Unit 4 Tutorials: Applications of Derivatives

INSIDE UNIT 4

What is a Maximum or a Minimum?

by Sophia

WHAT'S COVERED

In this lesson, you will learn about the different kinds of maximum and minimum points on a graph of a function. Specifically, this lesson will cover:

1. Definitions of Global and Local Maximum and Minimum Points
2. Finding Global and Local Maximum and Minimum Points

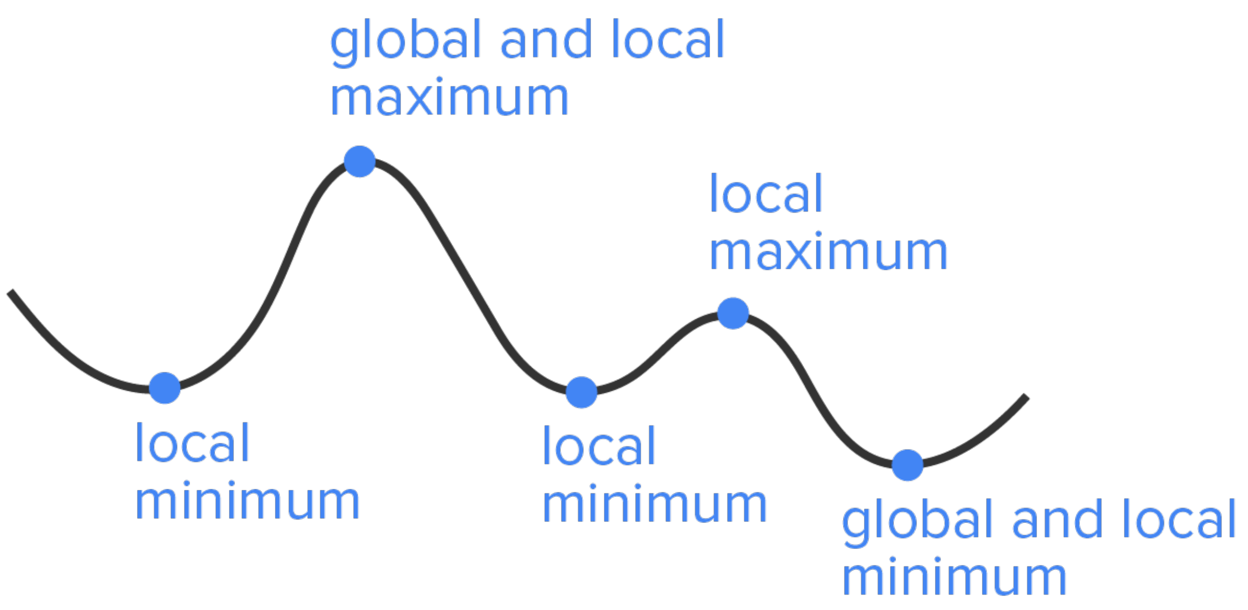
1. Definitions of Global and Local Maximum and Minimum Points

One of the main uses of derivatives is to find minimum and maximum values of a function, or more simply put, **extreme values** (or **extrema**) of a function.

A function can have any one of the following:

- **Global (or Absolute) Maximum**: A function $f(x)$ has a global (or absolute) maximum at $x=a$ if $f(a) \ge f(x)$ for all *x*. In other words, $f(a)$ is the largest value of a function $f(x)$, and occurs when $x=a$.
- **Global (or Absolute) Minimum**: A function $f(x)$ has a global (or absolute) minimum at $x=a$ if $f(a) \le f(x)$ for all *x*. In other words, $f(a)$ is the smallest value of a function $f(x)$, and occurs when $x=a$.
- **Local (or Relative) Maximum**: A function $f(x)$ has a local (or relative) maximum at $x=a$ if $f(a) \ge f(x)$ for all *x* close to $x=a$. In other words, $f(a)$ is the largest value of a function $f(x)$ for values near $x=a$.
- **Local (or Relative) Minimum**: A function $f(x)$ has a local (or relative) minimum at $x=a$ if $f(a) \le f(x)$ for all *x* close to $x=a$. In other words, $f(a)$ is the smallest value of a function $f(x)$ for values near $x=a$.

The graph shown here summarizes the differences between local and global extrema. Note that the second labeled point (from left to right) is both a local maximum and a global maximum because it meets both conditions: it is both the highest point on the graph and it is the highest point when compared to points immediately to the left and right.



TERMS TO KNOW

Extreme Values

The minimum or maximum values of a function.

Extrema

Another word for extreme values.

Global (or Absolute) Maximum

A function $f(x)$ has a global (or absolute) maximum at $x=a$ if $f(a) \ge f(x)$ for all *x*. In other words, $f(a)$ is the largest value of a function $f(x)$, and occurs when $x=a$.

Global (or Absolute) Minimum

A function $f(x)$ has a global (or absolute) minimum at $x=a$ if $f(a) \le f(x)$ for all *x*. In other words, $f(a)$ is the smallest value of a function $f(x)$, and occurs when $x=a$.

Local (or Relative) Maximum

A function $f(x)$ has a local (or relative) maximum at $x=a$ if $f(a) \ge f(x)$ for all *x* close to $x=a$. In other words, $f(a)$ is the largest value of a function $f(x)$ for values near $x=a$.

Local (or Relative) Minimum

A function $f(x)$ has a local (or relative) minimum at $x=a$ if $f(a) \le f(x)$ for all *x* close to $x=a$. In other words, $f(a)$ is the smallest value of a function $f(x)$ for values near $x=a$.

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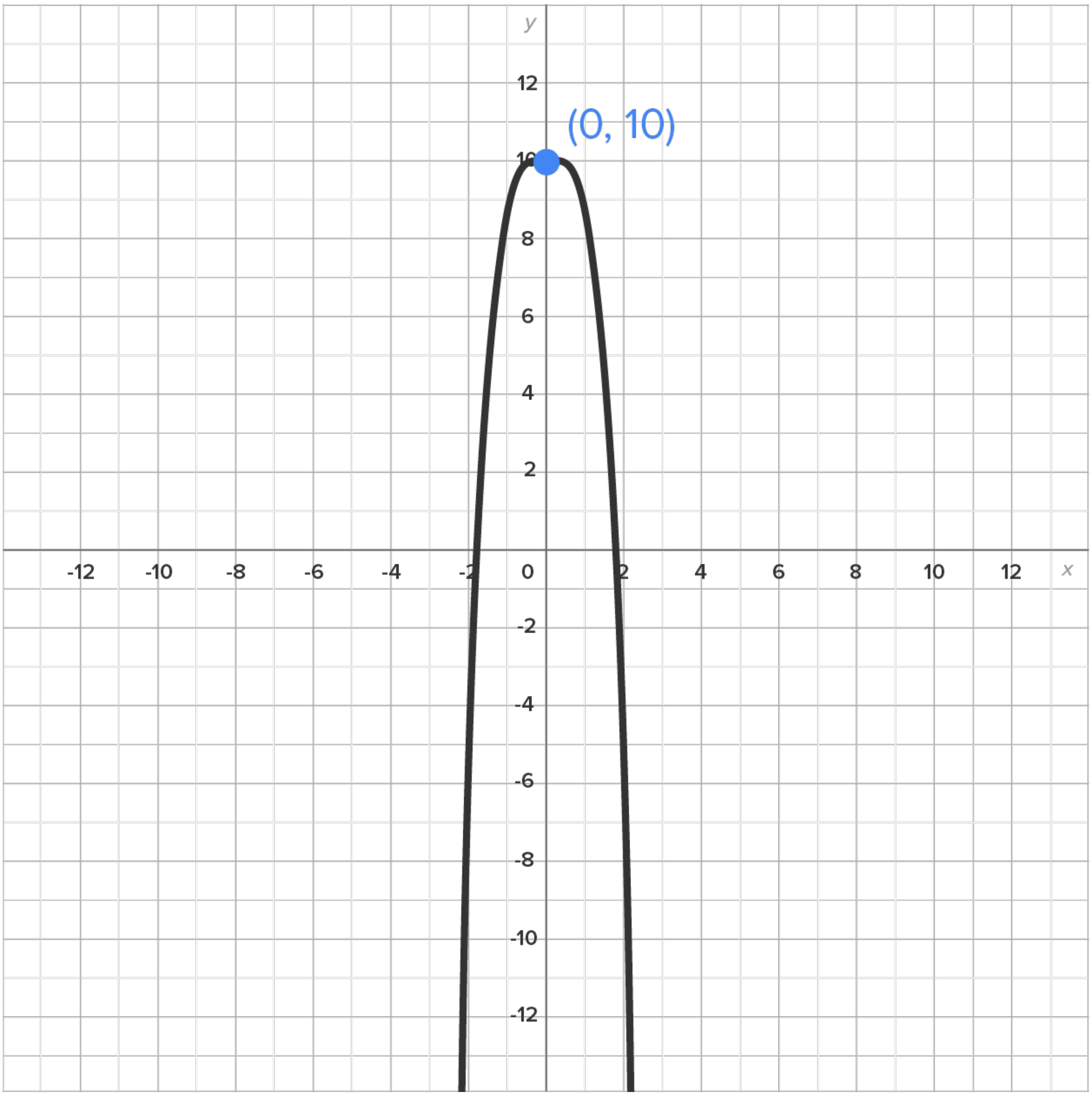
2. Finding Global and Local Maximum and Minimum Points

→ EXAMPLE Consider the function $f(x) = -x^4 + 10$ as shown in the graph below.

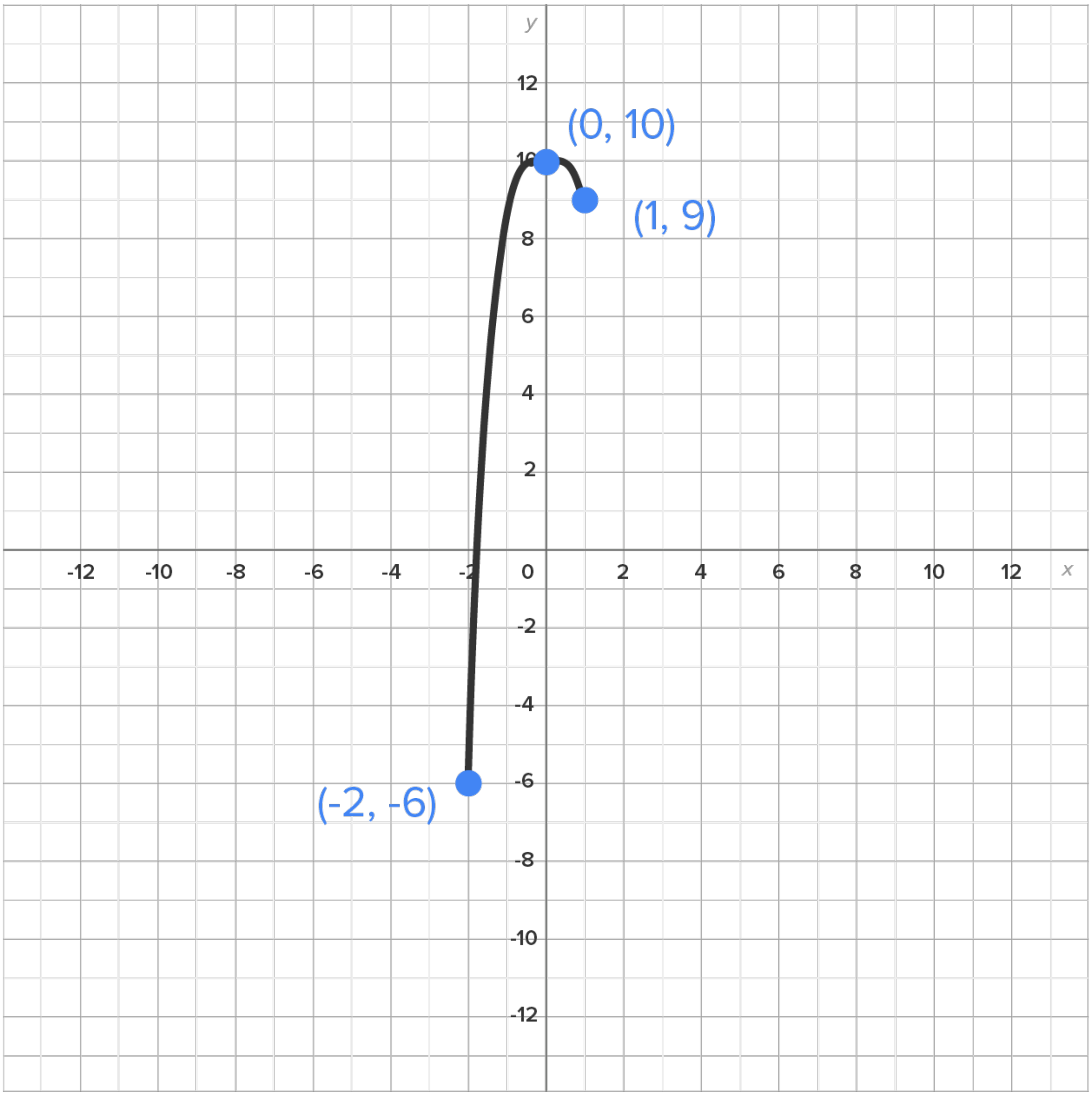


The highest point on the graph is (0, 10), while there is no lowest point. It is also the highest point when compared to other points around it.

Therefore, we say that $f(x)$ has a global maximum and local maximum at $x = 0$, and its value is 10.

There is no local or global minimum point.

↱ EXAMPLE Now consider the function $f(x) = -x^4 + 10$, but contained on the interval $[-2, 1]$.



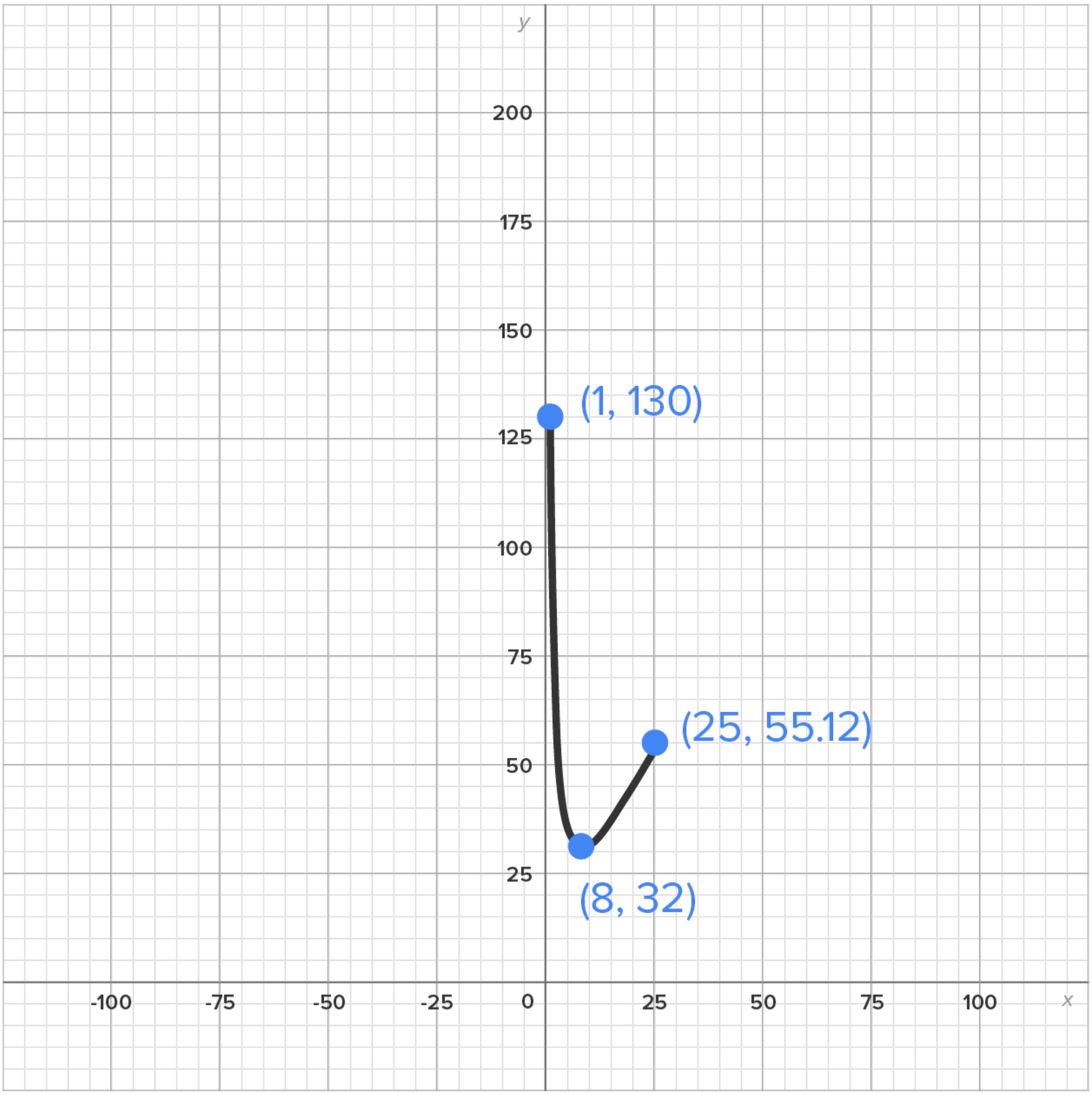
The highest point on the graph is (0, 10), which is also the highest point around (0, 10). Therefore, at $x = 0$, both a local and global maximum occurs and is equal to 10.

The lowest point on the graph is (-2, -6), which means that $f(x)$ has a global minimum at $x = -2$, which is equal to -6.

Neither (-2, -6) nor (1, 9) are considered local minimum values. This is because there is no graph on the other side of the points to compare.

In other words, having a local extreme point at $x = a$ means that $f(a)$ is the most extreme value on an open interval containing a (both sides of $x = a$).

TRY IT



Use the graph to determine all global and local maximum and minimum values of the function. +

The global maximum is 130 and it occurs at $x = 1$, and both a local and global minimum is 32 at $x = 8$.

SUMMARY

In this lesson, you learned that one of the main uses of derivatives is to find minimum and maximum values of a function. A function can have several types of extreme values, which can be identified from a graph. These **points** include: **global (or absolute) maximum**, **global (or absolute) minimum**, **local (or relative) maximum**, and **local (or relative) minimum**. Next, you explored using graphs to **find all global and local maximum and minimum values** of each respective function.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Extrema

Another word for extreme values.

Extreme Values

The minimum or maximum values of a function.

Global (or Absolute) Maximum

A function $f(x)$ has a global (or absolute) maximum at $x = a$ if $f(a) \geq f(x)$ for all *x*. In other words, $f(a)$ is the largest value of a function $f(x)$, and occurs when $x = a$.

Global (or Absolute) Minimum

A function $f(x)$ has a global (or absolute) minimum at $x = a$ if $f(a) \leq f(x)$ for all *x*. In other words, $f(a)$ is the smallest value of a function $f(x)$, and occurs when $x = a$.

Local (or Relative) Maximum

A function $f(x)$ has a local (or relative) maximum at $x = a$ if $f(a) \geq f(x)$ for all *x* close to $x = a$. In other words, $f(a)$ is the largest value of a function $f(x)$ for values near $x = a$.

Local (or Relative) Minimum

A function $f(x)$ has a local (or relative) minimum at $x = a$ if $f(a) \leq f(x)$ for all *x* close to $x = a$. In other words, $f(a)$ is the smallest value of a function $f(x)$ for values near $x = a$.

Critical Numbers

by Sophia

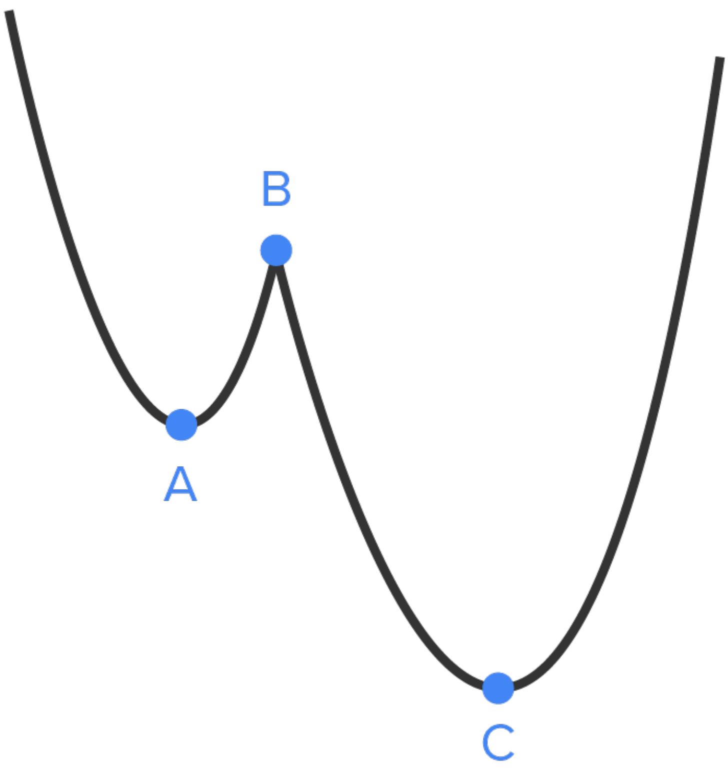
WHAT'S COVERED

In this lesson, you will now take a more analytical approach to locating maximum and minimum values of a function by finding critical values. Specifically, this lesson will cover:

1. Defining Critical Numbers of a Function
2. Finding Critical Numbers of a Function

1. Defining Critical Numbers of a Function

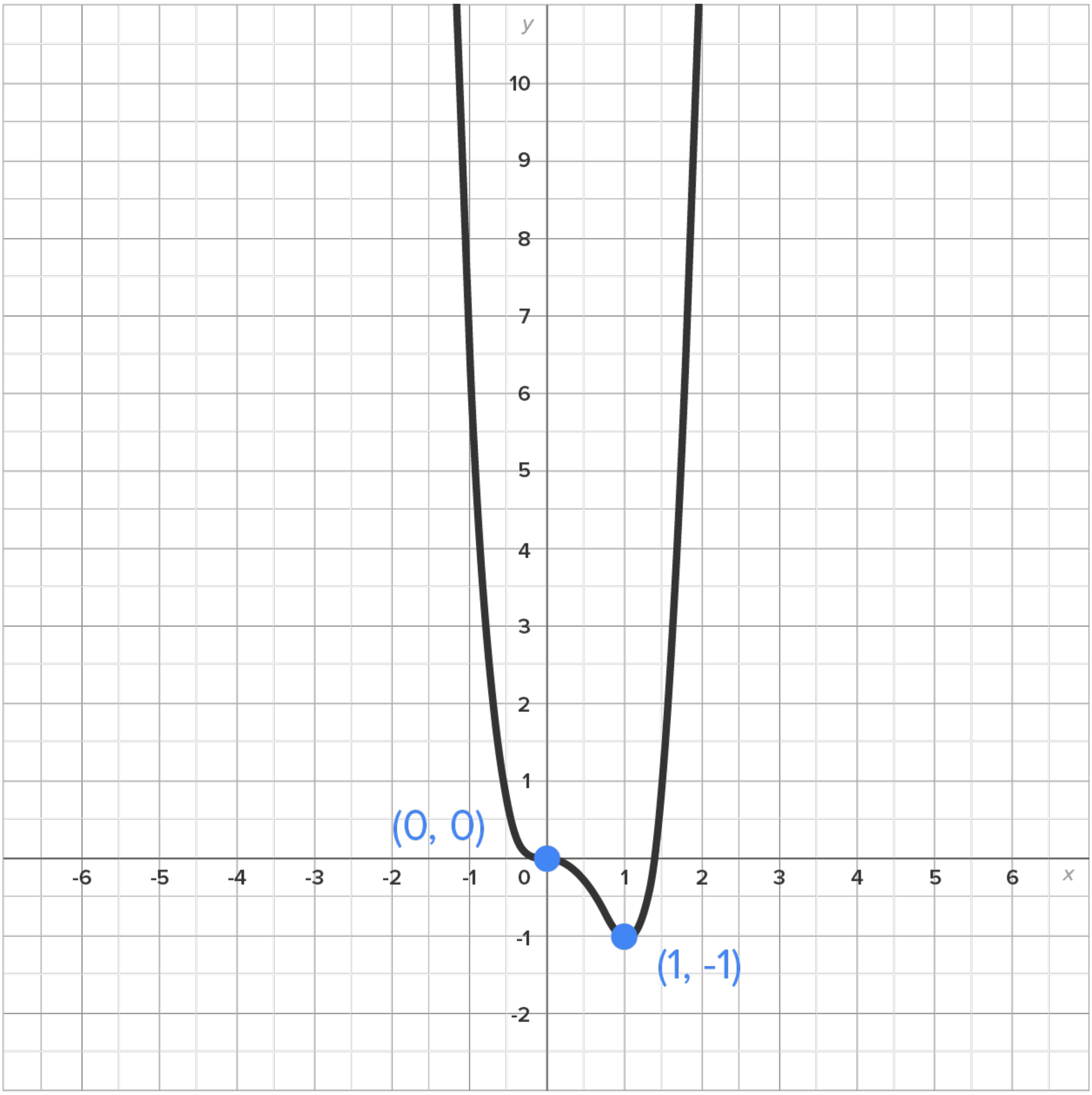
Consider the graph below, which shows the graph of a function $y = f(x)$.



- At point *A*, there is a local minimum, and the derivative is 0.
- At point *B*, there is a local maximum, and the derivative is undefined.
- At point *C*, there is a local minimum, and the derivative is 0.

The x-coordinates where the derivative is 0 or undefined can lead to finding maximum or minimum values. Since this is important, these values of *x* are called **critical numbers**.

It is not always the case that a critical number leads to a maximum or minimum value. Consider this graph:



Note how $f'(x)=0$ at both $x=0$ and $x=1$, but there is no minimum or maximum when $x=0$ (but there is a local and global minimum when $x=1$).

Thus, the critical numbers give information about where the minimum and maximum points could be, but further analysis will be required to determine the exact behavior at each critical number.

TERM TO KNOW

Critical Number

A value of c in the domain of $f(x)$ for which $f'(c)=0$ or $f'(c)$ is undefined, provided that $f(c)$ is defined.

2. Finding Critical Numbers of a Function

Now that we know what critical numbers are, let's get some practice finding the critical numbers of a few functions.

HINT

When your goal is to find critical numbers, first note the domain of the function. When $f'(x)$ is undefined, this will help you determine which x-values are critical numbers and which x-values are not.

EXAMPLE Consider the function $f(x) = -x^3 + 12x + 10$. Find the critical numbers of $f(x)$.

$f(x) = -x^3 + 12x + 10$	Start with the original function; the domain is all real numbers.
$f'(x) = -3x^2 + 12$	Find the derivative of *f*.
$-3x^2 + 12 = 0$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x)$ equal to 0 to find critical numbers.
$-3(x^2 - 4) = 0$	Factor out -3.
$-3(x+2)(x-2) = 0$	Factor $x^2 - 4$.
$x + 2 = 0$ or $x - 2 = 0$	Set each factor equal to 0.
$x = -2, x = 2$	Solve.

Conclusion: the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

EXAMPLE Find the critical numbers of $f(x) = 10\sqrt{x} - x$.

$f(x) = 10\sqrt{x} - x$	Start with the original function. Note that the domain is $[0, \infty)$.
$f(x) = 10x^{1/2} - x$	Rewrite as a power to take the derivative.
$f'(x) = 5x^{-1/2} - 1$	Find the derivative of *f*.
$f'(x) = \frac{5}{x^{1/2}} - 1$	Rewrite using positive exponents.
$f'(x) = \frac{5}{\sqrt{x}} - 1$	Then, rewrite as a radical.
$\frac{5}{\sqrt{x}} - 1 = 0$	The derivative is undefined when $x = 0$. Since $f(0)$ is defined, $x = 0$ is a critical number. Set equal to 0 to find other critical numbers.
$\frac{5}{\sqrt{x}} = 1$	Add 1 to both sides.
$5 = \sqrt{x}$	Multiply both sides by $\sqrt{x}$.
$25 = x$	Square both sides.

Thus, the critical numbers are $x = 0$ and $x = 25$.

TRY IT

Consider the function $f(x) = \frac{1}{4}x - \ln x$.

Find all critical numbers of the function. +

$x = 4$ (The derivative at $x = 0$ is undefined, but $f(0)$ is also undefined, which means $x = 0$ is not a critical number.)

SUMMARY

In this lesson, you learned how to **define critical numbers of a function**, which are values of *x* in the domain of the function where the derivative is either 0 or undefined, that give information where the local minimum and maximum points could be. Keep in mind, however, that further analysis is needed to determine the exact behavior at each critical number. Then, you used this newly acquired knowledge to practice **finding the critical numbers of a few functions**, noting that it is important to pay attention to the domain of the function, which will help you determine which x-values are critical numbers and which are not.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Critical Number

A value of c in the domain of $f(x)$ for which $f'(c) = 0$ or $f'(c)$ is undefined, provided that $f(c)$ is defined.

Finding Maximums and Minimums of a Function

by Sophia

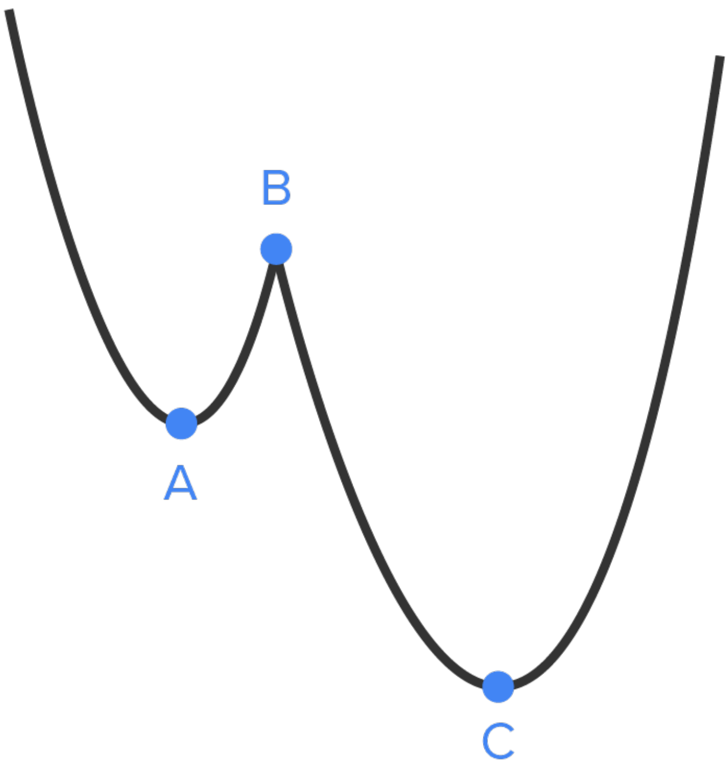
WHAT'S COVERED

In this lesson, you will use derivatives and critical numbers to find local maximum and local minimum values. Specifically, this lesson will cover:

1. The Relationship Between Critical Numbers and Local Extrema
2. Finding Local Extrema

1. The Relationship Between Critical Numbers and Local Extrema

Consider the graph of a function $y = f(x)$, shown here:



- At point *A*, $f'(x) = 0$.
- At point *B*, $f'(x)$ is undefined.
- At point *C*, $f'(x) = 0$.

As discussed in the previous tutorial, values of x in the domain of $f(x)$ where $f'(x) = 0$ or $f'(x)$ is undefined are called critical numbers.
Therefore, critical numbers can tell us where local maximum or minimum values could occur.

However, the only way to find out is through further analysis, which will be covered in challenge 4.3.

STEP BY STEP

To identify relative extrema:

1. Find all critical values.
2. Use a graph of the function to determine which critical numbers correspond to which relative extreme points.

Now that we know the connection between critical numbers and extrema, let's look at a few examples.

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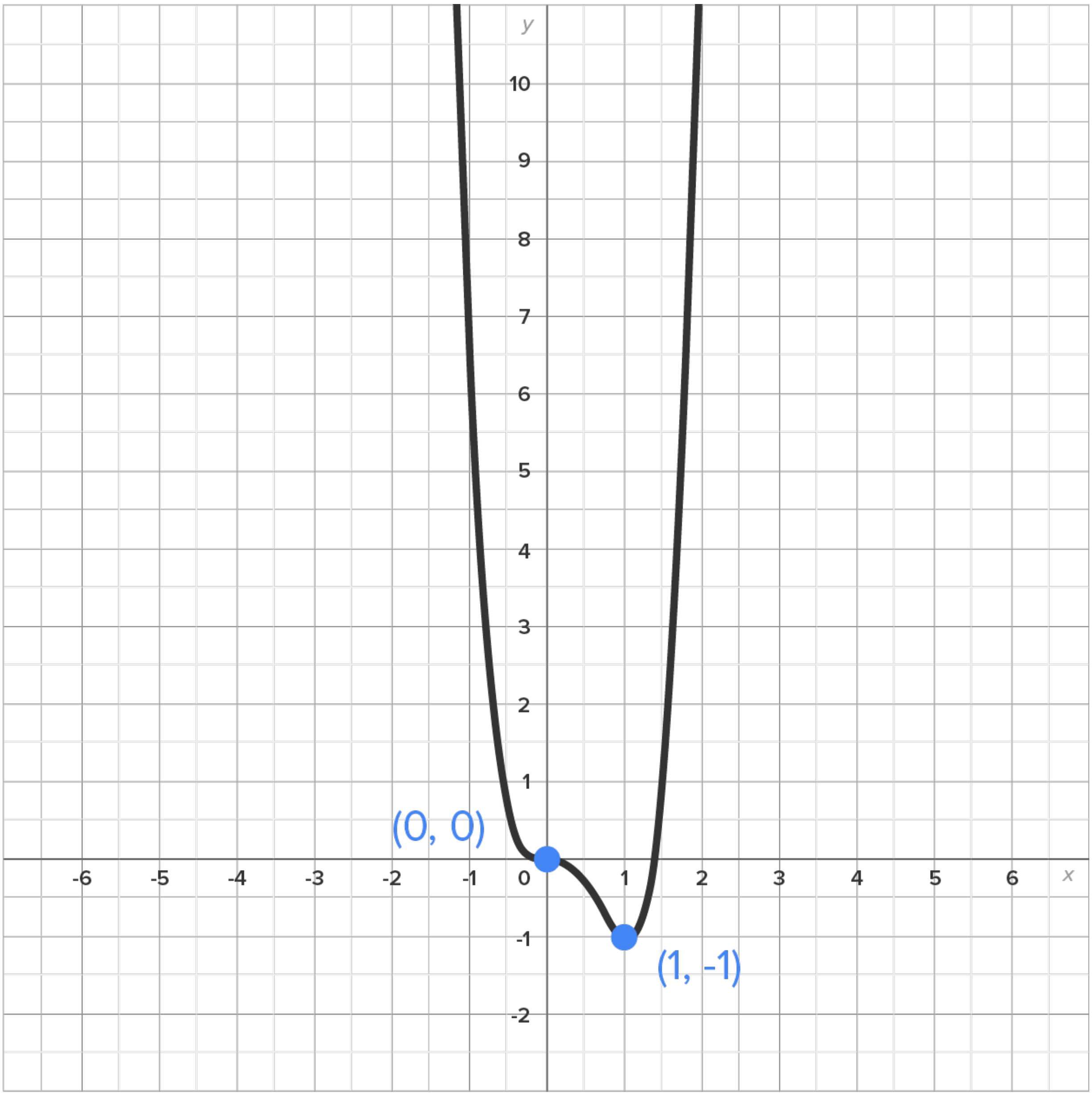
2. Finding Local Extrema

EXAMPLE Consider the function $f(x) = 3x^4 - 4x^3$. First, find all critical numbers:

$f(x) = 3x^4 - 4x^3$	Start with the original function; the domain is all real numbers.
$f'(x) = 12x^3 - 12x^2$	Take the derivative.
$12x^3 - 12x^2 = 0$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x) = 0$ and solve.
$12x^2(x-1) = 0$	Factor.
$12x^2 = 0,\ x - 1 = 0$	Set each factor equal to 0.
$x = 0,\ x = 1$	Solve.

Thus, the critical numbers are $x = 0$ and $x = 1$.

Now, the graph of $f(x)$ is shown.



The point (0, 0) is neither a local maximum nor a local minimum, while a local minimum (also a global minimum) occurs at (1, -1).

TRY IT

Consider the function $f(x) = -\frac{1}{2}x^4 + 9x^2 + 10$.

Find all critical numbers of *f*, then determine the local minimum and maximum points by using a graph.

+

Critical numbers are $x = 0$, -3, and 3.
Local minimum is at (0, 10) and local maximums are at (-3, 50.5) and (3, 50.5).

SUMMARY

In this lesson, you learned about **the relationship between critical numbers and local extrema**, which is that critical numbers help locate the coordinates of local maximum and minimum points. Understanding this connection between critical numbers and extrema, you practiced **finding local extrema** of a function by first determining all critical numbers. As you saw with one example, a critical number at $x = c$ doesn't automatically imply that there is a local maximum or minimum at $x = c$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Extreme Value Theorem - Endpoint Extremes

by Sophia

WHAT'S COVERED

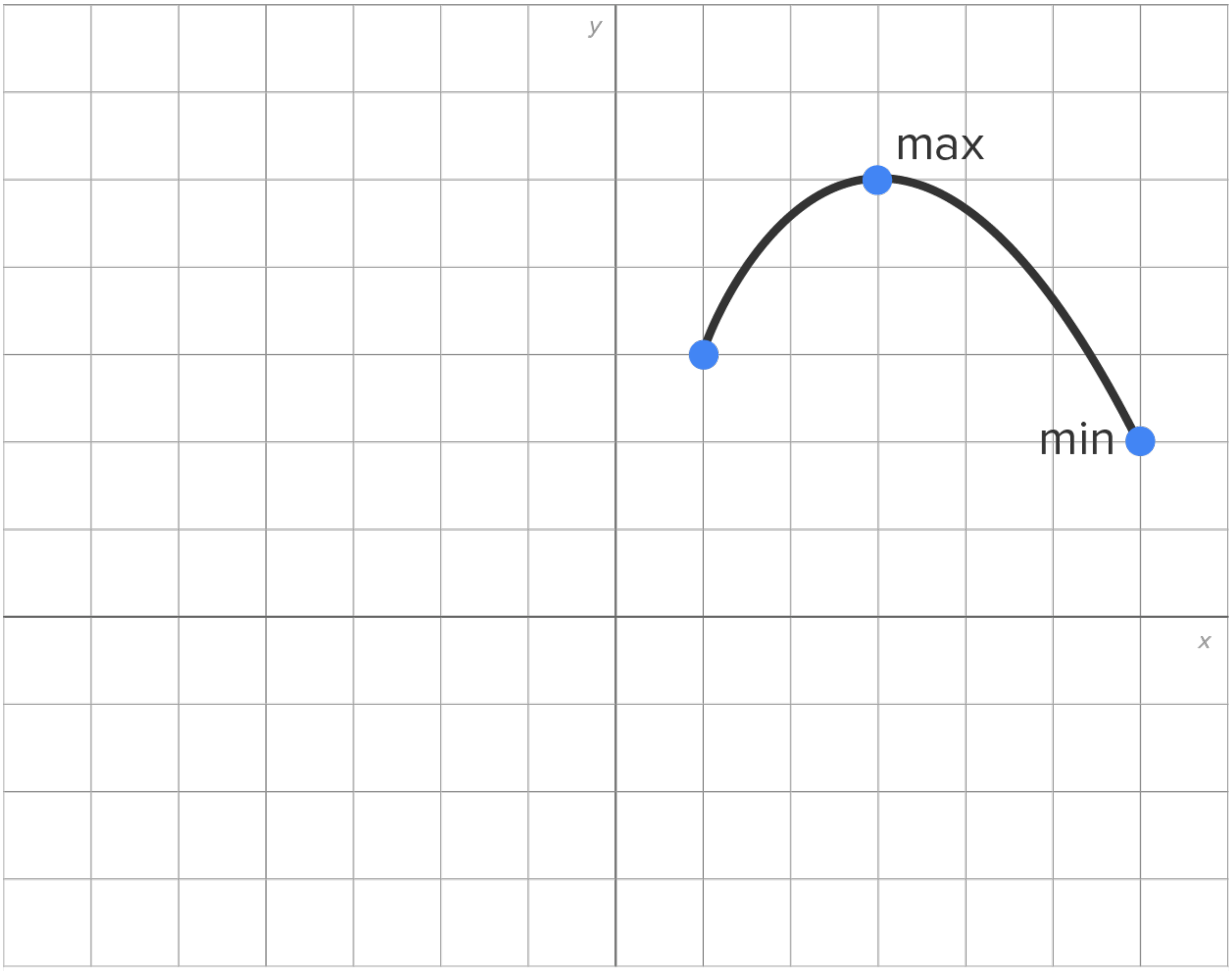
In this lesson, you will use critical numbers and endpoint analysis to determine the maximum and minimum values of a continuous function on some closed interval. Specifically, this lesson will cover:

1. The Extreme Value Theorem
2. Finding Extreme Values of a Continuous Function on a Closed Interval

1. The Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is guaranteed to have global maximum and global minimum values on the interval $[a, b]$. This is known as the **extreme value theorem**.

Here is an illustration of the extreme value theorem:

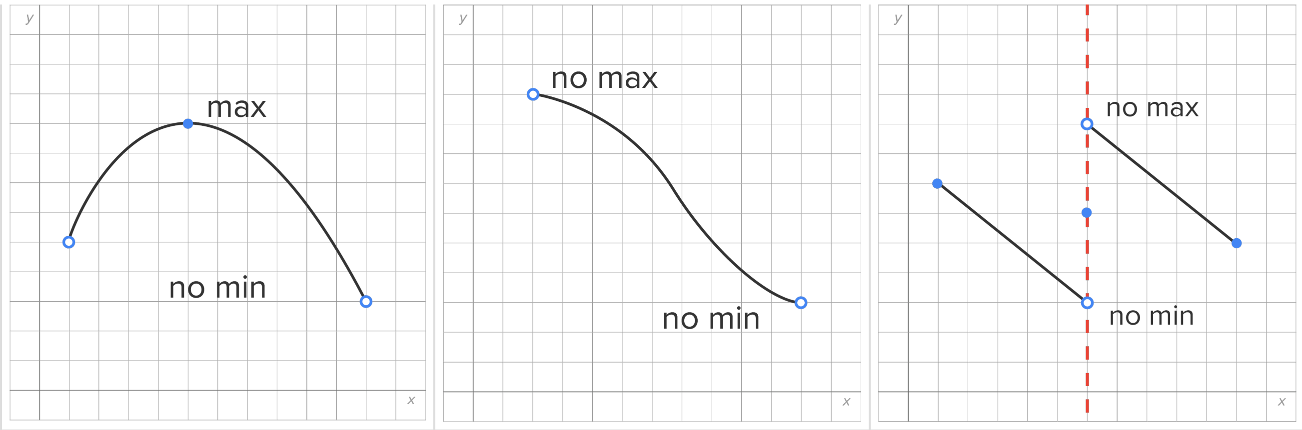


f continuous closed interval

- The function is continuous on the interval $[a, b]$.
- The maximum point occurs inside the interval.
- The minimum occurs at an endpoint.

The following are examples of situations in which one of the criteria is violated.

f Continuous Open Interval	f Continuous Open Interval	f Not Continuous Closed Interval



TERM TO KNOW

Extreme Value Theorem

If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

2. Finding Extreme Values of a Continuous Function on a Closed Interval

As a result of the theorem, here is what we need to do in order to find the global minimum and maximum values of $f(x)$ on a closed interval $[a, b]$.

1. Find all critical numbers of $f(x)$ that are in the interval $[a, b]$.
2. Evaluate $f(x)$ at each endpoint and each critical number. The largest value of f is the global maximum and the smallest value of f is the global minimum.

EXAMPLE Find the global maximum and minimum points of the function $f(x) = x^3 - 6x^2 + 5$ on the interval $[-1, 3]$.

First, find the critical numbers.

$f(x) = x^3 - 6x^2 + 5$	Start with the original function.
$f'(x) = 3x^2 - 12x$	Take the derivative.
$3x^2 - 12x = 0$ $3x(x-4) = 0$ $x = 0,\ x = 4$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x) = 0$ and solve for x.

Therefore, the critical numbers are $x = 0$ and $x = 4$.

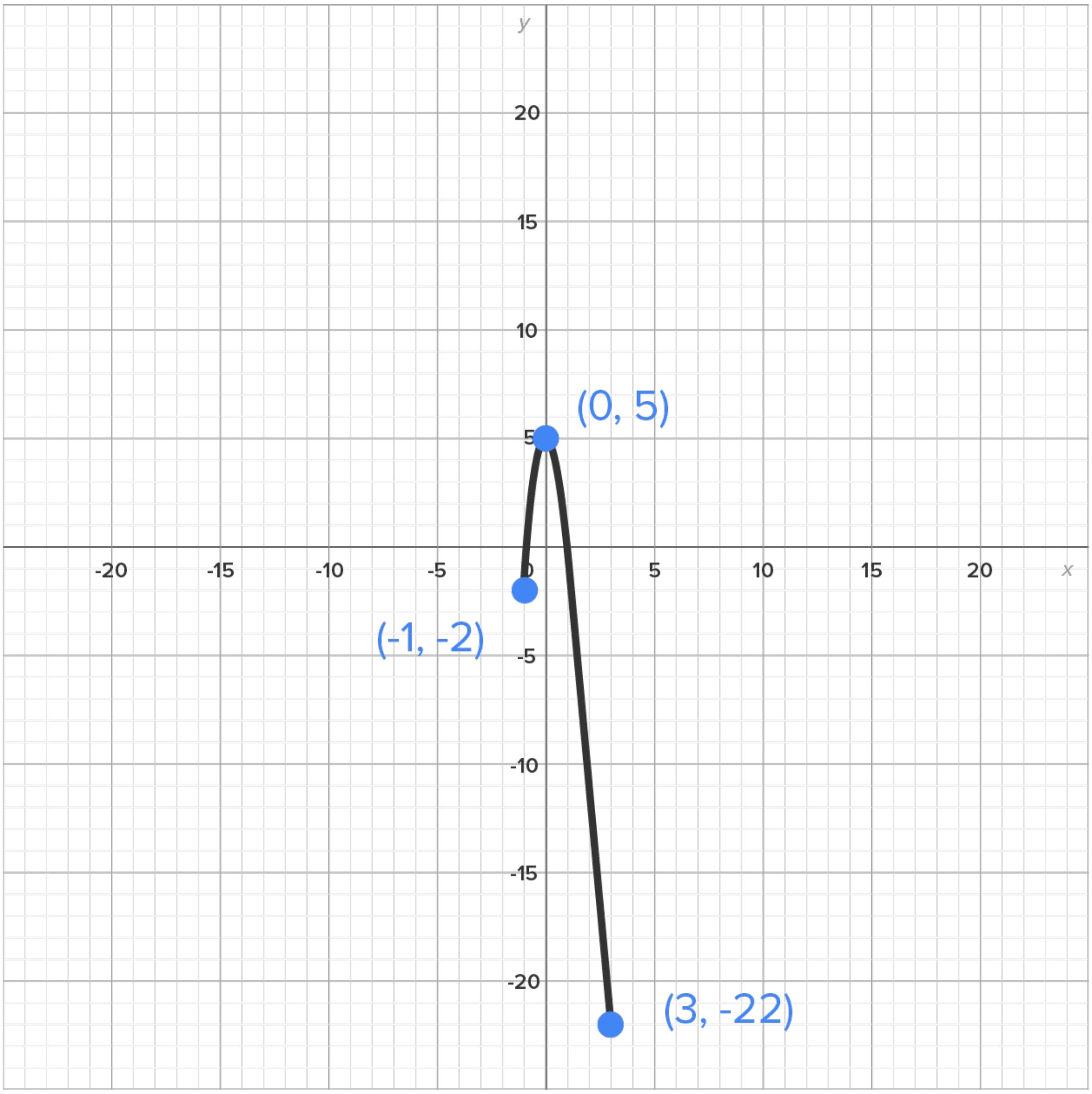
However, since only the closed interval $[-1, 3]$ is considered, the critical value $x = 4$ is not used.

Now, evaluate $f(x)$ at the endpoints, $x = -1$ and $x = 3$, and the remaining critical number, $x = 0$.

x	$f(x)$	Result
-1	$(-1)^3-6(-1)^2+5=-2$	Neither a Global Maximum or Global Minimum
0	$(0)^3-6(0)^2+5=5$	Global Maximum
3	$3^3-6(3)^2+5=-22$	Global Minimum

In conclusion, the global maximum occurs at the point (0, 5) and the global minimum occurs at the point (3, -22). In order words, the global maximum value is 5 and occurs when $x=0$; and the global minimum value is -22 and occurs when $x=3$.

The graph of the function on $[-1, 3]$ is shown below, which confirms the results.



WATCH

In this video, we'll find the global minimum and maximum values of $f(x)=10\sqrt{x}-x$ on the interval $[16, 64]$.

Video Transcription

Hello there. Welcome back. What we're going to look at in this video is the extreme value theorem, meaning that if you have a continuous function on a closed interval, there is a guaranteed absolute maximum value, or as we sometimes call global maximum value and a global minimum value.

And looking at our function here, our function is definitely continuous. There are no breaks in the graph. So therefore, we can use the theorem. So the way mins and maxes work, we definitely have to check the endpoints. So we have x equals 16 and x equals 64 as candidates for where maximum or minimum could occur.

But we also need to check the inside of the interval by finding the critical number. And those are values, remember, where the derivative is either 0 or undefined. And more importantly, those are values of x where we could see a minimum or a maximum. So it comes down to taking the derivative, setting it equal to 0, finding any critical numbers that are on the interval, and then comparing those values against the ones at the endpoints.

So I am going to start by making a table, since we are going to be substituting several values of x, and we already know that 16 is going to get substituted and we already know that 64 is going to get substituted. So let's see what else we can find here. So I'm going to first write the function in a form that's easier to find the derivative. And that is using exponents instead of radical symbols. And then the derivative is-- well, let's see.

1/2 times 10, is 5x to the negative 1/2 minus 1. And just looking at that, we have x to a negative power, which means 1 over x to a positive power. I'm going to write it that way so we can more clearly see what's happening. We see that this is undefined when x equals 0. But that value is of no concern to us, because that's outside of the interval that we're interested in. So we can ignore the fact that this is not defined when x equals 0.

So I'm going to focus on where the derivative is equal to 0. So then I'm going to add 1 to both sides. And I'm going to multiply both sides by x to the 1/2 power. Now, remember, x to the 1/2 is really the square root. So if it helps you to solve that, convert that to a square root. And then I'm going to square both sides and we're going to get 25 equals x. And that, remember, is what we call a critical number.

More importantly, it's inside of our interval, so that means there definitely could be a min or a max happening when x equals 25. So we'll substitute that there, or we'll place that there. Now we're ready to substitute all of our values. We're going to plug in 16, 64 and 25, and see what the biggest and smallest value of f are.

So plugging in 16, we get 10 square root of 16 minus 16. 10 times 4 is 40, minus 16 is 24. If we substitute 64, we have 10 root 64 minus 64. 10 times 8 is 80, minus 64 is 16. And finally, substituting 25, 10 root 25 minus 25.

Root 25 is 5 times 10 is 50, minus 25 is 25. So look at that. We have our maximum value here and our minimum value here. So that means that the global minimum value is 16, which occurs when x equals 64. So we can say at the point 64, comma 16. And the global maximum value is 25, which occurs at the point 25, 25. And there, we have the extreme value theorem.

SUMMARY

In this lesson, you learned that when $f(x)$ is continuous on a closed interval, **the extreme value theorem** guarantees a global minimum value and a global maximum value at some location within the closed interval. Then, you applied this theorem to **find extreme values of a continuous function on a closed interval**, by first finding all critical numbers of $f(x)$ that are in the interval $[a, b]$, then evaluating $f(x)$ at each endpoint and each critical number. This concept is going to be very useful once we use derivatives to solve optimization problems.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Extreme Value Theorem
If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

Rolle's Theorem

by Sophia

WHAT'S COVERED

In this lesson, you will learn about Rolle's theorem, a seemingly simple yet powerful theorem whose consequences are used in Unit 5 (Antiderivatives). Specifically, this lesson will cover:

1. Introduction to Rolle's Theorem
2. Applying Rolle's Theorem

1. Introduction to Rolle's Theorem

Let's say we have a function that passes through the points (1, 6) and (5, 6).

TRY IT

Take a piece of paper and draw the points (1, 6) and (5, 6). Connect the two points with a curve that is continuous and differentiable (something other than a horizontal line between them). This means that the graph has no break and no sharp turn.

What do you notice about your curve? Does your curve contain at least one horizontal tangent? +

Hopefully you have at least one horizontal tangent. As it turns out, under certain circumstances, this will always happen.

BIG IDEA

Rolle's theorem:

Let $f(x)$ be continuous on the closed interval $[a, b]$ with $f(a) = f(b)$, and differentiable on the open interval (a, b).

Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = 0$.

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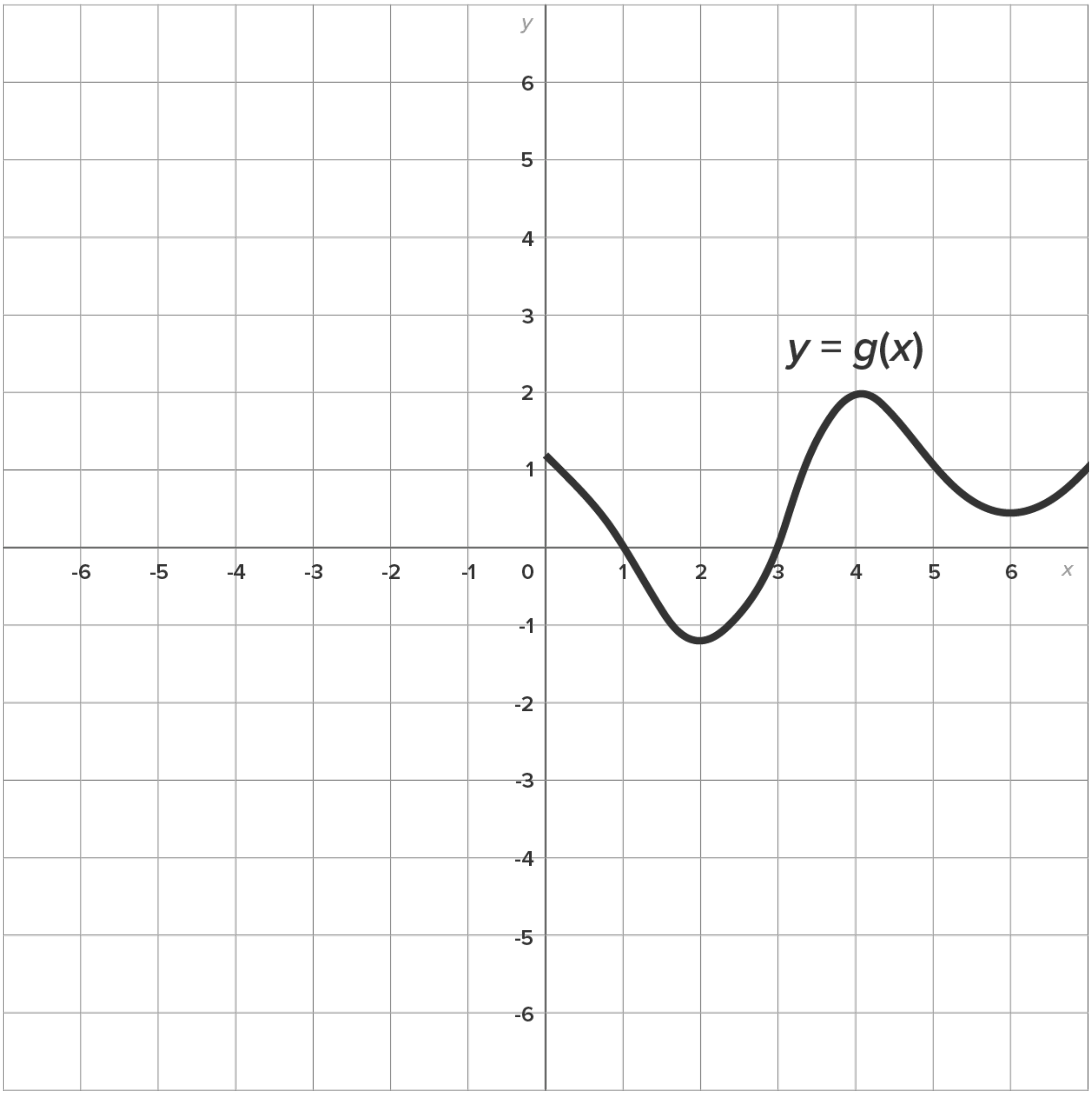
2. Applying Rolle's Theorem

Now, let's look at a few examples of how Rolle's theorem can be applied.

EXAMPLE Here is the graph of some function $y = g(x)$, where $g(0) = g(7)$.

Since $g(x)$ is continuous and differentiable, it follows by Rolle's theorem that there is at least one value of c between 0 and 7 where $f'(c)=0$.



In the graph, we can see there are three x-values where a horizontal tangent line occurs: $x=2$, $x=4$, and $x=6$. Therefore, the guaranteed values of c are 2, 4, and 6.

↱ EXAMPLE Consider the function $f(x)=3x+\frac{3}{x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

First, check requirements for Rolle's theorem.

$f(x)$ is continuous on any interval not including 0, and therefore is continuous on $\left[\frac{1}{2}, 2\right]$.

$f(x)$ is differentiable everywhere except where $x=0$, so $f(x)$ is certainly differentiable on $\left(\frac{1}{2}, 2\right)$.

$f\left(\frac{1}{2}\right)=3\left(\frac{1}{2}\right)+\frac{3}{\left(\frac{1}{2}\right)}=\frac{15}{2}$ and $f(2)=3(2)+\frac{3}{2}=\frac{15}{2}$. Therefore, $f(a)=f(b)$.

Thus, the conditions of Rolle's theorem have been met and there is at least one value of *c* between $\frac{1}{2}$ and 2 such that $f'(c) = 0$.

To find all values of *c*, take the derivative, then set equal to 0, then solve.

$f(x) = 3x + \frac{3}{x}$	Start with the original function.
$f(x) = 3x + 3x^{-1}$	Rewrite to use the power rule.
$f'(x) = 3 - 3x^{-2}$	Take the derivative.
$f'(x) = 3 - \frac{3}{x^2}$	Rewrite with positive exponents.
$3 - \frac{3}{x^2} = 0$	Set equal to 0.
$3 = \frac{3}{x^2}$	Add $\frac{3}{x^2}$ to both sides.
$3x^2 = 3$	Multiply both sides by x^2.
$x^2 = 1$	Divide both sides by 3.
$x = \pm 1$	Take the square root of both sides.

Since we want all values on the interval $\left(\frac{1}{2}, 2\right)$, the value guaranteed by Rolle's theorem is $c = 1$. (In other words, since $c = -1$ is not on the interval $\left(\frac{1}{2}, 2\right)$, it is not considered.)

WATCH

In this video, we will find all values of *c* guaranteed by Rolle's theorem for $f(x) = 20\sqrt{x} - 2x$ on the interval $[16, 36]$.

Video Transcription

Hi there, and welcome back. What we're going to look at, here is an example to illustrate Rolle's theorem, which remember Rolle's theorem says that if you have a continuous function on a closed interval, and the value of the function at each endpoint is the same, then we're guaranteed a horizontal tangent somewhere in the interval. So that's what we're going to check here.

Our function here, as we see, 20 square root of x minus 2x. It is continuous on the interval 16 to 36. There are no breaks. We can see that with a square root function. And it is differentiable on that interval as well. So the thing we have to verify is that f of 16 and f of 36 are the same.

So f of 16 is 20 square root of 16 minus 2 times 16. Square root of 16 is 4, times 20 is 80. So that is 80 minus 32, which is 48. And f of 36 is 20 square root 36 minus 2 times 36, which is 120 minus 72. 120 minus 72, which is also 48. So those two function values are equal.

So that means that somewhere in the middle of the interval, we are guaranteed a horizontal tangent. So what we do is we take the derivative of f And we set it equal to 0 to find where that horizontal tangent is. And we know that location should be somewhere between 16 and 36.

So let's find out. So f of x, I'm going to rewrite as 20x to the 1/2 minus 2x. And then I'm going to take the derivative. So 20 times 1/2 is 10x, to the-- take one away from the power. That's x to the negative 1/2, minus 2 equals 0.

Now, before I get to solving, I'm going to rewrite x to the negative 1/2 in a more familiar form, which means I'm going to write this as 10 over x to the 1/2 minus 2 equals 0, which means 10 over square root of x minus 2 equals 0. So then to solve this, I'm going to add 2 to both sides. And then multiply both sides by the square root of x. And then divide both sides by 2, just to isolate the square root of x to one side. And then finally, we're going to square both sides.

And as we can see, this is the value of c, even though we're calling it x. This is the value of c that is guaranteed by Rolle's theorem. The horizontal tangent occurs at 25. The endpoints were 16 and 36. 25 is right between them. And there we have Rolle's theorem.

SUMMARY

In this lesson, you learned that when a function is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a)=f(b)$, then **Rolle's theorem** guarantees that there is a value of *c* between *a* and *b* such that $f'(c)=0$, which means that there is a guaranteed horizontal tangent line at *c*. Then, you examined a few examples involving the**application of Rolle's theorem**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Mean Value Theorem for Derivatives

by Sophia

WHAT'S COVERED

In this lesson, you will learn another theorem called the mean value theorem, whose consequences are used in Unit 5 (Antiderivatives). Specifically, this lesson will cover:

1. The Mean Value Theorem for Derivatives and a Real-Life Connection
2. Applying the Mean Value Theorem for Derivatives

1. The Mean Value Theorem and a Real-Life Connection

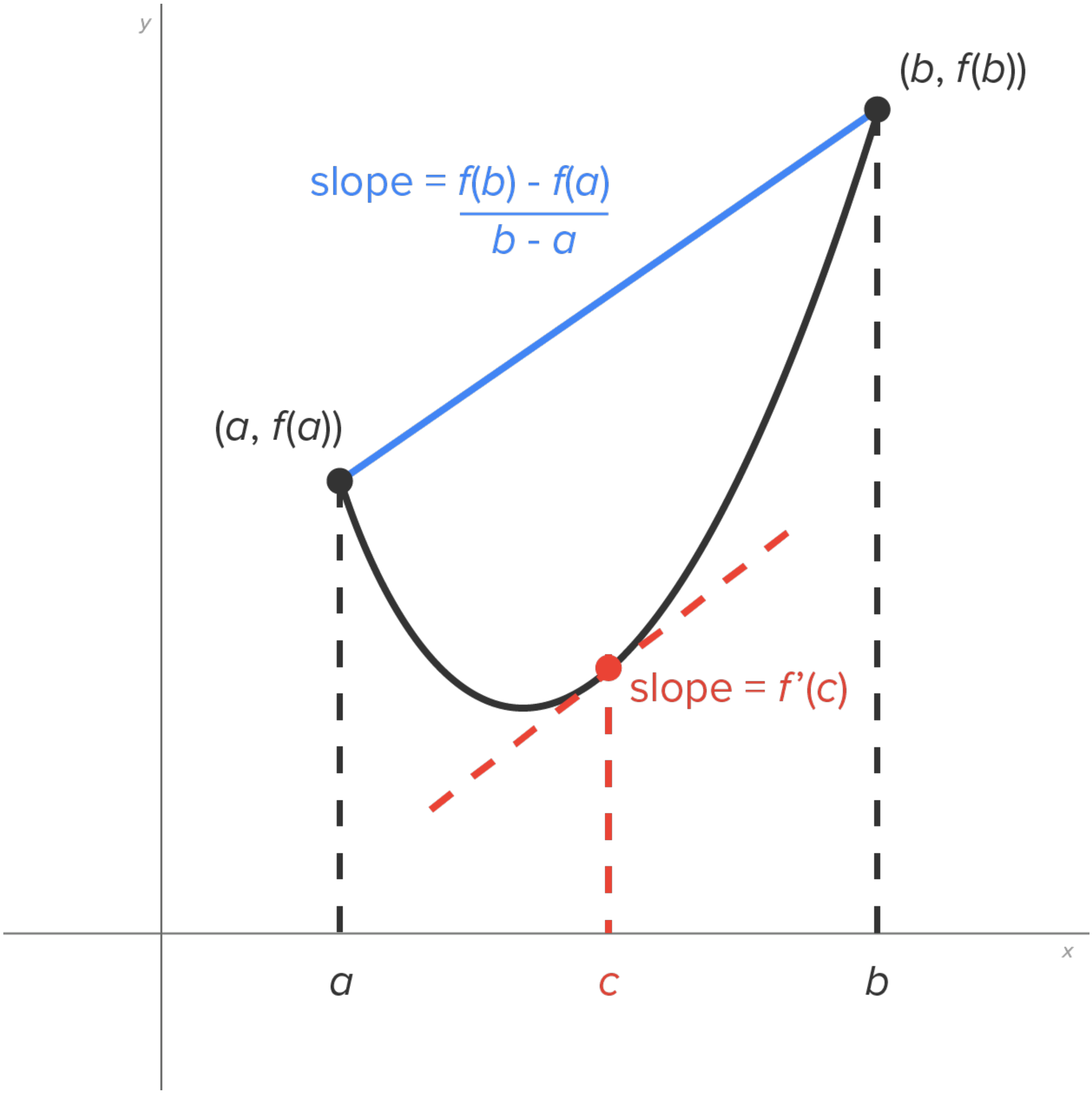
The distance between two toll booths on a major highway is 2 miles, and it took you 1.5 minutes to get from one toll booth to the other. Assume the speed limit on the highway is 65mph. Seeing no police cars on the route, you are surprised that you are being pulled over, and it turns out it is for speeding. What happened?

Let's look at this situation: It took 1.5 minutes to travel 2 miles. On average, your velocity was:

$$\frac{2 \text{ miles}}{1.5 \text{ minutes}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{120 \text{ miles}}{1.5 \text{ hours}} = \frac{80 \text{ miles}}{\text{hour}}$$

It stands to reason that at some point, you had to be traveling at a speed of 80 mph. Therefore, we are saying that there is a single point where the instantaneous velocity between the toll booths is equal to the average velocity traveled between the toll booths. This is the idea behind the **mean value theorem for derivatives**.

That is to say, there is a value of c where the instantaneous rate of change in f is equal to the average rate of change of f over the interval $[a, b]$.



As you can see, the tangent line at *c* and the slope of the line between the endpoints are identical.

TERM TO KNOW

Mean Value Theorem for Derivatives

Let $f(x)$ be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b). Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

---

2. Applying the Mean Value Theorem for Derivatives

Now that we have defined the mean value theorem for derivatives, let's see how it applies to some functions.

→ EXAMPLE Consider the function $f(x)=x^2-3x$ on the interval $[1, 3]$. Note that $f(x)$ is continuous on $[1, 3]$ and differentiable on the interval $(1, 3)$.

Then, we should be able to find a value of c between 1 and 3 where $f'(c)=\frac{f(3)-f(1)}{3-1}$.

First we'll compute $\frac{f(3)-f(1)}{3-1}$.

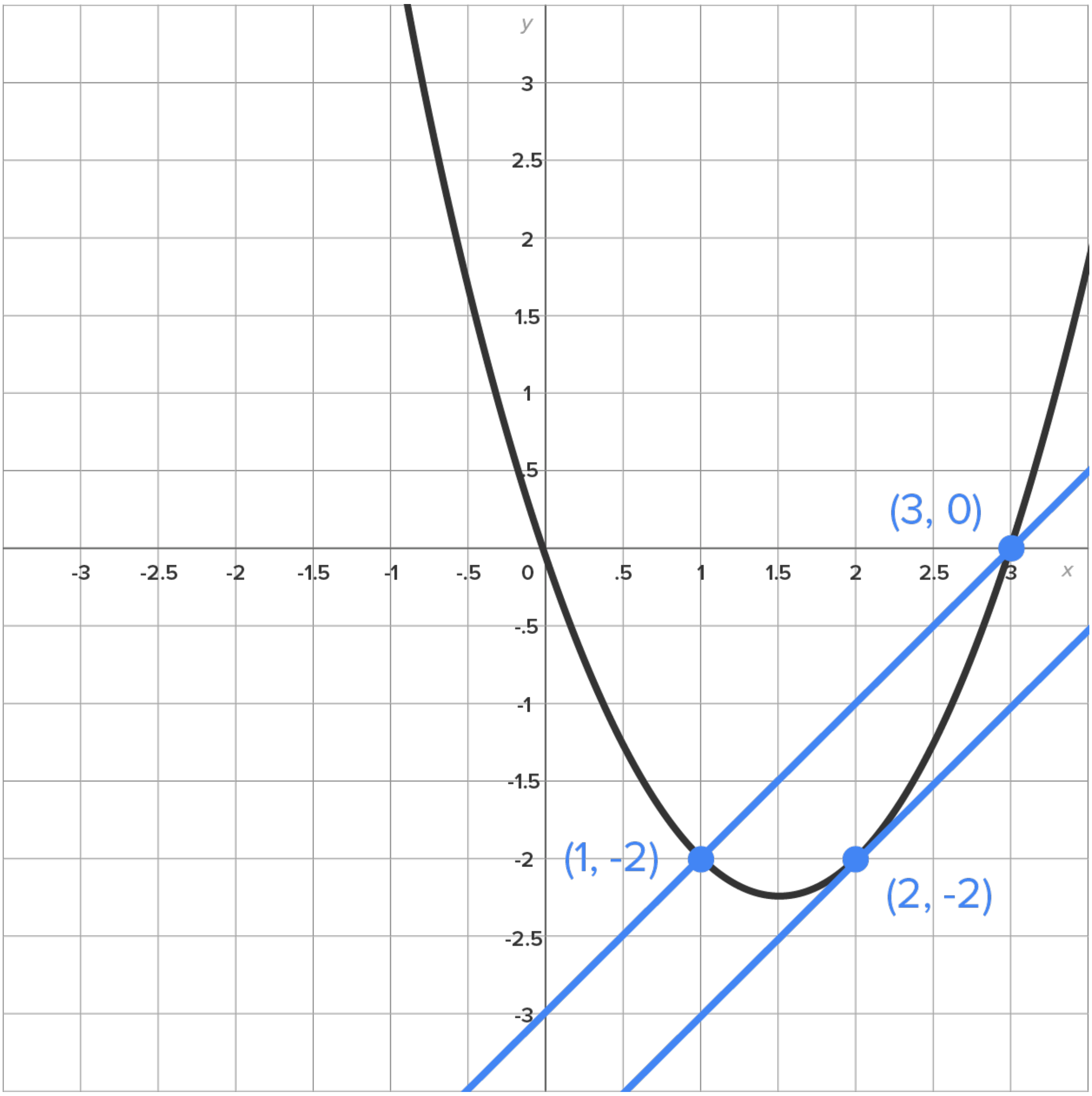
- We have $f(1)=-2$ and $f(3)=0$.
- Then, $\frac{f(3)-f(1)}{3-1}=\frac{0-(-2)}{3-1}=1$.

Now, we want to find the value of c guaranteed by the mean value theorem.

$f(x)=x^2-3x$	Start with the original function.
$f'(x)=2x-3$	Take the derivative.
$2x-3=1$	Set the derivative equal to $\frac{f(3)-f(1)}{3-1}$.
$2x=4$	Add 3 to both sides.
$x=2$	Divide both sides by 2.

Since $x=2$ is clearly inside the interval $(1, 3)$, $c=2$.

Conclusion: The value guaranteed by the mean value theorem for derivatives is $c=2$. The graph of this situation is shown here.



The curve is $f(x) = x^2 - 3x$.

Note how the slope of the tangent line at $x = 2$ is equal to the slope of the secant line between $x = 1$ and $x = 3$.

WATCH

In this video we will find the value(s) of c guaranteed by the mean value theorem for derivatives for the function $f(x) = x^3 - 2x$ on the interval $[1, 4]$.

Video Transcription

[MUSIC PLAYING] [MUSIC PLAYING]

Hello. It's good to see you again. In this video, we're going to look at the mean value theorem and use a function to illustrate the consequence of the theorem. Remember that the mean value theorem says that if a function is continuous on a closed interval and differentiable on the open interval, then there's a value of c somewhere between a and b-- a and b being the end point of the interval-- where this quantity right here holds.

So we're saying that the derivative at some point in the middle of the interval or within the interval is equal to the average rate of change over the interval. f of b minus f of a divided by b minus a. So we're going to start here with a function, f of x equals x cubed minus 2x. It is a polynomial, so we know that it is continuous, and it is differentiable on any interval we could use.

We are going to use the interval 1 to 4. So what we want to do is find the derivative and set it equal to the quantity f of b minus f of a over b minus a. So we're going to compute that first. So f of 4 minus f of 1 over 4 minus 1 is equal to-- well, let's see. f of 4 is equal to 4 to the third minus 2 times 4-- which is 64-- minus 8-- which is 56.

f of 1 is 1 to the third minus 2 times 1, which is negative 1. So you have 56 minus negative 1. And then 4 minus 1 is 3. So this is 57 over 3, which is 19. Interesting result. OK. So now what we do is we find the derivative of our function. So f prime of x is 3x squared minus 2. And the value of c guaranteed by the mean value theorem is where the derivative is equal to 19.

So now, we solve for x, and hopefully we get a number between 1 and 4. So let's see what happens there. We're going to add 2 to both sides. We get 3x squared equals 21. We can divide by 3. And that means that x is equal to plus or minus the square root of 7. But remember, we're focused on the interval 1 to 4.

So if we look at those numbers, negative square root of 7 versus the square root of 7, negative square root of 7 is not on the interval. It does solve the equation, it's just not a solution to our problem. So we're going to discount that. And the square root of 7 is between 2 and 3, so definitely between 1 and 4. So the square root of 7 is the value guaranteed by the mean value theorem.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that **the mean value theorem for derivatives** guarantees that for a continuous function on a closed interval $[a, b]$, the derivative (slope of the tangent line) at some point on the interval is equal to the average rate of change (slope of the secant line) over the entire interval. You examined a **real-life connection** by seeing how the MVT can be used to catch drivers who drive too fast, followed by several examples **applying the mean value theorem for derivatives** to some functions. This idea will be utilized in Unit 5 when we start to explore antiderivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Mean Value Theorem for Derivatives

Let $f(x)$ be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b).

Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

First Shape Theorem

by Sophia

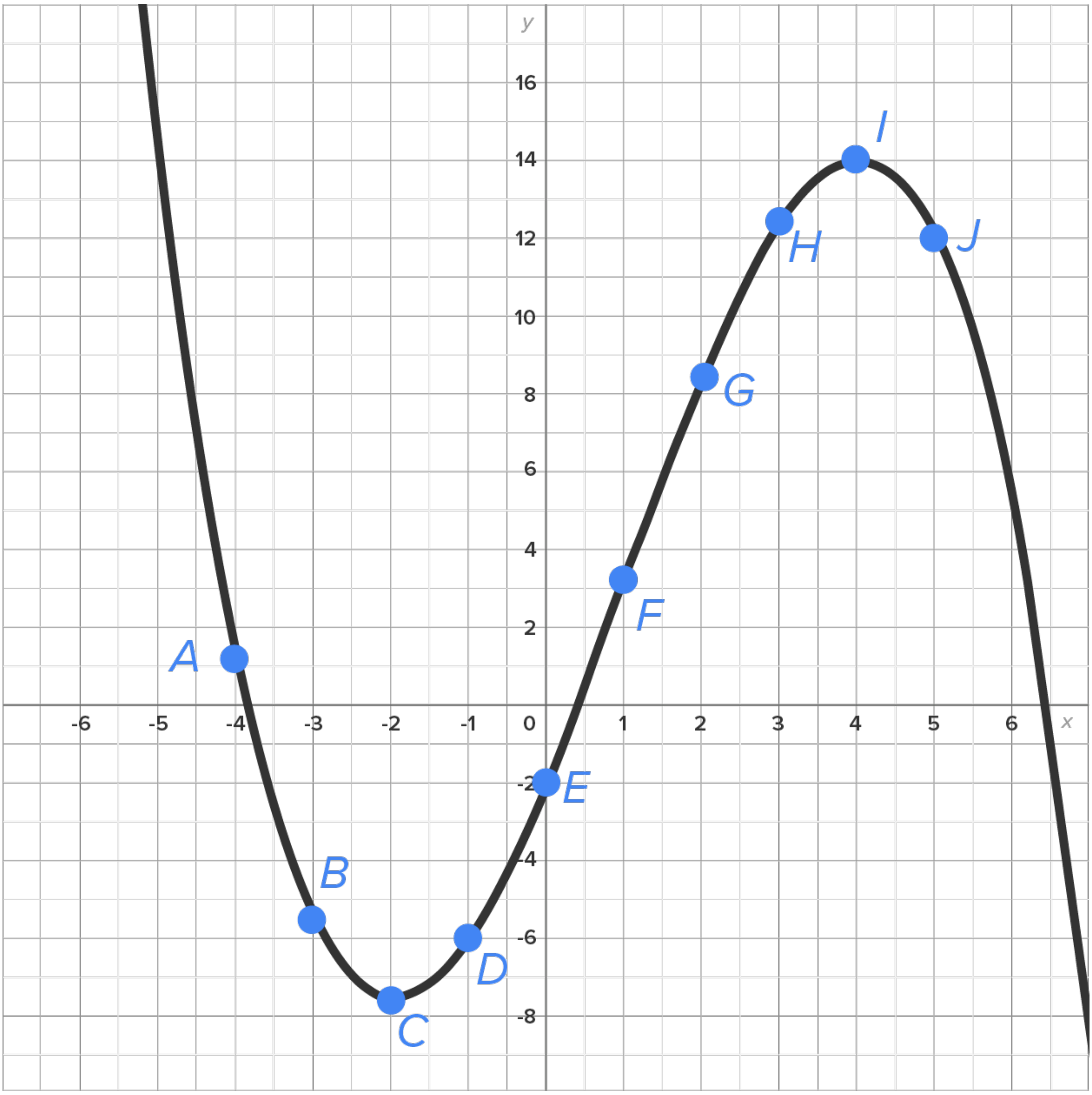
WHAT'S COVERED

In this lesson, you will use properties of a function $f(x)$ to sketch the graph of its derivative, $f'(x)$. Specifically, this lesson will cover:

1. What $f'(x)$ Tells Us About the Graph of $y = f(x)$
2. Using Slope to Graph $y = f'(x)$ Given $y = f(x)$

1. What $f'(x)$ Tells Us About the Graph of $y = f(x)$

Consider the graph of a function $y = f(x)$, shown below.



Note that the graph is decreasing at points *A*, *B*, and *J*. Notice also that the slopes of the tangent lines at each of these points are negative.

Note that the graph increases at points *D*, *E*, *F*, *G*, and *H*. Notice also that the slopes of the tangent lines at each of these points are positive.

Finally, points *C* and *I* are local maximum/minimum points. Notice also that the slope of the tangent line at each of these points is zero.

This leads to a very useful link between the behavior of $f(x)$ and the value of $f'(x)$.

BIG IDEA

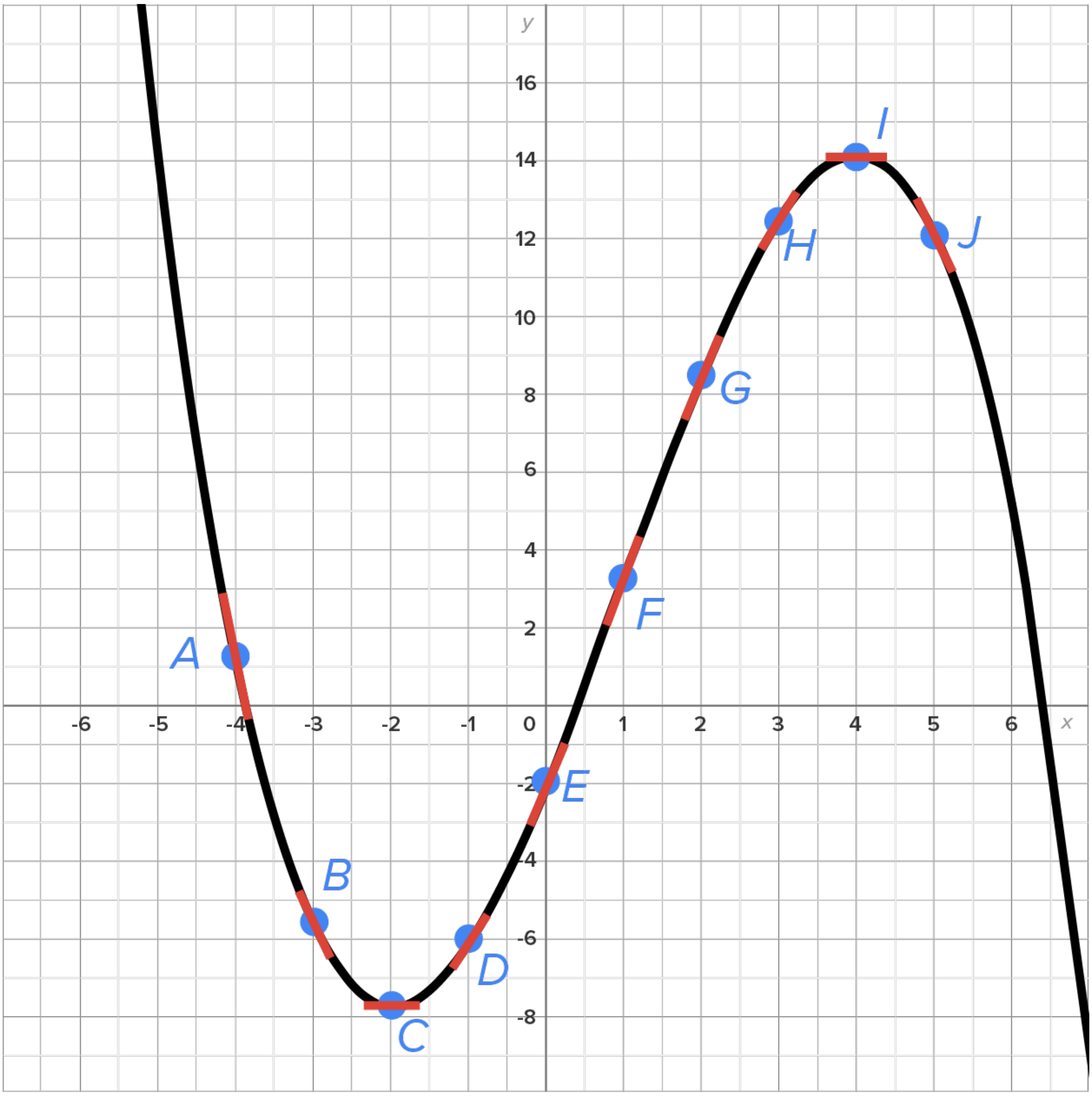
If $f(x)$ is increasing at $x = a$, then $f'(a) > 0$.
If $f(x)$ is decreasing at $x = a$, then $f'(a) < 0$.

---

2. Using Slope to Graph $y = f'(x)$ Given $y = f(x)$

Given what we know about $f'(x)$ when $f(x)$ is increasing or decreasing, we can get a rough sketch of the graph of $f'(x)$ when given the graph of $f(x)$.

↪ EXAMPLE Consider the graph of $y = f(x)$ shown below with tangent line segments at points *A* through *J*. Notice also the local minimum at point *C* and the local maximum at point *I*.

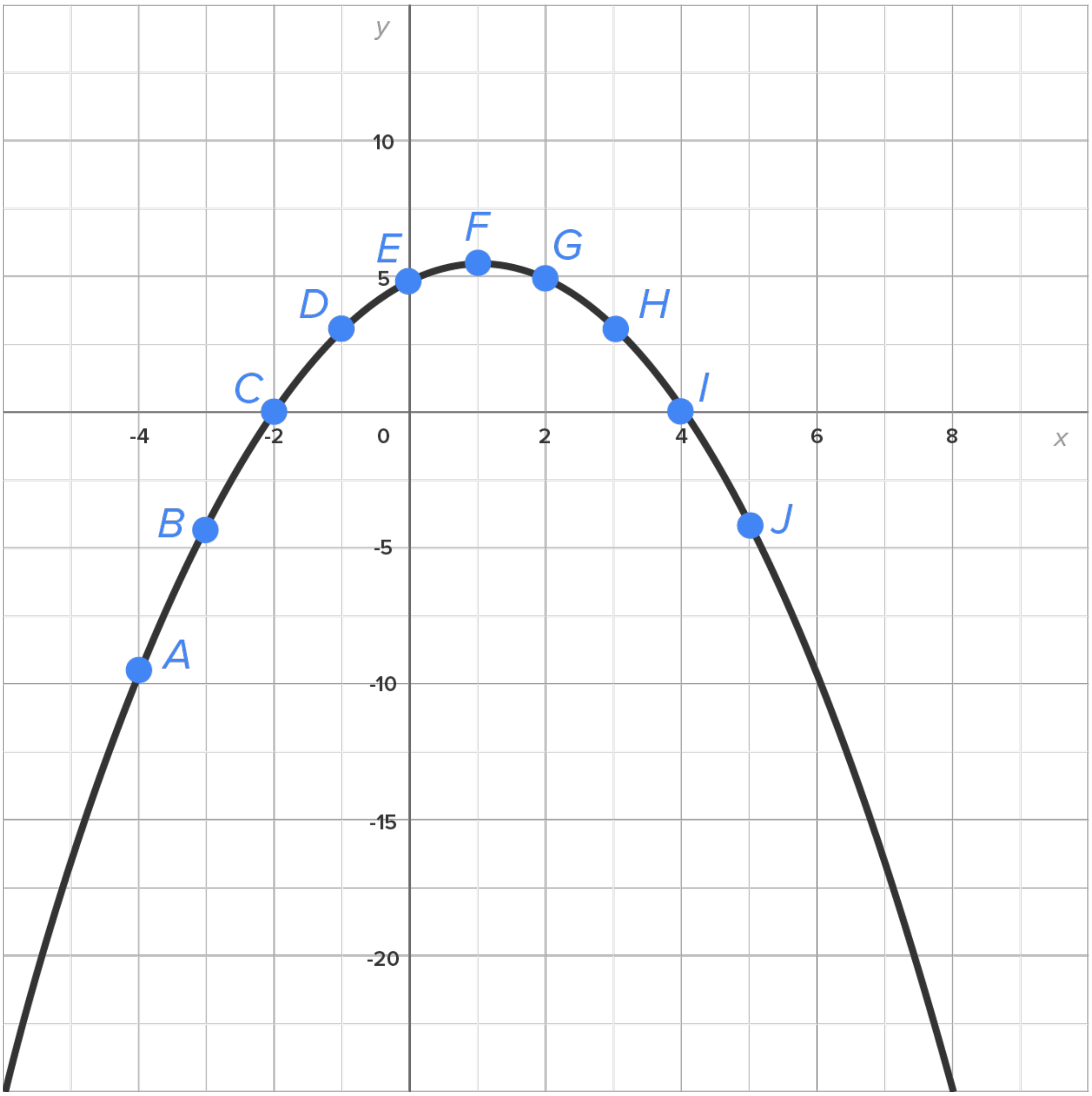


The behavior of $f'(x)$ can be summarized in the following table at each point. Remember that $m_{\tan}$ is the value of $f'(x)$ at any point.

Point	Value of $f'(x)$
A	$f'(x) < 0$
B	$f'(x) < 0$, but the value of $f'(x)$ is larger than its value at *A*

C	$f'(x) = 0$ (horizontal tangent line)
D	$f'(x) > 0$
E	$f'(x) > 0$, but its value is noticeably greater than the slope at point *D*
F	$f'(x) > 0$, but its value is slightly greater than the slope at point *E*
G	$f'(x) > 0$, but its value is slightly less than the slope at point *F*
H	$f'(x) > 0$, but its value is noticeably less than the slope at point *G*
I	$f'(x) = 0$ (horizontal tangent line)
J	$f'(x) < 0$

The graph of the derivative is shown here. Note that the points *A* through *J* have the same x-coordinates as those marked on the graph of $f(x)$.



WATCH

In this video, we'll sketch the derivative of a function given its graph.

Video Transcription

[MUSIC PLAYING] Hello there, and welcome back. What we're going to do is take the graph of y equals f of x and use it to sketch the graph of its derivative. Now remember, the derivative of f of x is really the slope of the tangent line at a value of x. So what we're basically doing is examining how the slopes of the tangent lines change as x changes.

So in order to get a sense of what's happening here-- I'm just going to choose x equals 0 here, and then we're going to change in both directions to see what our slopes are doing. So at x equals 0, notice that the slope is positive. It's not a very large positive number from what it looks like. We don't know the scale of the function, and that's OK. Because all we're doing is looking at what happens as we change.

So as we go to the right, x increasing, notice that the slopes are changing pretty sharply. Here's a slope that's larger than the first one, even larger than the second one, and even larger and larger and larger. So that means the values of those slopes are what I'm going to plot here. So starting at x equals 0, I'm going to say that that is the slope.

And then we're increasing quite sharply. So very similar to the actual function. We'll talk about that in a moment. And then as if-- if we choose values of-- negative values of x, well, let's see what's happening. As x goes to the left, those slopes are still positive, but they're getting closer and closer to 0.

So that means as x goes to the left, my derivative graph is also going to basically hug the x-axis. So it would be a horizontal asymptote there. And I didn't really do a good job of making it very close to the x-axis. Let me try that again. There. That's better. So there's the graph of our derivative, and notice it looks very similar to the graph of f.

And the graph of f really reminds me of an exponential function. And as we know-- let's just say it's e to the x-- we know the derivative of e to the x is, again, e to the x. So it looks very similar to its original function. So least we have that link to go back to too. But again, to graph a derivative, we look for things like how do the slopes change.

And as you go to the right, they're changing sharply. They're going way up. The slopes are going way up-- I should say-- which means that the derivative increases pretty sharply. And as we go to the left, my slopes are getting closer to 0, which means my derivative graph is getting close to 0 as x moves to the left. So there we have sketched in the graph of a derivative.

[MUSIC PLAYING]

WATCH

In this next video, we'll sketch the derivative of a function given its graph.

Video Transcription

[MUSIC PLAYING] Greetings and welcome back. What we're going to do in this video is having a look at the graph of this function y equals f of x. We're going to try to sketch the graph of its derivative. Remember, with these graphs we're only really focused on the shape of the graph. We're not necessarily focused on specific values of x, although that can be helpful. Notice how I have my axes positioned. I draw my blank graph below the graph of f, that way we can easily correspond the common values of x for the special points on the graph where we know the value of the derivative.

So remember that the easiest points to spot on the graph of a function are where its horizontal tangents are. I'm going to label them with points right here. So that means at the same value of x I know the value of the derivative is zero, which means at those points the graph of the derivative is right on the x-axis. So what else is happening on this?

Well, we notice that there is a cusp in between the horizontal tangents, which suggests that the derivative is going infinite undefined derivative. So that means that we're going to have a vertical asymptote at that same value of x. So let's reconcile what's happening up to the asymptote. So starting with the left hand horizontal tangent, notice that the derivative starts out at 0. The slope of the tangent line starts out at 0 and as you move toward the cusp these slopes are getting more, and more, and more negative. So that means that the graph of the derivative is under the x-axis and moving in a very sharp decreasing direction.

Now, what's happening between the cusp and the other horizontal tangent? Notice that if you start at the horizontal tangent the slope is zero and then these slopes are positive, but getting much more positive and much more sharp as you get closer to the asymptote. That suggests that the derivative graph is going to look like this. OK? So then what's happening on the other side of those maximum points, the horizontal tangent points?

As you go to the left of this horizontal tangent, notice that the slopes are positive and as you go to the left the slopes are getting steeper and they're positive. So that means that we're going to have something like this. The slopes are positive and getting steeper, which means they're increasing in value, and to the right of the right hand horizontal tangent notice that the slopes are negative. So that means this is going to punch through the x-axis. But then notice as x goes to the extreme right those slopes are getting closer to 0, which means there is a point where the slope is where we're going to say most negative, I guess you could say. Then the slopes come back up to get closer to 0.

So there is our best guess at what the graph of the derivative looks like because we don't know what the equation of the original function is, but based on the properties this is our best guess.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned about a useful link between the behavior of $f(x)$ and the value of $f'(x)$. Specifically, **given the graph of** $y=f(x)$, it is possible to sketch **the graph of** $y=f'(x)$ by **using slopes** of the tangent lines at given points and their respective behavior.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Second Shape Theorem

by Sophia

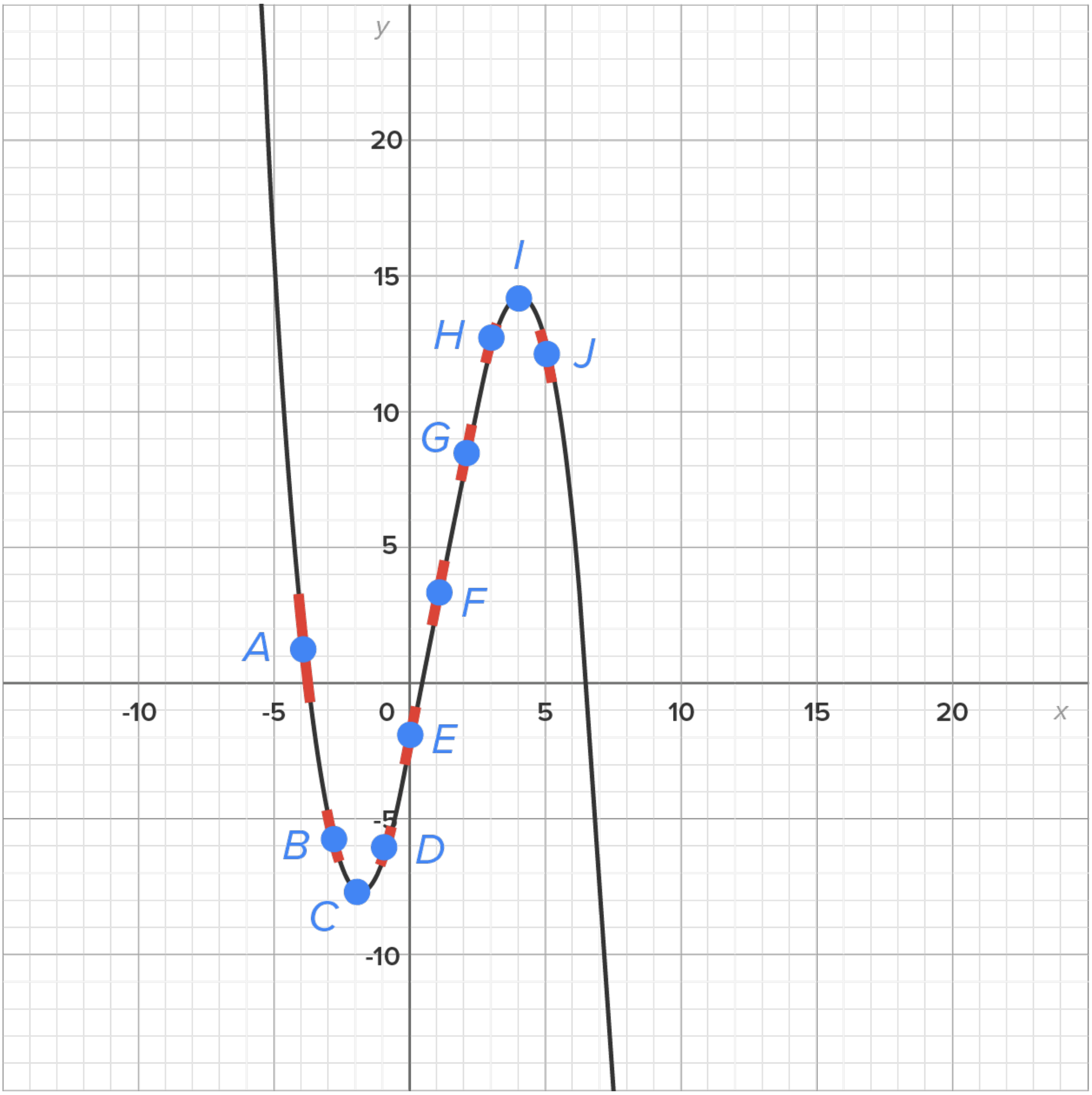
WHAT'S COVERED

In this lesson, you will sketch the shape of a function $f(x)$ given values of its derivative, $f'(x)$. Specifically, this lesson will cover:

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$
2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$

Consider the graph below of a function $y = f(x)$, which is the same as the graph in the last tutorial.



Notice that the graph of $f(x)$ is increasing at every point where its tangent line has a positive slope, and the graph of $f(x)$ is decreasing at every point where its tangent line has a negative slope. This means we can extend the ideas of increasing and decreasing from points to intervals.

BIG IDEA

If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that same interval.
If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that same interval.

Based on this fact, we can use the derivative to determine entire intervals over which a function $f(x)$ is increasing or decreasing.

WATCH

In the following video, we'll find all intervals over which $f(x) = -x^3 + 3x^2 + 24x + 10$ is increasing or decreasing. One of the main objectives of this video is to show how the information gets organized, so viewing this video is important!

Video Transcription

Hi there. I hope you're ready to do some math, because we're going to look at a function analytically to determine where it's increasing and decreasing. And remember that that information is obtained by looking at the first derivative. We know that a positive slope means that the graph is increasing, and we know that a negative slope means that the graph is decreasing.

So what we're going to do to do the analytic piece is we first want to figure out where the graph is transitioning. And that is as simple as figuring out where the derivative is 0, because if it's 0, I know that it's either positive or negative on one side and positive or negative on the other side. So it's easier to find those points first and then compare.

So here's what we do. We take our first derivative. And this is just the power rule-- negative 3x squared plus 6x plus 24. And we know that this transition point could occur either when the derivative is equal to 0 or undefined. Now we have a polynomial function here, which we know is never undefined. So we are going to stick with setting it equal to 0.

So we're going to solve, and that means I'm going to factor out a negative 3, like so. And I noticed that factor is quite nicely. So I'm going to factor that into x minus 4 and x plus 2. And by setting each factor equal to 0, I know I can ignore the negative 3 because negative 3 is never 0. This one means that x equals 4, and this one means that x equals negative 2. So these are our points where the graph could be transitioning between increasing and decreasing. We call those the critical numbers.

So what do we do with this? And the major point of this video is to see how to organize this information to get the most out of it. So we're going to make a sign graph, kind of a number line, and what we notice is this number line gets broken into three pieces. We have our critical number at negative 2. We have our other critical number at 4, which divides the number line into three intervals. We have negative infinity to negative 2, we have negative 2 up to 4, and we have 4 to infinity.

Now what is the what's the business with these intervals? Picking the first interval, negative infinity to negative 2, this tells me that whatever value of x I pick on in that interval. If I plug it in to f prime, I'm going to get the same sign-- sign. It's either going to always be positive or always be negative because the only way you could change from negative to positive or positive to negative is if you pass through 0 if it's a continuous function, which we know we have here. So we use test numbers for each interval to tell what the whole interval is behaving like.

So I'm going to select three values of x because there's three intervals. And there's no science behind what values you pick, but we're going to try to keep it simple for ourselves. So since the critical number is negative 2, I'm going to select negative 3. For the interval negative 2 to 4, I'm going to pick 0 because that's probably the easiest number to use. And on 4 to infinity, I'm going to select the number 5, just because it's close to 4.

And what we're going to do is substitute these into the first derivative to see if they're positive or negative. And that will, in turn, tell us whether or not the function is-- or whether the function is increasing or decreasing. So f prime of negative 3 is negative 3 times negative 3 squared. I'm basically using this right here, using the derivative before we've manipulated it-- plus 6 times negative 3 plus 24. And if you crunch that out, you get negative 21, which tells me I have a negative slope, which tells you that my function is decreasing.

So I tend to write it like this here-- just write a decreasing line just to give it a little bit of visual appeal. So plugging in 0-- now one thing we notice is that any of the terms are going to just drop out because 0 to any power is 0. But it's nice to show it out. So we have 24, which means graph is increasing on that interval.

And last but not least, at 5, we have negative 3 times 5 squared plus times 5 plus 24. And that is negative 21 again, so we're back to decreasing.

OK, so looking at that sign graph, there's actually a lot of information there, and it looks like we have a minimum point somewhere and a maximum point somewhere, but that's for further on down the line. We can write out the intervals over which this function is increasing and decreasing. So we could say this function is increasing on the interval, while the only interval where it's increasing is the one in the middle, negative 2 to 4 and it's basically decreasing everywhere else.

But remember, there's a certain way we write this. Decreasing on the interval now, we have negative infinity to negative 2. And we also have the interval 4 to infinity. Since those two intervals describe a set of values that have something in common-- in this case, where the function is decreasing-- we put a U in between them to represent that there's a union of those sets.

And that's our answer. So analytically, we were able to determine where the function was increasing and decreasing. And just so you have another visual to go along with this, I have the graph of the function right here. And as you can see, it is decreasing up to x equals negative 2, increasing between negative 2 and 4, and then decreasing again after x equals 4. So that more or less confirms our results.

Now that you see how to organize the information, let's look at more examples.

↱ EXAMPLE Let $f(x)=\frac{x}{x^2+1}$. Determine the intervals over which $f(x)$ is increasing or decreasing.
Since this information comes from the first derivative, we start there. Finding the derivative of this function requires the quotient rule.

$f(x)=\frac{x}{x^2+1}$	Start with the original function; the domain is all real numbers.
$f'(x)=\frac{(1)(x^2+1)-x(2x)}{(x^2+1)^2}$	Apply the quotient rule.
$f'(x)=\frac{x^2+1-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$	Simplify.

$f'(x)$ can only change its sign when $f'(x)=0$ or is undefined. It is never undefined since the expression in the denominator is never zero.

Now, set $f'(x)=0$ and solve.

$$\frac{1-x^2}{(x^2+1)^2}=0$$ Set $f'(x)=0$. Since $f'(x)$ is never undefined, this case is not considered.

$1-x^2=0$ Multiply both sides by $(x^2+1)^2$.

$1=x^2$ Solve for *x*.

$x=\pm 1$

This means that the real number line is broken into three intervals:

$(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$

Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2
Value of $f'(x)=\frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-1, 1)$ and decreasing on $(-\infty, -1)\cup(1, \infty)$.

Let's look at an exponential function now.

↱ EXAMPLE Let $f(x)=e^{-x^2}$. Determine the intervals over which $f(x)$ is increasing or decreasing.

Since this information comes from the first derivative, we start there. This derivative requires the chain rule.

$f(x)=e^{-x^2}$ Start with the original function.

$f'(x)=-2x\cdot e^{-x^2}$ Apply the chain rule.

$f'(x)$ can only change its sign when $f'(x)=0$ or is undefined. It is never undefined since there are no domain restrictions.

Now, set $f'(x)=0$ and solve.

$-2x\cdot e^{-x^2}=0$ The derivative is set to 0.

$-2x=0$ or $e^{-x^2}=0$ Set each factor equal to 0.

$x=0$ $-2x=0$ implies $x=0$.
$e^{-x^2}=0$ has no solution.

This means that the real number line is broken into two intervals:

$(-\infty, 0)$, $(0, \infty)$

Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	-1	1
Value of $f'(x) = -2xe^{-x^2}$	$2e^{-1}$	$-2e^{-1}$
Behavior of $f(x)$	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-\infty, 0)$ and decreasing on $(0, \infty)$.

---

2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

Consider the function $f(x) = \frac{x}{x^2+1}$. In part 1, we obtained the following sign graph for $f'(x)$. Notice that the bottom row has been added to show the direction the graph is moving when increasing or decreasing.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2
Value of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing
Direction	$\searrow$	$\nearrow$	$\searrow$

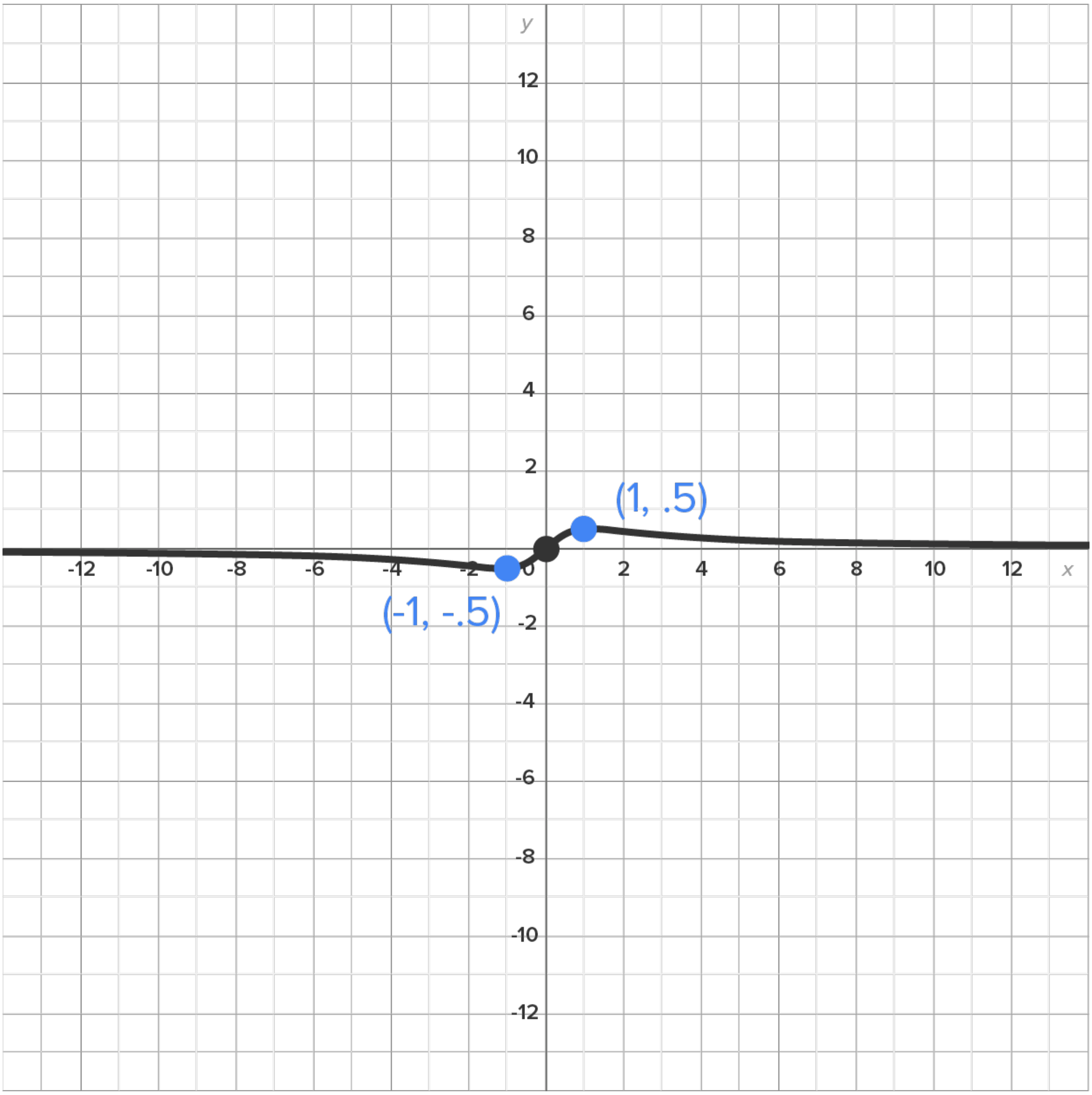
This means that when $x = -1$, the graph is transitioning from decreasing to increasing; and when $x = 1$, the graph is transitioning from increasing to decreasing. Since $f(x)$ is continuous at $x = -1$ and $x = 1$, this means that there is a local minimum at $x = -1$ and a local maximum at $x = 1$.

To find the coordinates of these points on the graph, substitute each value into $f(x)$:

- Local minimum: $(-1, f(-1))$, or $\left(-1, \frac{-1}{2}\right)$

- Local maximum: $(1, f(1))$, or $\left(1, \frac{1}{2}\right)$

The graph of $f(x)$ is shown here for reference:



Given a function $f(x)$, the first derivative, $f'(x)$, can be used to locate maximum and minimum values. To do so, use the same sign graph you used to determine where the function is increasing or decreasing. Then, observe the critical numbers where $f(x)$ transitions between increasing and decreasing. When $f(x)$ is continuous at these critical numbers, there is either a local minimum or local maximum point.

Performing this analysis to locate local minimum and maximum points is called the**first derivative test**.

> ↱ EXAMPLE Use the first derivative test to determine all local minimum and maximum points of the function $f(x) = x^4 - 12x^3$.
>
> First, find all critical numbers:

$f(x) = x^4 - 12x^3$	Start with the original function; the domain is all real numbers.
$f'(x) = 4x^3 - 36x^2$	Take the derivative.
$4x^3 - 36x^2 = 0$	Set $f'(x) = 0$. Since $f'(x)$ is a polynomial, there is no value of x for which $f'(x)$ is undefined, so this case is not considered.
$4x^2(x-9) = 0$ $4x^2 = 0,\ x - 9 = 0$ $x = 0,\ x = 9$	Solve by factoring.

Thus, the critical numbers are $x = 0$ and $x = 9$. Now we move to the first derivative test, which means making a sign graph, determining the intervals of increase and decrease, then observing which critical numbers produce a local maximum or local minimum.

Interval	$(-\infty, 0)$	$(0, 9)$	$(9, \infty)$
Test Value	-1	1	10
Value of $f'(x) = 4x^3 - 36x^2$	-40	-32	400
Behavior of $f(x)$	Decreasing	Decreasing	Increasing
Direction	↘	↘	↗

At $x = 0$, $f(x)$ does not change direction; it is decreasing on both sides. There is no local extreme value at $x = 0$.

At $x = 9$, the $f(x)$ transitions from decreasing to increasing, indicating that there is a local minimum value when $x = 9$.

Thus, there is a local minimum at $(9, f(9))$, or $(9, -2187)$.

BIG IDEA

There is no guarantee that a local extreme value occurs at a critical number. This is why we perform the first derivative test.

TRY IT

Consider the function $f(x) = xe^{-x}$.

Use the first derivative test to determine the local extrema of the function. +

Local maximum at $\left(1, \frac{1}{e}\right)$.

TERM TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.

SUMMARY

In this lesson, you learned that the derivative, $f'(x)$, can be used to provide information about where $f(x)$ is increasing or decreasing. In other words, you can **use values of $f'(x)$ to sketch the shape of the graph of $f(x)$**. You also learned that you can apply these ideas further, **using $f'(x)$ to detect local maximum and minimum values** by performing an analysis called the first derivative test.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.

Concavity

by Sophia

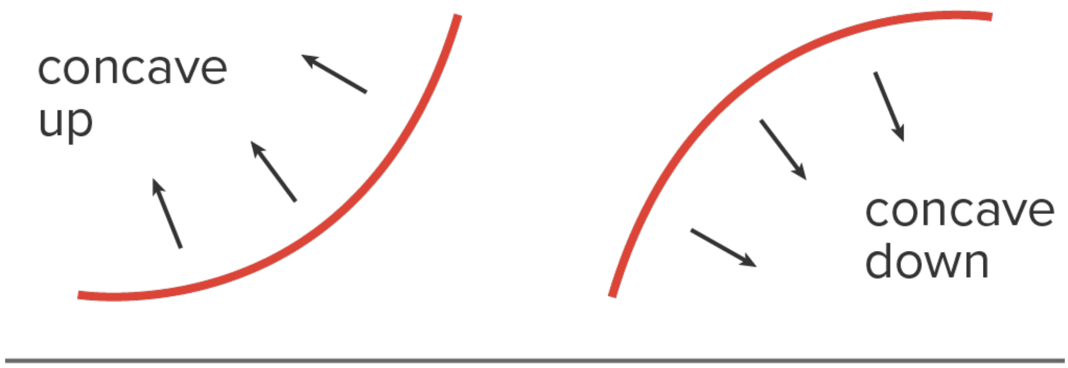
WHAT'S COVERED

In this lesson, you will learn how to use the second derivative to determine the direction that the graph of a function opens, also known as its concavity. Specifically, this lesson will cover:

1. Defining Concavity
2. Determining Where a Function Is Concave Up/Concave Down

1. Defining Concavity

Concavity refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward. A graph is **concave up** on an interval if it opens upward on that interval. A graph is **concave down** on an interval if it opens downward on that interval.



WATCH

This video shows how concavity relates to how slopes of tangent lines change.

Video Transcription

[MUSIC PLAYING] OK, so what we have in this picture is the graph of a function, f of x. And what we're going to do to try to motivate concavity and how it comes about is to analyze the slopes of tangent lines. So as you notice at this point here on the left, we have a slope, and we're about an x equals about negative 3.5 or so. The slope of the tangent line there is a positive 43, so pretty steep, OK? So we're going to do is analyze what happens to the slopes as we move along the curve, as x increases.

So looking at that, if I move along a little bit, you notice that the slopes are decreasing as we move along the curve here. So that's causing the curve to bend in a certain way, OK? And we keep moving to the right, and we hit the horizontal tangent. And the slopes keep decreasing. Now we're into the negative slopes.

And up until we hit this point, p-- now keep your eye on the values. The slope there is negative 14.25, negative 14 and almost 3/4, and then we hit this point p, and then our slope is negative 15. Now, as we move through p, going to the other side, notice the slopes are going another way, which is causing the curve to bend in a different way. If the slopes are increasing, it's causing the curve to bend upward. Because as you move to the right, the curve is starting to go further and further up more steeply.

So how does this relate to the shape of the curve? Well, let's look at this. So coming back to this part, when the slopes were decreasing, the curve was bending in a downward fashion. We call that concave down. Now let's think about how that relates to derivatives. So the slope, which is f prime, is decreasing. So remember that when a function is decreasing, its derivative is negative. So if f prime is decreasing, its derivative is negative. But the derivative of f prime is f double prime, OK? So this is where the second derivative comes into play.

When the second derivative is negative, that means the curve is concave down, because that comes from the slopes decreasing, OK? If we go to the other side of point p here, this is where the slopes are increasing, causing the curve to bend upward, just like you see here. And that's where f prime is increasing, which means its derivative, f double prime, is positive. So that is where the link between the second derivative and concavity comes from, as you're going to read about in the tutorial.

[MUSIC PLAYING]

BIG IDEA

Based on the video, we make the following observations:

- If $f''(x)>0$ on an interval, then the graph of $f(x)$ is concave up on the same interval.
- If $f''(x)<0$ on an interval, then the graph of $f(x)$ is concave down on the same interval.

HINT

Remember that a function can change between positive and negative when it is either equal to 0 or when it is undefined. Therefore, to determine where the graph of the function is concave up or concave down, find all values where $f''(x)=0$ or $f''(x)$ is undefined. Then, make a sign graph similar to what you did for the first derivative test.

TERMS TO KNOW

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.

Concave Up

When a graph opens upward on an interval.

Concave Down

Concave Down

When a graph opens downward on an interval.

---

2. Determining Where a Function Is Concave Up/Concave Down

→ EXAMPLE Determine the interval(s) over which the graph of $f(x)=x^3-3x^2+5$ is concave up or concave down. Since concavity is determined from the second derivative, we start there.

$f(x)=x^3-3x^2+5$	Start with the original function.
$f'(x)=3x^2-6x$	Take the first derivative.
$f''(x)=6x-6$	Take the second derivative.

Since $f''(x)$ is never undefined, we set it to 0 and solve:

$6x-6=0$	The second derivative is set to 0.
$6x=6$	Add 6 to both sides.
$x=1$	Divide both sides by 6.

Thus, $f(x)$ could be changing concavity when $x=1$. This means that at any x-value on the interval $(-\infty, 1)$, the concavity is the same. The same can be said for the interval $(1, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 1)$	$(1, \infty)$
Test Value	0	2
Value of $f''(x)=6x-6$	-6	6
Behavior of $f(x)$	Concave down	Concave up

Therefore, the graph of $f(x)$ is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

→ EXAMPLE Determine the interval(s) over which the graph of $f(x)=5x^2-18x^{5/3}$ is concave up or concave down. Note that the domain of $f(x)$ is all real numbers.

Since concavity is determined from the second derivative, we start there.

$f(x)=5x^2-18x^{5/3}$	Start with the original function.

$$f'(x) = 10x - 18 \cdot \frac{5}{3}x^{2/3}$$ Take the first derivative.
$$= 10x - 30x^{2/3}$$

$$f''(x) = 10 - 30\left(\frac{2}{3}\right)x^{-1/3}$$ Take the second derivative.
$$= 10 - 20x^{-1/3}$$
$$= 10 - \frac{20}{x^{1/3}}$$

Note that $f''(x)$ is undefined when $x = 0$.

To find other possible transition points, set $f''(x) = 0$ and solve:

$10 - \frac{20}{x^{1/3}} = 0$	The second derivative is set to 0.
$10x^{1/3} - 20 = 0$	Multiply everything by $x^{1/3}$.
$10x^{1/3} = 20$	Add 20 to both sides.
$x^{1/3} = 2$	Divide both sides by 10.
$x = 8$	Cube both sides.

Thus, $f(x)$ could be changing concavity when $x = 0$ or $x = 8$. This means that at any x-value on the interval $(-\infty, 0)$, the concavity is the same. The same can be said for the intervals $(0, 8)$ and $(8, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, 8)$	$(8, \infty)$
Test Value	-1	1	27
Value of $f''(x) = 10 - \frac{20}{x^{1/3}}$	30	-10	$\frac{10}{3}$
Behavior of $f(x)$	Concave up	Concave down	Concave up

Thus, the graph of $f(x)$ is concave up on $(-\infty, 0) \cup (8, \infty)$ and concave down on the interval $(0, 8)$.

WATCH

In this video, we'll determine the intervals over which the function $f(x) = \ln(x^2 + 1)$ is concave up or concave down.

Video Transcription

[MUSIC PLAYING] Hi there. Welcome back. What we're going to take a look at in this example is given the function f of x equals the natural log of x squared plus 1, we're going to determine, analytically, where the function is concave up and where the function is concave down.

So remember that concavity-- that information is found from the second derivative. So we're going to first take the second derivative and then do our analysis that way.

So first thing we'll do is take the first derivative. Derivative of the natural log function is 1 over the something, and then times the derivative of the inside, and that gives us 2x over x squared plus 1. And now the second derivative is going to be the derivative of 2x over x squared plus 1, which requires the quotient rule.

So what I'm going to do is, knowing that we need the derivative of each piece, I'm going to write the derivative as 2 here, and I'm going to write the derivative as 2x there. And then the second derivative is-- well, let's see. It's low d high, which means the denominator times the derivative of the numerator, so x squared plus 1 times 2 minus high d low so that is 2x as the numerator times 2x as the derivative of the denominator, and then all over low squared.

So naturally what we will do is simplify the numerator. So we have distributing the two, 2x squared plus 2 minus 4x squared, all over x squared plus 1 squared, which is 2 minus 2x squared over x squared plus 1 squared. And that looks simplified enough for now.

So what we first have to do is figure out where all the possible points of transition are and with any expression that is where it's either equal to 0 or where it's undefined. The good news is, with this one, there is no place where the derivative of the second derivative is undefined because x squared plus 1 can never be 0, which certainly means that the square can never be 0. So that means we don't have to worry about undefined here.

So we're just going to set this equal to 0. And there's two ways to look at this. If you have a fraction that's equal to 0, the only way for a fraction to be equal to 0 is if its numerator is 0. Or we could just go back to basics and multiply both sides by the denominator, which is x squared plus 1 the quantity squared. Either way, we end up with the equation 2 minus 2x squared equals 0.

So we'll solve for x. I'm going to add 2x squared to both sides. I'm going to divide both sides by 2 and I'm going to take the square root of both sides and get plus or minus 1. So those are the possible places where the graph is transitioning between being concave up and concave down.

So we are going to make a sine graph of f double prime, very similar to what we did in f prime in the last video to determine increasing and decreasing-- very similar concept here. So I know that I have a possible transition at negative 1 and another possible transition at 1. And what we're going to do is the same thing. We're going to pick some test numbers. We're going to plug them into the second derivative and try to tell what exactly is happening.

Now remember, the first thing is we divided the number line into three intervals. So let's just mark those down. This is negative infinity to negative 1. This is -1 to 1. And this is 1 to infinity. So for my test numbers, I'm going to pick negative 2 for this interval just because it's close to negative 1. You could pick anything you want to. On the interval -1 to 1, I'm going to pick 0. And on the interval 1 to infinity, I'm going to pick 2.

So we're going to plug all of those into the second derivative to see what's happening. So f double prime at negative 2 is-- and remember, we're going to use the last version of it before we started manipulating it, so we're going to plug it into that. So you have 2 minus 2 times negative 2 squared-- lots of 2's-- divided by negative 2 squared plus 1 squared.

Well, the numerator is, let's see-- negative 2 squared is 4. 2 times 4 is 8. 2 minus 8 is negative 6. The denominator is 4 plus 1 squared, which is 25, which all that tells me is I have a negative value, which means we are concave down.

So I'm just going to draw a picture that looks something like this-- just a concave down. At 0-- and we have 2 minus 2 times 0 squared all over 0 squared plus 1 squared. Now after you work with all the 0's there, the numerator is 2 and the denominator is 1, which is 2, which means we're concave up.

So we are definitely transitioning between concavity types there. And then plugging in 2-- we have 2 minus 2 times 2 squared all divided by 2 squared plus 1 squared. And turns out, that's the same as if we plugged in negative 2. That's still negative 6 over 25. So that result doesn't change. That means we are concave down.

So as far as writing the intervals, it's concave up on the interval. The only interval we have there is -1 to 1. And it's concave down on the interval-- well, let's see. We have negative infinity to negative 1. We also have 1 to infinity. And again, those are both intervals that describe the same set of values. So we put a union in between them-- same set of values that are-- the answer to our problem, I should say.

So there are the intervals where the function is concave up and concave down. And just to kind confirm the results, I do have a graph of the function y equals natural log x squared plus 1. And as you can see, at those points right there, where x is 1 and negative 1, we do indeed have inflection points. You notice it's concave up in the middle, and it switches to concave down as you go to the extremes.

SUMMARY

In this lesson, you learned that **concavity is defined** as the direction in which a graph opens, noting that a graph is concave up if it opens upward on an interval and concave down if it opens downward on an interval. You also learned that you can **determine where a function is concave up/concave down** by using the second derivative of the function $f(x)$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Concave Down
When a graph opens downward on an interval.

Concave Up
When a graph opens upward on an interval.

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.

Inflection Points

by Sophia

WHAT'S COVERED

In this tutorial, we will find inflection points that occur when a curve changes concavity. Inflection points are useful in modeling an epidemic. Specifically, this lesson will cover:

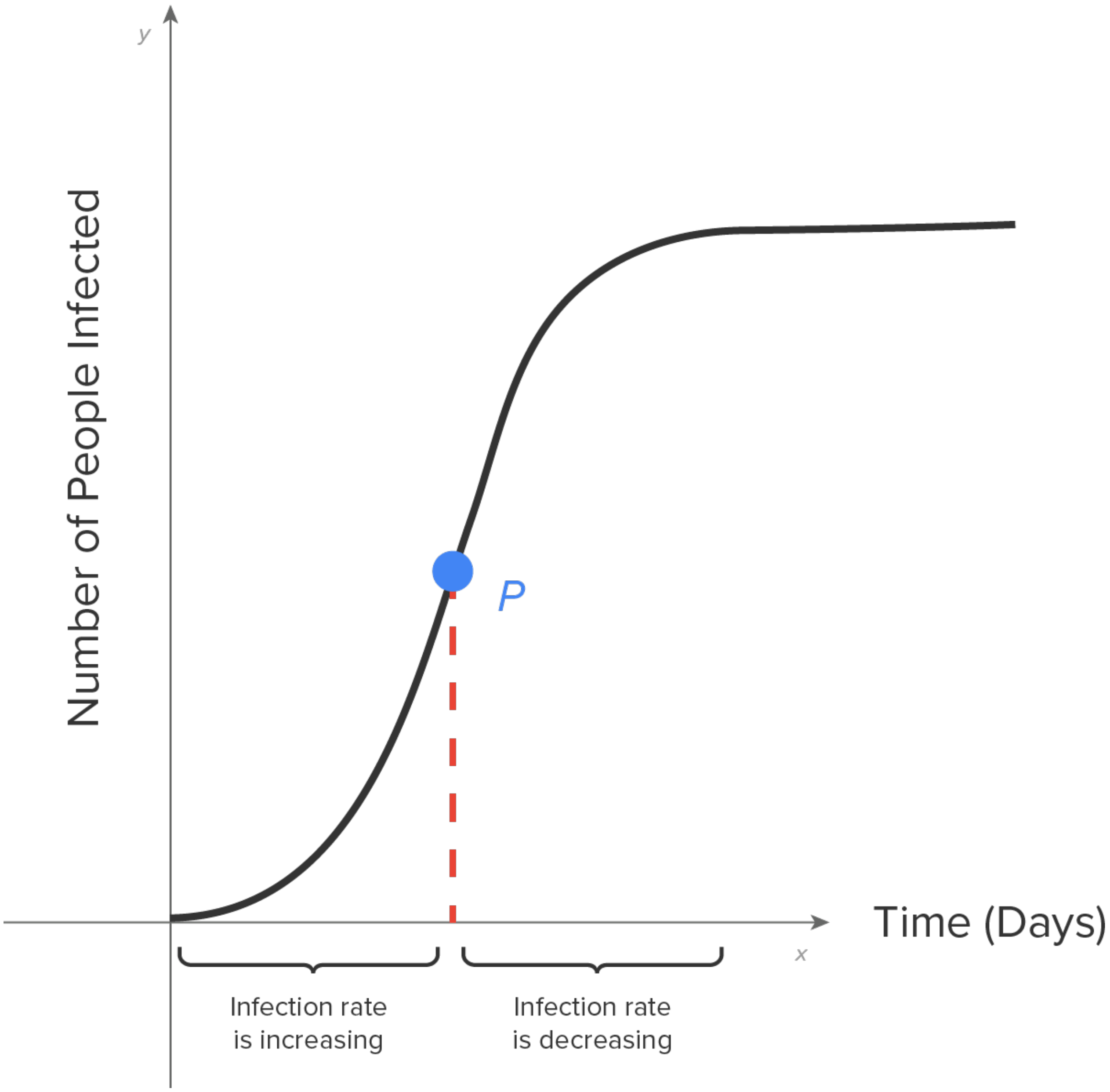
1. Defining the Point of Inflection
2. Determining the Inflection Points of a Function

1. Defining the Point of Inflection

In the last tutorial, we learned that $f''(x)$ gives information about the concavity of $f(x)$.

When a curve has a point where it transitions between being concave up and concave down, and the tangent line exists, the point is called an **inflection point** (or **point of inflection**).

Consider the graph shown below, which represents the number of people who have become infected with a disease.



As you can see, the number of cases is increasing over the entire domain.

To the left of point *P*, the slopes of the tangent lines are increasing. This means that the rate of infection is increasing.

To the right of point *P*, the slopes of the tangent lines are decreasing. This means that the rate of infection is decreasing.

The inflection point is the transition point between these two events. In terms of disease control, this point is important since it represents the point at which the disease is beginning to get under control.

In the last tutorial, you learned that the graph of $f(x)$ is concave down when $f''(x) < 0$ and the graph of $f(x)$ is concave up when $f''(x) > 0$.

BIG IDEA

As long as $f(x)$ is continuous at $x = c$, the graph of $f(x)$ could have a point of inflection when $f''(c) = 0$ or $f''(c)$ is undefined. To verify this, make a sign graph of $f''(x)$.

TERM TO KNOW

Inflection Point (Point of Inflection)

A point on a curve at which concavity changes.

---

2. Determining the Inflection Points of a Function

EXAMPLE Consider the function $f(x) = -2x^3 + 18x^2 + 30x - 40$. Find any points of inflection.

$f(x) = -2x^3 + 18x^2 + 30x - 40$	Start with the original function; the domain is all real numbers.
$f'(x) = -6x^2 + 36x + 30$	Take the first derivative.
$f''(x) = -12x + 36$	Take the second derivative.
$-12x + 36 = 0$	Any inflection points could occur when $f''(x) = 0$. (Note: $f''(x)$ is never undefined.)
$-12x = -36$	Subtract 36 from both sides.
$x = 3$	Divide both sides by -12.

Therefore, there could be a point of inflection when $x = 3$.

Now, select one number (called a *test value*) inside the intervals $(-\infty, 3)$ and $(3, \infty)$ to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 3)$	$(3, \infty)$
Test Value	0	4
Value of $f''(x) = -12x + 36$	36	-12
Behavior of $f(x)$	Concave up	Concave down

Therefore, $f(x)$ is concave up on the interval $(-\infty, 3)$ and concave down on the interval $(3, \infty)$. Thus, a point of inflection occurs when $x = 3$.

On the graph of $f(x)$, the inflection point is located at $(3, f(3)) = (3, 158)$.

Let's take a look at a different function.

EXAMPLE Consider the function $f(x) = x^4 - 2x$. Find any points of inflection.

$f(x) = x^4 - 2x$	Start with the original function; the domain is all real numbers.
$f'(x) = 4x^3 - 2$	Take the first derivative.
$f''(x) = 12x^2$	Take the second derivative.

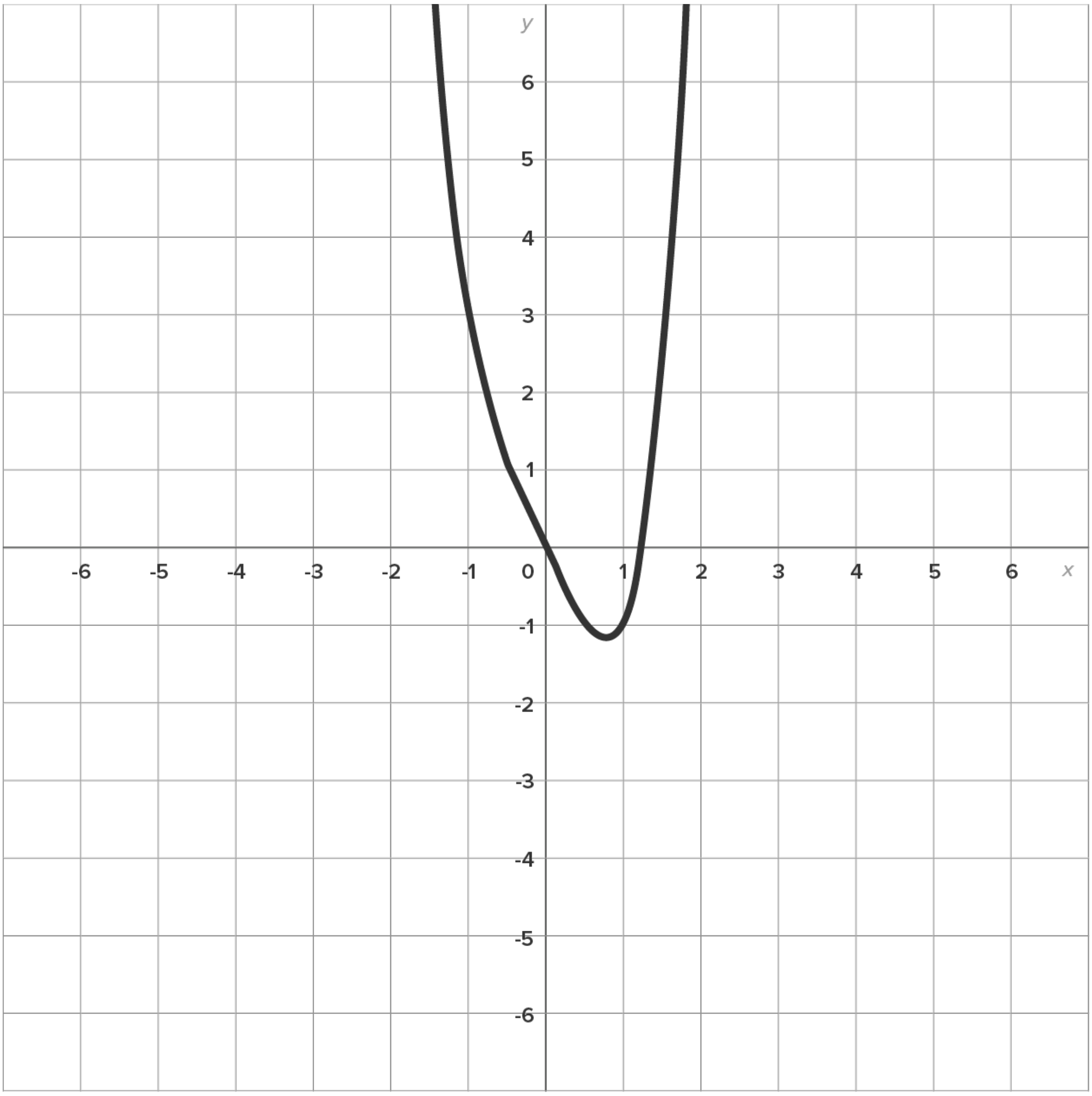
$12x^2 = 0$	Any inflection points could occur when $f''(x) = 0$. (Note: $f''(x)$ is never undefined.)
$x^2 = 0$	Divide both sides by 12.
$x = 0$	Take the square root of both sides.

Therefore, an inflection point possibly occurs when $x = 0$.

Now, select one number (called a *test value*) inside the intervals $(-\infty, 0)$ and $(0, \infty)$ to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	-1	1
Value of $f''(x) = 12x^2$	12	12
Behavior of $f(x)$	Concave up	Concave up

Since the concavity does not change at $x = 0$, there is no inflection point when $x = 0$. Since there were no other possible points of inflection, $f(x) = x^4 - 2x$ has no points of inflection. As you can see, the graph is always concave up.



WATCH

In this video, we'll analyze the function $f(x) = 2x - 3x^{2/3}$ for points of inflection.

Video Transcription

Hi there. Welcome back. In this video, we're going to find any inflection points that the graph of f of x equals 2x minus 3x to the 2/3 may have. And remember that the inflection point is where the graph changes concavity. And concavity, that information is found by using the second derivative. So we're going to find the second derivative of the function first, and then solve accordingly.

So first things first. First derivative, that's going to be 2 minus 2x to the negative 1/3, just using the power rule, and then taking the second derivative. Derivative of 2 is 0 minus-- now we have 2 times negative 1/3 x to the negative 4/3. And that looks like 2/3x to the negative 4/3.

Now, that might be a little bit difficult to see what's happening. So I'm going to rewrite this in terms of positive exponents, which just means writing the x to the negative 4/3 as x to the 4/3 in the denominator. And we can immediately see that this is undefined when x equals 0.

So that is definitely a candidate. But one thing we have to check is, you know, is it a candidate, because if it's not defined in the original function, there's no point to even talk about there. And looking at the original function, the domain is all real numbers, because the 3 in the denominator indicates a cubed root, and it's positive 2/3 power. So that means x equals 0 will be-- we can plug in x equals 0. The value of f of 0 is defined. So that means 0 is definitely a candidate for being a point of inflection.

So let's take a look. So we're going to make our sine graph for f double prime. We have to consider x equals 0 as a possible place where the graph changes concavity. And we're going to do our intervals. So remember, the intervals are going to be negative infinity to 0 and 0 to infinity. There's no other possible places. And we're going to do our test numbers.

I'm going to say negative 1, and I'm going to say 1. And then we're going to substitute those values into the second derivative. So you have 2 over 3 times negative 1 to the 4/3. Well, negative 1 to the 4/3, cubed root of negative 1 is negative 1 raised to the fourth is 1. So you have 2 over 3 times 1, which is 2/3, which means concave up. So it's concave up to the left of 0.

If we plug in 1, this is interesting. We have 2 over 3 times 1 to the 4/3. Well, 1 to any power is 1. So this is also 2/3, which means also concave up. So this graph does not have a point of inflection, because it is not changing concavity ever. It is always concave up, except possibly at x equals 0, where it kind of just levels off. So there is no point of inflection.

And there is the graph of the function. And as we can see, it comes up. There is a sharp corner at 0. It is concave up to the left. I don't think I included enough of the graph there, but it is concave up there. And then to the right, it continues to be concave up. It just comes to a sharp point in the middle there at x equals 0.

SUMMARY

In this lesson, you learned that **the point of inflection is defined** as the point on a curve where it transitions between being concave up and concave down (and the tangent line exists). This point is useful in modeling an epidemic, for example, since it can represent the point at which the disease is beginning to get under control. You also learned that to **determine the inflection points of a function**, first find all values in the domain of $f(x)$ where $f''(x)$ is either 0 or undefined, then use a graph of the signs of $f''(x)$ to determine the x-values where inflection points occur.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Inflection Point (Point of Inflection)
A point on a curve at which concavity changes.

f'' and Extreme Values of f

by Sophia

WHAT'S COVERED

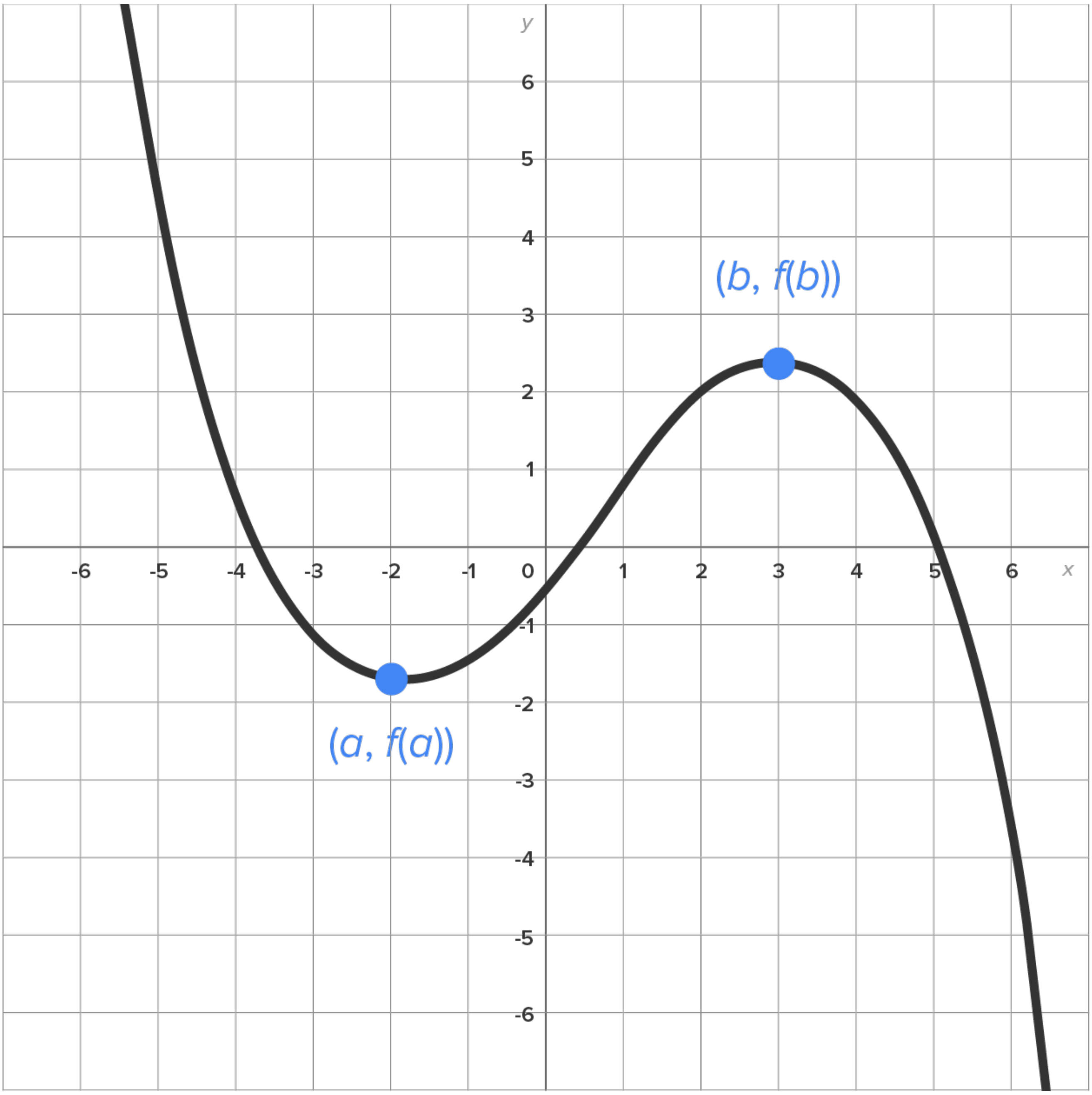
In this lesson, you will use the second derivative to determine if a maximum or minimum occurs at a critical number. Specifically, this lesson will cover:

1. The Second Derivative Test
2. Locating Maximum/Minimum With the Second Derivative Test

1. The Second Derivative Test

When a function has several critical numbers, and when the second derivative is relatively easy to get, the second derivative test is much more efficient to use than the first derivative test when locating maximum and minimum values. However, there are conditions under which the second derivative doesn't give enough information, and we need to use information from the first derivative to determine maximum and minimum points of a function.

Consider the graph shown below:



Observe the following at the local extreme points:

The point $(a, f(a))$ is a local minimum.

- Since there is a horizontal tangent at $x = a$, we know $f'(a) = 0$.
- Since the graph is concave up at $x = a$, we know that $f''(a) > 0$.

The point $(b, f(b))$ is a local maximum.

- Since there is a horizontal tangent at $x = b$, we know $f'(b) = 0$.
- Since the graph is concave down at $x = b$, we know that $f''(b) < 0$.

Based on this graph, there is a connection between the concavity of $f(x)$ when there is a horizontal tangent line and the type of local extrema at that point. Formally stated, this is called the **second derivative test**.

HINT

Only critical numbers where $f'(c)=0$ are considered for the second derivative test. If $f'(c)$ is undefined, then $f''(c)$ is also undefined, which means we cannot determine if $f''(c)$ is positive, negative, or zero.

That said, if $f(x)$ has critical numbers where $f'(c)$ is undefined, then the first derivative test will need to be used to determine if any local extrema occur at $x=c$.

TERM TO KNOW

Second Derivative Test

Suppose $f'(c)=0$, which means $f(x)$ has a horizontal tangent at $x=c$.

- If $f''(c)<0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.
- If $f''(c)>0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.
- If $f''(c)=0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.

2. Locating Maximum/Minimum With the Second Derivative Test

Let's look at a few examples of how the second derivative test is implemented.

EXAMPLE Determine the local maximum and minimum values of $f(x)=-2x^3+6x^2+15$.

First, find the values of *c* for which $f'(c)=0$.

$f(x)=-2x^3+6x^2+15$	Start with the original function; the domain is all real numbers.
$f'(x)=-6x^2+12x$	Take the derivative.
$-6x^2+12x=0$ $-6x(x-2)=0$ $-6x=0,\ x-2=0$ $x=0,\ x=2$	Set $f'(x)=0$ and solve.

The critical numbers are $x=0$ and $x=2$.

Now, take the second derivative using $f'(x)=-6x^2+12x$ and substitute $x=0$ and $x=2$.

$f'(x)=-6x^2+12x$	Take the first derivative.
$f''(x)=-12x+12$	Take the second derivative.
$f''(0)=-12(0)+12=12$	Substitute the critical number, $x=0$. Since $f''(0)$ is positive, $f(0)$ is a local minimum.

$f''(2) = -12(2) + 12 = -12$ Substitute the critical number, $x = 2$. Since $f''(2)$ is negative, $f(2)$ is a local maximum.

Therefore, the local minimum value is $f(0) = 15$ and the local maximum value is $f(2) = 23$. On the graph, the local minimum is located at (0, 15) and the local maximum is located at (2, 23).

Let's look at another example.

→ EXAMPLE Determine the local maximum and minimum values of $f(x) = 5x + \frac{20}{x}$.

First, find the values of c for which $f'(c) = 0$.

$f(x) = 5x + \frac{20}{x} = 5x + 20x^{-1}$	Start with the original function; rewrite to use the power rule. The domain is $(-\infty, 0) \cup (0, \infty)$.
$f'(x) = 5 - 20x^{-2}$	Take the derivative.
$5 - 20x^{-2} = 0$ $5 - \frac{20}{x^2} = 0$ $5x^2 - 20 = 0$ $5(x^2 - 4) = 0$ $5(x+2)(x-2) = 0$ $x = 2, x = -2$	Set $f'(x) = 0$ and solve.

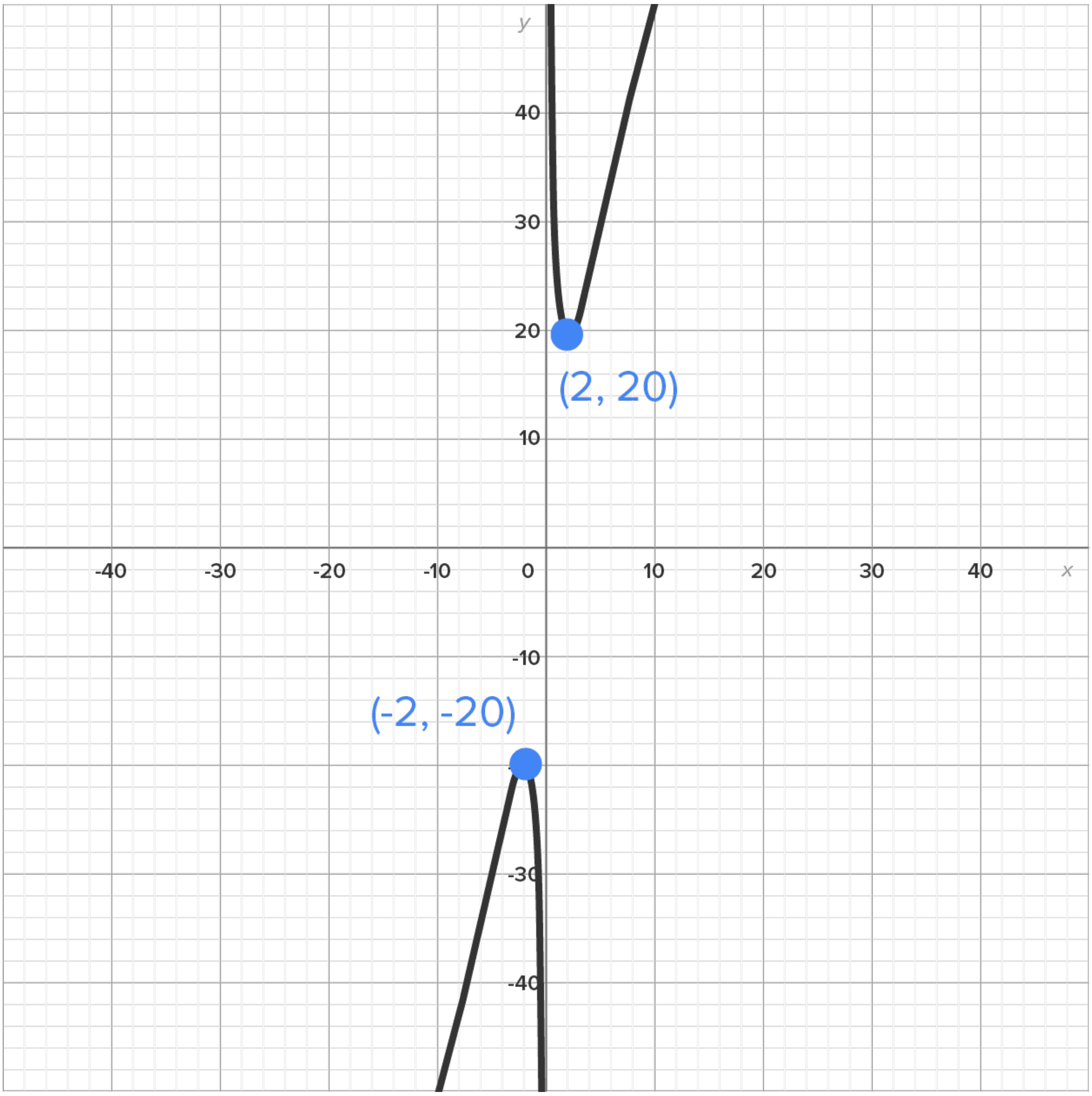
The critical numbers are $x = -2$ and $x = 2$.

Now, take the second derivative using $f'(x) = 5 - 20x^{-2}$, then substitute $x = -2$ and $x = 2$.

$f'(x) = 5 - 20x^{-2}$	Take the first derivative.
$f''(x) = 40x^{-3} = \frac{40}{x^3}$	Take the second derivative.
$f''(-2) = \frac{40}{(-2)^3} = -5$	Substitute the critical number, $x = -2$. Since $f''(-2)$ is negative, $f(-2)$ is a local maximum.
$f''(2) = \frac{40}{2^3} = 5$	Substitute the critical number, $x = 2$. Since $f''(2)$ is positive, $f(2)$ is a local minimum.

Therefore, the local maximum value is $f(-2) = -20$ and the local minimum value is $f(2) = 20$. On the graph, the local maximum is located at (-2, -20) and the local minimum is located at (2, 20).

Here is a graph of the function:



Let's now look at an example where the second derivative test cannot be used.

↱ EXAMPLE Determine the local maximum and minimum values of $f(x) = x^3 - 6x^2 + 12x + 10$.

First, find the values of c for which $f'(c) = 0$.

$f(x) = x^3 - 6x^2 + 12x + 10$	Start with the original function; the domain is all real numbers.
$f'(x) = 3x^2 - 12x + 12$	Take the derivative.
$3x^2 - 12x + 12 = 0$	Set $f'(x) = 0$ and solve.
$3(x^2 - 4x + 4) = 0$	
$3(x-2)(x-2) = 0$	
$x = 2$	

There is only one critical number, $x = 2$.

Now, take the second derivative using $f'(x) = 3x^2 - 12x + 12$, then substitute $x = 2$.

$f'(x) = 3x^2 - 12x + 12$	Take the first derivative.
$f''(x) = 6x - 12$	Take the second derivative.
$f''(2) = 6(2) - 12 = 0$	Substitute the critical number, $x = 2$.

This means that the second derivative cannot be used to determine if $f(x)$ attains a local maximum or minimum at $x = 2$.

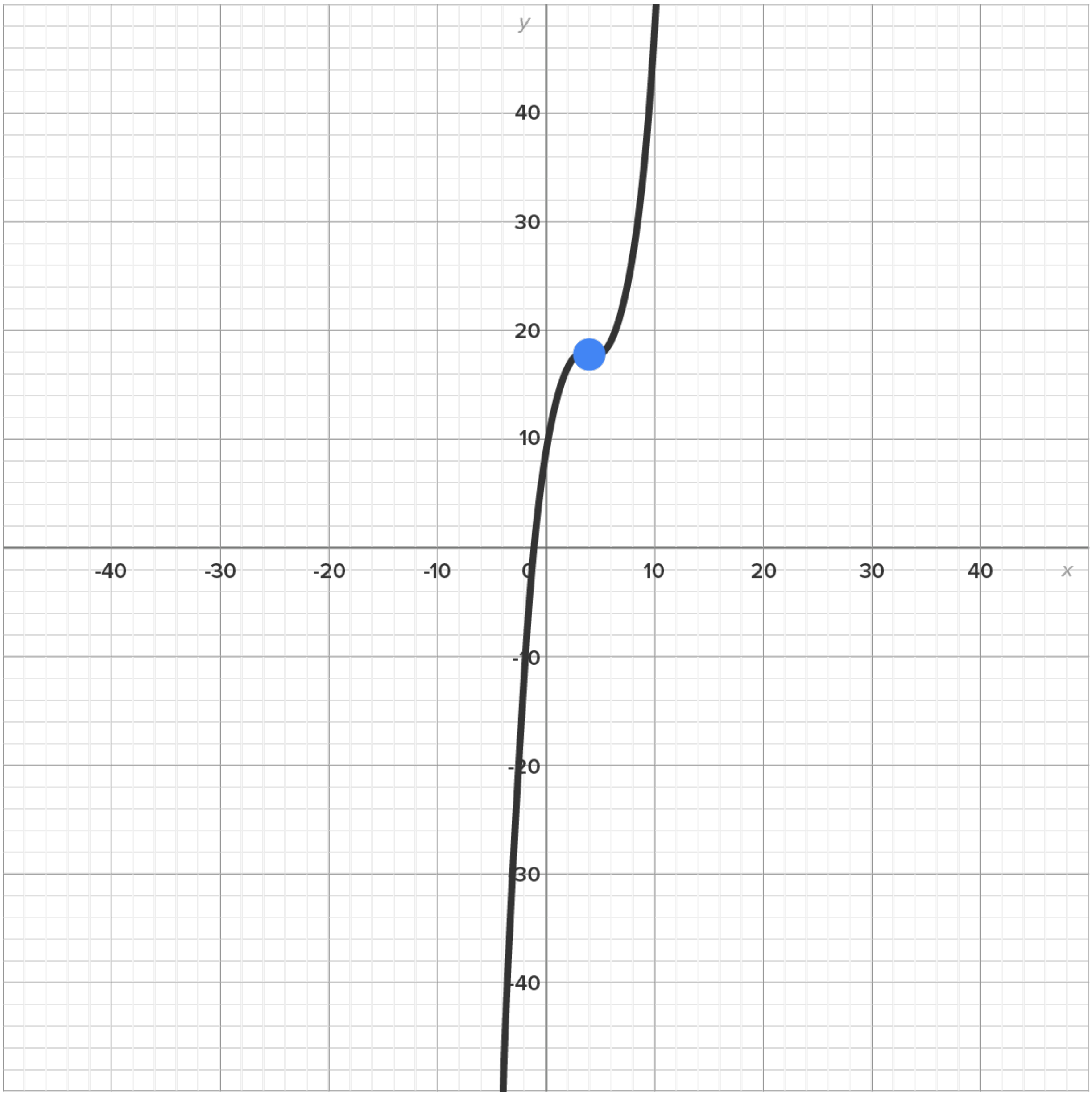
To determine the behavior of $f(x)$ at $x = 2$, we now turn back to the first derivative, $f'(x) = 3x^2 - 12x + 12$.

Evaluate on either side by finding $f'(1)$ and $f'(3)$:

- $f'(1) = 3(1)^2 - 12(1) + 12 = 3$ (increasing)
- $f'(3) = 3(3)^2 - 12(3) + 12 = 3$ (increasing)

Since $f(x)$ is increasing on both sides of $x = 2$, this means that there is no minimum or maximum when $x = 2$.

The graph of $f(x)$ is shown in the figure:



WATCH

In this video, we'll use the second derivative test to locate local maximum and minimum values of the function $f(x) = xe^{-0.5x}$.

Video Transcription

Well, hello there. Welcome back. We're going to continue our quest of maxes and mins and derivatives by looking at a function and using the second derivative test to determine if there's a minimum or a maximum at a value where there's a horizontal tangent, a.k.a., a critical number, but not where it's undefined.

So let's just look over to the right at the criteria for using the second derivative test. Let's say that f prime of c is equal to 0. If the second derivative happens to be positive at that value, we have a local minimum at c. And if the second derivative is negative at that value, then we have a local maximum at c.

So we need to first find the critical numbers where the first derivative is equal to 0, and then use those in the second derivative. So here, we have our function, and it does require a product rule to find the derivative. So I'm going to say x is the first and e to the negative 0.5x is the second.

So the derivative, as we know with the product rule, is the derivative of the first, which is 1, times the second, which is e to the negative 0.5x, plus the first x, times the derivative of the second. Now, the derivative of an exponential is e to the negative 0.5 x times negative 0.5 It's itself times the derivative of the exponent, the inside.

So I'm going to clean things up, and all that means is I'm not going to write the 1. And then I'm going to say, minus 0.5xe to the negative 0.5x. And actually, what I'm going to do is, before I say it's simplified, I am going to factor out e to the negative 0.5x. And that's going to leave you with 1 minus 0.5x. So this is my first derivative. So I want to know where that's equal to 0. So I set it equal to 0.

So if we need a point of reference here, this is finding critical numbers. And remember that since it's in factored form, we know that each factor could be 0. Now remember, an exponential never takes on a value of 0, because there is no exponent that creates 0. So this is no solution.

And on the right, if we add 0.5x to both sides and divide by 0.5, we have x equals 2. So x equals 2 is the critical number we're going to pay attention to. Now we move to the second derivative. So using the equation in the box, we're just going to use the products rule once again.

So if double prime is-- so the derivative of the first, which is e to the negative 0.5x, the derivative of that is e to the negative 0.5x times negative 0.5, and then times the second, which is 1 minus 0.5x, plus the first e to the negative 0.5x times the derivative of the second, which is 0 minus 0.5. The derivative of 1 is 0. The derivative of 0.5x is 0.5.

Now, at that point, we could simplify, but remember, we are just going to be substituting a value in to see what the second derivative is. So I'm going to leave it as is and just substitute 2. And we have a lot of 0.5s and 2s here. So let's just remember. Let's just look at this here. So 0.5 times 2 is 1.

So this is e to the negative 1 times negative 0.5 times-- well, this is 1. That makes it look like a fraction, though. There we go. So that's 1 minus 1. So that's 0. So that whole term drops out. And then we have minus 0.5e to the negative 1. So this whole term goes away, and we have -0.5 times 1 over e.

Now remember, e is a positive number, so this whole thing is negative. Therefore, there is a local maximum at x equals 2 by the second derivative test, what is the value of f at that point? Well, remember, the maximum value would then be f of 2 equals-- now remember, f of x was xe to the negative 0.5x. So that'd be 2e to the negative 0.5 times 2, which is 2 times e to the negative 1, which is 2 over 8.

So the local maximum point occurs at 2, comma 2 over e. And I have a graph here just to confirm our results. Here is the graph of y equals xe to the -0.5 x. And as you can see, the maximum point turns out to be a global max at x equals 2. And 0.736 is the decimal approximation of 1 over e. So there we have it. Another way to find mins and maxes is to use a second derivative test.

SUMMARY

In this lesson, you learned that under certain conditions, the second derivative can be used to determine if $f(x)$ has a local maximum or minimum at a critical value *c* for which $f'(c)=0$, known as **the second derivative test**. Next, you explored several examples **locating the maximum and minimum values with the second derivative test**; however, when those conditions aren't met, the first derivative test is used to determine the locations of local maximum and minimum values.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Second Derivative Test

Suppose $f'(c)=0$, which means $f(x)$ has a horizontal tangent at $x=c$.

• If $f''(c)<0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.

• If $f''(c)>0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.

• If $f''(c)=0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.

Applied Maximum and Minimum Problems

by Sophia

WHAT'S COVERED

In this lesson, you will apply your knowledge of derivatives to real-world maximization and minimization problems (which collectively are called *optimization problems*). Specifically, this lesson will cover:

1. Strategy for Solving Optimization Problems
2. Solving Applied Optimization Problems

1. Strategy for Solving Optimization Problems

An **optimization problem** is a problem in which the maximum or minimum value is sought, whichever is relevant.

STEP BY STEP

To solve an optimization problem:

1. Identify the function to be optimized. This is called the primary equation.
   a. If the goal is to maximize the area, the primary equation expresses the area as a function.
   b. If the goal is to minimize the amount of material used, then the primary equation gives the total amount of material used.
2. If your primary equation has more than one variable (for example: $A = xy$), you will need to form a secondary equation based on other information that is given in the problem.
3. If applicable, use the secondary equation in Step 2 to write the primary equation in Step 1 in terms of one independent variable. Also, state the domain of the function.
4. Find critical numbers.
5. Keeping the requirements in mind, use one of methods covered in this challenge to determine where the extreme points are located:
   a. Extreme Value Theorem
   b. First Derivative Test
   c. Second Derivative Test

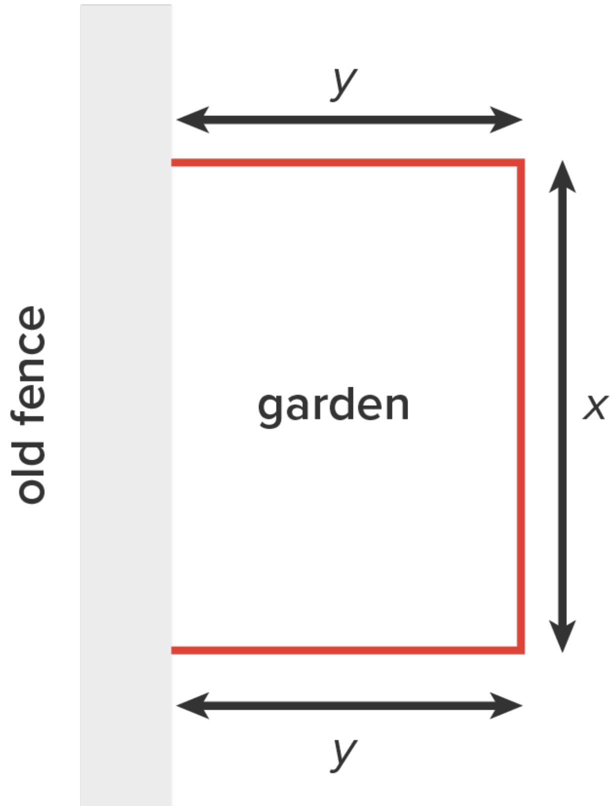
Now that we have a game plan, let's solve a few optimization problems.

TERM TO KNOW

Optimization Problem
A problem in which the maximum or minimum value is sought, whichever is relevant.

2. Solving Applied Optimization Problems

↪ EXAMPLE A garden is to be constructed against an old fence, as shown in the figure. Trim is to be placed around the garden on the three remaining sides but is not needed on the fence side. If 24 feet of trim is to be used, what is the largest area that can be enclosed?



As the figure suggests, let x = the side parallel to the fence and let y = the length of the other two sides.

We want to maximize the area of the garden, which means our primary equation is $A = xy$, but this equation has too many variables for us to use calculus just yet. Thus, there should be a secondary equation we can use from information in the problem.

We also know there is 24 feet of trim available, which means $y + y + x = 24$, or $2y + x = 24$. This is the secondary equation.

Since it is easier to solve for x, the equation can be written $x = 24 - 2y$.

Now, the area equation can be written $A = xy = (24 - 2y)y = 24y - 2y^2$, which leads us to:

The function to optimize (maximize) is $A(y) = 24y - 2y^2$.

The next thing we should look at is the domain of the function. Since y is a side of the rectangle, it must be nonnegative and can be no more than 12 since the total amount of fencing is 24 feet, and there are two sides with length *y*.

Thus, the domain is $0 \le y \le 12$.

To determine the maximum value, we first take the derivative (with respect to *y*) and find critical points. Since $A(y)$ is continuous on the closed interval $[0, 12]$, we can apply the extreme value theorem, which means evaluating $A(y)$ at its endpoints and at any critical numbers.

First, find the derivative and critical numbers:

$A(y) = 24y - 2y^2$	Start with the original function.
$A'(y) = 24 - 4y$	Take the derivative.
$24 - 4y = 0$ $24 = 4y$ $6 = y$	Set $A'(y) = 0$, then solve.

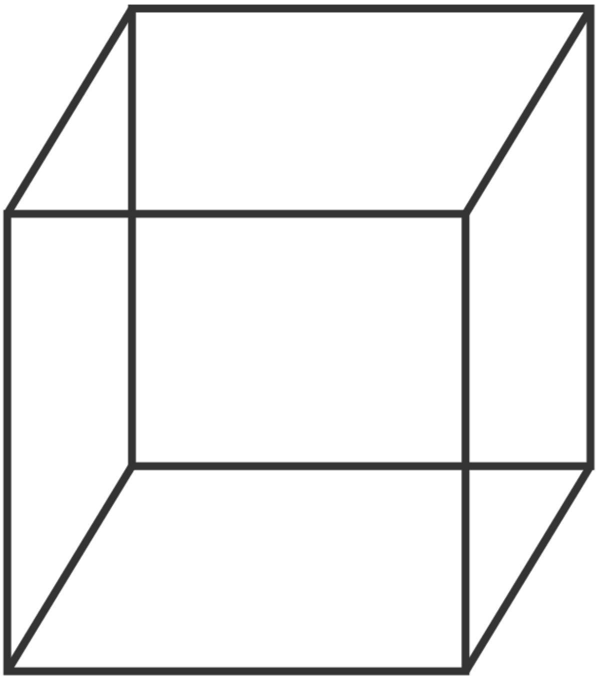
The critical number is $y = 6$, which is inside the interval $[0, 12]$. Now, evaluate $A(y)$ at each endpoint and at $y = 6$.

y	0	6	12
$A(y) = 24y - 2y^2$	0	72	0

Thus, the maximum area is 72 ft^2, which occurs when $y = 6$.

Let's now look at an example where we minimize the surface area of a rectangular box with known volume.

↱ EXAMPLE A rectangular box with a square base and no lid has volume 500 in^3. What is the least amount of material that could be used to construct such a box? (In other words, what is the minimum surface area?)



- Let x = the length of the base.
- Let h = the height of the box.

Primary Equation:
The surface area is the sum of the areas of all sides. Since this box has no lid, we do not count the area of the top.

The base has area x^2 and each of the four sides have area xh. This means that the surface area is $S=x^2+4xh$. This is the primary (optimization) equation.

Secondary Equation:
Since volume is (length)(width)(height), this translates to volume $=x \cdot x \cdot h=x^2 h$.

We are given that the volume is 500 in^3, so this is written $x^2 h=500$. This is the secondary equation that will be used to write the primary equation in terms of one variable.

To do so, it is easiest to replace h with an expression in terms of x. Using the volume (secondary) equation, rewrite $x^2 h=500$ as $h=\frac{500}{x^2}$.

Substituting into the surface area (primary) equation gives $S=x^2+4xh=x^2+4x\left(\frac{500}{x^2}\right)=x^2+\frac{2000}{x}$, which leads us to:

The function we want to optimize (minimize) is $S(x)=x^2+\dfrac{2000}{x}$.

Now, let's discuss the domain of this function. Looking at the equation $h=\dfrac{500}{x^2}$, the value of *h* will be positive (and therefore valid) for any positive value of *x*. Therefore, the domain is $(0, \infty)$.

Since this is an open interval (no endpoints), we will only focus on finding the minimum value by investigating the function at the critical values in the interval, not the endpoints.

Now, we find the derivative and all critical numbers.

$S(x)=x^2+\dfrac{2000}{x}=x^2+2000x^{-1}$	Start with the original function; rewrite so the power rule can be used.
$S'(x)=2x-2000x^{-2}$	Take the derivative.
$2x-2000x^{-2}=0$	Set $S'(x)=0$, then solve.
$2x-\dfrac{2000}{x^2}=0$	Rewrite with positive exponents.
$2x=\dfrac{2000}{x^2}$	Add $\dfrac{2000}{x^2}$ to both sides.
$2x^3=2000$	Multiply both sides by x^2.
$x^3=1000$	Divide both sides by 2.
$x=10$	Take the cube root of both sides.

Thus, there is a critical number at $x=10$. To determine if it is a minimum, use the second derivative test.

$S'(x)=2x-2000x^{-2}$	Take the first derivative.
$S''(x)=2+4000x^{-3}$	Take the second derivative.
$S''(10)=2+4000(10)^{-3}=6$	Evaluate $S''(10)$.

Since $S''(10)$ is positive, there is a minimum when $x=10$.

Thus, the minimum amount of material to build the box is $S(10)=10^2+\dfrac{2000}{10}=300\text{ in}^2$.

WATCH

In this video, we will determine the optimal route to lay cable across two terrains (underground and underwater, with water being more expensive).

Video Transcription

Hi there. In this video, what we're going to do is take all the knowledge we learned about derivatives and finding maximum and minimum points, and we're going to solve an optimization problem. And this particular problem, if you follow along the red, we're trying to find the optimal path to go from point A to point B by selecting a point along the shoreline so that we minimize the cost of laying the cable.

And what we know is that every mile on land costs $5,000, and every mile going underwater costs $8,000. And what we have is that the point A is 2 miles from the shoreline. And we're talking about a distance of 8 feet or 8 miles along the shoreline. So we're trying to find the optimal path. So let's just take a moment and just consider some easy paths just to see.

So one path might be, knowing that the water is the most expensive, one choice might be, well, why don't I go directly across the water and then directly down the shoreline. And while that might appear to be cost effective, the problem is, it's a longer distance than any other distance we could ever find. you know, we're going the whole way across and then the whole way down. That's a total of 10 miles. So it could be the optimal path. It's just not clear.

The other possible path is if we go directly from A to B. And while that is the shortest distance, the problem is it's also the most expensive when it comes to materials. $8,000 per mile, and we're using all of the distance underwater. So that can really drive the cost up.

So what we do is we pick a point along the shoreline for the cable to stop, and then we go the rest of the way down the shoreline. Maybe that'll end up with an optimal solution. We'll see. And of course, the calculus will help us figure out where on the shoreline that should be.

So looking at the two distances, we need distances in terms of x. Now, this straight-line distance here, the horizontal distance, since the entire horizontal distance is 8 and we're saying x is the distance we're going from the left, that means this distance here is 8 minus x. That one wasn't too bad.

The other one, the slanted distance, however, notice that kind of makes a right triangle. Well, it does make a right triangle where one side is x, the other side is 2, and the distance we want is the hypotenuse of that triangle. That, we have to use Pythagorean theorem to get. And just setting it up, we 2 squared plus x squared is equal to-- I'm going to call this side AC, because it's going from point A to point C.

And if we solve for AC, we get the square root of x squared plus 4. So this side here is the square root of x squared plus 4. So when we formulate our cost function, we have to keep those two distances in mind. One distance is the square root of x squared plus 4, and that's going to cost us $8,000 per mile. So the total cost of that piece is $8,000 times the square root of x squared plus 4.

The other piece is going to cost us $5,000 per mile, and we're going 8 minus x miles. There's our cost function. So now, here's an added bonus. We have a nice domain that we can find here too. We know that because x is the location along the shoreline, I know that x cannot be any more than 8. And I know it can't be any less than 0.

So that means we are finding the optimal value of a function on a closed interval. That means we can apply the extreme value theorem, which basically says if you have a continuous function on a closed interval, there is a minimum and there is a maximum guaranteed. So we're going to utilize that instead of using the second derivative test this time.

OK, so ready to find the derivative and find the critical numbers? And then off we go. So first thing you need to do before taking the derivative is to rewrite the square root, because we know we deal better when it's a 1/2 power. So that means C of x is 8,000. x squared plus 4 raised to the 1/2 plus-- and I'm going to go ahead and multiply the 5,000 through. So you have 40,000 minus 5,000x, OK.

So now we're ready to take the derivative. So C prime is-- OK. So 8,000 times something with the power rule. So it's 8,000 times the 1/2, which is 4,000, times the something to the negative 1/2 times the derivative of the inside. The derivative of 40,000 is 0, and the derivative of minus 5,000x is minus 5,000.

And we're going to clean up the derivative first before we do anything else with it. So this is 8,000x times x squared plus 4 to the negative 1/2 minus 5,000. And I'm going to rewrite the negative 1/2 power as dividing by the square root. Remember, that would be over the 1/2 power, and then a 1/2 power mean square root. So you have 8,000x over square root X squared plus 4 minus 5,000.

Now, remember that critical numbers occur when the derivative is either equal to 0 or is undefined. And you might be thinking, this is undefined because there's a denominator here. But remember, that denominator is x squared plus 4. There is no value of x that makes that 0. X squared is non-negative, and adding 4 just means my denominator is always at least 4, never 0.

So I'm going to set the derivative equal to 0 in hopes of finding a critical number. So what we'll do, we'll isolate the x terms to one side. So I'm going to add 5,000 to both sides. So you have 8,000x over the square root equals 5,000. And I'm going to go ahead and multiply both sides by the square root. So that means we have 8,000x equals 5,000 square root x squared plus 4.

And right away, I notice both sides have a common factor of 1,000. So I'm going to divide both sides by 1,000 right away. That's going to go a long way in making our computations much easier, see, if 8x equals 5 squared root x squared plus 4. And now to solve for x, this is an old trick you might remember from your algebra experience. We're going to square both sides.

And after squaring both sides, let's see what we end up with here. So this is 64x squared equals 5 squared is 25. And the square root squared is just what's under the radical. So we have that. Going to distribute the 25. And then subtract 25x squared from both sides. That means we have 39x squared equals 100, which means x squared is equal to 100 over 39. Kind of a weird number there.

So that means x is equal to-- now normally, it's plus and minus. But we know in this problem, we're only dealing with positives. X is a distance. So it's the square root of 100 over 39, which oddly enough, is just very close to 1.6. So that number is on our interval. So it does need to be checked.

So we substitute 1.6 into the cost function. We substitute 0 and 8, because those were the endpoints. And here is the analysis. So when x equals 0, meaning we're going straight across the river and then 8 miles down, the total cost is $56,000. At 1.6, the total cost is $52,490.

And at 8, then that would be corresponding to going directly across the river from point A to point B, the slanted distance. It turns out that is the most costly. It's almost $66,000. So the minimum cost occurs when x is equal to 1.6, and a minimum cost of $52,490.

→ EXAMPLE A 30-inch rod is to be cut into at most two pieces. One piece will be made into a square, and the other piece will be made into an equilateral triangle. How long should each piece be in order to maximize the combined area, and what is the maximum area?

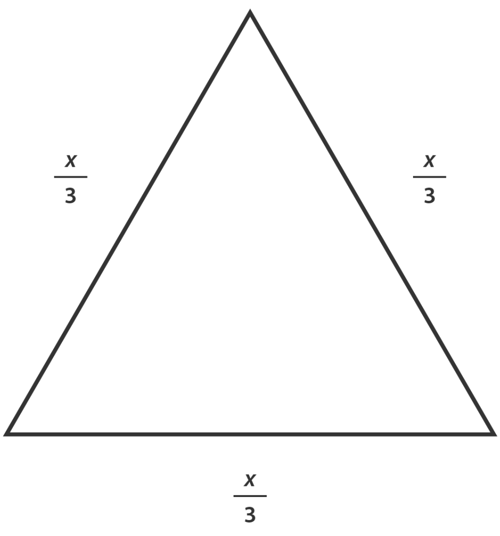
First, start by drawing a picture. Here is the rod being divided into two pieces:

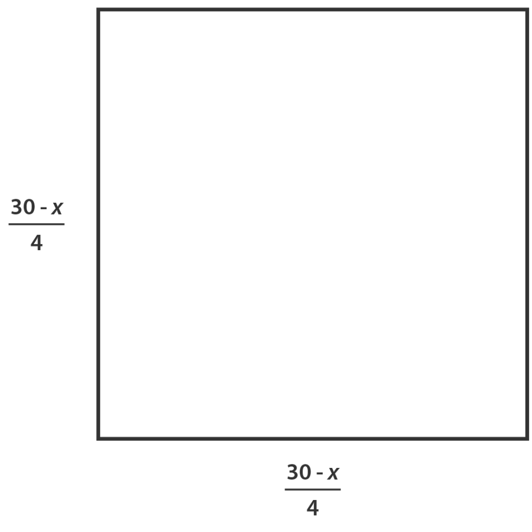
- Let x = the length of the piece that will be used for the triangle.
- Then, $30-x=$ the length of the piece that will be used for the square.

It is clear that x must be between 0 and 30 inches, therefore the domain is $[0, 30]$, a closed interval.

Now, let's form the expressions for the lengths of the sides.

- Triangle: Since the piece of the rod has length x, each of its 3 sides has length $\frac{x}{3}$.
- Square: Since the piece of the rod has length $30-x$, each of its 4 sides has length $\frac{30-x}{4}$.





Forming the area function:

The area of an equilateral triangle with a side of length s is $A=\frac{\sqrt{3}}{4}s^2$. Then, the area of our equilateral triangle is $A_T=\frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2=\frac{\sqrt{3}}{36}x^2$.

The area of a square with sides of length *s* is $A=s^2$. Then, the area of the square is
$A_S=\left(\frac{30-x}{4}\right)^2=\frac{900-60x+x^2}{16}=\frac{225}{4}-\frac{15}{4}x+\frac{1}{16}x^2$.

Since we want to find the maximum combined area, the optimization function is A_T+A_S, which leads us to:
$A(x)=\frac{\sqrt{3}}{36}x^2+\frac{225}{4}-\frac{15}{4}x+\frac{1}{16}x^2$ on the interval $[0, 30]$.

Now, take the derivative and find all critical numbers on the interval $[0, 30]$. Since $A(x)$ is continuous on the closed interval $[0, 30]$, the extreme value theorem can be used to determine the minimum and maximum values of $A(x)$.

$A(x)=\frac{\sqrt{3}}{36}x^2+\frac{1}{16}x^2+\frac{225}{4}-\frac{15}{4}x$	Start with the original function (place like terms next to each other).
$A'(x)=\frac{\sqrt{3}}{36}(2x)+\frac{1}{16}(2x)+0-\frac{15}{4}$	Take the derivative.
$A'(x)=\frac{\sqrt{3}}{18}x+\frac{1}{8}x-\frac{15}{4}$	Simplify.
$\frac{\sqrt{3}}{18}x+\frac{1}{8}x-\frac{15}{4}=0$	Set $A'(x)=0$. There is no possibility for $A'(x)$ to be undefined since it is a linear function.
$72\left(\frac{\sqrt{3}}{18}x+\frac{1}{8}x-\frac{15}{4}\right)=72(0)$ $4\sqrt{3}x+9x-270=0$	Multiply both sides by the LCD (72) to clear the fractions.
$4\sqrt{3}x+9x=270$ $(4\sqrt{3}+9)x=270$ $x=\frac{270}{4\sqrt{3}+9}\approx 16.95$	Solve for *x*. Since the exact value is complicated, use the approximate value.

Now, make a table to compare the values of $A(x)$ at the critical number as well as the endpoints.

x	0	30	16.95
$A(x)$	$\frac{225}{4}=56.25$	$25\sqrt{3}\approx 43.30$	24.47 (approx.)

The maximum area occurs when $x=0$ (which means the entire 30 inches will be used to make the square and none of it will be used to make the triangle).

Thus, the maximum possible area is 56.25 in^2.

SUMMARY

In this lesson, you learned about the **strategy for solving optimization problems**, which are problems in which the maximum or minimum value is sought (whichever is relevant). As you learned by examining several real-world examples, **solving applied optimization problems** can be particularly challenging since you have to come up with the function on your own. This takes practice, and drawing pictures or making tables is often helpful.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Optimization Problem

A problem in which the maximum or minimum value is sought, whichever is relevant.

Limits As x Becomes Arbitrarily Large ("Approaches Infinity")

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the behavior of a function as x gets arbitrarily large. Specifically, this lesson will cover:

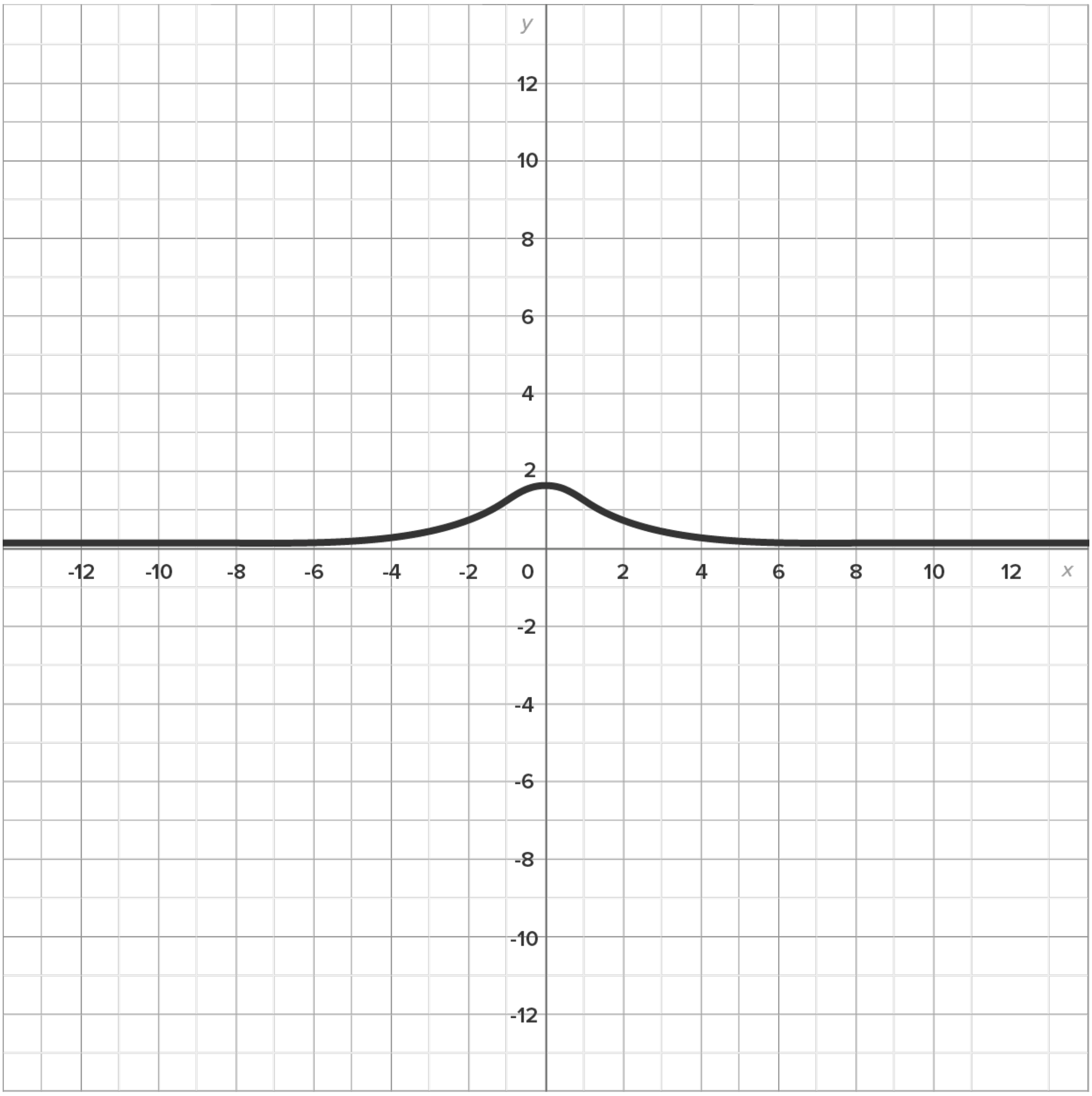
1. Graphically Finding Limits As $x \to \infty$ and As $x \to -\infty$
2. Numerically Finding Limits As $x \to \infty$ and As $x \to -\infty$
3. Analytically Finding Limits As $x \to \infty$ and As $x \to -\infty$

1. Graphically Finding Limits As $x \to \infty$ and As $x \to -\infty$

The notation $\lim_{x \to \infty} f(x)$ is used when we want to know what the value of $f(x)$ is approaching as x continues to increase, which we sometimes say increases without bound. The notation $\lim_{x \to -\infty} f(x)$ is used when we want to know what the value of $f(x)$ is approaching as x continues to decrease, which we sometimes say decreases without bound. In this tutorial, we will see how to find limits of the form $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ graphically, numerically, and algebraically.

Let’s look at finding this graphically first.

EXAMPLE Consider the graph of $f(x) = \frac{4}{x^2+3}$ shown below:



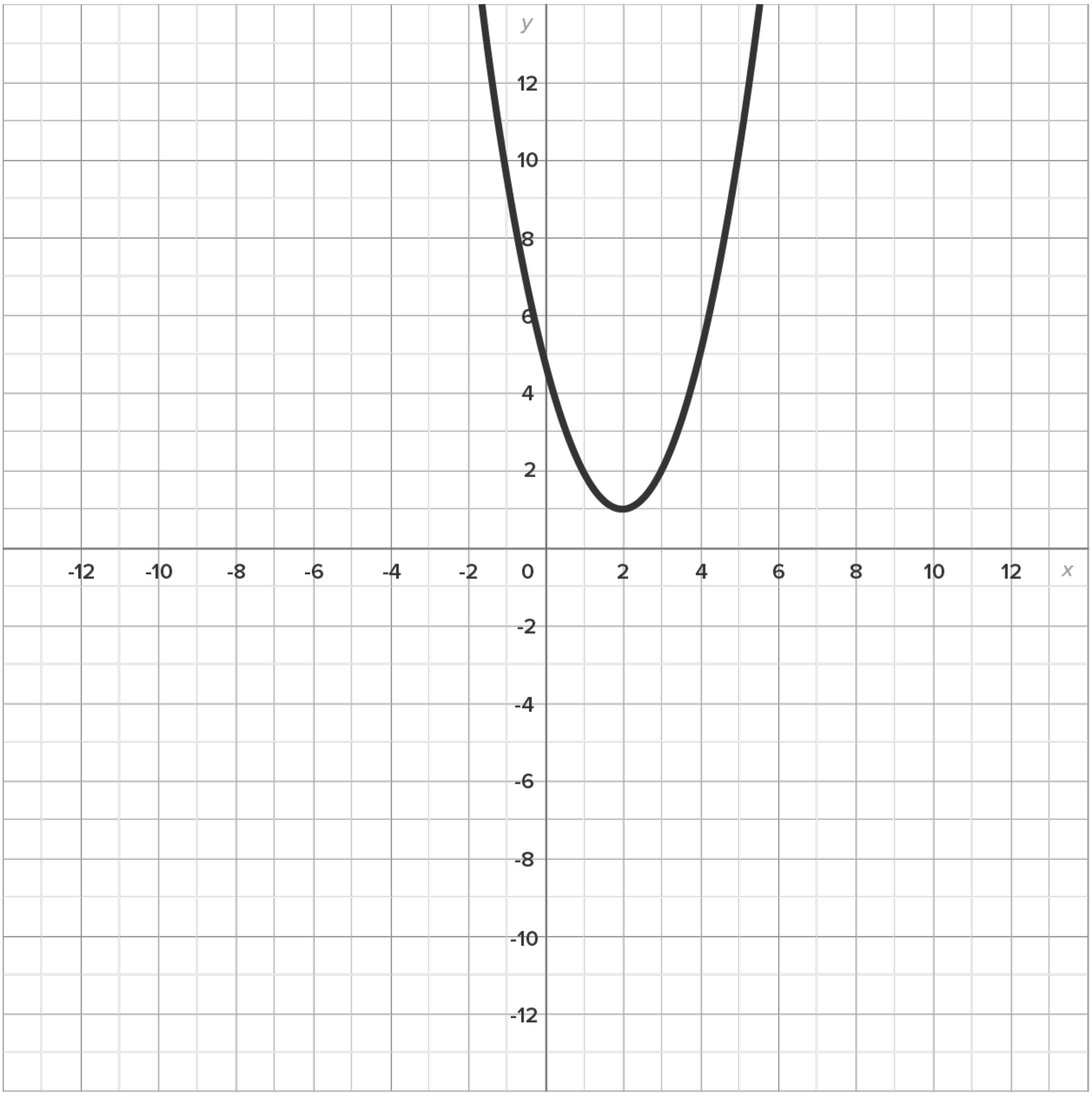
Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to get closer to 0.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to get closer to 0 as well.

Thus, we write $\lim_{x \to \infty} \frac{4}{x^2+3} = 0$ and $\lim_{x \to -\infty} \frac{4}{x^2+3} = 0$.

Let's now look at the limit of a quadratic function.

→ EXAMPLE Consider the graph of $f(x) = x^2 - 4x + 5$.



Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to increase indefinitely.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to increase indefinitely.

Thus, we could say $\lim_{x \to \infty} (x^2 - 4x + 5) = \infty$ and $\lim_{x \to -\infty} (x^2 - 4x + 5) = \infty$.

HINT

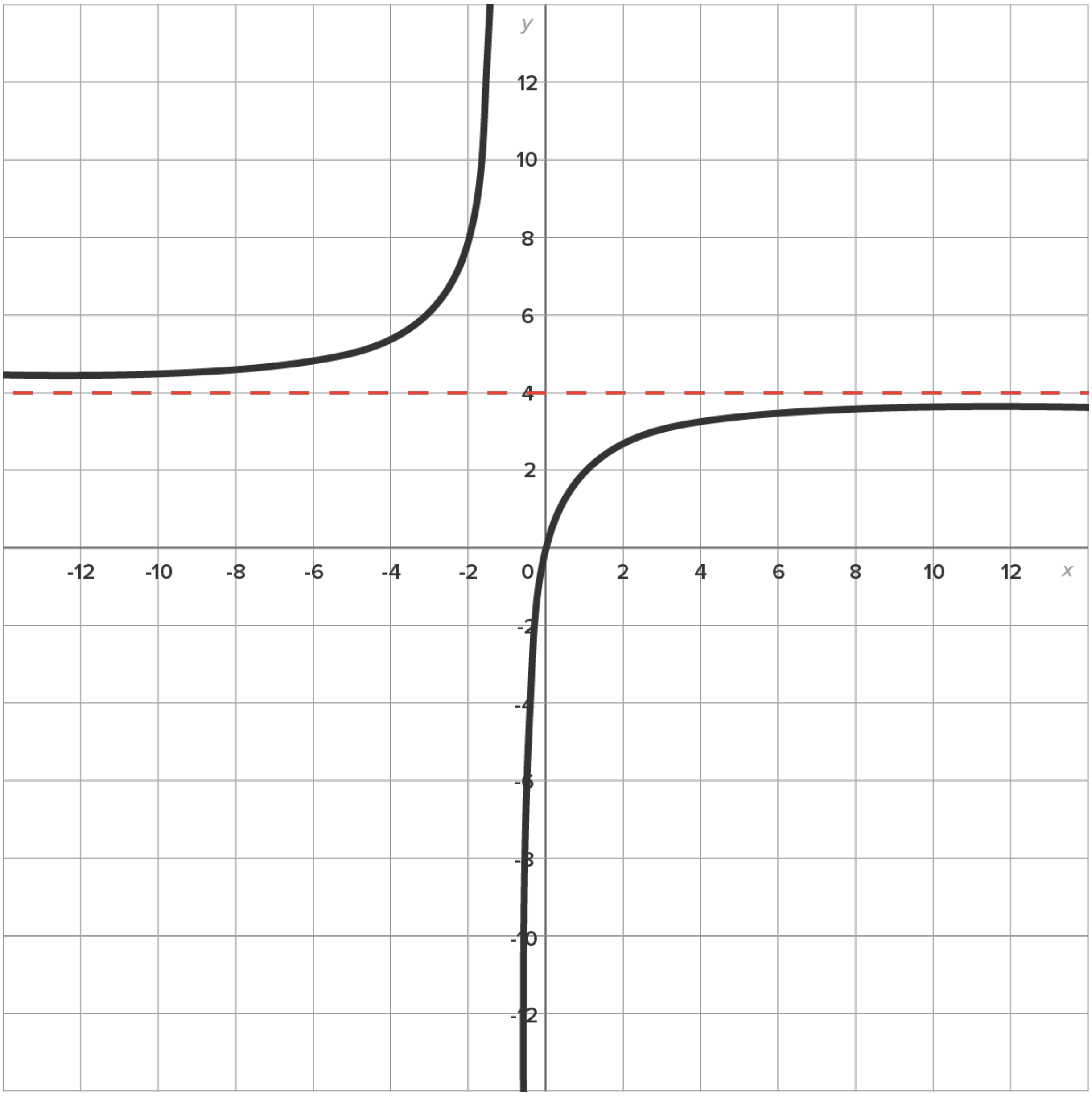
Stating that a limit is equal to infinity is contradictory since there actually is no limit.

There are cases where knowing the limit is "∞" or "$-\infty$" is useful, but when this happens, we can also say that the "limit does not exist" or "there is no limit."

For the purposes of this tutorial and this course, we will say "∞" and "$-\infty$" are acceptable answers.

Now let's look at a graph in which the limit is not as apparent.

→ EXAMPLE Consider the graph of $f(x) = \frac{4x}{x+1}$ shown below. The graph of $y = 4$ (dashed line) is also shown for reference:



Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to get closer to 4.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to get closer to 4 as well.

Thus, we write $\lim_{x \to \infty} \frac{4x}{x+1} = 4$ and $\lim_{x \to -\infty} \frac{4x}{x+1} = 4$.

You may note the use of the words "appears to get closer to" in the last two examples.

For instance, how do we really know that the limit in the last example is 4 and not 3.9 or something else "close to 4"?

When using a graph, we can only really estimate limits. This is why other techniques are used.

2. Numerically Finding Limits As $x \to \infty$ and As $x \to -\infty$

Recall that to find a limit numerically means to use a table of values to determine a pattern in the values of $f(x)$ in order to evaluate the limit.

↪ EXAMPLE Consider the function $f(x)=\frac{4}{x^2+3}$.

When evaluating $\lim_{x \to \infty} \frac{4}{x^2+3}$, we want to choose x-values that get progressively larger. While there is no set rule for choosing x-values, consider using powers of 10:

x	10	100	1,000	10,000
$f(x)=\frac{4}{x^2+3}$	0.03883495	0.00039988	0.00000400	0.00000004

By looking at the table, we can conclude that $\lim_{x \to \infty} \frac{4}{x^2+3}=0$.

Let's look at what happens when the limit itself is infinite.

↪ EXAMPLE Consider the function $f(x)=x^2-4x+5$.

We'll use the same tables to evaluate $\lim_{x \to \infty}(x^2-4x+5)$.

x	10	100	1,000	10,000
$f(x)=x^2-4x+5$	65	9,605	996,005	99,960,005

As we can see, the values of $f(x)$ are increasing rather quickly as x gets larger. We can conclude that $\lim_{x \to \infty}(x^2-4x+5)=\infty$.

To find $\lim_{x \to -\infty}(x^2-4x+5)$, we use a similar table:

x	-10	-100	-1,000	-10,000
$f(x)=x^2-4x+5$	145	10,405	1,004,005	100,040,005

Once again, the values of $f(x)$ are convincingly increasing rather quickly as x gets smaller (more negative). We can conclude that $\lim_{x \to -\infty}(x^2-4x+5)=\infty$.

TRY IT

Earlier, we found $\lim_{x \to \infty} \frac{4}{x^2+3}$ and concluded that the limit is equal to 0. Complete the following table and evaluate $\lim_{x \to -\infty} \frac{4}{x^2+3}$. (Notice that this is for negative infinity.)

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4}{x^2+3}$				

Round each value to 8 decimal places. +

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4}{x^2+3}$	0.03883495	0.00039988	0.00000400	0.00000004

What can we conclude about the limit? +

We can conclude that $\lim_{x \to -\infty} \frac{4}{x^2+3} = 0$.

Here is another for you to try.

TRY IT

Consider the function $f(x) = \frac{4x}{x+1}$.

Complete each table in order to evaluate $\lim_{x \to \infty} \frac{4x}{x+1}$ and $\lim_{x \to -\infty} \frac{4x}{x+1}$.

x	10	100	1,000	10,000
$f(x) = \frac{4x}{x+1}$				

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4x}{x+1}$				

Round each value to 6 decimal places. +

x	10	100	1,000	10,000
$f(x) = \frac{4x}{x+1}$	3.636364	3.960396	3.996004	3.999600

x	-10	-100	-1,000	-10,000

$f(x) = \frac{4x}{x+1}$	4.444444	4.040404	4.004004	4.000400

What can we conclude about the limit? +

We can conclude that $\lim_{x \to \infty} \frac{4x}{x+1} = 4$ and $\lim_{x \to -\infty} \frac{4x}{x+1} = 4$.

3. Analytically Finding Limits As x → ∞ and As x → - ∞

Evaluating limits numerically may be more convincing than using a graph, but it is still not as precise as using algebraic facts.

To get us started, here are a few properties of limits, if c is a constant and n is positive:

Limit as $x \to \infty$	Limit as $x \to -\infty$	Explanation
$\lim_{x \to \infty} \frac{c}{x^n} = 0$	$\lim_{x \to -\infty} \frac{c}{x^n} = 0$	As $x \to \pm\infty$, the denominator grows, making the overall value smaller and closer to 0.
$\lim_{x \to \infty} c = c$	$\lim_{x \to -\infty} c = c$	The limit of a constant is a constant.
$\lim_{x \to \infty} x = \infty$	$\lim_{x \to -\infty} x = -\infty$	As x itself increases or decreases without bound, the limit of x also does.
$\lim_{x \to \infty} x^n = \infty$	$\lim_{x \to -\infty} x^n = \infty$ if n is even $\lim_{x \to -\infty} x^n = -\infty$ if n is odd	A negative number raised to an even power is positive; a negative number raised to an odd power is negative.
If $\lim_{x \to \infty} f(x) = \pm\infty$, then $\lim_{x \to \infty} \frac{c}{f(x)} = 0$	If $\lim_{x \to -\infty} f(x) = \pm\infty$, then $\lim_{x \to -\infty} \frac{c}{f(x)} = 0$	If $f(x)$ grows larger and larger as $x \to \pm\infty$, then $\frac{c}{f(x)}$ gets smaller and tends toward 0. (Dividing by a larger number gives a smaller result.)

↱ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} e^{-3x}$

Since $e^{-3x} = \frac{1}{e^{3x}}$ and $e^{3x} \to \infty$ as $x \to \infty$, it follows that $\lim_{x \to \infty} e^{-3x} = \lim_{x \to \infty} \frac{1}{e^{3x}} = 0$.

STEP BY STEP

If $f(x)$ is a rational function, we can use the following technique to evaluate the limit:

1. Divide the numerator and denominator by x^n, where n is the highest power of x in the denominator.

2. Use properties of limits (including those shown above) to evaluate the limit.

↪ EXAMPLE Evaluate $\lim_{x \to \infty} \frac{4}{x^2+3}$ analytically.

$$\lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\frac{(x^2+3)}{x^2}}$$ Divide the numerator and denominator by the highest power of x in the denominator, which is x^2.

$$\lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\left(\frac{x^2}{x^2}+\frac{3}{x^2}\right)} = \lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\left(1+\frac{3}{x^2}\right)}$$ Expand the denominator, then simplify.

$$\frac{\lim_{x \to \infty}\left(\frac{4}{x^2}\right)}{\lim_{x \to \infty}\left(1+\frac{3}{x^2}\right)}$$ $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)}$$

$$\frac{0}{1+0} = 0$$ $$\lim_{x \to \infty}\left(\frac{4}{x^2}\right) = 0, \ \lim_{x \to \infty}\left(\frac{3}{x^2}\right) = 0, \ \lim_{x \to \infty} 1 = 1$$

Thus, analytically, $\lim_{x \to \infty} \frac{4}{x^2+3} = 0$.

TRY IT

Consider the following limit: $\lim_{x \to \infty} \frac{3x+2}{7x+1}$.

Evaluate the limit analytically. +

$\lim_{x \to \infty} \frac{3x+2}{7x+1} = \frac{3}{7}$ (Divide numerator and denominator by x.)

THINK ABOUT IT

In the problem you just tried, how would you have arrived at this answer either graphically or numerically without using an approximation?

Here is a limit in which the squeeze theorem is used.

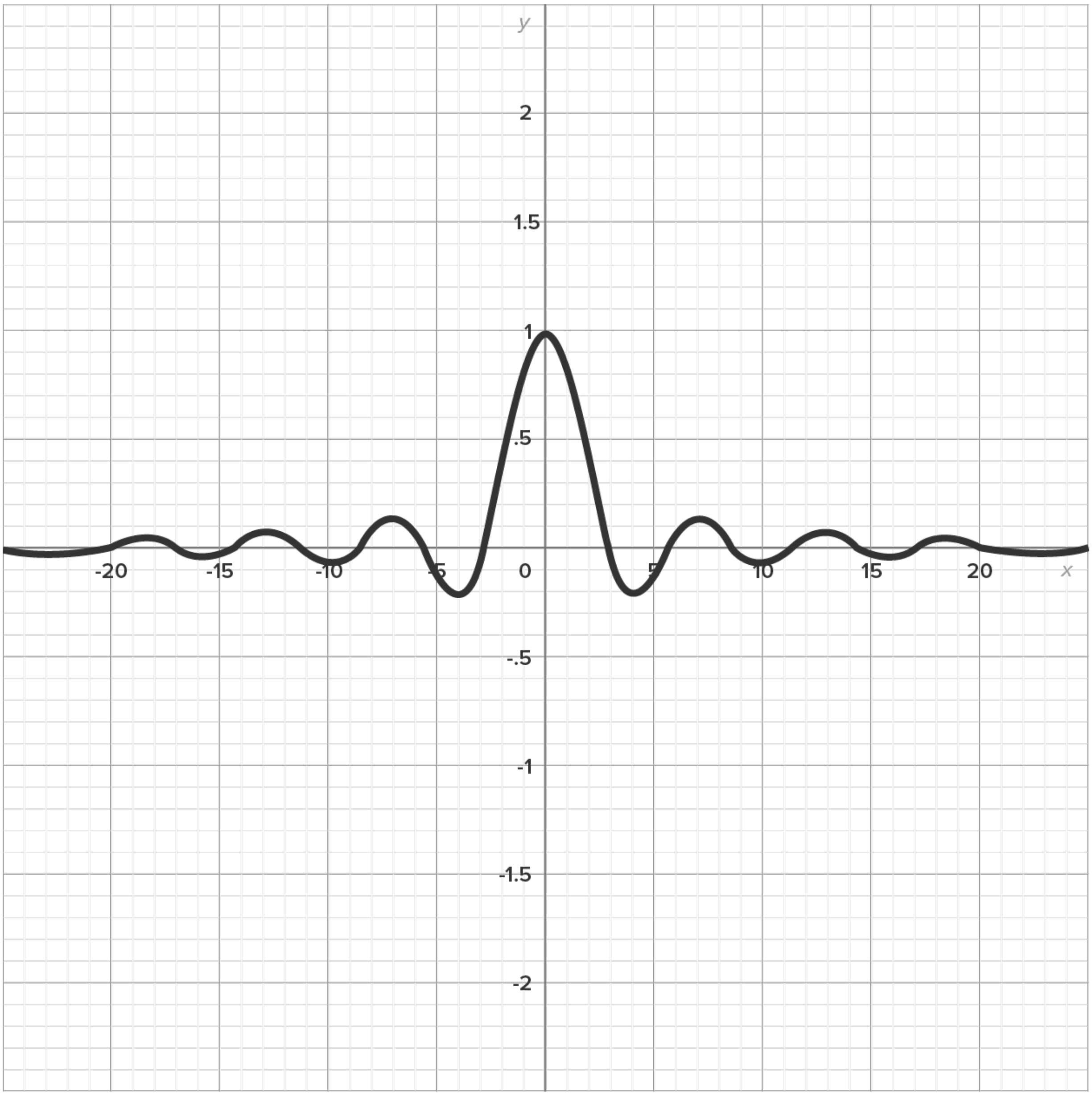
↪ EXAMPLE Evaluate $\lim_{x \to \infty} \frac{\sin x}{x}$. The method outlined earlier will not work since $\sin x$ is not a polynomial.

Recall that $-1 \le \sin x \le 1$ for all real numbers x. As $x \to \infty$, x is a positive number, so it is possible to divide all parts of the inequality by x: $\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

Using properties of limits, we know $\lim_{x \to \infty} \frac{-1}{x} = 0$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Since $\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$, it follows that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

Here is the graph of $f(x) = \frac{\sin x}{x}$, which helps to confirm this result:



WATCH

In this video, we'll evaluate $\lim_{x \to \infty} \frac{2x+1}{\sqrt{9x^2+1}}$.

Video Transcription

Hi there and welcome back. What we're going to do in this video is find a limit of a rational function, well, not quite a rational function, 2x plus 1 divided by the square root of 9x squared plus 1. And what we're going to find is that we can use the technique that we already know, which is dividing by the highest power in the denominator.

But we have to be really creative on what we do here. So looking at the denominator, it has a square root, which in the highest power under the square root is x squared. So I'm actually going to divide everything-- and what I mean by everything is the numerator and denominator by the square root of x squared.

Now, that's going to come in two forms. Remember that the square root of x squared is equal to x as long as x is positive. And since x is going to infinity, we are very certain that x is going to be a positive number. So what I'm going to do is I'm going to divide the numerator by x, and I'm going to divide the denominator by the square root of x squared. And that's so that the square roots align and the non-square roots align.

So looking at that expression, now we have some simplifying to do. So this is equal to the limit as x approaches infinity. Now, 2x plus 1 all divided by x is really 2x divided by x plus 1 divided by x. And in the denominator we can now combine those square roots as 9x squared plus 1 all over x squared. So now this is looking a little bit better.

In the numerator, I'm going to go-- whoops, I'm going to go this way. So we have the limit as x approaches infinity of 2 plus 1 over x-- I'm going to keep that in parentheses-- divided by. In the square root we really have 9x squared over x squared plus 1 over x squared. So the x squaredes are going to cross out and leave us with a 9 plus 1 over x squared.

Great, this is already looking better. And you notice at the top here, I wrote down the property of limits that's the most useful here. Limit as x approaches infinity of some constant divided by some positive power of x is always equal to zero. That denominator continues to grow. The larger the denominator, the smaller the actual value of the expression.

So one last thing we'll do is I'm going to rewrite this as the limit of the numerator, because it's easier to look at one thing at a time, of 2 plus 1 over x divided by the limit in the denominator square root 9 plus 1 over x squared. So now the limit of the numerator-- well, as x goes to infinity, 1 over x-- whoops, didn't mean to erase it, even though it kind of does erase.

1 over x does go to 0. And as x goes to infinity, 1 over x squared goes to 0. So we essentially have the limit of 2 over the limit of the square root of 9. Those are just constants so this limit is equal to 2 over the square root of 9, which is 2/3. And there's our limit.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that **evaluating limits as $x \to \infty$ and as $x \to -\infty$ can be done graphically, numerically, and analytically**. To evaluate graphically is probably the easiest, but it is sometimes difficult to get a precise answer. Using a numerical approach helps to see the patterns in how the values of $f(x)$ change, but again, it is sometimes difficult to get a precise answer. The analytical approach, while more lengthy, will produce a precise answer.

The graphical approach is best to use when the limit is either ∞ or $-\infty$ since this is fairly simple to spot from a graph. As long as the expression can be manipulated algebraically, the analytical approach will give a more precise and convincing answer.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

The Limit is Infinite

by Sophia

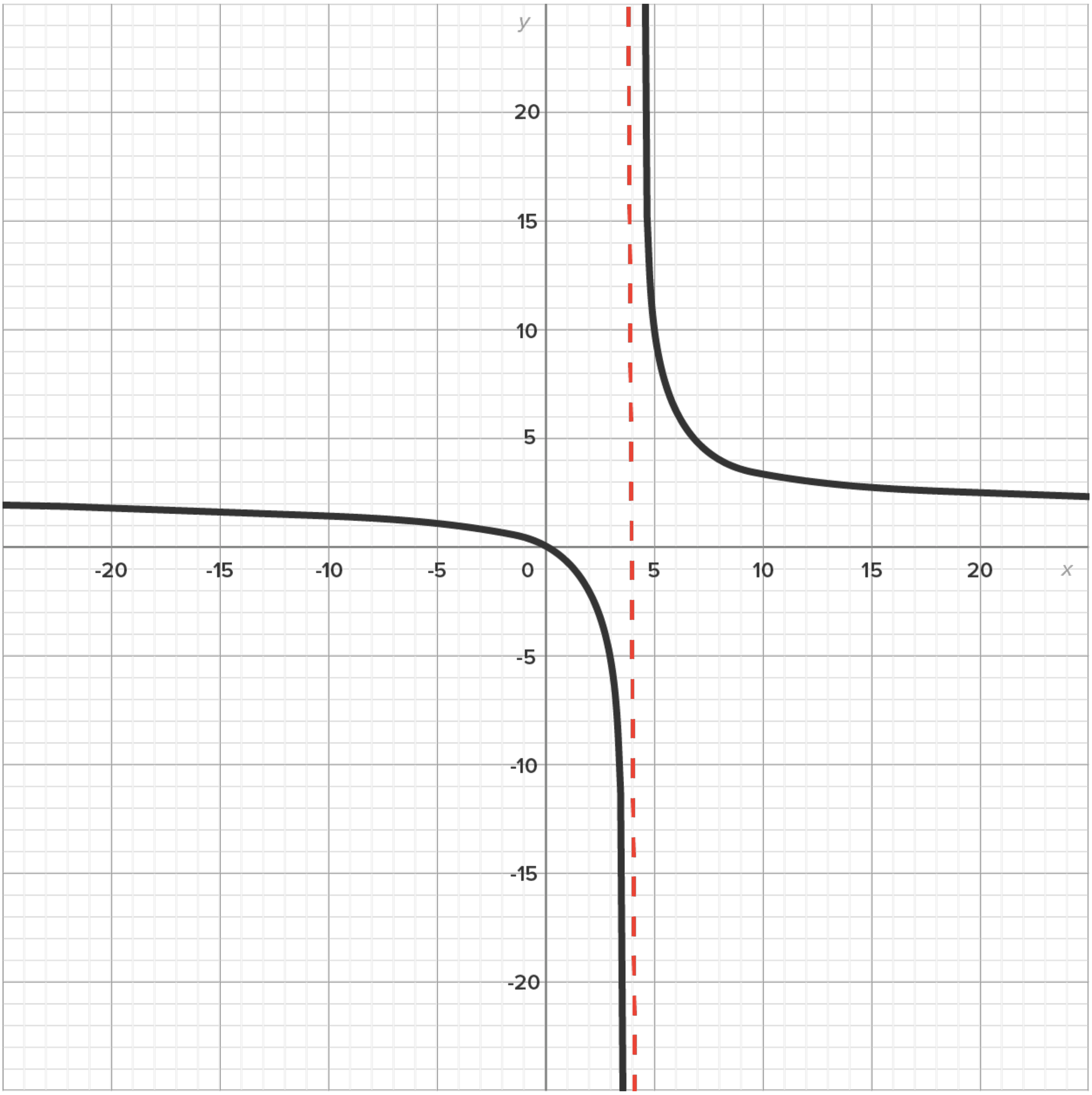
WHAT'S COVERED

In this lesson, you will analyze functions whose values get arbitrarily large as *x* approaches a finite value. Specifically, this lesson will cover:

1. Graphically Finding Infinite Limits
2. Numerically Finding Infinite Limits
3. Analytically Finding Infinite Limits

1. Graphically Finding Infinite Limits

Consider the graph of $f(x) = \frac{2x}{x-4}$, as shown in the figure (the dashed line $x = 4$ is drawn for reference).



Notice the behavior of the graph near $x = 4$.

As x gets closer to 4 from the left side, the graph decreases in value very quickly.

As a limit, this is written $\lim_{x \to 4^-} \frac{2x}{x-4} = -\infty$.

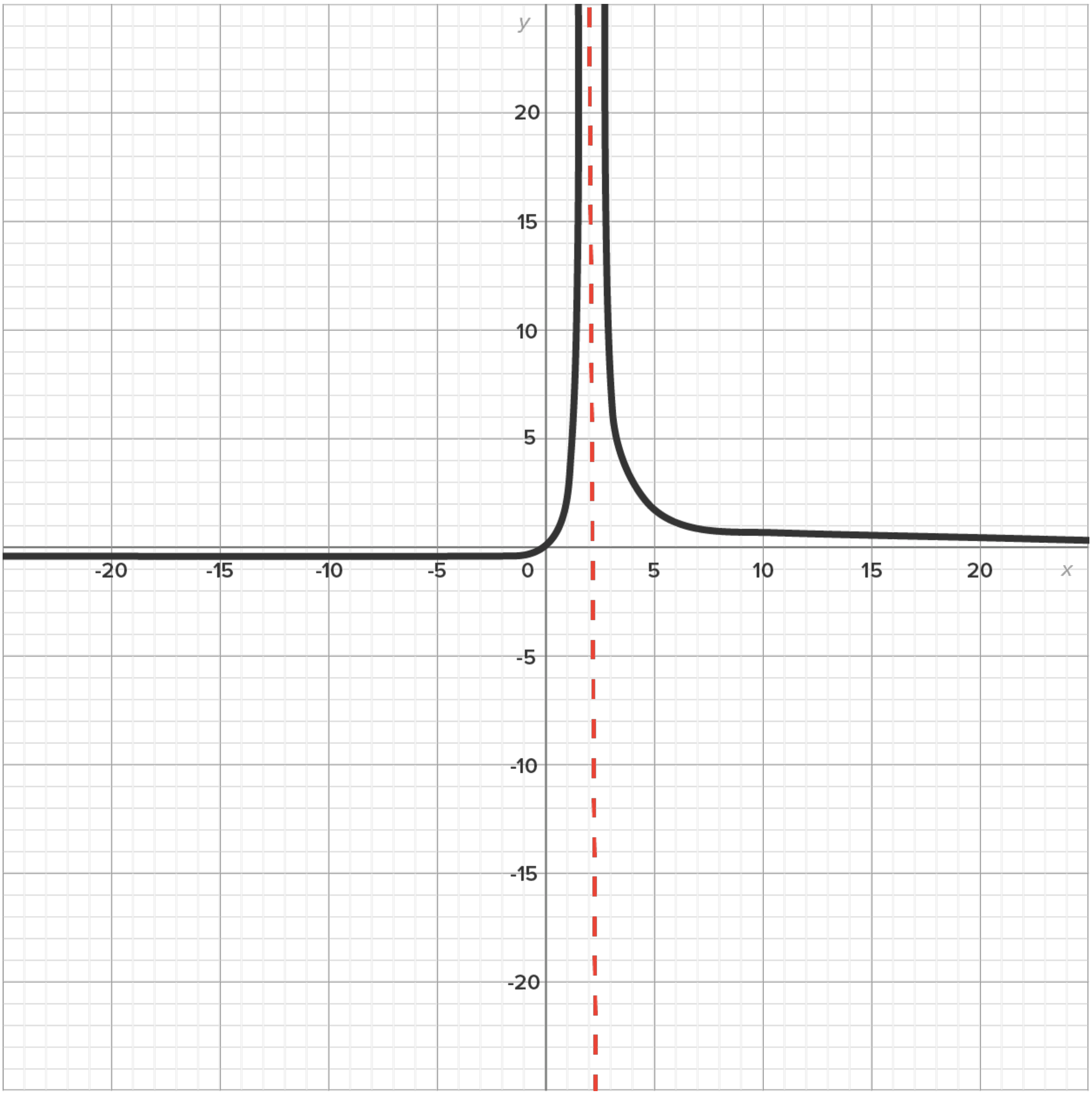
As x gets closer to 4 from the right side, the graph increases in value very quickly.

As a limit, this is written $\lim_{x \to 4^+} \frac{2x}{x-4} = \infty$.

Then, since the one-sided limits are not equal, the limit $\lim_{x \to 4} \frac{2x}{x-4}$ does not exist.

TRY IT

Consider the graph of $f(x) = \dfrac{3x}{(x-2)^2}$ below:



Suppose you want to find each of the following limits:

- $\lim\limits_{x \to 2^-} \dfrac{3x}{(x-2)^2}$
- $\lim\limits_{x \to 2^+} \dfrac{3x}{(x-2)^2}$
- $\lim\limits_{x \to 2} \dfrac{3x}{(x-2)^2}$

Evaluate each limit (if possible).

+

All limits are ∞.

2. Numerically Finding Infinite Limits

Using tables is another way to get some understanding about a limit.

↪ EXAMPLE Consider the function $f(x) = \frac{x+4}{x^2-1}$. Use a table of values to evaluate $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$.

In Unit 2, you may recall that the expression can be simplified before evaluating the limit. That is not the case here (if you factor the denominator, none of its factors cancel with the numerator).

To evaluate the one-sided limits, we choose x-values that successively get closer to 1 on each side:

From the left:

x	0.9	0.99	0.999	0.9999
$f(x) = \frac{x+4}{x^2-1}$	-25.7895	-250.7538	-2500.7504	-25000.7500

From the right:

x	1.1	1.01	1.001	1.0001
$f(x) = \frac{x+4}{x^2-1}$	24.2857	249.2537	2499.2504	24999.2500

As x gets closer to 1 from the left, the value of $f(x)$ decreases quickly. This means $\lim_{x \to 1^-} \frac{x+4}{x^2-1} = -\infty$.

As x gets closer to 1 from the right, the value of $f(x)$ increases quickly. This means $\lim_{x \to 1^+} \frac{x+4}{x^2-1} = \infty$.

As a result, since the one-sided limits do not agree, $\lim_{x \to 1} \frac{x+4}{x^2-1}$ does not exist.

TRY IT

Consider the function $f(x) = \frac{2x}{x^2-4x+3}$ and we want to find $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$, and $\lim_{x \to 3} f(x)$.

Use appropriate tables to evaluate the limits. +

From the left:

x	2.9	2.99	2.999	2.9999

$f(x) = \dfrac{2x}{x^2 - 4x + 3}$	-30.5263	-300.5025	-3000.5003	-30000.5000

From the right:

x	3.1	3.01	3.001	3.0001
$f(x) = \dfrac{2x}{x^2 - 4x + 3}$	29.5238	299.5025	2999.5002	29999.5000

$\lim\limits_{x \to 3^-} f(x) = -\infty$, $\lim\limits_{x \to 3^+} f(x) = \infty$, and $\lim\limits_{x \to 3} f(x)$ does not exist.

3. Analytically Finding Infinite Limits

To evaluate an infinite limit analytically, one-sided limits are used. Instead of making a table, the numerator and denominator are analyzed separately.

↱ EXAMPLE Evaluate $\lim\limits_{x \to 4} \dfrac{3x}{x-4}$ analytically.

Since $x = 4$ cannot be substituted directly and the function doesn't simplify, we'll examine the one-sided limits near $x = 4$.

Left-sided limit:
If $x \to 4^-$, then $x < 4$ but is getting closer to 4.

- The numerator, 3*x*, is close to 12.
- The denominator, $x - 4$, is a small negative number.
- Thus, we have $\dfrac{\textit{Close to 12}}{\textit{small negative}}$, which is equal to a large negative number.
- This means that $\lim\limits_{x \to 4^-} \dfrac{3x}{x-4} = -\infty$.

Right-sided limit:
If $x \to 4^+$, then $x > 4$ but is getting closer to 4.

- The numerator, 3*x*, is close to 12.
- The denominator, $x - 4$, is a small positive number.
- Thus, we have $\dfrac{\textit{Close to 12}}{\textit{small negative}}$, which is equal to a large positive number.
- This means that $\lim\limits_{x \to 4^+} \dfrac{3x}{x-4} = \infty$.

This means that $\lim\limits_{x \to 4} \dfrac{3x}{x-4}$ does not exist. Since the left- and right-sided limits do not have the same unbounded behavior, we cannot write the overall behavior.

WATCH

In this video, we'll evaluate $\lim_{x \to 2} \frac{4x}{(x-2)^2}$ analytically.

Video Transcription

Hello there. We're going to continue our study of limits by looking at a limit that goes to infinity. We have the limit as x approaches 2 of 4x divided by x minus 2 squared. And we are going to examine this limit analytically rather than using a graph or tables.

So remember that our general strategy with the limit that we suspect goes to infinity. And why do we suspect it goes to infinity? If we look at the numerator, the numerator is going to 8, and the denominator contains an x minus 2 squared. So as x gets closer to 2, that denominator is getting really, really close to 0, which suggests that we're dividing by a very small number, which means the result is going to be a very large number.

So that's more or less the analytical piece to this. So if we do the left-hand limit and the right-hand limit, so the left-hand limit, we're going to look at the limit as x approaches 2 from the left of 4x over x minus 2 squared. And we're going to examine what's going on in the numerator and the denominator. Now, as x approaches 2 from the left, this 4x, 4 times 2 is 8. The numerator is getting closer to 8.

And the denominator, now let's think about this. It's a number smaller than 2, which means x is less than 2, which means we have a small negative number squared, which is going to be 8 divided by an even smaller positive number. So as x approaches 2 from the left, this limit is going to positive infinity, again, kind of contradictory, limit, infinity. We'll just have the function as increasing without bounds as x approaches 2 from the left. It is going up indefinitely.

Now we'll look at the right-hand limit. We have the limit as x approaches 2 from the right-hand side. 4x over x minus 2, quantity squared. And it's a very similar result, because we have still 8 in the numerator. But now we have a small positive number, because we're taking a number slightly larger than 2 and subtracting 2, resulting in a positive.

So I'm going to call that small positive squared, which is 8 over a small positive, which again is positive infinity. So this limit actually agrees on both sides, and we can say that the limit as x approaches 2 of 4x over x minus 2 squared is equal to infinity.

SUMMARY

In this lesson, you learned that **infinite limits can be determined graphically, numerically, and analytically**. All methods produce reliable results, with the graphical approach being the simplest. When a graph is not available or is too difficult to read, it is useful to use either the numerical approach, using the appropriate tables to evaluate the limits, or the analytical approach, using one-sided limits and analyzing the numerator and denominator separately.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Horizontal and Vertical Asymptotes

by Sophia

WHAT'S COVERED

In this lesson, you will connect limits with horizontal and vertical asymptotes. Specifically, this lesson will cover:

1. Horizontal Asymptotes and Limits
2. Vertical Asymptotes and Limits

1. Horizontal Asymptotes and Limits

The graph of $f(x)$ has a **horizontal asymptote** $y = c$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

EXAMPLE Find all horizontal asymptotes of the graph of $f(x) = \frac{7x+1}{8x+3}$.

To find horizontal asymptotes, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$\lim_{x \to \infty} \frac{7x+1}{8x+3}$ Start with the required limit to evaluate.

$= \lim_{x \to \infty} \frac{\frac{7x}{x} + \frac{1}{x}}{\frac{8x}{x} + \frac{3}{x}}$ Divide each term by the highest power of x in the denominator, which is x.

$= \lim_{x \to \infty} \frac{7 + \frac{1}{x}}{8 + \frac{3}{x}}$ Simplify.

$= \frac{\lim_{x \to \infty}\left(7 + \frac{1}{x}\right)}{\lim_{x \to \infty}\left(8 + \frac{3}{x}\right)}$ $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)}$

$= \frac{7}{8}$ $\lim_{x \to \infty}\left(7 + \frac{1}{x}\right) = \lim_{x \to \infty} 7 + \lim_{x \to \infty} \frac{1}{x} = 7 + 0 = 7$

$\lim_{x \to \infty}\left(8 + \frac{3}{x}\right) = \lim_{x \to \infty} 8 + \lim_{x \to \infty} \frac{3}{x} = 8 + 0 = 8$

Thus, the graph of $f(x)$ has a horizontal asymptote at $y = \frac{7}{8}$.

Now, check $\lim_{x \to -\infty} \frac{7x+1}{8x+3}$.

$\lim_{x \to -\infty} \frac{7x+1}{8x+3}$	Start with the required limit to evaluate.
$= \lim_{x \to -\infty} \frac{\frac{7x}{x}+\frac{1}{x}}{\frac{8x}{x}+\frac{3}{x}}$	Divide each term by the highest power of x in the denominator, which is x.
$= \lim_{x \to -\infty} \frac{7+\frac{1}{x}}{8+\frac{3}{x}}$	Simplify.
$= \frac{\lim_{x \to -\infty}\left(7+\frac{1}{x}\right)}{\lim_{x \to -\infty}\left(8+\frac{3}{x}\right)}$	$\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to -\infty} f(x)}{\lim_{x \to -\infty} g(x)}$
$= \frac{7}{8}$	$\lim_{x \to -\infty}\left(7+\frac{1}{x}\right) = \lim_{x \to -\infty} 7 + \lim_{x \to -\infty} \frac{1}{x} = 7+0 = 7$ $\lim_{x \to -\infty}\left(8+\frac{3}{x}\right) = \lim_{x \to -\infty} 8 + \lim_{x \to -\infty} \frac{3}{x} = 8+0 = 8$

This produces the same result as the other limit, so there is no additional horizontal asymptote.

We can conclude that the graph of $f(x)$ has one horizontal asymptote at $y = \frac{7}{8}$. If you graph this function, you would see that the graph approaches the horizontal line $y = \frac{7}{8}$ as $x \to \pm\infty$.

TRY IT

Consider the function $f(x) = \frac{3x}{x^2+3}$.

Find all horizontal asymptotes of the graph of the function. +

$f(x)$ has one horizontal asymptote, and its equation is $y = 0$.

If $f(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are both polynomials, the following are results of limits:

- If $N(x)$ and $D(x)$ have the same degree, then the horizontal asymptote is $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If the degree of $N(x)$ is less than the degree of $D(x)$, then the horizontal asymptote is $y = 0$.
- If the degree of $N(x)$ is more than the degree of $D(x)$, then there is no horizontal asymptote, and this case is discussed in the next tutorial.

↪ EXAMPLE The function $f(x) = \frac{2x^2+4x+1}{3x^2+5x}$ has the horizontal asymptote $y = \frac{2}{3}$ since the degrees are the same.

↪ EXAMPLE The function $f(x) = \frac{3x}{x^2+3}$ has the horizontal asymptote as you saw in the Try It above, since the degree of the numerator is less than the degree of the denominator.

WATCH

In this video, we'll use limits to find the equations of all horizontal asymptotes of the function

Video Transcription

Hello there. Good to see you. What we're going to do in this video is considering the function f of x equals the absolute value of x divided by 2x plus 3. We're going to find the equations of the horizontal asymptotes. And we're used to functions having just one horizontal asymptote. We'll see if that's the case with this one here too.

So remember that a horizontal asymptote results from the limit as x approaches infinity of the function. And as well, the limit as x approaches negative infinity of the function. Now, remember that the rational functions we've dealt with in the past usually have the same horizontal asymptote on both sides.

And this isn't exactly a rational function because of the absolute value in the numerator. But one thing that's going to help us to evaluate these limits is, remember that the absolute value of x is equal to x if x is greater than or equal to 0, negative x if x is less than 0, which basically means the absolute value of x is the number underneath the bars if you have a non-negative number. And it's the opposite of the number underneath the bars if x is negative.

So let's look at each of these limits separately. So as x approaches infinity, we know that that means that x is positive. So this becomes the limit as x approaches infinity of x over 2x plus 3. Now that's looking more familiar. We go through, and we divide by the highest power in the denominator, which is x. So we do limit as x approaches infinity. We divide every term by x. So you have x over x over 2x over x plus 3 over x. And then we simplify.

So you have 1 over 2 plus 3 over x. Now, ordinarily, I would just write the limit of the numerator over the limit of the denominator. But in this case, we can see, here's what's going to happen. As x approaches infinity, this term is going to go to 0 and this limit is 1/2.

So that means that one of the horizontal asymptotes is y equals 1/2. Now, let's see if the other limit gives us the same thing. So as x approaches negative infinity, the absolute value of x is going to be negative x, because we're saying that x is less than 0 for sure. So this is the limit as x approaches negative infinity of negative x over 2x plus 3.

So we go through the same motions here. We have the limit as x approaches negative infinity. I'm going to divide everything by x again. So negative x over x over 2x over x plus 3 over x. And there's some simplification that can happen here. We have the limit as x approaches negative infinity. Negative 1 over

2 plus 3 over x.

And a very similar thing happens here as what happened in the other limit. We have negative 1 to the numerator for sure. We have 2 in the denominator for sure. 3 over x will tend to 0, and this limit is negative 1/2. So that means that another horizontal asymptote is y equals 2 negative 1/2. And if you were to graph the function, you would notice that to the right, the graph levels off to y equals 1/2. And to the extreme left, the graph levels off to y equals negative 1/2. Kind of cool.

TERM TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

2. Vertical Asymptotes and Limits

The graph of $f(x)$ has a **vertical asymptote** $x = a$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

The "$\pm\infty$" in the above definition means that the limit could be either $-\infty$ or ∞ for there to be a vertical asymptote when $x = a$.

HINT

If $f(x)$ is a rational function, the only values of x where a vertical asymptote could occur are those values where the denominator is equal to 0.

EXAMPLE Determine the vertical asymptotes of $f(x) = \dfrac{x^2 - 2x}{x^2 - 4x}$.

First, find all values of x for which the denominator is 0:

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0, x = 4$

Thus, the possible vertical asymptotes are $x = 0$ and $x = 4$. To determine which are vertical asymptotes, we need to evaluate a one-sided limit for each x-value. For this example, we'll choose right-sided limits.

Is $x = 0$ a vertical asymptote?

$\lim_{x \to 0^+} \dfrac{x^2 - 2x}{x^2 - 4x}$	Start with the required limit to evaluate.
$\lim_{x \to 0^+} \dfrac{x(x-2)}{x(x-4)} = \lim_{x \to 0^+} \dfrac{x-2}{x-4}$	Factor, then remove the common factor.

$$= \frac{0-2}{0-4} = \frac{1}{2}$$ Direct substitution works!

Since the limit is not $\pm\infty$, there is no vertical asymptote at $x = 0$. (Note: The left-sided limit would produce the same result.)

Is $x = 4$ a vertical asymptote?

$$\lim_{x \to 4^+} \frac{x^2 - 2x}{x^2 - 4x}$$ Start with the required limit to evaluate.

$$\lim_{x \to 4^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \to 4^+} \frac{x-2}{x-4}$$ Factor, then remove the common factor.

$$= \infty$$ As *x* approaches 4 from the right, *x* - 2 is around 2, and *x* - 4 is a small positive number.
$\frac{\text{around 2}}{\text{small positive number}} =$ a large number, so the limit is ∞.

We can conclude that there is a vertical asymptote at $x = 4$, but not at $x = 0$. If you were to graph the function, this would confirm this result.

TRY IT

Consider the function $f(x) = \dfrac{2x+1}{x^2 - 6x + 5}$.

Find the equations of all vertical asymptotes of the function. +

The vertical asymptotes are $x = 1$ and $x = 5$.

TERM TO KNOW

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

SUMMARY

In this lesson, you learned that **the horizontal and vertical asymptotes of a function are related to limits** of a function where infinity is involved. Specifically, a function $f(x)$ has a horizontal asymptote at $y = c$ if $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$, and a function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Other Asymptotes as x Approaches ∞ and -∞

by Sophia

WHAT'S COVERED

In this lesson, you will investigate other types of asymptotes that are neither horizontal nor vertical. Specifically, this lesson will cover:

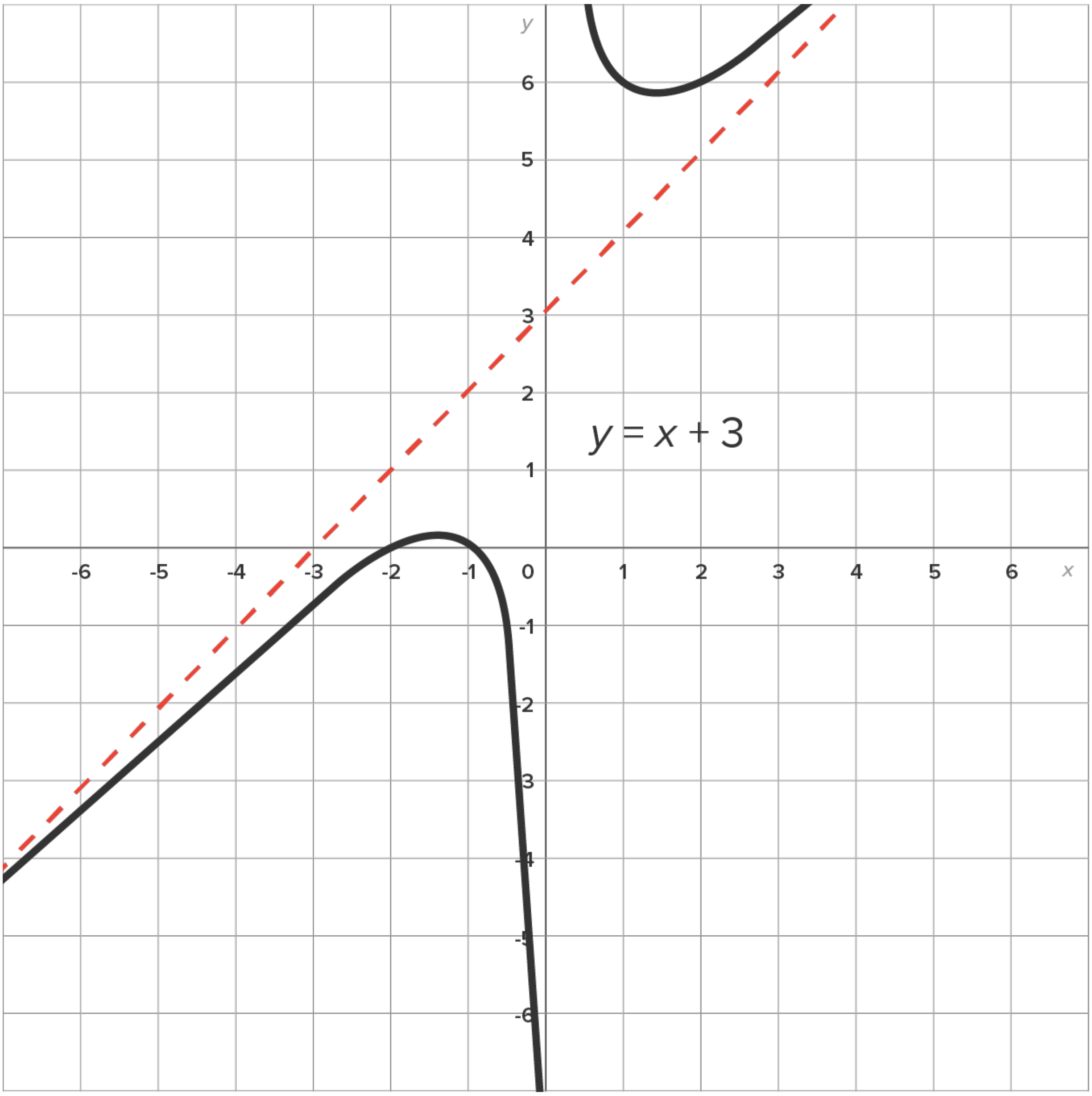
1. Slant (Oblique) Asymptotes
2. Other Nonlinear Asymptotes

1. Slant (Oblique) Asymptotes

When a rational function $f(x)$ doesn't have a horizontal asymptote, it could have a **slant asymptote**, which is a slanted line that the graph of $f(x)$ approaches as $x \to \pm\infty$.

To see how to recognize a slant asymptote, let's look at this first example.

> EXAMPLE Consider the function $f(x) = \dfrac{x^2+3x+2}{x}$.
>
> Performing the division, we have $f(x) = \dfrac{x^2}{x} + \dfrac{3x}{x} + \dfrac{2}{x} = x+3+\dfrac{2}{x}$.
>
> As $x \to \pm\infty$, $\dfrac{2}{x} \to 0$, which means the graph of $f(x)$ gets closer to the graph of $y = x+3$. Thus, the slant asymptote is $y = x+3$.
>
> The graph of $f(x)$ along with its slant asymptote (dashed) is shown in the figure. Note how the graph approaches its slant asymptote as $x \to \pm\infty$.



TRY IT

Consider the function $f(x) = \frac{3x^2 + 5x + 2}{x + 2} = 3x - 1 + \frac{4}{x + 2}$.

Identify the slant asymptote of the graph of this function. +

The graph of $f(x)$ has the slant asymptote $y = 3x - 1$ since $\frac{4}{x+2} \to 0$ as $x \to \pm\infty$.

TERM TO KNOW

Slant (Oblique) Asymptote

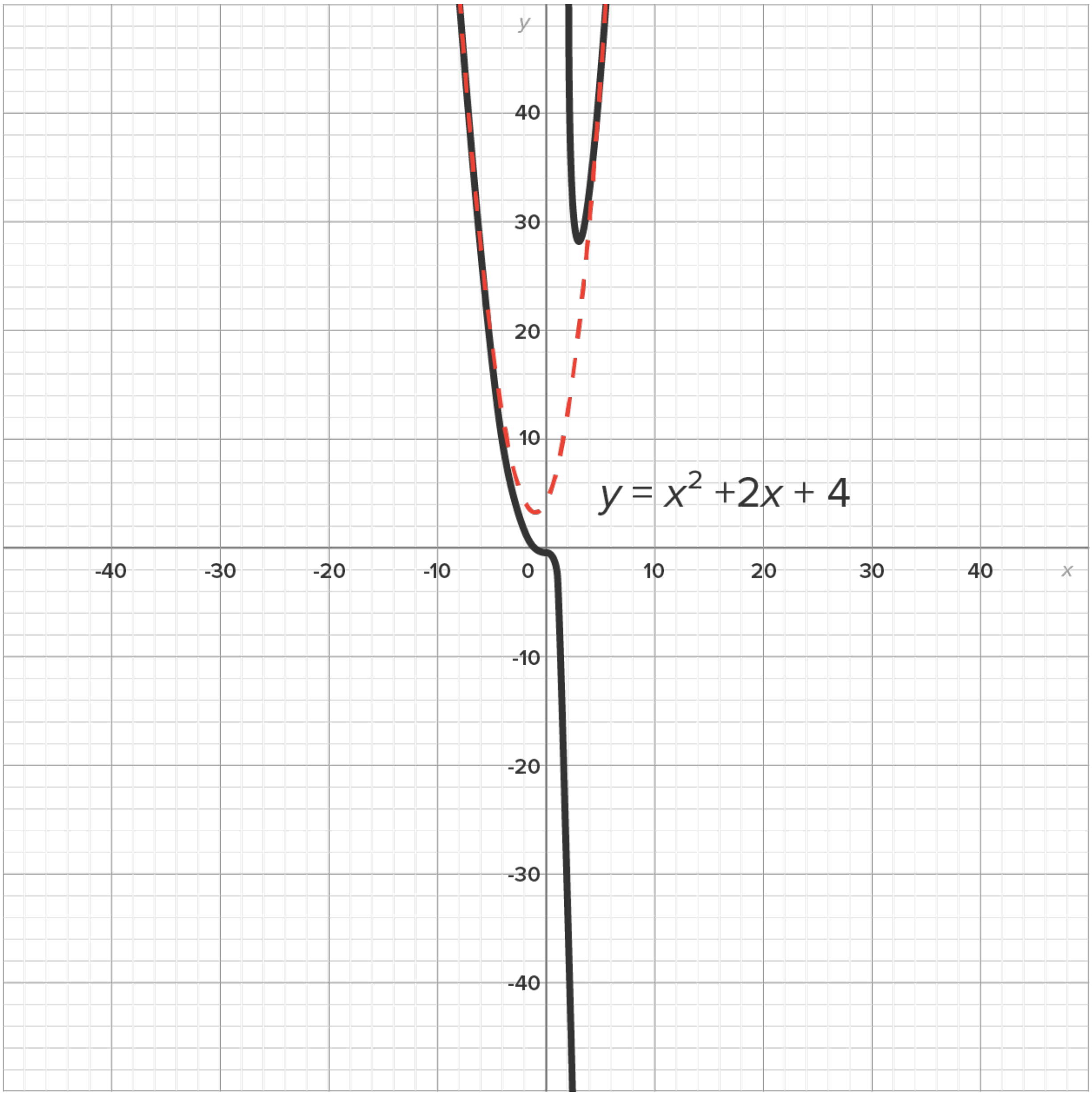
The slanted line that a graph approaches as $x \to \pm\infty$.

2. Other Nonlinear Asymptotes

A **nonlinear asymptote** is the curve that a graph approaches as $x \to \pm\infty$.

↪ EXAMPLE Consider the function $f(x) = \frac{x^3+1}{x-2} = x^2+2x+4+\frac{9}{x-2}$.

Since $\frac{9}{x-2} \to 0$ as $x \to \pm\infty$, the graph of $f(x)$ has a nonlinear asymptote $y = x^2+2x+4$. The graph of $f(x)$ along with the nonlinear asymptote (dashed) is shown in the figure.



TRY IT

Consider the function $f(x) = x^2 + \frac{x}{x^2+1}$.

Write the equation of the nonlinear asymptote of the function. +

The nonlinear asymptote is $y = x^2$.

TERM TO KNOW

Nonlinear Asymptote

The curve that a graph approaches as $x \to \pm\infty$.

SUMMARY

In this lesson, you learned that when a rational function doesn't have a horizontal asymptote, it could have either a **slant (oblique) asymptote**, which is a slanted line that the graph of $f(x)$ approaches as $x \to \pm\infty$, or a **nonlinear asymptote**, which is the curve that a graph approaches as $x \to \pm\infty$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Nonlinear Asymptote

The curve that a graph approaches as $x \to \pm\infty$.

Slant (Oblique) Asymptote

The slanted line that a graph approaches as $x \to \pm\infty$.

Putting It All Together: Sketching a Graph

by Sophia

WHAT'S COVERED

In this lesson, you will use properties of $f(x)$, $f'(x)$, and $f''(x)$, and limits to sketch the graph of a function. Specifically, this lesson will cover:

1. Graphing Functions: A General Strategy
2. Graphing Functions: Examples

1. Graphing Functions: A General Strategy

The following information is useful when graphing a function $y=f(x)$:

STEP BY STEP

1. Obtain the following information from $f(x)$:
   a. Find the domain of the function.
   b. Determine if there are any vertical, horizontal, slant, or nonlinear asymptotes by using limits.
   c. Find all x-intercepts (when convenient) and y-intercepts. Note that x-intercepts are not always easy to find without technology.
2. Obtain the following information from $f'(x)$:
   a. Find all critical numbers.
   b. Determine all intervals where $f(x)$ is increasing and decreasing.
   c. Use the first derivative test to locate all local maximum and minimum points.
3. Obtain the following information from $f''(x)$:
   a. Find all values where $f''(x)=0$ or is undefined.
   b. Determine all open intervals over which $f(x)$ is concave up or concave down.
   c. Determine any inflection points of $f(x)$.

---

2. Graphing Functions: Examples

EXAMPLE Use the techniques from this unit to sketch the graph of $f(x)=x^4-18x^2+32$.

Since both the first and second derivatives will be used, we'll find those first:

$f'(x)=4x^3-36x$

$f''(x)=12x^2-36$

1. Information from $f(x)$:
   a. The domain of the function is all real numbers.
   b. Since $f(x)$ is a polynomial, there are no asymptotes.
   c. The y-intercept is (0, 32). To find the x-intercepts, set $f(x)=0$ and solve.
      $x^4-18x^2+32=0$
      $(x^2-16)(x^2-2)=0$
      $x^2-16=0$ or $x^2-2=0$
      $x^2=16$ or $x^2=2$
      $x=\pm 4$, $x=\pm\sqrt{2}$
      Thus, the graph of $f(x)$ has 4 x-intercepts: $(\pm 4,\ 0)$, $(\pm\sqrt{2},\ 0)$
2. Information from $f'(x)$:
   a. Critical numbers:
      $4x^3-36x=0$
      $4x(x^2-9)=0$
      $4x(x+3)(x-3)=0$
      $4x=0$ or $x+3=0$ or $x-3=0$
      $x=0,\ -3,\ 3$

For parts b and c, use a sign graph for $f'(x)$:

Interval	$(-\infty,\ -3)$	(-3, 0)	(0, 3)	$(3,\ \infty)$
Test Number	-4	-1	1	4
Value of $f'(x)$	-112	32	-32	112
Behavior	Decreasing	Increasing	Decreasing	Increasing
Visual	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

   b. $f(x)$ is increasing on $(-3,\ 0)\cup(3,\ \infty)$.
      $f(x)$ is decreasing on $(-\infty,\ -3)\cup(0,\ 3)$.
   c. $f(x)$ has local minimums at $(-3,\ f(-3))$ and $(3,\ f(3))$. These points are (-3, -49) and (3, 49).
      $f(x)$ has a local maximum at $(0,\ f(0))$. This point is (0, 32).
3. Information from $f''(x)$:
   a. Set $f''(x)=0$ and solve:
      $12x^2-36=0$
      $12x^2=36$
      $x^2=3$
      $x=\pm\sqrt{3}$

   For parts b and c, use a sign graph for $f''(x)$:
   (Note: $\sqrt{3}\approx 1.73$.)

Interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, \sqrt{3})$	$(\sqrt{3}, \infty)$
Test Number	-2	0	2
Value of $f''(x)$	12	-36	12
Behavior	Concave up	Concave down	Concave up

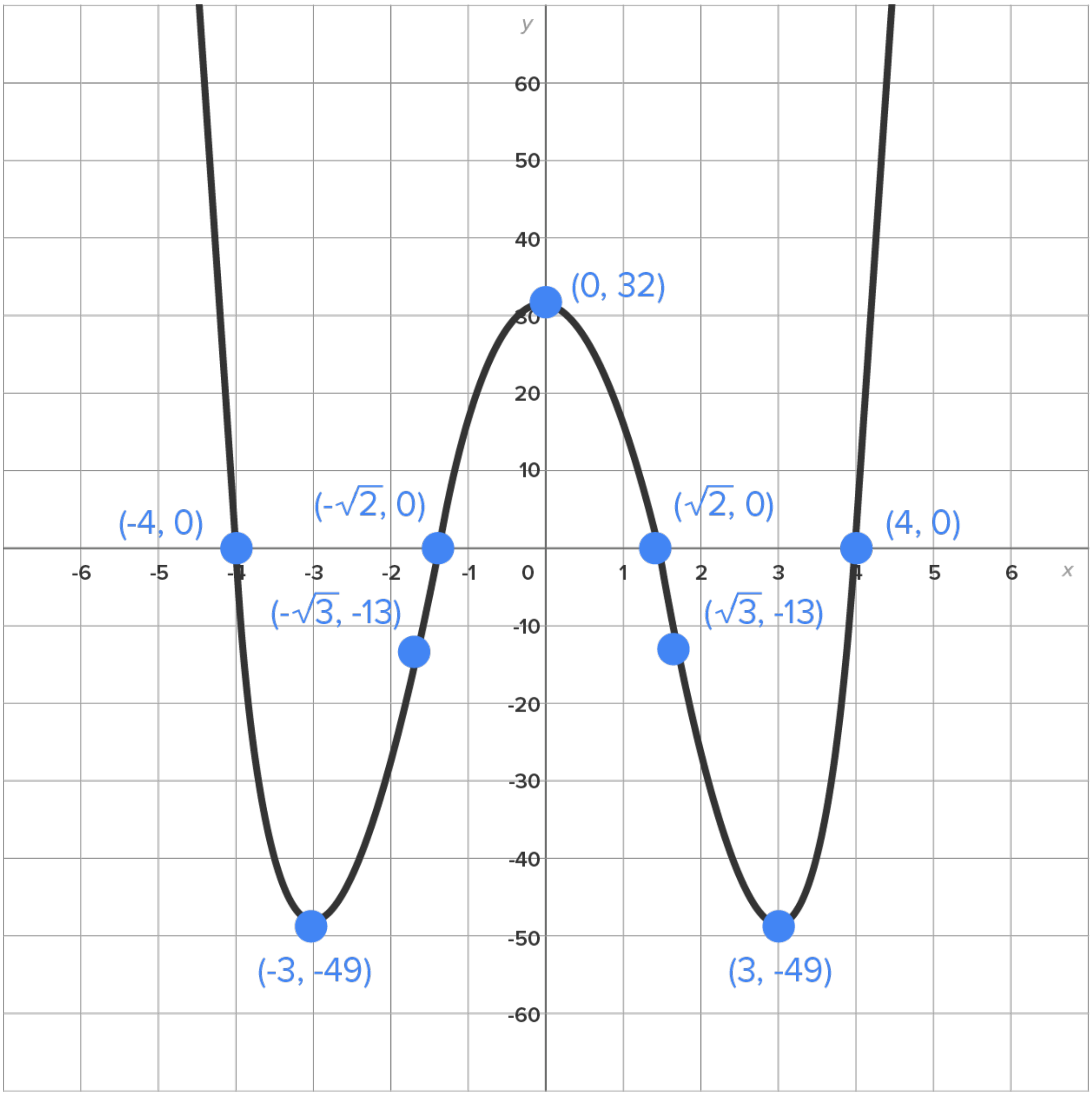
b. $f(x)$ is concave up on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

$f(x)$ is concave down on the interval $(-\sqrt{3}, \sqrt{3})$.

c. There are two inflection points: $(-\sqrt{3}, f(-\sqrt{3}))$ and $(\sqrt{3}, f(\sqrt{3}))$

These are the points $(-\sqrt{3}, -13)$ and $(\sqrt{3}, -13)$.

Pulling all this information together, the graph of the function with all important points labeled is shown in the figure.



WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x) = x - 6\sqrt{x-1}$. (Note: This video is over 10 minutes long.)

Video Transcription

Hi there, and welcome back. What we're going to do is use aspects of f, the derivative of f and the second derivative of f, all the properties with concavity, increasing and decreasing domain, to try to sketch the graph of a function. And the function we're going to look at right here is f of x equals x minus 6 times the square root of x minus 1.

So one of the most important aspects of a function is its domain-- what values of x are going to return real values? And notice we have a square root here. So the domain is all the values of x, the set of all values of x, such that the square root is defined. So that right here is going to be the interval 1 to infinity. Close bracket on 1 because x can equal 1. And there are no asymptotes to speak of here. So there's nothing going to happen with the asymptotes there.

Now ordinarily, we would also talk about intercepts. There is no y-intercept because x equals 0 is not in the domain of the function, and it's not very convenient to find the x-intercepts. We could set that equal to 0 and trying to solve-- it does get a little bit complicated. So we're just going to hold off on the intercepts until we actually get the graph of the function. We can approximate them.

So as far as critical numbers go, now remember, that comes from the first derivative. So we need to find the first derivative, and I'm going to rewrite the function f first to the 1/2 power, because that is more convenient to take a derivative with. So this means that the first derivative f prime is 1 minus 6 times 1/2 x minus 1 to the negative 1/2 , and then the derivative of the inside is 1. I'm not going to write that down necessarily here, but we have 1 minus 3 times x minus 1 to the negative 1/2.

So I'm going to save this for later because I know for sure that we're going to be taking another derivative and we're going to want the power version there with the negative 1/2 power. So thinking about critical numbers, usually we would set this equal to 0 and undefined. I'm going to rewrite the derivative in a more friendly form. Now remember, negative 1/2 power is 1/2 power in the denominator. And remember that 1/2 power really means the square root. So this is 1 minus 3 over square root x minus 1.

Clearly, this is undefined when x equals 1. It is the endpoint of the domain so it is something that we do need to consider in our number line, but we're not going to be able to substitute it because it's undefined. But it is also the endpoint of the domain, so there's nothing on the other side of it.

And we do also know-- so I'm just going to write here undefined when x equals 1. The other critical numbers can come from when the first derivative is equal to 0. So what I'm going to do is add the square root expression to both sides. Multiply both sides by the square root. Square both sides. So basically just did that, and then add 1 to both sides. And it looks like a critical number is 10. So x equals 1 end point, x equals 10, something could be happening there.

So what we do we make our sign graph for f prime. And x equals 1 is on the extreme left of that number line because we can't go to the left of 1. And we have 10 as our critical number. I should label that with a line not an x. So now what we're going to do is test values to the left and to the right of 10 to see what this derivative is doing. Is our function going to change from increasing to decreasing? Are we going to get an extrema? Or are we just going to get a point where the tangent line is horizontal and then just keeps going in the same direction?

So looking at our derivative again, f prime of x is 1 minus 3 over square root x minus 1. So I'm going to pick values of x so that when I subtract 1, I get a nice square root. So I notice that when I pick x equals 5, I end up with the square root of 4. So I'm going to pick 5, and I'm also going to pick-- well, see, if I want the number underneath the square root to be, say, 16, I want to plug in x equals 17 to make a nice square root.

So f prime of 5 is 1 minus 3 over square root of 4, which is 1 minus 3/2, which is negative 0.5. So that means that our function is decreasing on that interval. And then at 17, we have 1 minus 3 over square root of 16, which is 1 minus 3/4, which is 0.25, which means we're increasing. Positive derivative means increasing. Negative derivative means decreasing.

So that means that this function is increasing on the interval 10 to infinity, and it's decreasing on the interval 1 to 10. So I'm not including 1 in the decreasing interval because it is an end point. Open intervals are what we use for increasing and decreasing. There is no local max because we're not going from increasing to decreasing.

Now let's say we have a local min at 10. I'm just going to write f of 10. We'll figure out the value of that function later on when we look at the graph. So there we have all the information from the first derivative. Now remember, second derivative tells us about concavity. So I'm just going to separate this out here, and let's see what we have here.

So f prime at the very top of the screen there-- I'll write it down again-- f prime of x is 1 minus 3 times x minus 1 to negative 1/2. So that means that f double prime is-- OK, so the derivative of 1 is 0 minus 3 times negative 1/2 x minus 1 to the negative 3/2.

Now if we simplify that a bit, we get 3/2 times x minus 1 to the 3/2, which is 3 over 2x minus 1 to the 3/2. Now remember, the domain of the original function is x greater than or equal to 1. x equals 1 clearly makes this second derivative undefined, so we really only have one interval of interest here, and that is when x is greater than 1. So I'm going to draw my number line for my sign graph. And this is x equals 1 right here at the very end. I can't go before that.

And the numerator is never 0, which means the fraction is never 0, which means we have no possible places where this change is concavity. So we just have to test a value to see what the concavity is here. So I'm going to substitute-- since I can plug in anything after 1-- I'm going to plug in 5 because we know that worked before. And we see that f double prime at 5 is 3 over 2 times 4 to the 3/2, which ends up being 3 over 16.

The bottom line is it's a positive number. So that means that our function is always going to be concave up on its entire domain. So we can answer that question. It's concave up on the interval 1 to infinity. It's never concave down because we don't have any intervals there. And that means there's also no point of inflection. So that means this function is always concave up. It attains a minimum value at 10 comma f of 10. And we're increasing on one side, decreasing on the other. Let's take a look at what the graph looks like.

And there it is. So just a couple of things about it-- so you notice the x-intercept at 34.971, just about 35?

We have our minimum value at 10 and negative 8. And naturally, f of x was x minus 6 square root x minus 1. So f of 10 is 10 to the minus 6 square root 9, which is 10 to the minus 6 times 3, which is 10 minus 18, which is negative 8. So that all works right there. So that verifies that result.

Notice also this is not an asymptote, but if I want to know what that point is right there, this is 1 comma f of 1. When you plug in 1, you get 1. And you notice that the tangent line is going vertical there, and that that's reflected in where the derivative was undefined. So at least these aspects do make sense, and you notice that the graph is always concave up on its entire domain. So there is the graph of our function f of x.

WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x) = 10x^3 - 3x^5$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do is apply all of the graphical properties that we learned about for a function-- increasing, decreasing, concavity, et cetera. And we're going to try to graph the function f of x equals 10x to the 3rd minus 3x to the 5th. And you'll notice that I have the first and second derivatives right below it because we know we get information about a graph from the first derivative of the function and the second derivative of the function.

So the key is to organize this in a way that we get everything we can from each function. So first off, we have f of x. Now f of x is where we can get the domain by looking at the expression. This one happens to be a polynomial, which means the domain is all real numbers, so negative infinity to infinity.

And we can also get the intercepts of the graph, which are nice key points when they're convenient to get. So remember that the y-intercept is the point where x is equal to 0. And we get the point 0, 0. Our x-intercepts are the points where f of x is equal to 0. So we set that equal to 0, and we solve. So factor out x to the 3rd, and then set each factor equal to 0.

We get x equals 0 again, and we get x squared equals 10/3, which means x is equal to plus or minus the square root of 10/3, which I'm just going to leave unrationalized right now because we're just going to approximate it later when we get to the graph. So basically, what we know is so far, the domain is all real numbers and we have three x intercepts. And one of the x-intercepts also happens to be the y-intercept.

So not a lot to go on-- let's look at the first derivative. So here's our first derivative. And we set it equal to 0. Normally we would also look for undefined, but we don't have any values of x where this is undefined. It's a polynomial. So we factor again. And we still get 15x squared equals 0 over here, which means x equals 0. That x equals 0 keeps coming up.

And the other factor, 2 minus x squared equal to 0 means x squared equals 2, which means x is equal to plus or minus the square root of 2. Now for testing purposes for the first derivative test, I always like to get an approximation of any real number I might not really know the value of. And we know square root of 2 is about 1.41.

So that means when it comes to test numbers around the square root of 2, I would use 1 and 2 on the positive side and negative 1 and negative 2 possibly on the negative side. As you can see here, those are the exact test numbers that were selected here. And notice the intervals. That's on this line right here. The intervals that we're testing-- negative infinity to negative root 2, negative root 2 to 0, 0 to root 2, root 2 to infinity, and again, selecting numbers that are inside each of those intervals that are convenient to substitute into the first derivative.

So down here, we substituted each of those test numbers into the first derivative. Negative values mean negative slopes, mean the function is decreasing. Positive values mean positive slope, which means the function is increasing. And as you can see from the arrows I drew there, it's decreasing on the first interval, then increasing, then increasing yet again, and then decreasing.

So what this means is it's increasing-- the function, f of x, is increasing on the open intervals negative root 2 to 0 and 0 to root 2, as you see in the middle there, and it's decreasing on the other two intervals--negative infinity to negative 2 and root 2 to infinity, which then means there is a local min at x equals negative root 2. And I just wrote negative root 2 comma f of negative root 2. We will figure out that coordinate when we need to when we graph.

And there's a local maximum when x equals root 2, which means the point root 2 comma f of root 2, OK? So we're just kind of collecting all this information. Now we have a little bit more. We know that there's a min point and a max point, and we know which direction the graph has to go in between those min and max points. Great.

But there's still a little bit more we need to know. We need to know a little bit more about concavity. So we go to the second derivative. And that is right here. So we take our second derivative, set it equal to 0, again, ignoring undefined. Factor about 60x, and we get x equals 0 and x equals plus or minus 1. So now we have three possible transition points as far as concavity goes.

So we make our second derivative number line, again keeping in mind the intervals negative infinity to negative 1, negative 1 to 0, 0 to 1, 1 to infinity. And what we notice is when we plug in our test values--and you notice I substituted plus and minus 2, and also, plus or minus a half. There's nothing sacred about those values, but again, trying to keep it convenient.

Substitute each of those values in, and at negative 2, we get a positive value, which means concave up since it's the second derivative. So I'm just going to write in here it's up here, down here, up here, and down here. So let's write our open intervals for the concavity direction.

It's concave up on the interval negative infinity to negative 1, joined with 0 to 1. The function is concave down on the interval -1 to 0, joined with 1 to infinity. And we have three transition points for the concavity. It goes up to down at x equals negative 1. So we'll say negative 1 f of negative 1, 0 f of 0, which we do technically already know that point, and 1 f of 1.

So now we put this all together. We can plot the three points of inflection. We can plot the intercepts. We can plot the mins and maxes. And pay particular attention to which way the graph is moving-- increasing, decreasing with concavity. When we do so, here is the graph of the function. So let's just examine everything here.

So these two points right here, the two x-intercepts on the outskirts here, we have plus and minus 1.826.

That is your square root of 10/3. So really, this is square root of 10/3 comma 0, or the rationalized form. The min and max-- now notice the x-coordinates are plus and minus 1.414, which is your approximation for the square root of 2. So that definitely lines up with what we found.

When x equals negative 1 and when x equals 1, we have our points of inflection. And that is definitely apparent on the graph. At 1, 7 you notice it's opening upward to the left and opening downward to the right, and vice versa at negative 1, negative 7. And we have our other x-intercept, 0, 0, which also happened to be a point of inflection. So there's our three inflection points.

And remember all of the details that we found. It's increasing from negative root 2 to 0 and then 0 to root 2, so on this part of the graph. So increasing and concave down has that shape that we see here. And increasing and concave up has this shape here. And then it switches to concave down for the rest of the road to the right. And to the left of negative 1, negative 7, you notice the graph is always concave up. So all of the things that we found definitely are confirmed with this graph. And there we have it.

Let's review one last example, involving a function that has asymptotes.

↪ EXAMPLE Use the techniques from this unit to sketch the graph of $f(x)=x^2+\frac{8}{x}$.

First, let's find all derivatives, rewriting $f(x)$ first:

$f(x)=x^2+\frac{8}{x}=x^2+8x^{-1}$

$f'(x)=2x-8x^{-2}$

$f''(x)=2+16x^{-3}$

1. Information from $f(x)$:
   a. Domain: $(-\infty,0)\cup(0,\infty)$
   b. Asymptotes:
      Vertical asymptote: $x=0$
      Nonlinear Asymptote: $y=x^2$ (Since $\frac{8}{x}\to 0$ as $x\to\infty$)
   c. Intercepts:
      There is no y-intercept since $x=0$ is not in the domain of $f(x)$.
      To find x-intercepts, set $x^2+\frac{8}{x}=0$ and solve:

$x^3+8=0$	Multiply both sides by *x*.
$x^3=-8$	Isolate x^3 to one side.
$x=-2$	Take the cube root of both sides.

      Thus, there is an x-intercept at (-2, 0).
2. Information from $f'(x)$:
   a. Earlier, we calculated $f'(x)=2x-8x^{-2}=2x-\frac{8}{x^2}$. $f'(x)$ is undefined when $x=0$, which is not in the domain of *f*. Therefore, 0 is not a critical number.
   To find other critical numbers, solve $2x-\frac{8}{x^2}=0$.

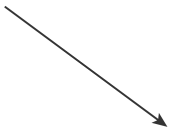
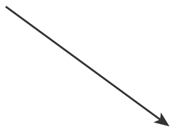
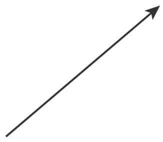
$2x^3 - 8 = 0$ Multiply both sides by x^2.

$2x^3 = 8$ Add 8 to both sides.

$x^3 = 4$ Divide both sides by 2.

$x = \sqrt[3]{4} \approx 1.59$ Take the cube root of both sides.

For parts b and c, use a sign graph for $f'(x)$. We have to consider possible changes in direction at $x = 0$ and $x = \sqrt[3]{4}$.

Interval	$(-\infty, 0)$	$(0, \sqrt[3]{4})$	$(\sqrt[3]{4}, \infty)$
Test Number	-1	1	2
Value of $f'(x) = 2x - \frac{8}{x^2}$	-10	-6	2
Behavior	Decreasing	Decreasing	Increasing
Visual			

b. Therefore, $f(x)$ is decreasing on $(-\infty, 0) \cup (0, \sqrt[3]{4})$ and increasing on $(\sqrt[3]{4}, \infty)$.

c. Remember that $f(x)$ is undefined when $x = 0$. Since $f(x)$ is defined when $x = \sqrt[3]{4}$, there is a local minimum point at $(\sqrt[3]{4}, f(\sqrt[3]{4}))$. Substituting, the local minimum point is approximately (1.59, 7.56).

3. Information from $f''(x)$:

a. Earlier, we computed $f''(x) = 2 + 16x^{-3} = 2 + \frac{16}{x^3}$. $f''(x)$ is undefined when $x = 0$, but $f(x)$ could still change concavity there.

To find possible inflection points, set $2 + \frac{16}{x^3} = 0$ and solve:

$2x^3 + 16 = 0$ Multiply both sides by x^3.

$x^3 = -8$ Subtract 16 from both sides, then divide both sides by 2.

$x = -2$ Take the cube root of both sides.

Now we make a sign graph for $f''(x)$, considering the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
Test Number	-3	-1	1
Value of $f''(x) = 2 + \frac{16}{x^3}$	$\frac{38}{27} \approx 1.41$	-14	18
Behavior	Concave up	Concave down	Concave up

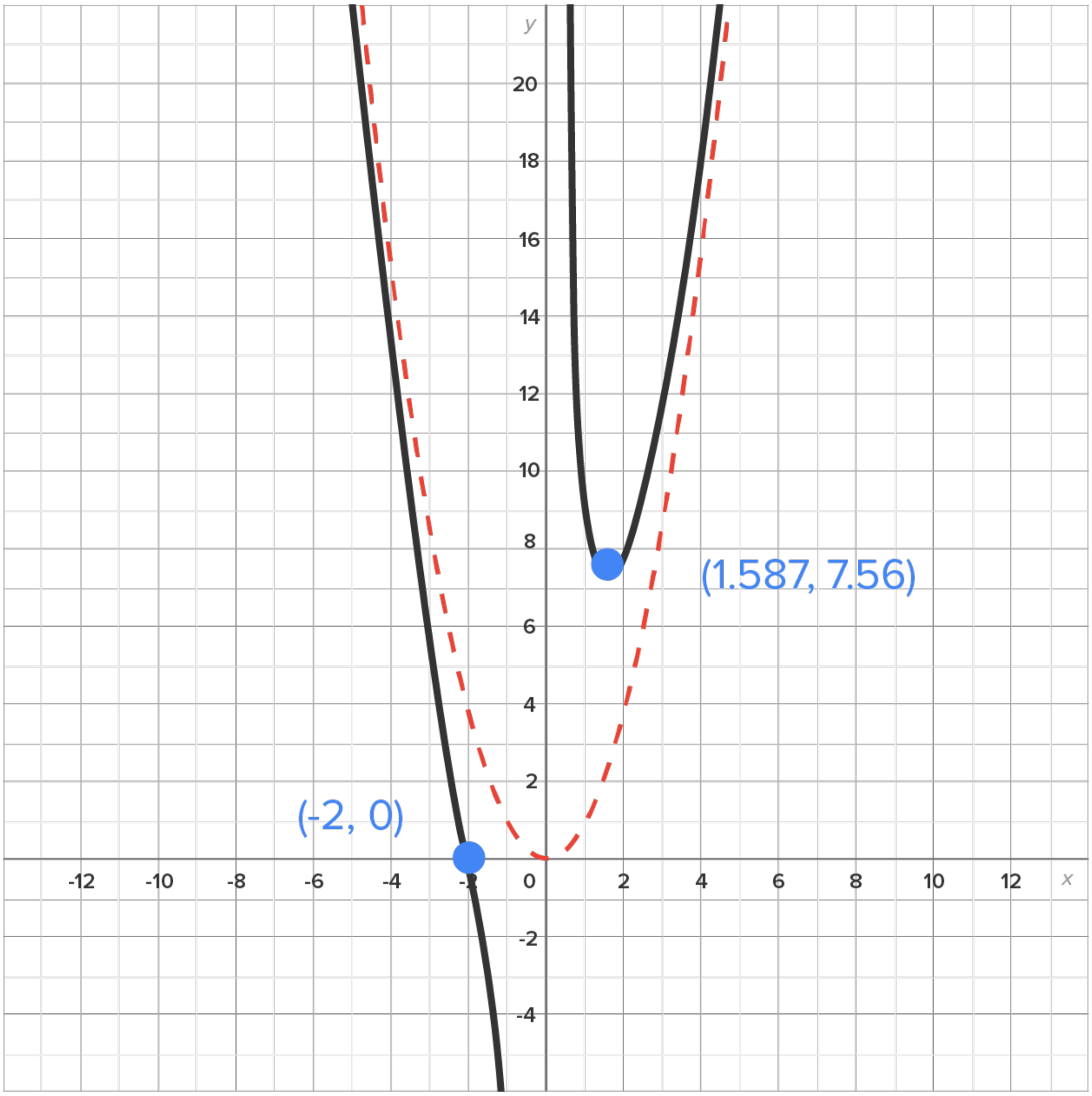
b. Thus, $f(x)$ is concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on the interval (-2, 0).

c. Since $f(-2)$ is defined, there is an inflection point at $(-2, f(-2))$, or (-2, 0), which we already found to be our x-intercept.

So, we know the following about the graph of $f(x)$:

- Vertical asymptote $x = 0$ and a nonlinear asymptote $y = x^2$
- x-intercept and inflection point at (-2, 0)
- Local minimum at (1.59, 7.56) (approximate coordinates)
- Decreasing on $(-\infty, 0) \cup (0, \sqrt[3]{4})$ and increasing on $(\sqrt[3]{4}, \infty)$
- Concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on (-2, 0)

Putting all the pieces together, here is the graph of $f(x)$, with a dashed curve to show the nonlinear asymptote.



SUMMARY

In this lesson, you learned **a general strategy** of using limits and properties of $f(x)$, $f'(x)$, and $f''(x)$

together to **graph a function** $y = f(x)$, followed by several **examples** of applying these techniques to **graph functions**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

What is L'Hopital's Rule?

by Sophia

WHAT'S COVERED

In this lesson, you will evaluate limits in special forms by using L'Hopital's rule. Specifically, this lesson will cover:

1. A Brief Review of Frequently Used Limits
2. Limits of the Form 0/0
3. Limits of the Form ∞/∞

1. A Brief Review of Frequently Used Limits

In this challenge, we will be seeing limits of exponential and logarithmic functions pretty often, so here is a chart that describes the behavior of some functions. To visualize these limits, use technology to graph the function.

Behavior of Common Functions	
$\lim_{x \to \infty} e^x = \infty$	$\lim_{x \to -\infty} e^x = 0$
$\lim_{x \to \infty} e^{-x} = 0$	$\lim_{x \to -\infty} e^{-x} = \infty$
$\lim_{x \to \infty} \ln x = \infty$	$\lim_{x \to 0^+} \ln x = -\infty$

2. Limits of the Form 0/0

Earlier in the course, we encountered limits such as $\lim_{x \to 2} \frac{x-2}{x^2-4}$ and $\lim_{x \to 3} \frac{5x-15}{7x-21}$.

Note that in each limit, the numerator and denominator are both zero when direct substitution is used. We say that these limits are in the form $"\frac{0}{0}"$. This is an example of an **indeterminate form**, since its value is not known until further analysis is done. There is no way to determine its value just by being $"\frac{0}{0}"$.

Case in point, if we evaluate these limits, we have:

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$$

$$\lim_{x \to 3} \frac{5x-15}{7x-21} = \lim_{x \to 3} \frac{5(x-3)}{7(x-3)} = \lim_{x \to 3} \frac{5}{7} = \frac{5}{7}$$

These limits both had the form $\frac{0}{0}$, but ended up having different values. This is why $\frac{0}{0}$ is called an indeterminate form.

While these limits were able to be manipulated easily using algebra, how would we go about evaluating $\lim_{x \to 0} \frac{e^{2x}-1}{x}$? This has the indeterminate form $\frac{0}{0}$, but there is no algebraic way to manipulate this expression. For limits like this, we have L'Hopital's rule.

BIG IDEA

Suppose $f(x)$ and $g(x)$ are differentiable on an open interval which contains $x = a$ and $g'(x) \neq 0$ except possibly at $x = a$. If $f(x)$ and $g(x)$ both approach 0 as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right exists.

Note: "a" can be replaced with either "$-\infty$" or "∞" to allow for limits as $x \to \pm\infty$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0} \frac{e^{2x}-1}{x}$

First check the requirements.

1. The numerator and denominator both approach 0 as $x \to 0$.
2. The numerator and denominator are both differentiable on any interval containing $x = 0$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 0} \frac{e^{2x}-1}{x}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0} \frac{2e^{2x}}{1}$	$D[e^{2x}-1] = 2e^{2x}$ and $D[x] = 1$
$= \lim_{x \to 0} \frac{2e^{2x}}{1} = \frac{2e^0}{1} = 2$	Use direct substitution.

We can conclude that $\lim_{x \to 0} \frac{e^{2x}-1}{x} = 2$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 2} \frac{x^5-32}{x^3-8}$

First, check the requirements.

1. The numerator and denominator both get closer to 0 as $x \to 2$.
2. The numerator and denominator are both differentiable on any interval containing $x = 2$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 2} \frac{x^5-32}{x^3-8}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 2} \frac{5x^4}{3x^2}$	$D[x^5-32] = 5x^4$ and $D[x^3-8] = 3x^2$

$= \lim_{x \to 2} \frac{5x^4}{3x^2} = \frac{5(2)^4}{3(2)^2} = \frac{20}{3}$ Use direct substitution.

Thus, $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8} = \frac{20}{3}$.

TRY IT

Consider the following limit: $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

Evaluate the limit. +

$\frac{3}{7}$

L'Hopital's rule can be applied more than once as long as the new limit is also in the form $\frac{0}{0}$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$

First, check the requirements. Let $f(x) = \cos(2x) - 1 + 2x^2$ and $g(x) = x^2$.

- $f(x)$ and $g(x)$ both get closer to 0 as $x \to 0$.
- $f'(x) = -2\sin(2x) + 4x$ and $g'(x) = 2x$
- $f(x)$ and $g(x)$ are both differentiable on any interval containing $x = 0$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0} \frac{-2\sin(2x) + 4x}{2x}$	$f'(x) = -2\sin(2x) + 4x$ and $g'(x) = 2x$
$= \lim_{x \to 0} \frac{-4\cos(2x) + 4}{2}$	The numerator and denominator both approach 0 as $x \to 0$, and the numerator and denominator are both differentiable. Therefore, L'Hopital's rule can be applied again: $D[-2\sin(2x) + 4x] = -4\cos(2x) + 4$ and $D[2x] = 2$
$= \lim_{x \to 0} \frac{-4\cos(2x) + 4}{2} = \frac{-4\cos(0) + 4}{2} = 0$	Direct substitution gives 0.

Conclusion: After applying L'Hopital's rule twice, $\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2} = 0$.

TERM TO KNOW

Indeterminate Form

A form of a limit, such as "$\frac{0}{0}$", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.

---

3. Limits of the Form ∞/∞

Consider the limit $\lim_{x \to \infty} \frac{2x+1}{3x+4}$, whose value is $\frac{2}{3}$, and $\lim_{x \to \infty} \frac{2x+1}{x^2+4}$, whose value is 0.

In each limit, the numerator and denominator go to ∞ as $x \to \infty$. This means that both limits are in the form $\frac{\infty}{\infty}$, which is another indeterminate form (since it leads to different values).

BIG IDEA

$\frac{\infty}{\infty}$ is another indeterminate form.

As it turns out, L'Hopital's rule can also be applied to this indeterminate form in a similar way:

BIG IDEA

Suppose $f(x)$ and $g(x)$ are differentiable on an open interval which contains $x = a$ and $g'(x) \neq 0$ except possibly at $x = a$. If $f(x)$ and $g(x)$ both tend toward ∞ or $-\infty$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right exists.

Note: "a" can be replaced with either "$-\infty$" or "∞" to allow for limits as $x \to \pm\infty$.

↪ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} \frac{2x+1}{x^2+4}$

Note that both the numerator and denominator are tending toward ∞ as $x \to \infty$. And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to \infty} \frac{2x+1}{x^2+4}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to \infty} \frac{2}{2x}$	$D[2x+1] = 2$, $D[x^2+4] = 2x$
$= \lim_{x \to \infty} \frac{1}{x} = 0$	Simplify the expression; then, the limit is 0 since $\lim_{x \to \infty} \frac{c}{x^n} = 0$.

Thus, $\lim_{x \to \infty} \frac{2x+1}{x^2+4} = 0$.

Let's look at an example that involves exponential functions.

↪ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} \frac{e^x+x^2}{x^3+8x}$

Note that both the numerator and denominator are tending toward ∞ as $x \to \infty$. And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to \infty} \frac{e^x + x^2}{x^3 + 8x}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to \infty} \frac{e^x + 2x}{3x^2 + 8}$	$D[e^x + x^2] = e^x + 2x$, $D[x^3 + 8x] = 3x^2 + 8$
$= \lim_{x \to \infty} \frac{e^x + 2}{6x}$	Numerator and denominator both tend toward ∞ as $x \to \infty$, and both are differentiable. Apply L'Hopital's rule again. $D[e^x + 2x] = e^x + 2$, $D[3x^2 + 8] = 6x$
$= \lim_{x \to \infty} \frac{e^x}{6}$	Numerator and denominator both tend toward ∞ as $x \to \infty$, and both are differentiable. Apply L'Hopital's rule again. $D[e^x + 2] = e^x$, $D[6x] = 6$
$= \lim_{x \to \infty} \frac{e^x}{6} = \infty$	L'Hopital's rule no longer applies since the denominator is constant while the numerator tends toward ∞. Since the numerator increases without bound, dividing by 6 doesn't affect this, and the limit is ∞.

Thus, $\lim_{x \to \infty} \frac{e^x + x^2}{x^3 + 8x} = \infty$.

TRY IT

Consider the following limit: $\lim_{x \to \infty} \frac{2x^3 + 8x + 5}{5x^3 + x}$

Evaluate the limit. +

$\frac{2}{5}$

WATCH

In this final example, we'll evaluate $\lim_{x \to \infty} \frac{\ln x}{x}$.

Video Transcription

[MUSIC PLAYING] Hi there and welcome back. What we're going to do is use L'Hopital's Rule as long as it applies to evaluate a limit with an indeterminate form. So in this one we're trying to find the limit as x approaches infinity of natural log of x divided by x.

So, remember, the two determinant forms we know so far are 0 over 0 and infinity over infinity, which just basically means if the numerator and denominator are both behaving the same way, we can apply a L'Hopital's Rule to try to evaluate the limit. So looking at the expression here, the natural log function does diverge off to infinity as x goes to infinity and x itself goes to infinity. So that means L'Hopital's rule can be used to evaluate this limit.

Now, remember, L'Hopital's Rule says that if this is equivalent to the limit as x approaches infinity of the derivative of the numerator over the derivative of the denominator-- and if we simplify this, we have the limit as x approaches infinity of 1 over x. And as x gets large, that's the denominator. We know that this limit is 0. So the conclusion is the limit as x approaches infinity of natural log x over x is equal to 0 by L'Hopital's rule.

[MUSIC PLAYING]

SUMMARY

In this lesson, you began with **a brief review of frequently used limits**. You also learned that if $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ (except possibly when $x = a$), L'Hopital's rule is a very convenient way to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$, that have the indeterminate **form 0/0** or the **form** ∞ / ∞.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Indeterminate Form

A form of a limit, such as "$\frac{0}{0}$", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.

Apply L'Hopital's Rule to the Indeterminate Forms "∞ - ∞" and "∞ * 0"

by Sophia

WHAT'S COVERED

In this lesson, you will learn strategies to use when evaluating limits that have other indeterminate forms. Specifically, this lesson will cover:

1. The Indeterminate Form ∞ – ∞
2. The Indeterminate Form ∞ · 0

1. The Indeterminate Form ∞ – ∞

The form $\infty - \infty$ occurs when there is a difference between two expressions that are both tending toward ∞ as $x \to a$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right)$

Since $\frac{1}{x} \to \infty$ and $\frac{1}{x^2} \to \infty$ as $x \to 0^+$, we have a limit of the form $\infty - \infty$. One strategy is to write it as a single fraction, since this is a more familiar scenario.

Since $\frac{1}{x} - \frac{1}{x^2} = \frac{x}{x^2} - \frac{1}{x^2} = \frac{x-1}{x^2}$, we have the following:

$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right)$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0^+} \left(\frac{x-1}{x^2} \right)$	Replace the expression with a single fraction.
$= \frac{\text{close to } -1}{\text{small positive number}}$	As x approaches 0 from the right, x - 1 approaches -1 and x^2 is a small positive number.
$= -\infty$	A negative number divided by a small positive number is a large negative number.

Thus, $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right) = -\infty$.

BIG IDEA

Many might think that a limit of the form $\infty - \infty$ should be 0 since you are "subtracting something from itself." As we can see, this is not the case. Once we see $\infty - \infty$ produce another value, we will see why it is an indeterminate form.

WATCH

In this video, we'll evaluate $\lim_{x \to \infty} \left(\sqrt{x^2+9x} - x\right)$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. We're going to do in this video is look at another indeterminate form, infinity minus infinity, and it's indeterminate because it does not always yield the same value. There are times when it ends up being 0. There's times when it ends up being infinity. There's times when it ends up being negative infinity. It could end up being one. There's just a lot of options here.

So the question is, how do we analytically go through this to figure out what the limit is? One of the strategies we like to use with radicals is to employ the conjugate because that helps us to rewrite the expression. I'm going to multiply this by the square root of x squared plus 9x, plus x, over the square root of x squared plus 9x, plus x. So let's see what we have here. And we'll put these in parentheses here as well. So we have the limit as x approaches infinity of-- Well, let's just see here.

So the denominator is now the square root of x squared plus 9x, plus x. Now the numerator, when these two get multiplied together the radicals just go away. So we have x squared plus 9x. The whole purpose of multiplying by the conjugate is to eliminate the outers and the inners and you notice we get x times the square root and then minus x times the square root. So they do go away. Then the last times the last is minus x squared.

What we notice is this has simplified quite considerably. So you have limit as x approaches infinity of 9x over square root, x squared plus 9x, plus x. Now you might remember before in an earlier challenge that we evaluated a limit just like this. The process is we divide by the highest power of x in the denominator. Now, what that means is I'm going to divide by square root of x squared, which remember is x. It just so happens that they're the same power in the denominator so we're going to use each one as it's relevant.

So I'm going to divide 9x by x. I'm going to divide the square root by the square root of x squared, and I'm going to divide x by x. Remember, x squared under the square root and x are the same because x is positive. We're going to positive infinity. This is the limit, x approaches infinity of 9 over-- Now that's a more complicated square root now. This is x squared plus 9 over x squared, plus 1. Now we'll simplify under the radical a little bit more. So that's equals the limit as x approaches infinity, 9 over.

Now, if I divide through by x squared, that is the square root of 1 plus 9 over x squared plus 1. And now if we take the limit as x approaches infinity, this term right here goes to 0. And now we're down to 9 over square root of 1 plus 1, which is 9/2. So maybe not an intuitive result, but that limit is 9/2. So infinity minus infinity could be 9/2. Could be 1, could be 4, it could be 0. It's indeterminate, which means we don't always know it's value. So there's our limit.

[MUSIC PLAYING]

Considering the results from these last two examples, it is clear now why $\infty - \infty$ is an indeterminate form. In one case, the result was $-\infty$, and in another case, the result was $\frac{9}{2}$.

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2. The Indeterminate Form ∞ · 0

The indeterminate form $\infty \cdot 0$ is handled in one of two ways.

Loosely speaking, we can say that a limit of the form $\frac{1}{\infty}$ will approach 0 and a limit of the form $\frac{1}{0}$ will approach $\pm\infty$.

That said, we can treat “0” and “∞” as reciprocals as far as limits are concerned.

This means that the indeterminate form $\infty \cdot 0$ could be rewritten as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, whichever is more convenient.

↪ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} x^2 e^{-2x}$

If we look at each factor separately, we see that $x^2 \to \infty$ and $e^{-2x} \to 0$ as $x \to \infty$. Thus, this limit has the form $\infty \cdot 0$.

To rewrite, consider the fact that $e^{-2x} = \frac{1}{e^{2x}}$, which means $\lim_{x \to \infty} x^2 e^{-2x} = \lim_{x \to \infty} x^2 \frac{1}{e^{2x}} = \lim_{x \to \infty} \frac{x^2}{e^{2x}}$, which now has the form $\frac{\infty}{\infty}$.

To evaluate, use L’Hopital’s rule.

$\lim_{x \to \infty} \frac{x^2}{e^{2x}}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to \infty} \frac{2x}{2e^{2x}}$	Since x^2 and e^{2x} are differentiable and the limit has the form $\frac{\infty}{\infty}$, L’Hopital’s rule is used. $D[x^2] = 2x$, $D[e^{2x}] = 2e^{2x}$
$= \lim_{x \to \infty} \frac{x}{e^{2x}}$	Remove the common factor of 2.
$= \lim_{x \to \infty} \frac{1}{2e^{2x}}$	Since $\lim_{x \to \infty} \frac{x}{e^{2x}}$ has the form $\frac{\infty}{\infty}$, continue to use L’Hopital’s rule. $D[x] = 1$, $D[e^{2x}] = 2e^{2x}$
$= 0$	Since the denominator grows very large as $x \to \infty$, the limit is 0.

Thus, $\lim_{x \to \infty} x^2 e^{-2x} = 0$.

BIG IDEA

If $\lim_{x \to a} f(x) \cdot g(x)$ has the form $\infty \cdot 0$, write $\lim_{x \to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)}$ or $\lim_{x \to a} \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$, then use L'Hopital's rule.

WATCH

In this video, we'll evaluate $\lim_{x \to 0^+} x^3 \cdot \ln x$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do in this video is evaluate a limit that has yet another indeterminate form. In this function we have x to the 1/3 times the natural log of x. As x gets close to 0, x to the third gets close to 0, and the natural log of x drives down to negative infinity. So this qualifies as a limit of the form 0 to infinity.

Remember our strategy for this kind of limit we rewrite one of them as a reciprocal so that it's a rational function, that we're able to use l'hopital's rule because 0 times infinity can either be written as 0 divided by 1 over infinity or infinity over 1 over 0. We consider 1 over infinity to be basically zero and we consider 1 over 0 to be basically infinity. So basically that just means flip one of them. So which one do we want to flip? Do we want to flip the x to the third or do we want to flip the natural log of x?

Thinking about derivatives, because l'hopital's rule involves a derivative, it's much easier to take the reciprocal of x to the third than it is to take the reciprocal of natural log of x. So we're going to write the natural log of x divided by 1 over x to the third, which remember, thinking ahead about derivatives is the limit as x approaches 0 from the right, natural log of x over x to the negative 3. So just to double check, by rewriting that what have we done? Natural log of x still drives off to negative infinity, but 1 over x to the third you get 1 over a very tiny positive number, which ends up being a large positive number. We have infinity over infinity and we're ready to use l'hopital's rule.

So to use l'hopital's rule we apply the derivative to the numerator and the denominator. So this is the limit as x approaches 0 from the right, 1 over x over negative 3x to the negative 4. It would seem that we're at a crossroads here, but remember, now we have powers of x. We can simplify this very nicely. So this is equal to the limit as x approaches 0 from the right, we have 1 over x over negative 3 over x to the fourth, which is the limit as x approaches 0 from the right. 1 over x times the reciprocal of that fraction, which turns out to be the limit as x approaches 0 from the right of x to the third over negative 3.

We can now use direct substitution for this limit. We get 0 to the third over negative 3, which is just 0. So the limit of that original function is 0. You can easily check that by looking at a graph. The graph would make it look like it's passing through 0,0. It's just getting so close to that point that it just looks like it. And there we have it.

[MUSIC PLAYING]

WATCH

In this video, we'll evaluate $\lim_{x \to \infty} x \cdot \sin\left(\frac{1}{x}\right)$.

Video Transcription

[MUSIC PLAYING] Hi there. What we're going to do is look at another limit that involves an indeterminate form. And we're going to try to rearrange this particular one and try to use L'Hospital's rule to evaluate the limit. So let's see what we come up with here.

So we have the limit as x approaches infinity of x times sin of 1/x. So it's a product of two things. So as x goes to infinity, the first term goes to infinity-- the first factor goes to infinity. The sine, however-- it's sine of 1 over [INAUDIBLE], which basically is sine of 0, which is 0. So this is the limit of the form infinity times 0.

Now, remember our strategy. The strategy is we write it as a fraction where we take the reciprocal of the factors. So it's either the reciprocal of x in the denominator, or it's the reciprocal of sin of 1/x in the denominator. I think it's going to be the reciprocal of x in the denominator. So we have the limit as x approaches infinity of sin of 1/x divided by 1/x.

And just to double check, this is going to 0. And 1/x is going to 0. So we're good to go in using L'Hospital's rule. So L'Hospital's rule says, take the derivative of the numerator. Take the derivative of the denominator. And just a little fun fact here, if we want the derivative of 1/x-- which, remember, is the derivative of x to the negative 1-- that is negative 1x to the negative 2, which is negative 1 over x squared.

That is one of those derivatives that we just happen to see often enough, it might be worth memorizing going from 1/x directly to negative 1 over x squared. So when we apply L'Hospital's rule here, the derivative of sin of 1/x is the cosine of the same of the something times the derivative of the inside.

And then the derivative of the denominator is negative 1 over x squared. And that's actually really good news, because the same thing showed up in the numerator and denominator, those essentially cross out, which means we have the limit as x approaches infinity of cosine 1/x.

But as x goes to infinity, this goes to 0. So this ends up being cosine of 0, which is 1. So that original limit is 1. And that we get by L'Hospital's rule.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that with the addition of new indeterminate forms, more strategies need to be used. Specifically, you learned that for **the indeterminate form** $\infty - \infty$, combining the fractions or rationalizing are the most common strategies; for **the indeterminate form** $\infty \cdot 0$, rewriting the expression using reciprocals then using L'Hopital's rule is the main strategy.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Limits with Variable Bases and Exponents

by Sophia

WHAT'S COVERED

In this lesson, you will learn strategies to evaluate indeterminate forms that have both variable bases and exponents. Specifically, this lesson will cover:

1. The Strategy for Evaluating Limits With Variable Bases and Exponents
2. Evaluating Limits With Variable Bases and Exponents

1. The Strategy for Evaluating Limits With Variable Bases and Exponents

Consider a function that has the form $y = f(x)^{g(x)}$. Of all the possible behaviors of $f(x)$ and $g(x)$ that could occur in a limit, there are three situations that lead to indeterminate forms.

Form	Explanation
0^0	The base and exponent both approach 0.
∞^0	The base grows without bound and at the same time, the exponent approaches 0.
1^∞	When the base approaches 1 and at the same time, the exponent increases without bound.

Since L'Hopital's rule can only be applied to limits with indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, limits with the indeterminate forms 0^0, ∞^0, or 1^∞ will need to be manipulated in order to use L'Hopital's rule.

To see how to start, consider the identity $a = e^{\ln a}$, which is valid as long as $a > 0$.

Replacing a with $f(x)^{g(x)}$, we can write $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}}$.

By the property of logarithms, we know that $\ln(f(x)^{g(x)}) = g(x) \cdot \ln f(x)$, which allows us to write $f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)}$.

This also means that $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)}$.

The limit on the right-hand side suggests that we can focus on the exponent $g(x) \cdot \ln f(x)$, which is a product, something that we have already handled using L'Hopital's rule.

BIG IDEA

If $\lim_{x \to a} g(x) \cdot \ln f(x) = L$, then the limit we seek is $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)} = e^L$.

To summarize, these steps will help to evaluate limits with indeterminate forms 0^0, ∞^0, or 1^∞.

STEP BY STEP

To evaluate a limit with an indeterminate form 0^0, 1^∞, or ∞^0:

1. Let $y = f(x)^{g(x)}$. Then, $\ln y = g(x) \cdot \ln f(x)$.
2. Find $\lim_{x \to a} \ln y$.
3. Assuming that $\lim_{x \to a} \ln y = L$, we know $\lim_{x \to a} y = e^L$, where $y = f(x)^{g(x)}$.

Let's see how this methodology is applied to specific examples.

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2. Evaluating Limits With Variable Bases and Exponents

Now that we have a strategy, let's evaluate a few limits that have one of these indeterminate forms.

→ EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+} x^x$

Note that this is a limit of the form 0^0, which will use our new strategy:

1. Take the natural logarithm of x^x: $\ln x^x = x \ln x$

2. Now find the limit:

$\lim_{x \to 0^+} x^x$	Start with the limit that needs to be evaluated.
$\lim_{x \to 0^+} x \ln x$	Evaluate the limit of the natural logarithm of the function. This has the form $0 \cdot (-\infty)$, which is another indeterminate form.
$= \lim_{x \to 0^+} \dfrac{\ln x}{\left(\frac{1}{x}\right)}$	The strategy here is to rewrite as either $\dfrac{x}{\left(\frac{1}{\ln x}\right)}$ or $\dfrac{\ln x}{\left(\frac{1}{x}\right)}$. The latter is preferable.
$= \lim_{x \to 0^+} \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$	The limit has the form $\frac{\infty}{\infty}$ and both numerator and denominator are differentiable, so L'Hopital's rule can be used. $D\left[\frac{1}{x}\right] = D[x^{-1}] = -x^{-2} = \frac{-1}{x^2}$, $D[\ln x] = \frac{1}{x}$
$= \lim_{x \to 0^+} (-x)$	Simplify $\dfrac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \frac{1}{x} \cdot \frac{x^2}{-1} = -x$.
$= 0$	Use direct substitution.

3. Then, the limit of the original function is $e^0 = 1$.

Thus, $\lim_{x \to 0^+} x^x = 1$.

WATCH

In this video, we will evaluate the limit $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$.

Video Transcription

[MUSIC PLAYING] Hi there-- good to see you again. What we're going to do in this video is take the limit of this function, which is going to end up being yet another one of our indeterminate forms. So remember, the goal is to rearrange-- rewrite-- the expression so that L'Hopital's rule can possibly be used. So let's look at the strategy for this one. So as x goes to infinity, we have a variable base and a variable exponent. So this is actually a limit of the form 1 to the infinity, which, again, doesn't tell us anything just by being 1 to the infinity. We have to analyze this a little more.

So remember that the strategy when you have a variable base and a variable exponent is, we take the natural log of 1 plus 2 over x to the x and then bring the power down. And that's the whole purpose of taking the natural log-- is to bring the power down. So then what we want is the limit as x approaches infinity of the logarithm of the expression, because, remember, we find the limit of the logarithm. We go through the paces. We find our limit. And then the original limit is e to whatever we get.

So, in this case, we have a infinity. And that is approaching-- let's see, natural log of 1 plus 0 is natural log of 1, which is 0. So we have an infinity times 0, which means, rewrite using a reciprocal. And here again, we're either going to take the reciprocal of x, which seems easier, versus the reciprocal of the natural log. I'm going to take the reciprocal of x.

So this is going to be the limit as x approaches infinity of the natural log of 1 plus 2 over x divided by 1 over x. And just to double-check, this still goes to 0. But now 1 over x also goes to 0. We are good to use L'Hopital's rule. So this is equal to the limit as x approaches infinity. Now here's where we have to be careful. Remember that the derivative of 1 over x is negative 1 over x squared. That showed up in a previous video.

So when we take the derivative of these expressions, we'll just keep that in mind so I don't have to keep writing as the negative power and doing that. You can if you want to. Derivative of natural log of something is 1 over the something times the derivative of the something. So the derivative of 1 plus 2 over x is negative 2 over x squared. And then the derivative of the denominator is negative 1 over x squared.

OK, so that actually majorly helps us here. So this is equal to the limit as x approaches infinity. Now let's just see what happens here. We can multiply by x squared over x squared. And they would just go away. We have a negative over a negative, which makes those positive. So we have 2 over 1 plus 2 over x. And as x goes to infinity, this 2 over x just goes to 0.

So now we have the limit of 2 over 1, which we know is just 2. Now, that's not the final answer. So remember the original limit. It was the limit as x approaches infinity 1 plus 2 over x to the x. Since this is the limit of the natural log, that means we have to counteract that by making it a base e. So that limit is e squared. And there we have it.

[MUSIC PLAYING]

TRY IT

Consider the following limit: $\lim_{x \to \infty} x^{3/x}$

Evaluate the limit. +

$\lim_{x \to \infty} x^{3/x} = 1$

SUMMARY

In this lesson, you learned **the strategy for evaluating limits with variable bases and exponents**. For instance, when evaluating $\lim_{x \to a} f(x)^{g(x)}$ and the limit results in one of the indeterminate forms (0^0, 1^∞, and ∞^0), the limit will need to be manipulated using logarithms in order to use L'Hopital's rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Terms to Know

Concave Down

When a graph opens downward on an interval.

Concave Up

When a graph opens upward on an interval.

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.

Critical Number

A value of c in the domain of $f(x)$ for which $f'(c) = 0$ or $f'(c)$ is undefined, provided that $f(c)$ is defined.

Extrema

Another word for extreme values.

Extreme Value Theorem

If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

Extreme Values

The minimum or maximum values of a function.

First Derivative Test

Used to identify possible local maximum and minimum points.

Global (or Absolute) Maximum

A function $f(x)$ has a global (or absolute) maximum at $x = a$ if $f(a) \geq f(x)$ for all x. In other words, $f(a)$ is the largest value of a function $f(x)$, and occurs when $x = a$.

Global (or Absolute) Minimum

A function $f(x)$ has a global (or absolute) minimum at $x = a$ if $f(a) \leq f(x)$ for all x. In other words, $f(a)$ is the smallest value of a function $f(x)$, and occurs when $x = a$.

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

Indeterminate Form

A form of a limit, such as "$\frac{0}{0}$", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.

Inflection Point (Point of Inflection)

A point on a curve at which concavity changes.

Local (or Relative) Maximum

A function $f(x)$ has a local (or relative) maximum at $x = a$ if $f(a) \geq f(x)$ for all *x* close to $x = a$. In other words, $f(a)$ is the largest value of a function $f(x)$ for values near $x = a$.

Local (or Relative) Minimum

A function $f(x)$ has a local (or relative) minimum at $x = a$ if $f(a) \leq f(x)$ for all *x* close to $x = a$. In other words, $f(a)$ is the smallest value of a function $f(x)$ for values near $x = a$.

Mean Value Theorem for Derivatives

Let $f(x)$ be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b).

Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Nonlinear Asymptote

The curve that a graph approaches as $x \to \pm\infty$.

Optimization Problem

A problem in which the maximum or minimum value is sought, whichever is relevant.

Second Derivative Test

Suppose $f'(c) = 0$, which means $f(x)$ has a horizontal tangent at $x = c$.

• If $f''(c) < 0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.

• If $f''(c) > 0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.

• If $f''(c) = 0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.

Slant (Oblique) Asymptote

The slanted line that a graph approaches as $x \to \pm\infty$.

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.