is also contained on the line.

Thus, the slope of the tangent line is approximately $m = \frac{6-4}{8-4} = \frac{2}{4} = \frac{1}{2}$. In fact, this is the exact slope of the tangent line.

TERM TO KNOW

Tangent Line

A line that touches (but does not cross) the graph of a function at a specific point.

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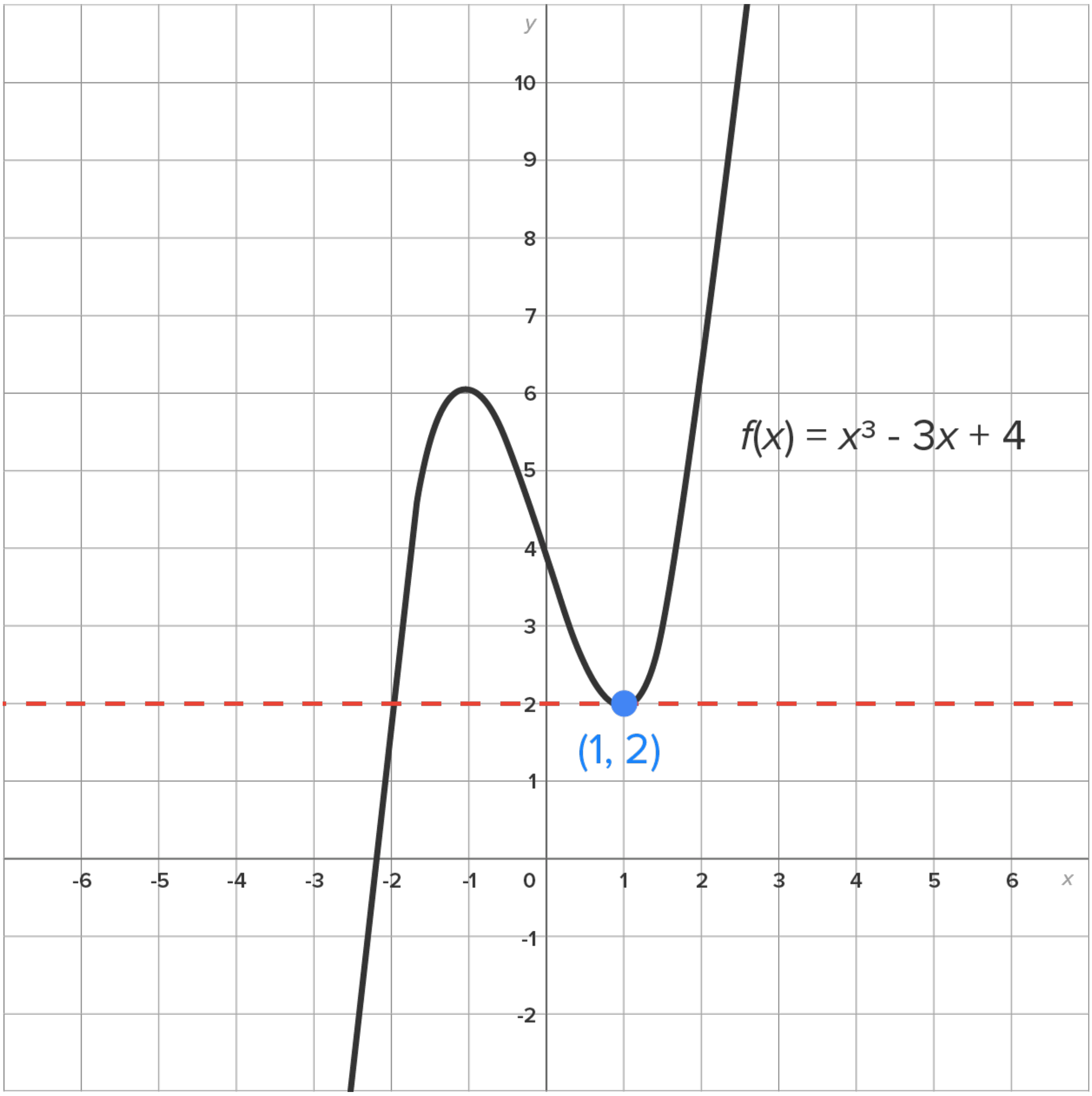
2. Horizontal Tangent Lines

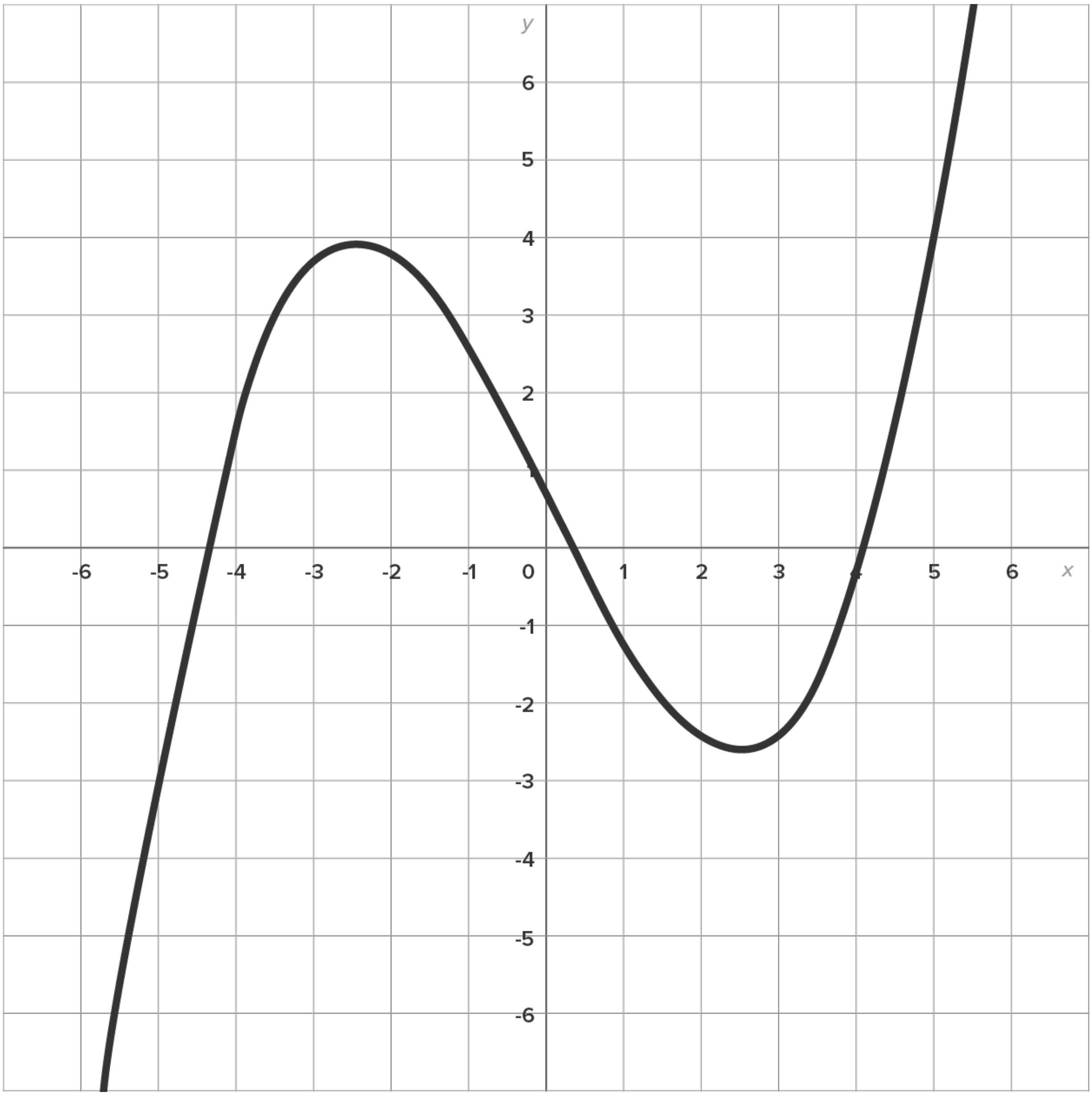
Tangent lines whose slopes are 0, also known as horizontal tangent lines, are very useful in calculus. It is important (and quite simple) to identify the places on a graph where the tangent line is horizontal.

↪ EXAMPLE Estimate the slope of the tangent line to the curve $f(x) = x^3 - 3x + 4$ at the point (1, 2). The graph of $f(x)$ and its tangent line are shown here:



The tangent line appears to be horizontal, which means its slope is zero.

↪ EXAMPLE Estimate all values of x for which the graph of $y = f(x)$ below has a horizontal tangent line.



Looking at the graph, the values of x for which the tangent lines are horizontal are about $x = -2.5$ and $x = 2.5$.

SUMMARY

In this lesson, you learned about tangent lines, which are lines that touch (but do not cross) the graph of a function at a specific point. You learned that given the graph of a function, you can visually **estimate the slope of the tangent line graphically.** This can be accomplished by estimating another point on the tangent line. You also learned that tangent lines whose slopes are 0 are known as **horizontal tangent lines**; these are very useful in calculus.

TERMS TO KNOW

Tangent Line

A line that touches (but does not cross) the graph of a function at a specific point.

Average Rate of Change

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to calculate the average rate of change of a function and how it relates to slope. Specifically, this lesson will cover:

1. Average Rate of Change
2. Secant Lines

1. Average Rate of Change

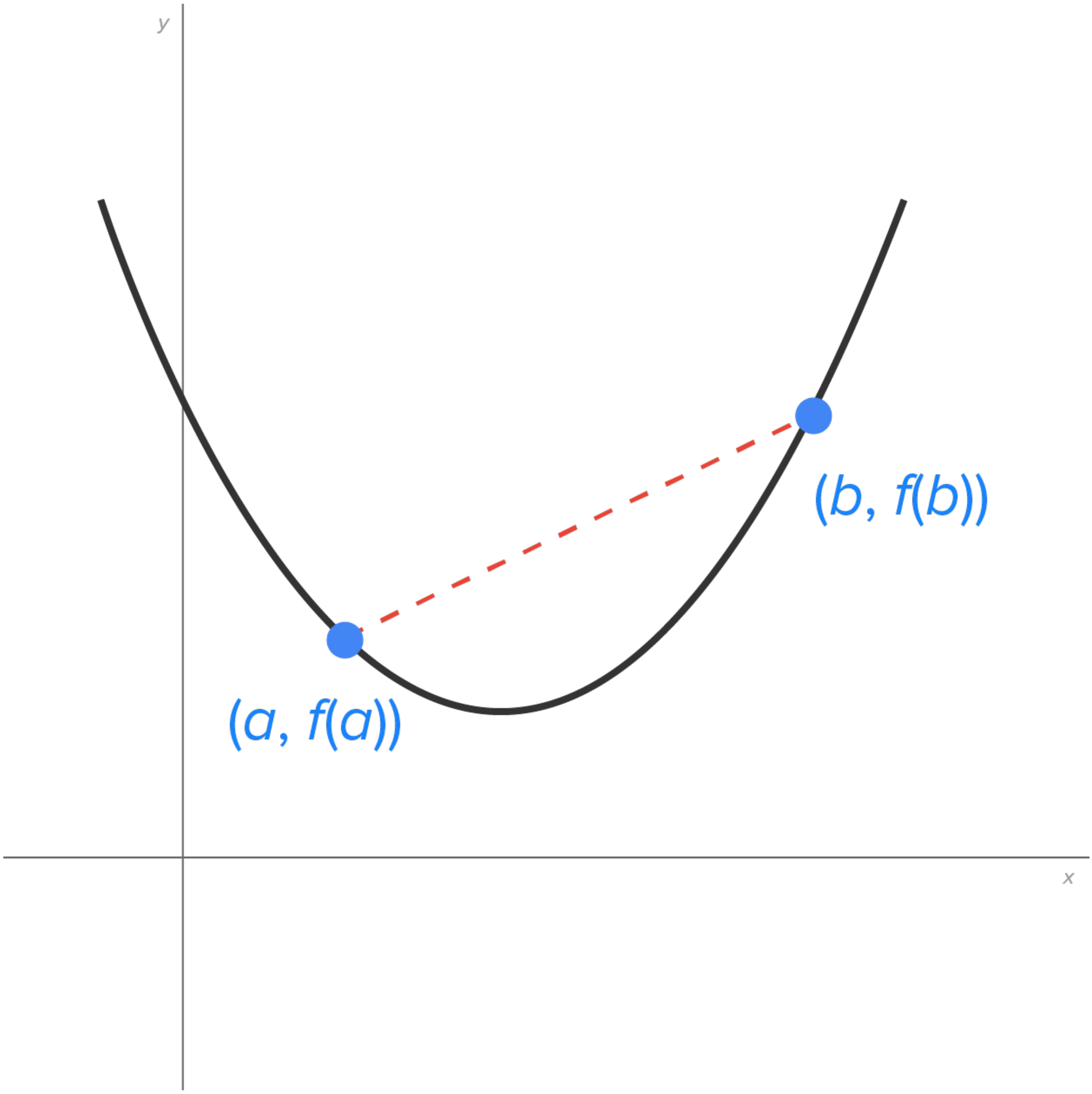
A 320-mile car ride takes 8 hours. We say that on average, the speed is 40 miles per hour. Naturally, there were times when the speed was slower (starting and stopping) and others when it was faster (highway driving). The "40 miles per hour" is an **average rate of change** over the 8 hours.

Given a function $y = f(x)$ on some interval $[a, b]$, we define the average rate of change as follows:

FORMULA

Average Rate of Change on the Interval [*a*, *b*]

$$\frac{\text{change in } f}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$



EXAMPLE Calculate the average rate of change of $f(x)=x^3-2x+4$ on the interval $[1, 3]$.

$f(1)=1^3-2(1)+4=3$, $f(3)=3^3-2(3)+4=25$

Average rate of change $=\dfrac{25-3}{3-1}=\dfrac{22}{2}=11$

This means that on average, the curve rises 11 units for every 1 unit of horizontal increase.

TRY IT

Consider the function $f(x)=4\sqrt{x+5}$.

Calculate the average rate of change of this function on the interval [4, 20]. +

$f(4) = 4\sqrt{9} = 12,\ f(20) = 4\sqrt{25} = 20$

Average rate of change $= \frac{20-12}{20-4} = \frac{8}{16} = \frac{1}{2}$

One important application of average rate of change is **velocity**.

↪ EXAMPLE An object is dropped off a tall building. Its height after t seconds is $h(t) = 1200 - 16t^2$ feet. What is the average rate of change of this object's height in its first 2 seconds of descent?

This translates to calculating the average rate of change from $t = 0$ to $t = 2$.

$h(0) = 1200 - 16(0)^2 = 1200,\ h(2) = 1200 - 16(2)^2 = 1136$

Average rate of change $= \frac{1136-1200}{2-0} = \frac{-64}{2} = -32$

Note that the unit of the average rate of change is feet/second. Thus, we can view the average rate of change as velocity. It is negative since the object is moving downward.

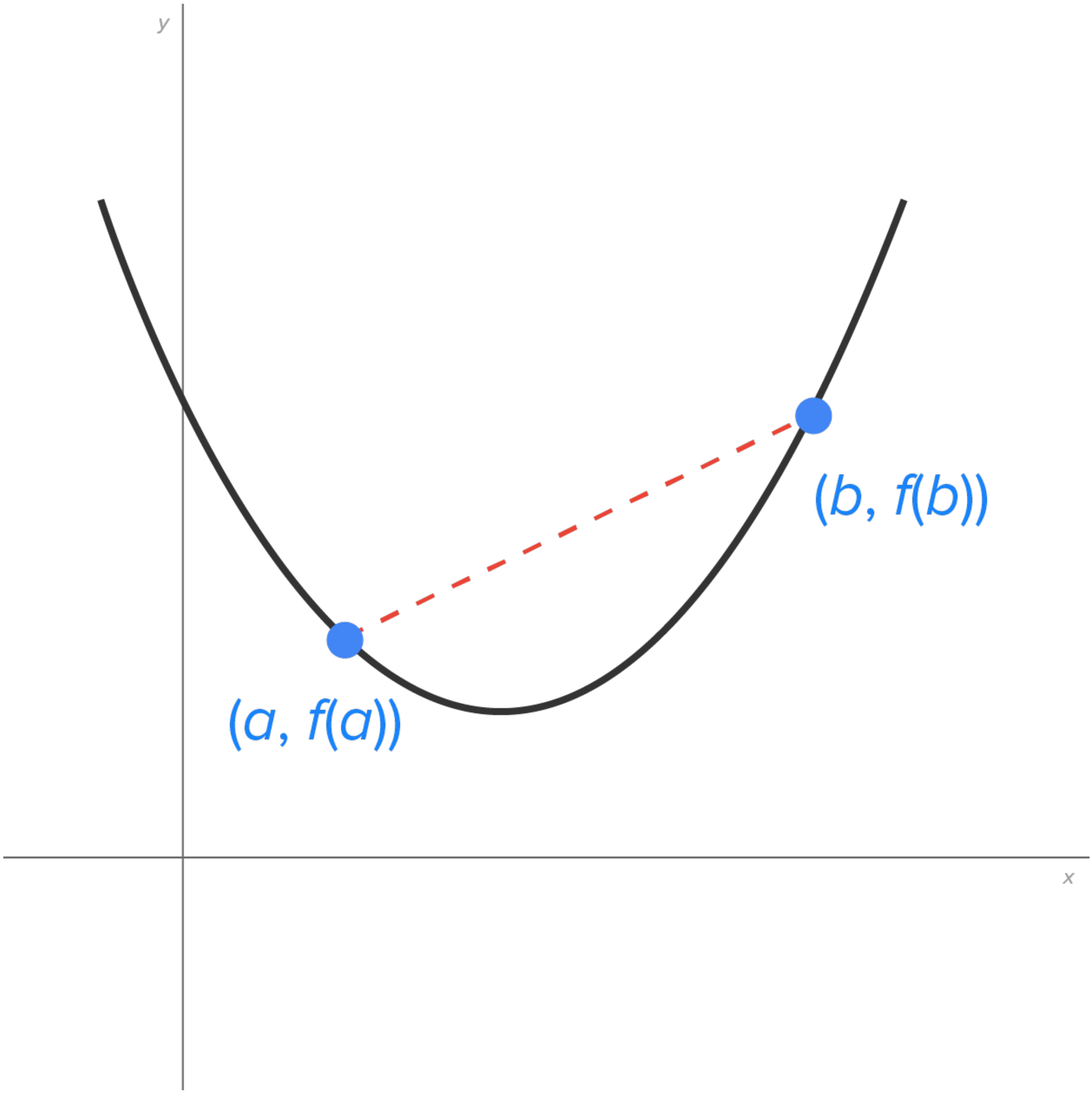
TERMS TO KNOW

Average Rate of Change
The net change divided by the length of the interval.

Velocity
The speed of some object relative to some starting point. Unlike speed, velocity can be negative.

2. Secant Lines

If you look closely at this picture, you might notice that the average rate of change is really the slope of the line connecting $(a, f(a))$ and $(b, f(b))$.



Recall that a tangent line touches the graph at one point. A secant line connects two points of the graph. The slope of the **secant line** is the average rate of change of the function between the two endpoints.

↱ EXAMPLE Calculate the slope of the secant line of $f(x)=10-2x-x^2$ between $x=1$ and $x=4$.

$$f(1)=10-2(1)-(1)^2=7,\ f(4)=10-2(4)-4^2=-14$$

$$Slope = Average\ rate\ of\ change = \frac{-14-7}{4-1} = -\frac{21}{3} = -7$$

TRY IT

Consider the function $f(x)=2^x$.

Find the slope of the secant line of this function between $x=0$ and $x=3$.

+

$f(0) = 2^0 = 1,\ f(3) = 2^3 = 8$

$$Slope = Average\ rate\ of\ change = \frac{8-1}{3-0} = \frac{7}{3}$$

TERM TO KNOW

Secant Line

A line that contains two points of the same function.

SUMMARY

In this lesson, you learned that the **average rate of change** of a function is the net change divided by the length of the interval. This can be visualized as the slope of the line between the two points on the graph, known as the **secant line**. You also learned that one important application of average rate of change is velocity, which, unlike speed, can be negative.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Average Rate of Change

The net change divided by the length of the interval.

Secant Line

A line that contains two points of the same function.

Velocity

The speed of some object relative to some starting point. Unlike speed, velocity can be negative.

FORMULAS TO KNOW

Average Rate of Change on the Interval [a, b]

$$\frac{change\ in\ f}{change\ in\ x} = \frac{f(b)-f(a)}{b-a}$$

Instantaneous Rate of Change

by Sophia

WHAT'S COVERED

In this lesson, you will compute and visualize the instantaneous rate of change of a function. Specifically, this lesson will cover:

1. Instantaneous Rate of Change
2. Computing Instantaneous Rate of Change

1. Instantaneous Rate of Change

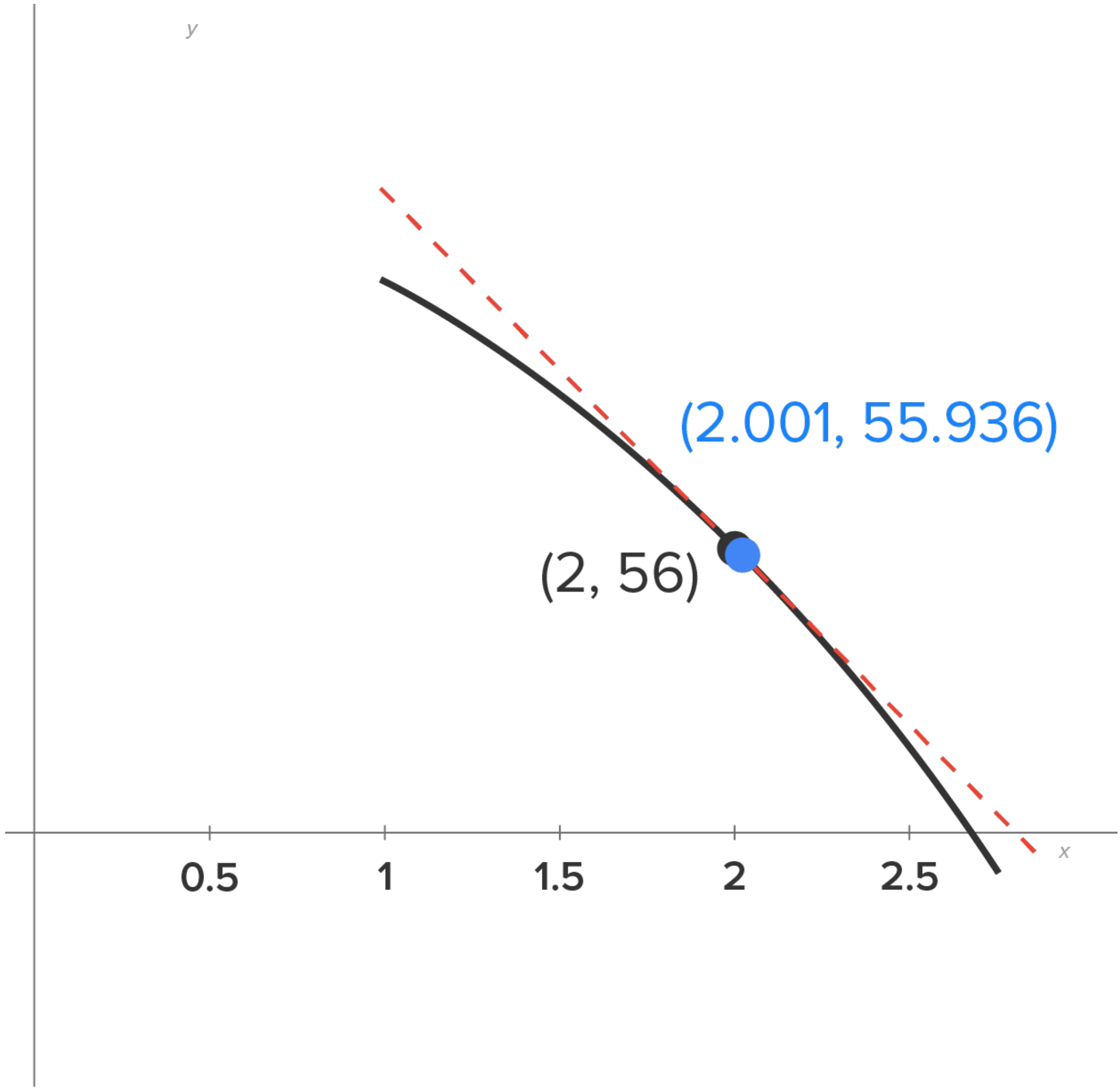
In the last section, we computed average rates of change, which require an interval of time. What if we want to compute the **instantaneous rate of change**, which is the rate of change at one specific point?

Let's say a tennis ball is dropped off the top of a building. Its height after t seconds is described by the function $y(t) = -16t^2 + 120$. Suppose we want to find the instantaneous rate of change at the instant that 2 seconds have passed.

To see if we can find a pattern, let's find average rates of change on small intervals of time starting with $t = 2$.

Interval	Length of Interval	Average Rate of Change
$[2, 2.1]$	0.1 seconds	$\frac{y(2.1)-y(2)}{2.1-2} = \frac{49.44-56}{0.1} = -65.6$ ft/sec
$[2, 2.01]$	0.01 seconds	$\frac{y(2.01)-y(2)}{2.01-2} = \frac{55.3584-56}{0.01} = -64.16$ ft/sec
$[2, 2.001]$	0.001 seconds	$\frac{y(2.001)-y(2)}{2.001-2} = \frac{55.935984-56}{0.001} = -64.016$ ft/sec

Notice that as the length of the interval decreases, the average rate of change seems to be approaching a value. It is safe to estimate the instantaneous rate of change as -64 ft per second (we will see later that this is the actual answer).



Let's examine the curve $y(t) = -16t^2 + 120$ and the secant line for the interval $[2, 2.001]$. (The curve is solid; the secant line is dashed).

Notice that the two points are nearly indistinguishable.

Furthermore, the secant line appears to only pass through one point instead of two (remember, because they are so close together), so the secant line actually looks like a tangent line!

BIG IDEA

The instantaneous rate of change of a function is represented graphically by the slope of the tangent line.

TERM TO KNOW

Instantaneous Rate of Change

The rate of change of a function at a specific point.

2. Computing Instantaneous Rate of Change

The problem we just completed in the previous section can be generalized by finding the average rate of change in $y(t) = -16t^2 + 120$ over the interval $[2, 2+h]$ for small values of *h*.

Since the instantaneous rate of change is the result of letting *h* get smaller and smaller, here is our plan to find the instantaneous rate of change for $y(t)$:

STEP BY STEP

1. Calculate the average rate of change on the interval $[2, 2+h]$.

The expression for the average rate of change is $\frac{y(2+h)-y(2)}{(2+h)-2} = \frac{y(2+h)-y(2)}{h}$.

First, find $y(2+h)$ and $y(2)$.

$$y(2+h) = -16(2+h)^2 + 120$$
$$= -16(4+4h+h^2) + 120$$
$$= -64 - 64h - 16h^2 + 120$$
$$= 56 - 64h - 16h^2$$

$$y(2) = -16(2)^2 + 120$$
$$= -16(4) + 120$$
$$= -64 + 120$$
$$= 56$$

Then, the average rate of change is $\frac{(56-64h-16h^2)-56}{h} = \frac{-64h-16h^2}{h} = -64 - 16h$.

2. The instantaneous rate of change is the average rate of change as the value of *h* gets smaller and smaller. In the simplified expression, substitute $h = 0$. This gives $-64 - 16(0) = -64$ ft/sec.

Conclusion: the instantaneous rate of change is -64 ft/sec.

BIG IDEA

To find the instantaneous rate of change of $f(x)$ at $x = a$, follow these steps.

1. Find and simplify $\frac{f(a+h)-f(a)}{h}$.

   Note, this should look familiar...it looks like a difference quotient!

2. Once simplified, substitute $h = 0$ to get the instantaneous rate of change.

EXAMPLE Compute the instantaneous rate of change of $f(x) = 2x^2 - 3x + 10$ when $x = 1$. We know the average rate of change is $\frac{f(1+h)-f(1)}{h}$.

$$f(1+h) = 2(1+h)^2 - 3(1+h) + 10$$
$$= 2(1+2h+h^2) - 3 - 3h + 10$$
$$= 2 + 4h + 2h^2 - 3 - 3h + 10$$
$$= 2h^2 + h + 9$$

$$\begin{aligned} f(1) &= 2(1)^2 - 3(1) + 10 \\ &= 2 - 3 + 10 \\ &= 9 \end{aligned}$$

Then, $\frac{f(1+h)-f(1)}{h} = \frac{(2h^2+h+9)-9}{h} = \frac{2h^2+h}{h} = 2h+1$. The instantaneous rate of change is $2(0)+1=1$.

Reminder: This is also the slope of the tangent line when $x=1$.

TRY IT

Consider the function $f(x)=x^3$.

Find the instantaneous rate of change of the function when $x=4$. +

First, find $f(4+h)$ and $f(4)$.

$$\begin{aligned} f(4+h) &= (4+h)^3 \\ &= (4+h)(4+h)(4+h) \\ &= (16+8h+h^2)(4+h) \\ &= h^3+12h^2+48h+64 \end{aligned}$$

$$\begin{aligned} f(4) &= 4^3 \\ &= 64 \end{aligned}$$

Then, $\frac{f(4+h)-f(4)}{h} = \frac{(h^3+12h^2+48h+64)-64}{h} = \frac{h^3+12h^2+48h}{h} = h^2+12h+48$.

The instantaneous rate of change is $0^2+12(0)+48=48$.

WATCH

The following video walks you through the process of calculating the instantaneous rate of change of $f(x)=\frac{1}{x+3}$ when $x=2$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome to the video on finding the instantaneous rate of change when I have a rational function at a specific x value. So for this one, I want to compute the instantaneous rate of change of f of x is equal to 1 over x plus 3, and I want to find that when x is equal to 2. Now, 2 is in the domain of the original function, so I'm able to go through the process with this.

Now, here we're going to recall that define the instantaneous rate of change of a function, we find the f of a plus h minus f of a all over h, difference quotient. And then once we have that simplified, we are going to plug in 0 for h, and then that will give us our instantaneous rate of change of that function there.

Now, the a that they're talking about in this formula is your specific x value. So in our example, our a is equal to 2. Now, recall to find this difference quotient-- and we'll go ahead and write it one more time where we have 2's where the As are. So this is f of 2 plus h minus f of 2 all over h that we're looking at for this problem.

And we do this in pieces. The first piece is to find f of 2 plus h. So we'll do that over to the side. Remember to find f of something, we go to the function's rule and take out the x and put in that something. And here, that something is the 2 plus h. So we are going to calculate f of 2 plus h by going to where the Xs are, take that out, and put in the 2 plus h instead. This gives us f of 2 plus h is equal to 1 over-- in my parentheses groupings, I don't have an action on it as, like, an exponent or a multiplication or any function acting on it.

So I really just have 2 plus h plus 3 in the denominator, which is 5 plus h. That expression, 1 over 5 plus h, goes where the f of 2 plus h is. We'll do that in just a minute. Next up, we want to look at the f of 2. So f of 2, we're going to go to our function and take out the Xs and put in 2 instead. So I have 1 over, take out the x and put in 2, and then plus 3. So my f of 2 is going to be 1/5, and that 1/5 is going to go where the green shaded expression is.

So we have-- take out the f of 2 plus h and put in our expression, 1 over 5 plus h, minus-- take out the f of 2 and put in our 1/5, and then that's all over h. Now, this is not simplify enough for me to be able to take out the Hs and put in zeros yet. And what we need to do here is we need to look at this complex fraction and rewrite it without the denominators in the numerator. So when I think about the fractions in the numerator, the length of those fraction bars are a grouping symbol.

So there's a grouping of 5 plus h, and then the other denominator is 5. So my lowest common denominator of the fractions in the numerator is 5 plus h times 5. And I want to multiply the numerator by that 5 plus h times 5. And then think of that over 1, so that you can see how to multiply it through. But that would change the fraction.

In order to make sure that I have the same original expression, I also need to multiply the denominator by that 5 plus h times 5. And I'm not going to put that one over 1. I certainly could, but I guess, actually, let's go ahead and put that over 1. And recall that my h is over 1 to realize that's in the same line of multiplication.

Now, when I multiply this 5 plus h times 5 times the first fraction, the 5 plus Hs will cancel, and I'll just be left with 5. When I multiply that 5 plus h times 5 times the second fraction following that minus, the 5's will cancel. And so they'll remove as a common factor pair and just leave me that parentheses of 5 plus h. And then that's all over-- and in the denominator, we don't want to multiply that out. We actually want to leave that factored as h times 5 plus h times 5.

Next, in the numerator, I have some more simplification to do. This negative 1 precedes a two term expression, so we need to multiply the negative 1 through the parentheses to remove those. So I have 5, and then minus 5 minus h over my h times 5 plus h times 5. Then, combining like terms in the numerator gives me a negative h over my h times 5 plus h times 5. And we're going to scoot this up a little bit, make it a little bit smaller to be able to still see the top.

When I look at the numerator, I notice that that's negative 1 times h and a common factor of h in the denominator. Those common factors of h will remove and leave me a negative 1 in the numerator. So I have negative 1 over the parentheses 5 plus h times 5. So there's where I have it in its simplest form, but now to find the instantaneous rate of change, we want to set h equal to 0.

So in that expression, take out the h and put in 0 instead. I have my negative 1 over 5 plus-- take out the h, put in 0, and then times 5. 5 plus 0 is 5, and 5 times 5 in the denominator is 25 in the denominator. So I end up with an instantaneous rate of change of my f of x equals 1 over x plus 3 when x is 2, comes out to be an instantaneous rate of change of negative 1 over 25. And that's how you find the instantaneous rate of change of a rational function at a specific x value.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that the **instantaneous rate of change** of a function gives the rate of change of the function at a single point (as opposed to average rate of change, which requires two points). The geometric interpretation of instantaneous rate of change is that it is the slope of the line tangent to $y=f(x)$ at that specific point. You also learned how to **compute the instantaneous rate of change**, which enables you to calculate instantaneous velocity at a specific point in time.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Instantaneous Rate of Change
The rate of change of a function at a specific point.

The Graph Method

by Sophia

WHAT'S COVERED

In this lesson, you will evaluate limits by using the graph of a function. Specifically, this lesson will cover:

1. Defining Limit Notation
2. Using Graphs to Evaluate Limits

1. Defining Limit Notation

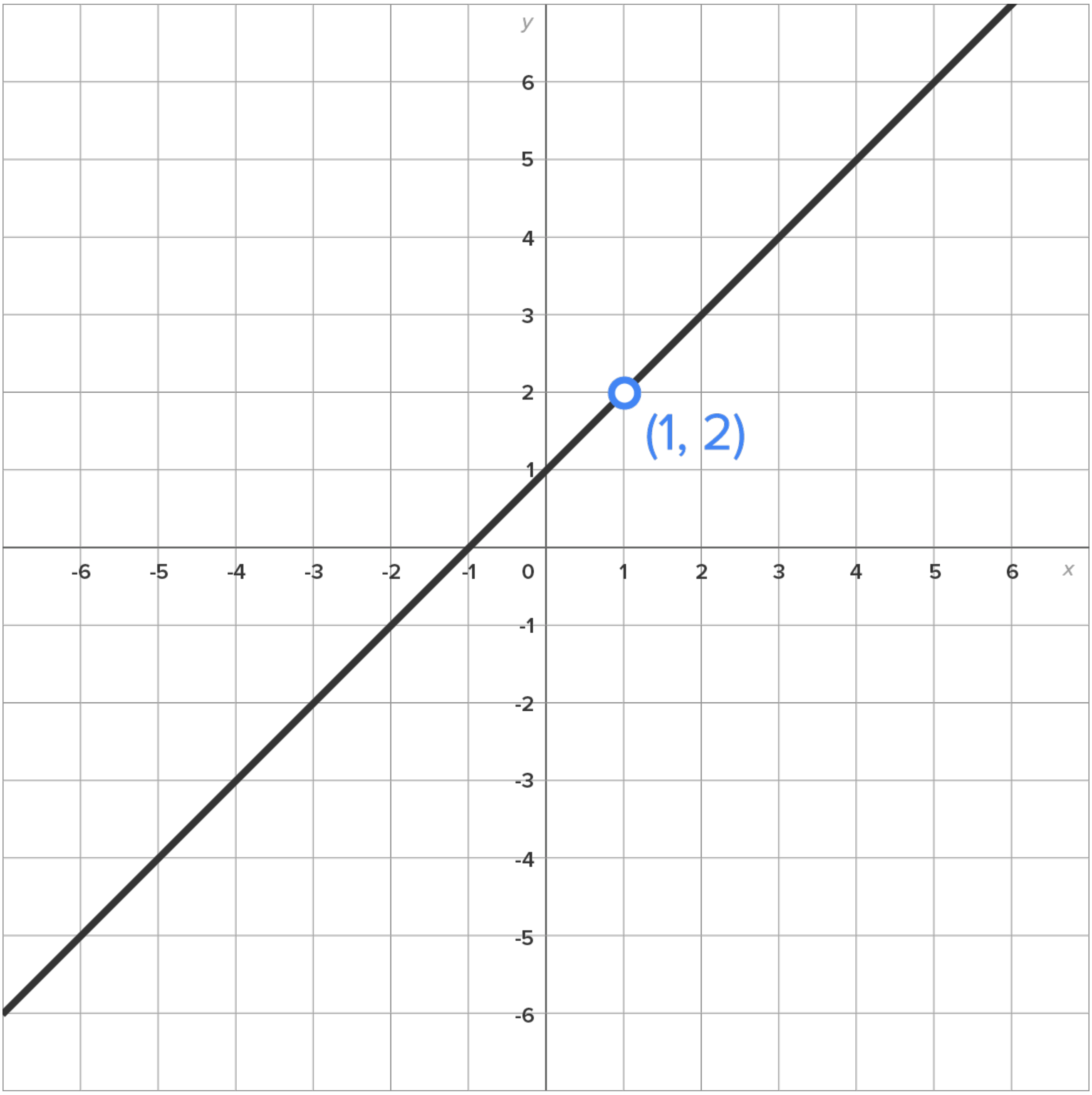
Consider the function $f(x) = \frac{x^2 - 1}{x - 1}$.

Notice that $f(x)$ is undefined when $x = 1$. However, we still may want to analyze the behavior of $f(x)$ around $x = 1$. The mathematical tool used to do this sort of analysis is called a limit.

BIG IDEA

$\lim_{x \to a} f(x) = L$ means "the limit of $f(x)$ as *x* gets closer to *a* is equal to *L*". In other words, as *x* gets closer to *a*, the value of $f(x)$ gets closer to *L*. We call *L* the limit of the function $f(x)$.

To see how this works graphically, shown below is the graph of $f(x) = \frac{x^2 - 1}{x - 1}$.



Notice that there is a hole in the graph at the point (1, 2), indicating that the graph of $f(x)$ is a line, but excludes the point (1, 2).

Since $f(x)$ is undefined when $x = 1$, we analyze the behavior of $f(x)$ by using limits.

That is, we want to evaluate $\lim_{x \to 1} f(x)$ or more specifically, $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

By examining the graph, it appears that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. Thus, we can write $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

TERM TO KNOW

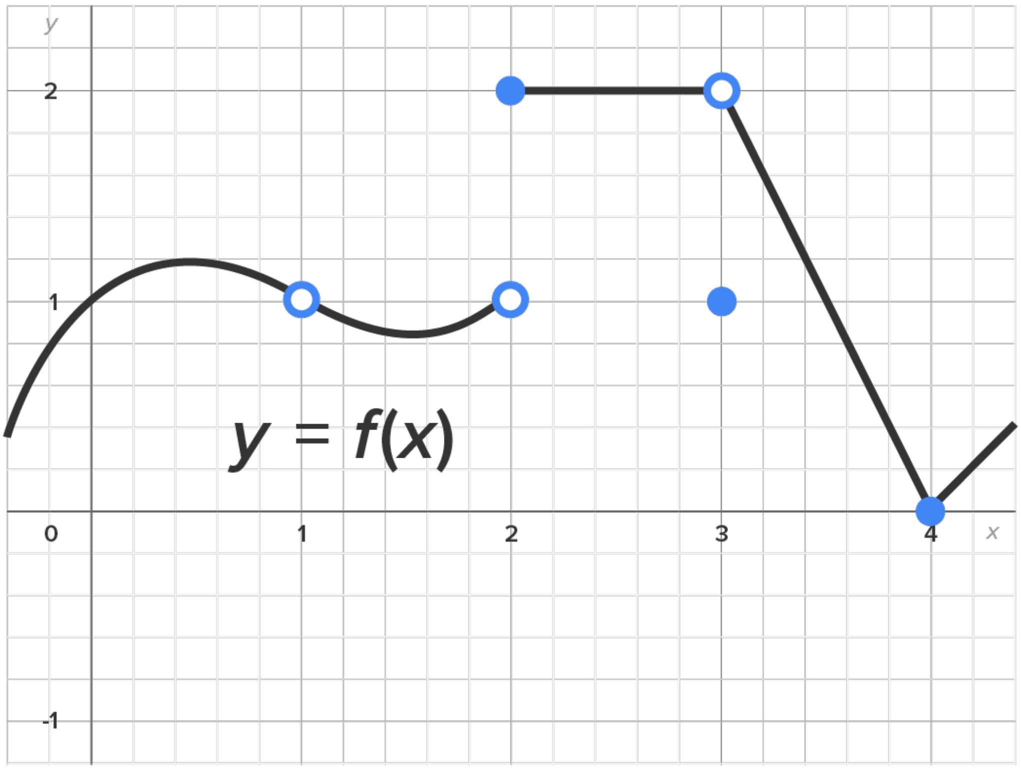
Limit

The value that a function $f(x)$ approaches as x gets closer to a specified number.

2. Using Graphs to Evaluate Limits

We can use the information from a graph to evaluate a limit.

↪ EXAMPLE Consider the graph of some function $y = f(x)$.

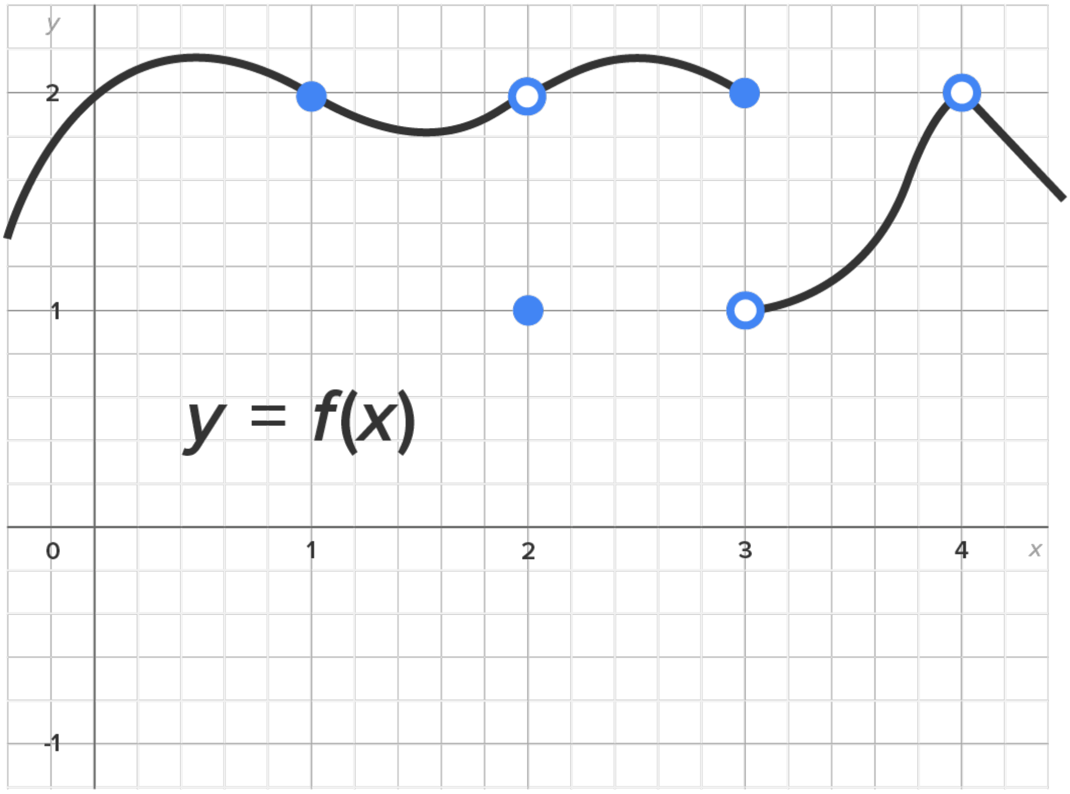


We can say the following:

Statement	Description
$\lim_{x \to 0} f(x) = 1$	As x gets closer to 0, $f(x)$ gets closer to 1.
$\lim_{x \to 1} f(x) = 1$	As x gets closer to 1, $f(x)$ gets closer to 1.
$\lim_{x \to 2} f(x)$ does not exist.	As x gets closer to 2 from the left (values smaller than 2), $f(x)$ gets closer to 1. However, as x gets closer to 2 from the right (values larger than 2), $f(x)$ gets closer to 2. Since $f(x)$ approaches two different values, as x approaches 2, we say the limit does not exist.
$\lim_{x \to 3} f(x) = 2$	As x gets closer to 3, $f(x)$ gets closer to 2. Note that the actual value of $f(3)$ is 1 (closed dot at $x = 3$), but the limit tells us what is happening as we get closer and closer to 3, not what is happening right at 3.
$\lim_{x \to 4} f(x) = 0$	As x gets closer to 4, $f(x)$ gets closer to 0.

TRY IT

Consider the graph pictured below.



Evaluate the function as *x* approaches 1. +

$$\lim_{x \to 1} f(x) = 2$$

Evaluate the function as *x* approaches 2. +

$$\lim_{x \to 2} f(x) = 2$$

Evaluate the function as *x* approaches 3. +

$\lim_{x \to 3} f(x)$ does not exist

Evaluate the function as *x* approaches 4. +

$$\lim_{x \to 4} f(x) = 2$$

SUMMARY

In this lesson, you learned about **defining limit notation**, or how the limit of a function is used to determine the behavior (or value) a function $f(x)$ approaches as x gets closer to some value. You also learned that you can **use the information from a graph to evaluate a limit**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Limit

The value that a function $f(x)$ approaches as x gets closer to a specified number.

The Table Method

by Sophia

WHAT'S COVERED

In this lesson, you will use tables to evaluate limits. Specifically, this lesson will cover:

1. Creating a Table of Values to Estimate a Limit
2. Using a Table of Values to Estimate a Limit

1. Creating a Table of Values to Estimate a Limit

Let's consider again the function $f(x) = \frac{x^2 - 1}{x - 1}$. This time though, we can use a table to estimate the value of $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

First, we must create the table. To do this, we need to use a sequence of x-values that get closer to 1 from both the left and the right.

From the left, you could use $x = 0.9, 0.99, 0.999$.

From the right, you could use $x = 1.001, 1.01, 1.1$.

Now, place the information into one table, also leaving a place for $x = 1$ as shown below:
(Notice the "---" in the place for $x = 1$. This is because $f(x)$ is undefined there.)

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x) = \frac{x^2 - 1}{x - 1}$				---			

Now, complete the table by substituting all x-values into the function.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x) = \frac{x^2 - 1}{x - 1}$	1.9	1.99	1.999	---	2.001	2.01	2.1

It appears that as x gets closer to 1 from either side, $f(x)$ gets closer to 2.

Thus, we can say $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$, just as we said in the graphing example in the previous part of this challenge.

WATCH

The following video walks you through the process of evaluating the limit numerically as *x* approaches -2 of the rational function $f(x)=\frac{x^3+8}{x+2}$.

Video Transcription

Hello, and welcome to a video on evaluating a limit as x approaches a specific value of an expression numerically.

Now, here, we're looking at the limit as x approaches negative 2 of x cubed plus 8 over x plus 2. When we are doing this numerically, we want to set up a table for our x values and our outputs of our expression values, our f of x values. And when you do set up the values, you want to make sure that, whatever your x is approaching, that number is in the middle of the table. Also, since I don't have a plus or minus to the negative two, that means I'm approaching negative two from both sides.

Now, I want to make sure that you think about how to set up these values. Remember on a number line. If we're looking at an x value of negative two and it's placement on the number line, if zero is to the far right here, then would come my negative one, and then my negative three, and then my negative three. So, as I'm approaching negative two from the left, on the left-hand side of negative two, then I'm looking at values of like -2.1 -2.01 -2.001.

So, numbers that are close to negative two. And then, you want to have values that are getting closer and closer on that side. As I approach negative two from the right, we're looking at spans of numbers that are in the span from -1 to -2, but close to -2, and that's why I've put in the numbers -1.9 -1.99 and -1.999. So again, as I'm coming from the outside edges towards the -2, the values of x are getting closer and closer to it.

Next step, what we do is we evaluate our function at each of these. So, in this specific case, my function, f of x, is that expression that you're taking the limit of. X-cubed plus 8 over x plus 2. So, if I'm finding for my first X equal to -2.1 I'm evaluating f of -2.1. So, that is my -2.1 cubed plus 8 over the -2.1 plus 2.

Now, when you run that through your calculator, you will see that you get a value of 12.61. And so, you put that across from of X underneath the -2.1. Let's do the next one also. So, f of -2.01-- I have f of -2.01 is my -2.01 cubed plus 8 over the -2.01 plus 2. And from that you get 12.06.

Now we're going to continue on with that process for each of these values. So if we put -2.001 and for the X's and the function, we would get an output of 12.006001. And then, coming from further away from the -2, coming from the right towards -2, plugging -1.9 through the function, I'll get 11.41. Plugging -1.99 through the function, I get 11.9401. And then, plugging -1.999 through the function, I get 11.994001.

And the limit, numerically, then we want to see, as the X's get closer to -2 from both sides, what are the outputs getting closer to? I know, right at -2, if I try to evaluate f of negative 2, we would get negative 2 cubed plus 8 over my negative 2 plus 2.

But that gives me a 0 in the numerator and a 0 in the denominator that indeterminate form. So, I can't get this by just plugging in our values. But, looking at what the outputs are getting closer and closer to, as you come in at -2 from both the left and the right, they're getting closer and closer to 12. So therefore, my limit, as x approaches negative 2 of that x cubed plus 8 over x plus 2 is equal to 12. And that is how you evaluate your limit numerically.

2. Using a Table of Values to Estimate a Limit

If a table is already created, we can use the information from the table to estimate the limit.

↪ EXAMPLE Evaluate $\lim_{x \to 0} \sqrt{x}$. Here is a table of values that represent x-values around $x = 0$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x) = \sqrt{x}$	undef.	undef.	undef.	0	0.03162	0.1	0.31623

From the left side, there is no limit since $\sqrt{x}$ is undefined when $x < 0$. From the right, it appears as if the limit is 0 since the values of $\sqrt{x}$ are trending toward 0.

Since the left-hand and right-hand sides do not match, we conclude that $\lim_{x \to 0} \sqrt{x}$ does not exist.

↪ EXAMPLE Use a table of values to evaluate $\lim_{x \to 0} \frac{\sin 4x}{x}$. The table with the values of $f(x)$ is shown below:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x) = \frac{\sin 4x}{x}$	3.89418	3.99893	3.99999	---	3.99999	3.99893	3.89418

It appears as if $f(x)$ is getting closer to 4 from either side. Therefore, we conclude that $\lim_{x \to 0} \frac{\sin 4x}{x} = 4$.

TRY IT

Consider the function $\frac{\sqrt{x} - 2}{x - 4}$. Answer the following questions to evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$.

Create a table of values for this function. +

x	3.9	3.99	3.999	0	4.001	4.01	4.1
$f(x) = \frac{\sqrt{x} - 2}{x - 4}$	0.25158	0.25016	0.25002	---	0.24998	0.24984	0.24846

What is the limit of the function as it approaches 4? +

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = 0.25$$

SUMMARY

In this lesson, you learned about another method to evaluate limits, by **creating a table of values to estimate a limit**. You also learned that it is very helpful to **use a table of values to estimate a limit**, since it shows patterns in how $f(x)$ changes as x approaches a number.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

The Algebra Method

by Sophia

WHAT'S COVERED

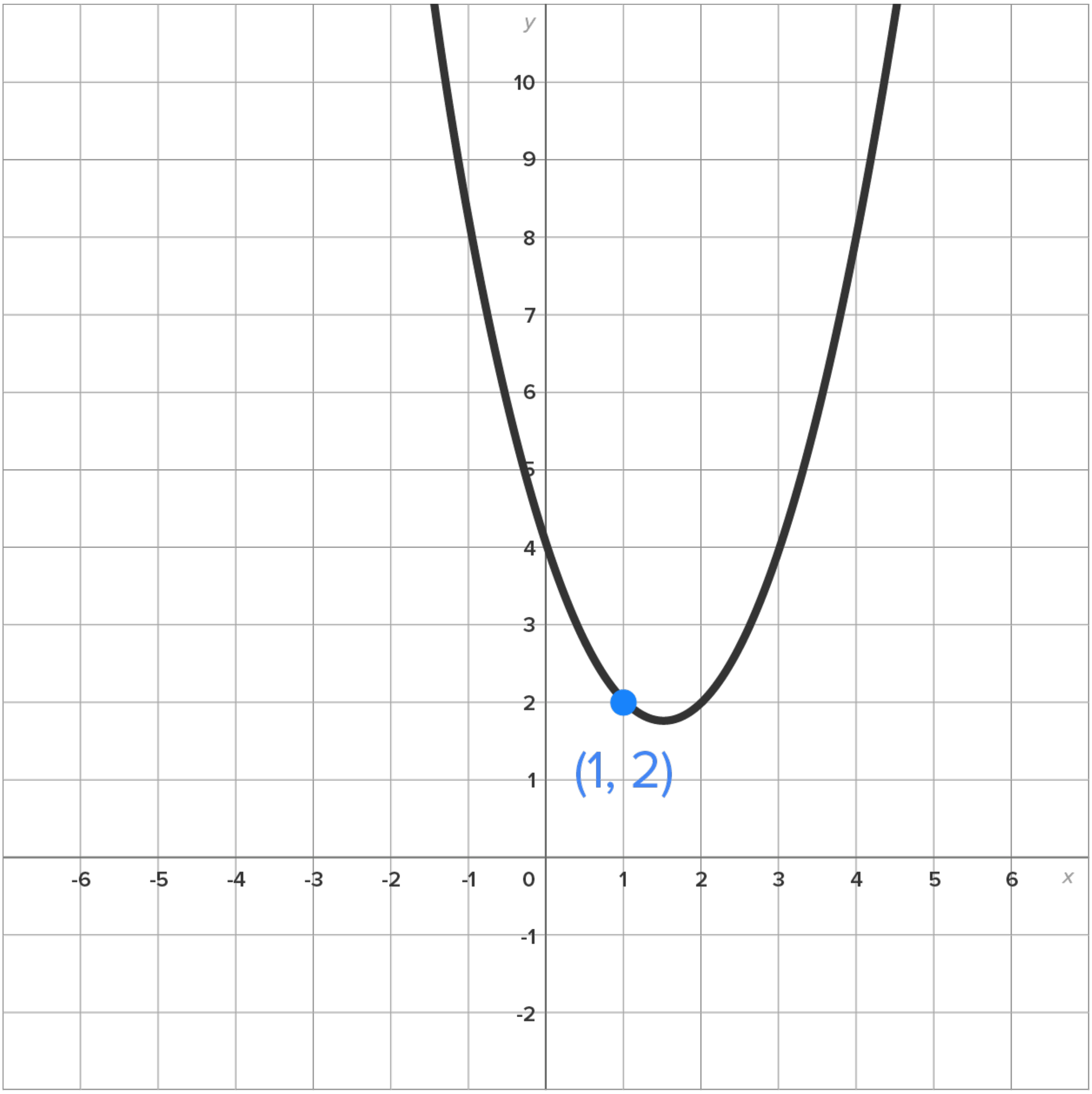
In this lesson, you will use algebraic techniques to evaluate limits. Specifically, this lesson will cover:

1. The Direct Substitution Method
2. Evaluating Limits by Simplifying the Expression

1. The Direct Substitution Method

Consider the function $f(x) = x^2 - 3x + 4$, with the goal of evaluating $\lim_{x \to 1} (x^2 - 3x + 4)$.

The graph of the function is shown below.



Things to notice:

- The graph contains the point (1, 2).
- As x gets closer to 1 from either side, the value of $f(x)$ gets closer to 2.

Thus, we can say that $\lim_{x \to 1} (x^2 - 3x + 4) = 2$.
We could have found this same answer by substituting 1 in for x in the function. This would have given us the point (1, 2) along with the value of the limit, 2.

BIG IDEA

When given a continuous function $f(x)$, one way to evaluate $\lim_{x \to a} f(x)$ is to substitute a in for x and simplify. This works when $f(x)$ is defined on both sides of $x = a$.

EXAMPLE Evaluate each limit below by using the direct substitution method.

Limit	Solution
$\lim_{x \to 3} \frac{x^2-4}{x+7}$	$\lim_{x \to 3} \frac{x^2-4}{x+7} = \frac{3^2-4}{3+7} = \frac{5}{10} = \frac{1}{2}$
$\lim_{x \to 2} (3x^4 - x^2)$	$\lim_{x \to 2} (3x^4 - x^2) = 3(2)^4 - 2^2 = 44$
$\lim_{x \to 1} \sin \pi x$	$\lim_{x \to 1} \sin \pi x = \sin \pi(1) = \sin \pi = 0$

TRY IT

Consider the function $f(x) = \frac{x^3-8}{x+1}$.

Evaluate the limit as x approaches 1. +

$$\lim_{x \to 1} \frac{x^3-8}{x+1} = -\frac{7}{2}$$

---

2. Evaluating Limits by Simplifying the Expression

What happens when direct substitution doesn't work? That is, what happens when $f(a)$ is undefined? There are other methods that could be helpful in evaluating the limit.

In previous parts of this challenge, we evaluated $\lim_{x \to 1} \frac{x^2-1}{x-1}$ by using graphs and tables. Both times we concluded that the limit is 2. While these methods were fairly straightforward to use, algebraic techniques are more convincing.

Remember that the limit $\lim_{x \to 1} \frac{x^2-1}{x-1}$ means that x is getting closer to 1, but not equal to 1.

Also notice that direct substitution will not work for this limit since $\frac{x^2-1}{x-1}$ is undefined when $x = 1$.

However, $\frac{x^2-1}{x-1}$ can be simplified, keeping in mind that $x \neq 1$.

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$$

This means $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 1+1 = 2$.

Once we simplified $\lim_{x \to 1} \frac{x^2-1}{x-1}$ to $\lim_{x \to 1} (x+1)$, we were able to use direct substitution and evaluate the limit.

The big question is: How do we know to simplify?

Notice that when using direct substitution with $\lim\limits_{x \to 1} \frac{x^2-1}{x-1}$, the numerator and denominator are both 0. This is a signal that the expression can be simplified.

↱ EXAMPLE Evaluate the limit: $\lim\limits_{x \to 5} \frac{x^2-6x+5}{x^2-25}$

Attempting direct substitution, notice that the numerator and denominator are both 0. This means we should try to simplify:

$\lim\limits_{x \to 5} \frac{x^2-6x+5}{x^2-25}$	Start with the original limit.
$= \lim\limits_{x \to 5} \frac{(x-5)(x-1)}{(x-5)(x+5)}$	Factor the numerator and denominator.
$= \lim\limits_{x \to 5} \frac{(x-1)}{(x+5)}$	Remove the common factor.
$= \frac{4}{10}$	Substitute $x=5$.
$= \frac{2}{5}$	Simplify the expression.

Conclusion: $\lim\limits_{x \to 5} \frac{x^2-6x+5}{x^2-25} = \frac{2}{5}$

WATCH

The following video walks you through the process of evaluating $\lim\limits_{x \to -2} \frac{x^3+8}{x^2-x-6}$ by simplifying the rational expression.

Video Transcription

Hello, and welcome to the video and evaluating the limit as x approaches a numerical value of a rational expression that is in indeterminate form and then by simplifying. So when we're looking at this limit question, I have the limit as x approaches negative 2 of x to the 1/3 plus 8, over x squared minus x minus 6.

Now, in doing these problems, the very first thing you want to do is you want to see what's happening with the denominator as x approaches that value. If the denominator approaches a numerical value that's not zero, we can just use simplification by-- we can just use substitution to get that. If however, my denominator approaches zero, then I need to look at what's happening in the numerator and then see if there's common factors that can be removed.

So let's look. Here, the limit as x approaches negative 2, while as x approaches negative 2, x squared approaches 4 and x approaches negative 2. So I have something approaching 4, minus something that's approaching negative 2, and then minus 6. So that would be 4, then minus a negative 2 is plus 2. So 4 plus 2 is 6, minus 6 gives me a 0 in the denominator.

Now let's see what's happening in the numerator. As x approaches negative 2, for my numerator, something approaching negative 2 cubed approaches negative 8, and then plus 8, that gives me a 0 in my numerator as well. So this is 0 over 0, which is called an indeterminate form. When we have that type of an expression, we need to investigate it a little bit closer to see what's really going on.

Now with a polynomial divided by a polynomial, I can actually look at the factorization of the numerator and the denominator because if a number run through a polynomial gives me 0 out, x minus that number is a factor.

So since negative 2 run through the numerator gives me a 0 out, x minus negative 2 or x plus 2 is a factor. And negative 2 through the x is in the denominator, did the same thing. It gave me 0 out. So I know that x plus 2 is also a factor in the denominator, but I do still need to find the remaining factors.

Well, let's investigate the factorization of that x cubed plus 8 in the numerator. I can use synthetic division in order to find the other factor. So I will take synthetic division, and I will divide with the negative 2. And remember, in synthetic division, you just use the numerical coefficients in front of the terms. So the numerical coefficient in front of x cubed is one.

There's no x squared, so the numerical coefficient in front of the x squared is zero. The numerical coefficient in front of x is 0, and then I have my constant term 8. Now with synthetic division, we bring that first number down, so one comes down, and then we multiply that negative 2 times that one. So negative 2 times 1 is negative 2, and we bring it up underneath the next number, and then we add. 0 plus negative 2 is negative 2, then we start over again.

We multiply, negative 2 times negative 2 gives me 4, and 0 plus 4 gives me 4. And then we multiply, negative 2 times 4 gives me negative 8. And then we add, 8 plus negative 8 is zero. And we knew we were going to get that 0 because negative 2 run through the polynomial gave me a 0 out. So my factors are x minus that negative 2, that x plus 2 that we already wrote down, and then the remaining factor is in descending order.

My power of my numerator was power 3, but I factored out an x to the first. So I'm going to have an x squared term, and its first coefficient is a one. So I have one x squared, then I have minus 2x, and then plus 4. So that x squared minus 2x plus 4 is the other factor of the numerator.

And specifically, this example, we wanted to do is a video because x cubed plus 8 isn't the easiest thing to factor, and it's nice to know that we can get that other factor using synthetic division. Now for the denominator, we could use synthetic division, but it's a much easier expression to find the other factor.

I need an x squared as my first term, so x times x is how I would get that x squared. I need a negative 6 as my last term, and positive 2 times negative 3 is how I would get that negative 6. And if you just double check with your outers and your inners, you'd have a negative 3x combined with a positive 2x would give you that negative x in the middle.

Now that we have it factored, we can remove the common factor pairs, and we're really just looking at the limit as x approaches negative 2 of that remaining factor x squared minus 2x plus 4, over x minus 3.

And limits are, as you get infinitely close to negative 2, not what's happening right at negative 2, and that's why we can evaluate the simplified function to get our limit.

So now as we put in negative 2 for the x, looking in the denominator first, I see that that is negative 2 minus 3, which gives me a negative 5 down there. I don't get a 0 in the denominator anymore. So I can continue the process of plugging in the negative 2 for the X's throughout the expression in the numerator as well.

So I have that quantity negative 2 squared minus times the negative 2, and then plus the 4. So that gives me 4 plus 4 plus 4 in the numerator, over a negative 5 in the denominator. And 4 plus 4 plus 4 gives me 12 in the numerator, and I get negative 5 in the denominator.

Now that doesn't simplify anymore, so I have the result that the limit as x approaches negative 2 of x cubed plus 8, over x squared minus x minus 6, is equal to negative 12/5.

↪ EXAMPLE Evaluate the limit: $\lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6}$

Attempting direct substitution, notice that the numerator and denominator are both 0. This means we should try to simplify:

$\lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6}$	Start with the original limit.
$= \lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6} \cdot \frac{6x}{6x}$	Since this is a complex fraction, multiply the numerator and denominator by the LCD, which is $6x$.
$= \lim_{x \to 6} \frac{6-x}{6x(x-6)}$	Distribute in the numerator and simplify. Leave the denominator in factored form.
$= \lim_{x \to 6} \frac{-(x-6)}{6x(x-6)}$	Factor out the negative factor in the numerator.
$= \lim_{x \to 6} \frac{-1}{6x}$	Remove the common factor.
$= -\frac{1}{36}$	Substitute $x=6$.

TRY IT

Consider the function $f(x) = \frac{14x-42}{x^2-8x+15}$.

Evaluate the limit as x approaches 3. +

$$\lim_{x \to 3} \frac{14x-42}{x^2-8x+15} = -7$$

SUMMARY

In this lesson, you learned that you can **evaluate limits** algebraically using two methods: **the direct substitution method** and by **simplifying the expression** first. Note that when $f(a)$ is undefined, direct substitution doesn't work. When using direct substitution with $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$, the numerator and denominator are both 0, which is a signal that the expression can be simplified. Both of these methods are useful since they require algebraic facts and will give exact answers rather than tables or graphs that often give approximations.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

One-Sided Limits

by Sophia

WHAT'S COVERED

In this lesson, we will explore limits by examining the left and right sides separately. Specifically, this lesson will cover:

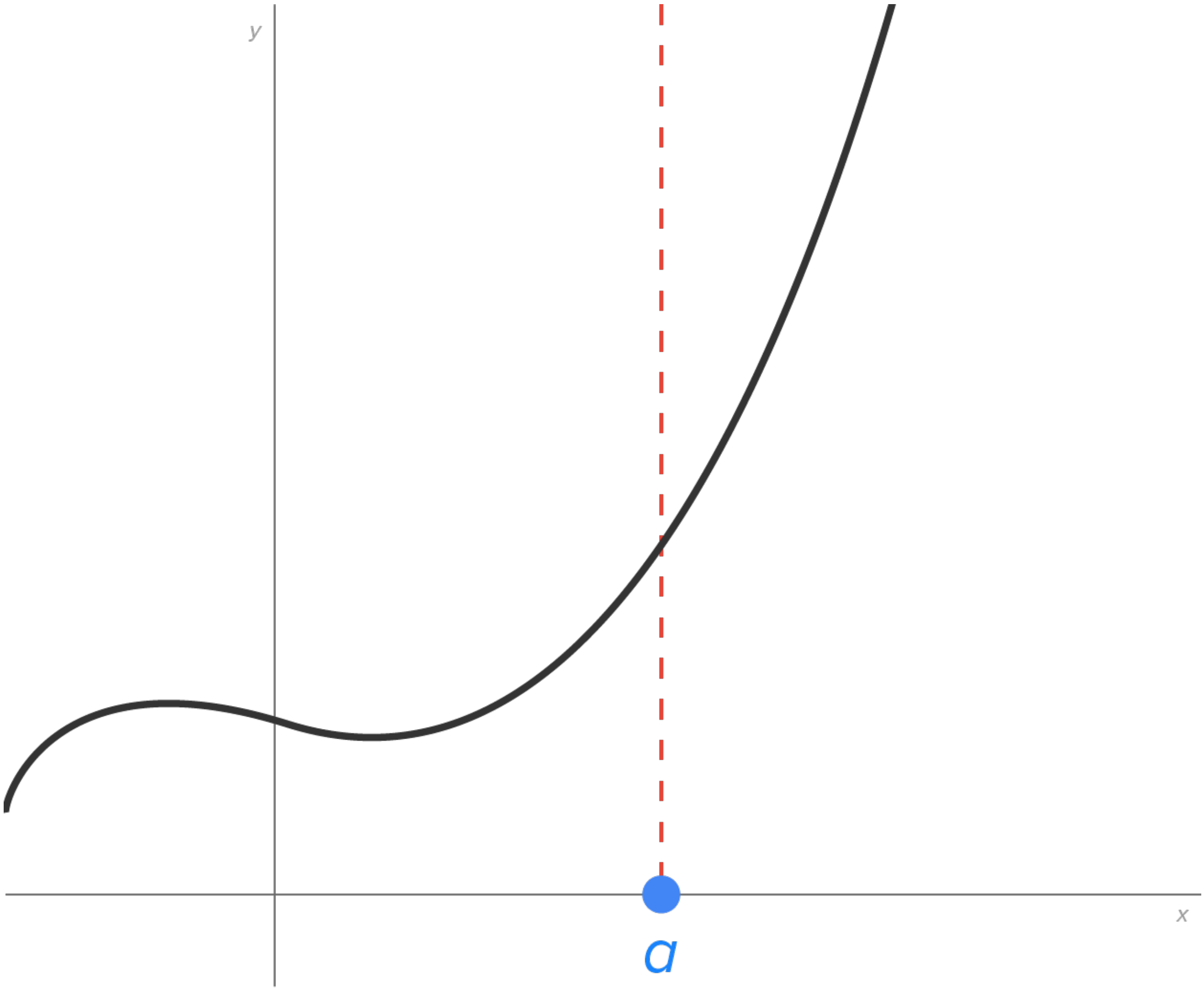
1. Notation Used for One-Sided Limits and Evaluating Them Graphically
2. Evaluating One-Sided Limits

1. Notation Used for One-Sided Limits and Evaluating Them Graphically

Recall that the notation $\lim_{x \to a} f(x)$ means to evaluate the limit of some function $f(x)$ as *x* gets closer to *a* from both sides. If the values from both sides don't match, $\lim_{x \to a} f(x)$ does not exist.

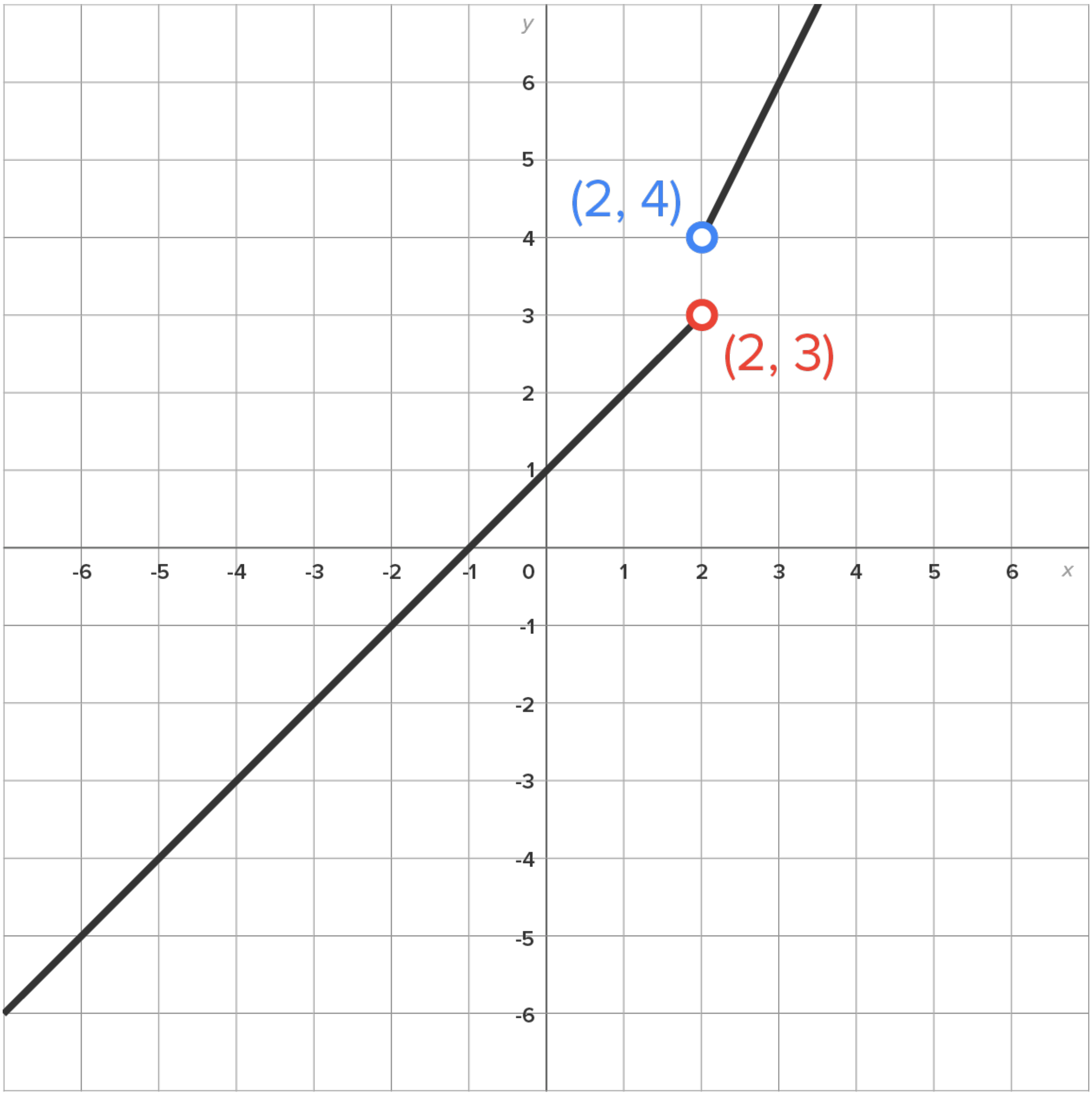
To that end, we define one-sided limits, meaning limits that focus on one side of $x = a$.

- Limit from the left: $\lim_{x \to a^-} f(x)$
- Limit from the right: $\lim_{x \to a^+} f(x)$



***x* approaches *a* from the left →** **← *x* approaches *a* from the right**

↳ EXAMPLE Consider the graph below, which shows some function $f(x)$.



We can say the following:

Statement	Description
$\lim_{x \to 2^-} f(x) = 3$	As x approaches 2 from the left, $f(x)$ gets closer to 3.
$\lim_{x \to 2^+} f(x) = 4$	As x approaches 2 from the right, $f(x)$ gets closer to 4.
$\lim_{x \to 2} f(x)$ does not exist	Since the left-hand and right-hand limits are not equal, $\lim_{x \to 2} f(x)$ does not exist.

BIG IDEA

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$, then $\lim_{x \to a} f(x) = L$. This means that if the left-hand and right-hand limits are both equal to the same value (L), the limit of the function is also equal to L as $x \to a$.

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

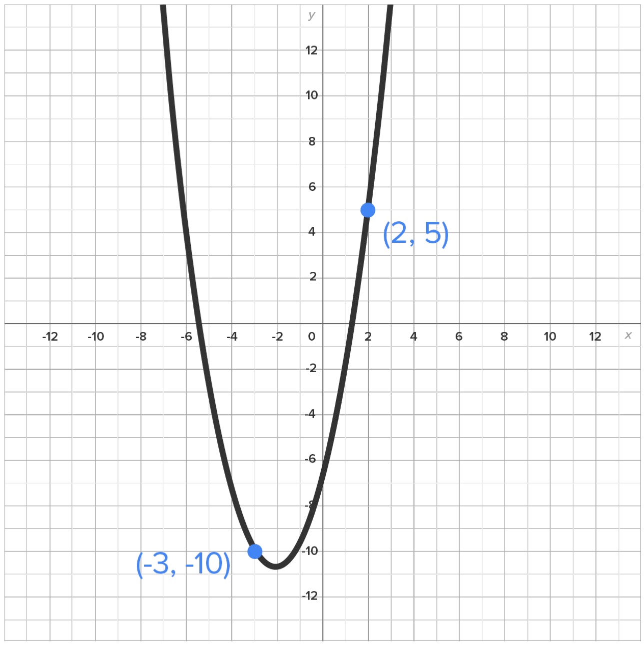
2. Evaluating One-Sided Limits

It turns out that one-sided limits can be evaluated using direct substitution.

Consider the graph of $f(x) = x^2 + 4x - 7$ shown in the graph to the right.

Notice the following:

- $\lim_{x \to -3^-} (x^2 + 4x - 7) = -10$
- $\lim_{x \to -3^+} (x^2 + 4x - 7) = -10$
- $\lim_{x \to 2^-} (x^2 + 4x - 7) = 5$
- $\lim_{x \to 2^+} (x^2 + 4x - 7) = 5$



Notice that in each case, the limits could have also been evaluated by direct substitution. Refer back to the "Direct Substitution" section earlier in this challenge to see which limits can be evaluated using this method.

EXAMPLE Evaluate the one-sided limits below by using the direct substitution method.

Limit	Solution
$\lim_{x \to 4^+} (x^2 + 4x + 5)$	$\lim_{x \to 4^+} (x^2 + 4x + 5) = 4^2 + 4(4) + 5 = 37$
$\lim_{x \to 1^-} \frac{x+2}{x^2+3}$	$\lim_{x \to 1^-} \frac{x+2}{x^2+3} = \frac{1+2}{1^2+3} = \frac{3}{4}$

TRY IT

Consider the function $\sqrt{2x^2 + 5x + 7}$.

Evaluate the limit as x approaches 2 from the left. +

$$\lim_{x \to 2^-} \sqrt{2x^2 + 5x + 7} = 5$$

Sometimes it is tempting to use direct substitution, even though it is technically not applicable.

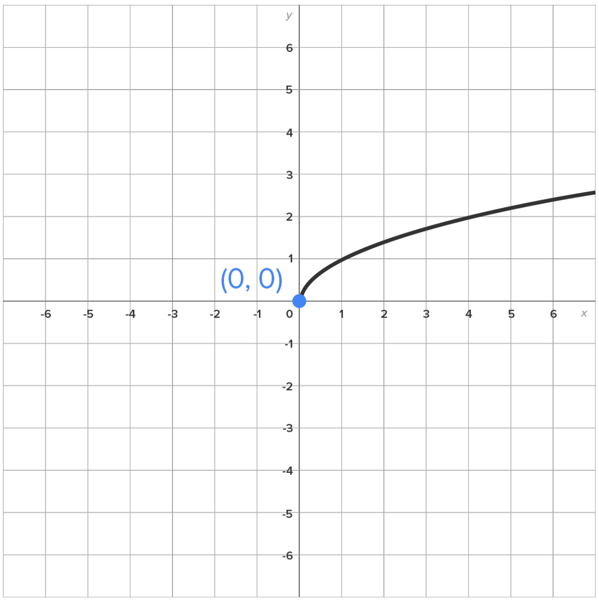
EXAMPLE

Consider $f(x) = \sqrt{x}$. Let's find the one-sided limits as $x \to 0$. To help visualize this, the graph is pictured to the right.

Left-Sided Limit: $\lim_{x \to 0^-} \sqrt{x}$ does not exist since $\sqrt{x}$ is undefined to the left of $x = 0$.

Right-Sided Limit: $\lim_{x \to 0^+} \sqrt{x} = 0$ since as x gets closer to 0 from the right, the value of $\sqrt{x}$ gets closer to 0.

Since the one-sided limits are not equal, this also means that $\lim_{x \to 0} \sqrt{x}$ does not exist.



It might have been tempting to use direct substitution since $\sqrt{0} = 0$. However, since $x = 0$ is also the endpoint of the domain of $f(x) = \sqrt{x}$, more care has to be taken when analyzing this function near $x = 0$.

BIG IDEA

When evaluating one-sided limits, be sure that $f(x)$ is defined on that side before using direct substitution.

To evaluate $\lim_{x \to a^-} f(x)$, make sure that $f(x)$ is defined for $x < a$.

To evaluate $\lim_{x \to a^+} f(x)$, make sure that $f(x)$ is defined for $x > a$.

When $f(x)$ is a piecewise function, one-sided limits are very useful in examining $f(x)$ at the x-values where the function changes definition.

EXAMPLE Consider the function $f(x) = \begin{cases} 2x+1 & if\ x < 3 \\ 10-x & if\ x \geq 3 \end{cases}$. Evaluate $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.

Since $f(x)$ changes definitions when at $x = 3$, evaluating $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$ takes some extra thought.

When evaluating $\lim_{x \to 3^-} f(x)$, we can replace $f(x)$ with $2x + 1$ and evaluate the limit:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x+1) = 7$$

When evaluating $\lim_{x \to 3^+} f(x)$, we can replace $f(x)$ with 10 - x and evaluate the limit:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (10-x) = 7$$

WATCH

The following video walks you through the process of evaluating $\lim_{x \to 5^-} f(x)$ and $\lim_{x \to 5^+} f(x)$ for $f(x) = \begin{cases} 4-x^2 & if\ x < 5 \\ x^2+x-1 & if\ x > 5 \end{cases}$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome to the video on evaluating one-sided limits, specifically, here, of a piecewise-defined function as x approaches the x values that are at the transition of the pieces of the function. Let's look at this example.

So here, we have f of x is equal to 4 minus x squared if x is less than 5 and x squared plus x minus 1 if x is greater than 5. And we want to find, first, the limit as x approaches 5 from the left of that function, f of x. And then after we get done working with that one, we will look at the limit as x approaches 5 from the right of f of x.

Now let's focus on the limit as x approaches 5 from the left of f of x. So when we look at numerical values that are left of 5 on the number line for the x's, those are numbers that are smaller than 5. So when we're doing values that are smaller than 5, the piece of the function that we use is this 4 minus x squared. So we are really looking at the limit as x approaches 5 from the left of the expression that is valid for those x values, which is that expression, 4 minus x squared.

And that expression we can evaluate by substitution. So we have our 4 minus, take out the x and put in our value of 5. Quantity square that. So I have four minus 25 for negative 21. So my limit, as x approaches 5 from the left of this piecewise-defined function f of x, is equal to negative 21.

Now let's look at the limit as x approaches 5 from the right. So as x approaches 5 from the right, those are numbers that are bigger than 5. So we're still getting closer and closer to 5, but on the right side from the right. So those are values that are bigger than 5. And our x squared plus x minus 1 is our expression that we have when our x's are greater than 5. So we are looking at the limit, as x approaches 5 from the right, of the expression x squared plus x minus 1.

Again, we can find this limit by direct substitution. So that is going to give us a value of 5 quantity squared, plus take out the x and put in my 5, and then minus 1. So I have 25 plus 5 minus 1, which is 29. So the limit, as x approaches 5 from the right of this piecewise-defined function, f of x, is equal to 29. And there, you have an example of how you find the one sided limits of a piecewise-defined function if you are approaching the x values at which the pieces transition.

[MUSIC PLAYING]

TRY IT

Consider the function $f(x)=\begin{cases} 5x-x^2 & \text{if } x\le 1 \\ 2x+3 & \text{if } x>1 \end{cases}$.

Evaluate the limit as *x* approaches 1 from the left. +

$$\lim_{x\to 1^-} f(x)=4$$

Evaluate the limit as *x* approaches 1 from the right.

+

$$\lim_{x \to 1^+} f(x) = 5$$

SUMMARY

In this lesson, you recalled that the notation $\lim_{x \to a} f(x)$ means to evaluate the limit of some function $f(x)$ as x gets closer to a from both sides. You also learned about the **notation used for one-sided limits**, specifically the notation $\lim_{x \to a^-} f(x)$ means to find the value $f(x)$ is approaching as x gets closer to a from values smaller than a, and $\lim_{x \to a^+} f(x)$ means to find the value of $f(x)$ is approaching as x gets closer to a from values larger than a. In general, **evaluating one-sided limits** is very similar to evaluating limits; one-sided limits can be **evaluated by graphing**, by tables, or by direct substitution, but be sure that $f(x)$ is defined on that side of x if you are using direct substitution.

When evaluating an overall limit, you learned that if $\lim_{x \to a^-} f(x)$ is not the same value as $\lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist. When dealing with piecewise functions and other functions that have restricted domains (for example, $f(x) = \sqrt{x}$ around $x = 0$), more care needs to be taken when evaluating both one-sided and overall limits.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Using Properties of Limits

by Sophia

WHAT'S COVERED

In this lesson, you will utilize limit properties to evaluate more complex limits. Specifically, this lesson will cover:

1. Limit Properties
2. Evaluating Limits Using Limit Properties

1. Limit Properties

Suppose we know that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then, we can establish the following properties of limits:

Property of Limits	Formula
Limit of a Constant	$\lim_{x \to a} k = k$
Limit of a Sum or Difference	$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$
Limit of a Product	$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LM$
Limit of a Quotient	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ (as long as $M \neq 0$)
Constant Multiple	$\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x) = kL$
Limit of a Power	$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = L^n$
Limit of an nth Root	$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ (If n is even, then $L > 0$.)

2. Evaluating Limits Using Limit Properties

Here are a few examples of using the limit properties to evaluate limits.

EXAMPLE Given $\lim_{x \to 3} f(x) = 20$ and $\lim_{x \to 3} g(x) = 4$, evaluate $\lim_{x \to 3} [2f(x) + 3]$, $\lim_{x \to 3} \frac{f(x)}{1 + g(x)}$, and $\lim_{x \to 3} \sqrt{f(x) - g(x)}$.

$\lim_{x \to 3} [2f(x) + 3]$ Start with the original limit.

$= \lim_{x \to 3} 2f(x) + \lim_{x \to 3} 3$	Apply the sum/difference property.
$= 2 \cdot \lim_{x \to 3} f(x) + \lim_{x \to 3} 3$	Apply the constant multiple property.
$= 2(20) + 3$	Apply the limit of a constant property: $\lim_{x \to 3} f(x) = 20$
$= 43$	Simplify the expression.

$\lim_{x \to 3} \frac{f(x)}{1+g(x)}$	Start with the original limit.
$= \frac{\lim_{x \to 3} f(x)}{\lim_{x \to 3} [1+g(x)]}$	Apply the quotient property.
$= \frac{\lim_{x \to 3} f(x)}{\lim_{x \to 3} 1 + \lim_{x \to 3} g(x)}$	Apply the sum/difference property.
$= \frac{20}{1+4}$	Apply the limit of a constant property: $\lim_{x \to 3} f(x) = 20$, $\lim_{x \to 3} g(x) = 4$
$= 4$	Simplify the expression.

$\lim_{x \to 3} \sqrt{f(x) - g(x)}$	Start with the original limit.
$= \sqrt{\lim_{x \to 3} [f(x) - g(x)]}$	Apply the nth root property.
$= \sqrt{\lim_{x \to 3} f(x) - \lim_{x \to 3} g(x)}$	Apply the sum/difference property.
$= \sqrt{20 - 4}$	$\lim_{x \to 3} f(x) = 20$, $\lim_{x \to 3} g(x) = 4$
$= \sqrt{16}$	Simplify the expression.
$= 4$	Simplify the expression.

TRY IT

Suppose $\lim_{x \to a} f(x) = -4$ and $\lim_{x \to a} g(x) = 12$ and you want to find $\lim_{x \to a} \frac{[f(x)]^2}{g(x)}$.

Evaluate this limit. +

$$\lim_{x \to a} \frac{[f(x)]^2}{g(x)} = \frac{4}{3}$$

SUMMARY

In this lesson, you learned that **limit properties** can be helpful in evaluating limits where other related limits are known. You also explored several examples of **evaluating limits using limit properties**. Future topics will illustrate the need to use these properties.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Comparing Limits of Functions: Squeeze Theorem

by Sophia

WHAT'S COVERED

In this lesson, you will evaluate more difficult limits by comparing them to other known limits. Specifically, this lesson will cover:

1. Defining the Squeeze Theorem
2. Evaluating Limits by Using the Squeeze Theorem

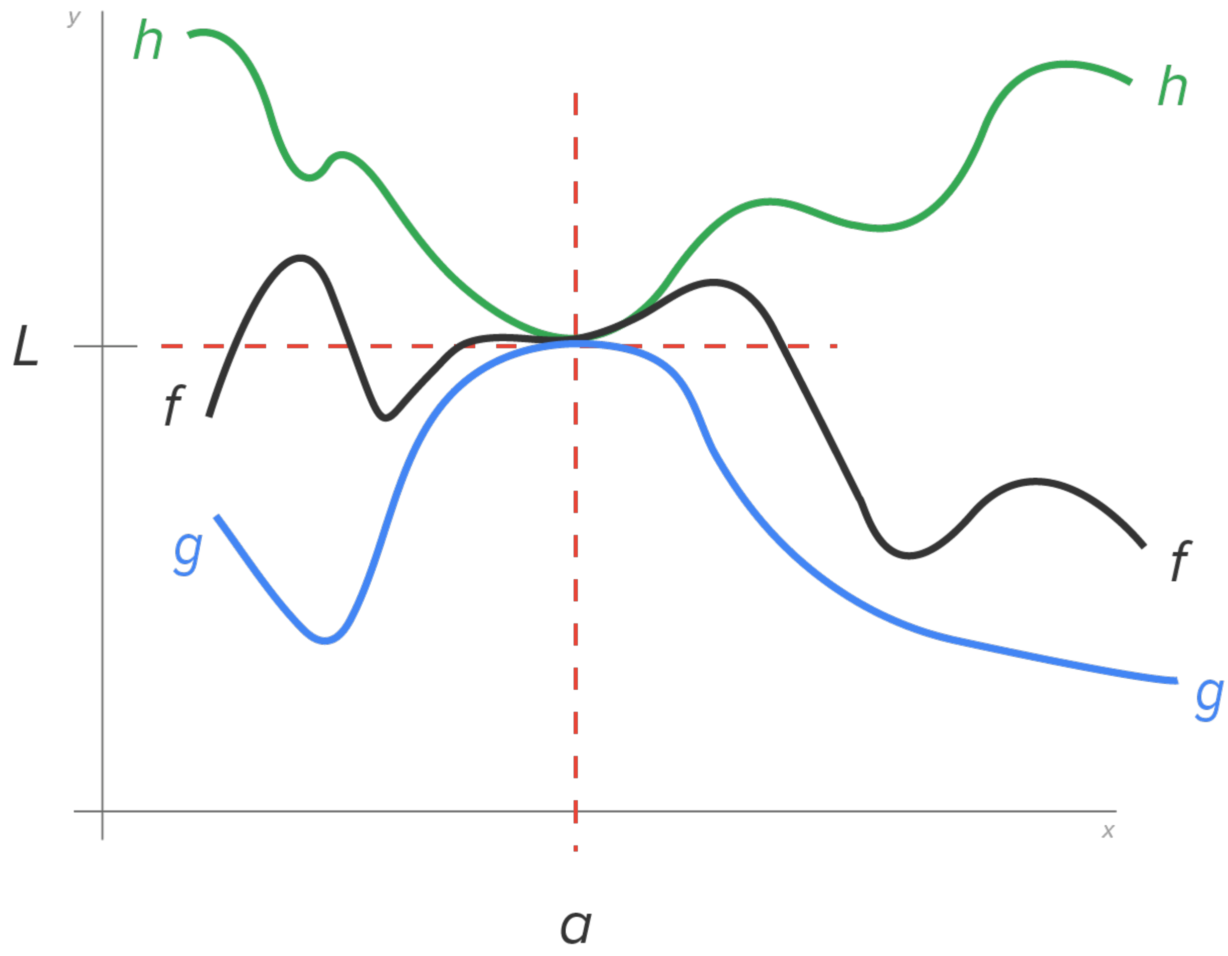
1. Defining the Squeeze Theorem

The squeeze theorem is a theorem that uses limit values and states the following:

CONCEPT TO KNOW

Suppose that $g(x) \leq f(x) \leq h(x)$ for all values of *x* near $x = a$, as shown in the figure below.

If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.



2. Evaluating Limits by Using the Squeeze Theorem

You can evaluate limits by using the squeeze theorem.

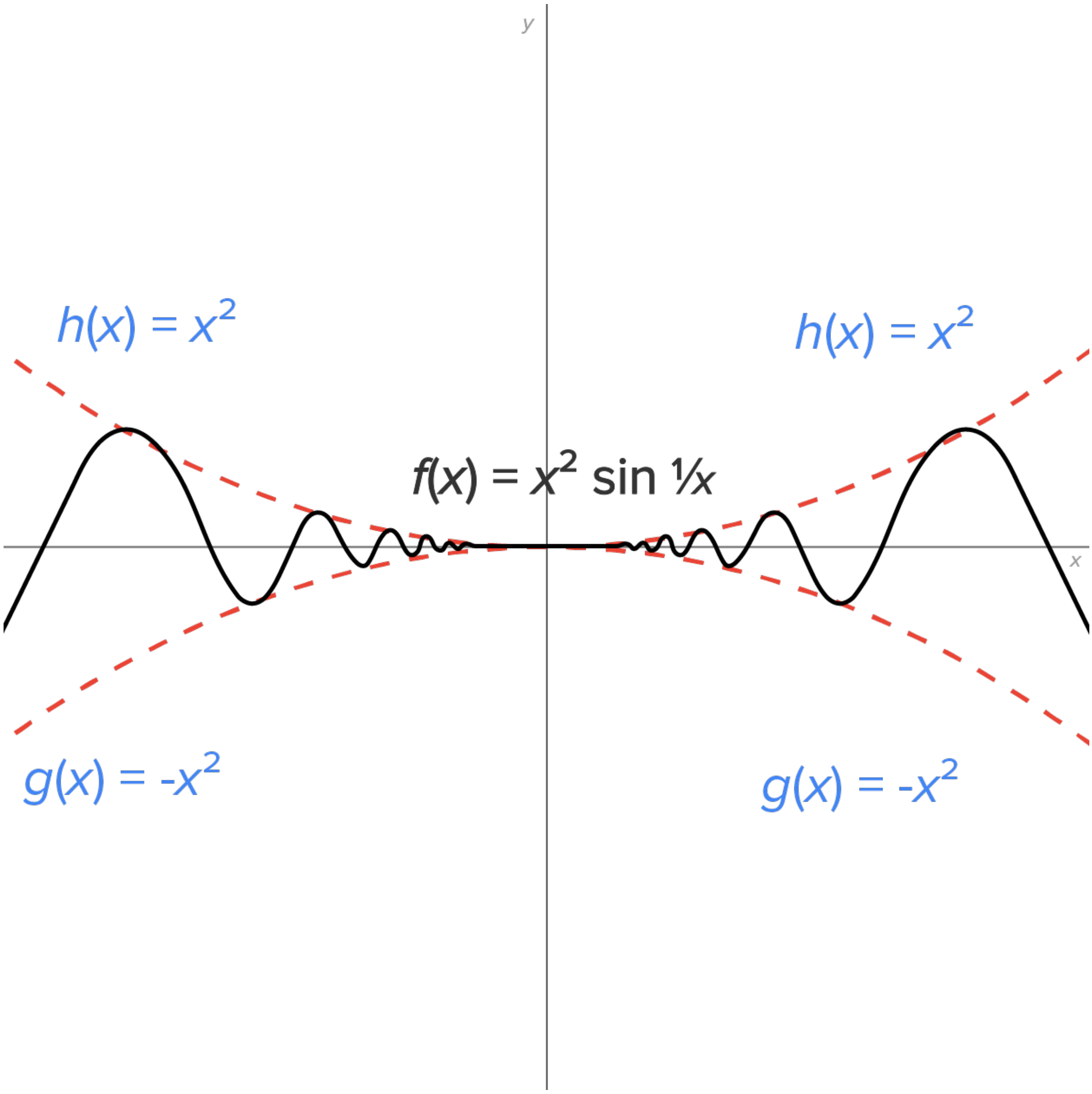
↱ EXAMPLE Consider the limit $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$. Note that direct substitution does not work since the function is undefined when $x = 0$.

Recall that the range of the sine function is $[-1, 1]$. This means for any choice of angle θ, $-1 \le \sin\theta \le 1$. This also means that $-1 \le \sin\left(\frac{1}{x}\right) \le 1 \; x \ne 0$.

Now, multiply all three parts of the inequality by x^2. Since $x^2 > 0$, the direction of the inequalities is preserved: $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 \; x \ne 0$

Let $g(x) = -x^2$, $h(x) = x^2$, and $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. Since $\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$, it follows by the squeeze theorem that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Here is a graph that helps to describe the situation. As you can see, the graph of $f(x)$ is always between the graphs of $g(x)$ and $h(x)$.



↪ EXAMPLE Suppose $4x-3 \le f(x) \le x^2+1$ for all *x* near $x=2$, except possibly at $x=2$. Let's evaluate $\lim_{x \to 2} f(x)$.

Since $\lim_{x \to 2}(4x-3) = 4(2)-3 = 5$ and $\lim_{x \to 2}(x^2+1) = 2^2+1 = 5$, it follows by the squeeze theorem that $\lim_{x \to 2} f(x) = 5$.

TRY IT

Consider the fact that $\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}$ near $x=0$. Suppose you want to find $\lim_{x \to 0} \frac{\sin x}{x}$.

Evaluate this limit. +

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

SUMMARY

In this lesson, you learned the **definition of the squeeze theorem**, which lets us find the limit of a function as x approaches a whose function values are between two other functions on both sides of a, and where the limits of the two other functions are the same as x approaches a. You learned that you can use the **squeeze theorem to evaluate limits** that are particularly difficult, with functions that have function values between two functions with known and equal limit values.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Continuous Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn what it means for a function to be continuous, including how limits are used in relation to continuity. Specifically, this lesson will cover:

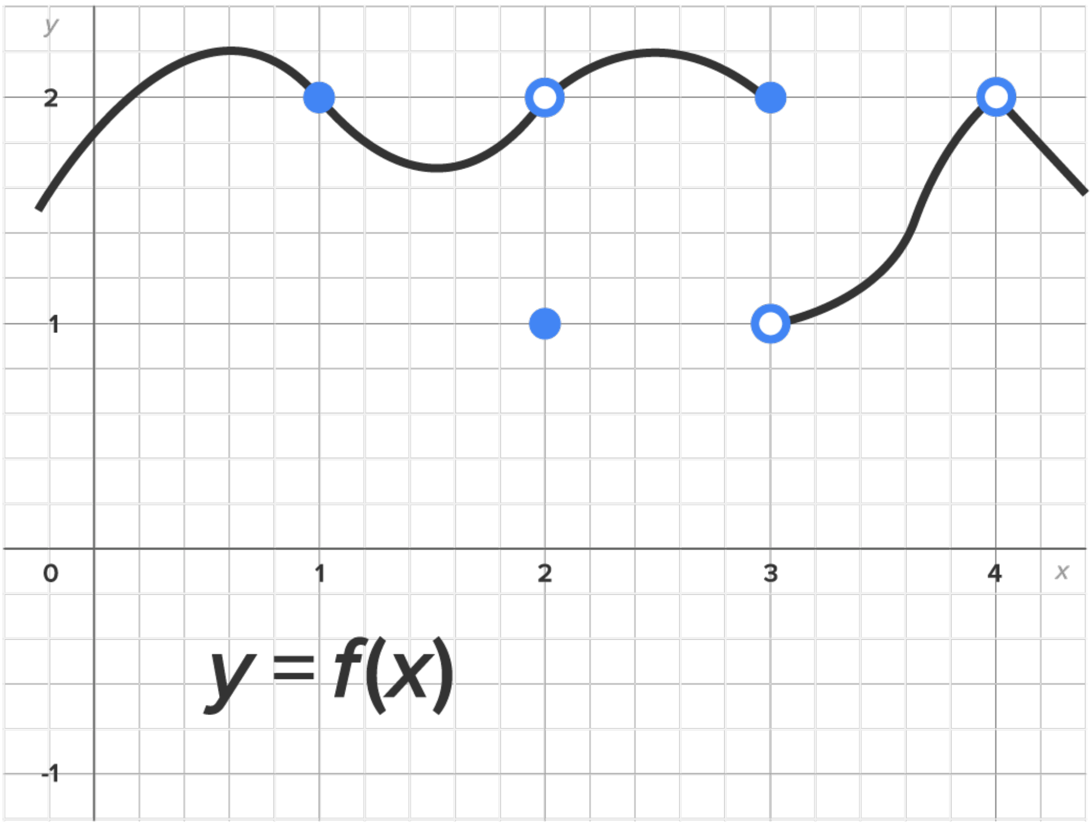
1. The Definition of Continuity
2. Determining if a Function Is Continuous at $x = a$
3. Determining Intervals Over Which a Function Is Continuous

1. The Definition of Continuity

A function is called **continuous** at a point where there is no break in the graph *at that point*.

That is, $\lim_{x \to a} f(x) = f(a)$.

Consider the graph of $y = f(x)$ shown below. We will examine the continuity of $f(x)$ when $x = 1, 2, 3$, and 4.



Given Point	Continuity of $f(x)$ at the Given Point
$x=1$	The graph of $f(x)$ is continuous when $x=1$ since there are no breaks in the graph at that point. Looking just before $x=1$, the graph passes through the point $(1, f(1))$ and continues to "flow" afterwards.
$x=2$	The graph of $f(x)$ is NOT continuous when $x=2$. There is a hole in the graph when $x=2$, meaning there is a break in the graph.
$x=3$	The graph is NOT continuous when $x=3$. There is a break in the graph.
$x=4$	The graph is NOT continuous when $x=4$. There is a hole in the graph.

Now, considering these 4 points, let's examine the limits at these points and the values of $f(x)$ at these points as well as whether or not the function is continuous at these points:

x-value	$\lim\limits_{x \to a} f(x)$	$f(a)$	Continuous?
$x=1$	$\lim\limits_{x \to 1} f(x)=2$	$f(1)=2$	Yes
$x=2$	$\lim\limits_{x \to 2} f(x)=2$	$f(2)=1$	No
$x=3$	$\lim\limits_{x \to 3} f(x)$ does not exist (the left-hand and right-hand limits are not equal).	$f(3)=2$	No
$x=4$	$\lim\limits_{x \to 4} f(x)=2$	$f(4)$ is not defined.	No

From this table, we can conclude the following:

- A function $f(x)$ is continuous at $x=a$ if $\lim\limits_{x \to a} f(x)=f(a)$. That is, $\lim\limits_{x \to a} f(x)$ exists and is equal to the value of $f(a)$.
- A function $f(x)$ is not continuous at $x=a$ if any of the following occur:
  - $\lim\limits_{x \to a} f(x)$ does not exist.
  - $f(a)$ is undefined.
  - $\lim\limits_{x \to a} f(x)$ exists, but is not equal to $f(a)$.

TERM TO KNOW

Continuous Function

A function that has no breaks in the graph. That is, $\lim\limits_{x \to a} f(x)=f(a)$.

2. Determining if a Function Is Continuous at $x=a$

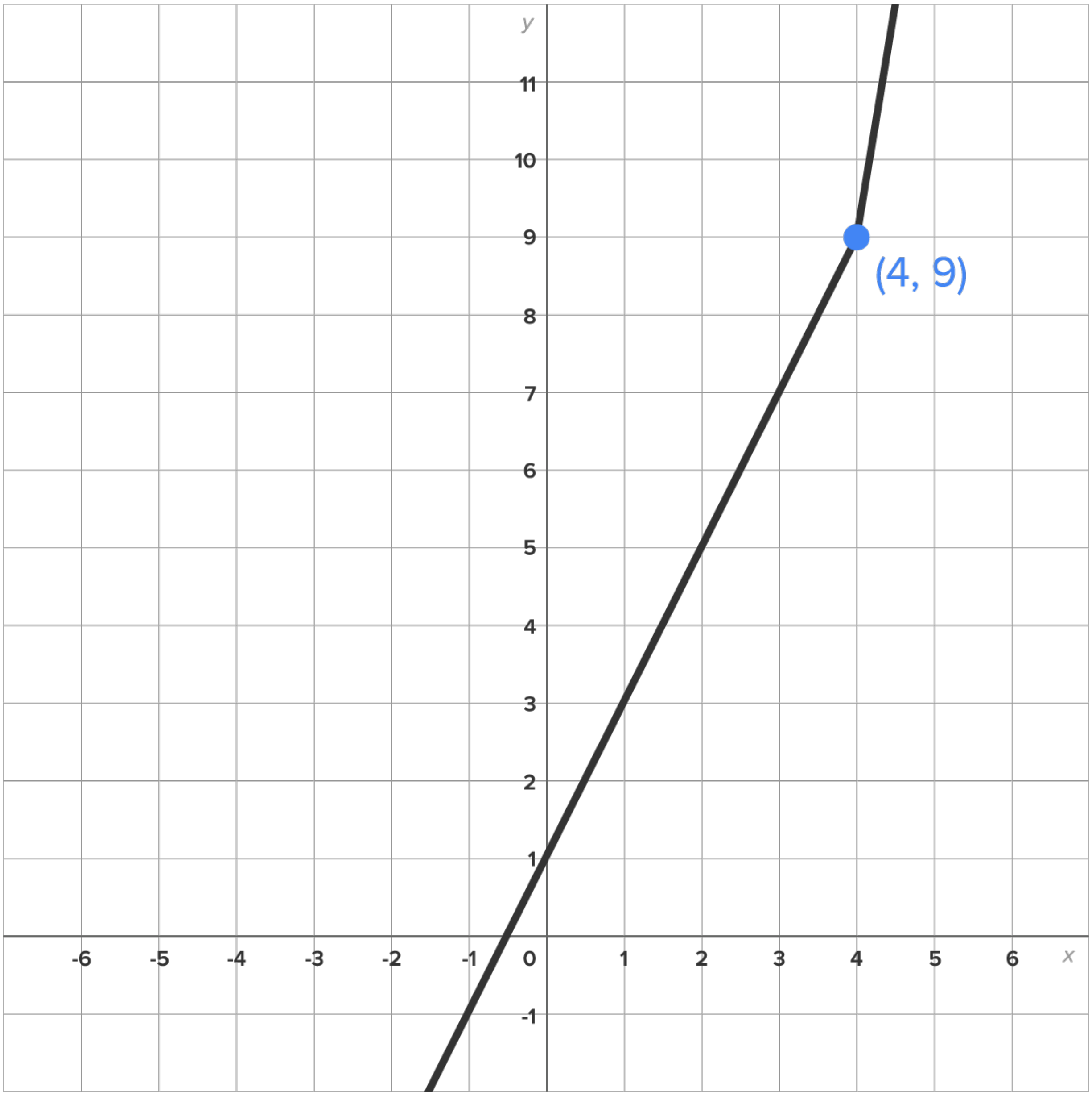
To determine if a function is continuous at $x=a$, we need to compare the values of $\lim\limits_{x \to a} f(x)$ and $f(a)$. While computing $f(a)$ is straightforward, computing $\lim\limits_{x \to a} f(x)$ requires more care, and sometimes requires one-sided limits.

↱ EXAMPLE Consider the function $f(x)=\begin{cases} 2x+1 & if\ x<4 \\ (x-1)^2 & if\ x\geq 4 \end{cases}$. Determine if $f(x)$ is continuous at $x=4$.

First, check to see if $\lim_{x\to 4} f(x)$ exists. Since $f(x)$ changes definition when $x=4$, we need to consider the one-sided limits:

- Left-sided limit: $\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}(2x+1)=2(4)+1=9$
- Right-sided limit: $\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}(x-1)^2=(4-1)^2=9$
- Conclusion: $\lim_{x\to 4} f(x)=9$, which means it exists and is equal to 9.

From looking at the function definition, $f(4)=(4-1)^2=9$. Thus, the limit and the function value are the same; therefore the function is continuous at $x=4$. Here is the graph of $f(x)$ to help visualize this:



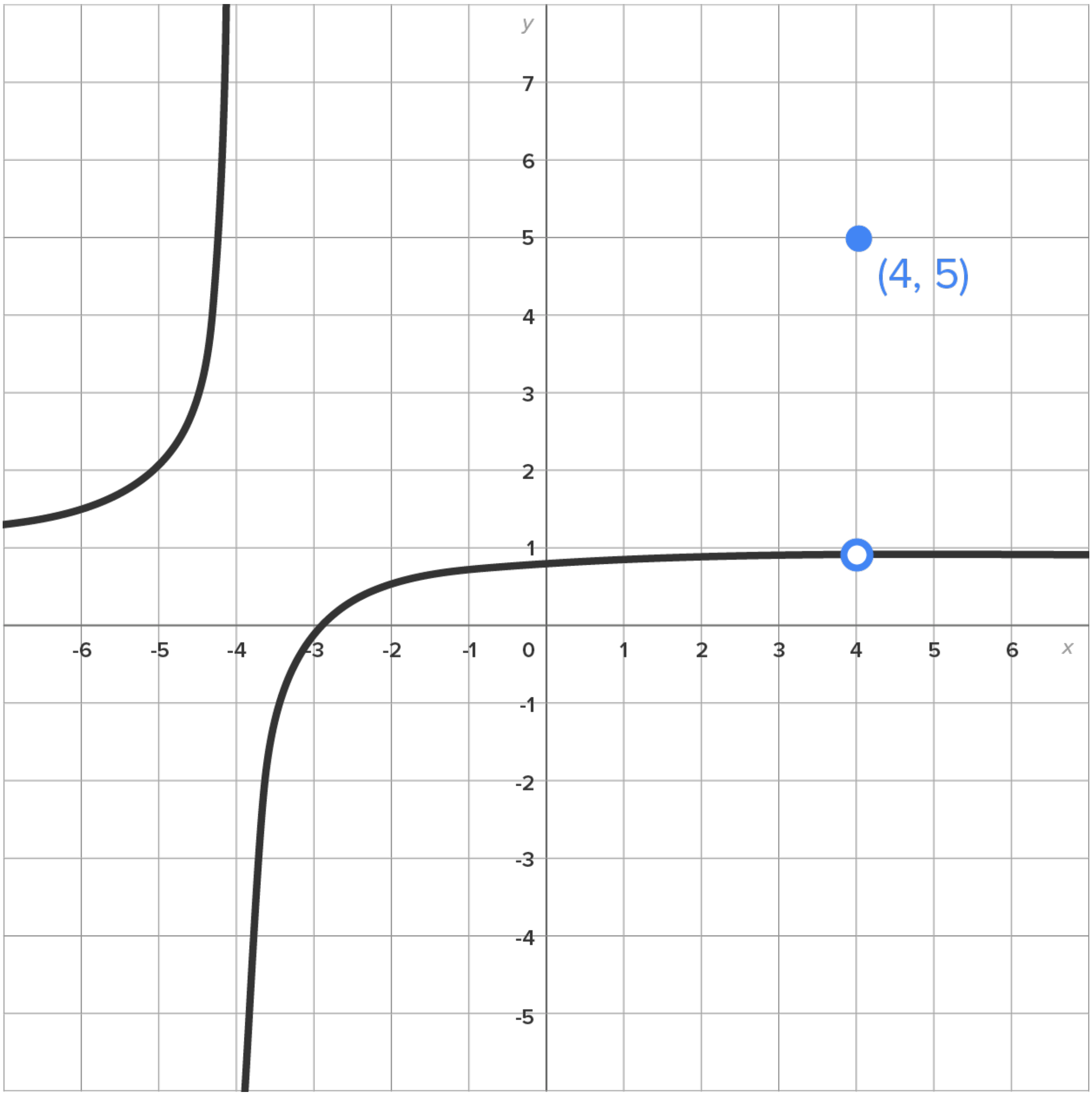
↱ EXAMPLE Consider the function $f(x)=\begin{cases} \dfrac{x^2-x-12}{x^2-16} & if\ x\neq 4 \\ 5 & if\ x=4 \end{cases}$. Determine if $f(x)$ is continuous at

$x = 4$.

First, evaluate $\lim\limits_{x \to 4} f(x)$. Since $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x^2 - 16} & \text{if } x \neq 4 \\ 5 & \text{if } x = 4 \end{cases}$ is defined on both sides of $x = 4$, there is no need to compute one-sided limits.

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x+3)}{(x+4)(x-4)} = \lim_{x \to 4} \frac{(x+3)}{(x+4)} = \frac{7}{8}$$

However, $f(4) = 5$. Since the limit and the function value are different, this function is not continuous at $x = 4$. Here is a graph to help visualize this:



TRY IT

Consider the function: $f(x) = \begin{cases} 3x + 4 & \text{if } x < 1 \\ \sqrt{x+8} & \text{if } x \geq 1 \end{cases}$

Determine if $f(x)$ is continuous when $x = 1$. +

The function is not continuous. $\lim_{x \to 1} f(x)$ does not exist.

---

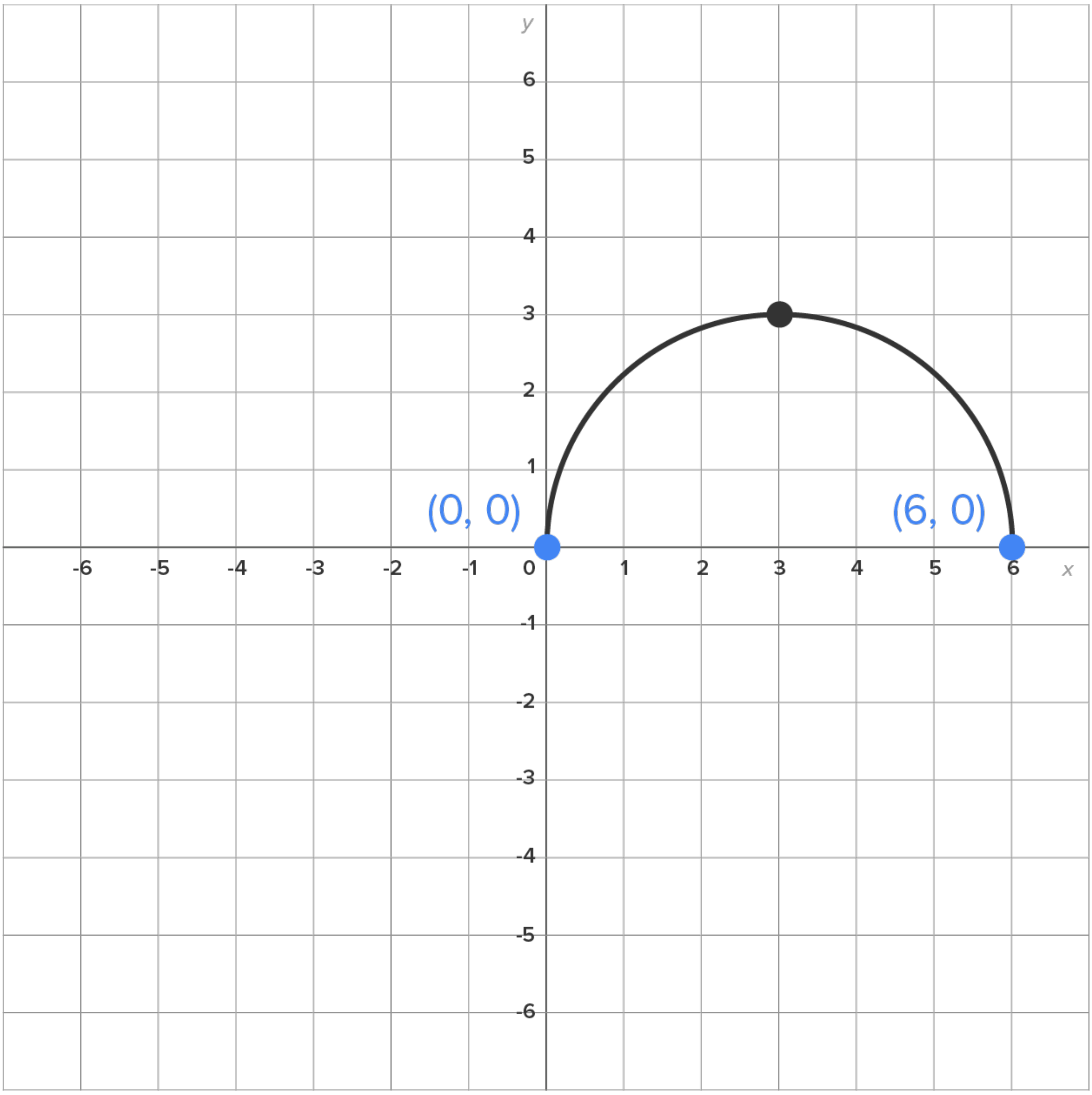
3. Determining Intervals Over Which a Function Is Continuous

For a function to be continuous on an interval of values, it has to be continuous at every point contained in the interval.

> ↪ EXAMPLE $f(x) = x^2 - 4x + 5$ is continuous at every real number. Thus, we say that $f(x)$ is continuous on the interval $(-\infty, \infty)$.

> ↪ EXAMPLE $f(x) = \frac{2}{x-1}$ is continuous at every value except $x = 1$. We can say that $f(x)$ is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. This can also be written as $(-\infty, 1) \cup (1, \infty)$.

It is also possible to define continuity at an endpoint. For example, consider $f(x) = \sqrt{6x - x^2}$, whose graph is shown below. Note that the domain of this function is $[0, 6]$.



This means that defining continuity at $x = 0$ and $x = 6$ takes a bit more care.

Consider the endpoint $x = 0$. It can only be approached from the right. Looking at the graph, observe that $\lim_{x \to 0^+} f(x) = 0$ and $f(0) = 0$.

Consider the endpoint $x = 6$. It can only be approached from the left. Looking at the graph, observe that $\lim_{x \to 6^-} f(x) = 0$ and $f(6) = 0$.

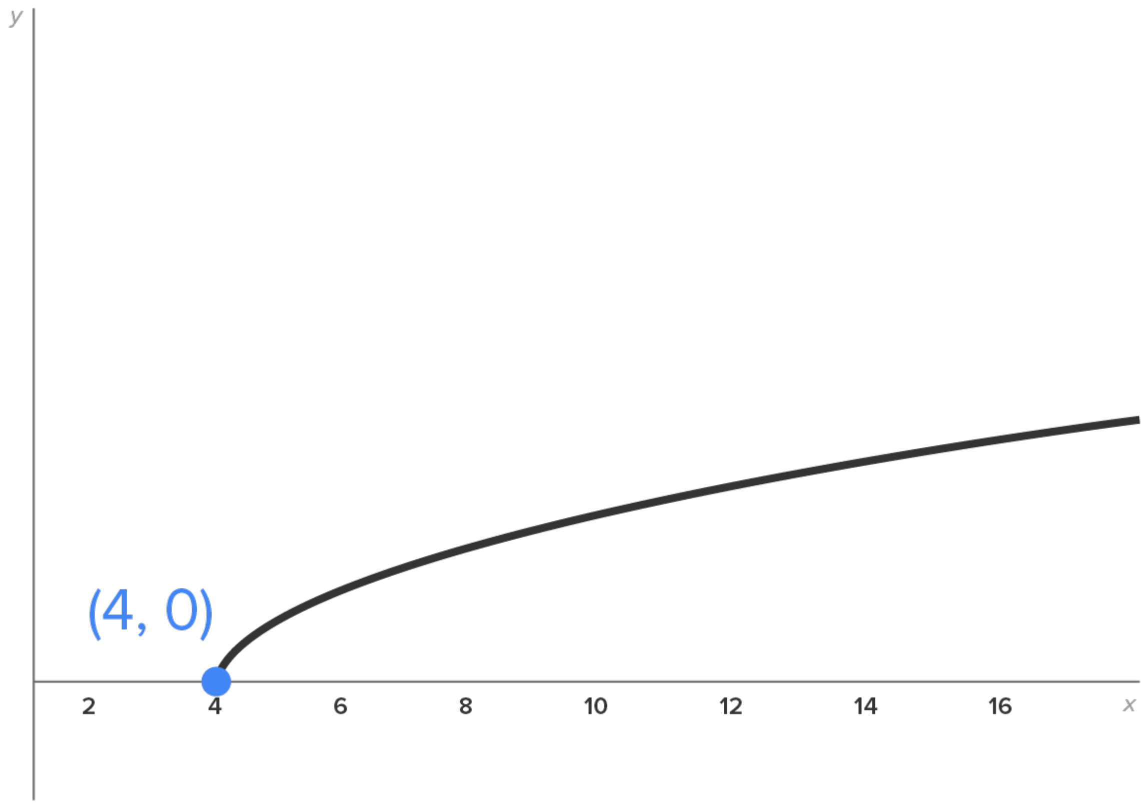
BIG IDEA

A function is **continuous from the left** at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

A function is **continuous from the right** at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Thus, in the previous problem, we can say that $f(x)$ is continuous from the left at $x = 6$ and continuous from the right at $x = 0$. This enables us to say that $f(x)$ is continuous for all values on the interval $[0, 6]$.

↪ EXAMPLE Determine the interval(s) over which $f(x)=\sqrt{x-4}$ is continuous. The graph is shown below.



Note that the domain of $f(x)$ is $[4, \infty)$. It follows that $f(x)$ is continuous on the interval $[4, \infty)$, noting that it is continuous from the right at $x=4$.

TRY IT

Consider the following table:

Function	Continuous Interval
$f(x)=3x-x^4$	?
$g(x)=\frac{x}{x+4}$	?
$h(x)=\sqrt{2x-1}$	?

Determine the interval(s) over which each function is continuous. +

Function	Continuous Interval
$f(x)=3x-x^4$	$(-\infty, \infty)$
$g(x)=\frac{x}{x+4}$	$(-\infty, -4)\cup(-4, \infty)$

$h(x)=\sqrt{2x-1}$	$\left[\frac{1}{2},\infty\right)$

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^-} f(x)=f(a)$.

Continuous From the Right

A function is continuous from the right at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$.

SUMMARY

In this lesson, you learned **the definition of continuity**, understanding that when given a graph, continuity is determined by locations where the graph has no breaks, jumps, or holes. A continuous function has no breaks in the graph; that is, $\lim\limits_{x\to a} f(x)=f(a)$. You learned that you can use limits to **determine if a function is continuous at** $x=a$ (a specific point) by comparing the values of $\lim\limits_{x\to a} f(x)$ and $f(a)$. It's important to note that while computing $f(a)$ is straightforward, computing $\lim\limits_{x\to a} f(x)$ requires more care, and sometimes requires one-sided limits. Lastly, you learned that by examining the domain of a function, you can use it to **determine the intervals over which a function is continuous**, noting that the function has to be continuous at every point contained in the interval in order to say the function is continuous on the interval.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^-} f(x)=f(a)$.

Continuous From the Right

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$.

Continuous Function

A function that has no breaks in the graph. That is, $\lim\limits_{x\to a} f(x)=f(a)$.

Which Functions are Continuous?

by Sophia

WHAT'S COVERED

In this lesson, you will explore how continuity can be applied to combinations of functions. Specifically, this lesson will cover:

1. Functions That Are Continuous for All Real Numbers
2. Rational Functions
3. Radical Functions
4. Combinations of Functions

1. Functions That Are Continuous for All Real Numbers

To gain an understanding of continuity, let's recall some non-piecewise functions that contain no breaks or holes in their graphs (i.e., no points where the function is undefined). In other words, these functions are continuous for every real number.

- Polynomial functions
- Sine and cosine functions
- Absolute value functions

Moreover, given that $f(x)$ and $g(x)$ are continuous at $x = a$, it follows that these functions are also continuous for $x = a$.

Function	Operations
$f \pm g$	Sum/difference
$f \cdot g$	Product
$\frac{f}{g}$	Quotient, provided $g(a) \neq 0$
$[f(x)]^n$, $n = 0, 1, 2, \ldots$	Raise $f(x)$ to a whole number power
$f(g(x))$	Composition of f and g

It follows that any of these combinations of polynomial, sine, cosine, and absolute value functions are continuous for all real numbers (provided that we do not create a denominator that could be zero).

The following functions are continuous for all real numbers.

Functions	Combinations
$f(x) = -2x^3 + 12x^2 + 5x - 8$	Polynomial
$g(x) = \sin^2 x - 2\cos x$	Powers of sine and cosine
$h(x) = x^3 \sin 2x$	Product of polynomial and sine, with composition
$j(x) = \dfrac{2x}{x^2+4}$	Quotient of two functions, but no value of x will make the denominator equal to 0

2. Rational Functions

Let's explore the quotient of two functions a bit more. We have to be careful though, since a function is undefined when its denominator is equal to 0.

A **rational function** has the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

↪ EXAMPLE $f(x) = \dfrac{2x}{x-1}$ is continuous for every real number except where $x - 1 = 0$, which means $x = 1$. The intervals over which $f(x)$ is continuous are $(-\infty, 1) \cup (1, \infty)$.

↪ EXAMPLE $g(x) = \dfrac{2x^2-3}{x^2-5x+6}$ is continuous for every real number except where $x^2 - 5x + 6 = 0$.

Writing in factored form, we have $(x-2)(x-3) = 0$, which has solutions $x = 2$ and $x = 3$. Thus, $g(x)$ is continuous for all real numbers except 2 and 3.

Using interval notation, $g(x)$ is continuous on the intervals $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

TERM TO KNOW

Rational Function

A function in the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

3. Radical Functions

Recall that the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$ when n is even (even root), and $(-\infty, \infty)$ when n is odd (odd root).

It follows that $f(x) = \sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd).

↪ EXAMPLE $f(x) = \sqrt[3]{x-1}$ is continuous for all real numbers.

↪ EXAMPLE $g(x)=\sqrt{x-4}$ is continuous when $x-4\geq 0$, which means the interval $[4, \infty)$.

4. Combinations of Functions

Consider the two functions $f(x)$ and $g(x)$. Given that we know where $f(x)$ and $g(x)$ are continuous, the following can be said for some combinations of $f(x)$ and $g(x)$:

- The functions $f(x)+g(x)$, $f(x)-g(x)$, and $f(x)\cdot g(x)$ are all continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous.
- The function $\frac{f(x)}{g(x)}$ is continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous, with the added condition that $g(x)\neq 0$.

↪ EXAMPLE Consider the function $f(x)=x^3\sqrt{x-4}$, which is a product of $y=x^3$ and $y=\sqrt{x-4}$.

- $y=x^3$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y=\sqrt{x-4}$ is continuous where $x-4\geq 0$, which is the interval $[4, \infty)$.

Therefore, $f(x)=x^3\sqrt{x-4}$ is continuous on the interval $[4, \infty)$.

↪ EXAMPLE Consider the function $f(x)=\frac{\cos x}{x-5}$, which is the quotient of $y=\cos x$ and $y=x-5$.

- $y=\cos x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y=x-5$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- Since the denominator is $x-5$, $f(x)$ is not continuous when $x-5=0$, which means $x=5$.

Therefore, $f(x)$ is continuous for all real numbers except $x=5$, which means $(-\infty, 5)\cup(5, \infty)$ in interval notation.

↪ EXAMPLE Consider the function $f(x)=\frac{2x}{\sqrt{x-3}}$, which is the quotient of $y=2x$ and $y=\sqrt{x-3}$.

- $y=2x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y=\sqrt{x-3}$ is continuous where $x-3\geq 0$, which is the interval $[3, \infty)$.
- Since the denominator is $\sqrt{x-3}$, $f(x)$ is also not continuous when $x-3=0$, which means $x=3$.

Therefore, $f(x)$ is continuous when $x>3$, which means the interval $(3, \infty)$.

TRY IT

Consider the function $f(x)=3x+\sqrt{2x-5}$.

Determine the interval(s) over which this function is continuous. +

$\left[\frac{5}{2}, \infty\right)$

WATCH

The following video goes through the examples determining the intervals of continuity for the functions $f(x)=4x^3\sqrt{x+15}$, $f(x)=\frac{x^2-16}{x+4}$, and $f(x)=\frac{9\sin x}{\sqrt{x-8}}$.

Video Transcription

[MUSIC PLAYING] Hey there. Welcome to today's video on using the properties of combinations of continuous functions to determine the intervals over which the given function is continuous. So here, I have three different examples to give you a little bit more insight into the process. Now, in part a, I have the function f of x is equal to 4x cubed times the square root of x plus 15.

Now, this function, f of x, is made up of two functions multiplied together. So I have the function g of x, which is my 4x cubed, and h of x, which is the square root of x plus 15. Now, these two functions are multiplied together, as I said, so we want to look at the interval for which each of these are individually continuous. And then the product of those two will be continuous over those values of x that meet both conditions.

So here, for g of x, it's equal to 4x cubed, which is a polynomial. And this polynomial-- polynomials in general-- are continuous over all reals. So that's the interval negative infinity to infinity.

Now, h of x is equal to the square root of x plus 15. Remember, if there's no little number in the crook, the index is assumed to be a 2 for square root. If there is a number in there, if it's an odd number in there, it would be continuous over all reals, but if it's an even number as an index, then it has to be that you have your even index root of a non-negative number. So whatever they're taking the square root of, in this case, has to be greater than or equal to 0.

So for h of x, this x plus 15 has to be greater than or equal to 0. And then subtracting 15 from both sides, I get x is greater than or equal to negative 15. The interval notation for that includes the negative 15, so we're going to have the bracket. And the x's that are greater than or equal to that are to the right of it, so off to infinity.

Now, the function f of x is continuous on the interval that meets both of those conditions. So it has to be the strictest of those. And that would be the interval negative 15 off to infinity. And remember, we always use a parenthese at the infinities for the numerical values if it's "or equal to," to include the endpoint. We use a bracket if it doesn't have that "or equal to." We use a parenthese.

Let's go on to part b. So for part b, the function f of x is made up of a quotient of two functions. It's the quotient of g of x is equal to x squared minus 16, and h of x, which is equal to x plus 4. Now, remember, for quotients, you have to make sure that you remember that the function is continuous for the restrictions of the two. Also, you have to make sure that the denominator is not equal to 0.

So while g of x equal to a polynomial-- x squared minus 16 is a polynomial-- is continuous through all real numbers, so negative infinity to infinity, h of x equal x plus 4, on its own, is also continuous from negative infinity to infinity. But because that is in the denominator for my function f of x is continuous for-- both of those are negative infinity to infinity, but I can't allow the denominator to be 0.

So x plus 4 can't be 0. That means x can't be negative 4. So when I write that, f of x is continuous on the interval negative infinity leading up to but not including negative 4. And we do that with a parenthese. Union, pick up on the other side. So parenthese, negative 4, and then off to infinity.

Part c. So for part c, I have a little bit of each of those cases. I have the square root of x minus 8, but that's also sitting in the denominator. So I also need to make sure my denominator is not 0. This function is made up of two different functions, g of x is equal to 9 sine x and h of x is equal to the square root of x minus 8.

Sine x is continuous throughout all real numbers, and multiplying that by 9 doesn't affect that situation of being continuous, so I have negative infinity to infinity for the interval that g of x is continuous. h of x has that same sort of situation that part had with the h of x.

It's a square root, so what's under the square root has to stay greater than or equal to 0. My x minus 8 has to be greater than or equal to 0, so my x has to be greater than or equal to 8 for h of x. Now, the x's that are greater are to the right of it. So we have our bracket at 8 off to infinity for our continuity for h of x.

Now, for f of x is continuous not only on the strictest of these cases, but also, I can't let the denominator be equal to 0. So I need to make sure that that x minus 8 is not 0. So x cannot be 8. That means, when I look at this interval bracket, 8 off to infinity, that I had for the strictest of the cases, but it is also the one in the denominator, I cannot include 8 either.

So instead of a squared bracket for the interval for f of x being continuous, it's going to be a parenthese at 8 and then off to infinity. So there, you have three more examples of how you use the properties of combinations of continuous functions to determine the intervals over which the given function is continuous. Hope this helped, and I'll see you in a future video.

TRY IT

Consider the following table:

Function	Continuous Interval
$f(x)=3x\cos(x^2)$	?
$g(x)=\dfrac{x^3-8}{x-2}$	?
$h(x)=\sqrt[5]{x^3-3x^2+10x-4}$	?

Determine the intervals over which each function is continuous. +

Function	Continuous Interval
$f(x)=3x\cos(x^2)$	All real numbers or $(-\infty,\infty)$

$g(x)=\frac{x^3-8}{x-2}$	All real numbers except 2 or $(-\infty, 2)\cup(2, \infty)$
$h(x)=\sqrt[5]{x^3-3x^2+10x-4}$	All real numbers or $(-\infty, \infty)$

SUMMARY

In this lesson, you learned about some non-piecewise **functions that are continuous for all real numbers**, including polynomial functions, sine and cosine functions, and absolute value functions. You explored the quotient of two functions, noting that you have to be careful since a function is undefined when its denominator is equal to 0. You learned about **rational functions**, which have the form $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials; a rational function is continuous at all real numbers except for those where $D(x)=0$. You also learned about **radical functions**, understanding that $f(x)=\sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd). Lastly, you learned that for **combinations of functions**, take special care with radical and rational functions. If $f(x)$ has no values where it is undefined, then $f(x)$ is continuous for all real numbers.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Rational Function

A function in the form $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x)=0$.

Intermediate Value Theorem

by Sophia

WHAT'S COVERED

In this lesson, you will analyze functions using the intermediate value theorem. Specifically, this lesson will cover:

1. The Intermediate Value Theorem
2. Real-World Applications

1. The Intermediate Value Theorem

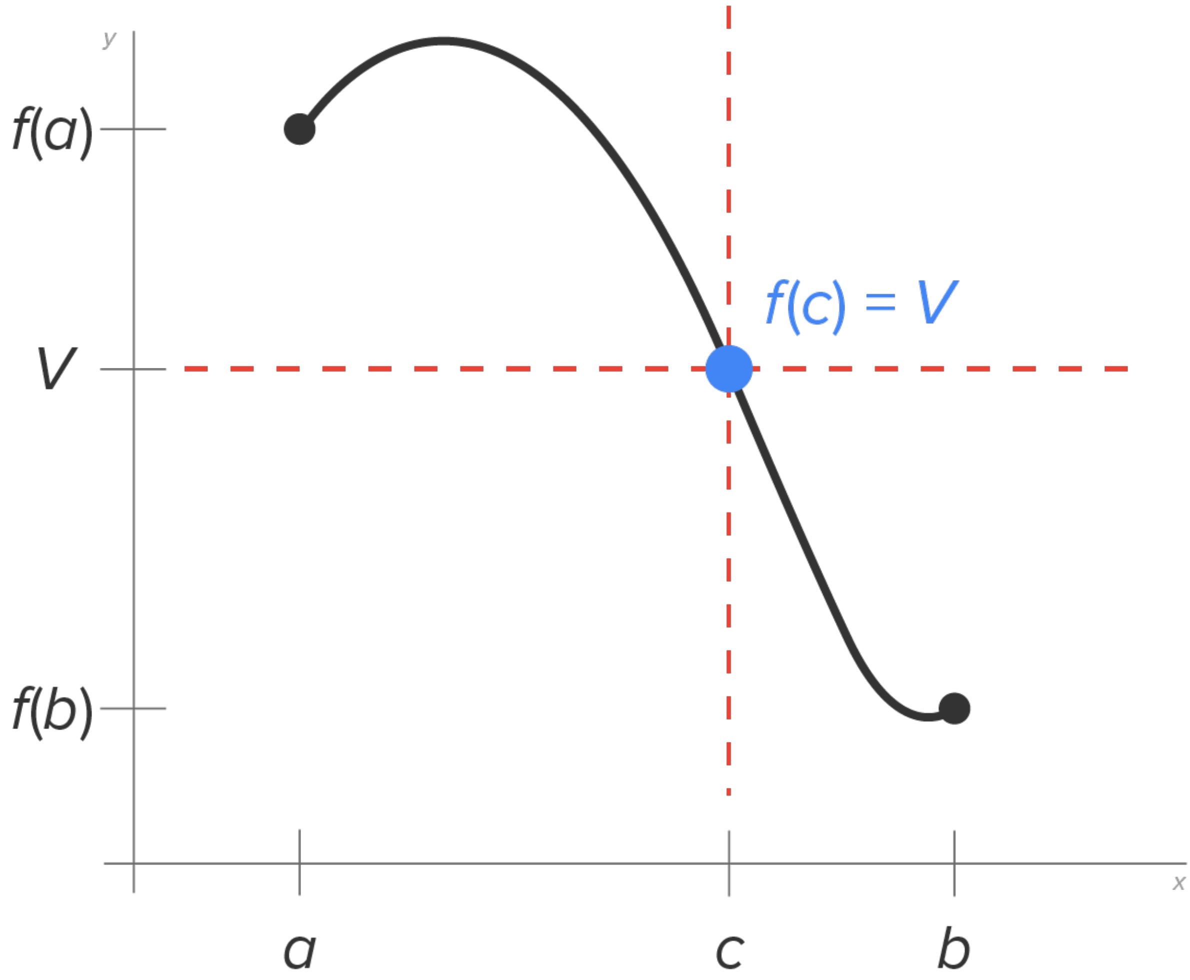
Suppose at 7 AM, you walk outside and it is 40°F. Then, at 11 AM, the temperature is 60°F. We know at some point between 7 AM and 11 AM, the temperature had to be 50°F. Why?

This is because temperature doesn't "jump" from one level to the next, meaning that the temperature is a continuous function of time.

Another way to visualize this:

1. Graph the points (7, 40) and (11, 60).
2. Connect the points with any continuous curve. Be creative.
3. Does your curve have a point where $y = 50$ between $x = 7$ and $x = 11$? The answer should be yes. Otherwise, your graph is not continuous.

This idea is generalized by the intermediate value theorem.
For the **intermediate value theorem (IVT)**, suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.



↱ EXAMPLE Consider the continuous function $f(x)=x^2+1$ on the closed interval $[1, 4]$. Note that $f(1)=1^2+1=2$ and $f(4)=4^2+1=17$.

Choose a value between 2 and 17, say, the value 8. By the IVT, this means that there is at least one value of c between 1 and 4 such that $f(c)=8$. Let's find this value.

Since we want $f(c)=8$, this means $c^2+1=8$, which means $c^2=7$, or $c=\pm\sqrt{7}$. Since $\sqrt{7}$ is between 1 and 4, this illustrates the existence of the value of c in the theorem.

Note that $-\sqrt{7}$ is not in the interval $[1, 4]$, so this value is not considered.

WATCH

An example of the IVT for the function $f(x)=x^2-7x$ on $[-3, 1]$ is presented in this video.

Video Transcription

[MUSIC PLAYING] Hi there. Welcome to today's video and the intermediate value theorem. Now, first I want to state the intermediate value theorem so it's accessible right there to use it as we're going through the example. So the intermediate value theorem says that, suppose you have a function f of x that is continuous on a closed interval bracket, a, comma, b, bracket, meaning that the end points are included.

Let capital V be a value between f of a and f of b. Then there is at least one c between the end points of the interval a and b such that f of c will equal V. So one of the things that's really important for you to remember here is that that value V is an output. It's an output between the outputs of the function at the left endpoint and the right endpoint. And your value c is an input, and it's an input that has to be between the endpoints of your interval a to b.

So here's our example. Let f of x equal x squared minus 7x on the closed interval negative 3 to 1, so f of x equaling x squared minus 7x. Well, that's a polynomial, and polynomials are continuous over the whole real number line. So definitely, it's continuous over the closed interval. So we have that aspect of the intermediate value theorem met.

Now, on the closed interval negative 3 to 1-- so squared brackets. It's a closed interval. So I also have our closed interval. I still need to see the rest of the question and see if we have the rest of the intermediate value theorem met. So it says, is there a solution to f of x equaling 18?

So our output is 18. That's corresponding to this V-value. In the interval between negative 3 and 1, same numbers that we had in the closed interval up here. And if so, find that value of c in the interval such that f of c is equal to our 18.

Well, the last part of the intermediate value theorem that we have to verify to see if we are guaranteed a solution is that this value, V, our 18, has to be between f of a and f of b. So we need to calculate those. So here, our function, f of x, is x squared minus 7x.

So f of something is equal to something squared minus 7 times something. And we want to start with that something being negative 3, the left endpoint of the interval. And when we go through and calculate that, we get that f of negative 3 is 9 plus 21, or 30.

Now, we want to calculate our f of 1. So again, we'll do f of something is equal to the something squared minus 7 times something. We are going to take out the x's and put in our something, which is our value of 1. So f of 1 is equal to 1 squared minus 7 times 1.

And we get for this that f of 1 is 1 minus 7 is negative 6. And remember, our value V has to be between those. And 18 is between 30 and negative 6. So yes, we are guaranteed at least one c-- one value in the interval that is a solution to f of x is equal to 18.

Next up, we want to find that value, because we've verified that we're guaranteed one. So we want to follow what this says such that f of c is equal to V. And remember, our V is 18. And our function is f of x is equal to x squared minus 7x. f of c means just take out the x's and put in c's instead. So instead of writing x squared minus 7x, I'm going to write c squared minus 7c is equal to our output of 18.

Now, that's quadratic in c. So to solve a quadratic equation, we want to have a 0 on the right-hand side. We have c squared minus 7c minus 18 is equal to 0. We look to see if that's factorable. And if not, we could use the quadratic formula. But this does factor. The factors of negative 18 that add to get negative 7 are negative 9 and positive 2.

So we have c minus 9 times c plus 2 is our factorization. That gives us c minus 9 is equal to 0, or c plus 2 is equal to 0. Solving for c, I get c is equal to 9 or c is equal to negative 2. But remember, the c's that we are interested in are in our interval. Our c has to be in the interval between my negative 3 and 1.

And when I compare those, 9 is not in that span of values, but negative 2 is. So yes, we are guaranteed a solution to f of c is equal to 18. And our solution is actually that c is equal to negative 2. Well, I hope that's been helpful in being able to get a little bit of a better handle on the intermediate value theorem. See you next time.

TERM TO KNOW

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.

---

2. Real-World Applications

Here is an example of a real-world application in which the IVT can be useful.

EXAMPLE Suppose a design requires a spherical shape with volume 200 in^3, but the radius of the sphere is to be between 3 and 4 inches. Is it possible to meet these requirements?

First, identify the function, which is the volume of a sphere: $V(r) = \frac{4}{3}\pi r^3$. This problem translates to: Is $V(r) = 200$ for some value in the interval $[3, 4]$?

Since this is a polynomial function, we know $V(r)$ is continuous. Now, evaluate $V(r)$ at the endpoints:

- $V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \approx 113.1 \text{ in}^3$
- $V(4) = \frac{4}{3}\pi(4)^3 = \frac{256}{3}\pi \approx 268.1 \text{ in}^3$

By the IVT, there is a value of r between 3 and 4 inches that produces a volume of 200 in^3.

One particularly useful application of the IVT is locating x-intercepts. Here is the important point:

BIG IDEA

If $f(a)$ and $f(b)$ have different signs (one is positive and one is negative), then there is a value of c in the interval (a, b) such that $f(c) = 0$.

EXAMPLE Let $f(x) = x - \cos x$. Show that there is an x-intercept on the interval $[0, 1]$.

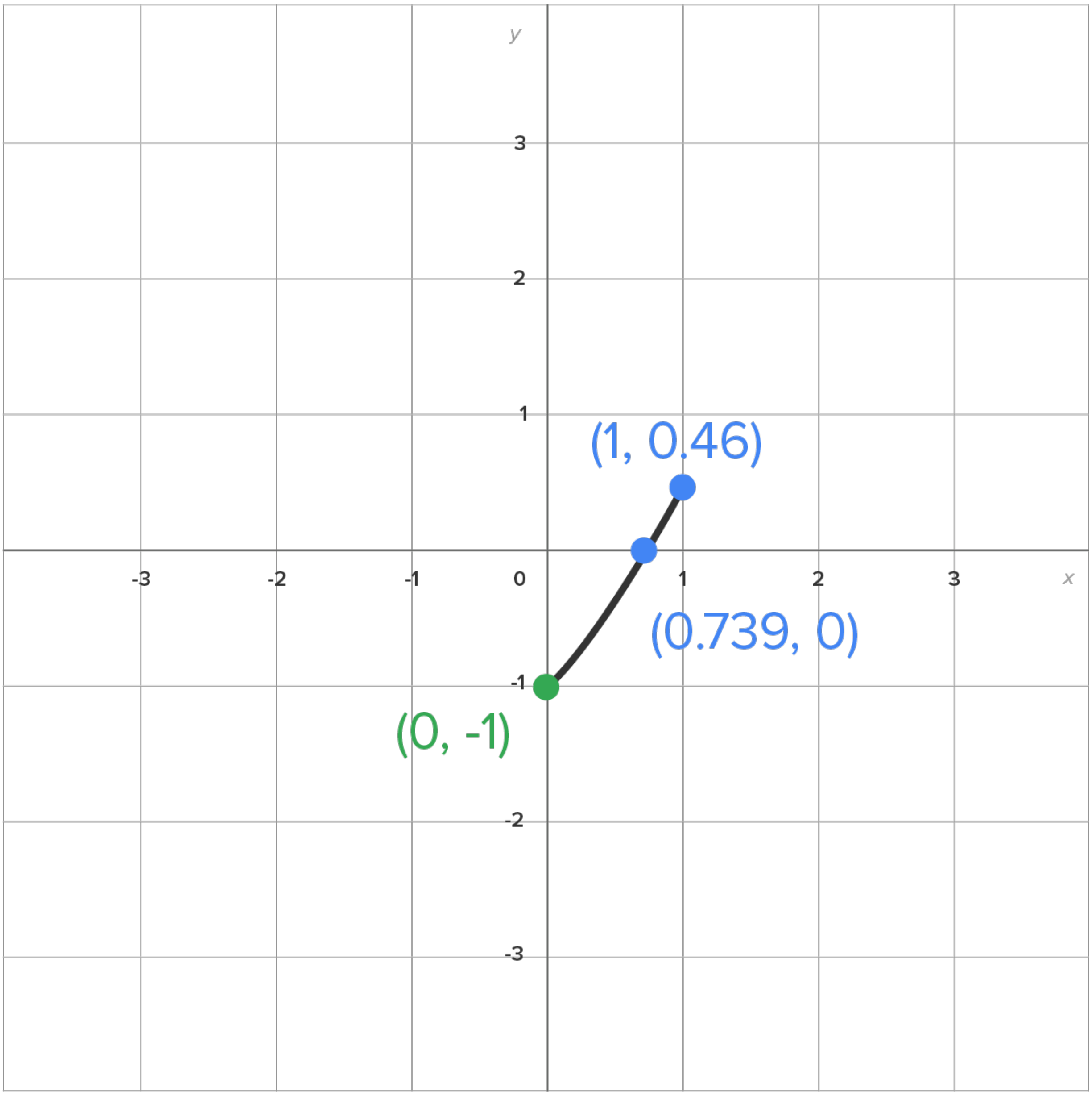
First, note that $f(x)$ is continuous. Next, evaluate the function at the endpoints:

- $f(0) = 0 - \cos 0 = -1$
- $f(1) = 1 - \cos 1 \approx 0.46$

Since $f(0)$ and $f(1)$ have opposite signs, it follows from the IVT that there is a value of x in the interval

$[0, 1]$ such that $f(x) = 0$.

Here is a graph to help illustrate. As you can see, the x-intercept occurs when $x \approx 0.739$, which is inside the interval $[0, 1]$.



TRY IT

Let $f(x) = x - 5\sqrt{x}$.

Use the IVT to determine if there is a guaranteed value of *x* for which *f* (*x*) = 20 on the interval [36, 100].

+

Since $f(x)$ is continuous on $[36, 100]$ with $f(36) = 6$ and $f(100) = 50$, there must be a value of *x* for which $f(x) = 20$ on the interval $[36, 100]$.

TRY IT

Let $f(x) = x - e^{-2x}$.

Use the IVT to determine if this function is guaranteed an x-intercept on the closed interval [0, 2]. +

Since $f(x)$ is continuous on $[0, 2]$ with $f(0) = -1$ and $f(2) \approx 1.98$, there must be a value of *x* for which $f(x) = 0$ on the interval $[0, 2]$.

SUMMARY

In this lesson, you learned about **the intermediate value theorem** (IVT), which is very useful in determining if an input is guaranteed in an interval (a, b) for which the output is *V* when you have a continuous function on a closed interval $[a, b]$. Specifically, the IVT states that if you have a continuous function on a closed interval $[a, b]$, and if *V* is between $f(a)$ and $f(b)$, you are guaranteed at least one input, *c*, in the interval $[a, b]$ for which $f(c) = V$.

You also learned about several useful **real-world applications** of the IVT, such as determining if x-intercepts exist on a closed interval. It is important to remember that if $f(a)$ and $f(b)$ have different signs (one is positive and one is negative), then there is a value of *c* in the interval (a, b) such that $f(c) = 0$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let *V* be a value between $f(a)$ and $f(b)$. Then, there is at least one value of *c* between *a* and *b* such that $f(c) = V$.

The Intuitive Approach

by Sophia

WHAT'S COVERED

In this lesson, you will demonstrate the definition of a limit by finding the value of δ that corresponds to a given ε for a specific limit. Specifically, this lesson will cover:

1. Finding the Value of δ That Corresponds to a Given Value of ε for a Linear Function
2. Finding the Value of δ That Corresponds to a Given Value of ε for a Nonlinear Function

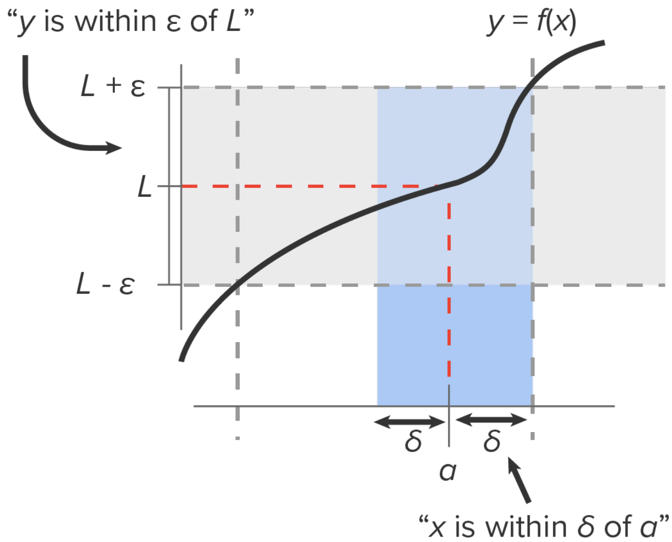
1. Finding the Value of δ That Corresponds to a Given Value of ε for a Linear Function

Recall a general limit statement: $\lim_{x \to a} f(x) = L$

Based on methods we talked about in this course so far, the general idea is that the value of $f(x)$ gets closer to L as x gets closer to a.

We now take a more analytical approach to establishing limits. Consult the figure on the right:

- The symbol ε is the Greek letter epsilon.
- The symbol δ is the Greek letter delta.



The idea illustrated here is that if the value of $f(x)$ is within ε units of the limit L, then there is a corresponding value of δ such that x is within δ units of a. Written as distances, we have the following:

- $f(x)$ is within ε units of the limit L: $|f(x) - L| < \varepsilon$
- x is within δ units of a: $|x - a| < \delta$

These ideas are used to establish the **Formal Definition of a Limit**, which states:
$\lim_{x \to a} f(x) = L$ means that for every given ε > 0, there exists δ > 0 so that:

- If x is within δ units of a (and $x \neq a$), then $f(x)$ is within ε units of L.

- This translates to $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

The goal in this part of the challenge will be to find the value of δ for a given value of ε.

HINT

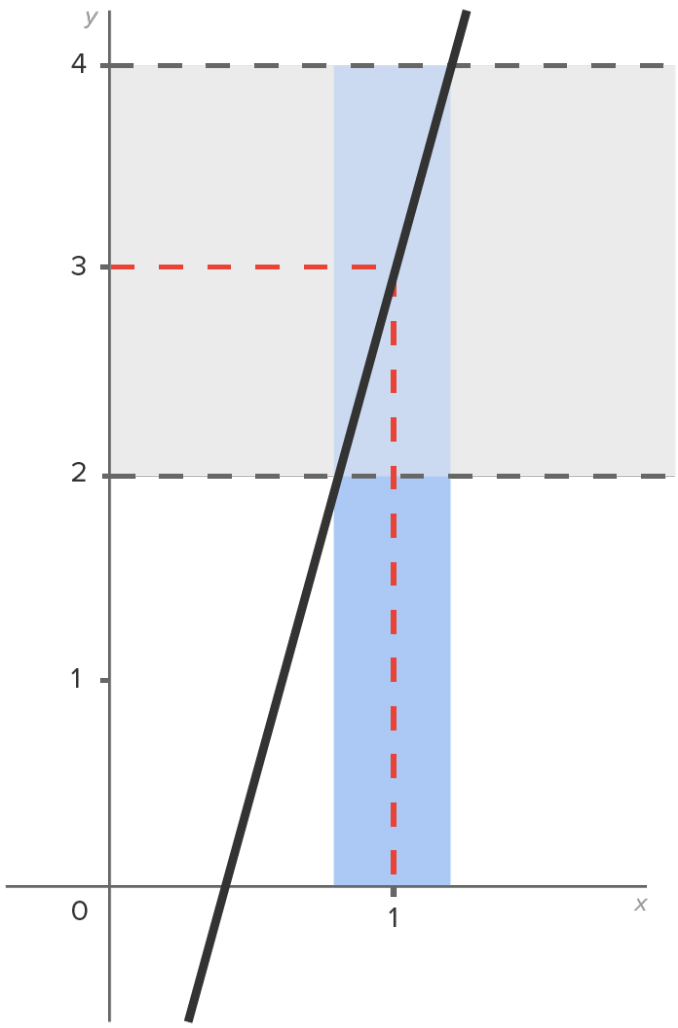
You may recall from algebra that $|x|<a$ is equivalent to saying $-a<x<a$ for any positive number *a*.

This means that $|f(x)-L|<\varepsilon$ can be rewritten $-\varepsilon<f(x)-L<\varepsilon$ and $|x-a|<\delta$ can be rewritten as $-\delta<x-a<\delta$.

These ideas are useful in determining the value of δ for a given ε.

EXAMPLE Consider the limit statement: $\lim_{x\to 1}(5x-2)=3$. What value of δ is required when $\varepsilon=1$?

Consider the picture shown below (the slanted line is the graph of $f(x)=5x-2$):



Remember that $\varepsilon=1$ means that we desire $f(x)$ to be within 1 unit of 3 (the limit). This means $|f(x)-3|<1$. Let's solve this:

$|5x-2-3|<1$ Replace $f(x)$ with $5x-2$.

$\lvert 5x-5\rvert<1$	Simplify the expression.
$-1<5x-5<1$	$\lvert x\rvert<a$ means $-a<x<a$.
$4<5x<6$	Add 5 to all three parts.
$0.8<x<1.2$	Divide all three parts by 5.

Thus, $\lvert f(x)-3\rvert<1$ implies that $0.8<x<1.2$.

So, what is the value of δ?

Recall that the goal is to find δ so that $\lvert x-a\rvert<\delta$. In this problem, $a=1$, so this can be written as $\lvert x-1\rvert<\delta$.

Recall from algebra that this means $-\delta<x-1<\delta$. Thus, it helps to get an inequality with $x-1$ in the middle. Then the left and right parts of the inequality give information as to what δ is.

We left off with $0.8<x<1.2$. To get $x-1$ in the middle, subtract 1 from all parts of the inequality. This gives $-0.2<x-1<0.2$. Thus, $\delta=0.2$.

In summary, we state the following: If *x* is within 0.2 units of 1, then $f(x)$ is within 1 unit of 3.

While a graph is helpful, let's try one now without the graph.

↱ EXAMPLE Consider the limit statement: $\lim_{x\to 3}(4x-5)=7$. Find the corresponding values of δ when ε = 0.5, 0.1, and 0.01.

For ε = 0.5, this means we want $\lvert 4x-5-7\rvert<0.5$. Now solve:

$\lvert 4x-12\rvert<0.5$	Simplify.
$-0.5<4x-12<0.5$	$\lvert x\rvert<a$ means $-a<x<a$.
$11.5<4x<12.5$	Add 12 to all three parts.
$2.875<x<3.125$	Divide all three parts by 4.
$-0.125<x-3<0.125$	Subtract 3 from all three parts to get $x-3$ in the middle.

Thus, δ = 0.125.

For ε = 0.1, this means we want $\lvert 4x-5-7\rvert<0.1$. Now solve:

$\lvert 4x-12\rvert<0.1$	Simplify.
$-0.1<4x-12<0.1$	$\lvert x\rvert<a$ means $-a<x<a$.
$11.9<4x<12.1$	Add 12 to all three parts.
$2.975<x<3.025$	Divide all three parts by 4.
$-0.025<x-3<0.025$	Subtract 3 from all three parts to get $x-3$ in the middle.

Thus, δ = 0.025.

For ε = 0.01, this means we want $|4x-5-7|<0.01$. Now solve:

$\|4x-12\|<0.01$	Simplify.
$-0.01<4x-12<0.01$	$\|x\|<a$ means $-a<x<a$.
$11.99<4x<12.01$	Add 12 to all three parts.
$2.9975<x<3.0025$	Divide all three parts by 4.
$-0.0025<x-3<0.0025$	Subtract 3 from all three parts to get $x-3$ in the middle.

Thus, δ = 0.0025.

HINT

Note that as the value of ε gets smaller, so does δ. This is the essence of a limit. As one distance gets smaller, the other does as well.

BIG IDEA

As the chosen values of ε get closer to 0, the corresponding value of δ also gets closer to 0.

When $f(x)$ is a linear function, finding the value of δ is fairly straightforward since the final inequality always has the form $-\delta<x-a<\delta$.

When $f(x)$ is a nonlinear function, this may not be the case, which means we have to think more critically to get the appropriate value of δ.

TERM TO KNOW

Formal Definition of a Limit

$\lim_{x\to a} f(x)=L$ means that for every given ε > 0, there exists δ > 0 so that:

- If *x* is within δ units of *a* (and $x\neq a$), then $f(x)$ is within ε units of *L*.
- This translates to $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

---

2. Finding the Value of δ That Corresponds to a Given Value of ε for a Nonlinear Function

The following are inequalities that may be useful. In each case, assume that *c* and *d* are nonnegative numbers.

- If $c<-x<d$, then $-d<x<-c$.
- If $c<x^2<d$, then $\sqrt{c}<x<\sqrt{d}$ (assuming *x* is positive).
- If $c<\sqrt{x}<d$, then $c^2<x<d^2$.

- If $c < \frac{1}{x} < d$, then $\frac{1}{d} < x < \frac{1}{c}$.

↱ EXAMPLE Consider the limit statement: $\lim_{x \to 64} \sqrt{x} = 8$. Let's find the corresponding value of δ when $\varepsilon = 2$.

We want $|\sqrt{x} - 8| < 2$.

$-2 < \sqrt{x} - 8 < 2$	$\lvert x\rvert < a$ means $-a < x < a$.
$6 < \sqrt{x} < 10$	Add 8 to all three parts.
$36 < x < 100$	Square all parts of the inequality.
$-28 < x - 64 < 36$	Subtract 64 from all three parts to get $x - 64$ in the middle.

Notice that this inequality is not "balanced." This makes it unclear what to select for δ. Is the answer 28 or 36? Remember what we are trying to say:

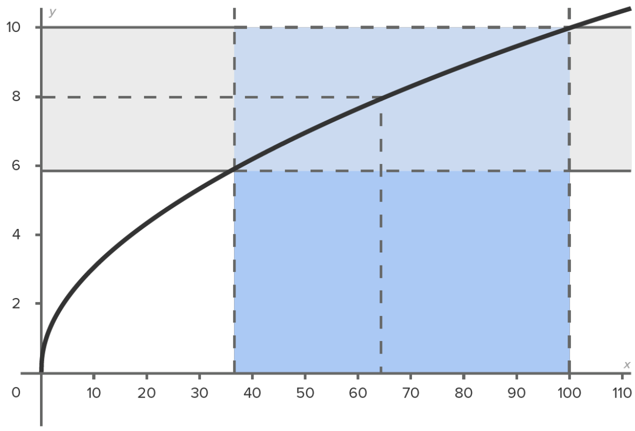
In order for $f(x)$ to be within 2 units of 8, x has to be within ______ units of 64.

Consider the graph shown to the right:

- The horizontal band shows that $6 < y < 10$.
- The vertical band shows that $36 < x < 100$.

The intersection is the "area of interest" for the limit.

If we move 28 units away from $x = 64$ in either direction, we stay inside the vertical band, which guarantees that $f(x)$ is within 2 units of the limit.



If we move 36 units away from $x = 64$ in either direction, we could fall outside the vertical band on the left-hand side, which does not guarantee that $f(x)$ is within 2 units of the limit.

To guarantee that $f(x)$ is within 2 units of the limit (8), x needs to be within 28 units of 64. Thus, when $\varepsilon = 2$, $\delta = 28$.

WATCH

The following video provides an example for a linear function and a radical function.

Video Transcription

[ELECTRONIC MUSIC] Hello, and welcome to today's video on the formal definition of a limit using the intuitive approach. Now, recall that, to do this, we are looking for a specific epsilon greater than 0. Find a delta greater than 0 so that the absolute value of F of x minus L is less than epsilon whenever 0 is less than the absolute value of x minus a is less than delta. Now, that 0 is less than just means that x cannot be equal to a. So that's why we have the double inequality for that.

Now, remember, here, the plan of attack is to start with what I've highlighted in blue and then transition that to looking like what I've highlighted in yellow. And when we get it to that point, the delta will be the number that we have on the right-hand side of that inequality.

So let's look at our first example, which is a linear case. The function that we're talking about is 2x plus 4, which is a linear function. Now, just to make sure that we understand each of the pieces of this, our function f of x is what you're taking the limit of. What that whole limit is equal to-- that's your L. And the value that x is approaching-- that is your a.

So now, when we look at what's highlighted in blue, we start with the absolute value of-- in this case, our function is 2x plus 4, and then minus our L here is a negative 6. So minus a negative 6 6-- that's what's in the absolute value, and it's less than the epsilon that they give us in this particular question. So our epsilon is 0.05.

Now, we're going to work through the steps. First, we're going to simplify what's inside the absolute value. So I have the absolute value of 2x. And then plus 4 minus a negative 6 is plus 6. That gives us plus 10. Is less than our 0.05. And then remember from algebra that, when we have the absolute value less than a positive number, I can write that as a double inequality of negative that 0.05 is less than-- drop the absolute values, 2x plus 10, which, in turn, is less than the 0.05.

Now, here's what we want to do next. We want to solve this double inequality so that we just have x in the middle. And then, once we get to just having x in the middle, we're going to focus on what we need for that yellow highlighted part, which means we need x minus the a. But we need to get it set where we just have x in the middle before we can do that.

So here, we're going to subtract 10 from all three parts. And when I subtract 10 from all three parts, I get our value of, -10.05 is less than 2x, which is less than-- 0.05 minus 10 is a negative 9.95.

Now I'm going to divide all three parts by 2. So -10.05 divided by 2 gives me a negative 5.025, which is less than x, which is less than negative 9.95 divided by 2 is a -4.975.

Once we've gotten it where we have just that x in the middle, we look again at that yellow highlighted-- we want, in the middle of this x minus whatever our a was, and our a is negative 5. So I need to subtract a negative 5 from all three parts. I'm going to write it like that at first, and then we'll simplify the sign. So it's a negative 5.025 minus my a, so minus my negative 5, less than x minus my a, so minus my negative 5 is less than the negative 4.975 minus my negative 5-- the a value.

Now, simplifying each of these, I get a negative 0.025 is less than x plus 5, which is less than 0.025. And now, notice how we have the-- the same but opposite values on the left of the double inequality and the right. So I can write this as the absolute value of x plus 5 is less than the 0.025. And now I have that matching that yellow highlighted part, and the value that's on the right of the inequality is your delta. So delta is equal to 0.025.

Next, let's do part B. Now, part B doesn't have a linear function that it's acting on. It's acting on the square root. So that's going to have to be considered a little bit more. But we're still going to start through with the same step. So think about the blue highlighted. Think about what each piece the roles play.

So f of x is your square root of x. 7 is your L. And 49 is your a. And then, of course, epsilon is 0.1.

So I start with the absolute value of F of x. That's square root of x minus the L, so minus 7, and that less than epsilon, so less than 0.1. Rewrite it without the absolute value. So I have a negative 0.1 is less than the square root of x minus 7 is less than 0.1.

Add the 7 to all three parts. I have 6.9 is less than the square root of x, which is less than 7.1. Now I've got that square root in the middle. I can square all three parts, and the inequality stays the same. So I have a 47.61 is less than x, which is less than a 50.41.

Now that we've cleared it where we have x in the middle, we go look at the yellow highlighted part. And I want x minus a, and here, my a is 49. So I'm going to subtract 49 from all three parts.

So I have 47.61 minus the 49 is less than x minus the 49, which is less than 50.41 minus the 49. Simplifying the left and the right, that gives me a negative 1.39 is less than my x minus 49, which, in turn, is less than 1.41.

So notice, here, I don't have that balance like I did with the linear case. So I have to look at the absolute value of each of those numbers and see which one is the smallest absolute value. And that is-- that absolute value is what I take from my delta. Well, the absolute value of negative 1.39 is 1.39. The absolute value of 1.41 is equal to 1.41. 1.39 is smaller than 1.41, so delta is equal to 1.39.

TRY IT

Consider the limit statement $\lim_{x \to 3} x^2 = 9$.

Find the value of δ that corresponds to ε = 0.5. Round δ to the nearest hundredth. +

We want $8.5 < x^2 < 9.5$, which means $2.915 < x < 3.082$, which in turn means $-0.085 < x - 3 < 0.082$. To find δ, compare $|-0.085|$ and $|0.082|$ since we are examining the distances between *x* and 3. Since $0.082 < 0.085$, use $\delta = 0.082$.

EXAMPLE Consider the limit statement: $\lim_{x \to 5} \frac{1}{2x} = \frac{1}{10}$. Let's find δ when $\varepsilon = 0.05$.

Start with $\left| \frac{1}{2x} - \frac{1}{10} \right| < 0.05$.

$-0.05 < \frac{1}{2x} - \frac{1}{10} < 0.05$ $\quad |x| < a$ means $-a < x < a$.

$\frac{1}{20} < \frac{1}{2x} < \frac{3}{20}$ $\quad$ Add $\frac{1}{10}$, and convert all to fractions.

$\frac{20}{3} < 2x < 20$	$c < \frac{1}{x} < d$ means $\frac{1}{d} < x < \frac{1}{c}$.
$\frac{10}{3} < x < 10$	Divide by 2.
$-\frac{5}{3} < x - 5 < 5$	Subtract 5.

It follows that $\delta = \frac{5}{3}$ since $\left| -\frac{5}{3} \right| = \frac{5}{3}$ and $\frac{5}{3}$ is smaller than 5.

SUMMARY

In this lesson, you learned that by using the formal definition of a limit, you can observe the relationship between ε and δ, which emphasizes the idea of "$f(x)$ getting closer to the limit as *x* gets closer to *a*." In this challenge, the goal was to **find the value of δ that corresponds to a given value of ε for a linear function and a nonlinear function**, and we observed that one getting smaller causes the other to get smaller. For linear functions, identifying δ is rather straightforward, but for nonlinear functions, more critical thinking is required to find the appropriate value of δ.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Formal Definition of a Limit

$\lim_{x \to a} f(x) = L$ means that for every given ε > 0, there exists δ > 0 so that:

- If *x* is within δ units of *a* (and $x \neq a$), then $f(x)$ is within ε units of *L*.
- This translates to $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

The Formal Definition of a Limit

by Sophia

WHAT'S COVERED

In this lesson, you will establish a relationship between ε and δ to formally prove the value of a limit. Specifically, this lesson will cover:

1. The Formal Definition of a Limit
2. Proving Limits (Finding δ in Terms of ε)

1. The Formal Definition of a Limit

This process is much like what we did in the last challenge, except this time we will have symbols instead of numbers. Since this process is more complicated with nonlinear functions, we will be focusing on linear functions only.

The formal definition of a limit is as follows:

CONCEPT TO KNOW

$\lim_{x \to a} f(x) = L$ means for every given ε > 0, there is a δ > 0 so that if *x* is within δ units of *a* (and $x \neq a$), then $f(x)$ is within ε units of *L*.

Equivalently, we can say $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

HINT

The use of the lower case Greek letters ε (epsilon) and δ (delta) in the definition is standard, and this definition is sometimes called the "epsilon-delta" definition of limit.

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2. Proving Limits (Finding δ in Terms of ε)

In order to prove a limit, we need to accomplish the following:

- Find an expression for δ in terms of ε, starting with $|f(x) - L| < \varepsilon$.
- For the expression you found, show that $0 < |x - a| < \delta$ leads to $|f(x) - L| < \varepsilon$.

EXAMPLE Prove the limit: $\lim_{x \to 4} (2x - 1) = 7$

Start with $|2x - 1 - 7| < \varepsilon$. Then, convert to an inequality with $x - 4$ in the middle (just as before).

$\lvert 2x-8 \rvert < \varepsilon$	Simplify the expression.
$-\varepsilon < 2x-8 < \varepsilon$	$\lvert x \rvert < a$ means $-a < x < a$.
$-\varepsilon + 8 < 2x < \varepsilon + 8$	Add 8 to all parts.
$-\frac{\varepsilon}{2} + 4 < x < \frac{\varepsilon}{2} + 4$	Divide all parts by 2.
$-\frac{\varepsilon}{2} < x-4 < \frac{\varepsilon}{2}$	Subtract 4 from all parts to get $x-4$ in the middle.

Thus, it appears that $\delta = \frac{\varepsilon}{2}$. Note also that the last inequality can be written $\lvert x-4 \rvert < \frac{\varepsilon}{2}$.

Now, to prove the limit, we basically just reverse the steps we did. While this is redundant, it is important since we need to establish that $-\frac{\varepsilon}{2} < x-4 < \frac{\varepsilon}{2}$ directly leads to a statement that $f(x)$ is within ε units of 7.

The goal is to convert this into an inequality with $2x-1$ in the middle.

Start with $-\frac{\varepsilon}{2} < x-4 < \frac{\varepsilon}{2}$.

$-\frac{\varepsilon}{2} + 4 < x < \frac{\varepsilon}{2} + 4$	Add 4 to all parts.
$-\varepsilon + 8 < 2x < \varepsilon + 8$	Multiply all parts by 2.
$-\varepsilon + 7 < 2x-1 < \varepsilon + 7$	Subtract 1 from all parts to get $2x-1$ in the middle.

This inequality is enough justification that $f(x) = 2x-1$ is within ε units of 7. This completes the proof.

Note: With $\delta = \frac{\varepsilon}{2}$, note that as ε gets closer to 0, so does δ. This is crucial for proving a limit. This is the mathematical version of saying “as *x* gets closer to *a*, $f(x)$ gets closer to *L*”.

↱ EXAMPLE Prove the limit: $\lim_{x \to 9} (50-4x) = 14$

First, find an expression for δ in terms of ε:

$\lvert 50-4x-14 \rvert < \varepsilon$	Start with the inequality.
$\lvert 36-4x \rvert < \varepsilon$	Simplify the expression.
$-\varepsilon < 36-4x < \varepsilon$	$\lvert x \rvert < a$ means $-a < x < a$.
$-\varepsilon - 36 < -4x < \varepsilon - 36$	Subtract 36 from all parts.
$\frac{\varepsilon}{4} + 9 > x > -\frac{\varepsilon}{4} + 9$	Divide all parts by -4 (note: inequalities change direction).
$-\frac{\varepsilon}{4} + 9 < x < \frac{\varepsilon}{4} + 9$	Rewrite the inequality.
$-\frac{\varepsilon}{4} < x-9 < \frac{\varepsilon}{4}$	Subtract 9 to get *x* - 9 in the middle.

At this point, it appears that $\delta = \frac{\varepsilon}{4}$. Now, reverse the steps.

$-\frac{\varepsilon}{4} < x - 9 < \frac{\varepsilon}{4}$	Start with the inequality.
$-\frac{\varepsilon}{4} + 9 < x < \frac{\varepsilon}{4} + 9$	Add 9 to all parts.
$\varepsilon - 36 > -4x > -\varepsilon - 36$	Multiply all parts by -4 (inequalities change direction).
$-\varepsilon - 36 < -4x < \varepsilon - 36$	Rewrite the inequality.
$-\varepsilon + 14 < -4x + 50 < \varepsilon + 14$	Add 50 to all parts.

At this point, we see that $f(x)$ is within ε units of 14. This completes the proof.

WATCH

The following video walks you through an example of finding δ to prove a limit for a linear function.

Video Transcription

[MUSIC PLAYING] Hi there. The topic of today's video is the formal definition of a limit.

Now recall the formal definition of a limit, we are looking at for a general epsilon greater than 0, we want to find a delta greater than 0 so that 0 less than the absolute value of x minus a is less than delta given that the absolute value of f of x minus L is less than epsilon. So we want to make sure that we keep track of this absolute value of f of x minus L is less than epsilon and this idea of 0 is less than the absolute value of x minus a is less than delta.

Now I want to just make a comment with the yellow highlighted part that 0 is less than part just actually tells us that x cannot equal a for that part of the inequality.

Now for our example of this process, we're looking at for epsilon greater than 0, find a delta greater than 0 necessary to prove that the limit as x approaches negative 4 of 8x plus 30 is equal to negative 2.

Now we are going to go through the same steps that we did when we had that intuitive approach and we had a specific numerical value for epsilon. But instead of having a specific numerical value for epsilon, we're just going to bring the notation for epsilon along the way.

So remember, we start with what's highlighted in blue, and we also want to look at what each of the components are in our specific example. So our 8x plus 30, that's our f of x. And what the limit is equal to, that's your L. And the value that the x is approaching, that is a.

So when we start with the blue highlighted, we have the absolute value of f of x is 8x plus 30 minus, from the blue highlighted part, our L is a negative 2, so we're going to have minus a negative 2. Close our absolute value and have that less than, and we're going to write the notation for epsilon.

Now simplifying what's inside the absolute value, we have the absolute value of 8x, and then 30 minus a negative 2 is 30 plus 2, or plus 32, is less than epsilon.

And now we can rewrite our absolute value of that expression is less than epsilon as a double inequality. Negative epsilon is less than 8x plus 32, which in turn is less than epsilon. So that allows us to write it without an absolute value and actually work a little bit more to transition this double inequality.

Now our goal is to get x alone in the middle first, and then be able to work towards what our yellow highlighted part is. So to get the x in the middle, we're going to subtract 32 from all three parts. So we'll have an epsilon, our negative epsilon, and then minus 32 is less than 8x plus 32 minus 32 less than epsilon minus 32.

So again, we're focusing on getting x alone in the middle. Simplifying this, we have negative epsilon minus 32 is less than 8x, which is less than epsilon minus 32.

Now I still need to remove that coefficient of 8. So we're going to divide all three parts by 8. And when we divide all three parts by 8, with each of the left side and the right side, we want to think of this as a negative epsilon over 8, and then for this left side negative 32 divided by 8 is a minus 4 is less than x, which is less than epsilon over 8 minus 4.

So our x is in the middle. Now look at what's highlighted in yellow. What we want in the middle is x minus a. Recall our a is what the x is approaching. So we want to subtract a negative 4 from all three parts.

So we have a negative epsilon over 8 minus 4, and then minus that negative 4 is less than x minus that negative 4, which in turn is less than epsilon over 8 minus 4, and again, minus that negative 4.

Now simplifying all of this, we have negative epsilon over 8 on the left because the minus 4 and the plus 4 will remove. I have an x plus 4 in the middle. And I have an epsilon over 8 on the right.

So this gives me negative epsilon over 8 is less than x plus 4, which is less than epsilon over 8. That has that balance between the left side and the right side just being opposites of each other.

So I can write that as the absolute value of x plus 4 is less than epsilon over 8. And that transitions to what we have in that yellow highlighted part. So our delta is equal to this epsilon over 8.

Now when we want to actually show this as why that works to prove our limit, and I'm going to do this over on the left in this open space, so this proves that limit, it works.

Delta equaling epsilon over 8 works since, start with your first line of what we began with on that right-hand side. We have, with what we began with on that left-hand side, the absolute value of 8x plus 30 minus negative 2.

Well, that's equal to the absolute value of 8x plus 32. And here, a common factor of 8 can be factored out of the absolute value. So that's 8 times the absolute value of x plus 4.

But the absolute value of x plus 4 is less than epsilon over 8 because it's less than delta and delta is epsilon over 8. So I have 8, that that is less than 8 times epsilon over 8, which is less than epsilon. And we get our original f of x minus L is less than epsilon when we have our delta to be chosen to be that epsilon over 8.

TRY IT

Consider $\lim_{x \to 3}(6x - 11) = 7$.

Prove this limit. +

Repeat the processes you saw in the first example. You should use $\delta = \frac{\varepsilon}{6}$.

SUMMARY

In this lesson, you learned that by using **the formal definition of a limit**, you can establish the true meaning of what it means for a limit to exist, by comparing δ and ε. You also learned that when **proving limits (finding δ in terms of ε)**, it crucial to note that as ε gets closer to 0, δ also gets closer to 0, and is the mathematical expression of “as *x* gets closer to *a*, $f(x)$ gets closer to *L*.”

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Terms to Know

Average Rate of Change

The net change divided by the length of the interval.

Continuous From the Left

A function is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Continuous From the Right

A function is continuous from the left at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Continuous Function

A function that has no breaks in the graph. That is, $\lim_{x \to a} f(x) = f(a)$.

Formal Definition of a Limit

$\lim_{x \to a} f(x) = L$ means that for every given ε > 0, there exists δ > 0 so that:

- If x is within δ units of a (and $x \neq a$), then $f(x)$ is within ε units of L.
- This translates to $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Instantaneous Rate of Change

The rate of change of a function at a specific point.

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.

Limit

The value that a function $f(x)$ approaches as x gets closer to a specified number.

Rational Function

A function in the form $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

Secant Line

A line that contains two points of the same function.

Tangent Line

A line that touches (but does not cross) the graph of a function at a specific point.

Velocity

The speed of some object relative to some starting point. Unlike speed, velocity can be negative.

Formulas to Know

Average Rate of Change on the Interval [a, b]

$$\frac{change\ in\ f}{change\ in\ x} = \frac{f(b)-f(a)}{b-a}$$

