Unit 3 Tutorials: The Derivative

INSIDE UNIT 3

Derivatives and Graphs

by Sophia

WHAT'S COVERED

In this lesson, you will explore a visual way to estimate the slope of the tangent line of a function $y=f(x)$. Specifically, this lesson will cover:

1. Estimating the Slope of a Tangent Line Graphically
2. The Derivative

1. Estimating the Slope of a Tangent Line Graphically

When you see the title of this challenge, it might sound familiar. That is because we explored how to estimate the slope of a tangent line in a previous challenge! In this challenge though, we will review those skills as well as introduce new notation and definitions to help go further into calculus.

Recall the following formula for slope:

FORMULA

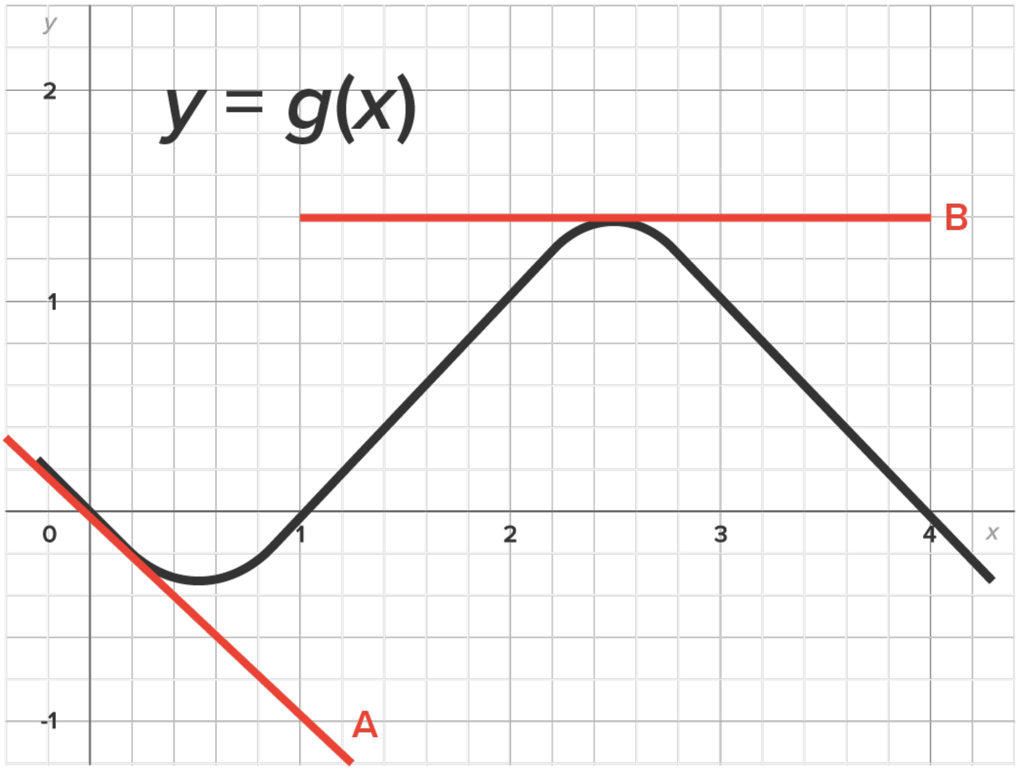
Slope of the Line Passing Through the Points (x_1, y_1) and (x_2, y_2)

$$m=\frac{y_2-y_1}{x_2-x_1}$$

HINT

The slope of the tangent line is often represented using $m_{\tan}$.

EXAMPLE Consider the function $y=g(x)$ whose graph is shown below.



The line marked A is the tangent line to the graph at $x = 0$. By the sketch, it passes through the points (0, 0) and (1, -1). Using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, the approximate slope is $m_{\tan} = \frac{-1 - 0}{1 - 0} = -1$.

The line marked B is tangent to the graph at $x = 2.5$. Since this line appears to be horizontal, $m_{\tan} = 0$.

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2. The Derivative

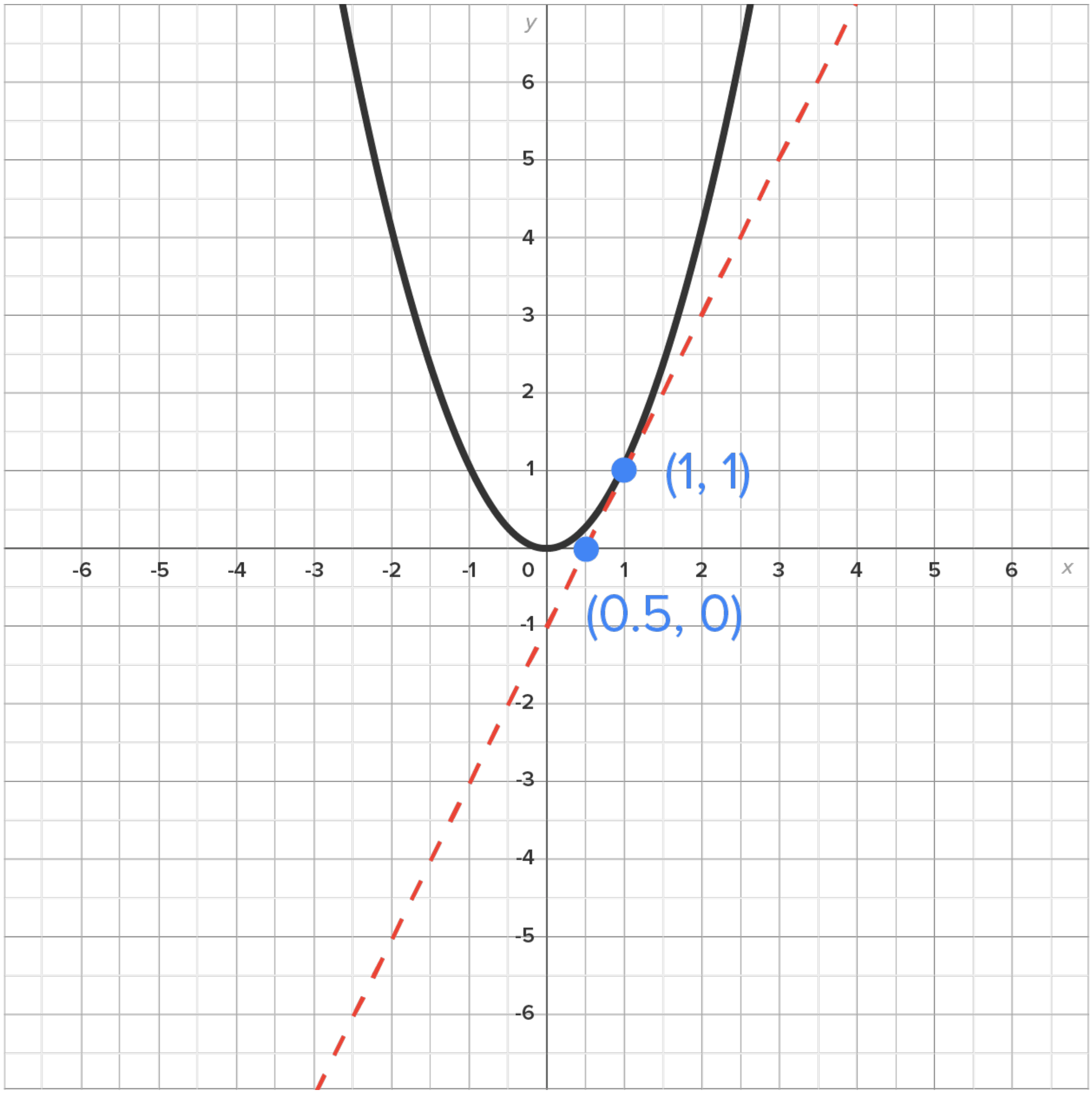
In the previous example, note how the slope of the tangent line changes as *x* changes. This tells us that the slope of the tangent line itself is a function of *x*. Instead of saying “the slope of a tangent line function,” we have a more elegant name for this important aspect of a function.

BIG IDEA

The slope of the tangent line at a point on the function is equal to the **derivative** of the function at the same point.

Now, let’s look at a more familiar function and the slopes of tangent lines at certain points.

EXAMPLE Consider the function $f(x) = x^2$. Let’s graphically estimate the derivative (slope of the tangent line) of $f(x)$ when $x = 1$.



The figure shows the graph of $f(x)=x^2$ (solid) and the tangent line (dashed) when $x=1$, which touches the graph at the point (1, 1).

Note that the tangent line when $x=1$ also passes through (0.5, 0). This means the slope of this line is 2.

So, in conclusion, the slope of the tangent line to $f(x)=x^2$ when $x=1$ is 2. Another way to say this is “the derivative of $f(x)$ when $x=1$ is 2.”

TRY IT

Using the graph of $f(x)=x^2$ from the previous example, estimate the derivative of $f(x)$ when $x=-2$.

Estimate the derivative. +

The slope should be -4. The tangent line at (-2, 4) should also pass through (-1, 0).

TERM TO KNOW

Derivative

The slope of the tangent line to the graph of a function at a point is also known as the derivative of the function at that point.

SUMMARY

In this lesson, you learned that the slope of the tangent line is also known as the **derivative**, which can be represented using $m_{\tan}$. You learned how to **estimate the slope of a tangent line** (derivative) graphically by drawing a tangent line at a given point, determining another point on the tangent line, then computing the slope using the formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Derivative

The slope of the tangent line to the graph of a function at a point is also known as the derivative of the function at that point.

FORMULAS TO KNOW

Slope of the Line Passing Through the Points (x_1, y_1) and (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Definition of Derivative

by Sophia

WHAT'S COVERED

In this lesson, you will algebraically compute the derivative function by using the limit definition of the derivative. Specifically, this lesson will cover:

1. Finding Derivatives by Limit Definition
2. The Derivative of a Constant Function $f(x)=k$
3. The Derivative of a Linear Function $f(x)=mx+b$

1. Finding Derivatives by Limit Definition

In the previous part of this challenge, we estimated the derivative of a function graphically at specific points on the graph of $y=f(x)$. We also noted that the derivative is itself a function since its value changes as *x* changes (and there is only one slope possible for a given x-value). Thus, it is important to be able to reliably compute the derivative function for a given function, which will be first done through limits and algebra.

The derivative of $f(x)$ is another function written $f'(x)$ and called "*f* prime of *x*." It represents the slope of the tangent line to $f(x)$ at a specific value of *x*.

Recall in earlier challenges that the slope of the secant line between the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by the expression $\dfrac{f(x+h)-f(x)}{h}$. This can also be expressed as $\dfrac{\Delta f}{\Delta x}$ or $\dfrac{\Delta y}{\Delta x}$ to emphasize that this is the ratio of the change in *y* to the change in *x*. Then, the slope of the tangent line is found by letting *h* get closer to 0.

FORMULA

Limit Definition of Derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The notation $f'(x)$ is used to emphasize that $f'(x)$ is related to the function $f(x)$. There are also other notations that are used in conjunction with taking the derivative $f(x)$.

Derivative Notation	Description
$f'(x)$	This is the most common notation and is called "*f* prime of *x*."
$D[f(x)]$	The "D" emphasizes that the derivative operation is being applied to $f(x)$.
$\dfrac{d}{dx}[f(x)]$	The "$\dfrac{d}{dx}$" emphasizes that the derivative operation is being applied to $f(x)$. The "*dx*" means that the derivative is taken with respect to *x*.

$\frac{df}{dx}$	This notation helps to visualize that the derivative is $\lim_{h \to 0} \frac{\Delta f}{\Delta x}$.
$\frac{dy}{dx}$	Similar to $\frac{df}{dx}$, this notation helps to visualize that the derivative is $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$.
y'	This is called "y prime," an alternate notation to $\frac{dy}{dx}$.

HINT

Any letter could be used for the independent variable, and any name can be used for a function. For example, given $y = g(t)$, we could write $\frac{dy}{dt}$, $g'(t)$, etc.

There are several notations used for the derivative, so you may see each of them used throughout the rest of this course.

Let's now use the limit definition to find the derivative of a popular function, $f(x) = x^2$.

EXAMPLE Use the limit definition of derivative to find $f'(x)$ when $f(x) = x^2$. The limit definition is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

First, compute $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$.

Then, evaluate the limit:

$f'(x) = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$	Replace $f(x+h)$ with $x^2 + 2hx + h^2$ and $f(x)$ with x^2.
$= \lim_{h \to 0} \frac{2hx + h^2}{h}$	Combine like terms in the numerator.
$= \lim_{h \to 0} \left(\frac{2hx}{h} + \frac{h^2}{h} \right)$	Separate the fractions.
$= \lim_{h \to 0} (2x + h)$	Remove the common factor of h in each fraction.
$= 2x$	Substitute 0 for h.

Thus, if $f(x) = x^2$, then $f'(x) = 2x$. This means that the slope of the tangent line to the curve $f(x) = x^2$ is $m_{\tan} = 2x$.

Let's check this with the results we got in the previous challenge. By using the graph, we estimated the slope of the tangent line at $x = 1$ to be 2.

Since $f'(x) = 2x$, evaluate this function when $x = 1$ to get the slope. Since $f'(1) = 2(1) = 2$, this tells us that the slope of the tangent line is 2.

From the "Try It" in tutorial 3.1.1, you (hopefully) estimated the slope to be 2 when $x = 1$. Substituting into the derivative, we have $f'(1)=2(1)=2$, which checks.

↪ EXAMPLE Find the derivative of $f(x)=2x^3$ by using the limit definition. As we know, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

First, note that $f(x+h)=2(x+h)^3=2(x^3+3hx^2+3h^2x+h^3)=2x^3+6hx^2+6h^2x+2h^3$.

Now, let's replace the function notations with expressions and evaluate the limit:

$f'(x)=\lim_{h\to 0}\frac{2x^3+6hx^2+6h^2x+2h^3-2x^3}{h}$	Replace $f(x+h)$ with $2x^3+6hx^2+6h^2x+2h^3$ and $f(x)$ with $2x^3$.
$=\lim_{h\to 0}\frac{6hx^2+6h^2x+2h^3}{h}$	Combine like terms in the numerator.
$=\lim_{h\to 0}\left(\frac{6hx^2}{h}+\frac{6h^2x}{h}+\frac{2h^3}{h}\right)$	Separate the fractions.
$=\lim_{h\to 0}(6x^2+6hx+2h^2)$	Remove the common factor of h in each fraction.
$=6x^2$	Substitute 0 for h.

Thus, if $f(x)=2x^3$, then $f'(x)=6x^2$. This means that the slope of the tangent line to the curve $f(x)=2x^3$ is $m_{\tan}=6x^2$.

Having the function for the derivative is an advantage since estimating the slope at a specific point can be difficult. In the previous example, we found that $f'(x)=6x^2$ when $f(x)=2x^3$. Let's say we want the slope of the tangent line to $f(x)$ when $x=2$. According to the derivative function, $f'(2)=6(2)^2=24$.

A slope of 24 might be difficult to obtain visually without a very carefully drawn graph. This is why having a function is important.

WATCH

The following video illustrates how to use the limit definition to find the derivative of $f(x)=\frac{4}{x+3}$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome back. What we're going to do in this video is find the derivative of f of x equals 4 over x plus 3 by using the limit definition of the derivative, which is this expression right here. So as usual, what we do is we substitute 4 over x plus 3 into the limit expression. And then our goal is to simplify, meaning writing an equivalent expression so that h is no longer in the denominator, because we want to be able to substitute 0 for h. OK?

So let's get started with this. So, according to our formula here, f prime of x is equal to the limit as h approaches 0 of 4 over-- now we're replacing x with x plus h, so that means we have x plus h plus 3. And notice I'm not using the grouping symbols, because if I had them here, that expression is really not changed, right? There's not a number in front of the parentheses. There's not a subtraction involved.

So the expression is equivalent whether we have the grouping symbols there or not. So it's just simpler to write it without the grouping symbols. And then we subtract off the original function, which is 4 over x plus 3, and then we divide the entire thing by h. And what we have now is a complex fraction, which means we have to multiply it by something that's going to wipe out the denominators that are in the numerator there.

So, the LCD between these two denominators is simply the product of the two. So it's x plus h plus 3 times x plus 3 upstairs. And remember, we have to also multiply in the denominator, because we want to make sure that we're multiplying by 1 at all times. Because we don't want to alter the expression, or alter its value, rather. We change the look but not the actual value. So this is now equal to the limit as h approaches 0 of, well, let's see.

In the denominator, we're just going to have h times x plus h plus 3 times x plus 3. Now, in the numerator, we're going to take that entire expression x plus h plus 3 times x plus 3 and multiply it to each fraction. When I multiply it to the first fraction, the x plus h plus 3 crosses out. So, we have 4 times x plus 3 minus-- and when we multiply it to the second fraction, the x plus 3's cross out. So we're left with 4 times x plus h plus 3.

So, we'll say we're getting somewhere. We do have some more simplification to do. This is equal to the limit as h approaches 0. Now, let's look at the numerator. I'm going to distribute above this in green just to save some space here. So we have 4x plus 12 minus 4x minus 4h minus 12.

And what we notice is that some terms cancel out. The 4x's cancel out, the twelfths cancel out, and we're left with negative 4h divided by h times x plus h plus 3 times x plus 3. And now we notice the single H's can cancel out now, because we have a factor of h in the numerator and a factor of h in the denominator.

So I'm going to go ahead and cancel those out, and now we have the limit as h approaches 0 of negative 4 over x plus h plus 3 times x plus 3. And now we're ready to substitute 0 for h. Now, x plus 0 plus 3 is just the same as x plus 3, so we have negative 4 over x plus 3 times x plus 3. We can write that more succinctly, or in more condensed form. x plus 3 the quantity squared. So the derivative is negative 4 over x plus 3 squared.

[MUSIC PLAYING]

Now, let's establish some general rules for derivatives of functions that have certain forms.

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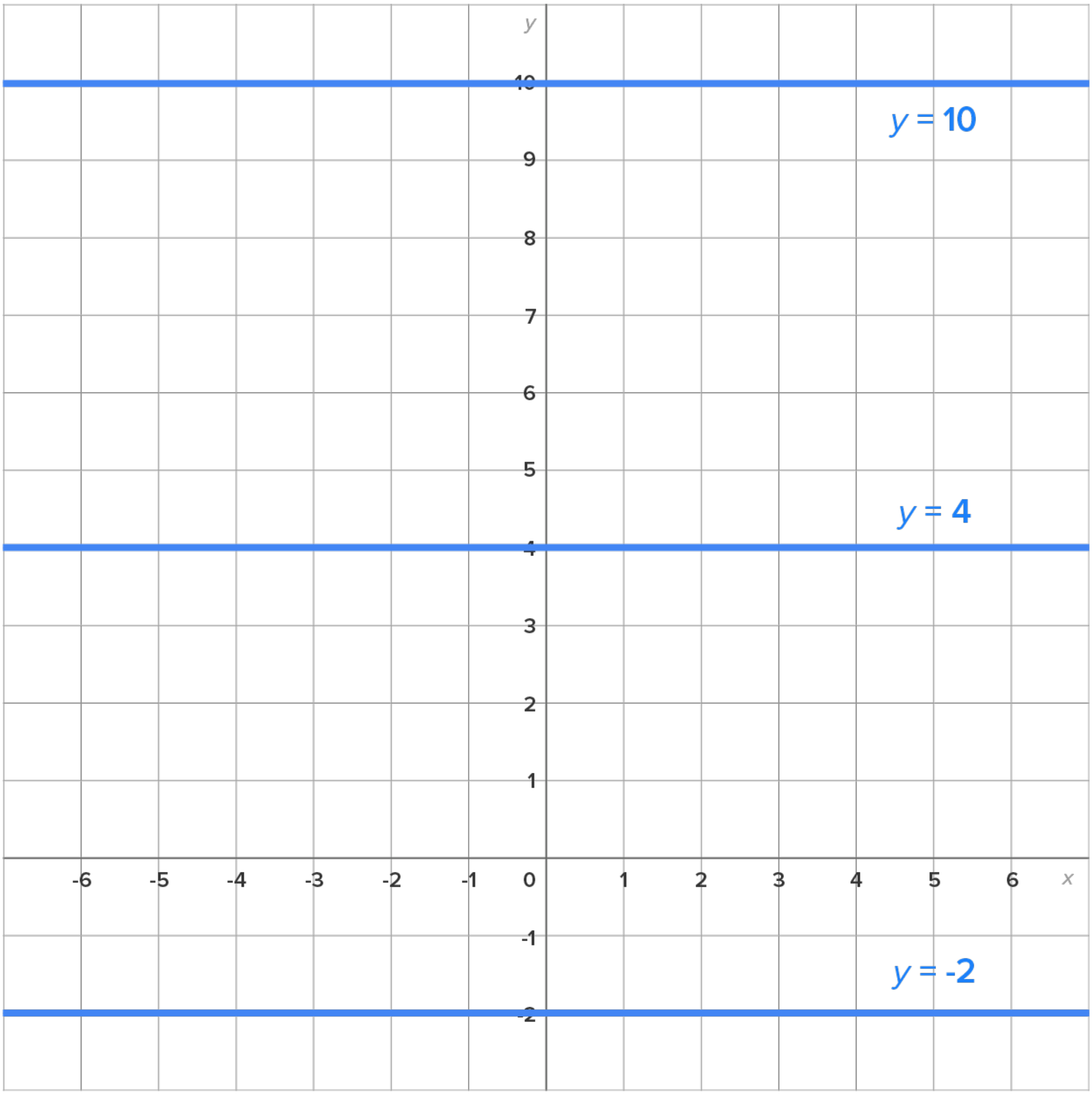
2. The Derivative of a Constant Function *f*(*x*) = *k*

If you graph the function $f(x)=k$ for any constant *k*, you would notice they all have something in common: their slopes are 0. Shown below are the graphs of $y = -2$, $y = 4$, and $y = 10$.



When speaking of the derivative of each function, it stands to reason that the derivative of each function is 0. Let's show this using the limit definition:

Let $f(x) = k$. Now, evaluate the limit:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	Apply the limit definition of a derivative.
$= \lim_{h \to 0} \frac{k-k}{h}$	Since $f(x) = k$, a constant, it follows that $f(x+h) = k$. Replace both $f(x)$ and $f(x+h)$ with k.
$= \lim_{h \to 0} \frac{0}{h}$	Simplify the numerator.
$= \lim_{h \to 0} 0$	The limit implies that $h \neq 0$, therefore $\frac{0}{h} = 0$.
$= 0$	The limit of a constant is the constant.

BIG IDEA

Thus, when $f(x)=k$, $f'(x)=0$. We often say "the derivative of a constant is 0."

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3. The Derivative of a Linear Function $f(x) = mx + b$

Since the slope of a linear function $f(x)=mx+b$ is m, it follows that the derivative of a linear function should also be m (since this is also the slope of a tangent line).

Let's use the limit definition of derivative to establish this:

Let $f(x)=mx+b$. Then, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

$=\lim\limits_{h\to 0}\dfrac{mx+mh+b-(mx+b)}{h}$	$f(x+h)=m(x+h)+b=mx+mh+b$ Also replace $f(x)$ with $mx+b$.
$=\lim\limits_{h\to 0}\dfrac{mh}{h}$	Simplify the numerator.
$=\lim\limits_{h\to 0} m$	Remove the common factor of h.
$=m$	The limit of a constant is the constant.

Thus, when $f(x)=mx+b$ (a linear function), its derivative is $f'(x)=m$.

TRY IT

Find each derivative function:

$D[9]$ +

0

$D[2x+9]$ +

2

$D[17 - 4.2x]$ +

-4.2

SUMMARY

In this lesson, you learned that the limit definition of the derivative produces the derivative function analytically (as opposed to estimating the derivative at specific points graphically). You used this

knowledge to **find derivatives by the limit definition**. You learned that the derivative function can be used to easily calculate the slopes of tangent lines for given values of x. You also learned special derivative rules that can be used to find **the derivative of a constant function** $f(x)=k$ and **the derivative of a linear function** $f(x)=mx+b$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Limit Definition of Derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Basic Derivative Rules

by Sophia

WHAT'S COVERED

In this lesson, you will learn more derivative rules for specific types of functions. Specifically, this lesson will cover:

1. Derivatives of x^n
2. Derivatives of $\sin x$ and $\cos x$
3. Derivatives of Absolute Value Functions
4. Finding the Slope of a Tangent Line

1. Derivatives of x^n

So far, here is what we know about the derivative of $f(x)=x^n$:

Value of n	$f(x)$	$f'(x)$
$n=1$	$f(x)=x$	$f'(x)=1$ (Derivative of linear function)
$n=2$	$f(x)=x^2$	$f'(x)=2x$ (Derived in last challenge)

Now, let's look at other values of n.

If $n=3$, then $f(x)=x^3$.

Also, $f(x+h)=(x+h)^3=x^3+3hx^2+3h^2x+h^3$.

Then, evaluate the limit:

$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ Apply the limit definition of a derivative.

$=\lim\limits_{h\to 0}\dfrac{x^3+3hx^2+3h^2x+h^3-x^3}{h}$ Replace $f(x+h)$ and $f(x)$ with their expressions.

$=\lim\limits_{h\to 0}\dfrac{3hx^2+3h^2x+h^3}{h}$ Simplify the numerator.

$=\lim\limits_{h\to 0}\left(\dfrac{3hx^2}{h}+\dfrac{3h^2x}{h}+\dfrac{h^3}{h}\right)$ Divide each term by h.

Remove the common factor of h in each fraction.

$$= \lim_{h \to 0} (3x^2 + 3hx + h^2) = 3x^2 \quad \text{Substitute 0 for } h.$$

Thus, when $f(x) = x^3$, its derivative is $f'(x) = 3x^2$.

Let's look at one more power:

If $n = 4$, then $f(x) = x^4$.

Also, $f(x+h) = (x+h)^4 = x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4$.

Then, evaluate the limit:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	Apply the limit definition of a derivative.
$= \lim_{h \to 0} \frac{x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4 - x^4}{h}$	Replace $f(x+h)$ and $f(x)$ with their expressions.
$= \lim_{h \to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h}$	Simplify the numerator.
$= \lim_{h \to 0} \left(\frac{4hx^3}{h} + \frac{6h^2x^2}{h} + \frac{4h^3x}{h} + \frac{h^4}{h} \right)$	Divide each term by h.
$= \lim_{h \to 0} (4x^3 + 6hx^2 + 4h^2x + h^3)$	Remove the common factor of h in each fraction.
$= 4x^3$	Substitute 0 for h.

Thus, when $f(x) = x^4$, its derivative is $f'(x) = 4x^3$.

Now, let's put the derivatives we've seen together:

Value of n	$f(x)$	$f'(x)$
$n = 1$	$f(x) = x$	$f'(x) = 1$
$n = 2$	$f(x) = x^2$	$f'(x) = 2x$
$n = 3$	$f(x) = x^3$	$f'(x) = 3x^2$
$n = 4$	$f(x) = x^4$	$f'(x) = 4x^3$

In these functions, it appears that the original exponent becomes the coefficient, while the new exponent is 1 less than the original exponent.

FORMULA

Power Rule

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

This is also written in other ways:

- $D[x^n] = n \cdot x^{n-1}$

- If $y = x^n$, then $y' = n \cdot x^{n-1}$ or $\frac{dy}{dx} = n \cdot x^{n-1}$

HINT

Recall the other functions that can be written with exponents:

- Radical functions: $f(x) = \sqrt[n]{x} = x^{1/n}$
- Reciprocal functions: $f(x) = \frac{1}{x^n} = x^{-n}$

EXAMPLE Find the derivative of $f(x) = x^7$.

Apply the power rule: $f'(x) = 7x^{7-1} = 7x^6$

EXAMPLE Find the derivative of $g(x) = \frac{1}{x^3}$.

First, rewrite as $g(x) = x^{-3}$.

Now apply the power rule: $g'(x) = -3x^{-3-1} = -3x^{-4}$

Since there is a negative exponent in the answer, this is not considered to be in simplest form. Using properties of exponents, write $g'(x) = \frac{-3}{x^4}$.

EXAMPLE Find the derivative of $h(x) = \sqrt{x}$.

First, rewrite as $h(x) = x^{1/2}$.

Now apply the power rule: $h'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$

Since there is a negative exponent in the answer, this is not considered to be in simplest form. Using properties of exponents, write $h'(x) = \frac{1}{2x^{1/2}}$. This could also be written as $h'(x) = \frac{1}{2\sqrt{x}}$.

TRY IT

Consider the functions $f(x) = x^{14}$, $g(x) = \frac{1}{x}$, and $h(x) = \frac{1}{\sqrt[3]{x}}$.

Find the derivative of *f*. +

$f'(x) = 14x^{13}$

Find the derivative of *g*. +

$g'(x) = -\frac{1}{x^2}$

Find the derivative of *h*. +

$$h'(x) = -\frac{1}{3x^{4/3}}$$

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2. Derivatives of sin *x* and cos *x*

In order to use the limit definition, let's keep the following identities and limits in mind:

- $\sin(x+h) = \sin x \cos h + \sin h \cos x$
- $\cos(x+h) = \cos x \cos h - \sin x \sin h$
- $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$
- $\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$

The limit definition of $f(x) = \sin x$ is worked out on page 4 of the textbook, linked at the beginning of this challenge. The result is $\frac{d}{dx}[\sin x] = \cos x$.

For the derivative of $f(x) = \cos x$, we set up the limit definition as usual:

$f'(x) = \lim\limits_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$	Apply the limit definition of a derivative.
$= \lim\limits_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$	Replace $\cos(x+h)$ with $\cos x \cos h - \sin x \sin h$.
$= \lim\limits_{h \to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$	Group cos *x* and sin *x* terms.
$= \lim\limits_{h \to 0} \frac{\cos x(\cos h - 1) - (\sin x)\sin h}{h}$	Factor cos *x*.
$= \lim\limits_{h \to 0} \left(\frac{\cos x(\cos h - 1)}{h} - \frac{(\sin x)\sin h}{h} \right)$	Write each part over *h*.
$= \lim\limits_{h \to 0} \left(\cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right) \right)$	Group "*h*" terms together.
$= \cos x(0) - \sin x(1)$	$\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$, $\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$
$= 0 - \sin x$	Simplify.
$= -\sin x$	Simplify.

Thus, if $f(x) = \cos x$, $f'(x) = -\sin x$.

FORMULA

Derivative of Sine

$$\frac{d}{dx}[\sin x] = \cos x$$

Derivative of Cosine

$$\frac{d}{dx}[\cos x] = -\sin x$$

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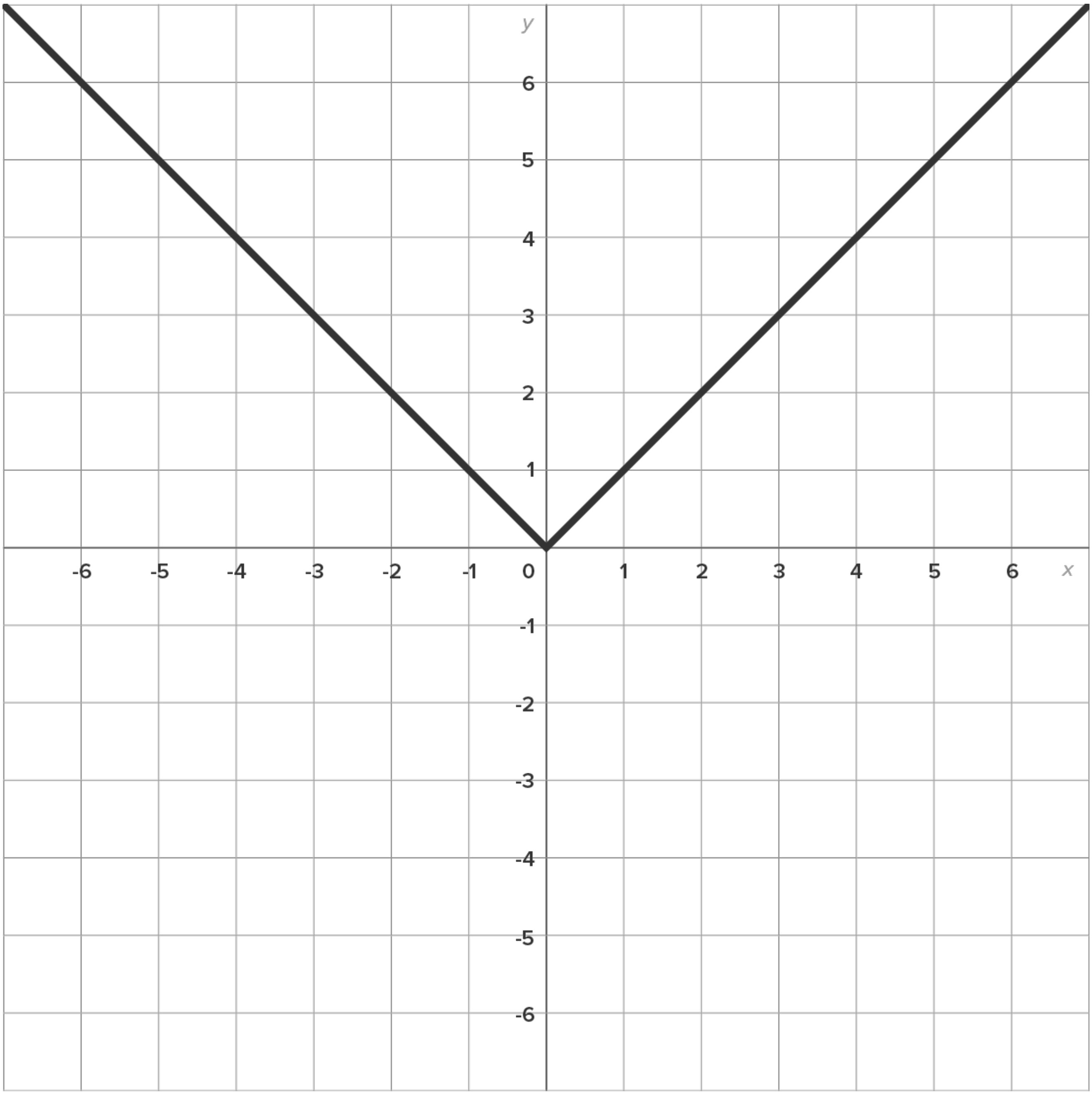
3. Derivatives of Absolute Value Functions

Recall the piecewise definition of $|x|$ and its graph:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$



- When $x < 0$, the graph is the line $y = -x$, which has slope -1.

- When $x > 0$, the graph is the line $y = x$, which has slope 1.
- When $x = 0$, the slope changes abruptly from -1 to 1, suggesting that there is no derivative when $x = 0$.

We can investigate this more closely using the limit definition of derivative for $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Now, let's examine the expression $\frac{|h|}{h}$ when $h < 0$ and when $h > 0$.

- When $h < 0$, $|h| = -h$, therefore $\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$.
- When $h > 0$, $|h| = h$, therefore $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$.

Thus, $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Since this limit is $f'(0)$, we also say that $f'(0)$ does not exist.

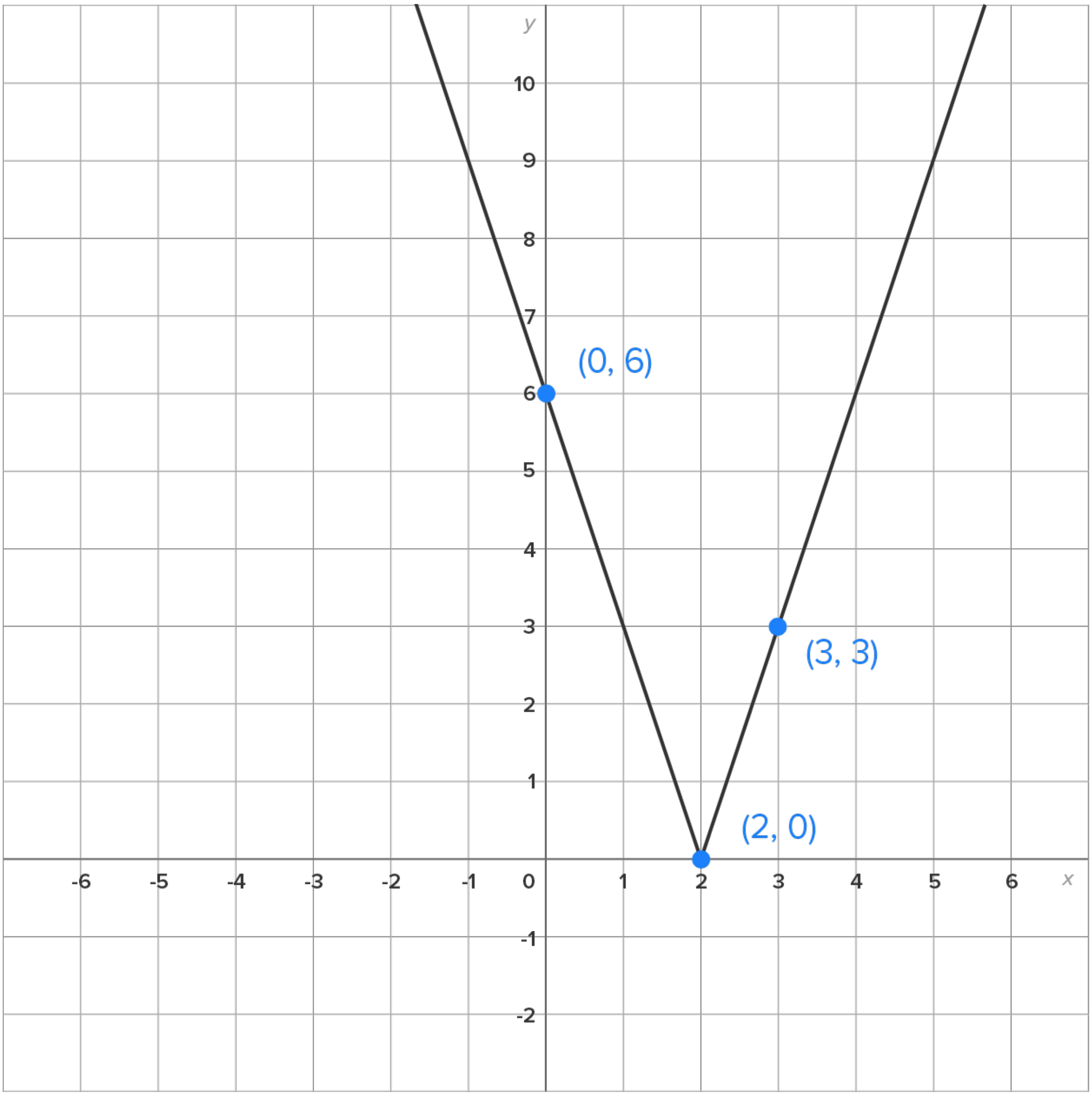
This means that the derivative of $f(x) = |x|$ is as follows:

$$D[|x|] \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

This idea can be applied to any absolute value function. We tend to analyze absolute value functions graphically rather than by using formulas.

↱ EXAMPLE Find the derivative of $f(x) = 3|x-2|$ graphically.

The graph of $f(x)$ is shown below.



When $x < 2$, the slope of the graph is -3.
When $x > 2$, the slope of the graph is 3.

When $x = 2$, the graph has a corner point and therefore the derivative is undefined there.

Therefore,

$$f'(x) = \begin{cases} -3 & \text{if } x < 2 \\ 3 & \text{if } x > 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$$

4. Finding the Slope of a Tangent Line

Now that we have some "shortcut" rules for finding derivatives, finding the slope of a tangent line is now a much easier process.

↪ EXAMPLE Find the slope of the tangent line to the graph of $f(x) = \frac{1}{x}$ when $x = 3$ and $x = 6$.

First, we need to find $f'(x)$. To do so, we need to rewrite $f(x) = \frac{1}{x} = x^{-1}$.

Now apply the power rule: $f'(x) = -1x^{-2} = \frac{-1}{x^2}$

The slope of the tangent line when $x = 3$ is $f'(3) = \frac{-1}{3^2} = -\frac{1}{9}$.

The slope of the tangent line when $x = 6$ is $f'(6) = \frac{-1}{6^2} = -\frac{1}{36}$.

SUMMARY

In this lesson, you learned that the limit definition of derivative is useful in establishing "shortcut" rules for finding **derivatives of** x^n**, sin*x*, cos*x*, and absolute value functions**. Using these rules enables us to solve problems involving derivatives and rates of change much more quickly and succinctly, such as **finding the slope of a tangent line**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of Cosine

$$\frac{d}{dx}[\cos x] = -\sin x$$

Derivative of Sine

$$\frac{d}{dx}[\sin x] = \cos x$$

Power Rule

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

Equations of Tangent Lines

by Sophia

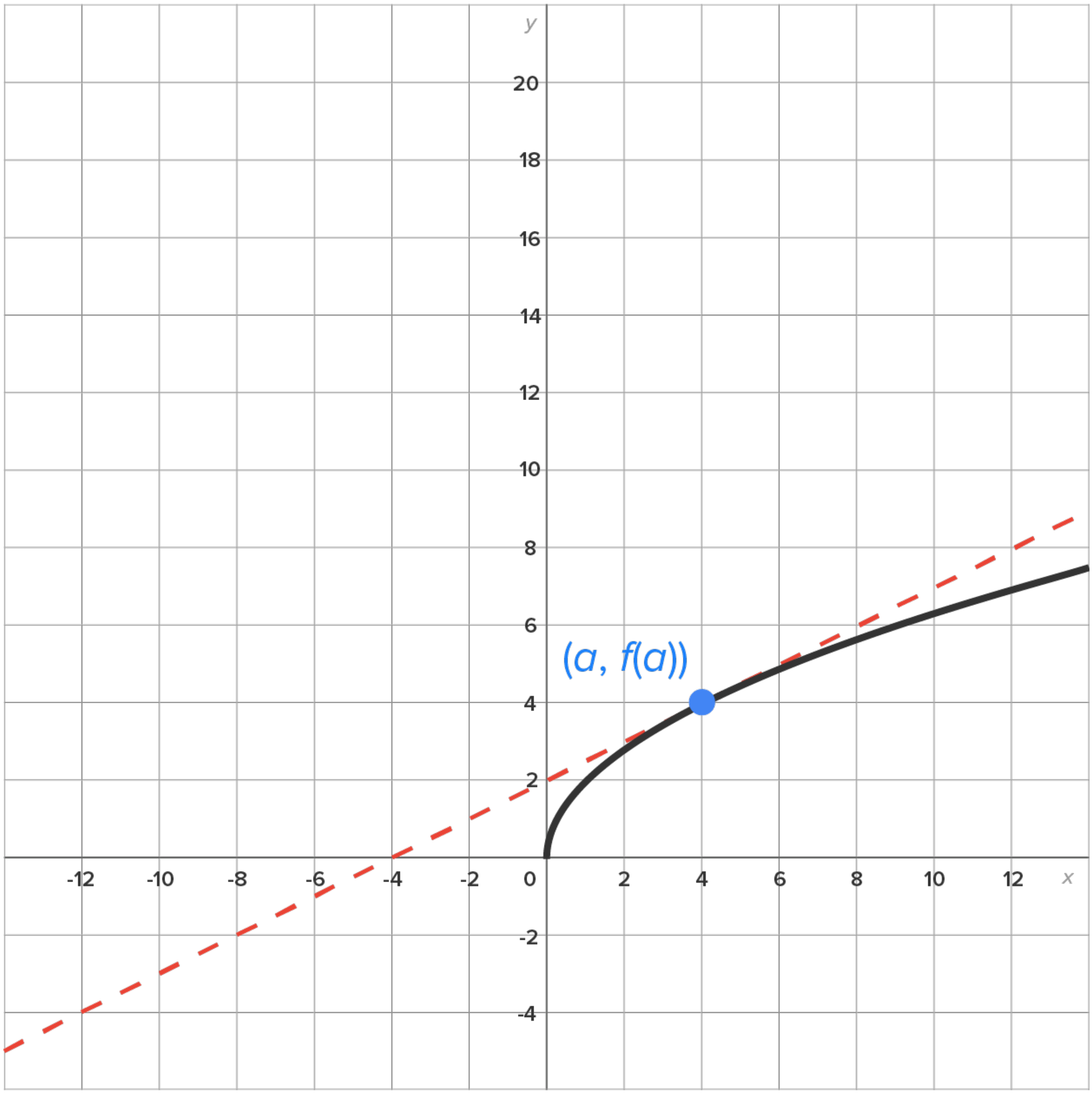
WHAT'S COVERED

In this lesson, you will use derivative rules to write the equation of a tangent line to a function $f(x)$. Specifically, this lesson will cover:

1. Writing the Equation of a Tangent Line at a Specific Point
2. Different Types of Functions
   a. Power Functions $(y=x^n)$
   b. $y=\sin x$ and $y=\cos x$

1. Writing the Equation of a Tangent Line at a Specific Point

Shown here is the graph of some function $y=f(x)$ and its tangent line at $(a, f(a))$.



Recall from Unit 1 that writing the equation of a line requires two things:

- The slope of the line
- A point on the line

Given a function $y = f(x)$, this information is known at $x = a$:

- The slope of the line is $f'(a)$.
- A point on the line is $(a, f(a))$.

For now, let's assume that $f'(a)$ is defined, meaning that the tangent line is nonvertical.
Now, use the point-slope form to write the equation of the tangent line:

$y - y_1 = m(x - x_1)$	Use the point-slope form.
$y - f(a) = f'(a)(x - a)$	$(x_1, y_1) = (a, f(a)),\ m = f'(a)$

$y=f(a)+f'(a)(x-a)$ Add $f(a)$ to both sides to solve for *y*.

FORMULA

Equation of a Tangent Line to $y=f(x)$ at $x=a$

$y=f(a)+f'(a)(x-a)$

---

2. Different Types of Functions

Now, let's focus on the mechanics required to write tangent lines for different types of functions.

2a. Power Functions ($y = x^n$)

EXAMPLE Write the equation of the line tangent to $f(x)=x^3$ when $x=2$.

First, the line is tangent to the graph at the point $(2, f(2))$, or (2, 8). The derivative is $f'(x)=3x^2$. Then, the slope of the tangent line is $f'(2)=3(2)^2=12$.

Now, use the tangent line formula:

$y=f(a)+f'(a)(x-a)$	Use the equation of a tangent line.
$y=f(2)+f'(2)(x-2)$	$a=2$
$y=8+12(x-2)$	$f(2)=8$ and $f'(2)=12$
$y=8+12x-24$	Distribute.
$y=12x-16$	Combine like terms.

In conclusion, the equation of the tangent line is $y=12x-16$.

EXAMPLE Write the equation of the line tangent to $f(x)=\frac{1}{x^2}$ when $x=1$. The line is tangent to the graph at the point $(1, f(1))$, or (1, 1).

First, rewrite $f(x)=\frac{1}{x^2}$ with a single exponent: $f(x)=x^{-2}$. By the power rule, $f'(x)=-2x^{-3}=\frac{-2}{x^3}$. Then, the slope of the tangent line is $f'(1)=\frac{-2}{(1)^3}=-2$.

Now, use the tangent line formula:

$y=f(a)+f'(a)(x-a)$	Use the equation of a tangent line.
$y=f(1)+f'(1)(x-1)$	$a=1$
$y=1-2(x-1)$	$f(1)=1$ and $f'(1)=-2$
$y=1-2x+2$	Distribute.

$y = -2x + 3$ Combine like terms.

In conclusion, the equation of the tangent line is $y = -2x + 3$.

TRY IT

Consider the function $f(x) = x^{3/2}$

Write the equation of the line tangent to the graph of this function at $x = 4$. +

$y = 3x - 4$

2b. $y = \sin x$ and $y = \cos x$

Let's look at an example involving a trigonometric function.

EXAMPLE Write the equation of the line tangent to the graph of $f(x) = \cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

First, recall that $f'(x) = -\sin x$. Then, the slope of the tangent line is $f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$.

Now, use the tangent line formula:

$y = f(a) + f'(a)(x - a)$	Use the equation of a tangent line.
$y = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$	$a = \frac{\pi}{2}$
$y = 0 + (-1)\left(x - \frac{\pi}{2}\right)$	$f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$
$y = -x + \frac{\pi}{2}$	Distribute and simplify.

Thus, the equation of the tangent line is $y = -x + \frac{\pi}{2}$.

SUMMARY

In this lesson, you learned how to **write the equation of the tangent line at a specific point**, noting that this equation can be found for a function $f(x)$ at $x = a$ as long as $f'(a)$ is defined. You also learned how to write tangent lines for **different types of functions**, such as **power functions ($y = x^n$)** and trigonometric functions (**$y = \sin x$** and **$y = \cos x$**). This is a gateway for a wider variety of applications that will be discussed later in this chapter once we learn how to find derivatives of more functions.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Equation of a Tangent Line to y = f(x) at x = a

$$y = f(a) + f'(a)(x - a)$$

Differentiability

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the differentiability of a function by using analytical techniques, which include a determination of continuity. Specifically, this lesson will cover:

1. Defining Differentiability
2. Determining Differentiability at $x = a$ Analytically
   a. Continuous but Not Differentiable
   b. Differentiable for All Real Numbers
   c. Not Continuous

1. Defining Differentiability

Differentiability is an important concept in calculus since it pertains to the "smoothness" of a curve. A function $y = f(x)$ is said to be **differentiable** at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

TERM TO KNOW

Differentiable
A function $y = f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

---

2. Determining Differentiability at *x* = *a* Analytically

The following statements are equivalent:

- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
- If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

How to interpret these statements:

- If $f(x)$ is not continuous at $x = a$, then it is never differentiable at $x = a$.
- If $f(x)$ is differentiable at $x = a$, then it is always continuous at $x = a$.

Note: This means that if $f(x)$ is continuous at $x = a$, $f(x)$ may or may not be differentiable at $x = a$.

HINT

Recall that the definition of continuity of $f(x)$ at $x = a$ is $\lim_{x \to a} f(x) = f(a)$.

2a. Continuous but Not Differentiable

Here is an example of a function that is continuous but not differentiable at a point.

EXAMPLE Determine if $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$. First, check for continuity at $x = 0$:

$$f(0) = \sqrt[3]{0} = 0$$

$$\lim_{x \to 0} \sqrt[3]{x} = 0$$

Since $\sqrt[3]{x}$ is defined for positive and negative real numbers, there is no need to use one-sided limits. Since the limit and $f(0)$ are equal, $f(x)$ is continuous at $x = 0$.

Now, let's check the derivative. Note that $f(x) = \sqrt[3]{x} = x^{1/3}$. Then, $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$.

Since $f'(0)$ is undefined (0 in the denominator), $f(x)$ is not differentiable at $x = 0$.

Note: In this case, this means that the slope of the tangent line is undefined, and that the tangent line is vertical.

2b. Differentiable for All Real Numbers

Here is an example of a function that is differentiable for all real numbers.

EXAMPLE Show that $f(x) = x^3$ is differentiable for all real numbers.

Check continuity: Since $f(x)$ is a polynomial function, it is continuous for all real numbers (this was established in Challenge 2.3).

Check the derivative: $f'(x) = 3x^2$, which is defined for all real numbers.

Thus, $f(x) = x^3$ is differentiable for all real numbers.

2c. Not Continuous

Here is an example of a function that is not continuous at a point, which means that it is also not differentiable at the point.

EXAMPLE Consider the function $f(x) = \frac{2}{x-1}$.

Since $f(x)$ is not continuous at $x = 1$, it is also not differentiable at $x = 1$.

TRY IT

Consider the following functions and x-values.

Function	Given x-value	Differentiable (Yes or No)?

$f(x) = \cos x$	$x = 0$	?
$g(x) = \sqrt{x}$	$x = 4$	?
$h(x) = \dfrac{x}{2x-1}$	$x = \dfrac{1}{2}$	?

Determine if each function is differentiable at the given x-value in the table. +

Function	Given x-value	Differentiable (Yes or No)?
$f(x) = \cos x$	$x = 0$	Yes
$g(x) = \sqrt{x}$	$x = 4$	Yes
$h(x) = \dfrac{x}{2x-1}$	$x = \dfrac{1}{2}$	No (not continuous, therefore not differentiable at $x = \frac{1}{2}$)

SUMMARY

In this lesson, you explored the first of two ways to **define differentiability**, noting that if a function is to be differentiable at $x = a$, it must be continuous at $x = a$ and $f'(a)$ needs to be defined. You learned how to **determine differentiability at $x = a$ analytically**, exploring examples of functions that are **continuous but not differentiable**, **differentiable for all real numbers**, and also **not continuous** and therefore not differentiable at the points of discontinuity.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Differentiable

A function $y = f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

Determining Differentiability Graphically

by Sophia

WHAT'S COVERED

In this lesson, you will look at what causes a function to not be differentiable and use graphical reasoning to determine differentiability, which is more straightforward than the analytical approach. Specifically, this lesson will cover:

1. Discontinuities
2. Vertical Tangents
3. Sharp Corners
4. Cusps

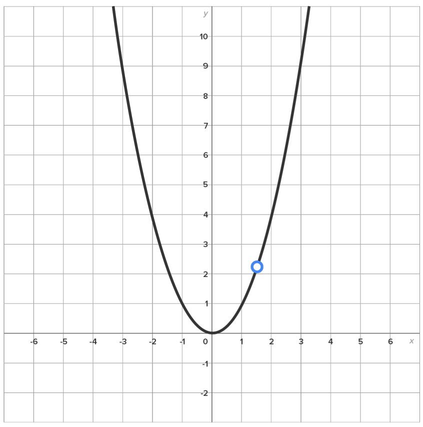
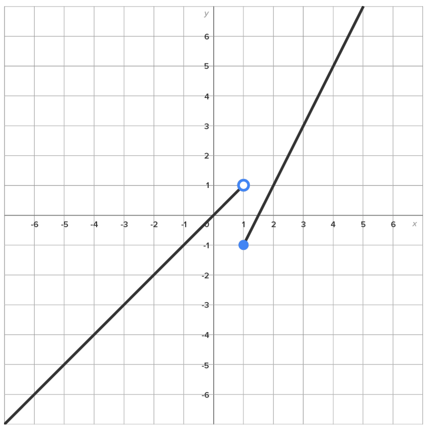
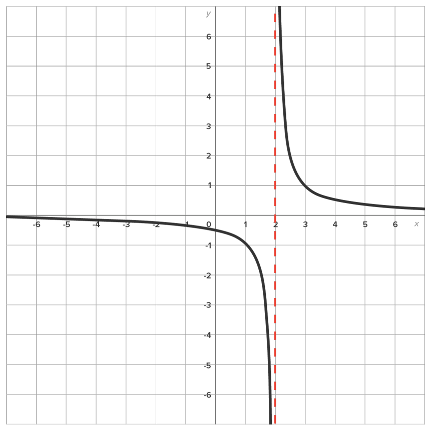
1. Discontinuities

As discussed in the previous section, a function is not differentiable at $x=a$ if it is discontinuous at $x=a$.

BIG IDEA

Therefore, if there is a break in the graph when $x=a$, the function is not differentiable at $x=a$.

There are three types of discontinuity:

Hole in the Graph	Graph Jumps	Vertical Asymptote
		

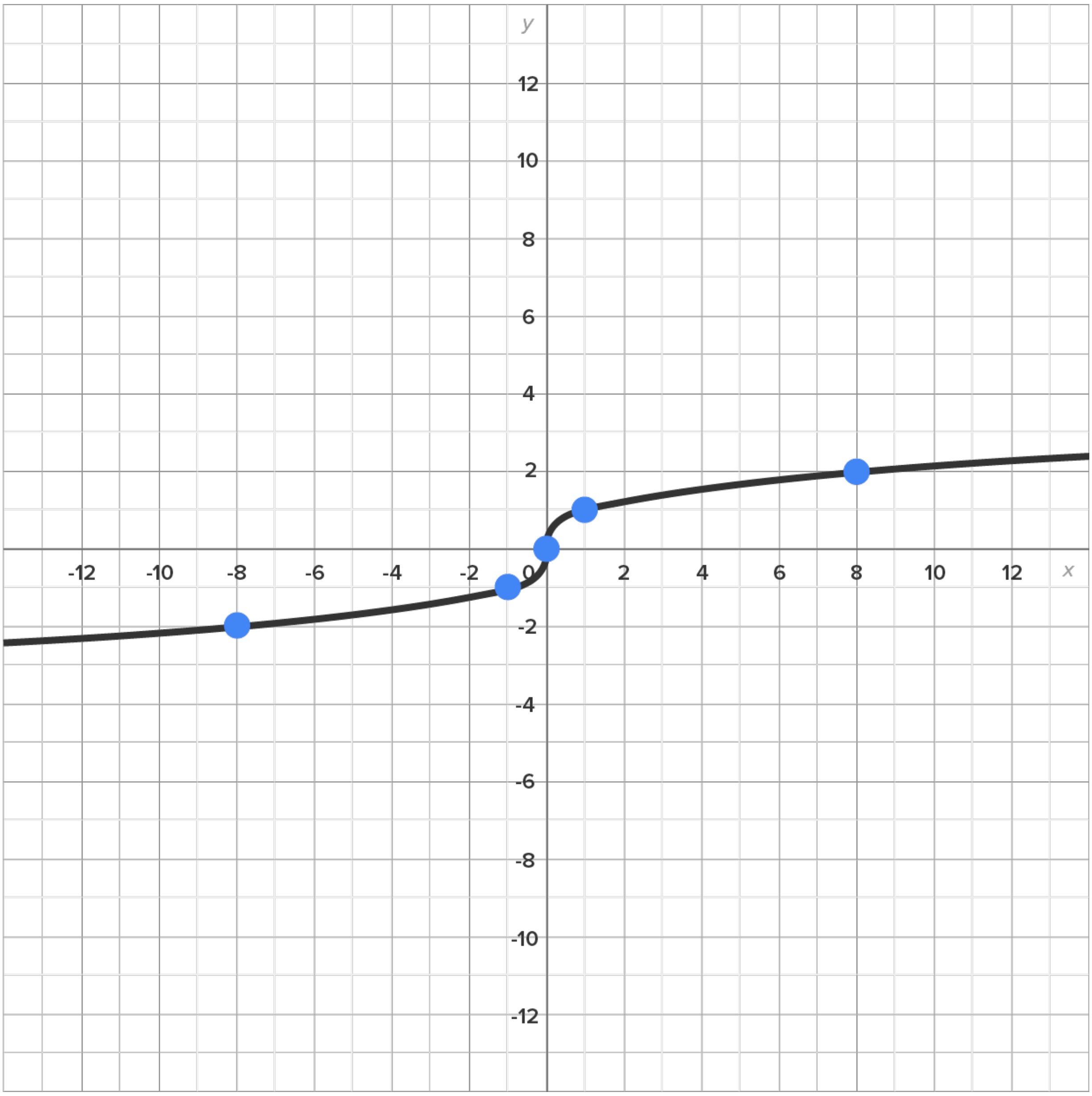
Since the continuity requirement isn't met at any discontinuity, it follows that a function is not differentiable at any x-value where $f(x)$ is discontinuous.

---

2. Vertical Tangents

When a tangent line is vertical, its slope is undefined. Since the derivative is the slope of a tangent line, a function is not differentiable at any point where there is a vertical tangent line.

↪ EXAMPLE Consider the graph of $y = \sqrt[3]{x}$, shown below:

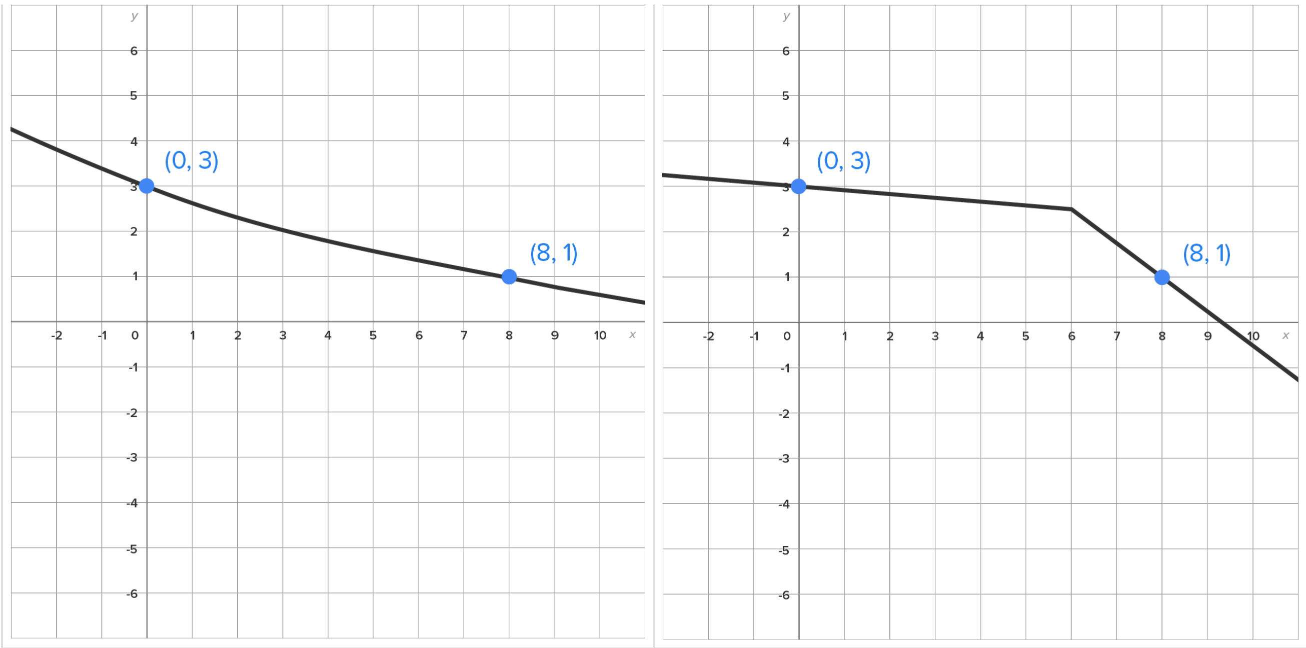


Note that when $x = 0$, the tangent line appears to be vertical. In fact, we can show this by finding $f'(x)$, which was calculated in 3.2.1: $f'(x) = \dfrac{1}{3x^{2/3}}$, which is undefined when $x = 0$, indicating an undefined slope (vertical line). Recall that in order for a function to be differentiable at a point, the derivative has to be defined at that point. Therefore, a vertical tangent line at a point is an indication that the function is not differentiable at that point.

3. Sharp Corners

Suppose you want to drive along a road from (0, 3) to (8, 1). Which graph provides a smoother ride?

Graph #1	Graph #2



Hopefully, you said the first one. The second graph shows a sudden transition (change in slope) when $x = 6$ (at the sharp corner), while the first graph changes smoothly from start to finish.

BIG IDEA

When the slope suddenly changes at $x = a$, then we say $f(x)$ is not differentiable when $x = a$. This is sometimes referred to as a sharp corner.

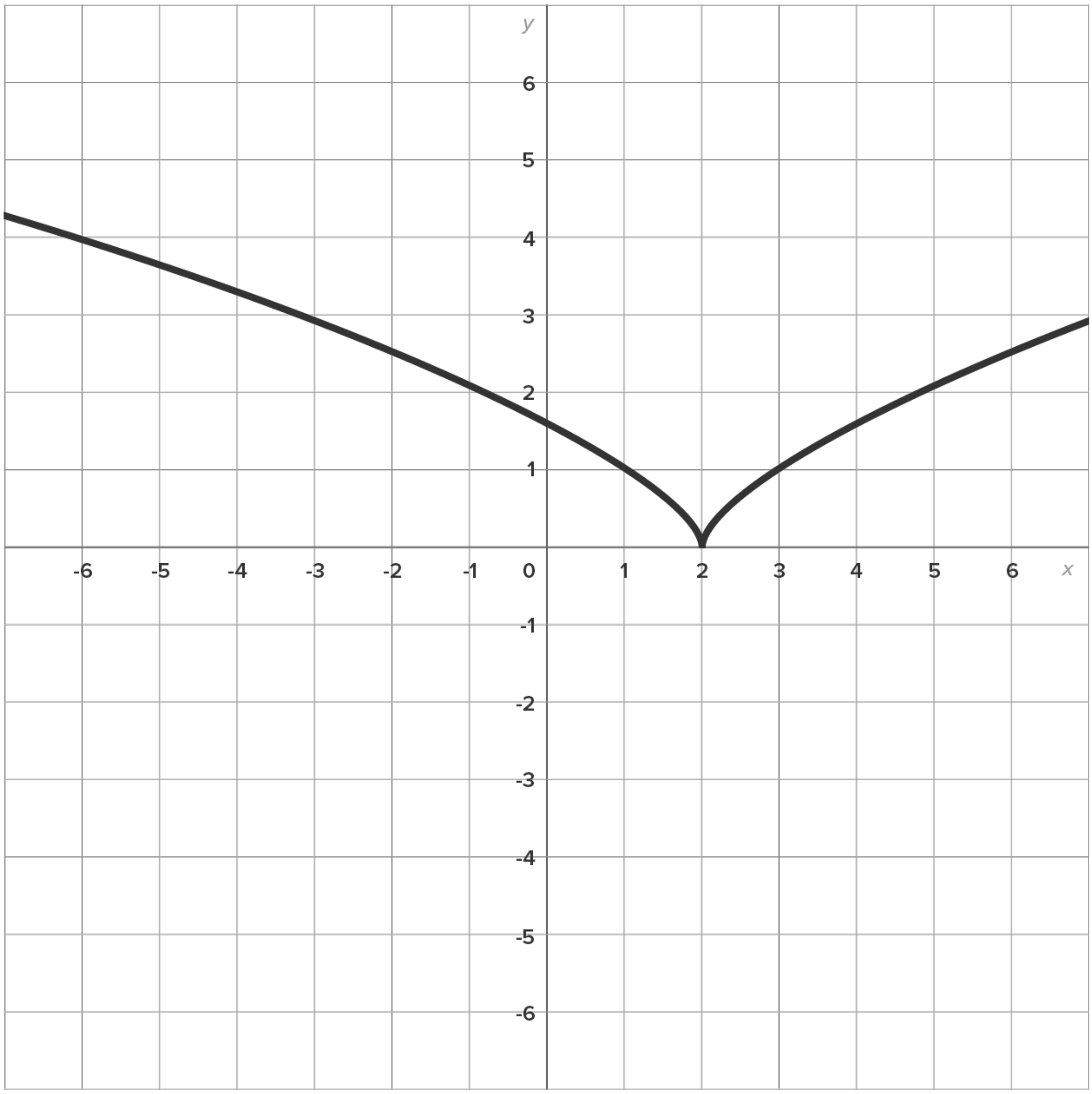
Thus, the function represented in the second graph is not differentiable when $x = 6$.

---

4. Cusps

We already have seen that a function is not differentiable at $x = a$ if there is a corner point at $x = a$. If the corner point happens to also have undefined slope, then that corner point is called a **cusp**. A cusp is a special type of corner point in that the slope of the tangent line at the cusp is undefined (vertical tangent line).

EXAMPLE Consider the graph of $y = f(x)$ shown below.



This graph shows a sharp corner at $x = 2$.

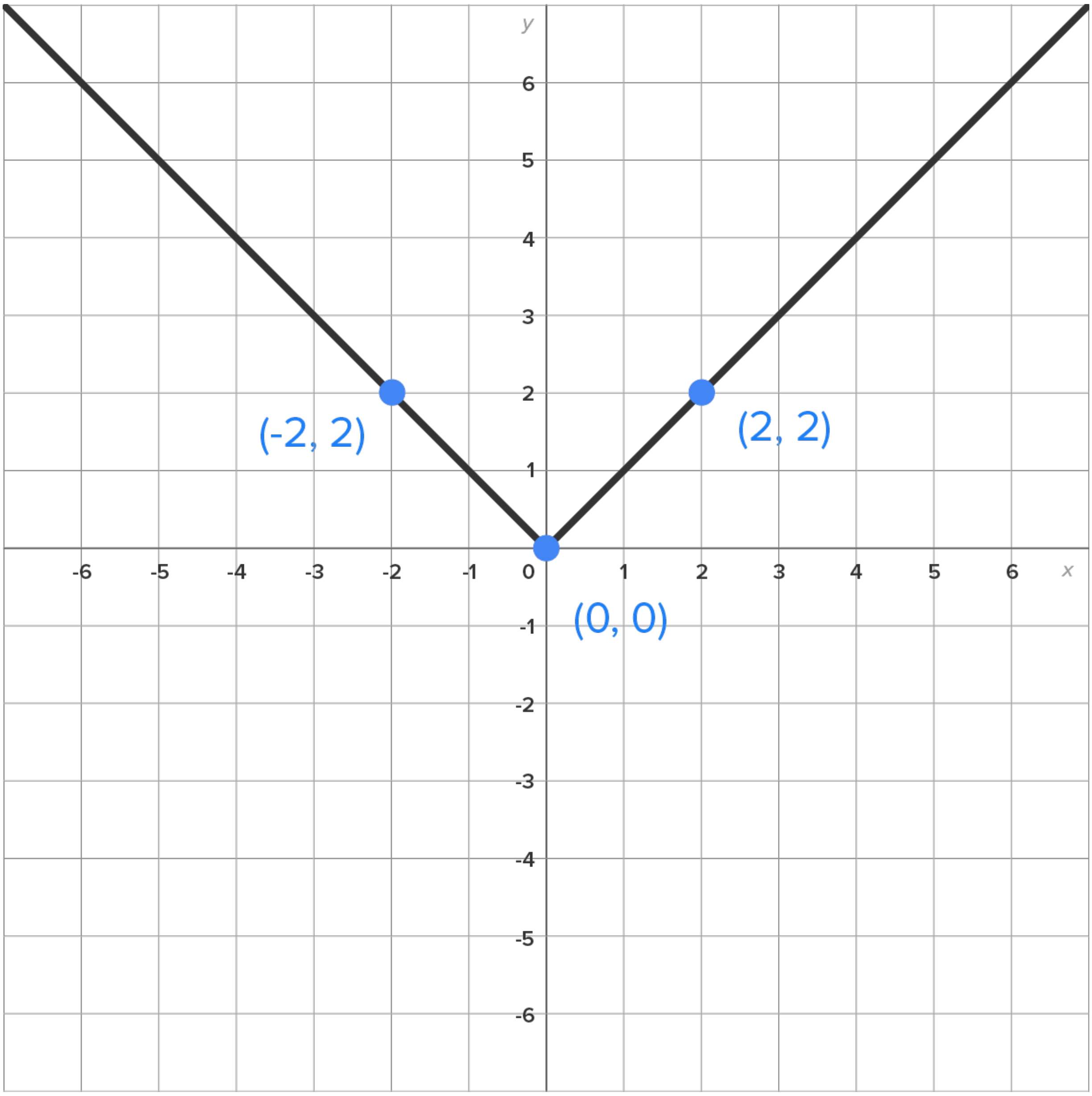
Notice that as *x* gets closer to 2 from either side, the slope gets steeper and steeper until becoming vertical. This implies that the derivative at $x = 2$ is undefined, meaning that this function is not differentiable when $x = 2$.

Since the tangent line is vertical at this point, we call this point a cusp.

THINK ABOUT IT

Imagine you are riding on a roller coaster. Obviously you would want the track to be continuous, but on a track full of corner points, the wheels would hit them, resulting in a very bumpy, loud, and dangerous ride.

→ EXAMPLE Consider the graph of $f(x) = |x|$.



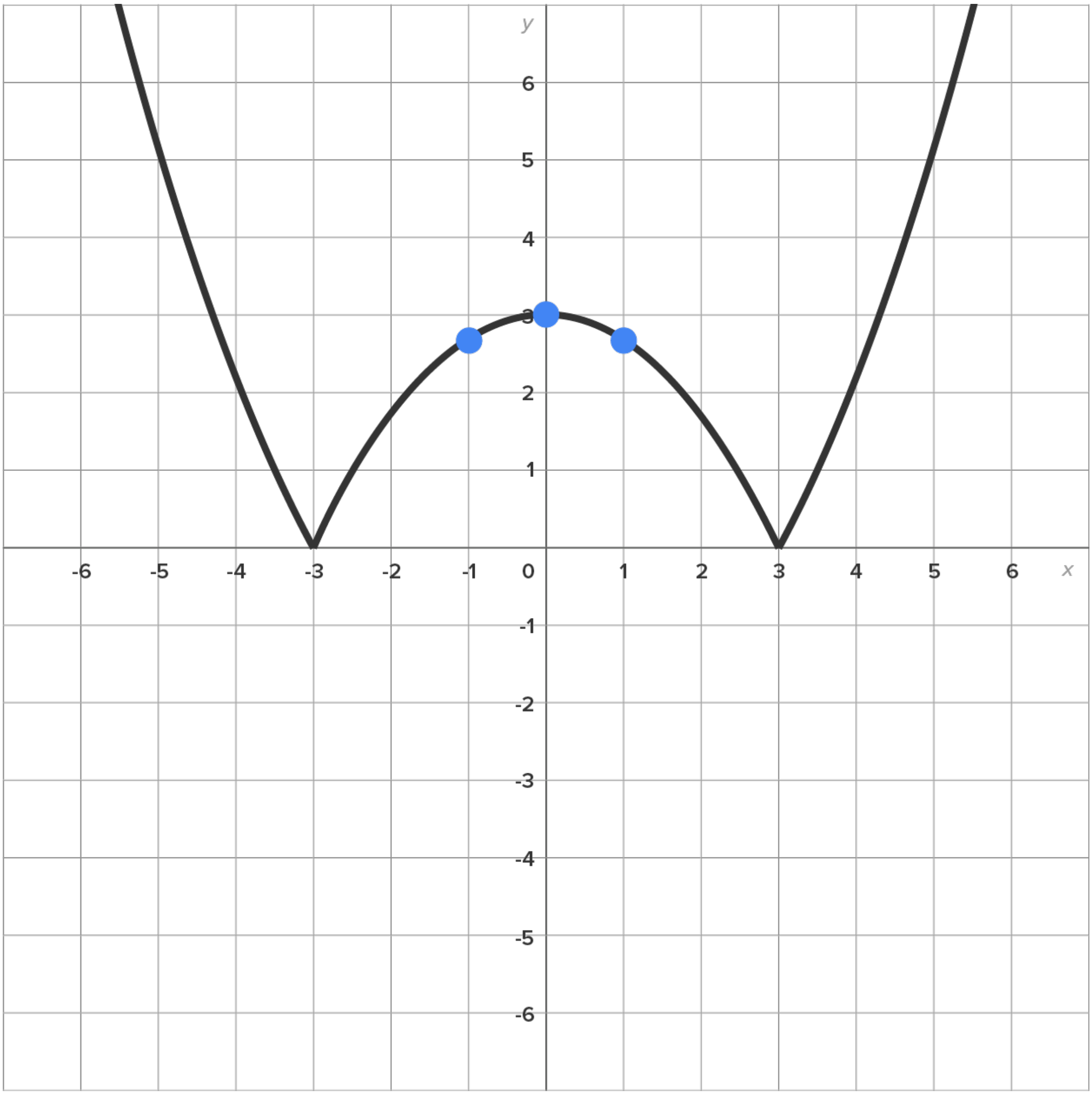
Notice that the graph is continuous everywhere (no breaks), but there is a sharp turn when $x = 0$. What could this mean? Let's explore this.

- To the left of (0, 0), the graph has slope -1.
- To the right of (0, 0), the graph has slope 1.
- At (0, 0), the slope changes directly from -1 to 1.

Thus, $f(x) = |x|$ is not differentiable at $x = 0$.

TRY IT

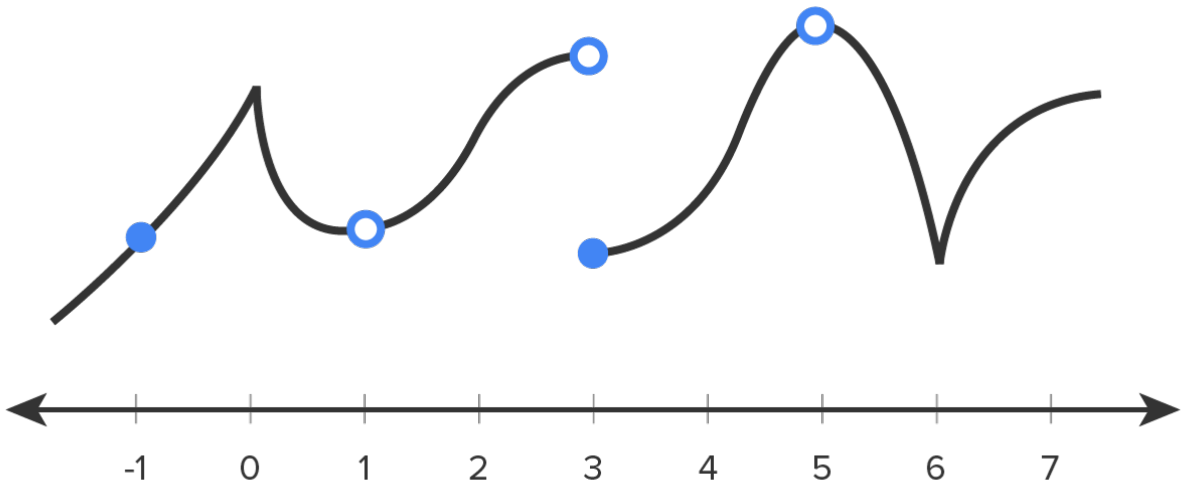
Shown below is the graph of $f(x) = |x^2 - 9|$.



Find all values of x for which $f(x)$ is not differentiable. +

$x = -3, x = 3$

TRY IT



Using the graph, determine the values of x for which $f(x)$ is not differentiable. +

$x = 0$ (cusp), 1 (hole), 3 (jump), 5 (hole), 6 (cusp)

TERM TO KNOW

Cusp

A pointed end where two parts of a curve meet at a vertical tangent.

SUMMARY

In this lesson, you learned that there are several graphical properties that indicate that a function is not differentiable: **discontinuities**, **vertical tangents**, **sharp corners**, and **cusps**. These are relatively easy to spot on a graph and therefore make the work of determining differentiability simpler. As you also saw, differentiability is necessary in circumstances in which smooth transitions are important, such as in the track of a roller coaster.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Cusp

A pointed end where two parts of a curve meet at a vertical tangent.

Derivative of Elementary Combinations of Functions

by Sophia

WHAT'S COVERED

In this lesson, you will use rules of differentiation to find derivatives of combinations of functions. One reason this is important to learn is to be able to analyze situations in which the function is more complex. For example, suppose that the height of a projectile t seconds after being launched is given by $h(t) = -16t^2 + 80t + 5$. If we learn some more derivative rules, we can analyze this function more efficiently without having to use the limit definition. Specifically, this lesson will cover:

1. Derivatives of Constant Multiples of Functions
2. Derivatives of Sums and Differences of Functions

1. Derivatives of Constant Multiples of Functions

To introduce this topic, let's say a jogger goes out for a run.

- Let $y(x) =$ the number of yards that a jogger runs in x minutes.
- Let $f(x) =$ the number of feet that a jogger runs in x minutes.

Since there are 3 feet in every yard, we also know that $f(x) = 3 \cdot y(x)$.
Now, let's think about the rates of change, namely $f'(x)$ and $y'(x)$.

If the jogger's rate of change at a certain instant is 40 yards per minute, then their rate is also 120 feet per minute (3 feet in every yard). Therefore, if we know the rate of change in the number of yards per minute, we just need to multiply by 3 to get the rate of change in the number of feet per minute.

Using symbols, here is the summary: If $f(x) = 3y(x)$, then $f'(x) = 3y'(x)$.

This leads us to the constant multiple rule for derivatives. If k is a constant, then we have the following rule: $D[k \cdot f(x)] = k \cdot D[f(x)]$. In other words, find the derivative of the variable part first, then multiply it by the constant.

FORMULA

Derivative of a Constant Multiple
$D[k \cdot f(x)] = k \cdot D[f(x)]$

EXAMPLE Find the derivative of $f(x) = 3x^4$.

$$f'(x) = 3D[x^4] \quad \text{Use the constant multiple rule.}$$

$$f'(x) = 3(4x^3) \quad \text{Use the power rule.}$$

$$f'(x) = 12x^3 \quad \text{Simplify.}$$

Thus, $f'(x) = 12x^3$.

↪ EXAMPLE Find the derivative of $f(x) = 12\sqrt{x}$.

First, rewrite as $\sqrt{x} = x^{1/2}$. Then:

$$f'(x) = 12D[x^{1/2}] \quad \text{Use the constant multiple rule.}$$

$$f'(x) = 12\left(\frac{1}{2}x^{-1/2}\right) \quad \text{Use the power rule.}$$

$$f'(x) = 6x^{-1/2} \quad \text{Simplify.}$$

$$f'(x) = \frac{6}{x^{1/2}} \quad \text{Write with positive exponents.}$$

Thus, $f'(x) = \frac{6}{x^{1/2}}$. Using radicals, this could also be written $f'(x) = \frac{6}{\sqrt{x}}$.

TRY IT

Consider the functions $f(x) = -10x^2$, $g(x) = 6\sqrt[3]{x}$, and $h(x) = \frac{5}{x^4}$.

Find the derivatives of the formulas above. +

$f'(x) = -20x$, $g'(x) = \frac{2}{x^{2/3}}$, and $h'(x) = \frac{-20}{x^5}$

2. Derivatives of Sums and Differences of Functions

Let's say that Fred and Gabby are baking cookies, where Fred makes 100 cookies per hour and Gabby makes 80 cookies per hour.

Then, the total number of cookies made per hour is 100 + 80 = 180 cookies per hour. Thus, the rate of change of the sum is the sum of the individual rates of change.

We can also say that Fred's rate of change is 20 cookies more per hour than Gabby's. Thus, the rate of change of their difference is the difference between the individual rates of change.

This leads to two more derivative rules:

FORMULA

Derivative of a Sum
$D[f(x)+g(x)] = D[f(x)] + D[g(x)]$

Derivative of a Difference
$D[f(x)-g(x)] = D[f(x)] - D[g(x)]$

EXAMPLE We know that $D[x^3] = 3x^2$ and $D[\sin x] = \cos x$.

Then, $D[x^3 + \sin x] = 3x^2 + \cos x$.

EXAMPLE Find the derivative of $f(x) = \frac{4}{x^6} - 7\cos x$.

Since this is a difference of functions, use the difference rule. To find $D\left[\frac{4}{x^6}\right]$:

$D\left[\frac{4}{x^6}\right] = D[4x^{-6}]$	Rewrite using negative exponents.
$= 4D[x^{-6}]$	Use the constant multiple rule.
$= 4(-6)x^{-7}$	Use the power rule.
$= -24x^{-7}$	Simplify.
$= -\frac{24}{x^7}$	Write using positive exponents.

To find $D[7\cos x]$:

$D[7\cos x] = 7D[\cos x]$	Use the constant multiple rule.
$= 7(-\sin x)$	$D[\cos x] = -\sin x$
$= -7\sin x$	Simplify.

Then, $D\left[\frac{4}{x^6} - 7\cos x\right] = -\frac{24}{x^7} - (-7\sin x) = -\frac{24}{x^7} + 7\sin x$.

Thus, $f'(x) = -\frac{24}{x^7} + 7\sin x$.

TRY IT

In the introduction, the function $h(t) = -16t^2 + 80t + 5$ was mentioned.

Find $h'(t)$. +

$h'(t) = -32t + 80$

TRY IT

Consider the function $f(x) = 6\sqrt{x} + 8x - 3\cos x$.

Find the derivative. +

$$f'(x) = \frac{3}{x^{1/2}} + 8 + 3\sin x$$

SUMMARY

In this lesson, you learned that the sum/difference and constant multiple rules for derivatives allow us to expand on the types of functions that can be differentiated by using rules rather than the limit definition. For example, you learned that to find the **derivative of constant multiples of functions**, you find the derivative of the variable part first, then multiply it by the constant. You also explored an example involving finding **derivatives of sums and differences of functions**, noting that the rate of change of the sum is the sum of the individual rates of change, and the rate of change of the difference is the difference between the individual rates of change. We're only getting started since there are several rules to discuss yet!

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of a Constant Multiple

$D[k \cdot f(x)] = k \cdot D[f(x)]$

Derivative of a Difference

$D[f(x) - g(x)] = D[f(x)] - D[g(x)]$

Derivative of a Sum

$D[f(x) + g(x)] = D[f(x)] + D[g(x)]$

The Product Rule

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to find derivatives of functions that are products of other functions. For example, $f(x)=x^2\sin x$ is the product of x^2 and $\sin x$. Specifically, this lesson will cover:

1. The Product Rule
2. Combining the Product Rule With Other Rules

1. The Product Rule

Suppose you want to find $D[f(x)\cdot g(x)]$.

At first glance, one might (incorrectly) think that $D[f(x)\cdot g(x)]=D[f(x)]\cdot D[g(x)]$. In other words, is the derivative of the product equal to the product of the derivatives?

Let's test this out. Let $f(x)=x^2$ and $g(x)=x^3$. Then, $f(x)\cdot g(x)=x^2\cdot x^3=x^5$.

Now, let's find $D[f(x)\cdot g(x)]$ and $D[f(x)]\cdot D[g(x)]$ to see if they are equal:

- $D[f(x)\cdot g(x)]=D[x^5]=5x^4$
- $D[x^3]\cdot D[x^2]=3x^2\cdot 2x=6x^3$

Oh no! These expressions are not equal!
In general, we can conclude that $D[f(x)\cdot g(x)]\neq f'(x)\cdot g'(x)$.

So, what is the correct product rule?

FORMULA

Product Rule for Derivatives

$D[f(x)\cdot g(x)]=D[f(x)]\cdot g(x)+f(x)\cdot D[g(x)]$

Using alternate notation: $\frac{d}{dx}[f(x)\cdot g(x)]=f'(x)\cdot g(x)+f(x)\cdot g'(x)$

In words: "The derivative of a product of two functions is the derivative of the first times the second, plus the first times the derivative of the second."

EXAMPLE Find the derivative of $f(x)=x^2\sin x$.

$f'(x)=D[x^2]\cdot\sin x+x^2\cdot D[\sin x]$	Apply the product rule.

$$f'(x) = 2x \cdot \sin x + x^2 \cdot \cos x$$ Use known derivatives.

$$f'(x) = 2x\sin x + x^2\cos x$$ Remove unnecessary symbols.

Thus, $f'(x) = 2x\sin x + x^2\cos x$.

↪ EXAMPLE Find the derivative of $f(x) = \cos^2 x$.

First, notice that $\cos^2 x$ can be rewritten as $\cos x \cdot \cos x$, so the product rule can be used to find its derivative.

$f'(x) = D[\cos x \cdot \cos x] = D[\cos x] \cdot \cos x + \cos x \cdot D[\cos x]$	Apply the product rule.
$f'(x) = (-\sin x)\cos x + \cos x(-\sin x)$	$D[\cos x] = -\sin x$
$f'(x) = -\sin x\cos x - \sin x\cos x$	Remove the grouping.
$f'(x) = -2\sin x\cos x$	Combine like terms.

Thus, $f'(x) = -2\sin x\cos x$.

Let's now look at one with a constant multiple.

↪ EXAMPLE Find the derivative of $f(t) = 4\sin t\cos t$.

This appears as a product of three functions, but the “4” is a constant. Let's write $f(t) = 4\sin t\cos t = (4\sin t)(\cos t)$. Then:

$f'(t) = D[4\sin t] \cdot \cos t + 4\sin t \cdot D[\cos t]$	Apply the product rule.
$f'(t) = 4\cos t \cdot \cos t + 4\sin t \cdot (-\sin t)$	Use the derivative rules for $\sin t$, $\cos t$, and the constant multiple.
$f'(t) = 4\cos^2 t - 4\sin^2 t$	Condense into exponential notation.

Thus, $f'(t) = 4\cos^2 t - 4\sin^2 t$.

Now, here are two examples for you to try.

TRY IT

Consider the function $g(x) = 4x^5\cos x$.

Find the derivative for the above formula. +

$$g'(x) = 20x^4\cos x - 4x^5\sin x$$

TRY IT

Consider the function $f(t) = 4\sin^2 t$.

Find the derivative for the above formula. +

$f'(t) = 8\sin t\cos t$

BIG IDEA

What we are starting to see is that finding derivatives involves building blocks rather than memorization. For example, the derivative of $f(x) = x^3 \sin x$ is not worth memorizing, but knowing the product rule, $D[x^3]$, and $D[\sin x]$, one can pretty quickly find $f'(x)$.

Now, let's look at an example where the product rule could be used, but isn't required.

EXAMPLE Find the derivative of $f(x) = (2x^2+5)(x-8)$.

This is clearly a product, so the product rule can be used to find the derivative:

$f'(x) = D[2x^2+5]\cdot(x-8)+(2x^2+5)\cdot D[x-8]$	Apply the product rule.
$f'(x) = (4x)(x-8)+(2x^2+5)(1)$	Take the derivatives.
$f'(x) = 4x^2-32x+2x^2+5$	Distribute.
$f'(x) = 6x^2-32x+5$	Combine like terms.

However, notice that $(2x^2+5)(x-8)$ can be multiplied before differentiation:

$$f(x) = (2x^2+5)(x-8)$$

$$f(x) = 2x^2(x)-8(2x^2)+5x-40$$

$$f(x) = 2x^3-16x^2+5x-40$$

Written this way, the derivative can be found with the sum/difference and constant multiple rules.

$f(x) = 2x^3-16x^2+5x-40$	Start with the original evaluated function.
$f'(x) = D[2x^3]-D[16x^2]+D[5x]-D[40]$	Use the sum/difference properties.
$f'(x) = 2(3x^2)-16(2x)+5(1)-0$	Apply the constant multiple and power rules.
$f'(x) = 6x^2-32x+5$	Simplify.

As you can see, the results are identical. It looks like simplifying the expression first enables us to use basic rules rather than the product rule.

BIG IDEA

When you need to find the derivative of a product of two functions, check first to see if the function can be manipulated/simplified first. You could avoid having to use the product rule in exchange for an easier rule.

TRY IT

Consider the function $f(x)=(x^2-4)(2x^2+3)$.

Find the derivative. +

$f'(x)=8x^3-10x$

2. Combining the Product Rule With Other Rules

As we learn more derivative rules, functions can get more complex. Here is an example of one such function.

EXAMPLE Find the derivative of $f(x)=x\sin x+\cos x$.

Since $f(x)$ is a sum of two functions, the sum/difference rule should be applied first.

$f'(x)=D[x\sin x]+D[\cos x]$	Apply the sum/difference rule.
$f'(x)=D[x]\cdot\sin x+x\cdot D[\sin x]+D[\cos x]$	Apply the product rule.
$f'(x)=1\cdot\sin x+x\cdot(\cos x)+(-\sin x)$	$D[x]=1$, $D[\sin x]=\cos x$, $D[\cos x]=-\sin x$
$f'(x)=\sin x+x\cos x-\sin x$	Simplify and remove excess symbols.
$f'(x)=x\cos x$	Combine like terms.

Thus, $f'(x)=x\cos x$.

The product rule can be extended to three or more functions (three is usually the most, though, as far as being optimistic).

WATCH

In the video below, we'll find the derivative of $f(x)=x\sin x\cos x$.

Video Transcription

[MUSIC PLAYING] Welcome back. It's good to see you. What we're going to do is take a look at a function that is a product of three functions and find its derivative using the product rule. So let's just remember that the derivative of the product of two functions, which we know already, is the derivative of the first function times the second plus the first function times the derivative of the second. So we're going to use that idea to find the derivative of this product of 3.

So the way we do it, though, is, since we know what the derivative of a product of two functions is, I'm going to treat the first two together as one function and this one as the second function. So that means that the derivative is equal to. So it's the derivative of x sine x times the second, which is cosine x, plus the first, which is x sine x, times the derivative of the second, which is cosine x. So we're going to go off to the side here and just focus on the derivative of x sine x, because that itself is a product, which is a bit more complicated. So let's take a look at that.

So the derivative of x sine x is the derivative of x times sine x plus the first, which is x, times the derivative of sine x. Well, the derivative of x is 1, and the derivative of sine x is cosine x. So writing it a little bit more simply, the derivative of x sine x is sine x plus x cosine x. So now we'll resume the original derivative-- so f prime of x is equal to.

So the derivative of x sine x is all of this, so we're going to use parentheses to place that in there. So we have sine x plus x cosine x times cosine x plus x sine x. And then the derivative of cosine x is negative sine of x And now we just have to pull some things together. So it looks like the first term here, we're going to be distributing a cosine of x.

So we have sine of x plus x cosine of squared x. And then in the second term here, we have x times sine x times negative sine x. So that's going to change the plus to a minus, and we're going to write x sine square root of x. And that is actually as simple as it gets. So we're going to call this f prime, and there is the expression. So that is how we handle the derivative of a product of three functions.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that with the knowledge of **the product rule**, you are able to find derivatives of even more combinations of functions, namely the product of two (or more) functions. It is important to note, however, that before automatically turning to the product rule to find the derivative of a product of two functions, always check first to see if the function can be manipulated or simplified first, allowing you to exchange the product rule for an easier rule. You also learned that as functions get increasingly complex, you may need to **combine the product rule with other rules** to find their derivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Product Rule for Derivatives

$$D[f(x)\cdot g(x)]=D[f(x)]\cdot g(x)+f(x)\cdot D[g(x)]$$

Using alternate notation: $\frac{d}{dx}[f(x)\cdot g(x)]=f'(x)\cdot g(x)+f(x)\cdot g'(x)$

The Quotient Rule

by Sophia

WHAT'S COVERED

In this lesson, you will find derivatives of functions that are quotients of other functions (for example, $f(x)=\frac{2x}{x^2+1}$). Specifically, this lesson will cover:

1. The Quotient Rule
2. Combining Derivative Rules

1. The Quotient Rule

Let's say that $Q(x)=\frac{f(x)}{g(x)}$. The goal is to find $Q'(x)$.

$Q(x)=\frac{f(x)}{g(x)}$	Start with the original function.
$Q(x)\cdot g(x)=f(x)$	Multiply both sides by $g(x)$.
$Q'(x)\cdot g(x)+Q(x)\cdot g'(x)=f'(x)$	Take the derivative. Apply the product rule on the left-hand side.
$Q'(x)\cdot g(x)=f'(x)-Q(x)\cdot g'(x)$	Subtract $Q(x)\cdot g'(x)$ from both sides.
$Q'(x)=\frac{f'(x)-Q(x)\cdot g'(x)}{g(x)}$	Divide both sides by $g(x)$ to solve for $Q'(x)$.
$Q'(x)=\frac{f'(x)-\left[\frac{f(x)}{g(x)}\right]\cdot g'(x)}{g(x)}$	Replace $Q(x)$ with $\frac{f(x)}{g(x)}$ to get an expression that involves only $f(x)$ and $g(x)$.
$Q'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$	Multiply by $\frac{g(x)}{g(x)}$ to simplify the complex fraction.

We now have an expression for $Q'(x)$, the derivative of $\frac{f(x)}{g(x)}$.

Note how the numerator is very similar to the product rule, but there is a subtraction between the terms instead of an addition sign.

Another way to remember this is to use the following saying:

$$D\left[\frac{high}{low}\right]=\frac{low\ dee\ high-high\ dee\ low}{low\ low}$$

In words, this is "low dee high minus high dee low over low low." The word *dee* is used for the derivative.

FORMULA

Quotient Rule for Derivatives

Using "Prime" Notation: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Using "D" Notation: $D\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot D[f(x)]-f(x)\cdot D[g(x)]}{[g(x)]^2}$

"High and Low" Version: $D\left[\frac{high}{low}\right]=\frac{low\ dee\ high-high\ dee\ low}{low\ low}$

EXAMPLE Find the derivative of $f(x)=\frac{2x}{x^2+1}$.

Apply the quotient rule formula and simplify:

$$f'(x)=\frac{(x^2+1)\cdot D[2x]-2x\cdot D[x^2+1]}{(x^2+1)^2}$$ Apply the quotient rule.

$$f'(x)=\frac{(x^2+1)(2)-2x\cdot 2x}{(x^2+1)^2}$$ $D[2x]=2,\ D[x^2+1]=2x$

$$f'(x)=\frac{2x^2+2-4x^2}{(x^2+1)^2}$$ Simplify on both sides of subtraction.

$$f'(x)=\frac{-2x^2+2}{(x^2+1)^2}$$ Simplify the numerator.

Thus, $f'(x)=\frac{-2x^2+2}{(x^2+1)^2}$. This is the simplest form. If you factor the numerator completely, you would get $f'(x)=\frac{-2(x+1)(x-1)}{(x^2+1)^2}$ and notice that there is no common factor between the numerator and denominator.

Now, try one for yourself. Be sure to simplify your answer as much as possible!

TRY IT

Consider the function $f(x)=\frac{3x^2-1}{2x^2+3}$.

Find the derivative. +

$$f'(x)=\frac{22x}{(2x^2+3)^2}$$

EXAMPLE Find the derivative of the function $f(x)=\frac{1}{3x^2+4}$.

Apply the quotient rule formula and simplify:

$$f'(x) = \frac{(3x^2+4)\cdot D[1]-(1)\cdot D\left[3x^2+4\right]}{(3x^2+4)^2}$$ Apply the quotient rule.

$$f'(x) = \frac{(3x^2+4)(0)-1\cdot 6x}{(3x^2+4)^2}$$ $D[1]=0,\ D\left[3x^2+4\right]=6x$

$$f'(x) = \frac{0-6x}{(3x^2+4)^2}$$ Simplify on both sides of subtraction.

$$f'(x) = \frac{-6x}{(3x^2+4)^2}$$ Simplify the numerator.

Thus, $f'(x) = \frac{-6x}{(3x^2+4)^2}$.

Let's look at an example that involves trigonometric functions.

WATCH

Here is a video to help illustrate the quotient rule by finding the derivative of $f(x) = \frac{\sin x}{x+1}$.

Video Transcription

Hello there. Hope you're having a great day today. What we're going to look at is the quotient rule. And a mnemonic that might be helpful for you to remember what the quotient rule is. So we know what the standard version is, the derivative of the numerator, and denominator minus the numerator times the derivative of the denominator, all over the denominator squared.

Well, over here, what we have is another version. If you read it through left to right, you have lo di hi minus hi di lo over lo lo. So lo refers to the denominator of the function, and hi refers to the numerator of the function. di hi refers to the derivative of the numerator, and di lo low refers to the derivative of the denominator,

OK. So writing that out, so let's just see what this looks like here. So we'll say hi is sine x, and lo is x plus 1. So their derivatives are cosine x and 1. And this is essentially di hi and di lo. So let's see what that looks like when we put it together in derivative form.

So f prime of x is equal to lo di hi. So lo is x plus 1. di hi is cosine x minus hi di lo is sine x times 1 all over lo squared, which is x plus 1 quantity squared. And believe it or not, that has pretty much already simplified. We have x plus 1 times cosine x. I was going to rewrite it without the extra multiply by 1. And that's all divided by x plus 1 squared. And that is our derivative.

Now, just a simplification note. You might be tempted to cross out a common factor of x plus 1 between the numerator and denominator. But remember, we can't do that because this term over here does not contain an x plus 1. So it has to be a factor of the entire numerator, not just one term. So that is considered simplified, and there's our quotient rule.

Here is an example for you to try on your own. Remember to simplify where possible!

TRY IT

Consider the function $f(x) = \dfrac{1 + \cos x}{1 - \cos x}$.

Find the derivative.

+

$$f'(x) = \frac{-2\sin x}{(1 - \cos x)^2}$$

Sometimes writing a derivative in simplest form requires the use of trigonometric identities. Here is an example.

↪ EXAMPLE Find the derivative of the function $f(x) = \dfrac{\sin x}{1 + \cos x}$.

Now, apply the quotient rule formula and simplify:

$f'(x) = \dfrac{(1+\cos x)\cdot D[\sin x] - \sin x \cdot D[1+\cos x]}{(1+\cos x)^2}$	Apply the quotient rule.
$f'(x) = \dfrac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$	$D[\sin x] = \cos x$, $D[1+\cos x] = -\sin x$
$f'(x) = \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$	Distribute $\cos x(1+\cos x)$ and simplify $-\sin x(-\sin x)$.
$f'(x) = \dfrac{\cos x + 1}{(1+\cos x)^2}$	Trigonometric Identity: $\sin^2 x + \cos^2 x = 1$
$f'(x) = \dfrac{1}{1+\cos x}$	Remove the common factor of $1+\cos x$.

Thus, $f'(x) = \dfrac{1}{1+\cos x}$.

Now, let's look at an example where the quotient rule could be used, but isn't required.

↪ EXAMPLE Find the derivative of the function $f(x) = \dfrac{6x^2 + 10x + 1}{3x^2}$.

This is clearly a quotient, so the quotient rule can be used to find the derivative:

$f'(x) = \dfrac{3x^2 \cdot D[6x^2 + 10x + 1] - (6x^2 + 10x + 1)\cdot D[3x^2]}{(3x^2)^2}$	Apply the quotient rule.
$f'(x) = \dfrac{(3x^2)(12x + 10) - (6x^2 + 10x + 1)(6x)}{9x^4}$	$D[6x^2 + 10x + 1] = 12x + 10$, $D[3x^2] = 6x$
$f'(x) = \dfrac{36x^3 + 30x^2 - 36x^3 - 60x^2 - 6x}{9x^4}$	Distribute on both sides of the subtraction sign.
$f'(x) = \dfrac{-30x^2 - 6x}{9x^4}$	Simplify the numerator.
$f'(x) = \dfrac{-30x^2}{9x^4} - \dfrac{6x}{9x^4}$	Write as single fractions.

$$f'(x) = \frac{-10}{3x^2} - \frac{2}{3x^3}$$ Simplify each fraction.

$$f'(x) = -\frac{10}{3x^2} - \frac{2}{3x^3}$$ Write negative in front of the fraction.

Thus, $f'(x) = -\frac{10}{3x^2} - \frac{2}{3x^3}$.

However, notice that the original expression can be rewritten by performing the division:

$$f(x) = \frac{6x^2 + 10x + 1}{3x^2} = \frac{6x^2}{3x^2} + \frac{10x}{3x^2} + \frac{1}{3x^2}$$ Split into individual fractions.

$$f(x) = 2 + \frac{10}{3x} + \frac{1}{3x^2}$$ Simplify each fraction.

$$f(x) = 2 + \frac{10}{3}x^{-1} + \frac{1}{3}x^{-2}$$ Write each term as a single exponent (to get ready for differentiation).

Now, $f(x)$ is a sum/difference of powers of x, which is much easier to differentiate:

$$f'(x) = D[2] + D\left[\frac{10}{3}x^{-1}\right] + D\left[\frac{1}{3}x^{-2}\right]$$ Apply the sum/difference rules.

$$f'(x) = 0 + \frac{10}{3}(-1)x^{-2} + \frac{1}{3}(-2)x^{-3}$$ Apply the power and constant multiple rules.

$$f'(x) = -\frac{10}{3}x^{-2} - \frac{2}{3}x^{-3}$$ Simplify.

$$f'(x) = -\frac{10}{3x^2} - \frac{2}{3x^3}$$ Write using positive exponents.

As you can see, the results are identical.

BIG IDEA

When you need to find the derivative of a quotient of two functions, check to see if the function can be manipulated/simplified first. You could avoid having to use the quotient rule in exchange for an easier rule.

---

2. Combining Derivative Rules

As we continue to learn more derivative rules, it is important to see how the rules work together in more complicated functions.

→ EXAMPLE A company has determined that the total cost of producing x items is modeled by the function $C(x) = 20x + \frac{50x}{x+1}$, where $C(x)$ is measured in dollars.

The rate at which the total cost changes is called the marginal cost, and is found by computing $C'(x)$. The production manager is interested in knowing the marginal cost at the point when 9 units are produced.

First, find the derivative. Since $C(x)$ is the sum of two functions, we know $C'(x)=D[20x]+D\left[\frac{50x}{x+1}\right]$. By the power rule, $D[20x]=20$.

Now, we use the quotient rule to find $D\left[\frac{50x}{x+1}\right]$:

$D\left[\frac{50x}{x+1}\right]=\frac{(x+1)\cdot D[50x]-50x\cdot D[x+1]}{(x+1)^2}$	Use the quotient rule.
$D\left[\frac{50x}{x+1}\right]=\frac{(x+1)(50)-50x\cdot(1)}{(x+1)^2}$	$D[50x]=50$, $D[x+1]=1$
$D\left[\frac{50x}{x+1}\right]=\frac{50}{(x+1)^2}$	$50(x+1)-50x=50x+50-50x=50$

Thus, $C'(x)=D[20x]+D\left[\frac{50x}{x+1}\right]=20+\frac{50}{(x+1)^2}$.

Now, we seek the marginal cost at 9 units, which is $C'(9)=20+\frac{50}{(9+1)^2}=20.5$, which means the cost is rising at a rate of \$20.50 per additional unit when 9 units are produced.

SUMMARY

In this lesson, you explored finding the derivative using **the quotient rule**, although it's important to check if the function can be manipulated or simplified first, as you may be able to use an easier rule. At this point, you are able to find derivatives involving, sums, differences, products, and quotients. As you learn more derivative rules, you are able to take derivatives of more functions, including those that **combine these derivative rules**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Quotient Rule for Derivatives

Using "Prime" Notation: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Using "D" Notation: $D\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot D[f(x)]-f(x)\cdot D[g(x)]}{[g(x)]^2}$

"High and Low" Version: $D\left[\dfrac{high}{low}\right] = \dfrac{low\ dee\ high - high\ dee\ low}{low\ low}$

The General Power Rule for Functions

by Sophia

WHAT'S COVERED

In this lesson, you will expand upon your derivative knowledge even further by examining powers of functions whose derivatives we know. For example, $f(x) = (3x+1)^5$ and $y = \sin^4 x$. This idea will also help in finding the derivatives of some other commonly used functions. Specifically, this lesson will cover:

1. Derivatives of Functions of the Form $y = [f(x)]^n$
2. Combining Derivative Rules

1. Derivatives of Functions of the Form $y = [f(x)]^n$

Derivatives of powers of a function have several uses, as we will see once we get to applications of derivatives. To establish a pattern for this type of derivative, we'll consider the functions $y = f^2$, $y = f^3$, and $y = f^4$, where f is being used to represent some function $f(x)$.

First, consider the function $y = f^2 = f \cdot f$.

By the product rule, we have:

$$\begin{aligned} y' = D[f^2] &= D[f] \cdot f + f \cdot D[f] \\ &= f' \cdot f + f \cdot f' \\ &= 2f \cdot f' \end{aligned}$$

Now consider the function $y = f^3 = f^2 \cdot f$.

By the product rule again, we have:

$y' = D[f^3] = D[f^2] \cdot f + f^2 \cdot D[f]$	Apply the product rule.
$= (2f \cdot f') \cdot f + f^2 \cdot f'$	Replace $D[f^2]$ with $= 2f \cdot f'$.
$= 2f^2 \cdot f' + f^2 \cdot f'$	Combine $f \cdot f = f^2$.
$= 3f^2 \cdot f'$	Combine like terms.

Next, consider $y = f^4 = f^3 \cdot f$.

$D[f^4] = D[f^3] \cdot f + f^3 \cdot D[f]$ Apply the product rule.

$= (3f^2 \cdot f') \cdot f + f^3 \cdot f'$ Replace $D[f^3]$ with $= 3f^2 \cdot f'$.

$= 3f^3 \cdot f' + f^3 \cdot f'$ Combine $f^2 \cdot f = f^3$.

$= 4f^3 \cdot f'$ Combine like terms.

By looking at this pattern, it seems as though the derivative of f^n is $n \cdot f^{n-1}$ (looks like the power rule), but then also multiplied by f'.

FORMULA

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D[[f(x)]^n] = n \cdot [f(x)]^{n-1} \cdot f'(x)$.

EXAMPLE Earlier, we found the derivative of $f(x) = \cos^2 x$ by using the product rule. Let's use the power rule and compare.

First, note that this can be written as $f(x) = (\cos x)^2$.

By the power rule, we have the following:

$f'(x) = 2(\cos x) \cdot D[\cos x]$ Apply the power rule.

$= 2(\cos x)(-\sin x)$ $D[\cos x] = -\sin x$

$= -2\sin x \cos x$ Combine and eliminate parentheses.

This matches the answer obtained in challenge 3.2.4.

EXAMPLE Find the derivative of the function $f(x) = (5x+1)^{10}$.

By the power rule, we have the following:

$f'(x) = 10(5x+1)^9 \cdot D[5x+1]$ Apply the power rule.

$= 10(5x+1)^9(5)$ $D[5x+1] = 5$

$= 50(5x+1)^9$ Combine $10 \cdot 5$.

A common mistake to make here is to multiply $50(5x+1)$ to get $250x + 50$, and subsequently $(250x+50)^9$. This is not correct since the $(5x+1)$ is raised to the 9th power and the 50 is not; therefore, they cannot be combined this way. The final answer is $f'(x) = 50(5x+1)^9$.

TRY IT

Consider the function $y = (x^2 - 9x + 20)^4$.

Find the derivative. +

$$\frac{dy}{dx}=4(2x-9)(x^2-9x+20)^3$$

Remember the other expressions that can be written as powers of *x*.

↪ EXAMPLE Find the derivative of the function $f(x)=\sqrt{3x^2+8}$.

Remember that $\sqrt{u}=u^{1/2}$. Then the power rule can be used.

$f(x)=\sqrt{3x^2+8}=(3x^2+8)^{1/2}$	Rewrite the radical using a power.
$f'(x)=\frac{1}{2}(3x^2+8)^{-1/2}\cdot D[3x^2+8]$	Use the power rule for derivatives.
$f'(x)=\frac{1}{2}(3x^2+8)^{-1/2}\cdot 6x$	$D[3x^2+8]=6x$
$f'(x)=3x(3x^2+8)^{-1/2}$	$\frac{1}{2}\cdot 6x=3x$
$f'(x)=\frac{3x}{(3x^2+8)^{1/2}}$	Rewrite with nonnegative exponents.

Thus, $f'(x)=\frac{3x}{(3x^2+8)^{1/2}}$, which could also be written $f'(x)=\frac{3x}{\sqrt{3x^2+8}}$ if radical notation is desired.

↪ EXAMPLE Find the derivative of the function $f(x)=\frac{1}{(5x+\cos x)^3}$.

$f(x)=\frac{1}{(5x+\cos x)^3}=(5x+\cos x)^{-3}$	Rewrite so that the power rule can be used.
$f'(x)=-3(5x+\cos x)^{-4}\cdot D[5x+\cos x]$	Apply the power rule.
$f'(x)=-3(5x+\cos x)^{-4}\cdot(5-\sin x)$	$D[5x+\cos x]=5+(-\sin x)=5-\sin x$
$f'(x)=-3(5-\sin x)(5x+\cos x)^{-4}$	Rearrange the factors.
$f'(x)=\frac{-3(5-\sin x)}{(5x+\cos x)^4}$	Rewrite with nonnegative exponents.

Thus, $f'(x)=\frac{-3(5-\sin x)}{(5x+\cos x)^4}$.

TRY IT

Consider the function $g(x)=\sqrt[3]{6x^4+5}$.

Find the derivative. +

$$g'(x) = \frac{8x^3}{(6x^4+5)^{2/3}}$$

↪ EXAMPLE The distance (measured in feet) from a moving camera to an object positioned at the point (1, 4) is given by the function $f(t) = \sqrt{2t^2 - 2t + 1}$, where t is measured in seconds. At what rate is the distance changing after 3 seconds?

Mathematically speaking, we want to compute $f'(3)$.

To find the derivative, we first need to rewrite $f(t)$:

$f(t) = \sqrt{2t^2 - 2t + 1} = (2t^2 - 2t + 1)^{1/2}$	Write the radical as $\frac{1}{2}$ power.
$f'(t) = \frac{1}{2}(2t^2 - 2t + 1)^{-1/2} \cdot D[2t^2 - 2t + 1]$	Apply the power rule.
$f'(t) = \frac{1}{2}(2t^2 - 2t + 1)^{-1/2} \cdot (4t - 2)$	$D[2t^2 - 2t + 1] = 4t - 2$
$f'(t) = \frac{1}{2}(4t - 2) \cdot (2t^2 - 2t + 1)^{-1/2}$	Rearrange the terms.
$f'(t) = (2t - 1) \cdot (2t^2 - 2t + 1)^{-1/2}$	Distribute $\frac{1}{2}(4t - 2) = 2t - 1$
$f'(t) = \frac{2t - 1}{(2t^2 - 2t + 1)^{1/2}}$	Rewrite with nonnegative exponents.

Now, we desire the rate of change when $t = 3$, so we substitute 3.

$$f'(3) = \frac{2(3) - 1}{(2(3)^2 - 2(3) + 1)^{1/2}} = \frac{5}{(13)^{1/2}} \approx 1.39 \text{ feet per second}$$

2. Combining Derivative Rules

Now that we are building up our derivative rules, we can find derivatives of more complex functions.

↪ EXAMPLE Find the derivative of the function $f(x) = 4x\sqrt{2x + 1}$.

At this point, we are conditioned to write radicals as fractional powers (to use the power rule).

$f(x) = 4x\sqrt{2x + 1} = 4x(2x + 1)^{1/2}$	Rewrite the square root as $\frac{1}{2}$ power.
$f'(x) = D[4x] \cdot (2x + 1)^{1/2} + 4x \cdot D[(2x + 1)^{1/2}]$	Apply the product rule.
$f'(x) = 4 \cdot (2x + 1)^{1/2} + 4x \cdot \frac{1}{2}(2x + 1)^{-1/2}(2)$	$D[4x] = 4$, $D[(2x + 1)^{1/2}] = \frac{1}{2}(2x + 1)^{-1/2}(2)$

$$f'(x)=4(2x+1)^{1/2}+4x(2x+1)^{-1/2}$$ $\frac{1}{2}\cdot 2=1$; remove excess symbols.

$$f'(x)=4(2x+1)^{1/2}+\frac{4x}{(2x+1)^{1/2}}$$ Rewrite with positive exponents.

At this point, $f'(x)$ is reasonably simplified. Thus, $f'(x)=4(2x+1)^{1/2}+\frac{4x}{(2x+1)^{1/2}}$.

It is possible to go further by forming a common denominator and combining the fractions. Let's see how this plays out:

$$f'(x)=\frac{4(2x+1)^{1/2}}{1}\cdot\frac{(2x+1)^{1/2}}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$$ The common denominator is $(2x+1)^{1/2}$. Write $4(2x+1)^{1/2}$ over 1 so it "looks" like a fraction.

$$f'(x)=\frac{4(2x+1)}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$$ Perform multiplication. $(2x+1)^{1/2}\cdot(2x+1)^{1/2}=(2x+1)^1=2x+1$

$$f'(x)=\frac{8x+4}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$$ Distribute $4(2x+1)=8x+4$.

$$f'(x)=\frac{12x+4}{(2x+1)^{1/2}}$$ Combine the numerators.

As you can see, the expression simplified nicely to one single fraction. That said, writing $f'(x)=4(2x+1)^{1/2}+\frac{4x}{(2x+1)^{1/2}}$ is equally acceptable.

WATCH

Sometimes factoring is very useful in obtaining a nicer form of the derivative. In the following video, we'll take the derivative of $f(x)=(4x-1)^3(2x+5)^4$ and write it in factored form.

Video Transcription

[MUSIC PLAYING] Hello, and thank you for joining us today. What we're going to look at is an example of how the product rule and the generalized power rule can be used together to find the derivative of a function. So over to the right just a couple reminders.

Remember that the derivative of f times g is the derivative of f times g plus f times the derivative of g. And remember that the derivative of a power of f is n times f to the n minus 1. Notice that kind of looks like the power rule. But then we multiply by the derivative of the inner function, the f.

So here we have f of x equals 4x minus 1 to the 1/3 times 2x plus 5 to the fourth and our goal is to find the derivative of this function. And we're also going to manipulate the derivative algebraically to show that there is a useful form of this derivative. So on the surface what we have is a product.

So we know that f prime of x is equal to-- so it's the derivative of 4x minus 1 to the 1/3 times 2x plus 5 to the fourth plus 4x minus 1 to the 1/3 times the derivative of 2x plus 5 to the fourth. There we go, just barely fit that there too.

OK, so the derivative of 4x minus 1 to the third. So the derivative of something to the third is 3 times the something squared-- that's using this generalized power rule here-- and then multiplied by the derivative of the something. Now the derivative of 4x minus 1 is 4. So I'm going to save times 4, and then that's times 2x plus 5 to the fourth plus-- now we have a 4x minus 1 to the third and the derivative of something to the fourth is 4 times something to the third times the derivative of the inner something which is 2. OK, so that's our product rule as well as our power rule.

OK, so continuing on here let's pull together some like terms here. Notice that here we have a 3 times a 4. That can easily pulled together as 12. That is an equal sign there, 4x minus 1 squared times 2x plus 5 to the fourth plus and over here we have a 4 and a 2. So we're going to call that 8 and then times 4x minus 1 to the 1/3 times 2x plus 5 to the third.

Now, ordinarily that would be considered sufficiently simplified, but we're going to take it one step further because coming soon in this course, we're going to be taking derivatives and setting them equal to 0, which means that the factored form of the derivative is going to be very useful to us as well. So I'm just going to put a little hatch mark here to remind you this is going to be OK for now. But we're going to want to go further later.

So in looking at this, if we're going to factor, we have to factor something common from each side of the addition sign. We have a 12 and an 8. So that's going to be a 4. I'll put the equals here. Looking at the factors of 4x minus 1, we have a square and we have a cube.

So I'm going to call that 4x minus 1 squared. And over here we have powers of 2x plus 5 to the fourth and 2 the third. So I can factor out 2x plus 5 to the third. And now let's see what remains.

Now, since there is a chance for parentheses inside, I'm going to use brackets to set this off. So if I factor of 4 from the 12, that leaves me with a 3. We're factoring out both factors of 4x minus 1. So there's not going to be any 4x minus 1's there.

And we're factoring out 2x plus 5 to the third from 2x plus 5 to the fourth, which is going to leave us with 2x plus 5 and then plus-- now we're factoring a 4 from the 8 which leaves us with two. We're factoring out two factors of 4x minus 1, which leaves us with one behind, because that was a third, and then all three factors of 2x plus 5. So that's done.

And now what we're going to do is simplify within the brackets because, remember, a simplified expression is something that's either completely factored or completely multiplied out. The expression in the brackets is neither. So the easiest way to handle that is to just multiply everything out.

So looking at that now, I'm going to go back to parentheses here because I'm going to simplify below this here. So this is 6x plus 15 plus 8x minus 2. So that looks to me like-- go back to the original color, 14x plus 13. And this is the factored form of the derivative.

[MUSIC PLAYING]

TRY IT

Consider the function $f(x)=(x+1)^4(2x+1)^3$.

Find the derivative and write your final answer in factored form. +

$f'(x)=2(x+1)^3(2x+1)^2(7x+5)$

SUMMARY

In this lesson, you learned how to apply the general power rule for **derivatives of functions, such as the form** $y=[f(x)]^n$. As you develop your repertoire of derivative formulas, you are able to**combine derivative rules** to find derivatives of more complex functions, such as the ones explored in this unit.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D\left[[f(x)]^n\right]=n\cdot[f(x)]^{n-1}\cdot f'(x)$.

Derivatives of Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn the derivatives of the remaining four trigonometric functions, and then incorporate these rules to find derivatives of combinations of functions. Specifically, this lesson will cover:

1. The Derivatives of tan*x*, sec*x*, cot*x*, and csc*x*
2. Summary of the Derivatives of the Trigonometric Functions
3. Derivatives of Combinations of Functions With tan*x*, sec*x*, cot*x*, and csc*x*

1. The Derivatives of tan*x*, sec*x*, cot*x*, and csc*x*

Recall the following identities:

- $\tan x = \frac{\sin x}{\cos x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$
- $\csc x = \frac{1}{\sin x}$

Notice that all four functions are quotients related to sin*x* and/or cos*x*. Since the derivatives of sin*x* and cos*x* are known, establishing rules for the other four functions should be fairly straightforward. The derivatives of each of the remaining trigonometric functions follows.

FORMULA

Derivative of Tangent

$D[\tan x] = \sec^2 x$

$D[\tan x] = D\left[\frac{\sin x}{\cos x}\right]$	Use the identity for tan*x*.
$= \frac{\cos x \cdot D[\sin x] - \sin x \cdot D[\cos x]}{(\cos x)^2}$	Apply the quotient rule.
$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2}$	$D[\sin x] = \cos x$, $D[\cos x] = -\sin x$
$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$	Simplify the numerator and rewrite the denominator.

$= \dfrac{1}{\cos^2 x}$ Use another trigonometric identity: $\sin^2 x + \cos^2 x = 1$

$= \sec^2 x$ Since $\sec x = \dfrac{1}{\cos x}$, $\sec^2 x = \dfrac{1}{\cos^2 x}$.

FORMULA

Derivative of Cotangent

$D[\cot x] = -\csc^2 x$

$D[\cot x] = D\left[\dfrac{\cos x}{\sin x}\right]$ Use the identity for cot*x*.

$= \dfrac{\sin x \cdot D[\cos x] - \cos x \cdot D[\sin x]}{(\sin x)^2}$ Apply the quotient rule.

$= \dfrac{(\sin x)(-\sin x) - \cos x \cdot (\cos x)}{(\sin x)^2}$ $D[\sin x] = \cos x$, $D[\cos x] = -\sin x$

$= \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$ Simplify the numerator and rewrite the denominator.

$= \dfrac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x}$ Factor out -1 from the numerator.

$= \dfrac{-1}{\sin^2 x}$ Use another trigonometric identity: $\sin^2 x + \cos^2 x = 1$

$= -\csc^2 x$ Since $\csc x = \dfrac{1}{\sin x}$, $\csc^2 x = \dfrac{1}{\sin^2 x}$.

FORMULA

Derivative of Secant

$D[\sec x] = \sec x \tan x$

$D[\sec x] = D\left[\dfrac{1}{\cos x}\right]$ Use the identity for sec*x*.

$= \dfrac{\cos x \cdot D[1] - 1 \cdot D[\cos x]}{(\cos x)^2}$ Apply the quotient rule.

$= \dfrac{\cos x(0) - 1 \cdot (-\sin x)}{(\cos x)^2}$ $D[1] = 0$, $D[\cos x] = -\sin x$

$= \dfrac{\sin x}{\cos^2 x}$ Simplify the numerator and rewrite the denominator.

At this point, it might appear that we should stop. However, there is a slightly simpler way to express this by using some trigonometric identities:

$= \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x}$ Split the fraction into two factors.

$= \tan x \cdot \sec x$ Use these trigonometric identities: $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$

$= \sec x \tan x$ It's more common to place sec*x* first.

The final result of $\sec x \tan x$ is easier to remember than $\frac{\sin x}{\cos^2 x}$, even though they are equivalent.

FORMULA

Derivative of Cosecant

$D[\csc x] = -\csc x \cot x$

$D[\csc x] = D\left[\frac{1}{\sin x}\right]$ Use the identity for csc*x*.

$= \frac{\sin x \cdot D[1] - 1 \cdot D[\sin x]}{(\sin x)^2}$ Apply the quotient rule.

$= \frac{\sin x(0) - 1 \cdot (\cos x)}{(\sin x)^2}$ $D[1] = 0$, $D[\sin x] = \cos x$

$= \frac{-\cos x}{\sin^2 x}$ Simplify the numerator and rewrite the denominator.

At this point, it might appear that we should stop. However, there is a slightly simpler way to express this by using some trigonometric identities:

$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$ Split the fraction into two factors.

$= -\cot x \csc x$ Use these trigonometric identities: $\cot x = \frac{\cos x}{\sin x}$, $\csc x = \frac{1}{\sin x}$

$= -\csc x \cot x$ It's more common to place csc*x* first.

The final result of $-\csc x \cot x$ is easier to remember than $\frac{-\cos x}{\sin^2 x}$, even though they are equivalent.

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2. Summary of the Derivatives of the Trigonometric Functions

Here are all the trigonometric functions and their derivatives:

- $D[\sin x] = \cos x$
- $D[\cos x] = -\sin x$
- $D[\tan x] = \sec^2 x$
- $D[\sec x] = \sec x \tan x$
- $D[\csc x] = -\csc x \cot x$
- $D[\cot x] = -\csc^2 x$

Notice these similarities:

- tan*x* and cot*x* have similar derivatives.
- sec*x* and csc*x* also have similar derivatives.
- If the function name begins with a "c", then its derivative has a negative sign.

3. Derivatives of Combinations of Functions with tan*x*, sec*x*, cot*x*, and csc*x*

Now that we have more derivative rules, we can see how they are used when combined with other functions.

↪ EXAMPLE Find the derivative: $y=4x^2-2\csc x+\frac{6}{x^2}$

First, rewrite $\frac{6}{x^2}$ as $6x^{-2}$ so that the power rule can be used.

Then, your function is: $y=4x^2-2\csc x+6x^{-2}$

$y'=D[4x^2]-D[2\csc x]+D[6x^{-2}]$	Use the sum/difference rules.
$y'=4D[x^2]-2D[\csc x]+6D[x^{-2}]$	Apply the constant multiple rules.
$y'=4(2x)-2(-\csc x\cot x)+6(-2)x^{-3}$	Apply the power rule and $D[\csc x]=-\csc x\cot x$.
$y'=8x+2\csc x\cot x-12x^{-3}$	Simplify.
$y'=8x+2\csc x\cot x-\frac{12}{x^3}$	Write the last term with a positive exponent.

Thus, $y'=8x+2\csc x\cot x-\frac{12}{x^3}$.

↪ EXAMPLE Find the derivative of the function $f(x)=3x\tan x$.

This is the product of two functions, so the product rule will be used:

$f'(x)=D[3x]\cdot\tan x+3x\cdot D[\tan x]$	Apply the product rule.
$f'(x)=3\cdot\tan x+3x\cdot\sec^2 x$	$D[3x]=3$, $D[\tan x]=\sec^2 x$
$f'(x)=3\tan x+3x\sec^2 x$	Eliminate unnecessary symbols.

Thus, $f'(x)=3\tan x+3x\sec^2 x$.

↪ EXAMPLE Find the derivative of the function $f(x)=4\sec^2 x$.

Since $4\sec^2 x=4(\sec x)^2$, we can use the general power rule.

$f'(x)=4\cdot D[(\sec x)^2]$	Use the constant multiple rule.
$f'(x)=4\cdot 2(\sec x)^1\cdot\sec x\tan x$	Apply the general power rule: $D[u^2]=2u\cdot u'$ $D[\sec x]=\sec x\tan x$

$$f'(x) = 8\sec x \cdot \sec x \tan x$$ Simplify.
$$f'(x) = 8\sec^2 x \tan x$$

Thus, $f'(x) = 8\sec^2 x \tan x$.

TRY IT

Consider the function $f(x) = 4x + x\cot x$.

Find the derivative. +

$f'(x) = 4 + \cot x - x\csc^2 x$

SUMMARY

In this lesson, you learned **the derivatives of tan*x*, sec*x*, cot*x*, and csc*x*,** the four remaining trigonometric functions, followed by a **summary of all six derivatives of the trigonometric functions**. By adding these to the mix of functions you can take derivatives of using basic rules instead the limit definition, you explored how they can be used to find the **derivatives of combinations of functions involving tan*x*, sec*x*, cot*x*, and csc*x*.**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of Cosecant
$D[\csc x] = -\csc x \cot x$

Derivative of Cotangent
$D[\cot x] = -\csc^2 x$

Derivative of Secant
$D[\sec x] = \sec x \tan x$

Derivative of Tangent
$D[\tan x] = \sec^2 x$

Higher-Order Derivatives

by Sophia

WHAT'S COVERED

In this lesson, you will take all the rules you have learned about derivatives and apply them to higher-order derivatives. There are situations in which the change in the rate of change is important and useful. Specifically, this lesson will cover:

1. Higher-Order Derivatives: Definitions and Notation
2. Finding Higher-Order Derivatives
3. Interpreting Higher-Order Derivatives

1. Higher-Order Derivatives: Definitions and Notation

When you take the derivative of a function, you actually are taking the first derivative of a function. That said, the second derivative of a function is the derivative of the first derivative. Then, the third derivative is the derivative of the second derivative, and so on.

Here is a table that shows past notations used with first derivatives, as well as the corresponding notation used for higher-order derivatives.

Function Representation	*y*-notation		$f(x)$-notation	
1st Derivative	y'	$\frac{dy}{dx}$	$f'(x)$	$D[f(x)]$
2nd Derivative	y''	$\frac{d^2y}{dx^2}$	$f''(x)$	$D^2[f(x)]$
3rd Derivative	y'''	$\frac{d^3y}{dx^3}$	$f'''(x)$	$D^3[f(x)]$
4th Derivative	$y^{(4)}$	$\frac{d^4y}{dx^4}$	$f^{(4)}(x)$	$D^4[f(x)]$
n^{th} Derivative	$y^{(n)}$	$\frac{d^ny}{dx^n}$	$f^{(n)}(x)$	$D^n[f(x)]$

HINT

Note that in the “prime” notations, the order of the derivative is written as a number enclosed in parentheses. Anything more than three prime symbols would be difficult to read.

2. Finding Higher-Order Derivatives

To find a second derivative, we need to find the first derivative, then differentiate a second time.

↪ EXAMPLE Find the second derivative of the function $f(x)=3x^4-9x^3+3x^2+8x-12$.

First derivative:

$f'(x)=3(4x^3)-9(3x^2)+3(2x)+8(1)-0$	Apply the sum/difference and constant multiple rules.
$f'(x)=12x^3-27x^2+6x+8$	Simplify.

Second derivative:

$f''(x)=D[f'(x)]=D[12x^3-27x^2+6x+8]$	Substitute the first derivative.
$f''(x)=12(3x^2)-27(2x)+6+0$	Apply the sum/difference and constant multiple rules.
$f''(x)=36x^2-54x+6$	Simplify.

Thus, $f''(x)=36x^2-54x+6$.

↪ EXAMPLE Find the 5th derivative of the function $y=4\cos x$.

Note that in each step, the constant multiple rule is used.

$y=4\cos x$	Start with the original function, $f(x)$.
$y'=4(-\sin x)=-4\sin x$	Find the first derivative.
$y''=-4(\cos x)=-4\cos x$	Find the second derivative.
$y'''=-4(-\sin x)=4\sin x$	Find the third derivative.
$y^{(4)}=4(\cos x)=4\cos x$	Find the fourth derivative (notice this is the same as $f(x)$).
$y^{(5)}=4(-\sin x)=-4\sin x$	Lastly, find the fifth derivative (notice this is the same as $f'(x)$).

Thus, $y^{(5)}=4(-\sin x)=-4\sin x$.

TRY IT

Consider the function $y=10x^2+2\sin x$.

Find the 6th derivative of the above formula. +

$y^{(6)}=-2\sin x$

↱ EXAMPLE Find the third derivative of $f(x)=4x^2-\frac{4}{x^3}$.

First, rewrite using negative exponents to make use of the power rule: $f(x)=4x^2-\frac{4}{x^3}=4x^2-4x^{-3}$

Now, take the appropriate derivatives:

$f'(x)=4(2x)-4\left(-3x^{-4}\right)$	Apply the sum/difference and power rules.
$f'(x)=8x+12x^{-4}$	Simplify.

Since we are finding more derivatives, there is no need to rewrite with positive exponents just yet. We will save this for when all derivatives are taken.

$f''(x)=8+12\left(-4x^{-5}\right)$	Apply the sum/difference and power rules.
$f''(x)=8-48x^{-5}$	Simplify.

And now the third derivative:

$f'''(x)=0-48\left(-5x^{-6}\right)$	Apply the sum/difference and power rules.
$f'''(x)=240x^{-6}$	Simplify.
$f'''(x)=\frac{240}{x^6}$	Rewrite with positive exponents.

Thus, $f'''(x)=\frac{240}{x^6}$.

Sometimes different rules are needed to find higher-order derivatives.

↱ EXAMPLE Find the second derivative of the function $y=5\tan x$.

First derivative: $\frac{dy}{dx}=D[5\tan x]=5\sec^2 x$

Second derivative:

$\frac{d^2y}{dx^2}=D\left[5\sec^2 x\right]=5D\left[\sec^2 x\right]$	Apply the constant multiple rule.
$\frac{d^2y}{dx^2}=5(2\sec x\cdot D[\sec x])$	Apply the power rule.
$\frac{d^2y}{dx^2}=5(2\sec x\cdot\sec x\tan x)$	$D[\sec x]=\sec x\tan x$

$\frac{d^2y}{dx^2} = 10\sec^2 x\tan x$ Perform multiplications.

Thus, $\frac{d^2y}{dx^2} = 10\sec^2 x\tan x$.

TRY IT

Consider the function $f(x) = 2\sec x$.

Find the second derivative. +

$f''(x) = 2\sec x\tan^2 x + 2\sec^3 x$

3. Interpreting Higher-Order Derivatives

Recall that $f'(x)$ is the (instantaneous) rate of change of $f(x)$. Then:

- $f''(x)$ is the rate of change of $f'(x)$.
- $f'''(x)$ is the rate of change of $f''(x)$.
- And so on...

The interpretation of higher-order derivatives depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then $h'(t)$ is the change in distance with respect to time. This is called the **velocity** of the object.
Furthermore, we are also interested in how the velocity changes, which is $h''(t)$. This is called the **acceleration** of the object. The rate of change in acceleration, called **jerk**, is $h'''(t)$.

↱ EXAMPLE A tennis ball is launched off the top of a building. Its height (in feet) after t seconds is modeled by the function $h(t) = -16t^2 + 20t + 40$. Let's find the height, velocity, and acceleration at the precise moment the object was in the air for 2 seconds.

Remember units:
- $h(t)$ = height after t seconds
- $h'(t)$ = velocity = change in height per second (feet/sec)
- $h''(t)$ = acceleration = change in velocity per second (ft/sec)/sec

Height: $h(2) = -16(2)^2 + 20(2) + 40 = 16$ feet

Velocity: $h'(t) = -32t + 20$
Then, $h'(2) = -32(2) + 20 = -44$ feet per second

Acceleration: $h''(t) = -32$

This means that $h''(t) = -32$ regardless of t. Therefore, $h''(2) = -32$ feet per second per second, meaning that the velocity is changing by -32 feet per second each second.

WATCH

In this video, we will examine the rate of change in the derivative (slope) of $f(x) = \sin^2 x$ at $x = 0$.

Video Transcription

[MUSIC PLAYING] Hello there. Good to see you again. What we're going to do in this video is investigate the rate of change of the slope of the graph of f of x equals sine squared x at x equals 0. Now, remember, the slope of the graph is another way of saying the slope of the tangent line of the graph. So looking at our graph here, now, when x equals 0, to the left we have negative slopes, and to the right we have positive slopes.

So the slopes are definitely changing. But what is the measure that we need to look at in order to look at the slope of the rate of change in the slope. And as we look at this, we know the slope itself is f prime, and the rate of change means I'm looking at a derivative. So it's the derivative of f prime, which is f double prime.

So the second derivative tells us information about how the slopes of the tangent lines are changing, and we want to know what's happening at 0. So the answer to this question is when we substitute 0 into f double prime. So you have a little bit of work ahead of us here.

Now for the function sine square root of x we know in order to use derivatives we like to write this as sine of x quantity squared. So that's going to be where we start, sine of x quantity squared. So f prime of x is equal to-- well, the derivative of something squared is 2 times the something times the derivative of the inside.

And there is basically nothing to do there. There is no way to rewrite that in order to move forward. So what we're going to do at this point is treat this as a product rule. I'm going to treat 2 sine x as the first function and cosine x as the second function.

Now what I like to do to organize my work is to take the derivative of each piece and put it pretty close to the original function there. So the derivative of 2 sine x is 2 cosine x. And the derivative of cosine x is negative sine x.

And then, remember, the derivative of a product is the crisscross pattern that I'm about to show you here. So it's the derivative of the first times the second. So we have 2 cosine x times cosine x plus the first times the derivative of the second.

So we have 2 sine x times negative sine x. And, yes, we are going to clean this up just a little bit here. So this is 2 cosine squared x minus 2 sine squared x. Remember, that's our second derivative. That's the function that we're looking for here. So then add 0.

What's happening here? I have two. Now cosine of 0 is 1. So it's basically 2 times 1 squared. Sine of 0 is 0. So that's 2 times 0 squared, and that just evaluates to be two. So that means at that moment, the slope is changing by 2 units for every unit that we increase on x.

And as you look at this, it makes sense because the slopes are going from negative to positive. So the second derivative at 0 really should be a positive number. And this confirms that it is positive. So there we have it.

TERMS TO KNOW

Velocity
An object's change in distance with respect to time.

Acceleration
An object's change in velocity with respect to time.

Jerk
An object's change in acceleration with respect to time.

SUMMARY

In this lesson, you learned the **definitions and notation of higher-order derivatives**, understanding that **finding higher-order derivatives** involves using rules of derivatives repeatedly. You learned that the **interpretation of higher-order derivatives** depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then the first derivative, $h'(t)$, is the velocity of the object; the second derivative, $h''(t)$, is the acceleration of the object; and the third derivative, $h'''(t)$, is the rate of change in acceleration, or jerk. We will explore the geometrical meanings of the first and second derivatives in unit 4.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Acceleration
An object's change in velocity with respect to time.

Jerk
An object's change in acceleration with respect to time.

Velocity
An object's change in distance with respect to time.

The Chain Rule

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to find derivatives of general composite functions by using the chain rule. In a previous tutorial, you learned how to find derivatives of $y=(f(x))^n$, which is a composite function, but this doesn't cover all composite functions. For example, we are still unable to find the derivative of functions such as $f(x)=\sin(3x)$ and $y=\tan(x^2)$. In this tutorial, we'll learn the techniques necessary to find derivatives of said functions. Specifically, this lesson will cover:

1. Motivation for the Chain Rule
    a. Composite Functions
    b. Examining Rates of Change
2. Applying the Chain Rule
    a. Basic Functions
    b. Applying the Chain Rule Twice
    c. Combining the Chain Rule With Other Rules

1. Motivation for the Chain Rule

1a. Composite Functions

Recall that a composite function has the form $y=f(g(x))$. We call $g(x)$ the "inner function" since it is plugged into the function *f*.

To find derivatives of composite functions, it will be helpful to first identify the inner function. Given $y=f(g(x))$, let $u=g(x)$. Then $y=f(u)$, a less complicated function. Here are a few examples:

Function	Inner Function	Written in Terms of *u*
$y=\sqrt{2x^2+5x}$	$u=2x^2+5x$	$y=\sqrt{u}$
$y=\sin(3x)$	$u=3x$	$y=\sin u$
$y=\dfrac{3}{(2+\sin x)^4}$	$u=2+\sin x$	$y=\dfrac{3}{u^4}$

1b. Examining Rates of Change

The chain rule is a derivative technique that uses several rates of change in one problem. Here is a real-life consideration:

A factory can produce 30 units of a certain item per hour at a cost of $20 per item. What is the cost per hour of producing the items?

If you think the answer is $600, great! That is correct. But let's look at this more closely so that we can understand the rates of change.

Let x = the number of hours, u = the number of units produced, and C, the cost.

We can translate the given information into rates of change:

- The factory can produce 30 units per hour: Slope $= \frac{\Delta u}{\Delta x} = 30$
- The units cost $20 each to produce: Slope $= \frac{\Delta C}{\Delta u} = 20$

Then, the cost per hour is $\frac{\Delta C}{\Delta x}$, which can be written as $\frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = (20)(30) = \$600$ per hour.

Generally speaking, let's say that C is a function of u, and u is a function of x. Then, C is also a function of x, and $\frac{\Delta C}{\Delta x} = \frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$.

While not a proof, this idea can be extended to derivatives (instantaneous rates of change). That is, given C is a function of u and u is a function of x, $\frac{dC}{dx} = \frac{dC}{du} \cdot \frac{du}{dx}$. This derivative rule applies to composite functions and is called the chain rule.

---

2. Applying the Chain Rule

2a. Basic Functions

The chain rule can be expressed with the following formula:

FORMULA

Chain Rule

Suppose $y = f(u)$, a composite function, where u is a function of x.

Then, $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

Using "prime" notation, we can write $y' = f'(u) \cdot u'$.

Using "D" notation, we can write $D[y] = f'(u) \cdot D[u]$.

BIG IDEA

With the chain rule in mind, we can write the derivative of each basic function.

- $D[u^n] = n \cdot u^{n-1} \cdot D[u]$
- $D[\sin u] = \cos u \cdot D[u]$
- $D[\cos u] = -\sin u \cdot D[u]$
- $D[\tan u] = \sec^2 u \cdot D[u]$
- $D[\cot u] = -\csc^2 u \cdot D[u]$

- $D[\sec u] = \sec u \tan u \cdot D[u]$
- $D[\csc u] = -\csc u \cot u \cdot D[u]$

HINT

$D[u^n] = n \cdot u^{n-1} \cdot D[u]$ is the power rule from **The General Power Rule for Functions**.

EXAMPLE Consider the function $y = \sin(3x)$. Find its derivative.

$y = \sin u$	Let $u = 3x$, the inner function.
$\frac{dy}{dx} = \cos u \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{dy}{dx} = \cos u \cdot 3$	$D[u] = D[3x] = 3$
$\frac{dy}{dx} = 3\cos 3x$	Rewrite “3” in front and replace u with $3x$.

Thus, $\frac{dy}{dx} = 3\cos 3x$.

In challenge 3.2, you took derivatives of functions of the form $y = [f(x)]^n$, which is a specific form of the chain rule ($y = u^n$ where $u = f(x)$). Here is a reminder of a power function that requires the chain rule.

EXAMPLE Consider the function $y = \sqrt[3]{4x^2 + \sin x}$. Find its derivative.

$y = \sqrt[3]{u} = u^{1/3}$	Let $u = 4x^2 + \sin x$, the inner function.
$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot (8x + \cos x)$	$D[u] = D\left[4x^2 + \sin x\right] = 8x + \cos x$
$\frac{dy}{dx} = \frac{1}{3}(4x^2 + \sin x)^{-2/3}(8x + \cos x)$	Replace u with $4x^2 + \sin x$.
$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{(4x^2 + \sin x)^{2/3}}(8x + \cos x)$	Rewrite with positive exponents.
$\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$	Combine fractions.

Thus, $\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$.

EXAMPLE Consider the function $y = \tan(x^2 + 1)$. Find its derivative.

$y = \tan u$	Let $u = x^2 + 1$, the inner function.
$\frac{dy}{dx} = \sec^2 u \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = f'(u) \cdot D[u]$

$\frac{dy}{dx} = (\sec^2 u) \cdot 2x$ $\quad D[u] = D[x^2+1] = 2x$ (parentheses added to show separation)

$\frac{dy}{dx} = 2x \cdot \sec^2(x^2+1)$ $\quad$ Rewrite "2*x*" in front and replace *u* with x^2+1.

Thus, $\frac{dy}{dx} = 2x \cdot \sec^2(x^2+1)$.

TRY IT

Consider the function $y = \cos(2x^3)$.

Find its derivative.

+

$$\frac{dy}{dx} = -6x^2 \sin(2x^3)$$

WATCH

Here is a video in which we find the derivative of $y = \sec(3x+1)$.

Video Transcription

[MUSIC PLAYING] Hello there. Good to see you again. What we're going to do in this video is find the derivative of a composite function where the secant function is involved. So we have f of x equals the secant of 3x plus 1. And we want to find its derivative.

Now since there's a more complicated expression inside of the secant function, this requires the chain rule. So let's just remember the derivative rule for secant of a function u. So the derivative of secant is secant tangent. So that part is consistent with the derivative rule we learned a while ago. But then chain rule says I take that and I multiply by the derivative of u, so the derivative of the inside.

So that means here f prime of x is equal to-- so it's the secant of 3x plus 1 times the tangent of 3x plus 1 and then times the derivative of 3x plus 1. So if u is equal to 3x plus 1, this means that u prime is equal to 3. And then one thing we normally do, we write the simpler factors out front followed by the more complicated factors, which are, in this case, the trig functions. And this is the derivative of f of x equals secant of 3x plus 1.

[MUSIC PLAYING]

2b. Applying the Chain Rule Twice

When the inner function is itself a composite function, the chain rule is applied more than once. Rather than assign a letter to each inside function, we'll present another way to organize this.

→ EXAMPLE Consider the function $y = \sin^2 3x$. Find its derivative.

$$y = (\sin(3x))^2$$ Rewrite as a quantity squared.

$$\frac{dy}{dx} = 2(\sin(3x)) \cdot D[\sin(3x)]$$ Apply the chain rule: $D[u^2] = 2u \cdot u'$

$$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot D[3x]$$ Apply the chain rule again: $D[\sin u] = \cos u \cdot u'$

$$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot 3$$ $D[3x] = 3$

$$\frac{dy}{dx} = 6\sin 3x \cos 3x$$ $2 \cdot 3 = 6$; omit unnecessary parentheses.

Thus, $\frac{dy}{dx} = 6\sin 3x \cos 3x$.

TRY IT

Consider the function $y = \tan^4 2x$.

Find its derivative. +

$$\frac{dy}{dx} = 8\tan^3 2x \sec^2 2x$$

2c. Combining the Chain Rule With Other Rules

Sometimes sums, differences, products, and quotients contain composite functions. The key is to approach the derivative carefully.

EXAMPLE Consider the function $y = x\sin 4x$. Find its derivative.

$$y = x\sin 4x$$ Start with the given function.

$$\frac{dy}{dx} = D[x] \cdot \sin 4x + x \cdot D[\sin 4x]$$ Apply the product rule.

$$\frac{dy}{dx} = 1 \cdot \sin 4x + x \cdot \cos 4x \cdot D[4x]$$ $D[x] = 1$, $D[\sin u] = \cos u \cdot D[u]$

$$\frac{dy}{dx} = 1 \cdot \sin 4x + x \cdot \cos 4x \cdot (4)$$ $D[4x] = 4$

$$\frac{dy}{dx} = \sin 4x + 4x\cos 4x$$ Omit unnecessary grouping symbols.

Thus, $\frac{dy}{dx} = \sin 4x + 4x\cos 4x$.

SUMMARY

In this lesson, you began by understanding the **motivation for the chain rule**, a derivative technique used to compute derivatives of **composite functions**, namely that it expands your ability to find **rates of change**. You learned how to **apply the chain rule**, exploring the formula and the derivative of each **basic function**. You also learned that when the inner function is itself a composite function, you need to **apply the chain rule twice**. Lastly, you examined an example of finding a derivative by **combining the chain rule with other rules**, such as the product rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Chain Rule

Suppose $y=f(u)$, a composite function, where *u* is a function of *x*.

Then, $\frac{dy}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$.

Using "prime" notation, we can write $\frac{dy}{dx}=f'(u)\cdot u'$.

Using "D" notation, we can write $\frac{dy}{dx}=f'(u)\cdot D[u]$.

Derivative of y = e^x

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to take derivatives of exponential functions. These functions are important since they model population growth, the decay of materials, the temperature change of an object, and much more. Specifically, this lesson will cover:

1. Derivatives of $y=e^x$ and Combinations of Functions With $y=e^x$
2. Derivatives of $y=e^u$ and Combinations of Functions With $y=e^u$, Where *u* Is a Function of *x*

1. Derivatives of $y = e^x$ and Combinations of Functions With $y = e^x$

Consider the function $f(x)=e^x$. Using the limit definition of the derivative, we can find $f'(x)$.

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	Use the limit definition of the derivative.
$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$	$f(x)=e^x$, $f(x+h)=e^{x+h}$
$f'(x)=\lim_{h\to 0}\frac{e^x e^h-e^x}{h}$	Apply the property of exponents: $e^a e^b=e^{a+b}$
$f'(x)=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}$	Remove the common factor of e^x.
$f'(x)=e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}$	Since e^x is a constant relative to *h*, it can be factored outside the limit.

Now, let's focus on the limit, which cannot be manipulated algebraically (there is no way to simplify e^h-1). Thus, we will use a table and hopefully be able to get a nice approximation for the limit. The following table shows the behavior of $\frac{e^h-1}{h}$ as $h\to 0$.

h	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$\frac{e^h-1}{h}$	0.95163	0.99502	0.99950	0.99995	—	1.00005	1.00050	1.00502	1.05171

The table suggests that $\lim_{h\to 0}\frac{e^h-1}{h}=1$. Note that this is not a formal proof, but it is convincing.

It follows that $f'(x) = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^x(1) = e^x$, which means that e^x is its own derivative!

FORMULA

The Derivative of e^x

$D[e^x] = e^x$

Now we'll incorporate this new derivative rule into the others we already know.

EXAMPLE Consider the function $f(x) = -2x^2 + 3e^x + 6$. Find its derivative.

$f(x) = -2x^2 + 3e^x + 6$	Start with the original function.
$f'(x) = D[-2x^2] + D[3e^x] + D[6]$	Use the derivative of a sum/difference rule.
$f'(x) = -2D[x^2] + 3D[e^x] + D[6]$	Use the constant multiple rule.
$f'(x) = -2(2x) + 3(e^x) + 0$	$D[x^2] = 2x$, $D[e^x] = e^x$, $D[6] = 0$
$f'(x) = -4x + 3e^x$	Simplify.

Thus, $f'(x) = -4x + 3e^x$.

EXAMPLE Write the equation of the line tangent to the graph of $f(x) = \frac{e^x - 1}{e^x + 1}$ at $x = 0$.

Recall that the equation of a tangent line at $x = 0$ is $y = f(0) + f'(0)(x - 0)$. $f'(0)$ will be computed once we find the derivative.

$$f(0) = \frac{e^0 - 1}{e^0 + 1} = \frac{1 - 1}{1 + 1} = 0$$

Let's find $f'(x)$:

$f(x) = \frac{e^x - 1}{e^x + 1}$	Start with the original function.
$f'(x) = \frac{(e^x + 1) \cdot D[e^x - 1] - (e^x - 1) \cdot D[e^x + 1]}{(e^x + 1)^2}$	Apply the quotient rule.
$f'(x) = \frac{(e^x + 1) \cdot e^x - (e^x - 1) \cdot e^x}{(e^x + 1)^2}$	$D[e^x - 1] = D[e^x] - D[1] = e^x$ $D[e^x + 1] = D[e^x] + D[1] = e^x$
$f'(x) = \frac{e^{2x} + e^x - e^{2x} + e^x}{(e^x + 1)^2}$	Distribute $e^x e^x = e^{x+x} = e^{2x}$.
$f'(x) = \frac{2e^x}{(e^x + 1)^2}$	Combine like terms.

Then, the slope is $f'(0)=\frac{2e^0}{(e^0+1)^2}=\frac{2(1)}{(1+1)^2}=\frac{2}{4}=\frac{1}{2}$.

So, the equation of the tangent line is $y=0+\frac{1}{2}(x-0)$, which simplifies to $y=\frac{1}{2}x$.

WATCH

Consider the function $f(x)=\sqrt{e^x+4x}$. Find its derivative.

Video Transcription

[MUSIC PLAYING] [MUSIC PLAYING]

Hi there. Let's continue our journey with derivatives. What we have here is f of x equals the square root of e to the x plus 4x, and we wish to find its derivative. And again, since this is a composite function, the chain rule is going to be required. Now, remember with square roots, we usually fare better if it's written as an exponent. And remember that the square root is really a 1/2 power.

So we have f of x equals e to the x plus 4x raised to the 1/2 power. So to take the derivative, we just have to remember that the derivative of u to the half is 1/2 u to the negative 1/2, and then times the derivative of the inside. So we're going to keep that in mind when we take this derivative here.

So f prime of x is equal to 1/2-- keep the inside the same-- to the negative 1/2. And then times the derivative of e to the x plus 4x, which I'm going to write in D notation. And now let's simplify. So you have 1/2 e to the x plus 4x to the negative 1/2 times-- now, the derivative of e to the x plus 4x. the derivative of e to the x is e to the x.

And the derivative of 4x is 4. So we have that. And now following the convention, I'm going to write the simpler factor out front with the 1/2. And I'm going to have e to the x plus 4x to the negative 1/2. Now, that is an OK answer. But if we desire to have positive exponents in the answer, we do have to keep going.

So just remembering what all this means. We have 1/2 times e to the x plus 4. Something to the negative power is 1 over the same something to the positive power. So we have that. And then I can write that as one cohesive fraction. So remember that the e to the x plus 4 here is really over 1.

So we really have e to the x plus 4 over 2 times e to the x plus 4x to the 1/2. And if we really desire, we can convert that 1/2 power to a square root. So this final answer is acceptable, as is this one. And if you only care about the answer without writing it in positive exponents, that could also be acceptable as well. Just depends on your instructions when finding derivatives. And there we have it.

[MUSIC PLAYING]

TRY IT

Find the slope of the tangent line to the function $g(x) = xe^x$ when $x = 2$.

What is the slope?

+

Recall that the slope of a tangent line to a function at a point is the derivative value at that point.

Therefore, the slope of the tangent line when $x = 2$ is $g'(2) = 3e^2$.

---

2. Derivatives of *y* = *e*^*u* and Combinations of Functions With *y* = *e*^*u*, Where *u* Is a Function of *x*

As a result of the chain rule, we have the following derivative rule for e^u:

FORMULA

The Derivative of e^u, Where *u* Is a Function of *x*

$D[e^u] = e^u \cdot u'$

EXAMPLE Consider the function $f(x) = e^{-x^2}$. Find its derivative.

$f(x) = e^{-x^2}$	Start with the original function.
$f'(x) = e^{-x^2}(-2x)$	$u = -x^2,\ u' = -2x$ $f'(x) = e^u \cdot u'$
$f'(x) = -2xe^{-x^2}$	Write "-2*x*" in front. It is conventional to write the simpler factors before the exponential function.

Thus, $f'(x) = -2xe^{-x^2}$.

TRY IT

Consider the function $f(x) = e^{2\sin x}$.

Find its derivative.

+

$f'(x) = 2cosx \cdot e^{\backslash 2sinx}$

EXAMPLE Consider the function $f(x) = x^2e^{-3x}$. Find its derivative.

$f(x) = x^2e^{-3x}$	Start with the original function.
$f'(x) = D[x^2] \cdot e^{-3x} + x^2 \cdot D[e^{-3x}]$	Use the product rule.

$$f'(x) = 2x \cdot e^{-3x} + x^2 \cdot e^{-3x}(-3) \qquad D[x^2] = 2x$$
$$D[e^u] = e^u \cdot u' \text{ (with } u = -3x)$$
$$D[e^{-3x}] = e^{-3x}(-3)$$

$$f'(x) = 2xe^{-3x} - 3x^2e^{-3x} \qquad \text{Simplify.}$$

Now let's see how these rules are applied within combinations of functions.

↱ EXAMPLE After being attached to a spring, the height of an object (in feet) is modeled by the function $f(t) = 4e^{-2t}\cos(3t)$, where t is the number of seconds since the object was set into motion. Find the initial velocity, which is the velocity when $t = 0$.

Recall that the velocity is $f'(t)$.

$f(t) = 4e^{-2t}\cos(3t)$	Start with the original function.
$f'(t) = D[4e^{-2t}] \cdot \cos(3t) + 4e^{-2t} \cdot D[\cos(3t)]$	Use the product rule.
$f'(t) = 4(e^{-2t})(-2) \cdot \cos(3t) + 4e^{-2t}(-\sin(3t))(3)$	$D[e^u] = e^u \cdot u'$ (with $u = -2t$) $D[\cos u] = -\sin u \cdot u'$ (with $u = 3t$)
$f'(t) = -8e^{-2t}\cos(3t) - 12e^{-2t}\sin(3t)$	Simplify.

Then, the velocity when $t = 0$ is $f'(0) = -8e^{-2(0)}\cos(3 \cdot 0) - 12e^{-2(0)}\sin(3 \cdot 0) = -8(1)(1) - 12(1)(0) = -8$ ft/s.

TRY IT

Consider the function $f(t) = \sin(2e^{-4t})$.

Find its derivative. +

$$f'(t) = -8e^{-4t}\cos(2e^{-4t})$$

SUMMARY

In this lesson, you learned how to take **derivatives of exponential functions (**$y = e^x$**) and combinations of functions with** $y = e^x$. You learned that the function $f(x) = e^x$ is quite unique since it is its own derivative. You also learned the **derivative rule for** $y = e^u$, as a result of the chain rule, as well as **combinations of functions with** $y = e^u$**, where *u* is a function of *x*.** Knowing both of these derivatives enables you to expand on the types of functions whose derivatives can be found. In a future challenge, we will be exploring more applications in which exponential functions are involved.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

The Derivative of e^x

$D[e^x] = e^x$

The Derivative of e^u, Where u Is a Function of x

$D[e^u] = e^u \cdot u'$

Derivative of y = a^x

by Sophia

WHAT'S COVERED

In this lesson, you will find derivatives of exponential functions with any base. For example, $f(x) = 2^x$, $g(x) = 3^{-x^2}$, and $A(t) = \left(\frac{1}{2}\right)^{t/300}$. Sometimes it is more convenient to model situations with bases other than *e*, so it is important that we learn about the derivatives of $y = a^x$ and $y = a^u$. Specifically, this lesson will cover:

1. Derivatives of $y = a^x$ and Combinations of Functions With $y = a^x$
2. Derivatives of $y = a^u$ and Combinations of Functions With $y = a^u$, Where *u* Is a Function of *x*

1. Derivatives of *y* = *a*^{x} and Combinations of Functions With *y* = *a*^{x}

WATCH

Please view this video to see how we arrive at the derivative formula for $f(x) = a^x$, where $a > 0$.

Video Transcription

[MUSIC PLAYING] Hello. And in this video, we're going to take a little bit of a different track. We're going to derive a derivative formula based on facts we already know. So here, we have f of x equals a to the x. So a could be any base. It doesn't have to be e. But we're going to use the fact that we know the derivative of e to the x to find the derivative of a to the x.

So the first thing to realize is that e to the natural log of a is equal to a. That's one of the logarithm properties discussed way back in unit 1. And we're going to replace a with e to the natural log of a. So we have e to the natural log of a raised to the x. And then by properties of exponents, we know that that's equal to e to the natural log of a times x. So all three of these things are equivalent, and that's going to be important later on in the problem.

So we know that the derivative of e to the u is e to the u times the derivative of u, And we're going to use that fact to find the derivative of this expression. So we have f prime of x is equal to-- now, remember, this natural log of a is basically just a constant. So we're going to use that when we find our derivative. So that means the derivative of e to the natural log of a times x is e to the natural log of a times x-- e to the something-- times the derivative of the something, OK?

So that's equal to-- and I'm going to leave this the way it is-- e to the natural log of a times x times-- now, remember, this is a constant. So that means we're basically taking the derivative of a constant times x, and that is just the constant. And now to finish this off, remember that e to the natural order of a times x, which is right here, is equivalent to a to the x. All three of those things in the first three lines or equivalent. So I'm going to rewrite that as a to the x, because, one, it's simpler, and it's related to what was written as the primary function, which was a to the x. So that means that the derivative is a to the x times the natural log of a.

[MUSIC PLAYING]

So, we can say the derivative of a^x can be expressed with the following formula:

FORMULA

The Derivative of a^x

$D[a^x] = a^x \cdot \ln a$

For instance, this means that $D[3^x] = 3^x \cdot \ln 3$ and $D\left[\left(\frac{1}{2}\right)^x\right] = \left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right)$.

Let's look at a few examples where $f(x) = a^x$ is combined with other functions.

EXAMPLE Consider the function $f(x) = x \cdot 10^x$. Find its derivative.

$f(x) = x \cdot 10^x$	Start with the original function.
$f'(x) = D[x] \cdot 10^x + x \cdot D[10^x]$	Use the product rule.
$f'(x) = (1) \cdot 10^x + x \cdot (10^x \cdot \ln 10)$	$D[x] = 1$, $D[10^x] = 10^x \cdot \ln 10$
$f'(x) = 10^x + x \cdot 10^x \cdot \ln 10$	Remove extra grouping symbols.

Thus, $f'(x) = 10^x + x 10^x \ln 10$.

This could also be rewritten by factoring out 10^x: $f'(x) = 10^x(1 + x\ln 10)$

2. Derivatives of $y = a^u$ and Combinations of Functions With $y = a^u$, Where u Is a Function of x

As a result of the chain rule, we have the following derivative formula:

FORMULA

The Derivative of a^u, Where u Is a Function of x

$D[a^u] = (a^u \cdot \ln a) \cdot u'$

↱ EXAMPLE Consider the function $f(x)=3^{-x^2}$. Find its derivative.

$f(x)=3^{-x^2}$	Start with the original function.
$f'(x)=\left(3^{-x^2}\cdot\ln 3\right)\cdot(-2x)$	$D[3^u]=(3^u\cdot\ln 3)\cdot u'$ Here, $u=-x^2$.
$f'(x)=-2x3^{-x^2}\ln 3$	Write "-2x" in front and remove unnecessary grouping symbols.

Thus, $f'(x)=-2x3^{-x^2}\ln 3$.

TRY IT

Consider the function $f(x)=\sqrt{5^x+2}$.

Find its derivative.

+

$f'(x)=\frac{1}{2}(5^x+2)^{-1/2}(5^x\cdot\ln 5)$ or $f'(x)=\frac{5^x\ln 5}{2\sqrt{5^x+2}}$

↱ EXAMPLE A drug has a half-life of 6 hours, which means that after 6 hours in the bloodstream, half of the original amount remains. When 40mg of this drug is introduced into the bloodstream, the amount remaining after t hours is $A(t)=40\left(\frac{1}{2}\right)^{t/6}$.

At what rate is the amount of drug in the bloodstream changing after 8 hours?

In this problem, we want to find $A'(8)$. So, let's first find $A'(t)$.

$A(t)=40\left(\frac{1}{2}\right)^{t/6}$	Start with the original function.
$A'(t)=40\left[\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)\right]\cdot\frac{1}{6}$	$D[a^u]=a^u\cdot\ln a\cdot u'$ $u=\frac{t}{6}=\frac{1}{6}t,\ u'=\frac{1}{6}$
$A'(t)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)$	$40\left(\frac{1}{6}\right)=\frac{40}{6}=\frac{20}{3}$ Remove extra symbols.

Then, $A'(8)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{8/6}\cdot\ln\left(\frac{1}{2}\right)\approx-1.83384$. This means that the amount of drug in the bloodstream is decreasing at a rate of about 1.83 mg/hr.

SUMMARY

In this lesson, you explored finding the **derivatives of** $y = a^x$ **and combinations of functions with** $y = a^x$. You also learned how to find the **derivatives of the general exponential function** $y = a^u$**, where *u* is a function of *x* (and related combinations of functions)**, which allows you to explore even more functions and applications. Remember that the derivative rule for $y = a^u$ is very similar to that of $y = e^u$ where *u* is a function of *x*, but with an extra factor of $\ln a$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

The Derivative of a^x

$D[a^x] = a^x \cdot \ln a$

The Derivative of a^u, Where u Is a Function of x

$D[a^u] = (a^u \cdot \ln a) \cdot u'$

Derivatives of Natural Logarithmic Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to differentiate logarithmic functions. Recall that a logarithmic function is the inverse of an exponential function. Thus, in any situation in which the rate of change of an exponential function is desired, it makes sense to also discuss the rates of change of logarithmic functions. Specifically, this lesson will cover:

1. The Derivative of $f(x) = \ln x$ and Functions Involving ln*x*
2. The Derivative of $f(u) = \ln u$ and Functions Involving ln*u*, Where *u* Is a Function of *x*
3. Using Properties of Logarithms Before Differentiating

1. The Derivative of *f*(*x*) = ln*x* and Functions Involving ln*x*

WATCH

Please view this video to see how to derive a formula for the derivative of $f(x) = \ln x$.

Video Transcription

[MUSIC PLAYING] Hello and welcome back. What we're going to do with this video is another derivation of a derivative rule. We're going to focus on the function natural log of x. And, again, using known properties, we'll be able to come up with a formula for the derivative.

So one other known fact, we used it in the last video as well, is that e to the natural log of x is equal to x. So those are two functions that are equivalent to each other. So we can deduce then that the derivative of each function is the same.

So you're probably wondering at this point, how does that relate to what we're trying to figure out here? Well, let's keep taking a look here. So the derivative of e to the something, I'm going to write that off to the side. The derivative of e to the u is, remember, itself times the derivative of the inside.

So the derivative of e to the natural log of x is e to the natural log of x times the derivative of the natural log of x. Now, that is the thing we're trying to find. We're trying to figure out what the derivative of natural log of x is. So how are we going to get there? Stay tuned.

On the right-hand side, the derivative of x is just 1. Now, remember from our relationship up here, e to the natural log of x is equivalent or is equal to x. So I'm going to replace e to the natural log of x with x, because that is much simpler.

So we have x times the derivative of natural log of x equals 1. And I'm gonna divide both sides by x, because that allows me to solve for the derivative expression. And this is why the derivative of the natural log of x is 1 over x, a very simple derivative formula. So we're going to use this to differentiate functions dealing with natural logarithms.

So, we can say the derivative of the natural log function can be expressed with the following formula:

FORMULA

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

With this new derivative rule, let's compute a few derivatives.

EXAMPLE Consider the function $f(x) = x^2 \ln x$.

$f(x) = x^2 \ln x$	Start with the original function.
$f'(x) = D[x^2] \cdot \ln x + x^2 \cdot D[\ln x]$	Use the product rule.
$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$	$D[x^2] = 2x$, $D[\ln x] = \frac{1}{x}$
$f'(x) = 2x \ln x + x$	Simplify $x^2 \cdot \frac{1}{x} = x$ and remove extra symbols.

Thus, $f'(x) = 2x \ln x + x$.

TRY IT

Consider the function $f(x) = \frac{\ln x}{x}$.

Find its derivative. +

$$f'(x) = \frac{1 - \ln x}{x^2}$$

HINT

Similar to trigonometric functions, powers of natural logarithmic functions are sometimes written with the power after the "ln". For example, $\ln^4 x$ means $(\ln x)^4$.

EXAMPLE Consider the function $f(x) = \ln^3 x$. Find its derivative.

$f(x) = \ln^3 x = (\ln x)^3$	Start with the original function. Rewrite in a more recognizable form.
$f'(x) = 3(\ln x)^2 \cdot \frac{1}{x}$	$D[u^3] = 3u^2 \cdot u'$ (Apply the chain rule.)

$$f'(x)=\frac{3(\ln x)^2}{x}$$ Combine as a single fraction.

Thus, $f'(x)=\frac{3(\ln x)^2}{x}$. It is also acceptable to write $f'(x)=\frac{3\ln^2 x}{x}$.

---

2. The Derivative of $f(u) = \ln u$ and Functions Involving $\ln u$, Where u Is a Function of x

In step with the chain rule, and the fact that $D[\ln x]=\frac{1}{x}$, we have the following rule for the derivative of $\ln u$:

FORMULA

Derivative of $\ln u$, Where u Is a Function of x

$$D[\ln u]=\frac{1}{u}\cdot u'$$

EXAMPLE Consider the function $f(x)=\ln(x^2+1)$. Find its derivative.

$f(x)=\ln(x^2+1)$	Start with the original function.
$f'(x)=\frac{1}{x^2+1}\cdot 2x$	$D[\ln u]=\frac{1}{u}\cdot u'$
$f'(x)=\frac{2x}{x^2+1}$	Rewrite as a single fraction.

Thus, $f'(x)=\frac{2x}{x^2+1}$.

HINT

The rule $D[\ln u]=\frac{1}{u}\cdot u'$ can also be written as $D[\ln u]=\frac{u'}{u}$.

EXAMPLE Consider the function $f(x)=\ln(\cos x)$. Find its derivative.

$f(x)=\ln(\cos x)$	Start with the original function.
$f'(x)=\frac{-\sin x}{\cos x}$	$D[\ln u]=\frac{u'}{u}$
$f'(x)=-\tan x$	Use this trigonometric identity: $\frac{\sin x}{\cos x}=\tan x$

Thus, $f'(x)=-\tan x$.

TRY IT

Consider the function $f(x)=\ln(2+\sin x)$.

Find its derivative. +

$$f'(x) = \frac{\cos x}{2+\sin x}$$

WATCH

The video below illustrates how to find the derivative of $f(x) = x \cdot \ln(x^3+1)$, which requires a combination of the product and chain rules.

Video Transcription

[MUSIC PLAYING] Hello, and welcome back to more derivatives with natural algorithms. In this video, we're going to find the derivative of a function that combines a few rules. As you can see here, we have x times the natural log of x to the third plus 1. So that's going to involve a product rule. So just to remember what the products rule is, the derivative of-- I guess we'll call it u times v-- is the derivative of the first times the second plus the first times the derivative of the second.

So what we have here is our first and our second. So the derivative is the derivative of x, which is 1-- so I'm just going to actually write that as the derivative of x-- times the natural log of x to the third plus 1 plus the first times the derivative of the natural log of x to the third plus 1.

So now looking at each derivative, the derivative of x-- that's a nice, simple one-- that's 1. So you have 1 times natural log of x to the third plus 1. And then we have x times-- now, the derivative of natural log of something-- I'm going to write that up here. Derivative of natural log of u, is 1 over u times u prime or u prime over u, as we have used a couple of times.

So derivative of natural log of x to the third plus 1. So we know it's something over x to the third plus 1. And the derivative of x to the third plus 1 is 3x squared. So there's the derivative of that. And now, we'll just clean things up a bit here. f prime of x is equal to the natural log of x to the third plus 1 plus-- now, remember, to pull these two expressions here together, this is really over 1.

So we have x to the third-- I'm sorry-- 3x to the third over x to the third plus 1. And that is the derivative of our function. So combining the product rule and the chain rule.

---

3. Using Properties of Logarithms Before Differentiating

↪ EXAMPLE Consider the function $f(x) = \ln(x \cdot e^{-2x})$. Find the derivative of this function.

$$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot D[x \cdot e^{-2x}] \qquad D[\ln u] = \frac{1}{u} \cdot u'$$

$$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot [D[x] \cdot e^{-2x} + x \cdot D[e^{-2x}]]$$ Use the product rule.

$$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot \left[1 \cdot e^{-2x} + x \cdot e^{-2x}(-2)\right] \quad D[x] = 1,\ D\left[e^{u}\right] = e^{u} \cdot u'$$

$$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot \left[e^{-2x} - 2x \cdot e^{-2x}\right] \quad \text{Simplify and remove unnecessary symbols.}$$

$$f'(x) = \frac{e^{-2x}}{x \cdot e^{-2x}} - \frac{2xe^{-2x}}{x \cdot e^{-2x}} \quad \text{Distribute.}$$

$$f'(x) = \frac{1}{x} - 2 \quad \text{Remove the common factors.}$$

Thus, $f'(x) = \frac{1}{x} - 2$.

This process was quite cumbersome. However, if we use the properties of logarithms that we reviewed in Unit 1, this can be made simpler.

FORMULA

Product Property
$\ln(ab) = \ln a + \ln b$

Quotient Property
$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Power Property
$\ln(a^{b}) = b \cdot \ln a$

EXAMPLE Consider the function $f(x) = \ln\left(x \cdot e^{-2x}\right)$. Find its derivative by first using logarithm properties.

Since, $\ln\left(x \cdot e^{-2x}\right)$ is the logarithm of a product, use properties of logarithms to rewrite:

$f(x) = \ln\left(x \cdot e^{-2x}\right)$	Start with the original function.
$f(x) = \ln x + \ln\left(e^{-2x}\right)$	$\ln(ab) = \ln a + \ln b$
$f(x) = \ln x + (-2x)\ln e$	$\ln(a^{b}) = b \cdot \ln a$
$f(x) = \ln x - 2x$	$\ln e = 1,\ -2x(1) = -2x$

So, in expanded (and simpler) form, $f(x) = \ln x - 2x$.

Then, $f'(x) = D[\ln x] - D[2x] = \frac{1}{x} - 2$.

HINT

To find the derivative of $\ln u$, where u is a product, quotient, or power (or any combination of them), use logarithm properties before finding the derivative. This results in simpler derivatives.

WATCH

In this video, we'll use properties of logarithms to find the derivative of $f(x)=\ln\left(\frac{x}{\sqrt{2x+1}}\right)$.

Video Transcription

Hello. I hope all is well in your world. What we're going to do is, in this video, look at the derivative of a natural log function where properties can be used to find the derivative. And this is a prime example, no pun intended, of a function where you don't want to really deal with the function as it's written. Because remember, the derivative of natural log-- well, let's review that.

The derivative of the natural log of u is 1 over u times the derivative of u. I really don't want to deal with the derivative of x divided by the square root of 2x plus 1 because that's going to involve a quotient rule. That's quite complicated. There's the quotient rule. There's a chain rule. There's a whole lot going on there.

So one thing we can do if our goal is just to find a derivative expression is to rewrite the logarithm using properties. So the first thing to remember is that we have the natural logarithm of a quotient. So remember that is the natural log of the numerator minus the natural log of the denominator.

So already that's looking better because we can handle the natural log of x no problem. Natural log of the square root of 2x plus 1, that might be a little challenging. But remember, square roots can be written as powers. So that's really the natural log of 2x plus 1 to the 1/2. And remember another logarithm property is that the power, the 1/2 power, can be brought down in front of the natural logarithm.

And that is as much as this simplify. So remember, this is all just y. We have not taken a derivative yet. So now our goal is to find the derivative of this function. But now, so we have dy dx equals the derivative of the natural log of x, which is 1 over x, minus-- now that's 1/2 times a natural log. So that means the constant multiple rule is in effect here. It's going to be 1/2 times whatever the derivative of natural log of 2x plus 1 is.

And according to our formula up here, the derivative of natural log of something is 1 over the something times the derivative of the something. So this is 1 over the something, 1 over 2x plus 1, times the derivative of 2x plus 1, which is 2. So this 2 comes from the derivative of 2x plus 1. Now all we have to do is simplify, and then we are done.

So dy dx is equal 2. Also the 1 over x is out there on its own. If we look here, we have a 1/2 times 1 over 2x plus 1 times a 2. I know that 1/2 times 2 is 1, so those cancel. And we're left with 1 over x minus 1 over 2x plus 1. And that is a perfectly simplified expression for the derivative.

And there we have it. Using logarithm properties makes derivatives much simpler, at least when we have a product, quotient, or power inside of the natural algorithm.

SUMMARY

In this lesson, you learned how to find **the derivative of a natural logarithmic function** (represented by $f(x) = \ln x$) and, given the chain rule, **the derivative of** $f(u) = \ln u$. These are the latest additions to our library of derivatives, and you've seen through examples and videos the different way the natural logarithmic function can be combined with other functions. You also learned that since logarithms have special properties, it is more advantageous to **use properties of logarithms before differentiating** functions that involve products, quotients, and powers.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of lnu, Where u Is a Function of x

$$D[\ln u] = \frac{1}{u} \cdot u'$$

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

Power Property

$$\ln(a^b) = b \cdot \ln a$$

Product Property

$$\ln(ab) = \ln a + \ln b$$

Quotient Property

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

Derivatives of Non-Natural Logarithmic Functions

by Sophia

WHAT'S COVERED

In this lesson, you will find derivatives involving logarithmic functions that deal with bases other than *e*. Since there were reasons to use exponential functions with bases other than *e*, it makes sense to discuss the corresponding logarithmic functions. Specifically, this lesson will cover:

1. The Derivatives of $y=\log_a x$ and $y=\log_a u$, Where *u* Is a Function of *x*
2. Derivatives of Functions Involving $\log_a x$ and $\log_a u$, Where *u* Is a Function of *x*

1. The Derivative of $y=\log_a x$ and $y=\log_a u$, Where u Is a Function of x

Consider the function $y=\log_a x$, where *a* is any positive number except 1.

If we apply the change of base formula, we have $\log_a x=\frac{\ln x}{\ln a}=\left(\frac{1}{\ln a}\right)\cdot\ln x$.

Then, $D[\log_a x]=D\left[\left(\frac{1}{\ln a}\right)\cdot\ln x\right]=\left(\frac{1}{\ln a}\right)\cdot\frac{1}{x}=\frac{1}{x\cdot\ln a}$.

So, we can say the derivative of a logarithm function with base *a* can be expressed with the following formula:

FORMULA

Derivative of a Logarithm Function, Base *a*

$$D[\log_a x]=\frac{1}{x\cdot\ln a}$$

When *x* is replaced with *u* (a function of *x*), the chain rule is used.

FORMULA

Derivative of a Composite Logarithm Function, Base *a*

$$D[\log_a u]=\frac{1}{u\cdot\ln a}\cdot u'=\frac{u'}{u\cdot\ln a}$$

HINT

Notice that this derivative formula is the same as the one for $\ln x$, but there is also a factor of $\ln a$ in the denominator.

2. Derivatives of Functions Involving $\log_a x$ and $\log_a u$, Where u Is a Function of x

Since the derivatives of $\log_a x$ and $\ln x$ are similar, we'll look at various functions.

↪ EXAMPLE Consider the function $f(x) = \log_2(x^3 + 5x)$. Find its derivative.

$f(x) = \log_2(x^3 + 5x)$	Start with the original function.
$f'(x) = \dfrac{1}{(x^3 + 5x)\ln 2} \cdot (3x^2 + 5)$	$D[\log_a u] = \dfrac{1}{u \cdot \ln a} \cdot u'$
$f'(x) = \dfrac{3x^2 + 5}{(x^3 + 5x)\ln 2}$	Write as a single fraction.

Thus, $f'(x) = \dfrac{3x^2 + 5}{(x^3 + 5x)\ln 2}$.

TRY IT

Consider the function $f(x) = x^2 \log_3(2x + 1)$.

Find its derivative. +

$$f'(x) = 2x \log_3(2x + 1) + \frac{2x^2}{(2x + 1)\ln 3}$$

WATCH

The following video illustrates the use of properties of logarithms to find the derivative of $f(x) = \log\left(\dfrac{x \cdot \sin x}{3x^2 + 1}\right)$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome to the final video on derivatives of logarithmic functions. In this video, we're going to look at how to find the derivative of a logarithm function that has a base other than e. And then we're going to apply properties of logarithms to help us find the derivative of a more complicated function.

So the first thing to remember is that when you see a log by itself as opposed to ln, it really means there's a base of 10. And, remember, that makes a difference when we're finding derivatives. So there's really a 10 there.

Now to find the derivative, the first thing we're going to do is use properties if we can because, again, the derivative of a log function is-- well, let's look at that. So the derivative of log base a of u is 1 over u times u prime. And what I forgot to put here is-- there is really another factor here in the denominator, the natural log of a.

So there's a base other than e. We put a natural log of a in the denominator. So first thing we'll do is use properties. So this is still f of x. We're not finding a derivative yet. I notice there's a fraction here. So this is log base 10 of the numerator minus log base 10 of the denominator.

And then in the first term we have log base 10 of x plus log base 10 of sine x because that was a product and then minus log base 10 of 3x squared plus 1. So now to find the derivative-- oops, that's just f of x.

Now, to find the derivative, we just find the derivative of each term. So derivative of log base 10 of x, that's the basic function so no chain rule needed. That derivative is 1 over-- ordinarily it would be 1 over x. But since there's a different base, we put a natural log of 10 in the denominator.

In the second one, we have 1 over sine x times natural log of 10. And then times the derivative of sine x which is cosine x and then minus 1 over 3x squared plus 1 times the natural log of 10 and 10 times the derivative of the inside.

Now, the derivative of 3x squared plus 1 is 6x. So just as a reminder, this is the derivative of 3x squared plus 1 just like cosine is the derivative of sine. And now let's pull some things together, so still f prime of x. This is 1 over x ln 10 plus-- now we have cosine x over sine x natural log 10 minus 6x over 3x squared plus 1 natural log 10.

One thing I notice is that there's a natural log of 10 in every denominator. So I'm going to pull out 1 over natural log 10 to make this look a little bit easier. So we have 1 over x plus cosine x over sine x minus 6x over 3x squared plus 1.

Now this is for all intents and purposes finished. But there is one tiny detail here. And that is, cosine divided by sine is actually another trig function. It's the cotangent. And there we have the derivative, a little bit involved with the details there. But there we have the derivative of the original function f of x equals log base 10 x sine x over 3x squared plus 1. And, again, making good use of those logarithm properties made the derivative a lot simpler than it could have been.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned to find **the derivatives of** $y=\log_a x$ **and** $y=\log_a u$**, where *u* is a function of *x*,** and **derivatives of functions involving** $\log_a x$ **and** $\log_a u$**, where *u* is a function of *x*.** Remember that the derivative of the "base *a*" logarithmic function is very similar to that of the natural logarithmic function with a factor of $\ln a$ in the denominator. With this function added to your toolbox, you now have the ability to find derivatives of any combination of polynomial, trigonometric, exponential, and logarithmic functions.

FORMULAS TO KNOW

Derivative of a Composite Logarithm Function, Base a

$$D\left[\log_a u\right] = \frac{1}{u \cdot \ln a} \cdot u' = \frac{u'}{u \cdot \ln a}$$

Derivative of a Logarithm Function, Base a

$$D\left[\log_a x\right] = \frac{1}{x \cdot \ln a}$$

Applications of Rates of Change

by Sophia

WHAT'S COVERED

In this lesson, you will examine some real-world applications in which derivatives are required to describe certain ideas. Specifically, this lesson will cover:

1. Vertical Motion
2. Growth and Decay
3. Applications to Business
   a. Total Cost and Marginal Cost
   b. Average Cost

1. Vertical Motion

In challenge 3.2, we discussed velocity and acceleration of an object that is traveling vertically. Recall the following, which are measured after t seconds of travel:

- $h(t) =$ the height of the object
- $v(t) = h'(t) =$ the velocity of the object
- $a(t) = h''(t) =$ the acceleration of the object

TRY IT

A tennis ball is launched off of a building that is 60 feet high at a velocity of 30 feet per second. Its height after t seconds is $h(t) = -16t^2 + 30t + 60$.

Determine if each question can be answered with or without calculus. Then, answer each question.

How high is the object after 1 second? +

No calculus required.
Answer: $h(1) = 74$ feet

What is the velocity of the object after 1 second? +

Calculus is required.
$v(t) = h'(t) = -32t + 30$
$v(1) = h'(1) = -32(1) + 30 = -2$ ft/sec

When does the object strike the ground? +

No calculus required.
Set $h(t) = 0$.
$h(t) = -16t^2 + 30t + 60$
Answer: $t \approx 3.089$ seconds

---

2. Growth and Decay

The function $f(t) = Ae^{kt}$ is used to model the exponential growth or decay of a substance.

- If $k > 0$, it models growth.
- If $k < 0$, it models decay.

Naturally, $f'(t)$ is used to find instantaneous rates of growth (or decay) at specific values of t.

TRY IT

The amount of bacteria present in a medium after t hours is given by $A(t) = 50e^{0.1t}$.

How many bacteria are present at the beginning? +

$A(0) = 50e^{0.1(0)} = 50e^0 = 50$ bacteria

How many bacteria are present after 7 hours? +

$A(7) = 50e^{0.1(7)} = 50e^{0.7} \approx 100.68$, so about 101 bacteria

At what rate is the amount of bacteria changing after 8 hours? +

We want to find $A'(8)$. First, find $A'(t)$:

$A(t) = 50e^{0.1t}$	Start with the original function.
$A'(t) = D[50e^{0.1t}] = 50D[e^{0.1t}]$	Use the constant multiple rule.
$A'(t) = 50e^{0.1t} \cdot 0.1$	$D[e^u] = e^u \cdot u'$
$A'(t) = 5e^{0.1t}$	Simplify.

Then, the rate of change is $A'(8) = 5e^{0.1(8)} = 5e^{0.8} \approx 11.13$ bacteria per hour.

---

3. Applications to Business

3a. Total Cost and Marginal Cost

In business, particularly in manufacturing, there are two functions that are analyzed the most often to help minimize costs:

- The total cost of producing x items, usually represented by $C(x)$.
- The average cost of producing x items, usually represented by $AC(x)$.

Note: $AC(x) = \frac{C(x)}{x}$.

The **fixed cost** is the cost before any items are produced. This is also called overhead.

Given a specific production level, it is useful to have an estimate for the cost of producing the*next* item.

EXAMPLE Given $C(x) = 0.2x^2 + 14x + 500$:

- The total cost of producing 9 units is $C(9) = 0.2(9)^2 + 14(9) + 500 = \$642.20$.
- The total cost of producing 10 units is $C(10) = 0.2(10)^2 + 14(10) + 500 = \$660$.

Then, the cost of producing the 10th item is \$660 - \$642.20 = \$17.80.

Thus, given a production level of 9 units, the cost of producing the next item is \$17.80.

TRY IT

Consider the cost function $C(x) = 0.2x^2 + 14x + 500$.

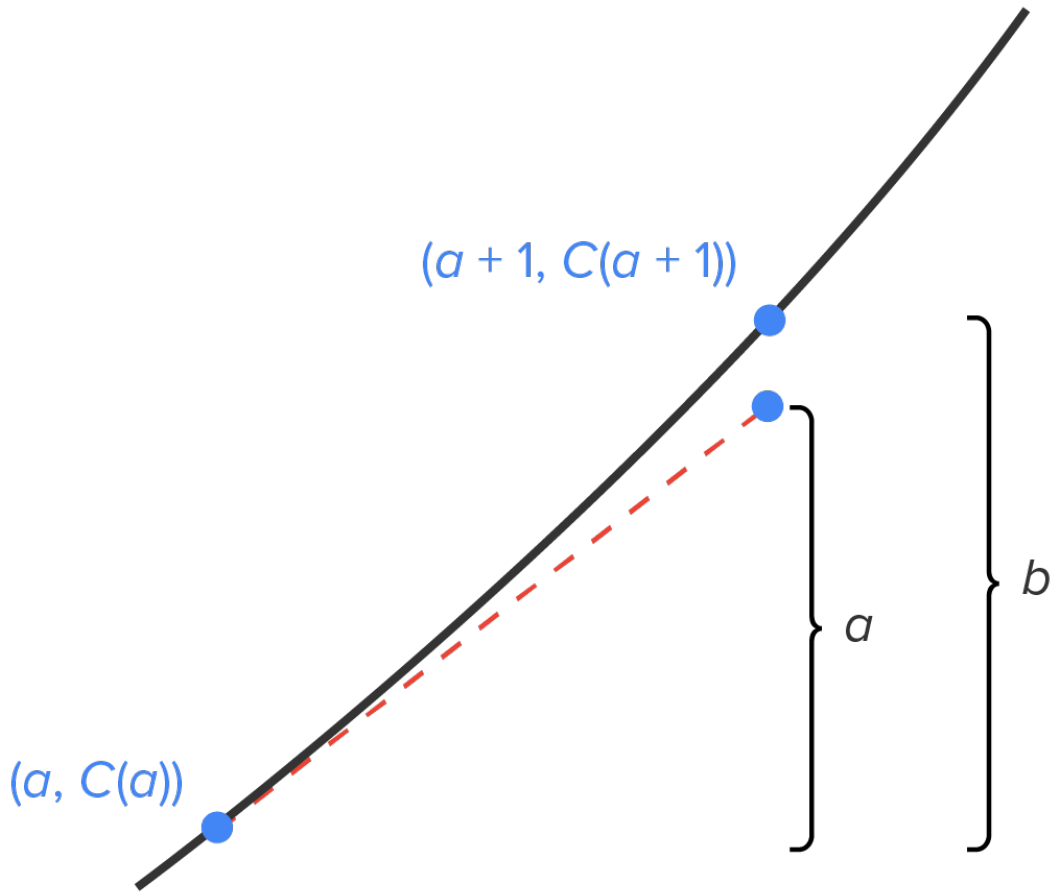
Find the cost of producing the 7th item. Then, compare it to the cost of producing the 10th item in the previous example. What do you notice?

+

\$16.60. Which is less than (not the same as) the cost of the 10th item. This suggests that each item costs a different amount.

Another way to approximate the cost of producing the next unit is to use the derivative of the cost function, which is called the **marginal cost function**. This makes sense since the rate of change in the cost would be measured in dollars per unit.

Here is a graph to show the reasoning, where the solid graph is the cost function, and the dotted graph is the tangent line:



As we can see, the tangent line at $(x, C(x))$ is a good approximation for the cost (and as a result, the change in cost) between production levels of x and $x + 1$. Therefore, the marginal cost function at x approximates the cost of the $(x + 1)$st unit.

↪ EXAMPLE Using $C(x) = 0.2x^2 + 14x + 500$, approximate the cost of producing the 10th and 7th units by using the marginal cost.

First, the marginal cost is the derivative of cost: $C'(x) = 0.4x + 14$

The approximate cost of producing the 10th unit is $C'(9) = 0.4(9) + 14 = 17.6$.
The approximate cost of producing the 7th unit is $C'(6) = 0.4(6) + 14 = 16.4$.

Thus, the 10th unit costs approximately \$17.60 and the 7th costs approximately \$16.40. Compared to previous results (\$17.80 and \$16.60 respectively), the derivative provides a good estimate.

TERMS TO KNOW

Fixed Cost (or Overhead)
The costs that are incurred before any items are produced. Mathematically, it is the total cost of producing 0 items.

Marginal Cost Function

Marginal Cost Function

The derivative (rate of change) of the cost function. Given a production level x, it approximates the cost of the next item.

3b. Average Cost

The average cost function, $AC(x)$, is the total cost to produce x items, divided by the number of items.

FORMULA

Average Cost Function

$$AC(x) = \frac{C(x)}{x}$$

Since a total cost function includes a fixed cost, It appears that the first few units cost more to make, since the fixed costs affect the total costs more.

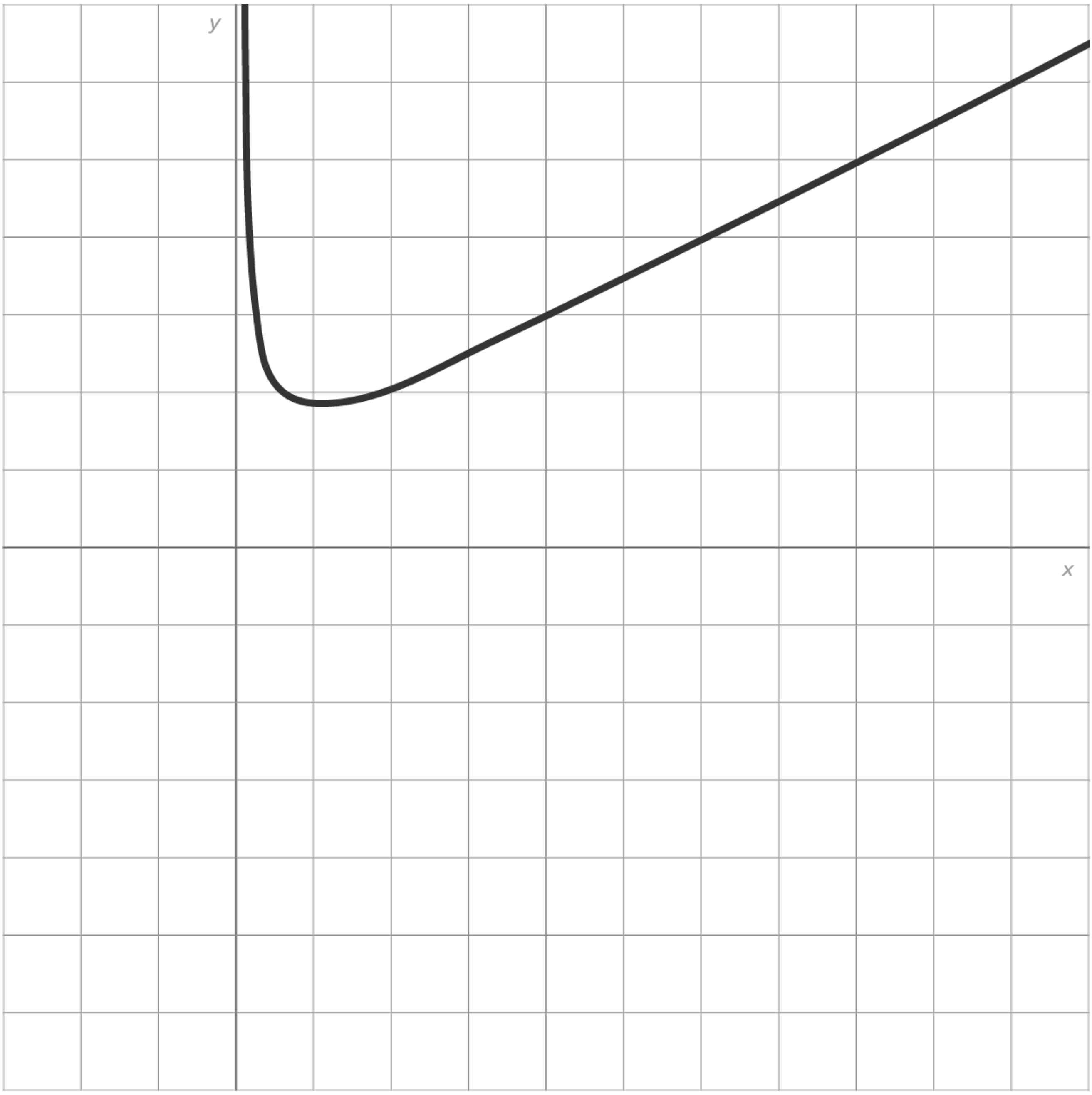
For example, consider the total cost of producing x items, $C(x) = 0.5x^2 + 40x + 600$.

In the table, we have the number of items, the total cost of x items, and the average cost.

x = # items	Total Cost, $C(x)$	Average Cost, $\frac{C(x)}{x}$
1	$640.50	$640.50
2	$682	$341
3	$724.50	$241.50
4	$768	$192

As you can see, as x increases, the average cost function drops sharply as x increases, but this isn't the whole story.

A typical average cost curve looks like the graph provided.



To the left of the minimum point, the average cost falls sharply, meaning that its derivative is negative.

To the right of the minimum point, the graph rises fairly steadily, meaning that its derivative is positive.

Knowing the rate of change of the average cost can help determine if a production strategy is cost effective when compared to revenue. For example, if the average cost to produce each unit is $50, but the slope is negative, this means that the average cost per unit could be made smaller if production is increased. This is a wide idea, assuming that the additional items could be sold.

> ↪ EXAMPLE Given a total cost function $C(x) = 0.2x^2 + 14x + 500$, find the rate at which the average cost is changing after 10 units are produced.
>
> First, write the average cost function:
>
> $AC(x) = \frac{C(x)}{x} = \frac{0.2x^2 + 14x + 500}{x}$ Start with the average cost function.

$AC(x) = \frac{0.2x^2}{x} + \frac{14x}{x} + \frac{500}{x}$ Divide each term by x.

$AC(x) = 0.2x + 14 + 500x^{-1}$ Perform simplifications; write the last term to prepare for the derivative.

Now, find the derivative:

$AC'(x) = 0.2(1) + 0 + 500(-1)x^{-2}$ Use the constant multiple rule.

$AC'(x) = 0.2 - \frac{500}{x^2}$ Simplify and rewrite with positive exponents.

Then, the rate of change at a production level of 10 units is $AC'(10) = 0.2 - \frac{500}{10^2} = 0.2 - 5 = -4.80$.

Thus, the average cost is declining at a rate of $4.80 per item, per item.

TRY IT

The total cost of producing x items is $C(x) = 4x^2 + 60x + 1200$.

At what rate is the average cost changing at a production level of 20 units? +

$1 per unit, per unit.

A question that is left for the next unit is, “What is the minimum average cost, and what is the production level required to achieve it?”

SUMMARY

In this lesson, you learned how derivatives are used in a wide range of applications, including **vertical motion**, exponential **growth and decay**, and some **applications to business**. You learned that in business, particularly in manufacturing, the two functions that are analyzed most often to help minimize costs are **total cost** and **average cost**. Given a specific production level, it is also useful to have an estimate for the cost of producing the next item, which can be determined by finding the **marginal cost**, or derivative of the cost. As you progress through the course, you will learn more applications of the derivative.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Fixed Cost (or Overhead)
The costs that are incurred before any items are produced. Mathematically, it is the total cost of producing 0 items.

Marginal Cost Function

The derivative (rate of change) of the cost function. Given a production level x, it approximates the cost of the next item.

FORMULAS TO KNOW

Average Cost Function

$AC(x) = \frac{C(x)}{x}$

Linear Approximation

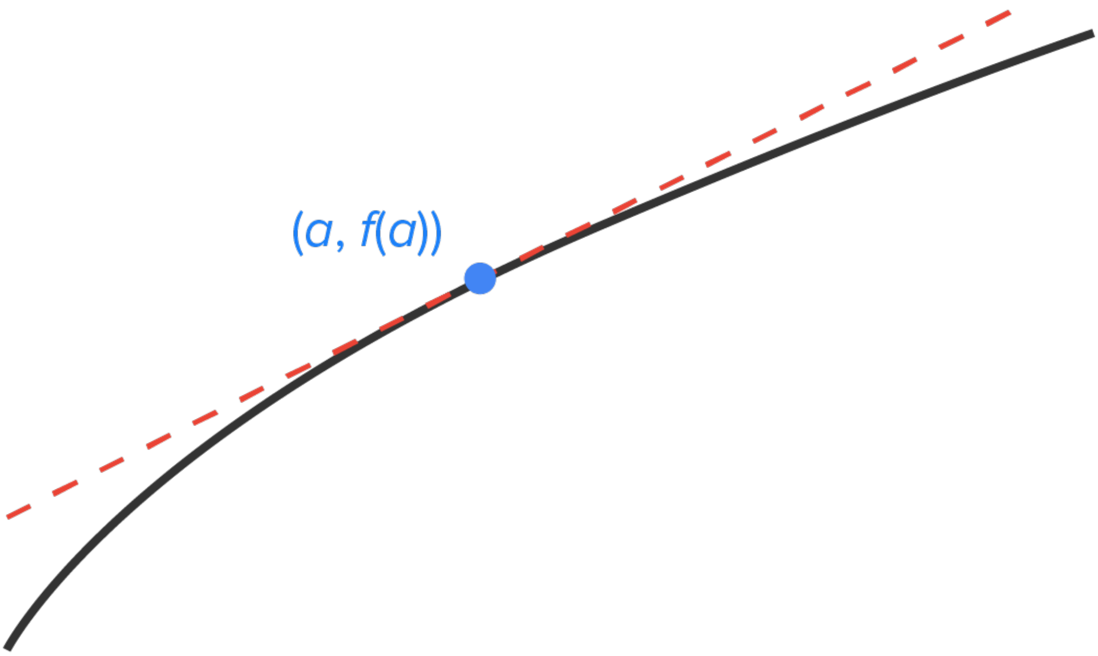
by Sophia

WHAT'S COVERED

In this lesson, you will see how the derivative can be used to approximate the values of various types of functions that would be difficult to compute otherwise, such as power, root, logarithmic, exponential, and trigonometric functions. For example, a well-placed tangent line for the function $f(x)=\sqrt{x}$ can be used to approximate $\sqrt{17}$. Specifically, this lesson will cover:

1. The Linear Approximation of $f(x)$ at $x=a$
2. Approximating Values of $f(x)$

1. The Linear Approximation of $f(x)$ at $x = a$



Consider the graph above.
The solid curve is the graph of $y=f(x)$.
The dashed line is the tangent line at $x=a$.

Notice how the tangent line and the graph of $y=f(x)$ are very close to each other near $x=a$. This means that the expression for the tangent line can be used to approximate the value of $y=f(x)$ near $x=a$. As a result, the tangent line at $x=a$ is called the **linear approximation of $f(x)$ at $x=a$**.

In the context of finding a linear approximation, the equation for the tangent line is written as $L(x)=f(a)+f'(a)(x-a)$. Note that the only change is using $L(x)$ to represent the output instead of *y*.

↪ EXAMPLE Find the linear approximation of $f(x)=\sqrt{x}$ when $x=16$. Remember that this is really the equation of the tangent line at $x=16$.

Find the derivative.

$f(x)=\sqrt{x}=x^{1/2}$	Rewrite as a power of *x*.
$f'(x)=\frac{1}{2}x^{-1/2}$	Find the derivative.
$f'(x)=\frac{1}{2x^{1/2}}=\frac{1}{2\sqrt{x}}$	Rewrite with a positive exponent, then the radical.
$f'(16)=\frac{1}{2\sqrt{16}}=\frac{1}{8}$	Substitute 16 in for *x* and simplify.

Now, form the linear approximation:

$L(x)=f(a)+f'(a)(x-a)$	Use the equation for the tangent line.
$L(x)=f(16)+f'(16)(x-16)$	$a=16$
$L(x)=4+\frac{1}{8}(x-16)$	$f(16)=4$ and $f'(16)=\frac{1}{8}$

The linear approximation at $x=16$ is $L(x)=4+\frac{1}{8}(x-16)$. This means that $L(x)$ approximates the value of $f(x)$ near $x=16$.

Note: Traditionally, we would simplify $L(x)$, but in this context, we are using $L(x)$ to approximate values of $f(x)$ near $x=16$. Thus, the "$x-16$" serves as a reminder that the linear approximation is based on the tangent line at $x=16$.

↪ EXAMPLE Find the linear approximation to $f(x)=e^x$ at $x=0$.

First, find the derivative: $f(x)=e^x$, $f'(x)=e^x$

Now, use the linear approximation:

$L(x)=f(a)+f'(a)(x-a)$	Use the equation for the tangent line.
$L(x)=f(0)+f'(0)(x-0)$	$a=0$
$L(x)=1+1(x-0)$	$f(0)=1$ and $f'(0)=1$
$L(x)=1+x$	Simplify.

This means that $L(x) = 1 + x$ approximates the value of $f(x) = e^x$ near $x = 0$.

WATCH

Here is a video example finding the linear approximation of $f(x) = \sqrt[3]{x}$ when $x = -64$.

Video Transcription

Hello, and welcome to the video on how to find a linear approximation of the function f of x equals the cubed root of x when x is equal to negative 64. Now remember, when we are looking at finding the linear approximation equation of a function, we're really just finding the equation of the tangent line to that function at that x value.

In order to do this, we want to remember that our derivative is the formula that gives us the slope of the tangent to a curve at any point. So if we find the derivative and evaluate it at our x value, that's the slope of the tangent line there. So the first thing we want to do when we're finding this linear approximation is to find the derivative of the function.

So our function is f of x is equal to the cubed root of x. But in order to take the derivative, we're going to write that cubed root of x as x to its fractional exponent 1/3, which means the same thing as the cubed root of x. Next, we are going to apply our power rule. So our derivative f prime of x is-- well, remember if we have letter base to number power. We bring the number power as a factor in front, keep the base and subtract 1 from the exponent.

1/3 subtract 1 is a negative 2/3. Now, because we are going to evaluate this derivative at x equals negative 64, we have to get it in the form that it's easier to evaluate when we plug in a number. So we want to make sure we first write it without a negative exponent. So f prime of x is 1 over 3 times x to the 2/3 power.

When we write it in the other part of the fraction, it changes the sign of the exponent. And then we also want to write it as the radical notation. So again, it's just a little bit easier to think through the simplification when we plug in a value for x. So this is 1 over 3 times. Well, the denominator of 3 in the exponent on the base of x means the cubed root of x. And then the numerator is our exponent that we take it to.

Many times, when we write it in a simplified form, we would write it as 1 over 3 times the cubed root of x squared. And the square would be on the x under the radical. But when we want to plug a number in for it so that our values don't get too large for us to be able to work with easily, we want to find our root first and then take it to the power.

So now let's look at our f prime evaluated at our x value of negative 64. So that's 1 over 3 times the cubed root of negative 64. And then close the parentheses and square that. So our f prime of negative 64 is equal to 1 over 3 times-- open our parentheses-- the cubed root of negative 64 is negative 4. And then we still have to square that.

Following order of operations, our f prime of negative 64 is 1 over 3 times-- well, negative 4 quantity squared is a 16. So that gives me f prime of negative 64 is 1 over, and 3 times 16 is 48. So that's the slope of our tangent line at x equal negative 64.

Now, we want to form our linear approximation. And our linear approximation, L of x, is equal to, it's f of a plus f prime at a times the quantity x minus a. Now, that a that we're talking about here is the x value that we are finding our linear approximation at. So a is negative 64.

Now we're going to see what we can do for our replacements. We just calculated our f prime at negative 64. And we're going to recall that f prime at negative 64 came out to be 1 1/48. Our f of the negative 64 is equal to-- look at your function. Follow your notation. The function is the cubed root of x. So that's the cubed root of negative 64, which gives us negative 4.

And then our a value is negative 64. So we're ready to plug everything in. Our linear approximation, L of x, is equal to f of a. So f of negative 64 is negative 4 plus f prime of a, the f prime of negative 64. So we have our f prime of a we're doing now. That's f prime of negative 64 is 1/48. And then open parentheses times x minus, and then minus our a. And our a is negative 64.

So there is just a little bit more that we want to simplify, but we don't want to simplify too much. So we will go ahead and take care of that double negative. L of x is equal to negative 4 plus 1/48 times x minus a negative 64 is an x plus 64.

And that's as far as we want to go with our simplification, because we want to remember that if we use this linear approximation to approximate for an x value, so output of this function using the linearization, we need to make sure we only do it for x's that are close to the x values that we created the linear approximation equation, the tangent line.

And so I want to make sure I only use this one for x values that are close to negative 64. And by leaving these parentheses and not actually distributing the 1/48 and collecting like terms, we are giving a visual reminder to only use this linear association to approximate the cubed root of the value for values that are close to negative 64.

Well, thank you for listening today, and I hope you've found this video helpful in the process of finding a linear approximation of a function when you have a specific x value.

TERM TO KNOW

Linear Approximation of *f*(*x*) at $x = a$

The tangent line to the graph of $f(x)$ at $x = a$.

---

2. Approximating Values of *f*(*x*)

The point of the linear approximation is to approximate the value of $f(x)$ for values close to $x = a$. Thus, in the previous example, $L(x) = 4 + \frac{1}{8}(x - 16)$ will be used to approximate values of $f(x) = \sqrt{x}$ for *x* near 16.

EXAMPLE Considering the last example, use the linear approximation to approximate $f(17) = \sqrt{17}$.

The linear approximation we found in the last example gives $L(17)=4+\frac{1}{8}(17-16)=4.125$.

From a calculator, $\sqrt{17}\approx 4.123$. The linear approximation for $\sqrt{17}$ is very close.

TRY IT

Consider the function $f(x)=\sqrt[3]{x}$.

Find the linear approximation of this function when $x=8$. +

$$L(x)=\frac{1}{12}(x-8)+2$$

Suppose you want to use this above linear approximation to approximate $\sqrt[3]{9}$.

Approximate to three decimal places. +

$$\sqrt[3]{9}\approx L(9)=2.083$$

SUMMARY

In this lesson, you learned that the tangent line to a function at $x=a$ can be used to find the **the linear approximation of $f(x)$ at $x=a$.** Then, you were able to apply this knowledge to explore a few examples **approximating values of $f(x)$** at x values near $x=a$. Later in this challenge, we will discuss the errors involved in using the linear approximation and how using other values of x affect the approximation.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Linear Approximation of f(x) at x = a
The tangent line to the graph of $f(x)$ at $x=a$.

The Linear Approximation Error | f(x) – L(x) |

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the error in using a linear approximation to approximate the value of a function. Specifically, this lesson will cover:

1. Calculating the Error in Using *L*(*x*) to Approximate *f*(*x*)
2. Analyzing the Error as *x* Gets Further Away From *a*

1. Calculating the Error in Using *L*(*x*) to Approximate *f*(*x*)

It is no surprise that using the linear approximation will produce some error.

The **linear approximation error** is the difference between the actual function value and the value obtained through the linear approximation.

FORMULA

Linear Approximation Error
$\text{Error} = |f(x) - L(x)|$

HINT

The error formula has an absolute value since we are only concerned with the size of the error, not necessarily which of $L(x)$ or $f(x)$ is larger. With absolute value, the difference is nonnegative regardless.

EXAMPLE Consider the function $f(x) = \ln x$ near $x = 1$. We'll use the linear approximation to estimate $\ln 1.2$, then find the linear approximation error.

To find the linear approximation, first find the derivative: $f(x) = \dfrac{1}{x}$

Now, form the linear approximation:

$L(x) = f(a) + f'(a)(x-a)$	Use the equation for the tangent line.
$L(x) = f(1) + f'(1)(x-1)$	$a = 1$
$L(x) = 0 + 1(x-1)$	$f(1) = 0$ and $f'(1) = 1$
$L(x) = x - 1$	Simplify.

The linear approximation tells us that $\ln 1.2 \approx L(1.2) = 1.2 - 1 = 0.2$.

The actual value of $\ln 1.2$ is 0.1823 (to 4 decimal places).

Then, the linear approximation error is $|f(1.2) - L(1.2)| \approx |0.1823 - 0.2| = 0.0177$.

TRY IT

Consider the function $f(x) = x^4$. Use the linear approximation at $x = 1$ to estimate $f(1.06)$. Then, find the linear approximation error to 4 decimal places.

What is the linear approximation? +

$L(1.06) = 1.24$

What is the linear approximation error? +

0.0225

2. Analyzing Error as *x* Gets Further Away From *a*

In 3.4.1, we used $L(x) = 4 + \frac{1}{8}(x - 16)$ to approximate $f(x) = \sqrt{x}$ for values of x near 16.

Here is a table of values (rounded to four decimal places) to illustrate what happens to the error as x moves away from 16:

x	$L(x) = 4 + \frac{1}{8}(x - 16)$	$f(x) = \sqrt{x}$	Error $= \lvert f(x) - L(x) \rvert$
16.25	4.0313	4.0311	0.0002
16.5	4.0625	4.0620	0.0005
16.75	4.0938	4.0927	0.0011
17	4.1250	4.1231	0.0019
17.25	4.1563	4.1533	0.0030
17.5	4.1875	4.1833	0.0042
17.75	4.2188	4.2131	0.0057
18	4.2500	4.2426	0.0074

As you can see, even though the errors are relatively small, they are increasing as x increases from 16. We would see a similar pattern if x were to decrease from 16 (15.75, 15.5, etc.).

SUMMARY

In this lesson, you learned how to **calculate the linear approximation error, or the error in using *L*(*x*) to approximate *f*(*x*).** The linear approximation error is found by calculating the positive difference between the linear approximation and the actual value. You also learned that when considering the linear approximation $L(x)=f(a)+f'(a)(x-a)$, the **error gets larger as *x* gets further from *a*.**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Linear Approximation Error

$\text{Error} = |f(x)-L(x)|$

The Differential of f

by Sophia

WHAT'S COVERED

In this lesson, you will express linear approximations in terms of differentials. Specifically, this lesson will cover:

1. Defining the Differential of *f*
2. Calculating the Differential of *f*

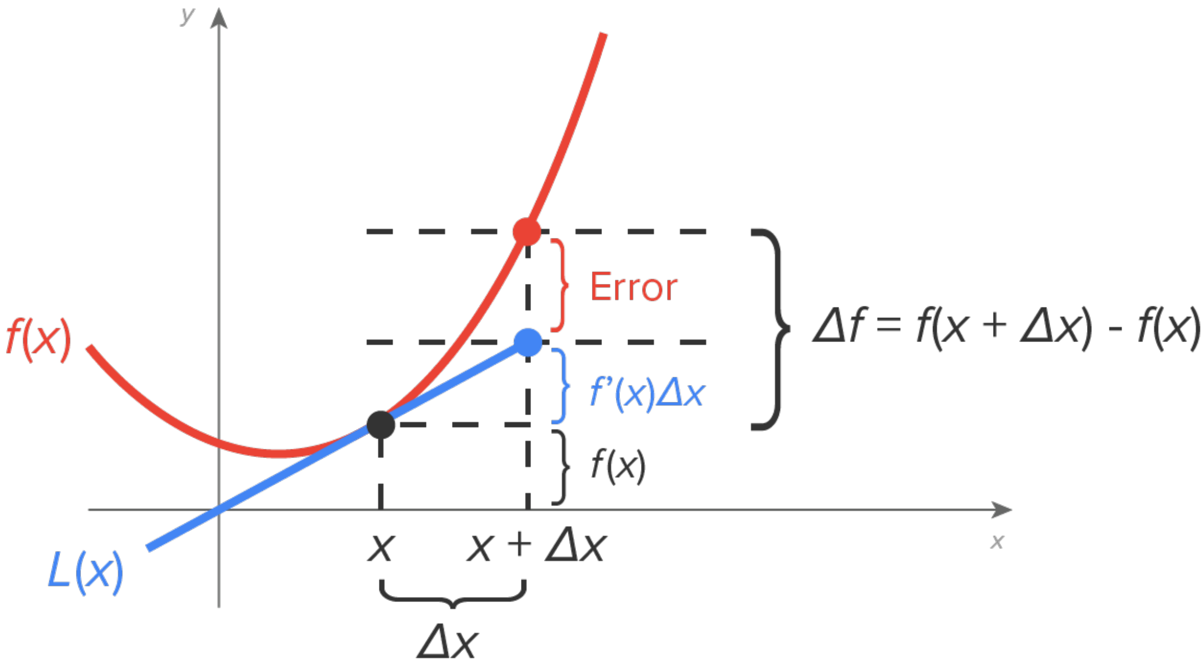
1. Defining the Differential of *f*

From when we first discussed rates of change and the derivative, recall the following quantities:

- $\Delta x =$ change in *x* (horizontal change)
- $\Delta y =$ change in *y* (vertical change)

When $y = f(x)$, Δy can be replaced with Δf to show that this is the change in function *f*.
Another goal in linear approximation is to find the change in *f* for a corresponding change in *x*.

Consider the image below:



Let $\Delta x =$ the horizontal change in x-values. This was "$x-a$" in the linear approximation formula.

Let $\Delta f =$ the actual change in *f* when moving from *x* to $x+\Delta x$. Then, $\Delta f = f(x+\Delta x) - f(x)$.

Now, let $A =$ the approximate change in *f* along the tangent line, which can be found as follows:

- Slope $= \frac{rise}{run} = f'(x) = \frac{A}{\Delta x}$
- Then, solving for *A*, we get $A = f'(x) \cdot \Delta x$.

Since *A* is approximating Δf, we can also say that $\Delta f \approx f'(x) \cdot \Delta x$.
This means that the change in *f* (when moving from *x* to $x+\Delta x$) can be approximated by multiplying the slope $f'(x)$ by Δx, the change in *x*.

This leads to the definition of the **differential of *f*.**

FORMULA

Differential of *f*

$df = f'(x)dx$ for any choice of *x* and any real number *dx*.

When $y = f(x)$, we can also write $dy = f'(x)dx$.

HINT

The differential uses the derivative at an x-value to give the approximate change in *f* when *x* changes to $x+\Delta x$.

When approximating the change in *y*, we use $dx = \Delta x$, but *dy* is an approximation of Δy.

2. Calculating the Differential of *f*

↱ EXAMPLE Given $f(x) = 4x^2 + 7x$, find the differential *df*.

Since $f'(x) = 8x + 7$, the differential is $df = (8x + 7)dx$.

↱ EXAMPLE Given $y = \ln(x^2 + 3)$, find the differential *dy*.

Since $\frac{dy}{dx} = \frac{1}{x^2+3} \cdot 2x = \frac{2x}{x^2+3}$, the differential is $dy = \frac{2x}{x^2+3}dx$.

TRY IT

Let $f(x) = x^2\sin(2x)$.

Find the differential *df*. +

$df = (2x\sin(2x) + 2x^2\cos(2x))dx$

WATCH

Here is a video in which we find the differential *dy* of $y = e^{-4x}\cos(7x)$.

Video Transcription

Hello, and welcome to the video on how to find a differential dy of a function that is a product of two functions. Now we've differentiated a product of two functions before. But it's always good to have more examples so that we get better and better at the process. This also has factors that are composition of functions. So we'll also use the chain rule. Now recall for the differential dy. My differential dy is equal to f prime of x dx.

Now when I'm looking at the idea of this what, I need to do is just find the expression for f prime of x, I'll put it into its slot in the differential notation and have our differential. We have our function y is equal to e to the negative 4x cosine of 7x. So I have a product between those two functions, And that's why I need to use the product rule when differentiating.

I also have a negative 4x in the exponent on e. So I'll have to use the chain rule there. And I'm taking the cosine of 7x. So I'll have to use a chain rule in that spot as well. We'll recall from the derivative of a product the dydx derivative is equal to-- well, we have the derivative of the first factor e to the negative 4x times-- keep the second factor the way it is-- cosine of 7x and then plus-- keep the first factor-- e to the negative 4x and multiply by the derivative of the second factor cosine of 7x.

So my dy/dx is equal to-- well, the derivative of e to the negative 4x. Well, Remember, the derivative of e to a power is e to the-- keep the power, but then multiply by the derivative of the power. The derivative of negative 4x is negative 4. So I'm going to multiply that in front of the expression. Now I'm going to have that times my cosine of 7x and then plus e to the negative 4x.

And this is going to be multiplied to the derivative of cosine of 7x. Remember, the derivative of cosine of an angle is negative sine, the angle. So I keep that 7x in there but then multiply by the derivative of 7x for the chain rule, which is multiplied by 7. So my dy/dx derivative is negative 4 to the negative 4x cosine of 7x.

And then when I multiply e to the negative 4x times negative sine 7x, that's negative e to the negative 4x sine 7x but then also multiply by 7. I'll have a negative 7 e to the negative 4x sine of 7x. So this dy/dx notation is just another way of also writing the derivative.

So our dy/dx notation is just also meaning my f prime of x. So this expression is what we'll put in for the f prime of x. My dy is equal to-- it's more than one term, so we need to put it in parentheses-- negative 4 e to the negative 4x cosine of 7x minus 7 e to the negative 4x sine of 7x, close the parentheses, and then dx.

So that is our differential. Notice that when we did the derivative, we had dy/dx is equal to our derivative. When we're doing the differential, it's dy is equal to the derivative and then times dx.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned how to **define and calculate the differential of *f*,** which is an approximation for the change in *f* when *x* changes by *dx* units.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Differential of f

$df = f'(x)dx$ for any choice of *x* and any real number *dx*.

When $y = f(x)$, we can also write $dy = f'(x)dx$.

Approximation of Measurement Error Using Differentials

by Sophia

WHAT'S COVERED

In this lesson, you will apply differentials to situations in which there could be measurement errors. For example, we can examine the effect that a measurement error on each side of a square could have on its area. Specifically, this lesson will cover:

1. Comparing the Differential to the Maximum Error
2. Applying Differentials to Situations Involving Measurement

1. Comparing the Differential to the Maximum Error

Let's say a square piece of material is to have sides 10” long, but each side could have a measurement error of at most 0.25”. What is the greatest possible error in measuring its area?

To answer this question, we need to look at a few scenarios:

- If the sides are all 0.25” too large, then the area would be $(10.25)^2 = 105.0625 \text{ in}^2$.
- If the sides are all 0.25” too small, then the area would be $(9.75)^2 = 95.0625 \text{ in}^2$.
- If the sides are all perfectly measured, then the area would be $10^2 = 100 \text{ in}^2$.
- When the sides are all 0.25” too large, the error is $105.0625 - 100 = 5.0625 \text{ in}^2$.
- When the sides are all 0.25” too small, the error is $100 - 95.0625 = 4.9375 \text{ in}^2$.

This means that the maximum error is 5.0625 in^2.
Think about how differentials are related to this situation:

- If the sides were measured accurately, the area would be $10^2 = 100 \text{ in}^2$.
- The goal is to determine the change in *A* when the side changes by at most 0.25”.

Now, let's examine this situation using differentials.
Let $A(x) = x^2$, which is the area of a square whose length is *x*.

Then, the differential of *A* is $dA = 2x dx$.

The situation above suggests that we want to find *dA* when $x = 10$ (length of a side) and $dx = 0.25$ (change in

x).

This gives $dA = 2(10)(0.25) = 5\text{ in}^2$, which is very close to the errors 4.9375 in^2 and 5.0625 in^2.

BIG IDEA

If $f(x)$ is a function that depends on an x-variable that has a possible error of *dx* units, then the differential *df* will provide an estimate of the maximum error in computing *f*.

2. Applying Differentials to Situations Involving Measurement

Now that we see how useful the differential is, let's apply them to situations involving measurement.

EXAMPLE A sphere is to be designed with a radius of 2.5 feet. What is the estimate for the maximum error when measuring the volume of the sphere if the possible error in measuring the radius is 0.1 feet?

The volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$, which has derivative $V'(r) = 4\pi r^2$. Thus, the differential is $dV = 4\pi r^2 dr$.

Now, let $r = 2.5$ and $dr = 0.1$. Then, the maximum error in estimating the volume is $dV = 4\pi(2.5)^2(0.1) \approx 7.854\text{ ft}^3$.

TRY IT

Suppose you have a cube with sides that are 8 inches.

Use differentials to estimate the maximum error in the surface area of this cube with an error of no more than 0.2" on each side. +

$S = 6x^2$, $dS = 12x dx$, $dS = 19.2\text{ in}^2$

WATCH

In this video, differentials are used to estimate the maximum error in the volume of a cube.

Video Transcription

[MUSIC PLAYING] Hello, and welcome to the video on using differentials to estimate the maximum error when measuring the volume of a cube if the possible error in measuring the sides is given. This is an application of differentials, as I mentioned, and our first step here is to look at the information in the problem, see if there is a geometrical shape that's being described, and if so, what the formula is that relates the shape to the other information in the problem.

Here the shape is a cube and we're relating the volume of the cube to the lengths of it's sides. From geometry, that formula is that the volume is equal to the length of the sides cubed. Now, we look at this and we want to say, well, what is the estimate of maximum error measuring the volume if we have the possible error of the sides. That's related with, as I mentioned before, the differentials.

So recalling that the derivative dv, ds, is here. We're going to use the general power rule, s cubes derivative is 3s squared. I can then go and say, well, our differential relationship is that dv is equal to 3s squared ds. Now, one of the important things to remember when doing these problems is you don't want to plug any numbers in for an entity in the problem that could be changing until after you find the differential. So now that we have the differential we can go ahead and plug-in the information that we know.

We know we want to have the side length to be 10 inches, so s is 10 inches. The possible error in measuring the sides is 0.25 inches. That's your ds. So now we can go ahead and plug that information in. We're going to plug the 10 inches in for the s and we are going to plug the 0.25 inches in for the ds.

We have dv is equal to 3 times 10 inches squared times 0.25 inches following our order of operations, we have 3 times 100 square inches times 0.25 inches, and that gives us that dv is equal to-- Well, 3 times 100 times 0.25 is 75, and inch squared times inch is cubic inches.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that **differentials can be used to estimate the maximum error** in computing *f* when its input variable, *x*, has known maximum errors. Specifically, if $f(x)$ is a function that depends on an x-variable that has an error of *dx* units, then the differential *df* will provide an estimate of the maximum error in computing *f*. Next, you **applied differentials to situations involving measurement**, such as estimating the error in calculating the volume of a sphere and surface area of a cube.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

The Algorithm for Newton's Method

by Sophia

WHAT'S COVERED

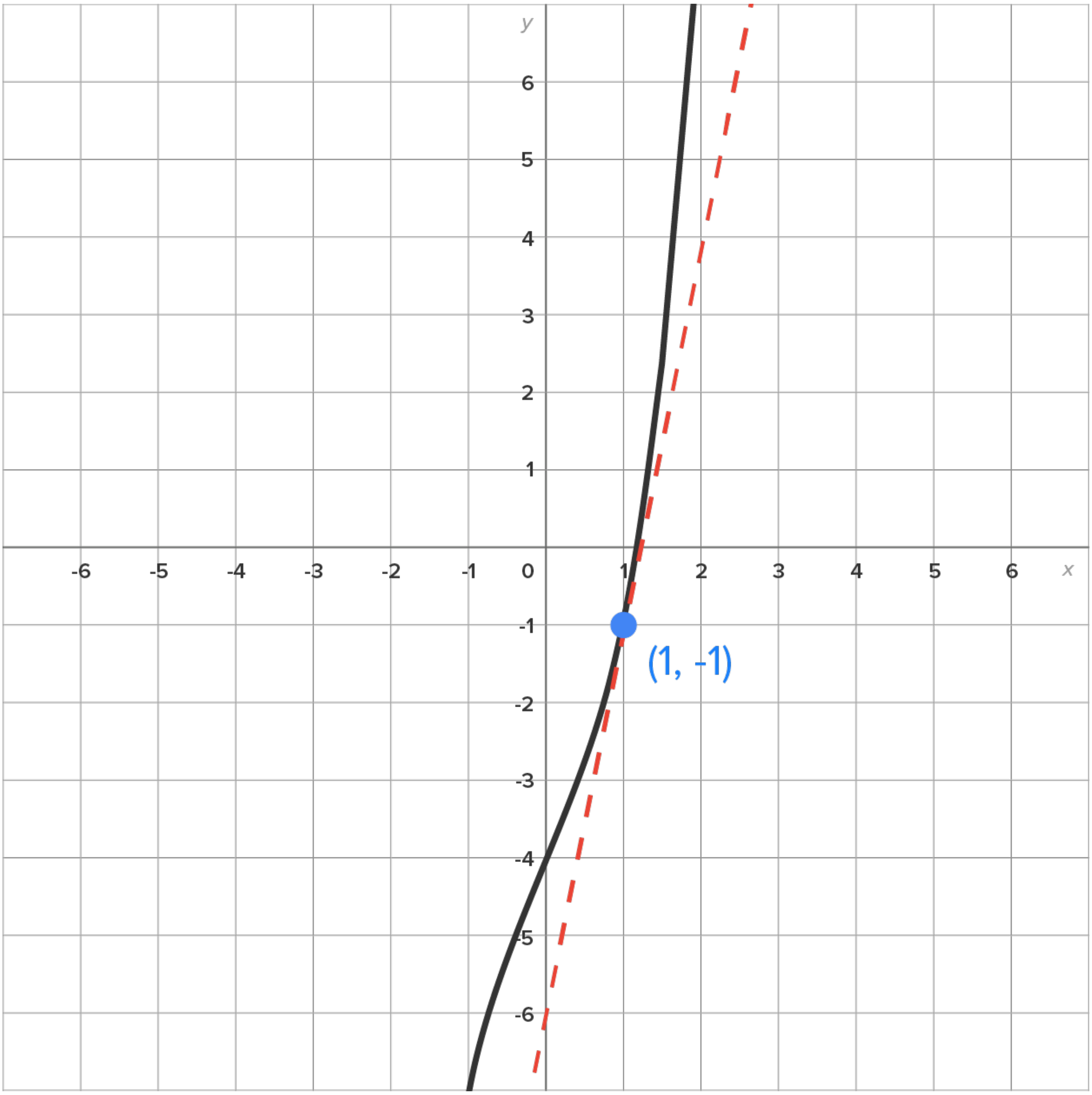
As you have seen in this challenge, the tangent line to a function at $x = a$ provides a good estimate to $f(x)$ near $x = a$. Another way to use the tangent line is to find its x-intercept to approximate the x-intercept of $f(x)$. In this lesson, you will learn Newton's method, which uses successive tangent lines to approximate an x-intercept. Most graphing utilities use Newton's method to locate x-intercepts and points of intersections of graphs. Specifically, this lesson will cover:

1. The Idea Behind Newton's Method
2. Applying Newton's Method
   a. Newton's Method: The Algorithm
   b. Approximating x-Intercepts with Newton's Method

1. The Idea Behind Newton's Method

The goal of Newton's method is to use tangent lines to approximate an x-intercept of the graph of $y = f(x)$. In other words, the goal is to solve the equation $f(x) = 0$.

Consider the function $f(x) = x^3 + 2x - 4$.



Now, consider the picture shown above, which has two graphs:

- The solid curve is the graph of $f(x)$.
- The dashed line is the tangent line at $x=1$ (this corresponds to our "guess").

To start the process for Newton's method, we're going to "guess" $x=1$ as the x-intercept.
Notice that the x-intercept of $f(x)$ is very close to the x-intercept of the tangent line. The advantage of using the tangent line is that it is much easier to solve a linear equation than it is a cubic equation.

First step: Find the equation of the tangent line at $x=1$.

Given $f(x)=x^3+2x-4$, the derivative is $f'(x)=3x^2+2$. Then, the slope of the tangent line is $f'(1)=3(1)^2+2=5$.

Then, the equation of the tangent line is:

$$y = f(1) + f'(1)(x - 1)$$
$$y = -1 + 5(x - 1)$$
$$y = -1 + 5x - 5$$
$$y = 5x - 6$$

Then, the x-intercept of the tangent line is found by letting $y = 0$ and solving for *x*:

$$0 = 5x - 6$$
$$6 = 5x$$
$$\frac{6}{5} = x \text{ (or 1.2 in decimal form)}$$

Thus, our approximation for the x-intercept is (1.2, 0).
So, where would we go from here?

We now have a new “guess” for the x-intercept of the graph of $f(x)$. To continue with this process, find the equation of the tangent line to $f(x)$ at $x = 1.2$, then find its x-intercept. We’ll formalize this process and then complete this problem in the next part of this challenge.

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2. Applying Newton’s Method

Consider a function $y = f(x)$ and let x_0 be the first guess for its x-intercept.

Write the equation of the tangent line at $x = x_0$: $y = f(x_0) + f'(x_0)(x - x_0)$.

Find the x-intercept of the tangent line, which means $y = 0$:

$0 = f(x_0) + f'(x_0)(x - x_0)$	Replace *y* with 0.
$-f(x_0) = f'(x_0)(x - x_0)$	Subtract $f(x_0)$ from both sides.
$-\dfrac{f(x_0)}{f'(x_0)} = x - x_0$	Divide both sides by $f'(x_0)$.
$x_0 - \dfrac{f(x_0)}{f'(x_0)} = x$	Add x_0 to both sides.

Now, this x-intercept is the next guess for the intercept, which under normal conditions, is a closer estimate than x_0. Since this process will continue, let’s call the x-intercept of the tangent line x_1. Then, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Now, suppose we want to continue this process:

- Find the equation of the tangent line at $x = x_1$.
- Find the x-intercept of the tangent line and call it x_2. Then, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

If we continue this process, we get a sequence of estimates x_0, x_1, x_2, ... for the estimates of the x-intercept that get closer to some number (which would be the actual x-intercept). Performing these iterations is what is known as Newton’s method.

2a. Newton's Method: The Algorithm

Suppose the goal is to find an approximation to an x-intercept of a function $y=f(x)$, which is equivalent to finding a solution to $f(x)=0$. Starting with an initial guess at $x=x_0$, the sequence of guesses x_1, x_2, x_3, ... is generated by the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

The process stops when one of two things occurs:

- Two consecutive x-values are "close enough" together.
- The x-values are jumping around to the point where they aren't getting closer to a common number.

FORMULA

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

2b. Approximating x-Intercepts with Newton's Method

EXAMPLE Let's pick back up with the function $f(x)=x^3+2x-4$. When we left off, we had $x_0=1$ and $x_1=1.2$. Let's perform two more iterations of Newton's Method to get a better approximation of the x-intercept. To use Newton's method, it is best to organize the information into a table:

Note: $f(x)=x^3+2x-4$ and $f'(x)=3x^2+2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
0	1	-1	5	1.2
1	1.2	0.128	6.32	1.179746835
2	1.179746835	0.001468379	6.175407787	1.179509057
3	1.179509057	0.0000002	6.173724847	1.179509025

The last two estimates are identical to 6 decimal places, so we conclude that the x-intercept to six decimal places of $f(x)$ is (1.179509, 0). This also means that the equation $x^3+2x-4=0$ has the solution $x\approx 1.179509$.

WATCH

Use Newton's method to find the approximate solution to $x-\cos x=0$.

Video Transcription

Hello, and welcome to today's video on how to complete three iterations of Newton's method to find an approximate solution to the equation x minus the cosine of x is equal to 0. Now, Newton's method is a method to approximate the x-coordinates of the x-intercepts of a graph.

And it's an iterative method, meaning that you run a value through the process, and you take the value you get out after running it through the process and run it through the process again, getting hopefully a more refined or closer approximation than the previous one. And you can keep doing that over and over to get closer and closer approximations to your original equation or x-coordinate of your x-intercept.

There are some situations where it actually does not get us closer to a solution, but for all of the problems that we will do in this course, Newton's method will work. So how do we do this? We will take our initial guess and put it in our table. And then what we'll do after that is run that initial guess through the function, run the initial guess through the derivative of the function.

And then our Newton's method process is, we take that initial guess and subtract off of it the function evaluated at that initial guess divided by the derivative evaluated at that initial guess. And we'll get a value out that will be our first iteration. Then we will bring that down into the next row of the table and do the process all over again.

So when we're starting this and we have our equation that we want to use Newton's method on, that x minus cosine x is equal to 0, because we want to apply the process of Newton's method, we need to make sure that our equation is set up so that everything is on one side equal to 0 on the other side.

So it's nice that this is already set up in that manner. When we have it that way, then our function we take is just the expression that's on the side opposite of 0. So we have our x minus cosine x. Now, I've drawn the graph of f of x equal x minus cosine x just to the left of what I just wrote. And if you notice, the curve is crossing the x-axis really close to an x value of 1.

So our initial guess is 1. So you want to get that from, actually, just drawing the graph of the function and seeing the integer x value that is really close to where the graph crosses. Now we want to find f of our x value, our first guess. And f of 1 is equal to 1 minus the cosine of 1.

Make sure you set your calculator in radian mode, and when you calculate 1 minus the cosine of 1, you'll get 0.4596976941. And that's what we'll put in our table right underneath the f of x sub n. So we have our 0.4596976941.

Next, we want f prime of x sub n. Well, f of x is x minus cosine x. So f prime of x is equal to-- well, the derivative of x is 1, minus the derivative of cosine x is negative sine x. So the derivative f prime of x is equal to 1 plus sine x. For our f prime of 1, we get 1 plus the sine of 1. And f prime of 1 then is about equal to, approximately equal to, 1.841470985.

And we don't want to round these at all. We want to use this with all the digits in place, or we cause the problem not to be able to get the accuracy that we can get from Newton's method. So that's going to go into our table under the f prime of x sub n. So we've got our 1.841470985.

And now for our x sub 1, we are going to get that by taking x sub 0, which is 1, minus f of x sub 0, which is the 0.4596976941, divided by f prime of 1, so divided by 1.841470985. And when you calculate that, you'll have 0.7503638678.

And that's what we will put down underneath the x sub n and across from the 1, meaning that it's x sub 1.

Now we're going to go through the process again. So we are going to take f of this 0.7503638678. And once we run it through the function, we'll get our value out of 0.0189230738.

When we run it through the derivative, so remember, we're running that 0.7503638678 through the derivative as well, and you'll get 1.68190495291. And then we will take our 0.7503638678 minus the quotient of 0.0189230738 and 1.68190495291. And you'll have the next iteration approximation of the 0.7391128909.

So now, I'm going to just go ahead and fill out this table. And then you'll see what we have in our box under the x sub n across from the 3 will be our value for our third iteration. So here, I've filled out the rest of the table, and our final approximation after three iterations is reporting that entire decimal expression there.

So we have our x sub 3 is approximately equal to the 0.7390851334. And that's Newton's method to approximate the solution of x minus cosine x equals 0 and completing three iterations.

SUMMARY

In this lesson, you learned **the idea behind Newton's method**, which is to use tangent lines to approximate an x-intercept of the graph of $y=f(x)$. Newton's method is a very straightforward approximation method designed to solve equations of the form $f(x)=0$ (equivalent to finding the x-intercepts of the graph of $y=f(x)$). You learned how to **apply Newton's method** using its **algorithm**, by starting with an initial guess at $x=x_0$, then generating a sequence of guesses x_1, x_2, x_3, ... to arrive at a close **approximation of the x-intercept**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

Implicit Differentiation

by Sophia

WHAT'S COVERED

In this lesson, you will apply techniques of derivatives when the equation defines y implicitly. So far, we know how to find derivatives when y is explicitly a function of x, meaning $y=f(x)$. The equation $x^2+2y^2=22$ is an example of an equation where y is defined implicitly, meaning y is not isolated to one side. The equation still defines a curve, so it makes sense to discuss the derivative and slopes of tangent lines, etc. Specifically, this lesson will cover:

1. Implicit Differentiation
2. Slopes and Equations of Tangent Lines

1. Implicit Differentiation

If y is some function of x, we know that the derivative of y is $\frac{dy}{dx}$.

Then, by the chain rule, we know the following:

$$\frac{d}{dx}[y^2]=2yD[y]=2y\frac{dy}{dx}$$

$$\frac{d}{dx}[\sin y]=\cos yD[y]=\cos y\frac{dy}{dx}$$

$$\frac{d}{dx}[\ln y]=\frac{1}{y}D[y]=\frac{1}{y}\frac{dy}{dx}$$

Now, consider the equation $x^2+2y^2=22$, where y is some function of x. If we take the derivative of both sides of the equation with respect to x, we get:

$\frac{d}{dx}[x^2]+\frac{d}{dx}[2y^2]=\frac{d}{dx}[22]$	Use the sum/difference rules.
$2x+4y\frac{dy}{dx}=0$	$D[x^2]=2x$, $D[2y^2]=4y\frac{dy}{dx}$, $D[22]=0$

At this point, notice that $\frac{dy}{dx}$ is a quantity in the equation. In order to get an expression for $\frac{dy}{dx}$, we solve for it as if it were a variable.

$2x+4y\frac{dy}{dx}=0$	Start where we left off.
	Subtract $2x$ from both sides.

$4y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{2x}{4y}$ Divide both sides by $4y$.

$\frac{dy}{dx} = -\frac{x}{2y}$ Simplify the fraction to its lowest terms.

This means that $\frac{dy}{dx} = -\frac{x}{2y}$. Note that the expression is written in terms of both x and y. This is very common with implicit differentiation.

STEP BY STEP

To find $\frac{dy}{dx}$ implicitly, perform these steps to the equation.

1. Differentiate both sides with respect to x.
2. Collect all terms with $\frac{dy}{dx}$ to one side.
3. Solve for $\frac{dy}{dx}$.

↪ **EXAMPLE** Now, let's look at another example. Given $2x^2+3xy+4y^2=100$, compute $\frac{dy}{dx}$.

$2x^2+3xy+4y^2=100$	Start with the original relation.
$\frac{d}{dx}[2x^2]+\frac{d}{dx}[3xy]+\frac{d}{dx}[4y^2]=\frac{d}{dx}[100]$	Apply the derivative to each term (use the sum/difference rule).
$4x+3(y)+3x\frac{dy}{dx}+8y\frac{dy}{dx}=0$	$D[2x^2]=4x$ $D[3xy]=D[3x\cdot y]=(3)y+3x\frac{dy}{dx}$ (product rule) $D[4y^2]=8y\frac{dy}{dx}$
$3x\frac{dy}{dx}+8y\frac{dy}{dx}=-4x-3y$	Subtract $4x$ and $3y$ from both sides.
$(3x+8y)\frac{dy}{dx}=-4x-3y$	Factor out $\frac{dy}{dx}$.
$\frac{dy}{dx}=\frac{-4x-3y}{3x+8y}$	Divide both sides by $3x+8y$.

Thus, $\frac{dy}{dx}=\frac{-4x-3y}{3x+8y}$.

TRY IT

Consider the equation $10x^2y^2+4x^3-3y^5=11$.

Find the derivative implicitly. +

$$\frac{dy}{dx}=\frac{-20xy^2-12x^2}{20x^2y-15y^4}$$

WATCH

Here is a video in which we find $\frac{dy}{dx}$ of $\cos(xy) = -\frac{1}{2} + e^y$.

Video Transcription

Hi, there. Welcome to the video and how to use implicit differentiation to find dy dx when I have a mathematical relation where the variables are mixed together and one's not sorted out on one side of the equal sign?

Well, we can still find the derivative without having to go through and trying to solve it by this implicit differentiation. Here, our mathematical relation is the cosine of xy is equal to negative 1/2 plus e to the y.

Now when we are using implicit differentiation, it's really important to look at what they're asking you to find. When we are asked to find the derivative, we want to note that what's in our numerator, dy, that's telling me that y is the dependent variable. And the denominator tells me the variable x is the independent variable. So it's very important to pay attention to that so that you know the roles that they play.

Now implicit differentiation-- we need to be careful as we go through this to use our derivative rules that we've already learned. And when we are differentiating the independent variable-- here the x-- that we will just go ahead with our rules. We don't need to do any extra factor there. But when we are differentiating and the process involves doing that on the dependent variable, we don't know what the dependent variable is equal to in terms of the independent variable, so we need to use the chain rule, which is to also multiply by the notation of the dy dx.

So let's get started and see how that plays out. So on the left-hand side of the equal sign, I have the cosine of the angle xy. Now remember, the derivative of cosine of an angle is negative sine the angle times the derivative of the angle. Now the derivative of xy is the derivative of a product. So I'm going to need to apply the product rule when I'm differentiating that inside product. So remember, that's the derivative of the first factor-- the derivative of x is 1-- and it's the x variable times the second factor unchanged plus the first factor times the derivative of the second factor.

Well, the second factor here is the letter y. The derivative of y is 1, but it's the dependent variable. So by the chain rule, I also have to multiply by dy dx. And this is equal to. On the right-hand side, the first term is just a constant, just a number. The derivative of a number is 0, and then plus the derivative of e to the y. Well, the derivative of e to the power is e to the power times the derivative of the power. And the derivative of y is 1, but it's the dependent variable, so I have to multiply by dy dx.

Now let's just rewrite that so it's a little bit more manageable to look at. So I have negative sine of xy times y plus x dy dx is equal to e to the dy dx. Next, what we need to do is to remove our parentheses that we can and sort all the terms with the dy dx on one side of the equation and all the terms without the dy dx on the other side. So I have negative sine xy times y. That's a negative y sine of xy. And then negative sine of xy times x dy dx-- well, negative times positive is a negative. x sine of xy times dy dx is equal to e to the y dy dx.

Now when I look at these terms, the first term does not have a dy dx in it. The second term does have a dy dx. And then I have my equal sign, and then on the right-hand side, I also have a dy dx. I want all the dy dx terms on the same side of the equal sign. So we are going to add this x sine of xy dy dx to both sides.

So I have negative y sine of xy is equal to e to the y dy dx plus x sine of xy dy dx. And have there been any terms on the right that didn't have the dy dx, I would have moved those over to the left.

Now what we've done by this is we've forced a common factor of dy dx in each of the terms on the right-hand side. So we are going to factor the did x out of the terms on the right-hand side. And we're going to factor it out the back. So we have e to the y plus x sine of xy and then dy dx. And then to solve for dy dx, divide both sides by that, we'll just multiply to it. So we're going to divide both sides by the e to the y plus x sine of xy.

And so our final result is that dy dx is equal to negative y sine of xy divided by e to the y plus x sine of xy. And that is how you use implicit differentiation to find dy dx for cosine of xy is equal to negative 1/2 plus e to the y.

Once we know the derivative, it is possible to find the slope of the tangent line, then the equation of the tangent line.

Since the implicit derivatives use the notation $\frac{dy}{dx}$ for the derivative, we need a way to show that we are evaluating the derivative at a point.

BIG IDEA

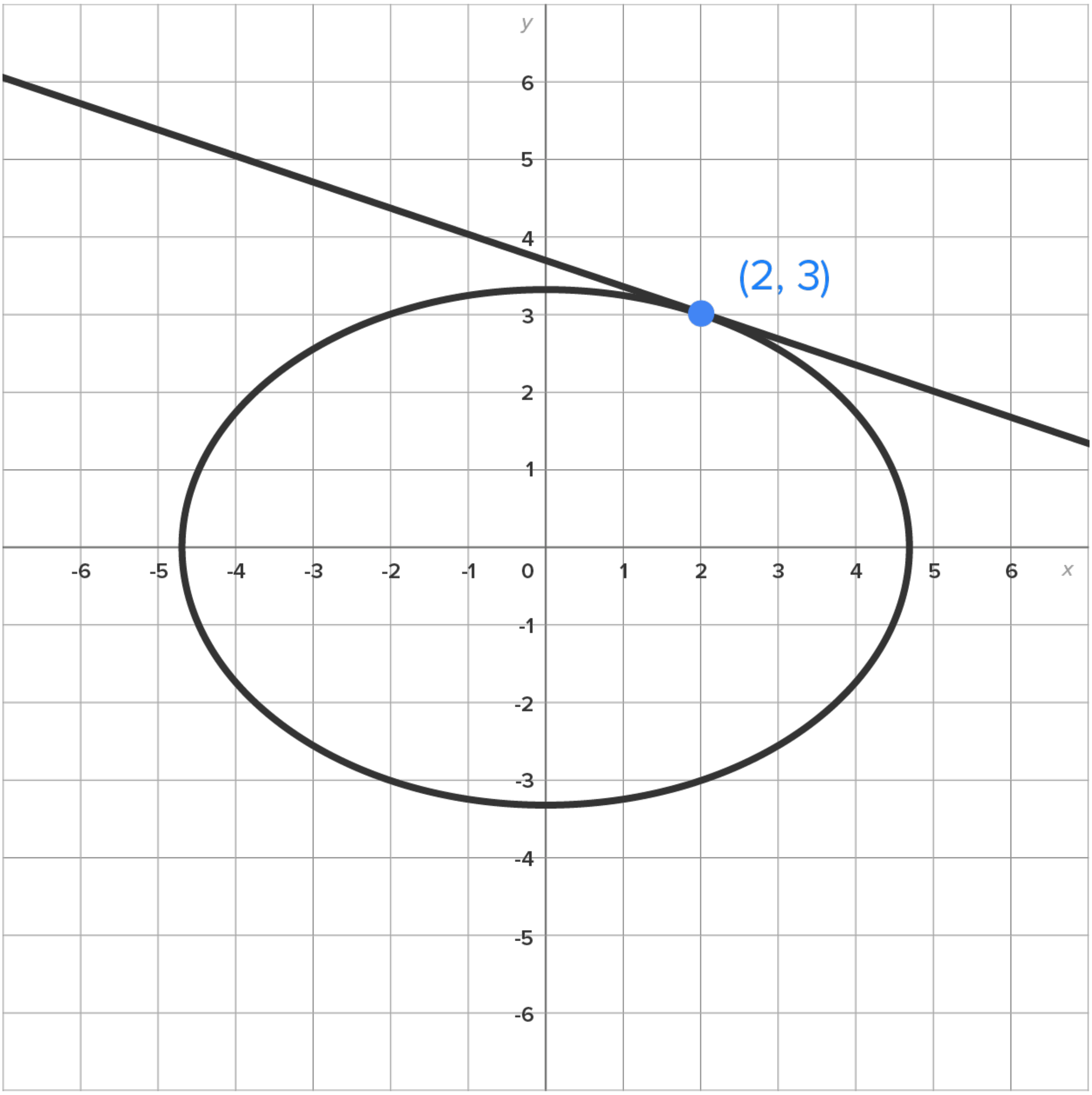
The notation $\left.\frac{dy}{dx}\right|_{(a,\ b)}$ means to evaluate $\frac{dy}{dx}$ when $x = a$ and $y = b$.

Now, we are ready to find slopes of tangent lines with implicit functions.

2. Slopes and Equations of Tangent Lines

Earlier in this challenge, we computed $\frac{dy}{dx}$ for the curve $x^2 + 2y^2 = 22$.

Shown in the graph below is the curve (the ellipse), and its tangent line at the point (2, 3).



The derivative formula we calculated earlier is $\frac{dy}{dx} = -\frac{x}{2y}$.

Then, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(2,\ 3)} = -\frac{2}{2(3)} = -\frac{1}{3}$.

To write the equation of the tangent line, we normally need $f(a)$ and $f'(a)$. Since y is defined implicitly, we do not have the "f" notation. That being the case, we'll make use of the point-slope form of a line.

Now, let's find the equation of the tangent line.

$y - y_1 = m(x - x_1)$	Use the point-slope form.
$y - 3 = -\frac{1}{3}(x - 2)$	The line passes through (2, 3) and has slope $-\frac{1}{3}$.
$y - 3 = -\frac{1}{3}x + \frac{2}{3}$	Distribute $-\frac{1}{3}$.

$y = -\frac{1}{3}x + \frac{11}{3}$ Add 3 to both sides.

The equation of the tangent line is $y = -\frac{1}{3}x + \frac{11}{3}$.

TRY IT

Consider the curve $2x^2 + 3xy + 4y^2 = 100$ with $\frac{dy}{dx} = \frac{-4x - 3y}{3x + 8y}$.

Write the equation of the line tangent to this curve at the point (0, 5). +

$y = -\frac{3}{8}x + 5$

WATCH

Watch this video to see an example of writing an equation of a tangent line to $x^2 + 2xy + 4y^2 = 12$ at the point (2, 1).

Video Transcription

[BRIGHT MUSIC] Welcome to the video on how to write the equation of a tangent line to a mathematical relation at a given point. So for this problem, we are asked to write the equation of the line tangent to x squared plus 2xy plus 4y squared equal 12 at the point 2,1. Now recall, to write the equation of a line, what we need is first, the slope, second, a point that the line goes through, and then we plug that information into our point slope form, y minus y sub 1 is equal to m times x minus x sub 1.

Let's look at the first thing, the slope. We'll recall that the slope of a tangent to a curve is given by the value of the derivative there. So the derivative is the equation that you put your values in, and you get your slope of your tangent out. So what we need to do, then, is find the derivative of our mathematic relation. Now here, the x's and the y's are intermixed in the equation. And so the relation is implied between the x's and y's, meaning that when we do our derivative, we can use our implicit differentiation.

Recall implicit differentiation. We're going to follow all of our derivative rules, and when we're differentiating the independent variable, we will just do the derivative rule. When we are taking the derivative of the dependent variable, once we do the derivative rule, we need to multiply by dy, dx for the chain rule. So let's get started. Here, I'm going to differentiate term by term.

The derivative of x squared is 2x, and that's the independent variable I differentiated. So we just do the rule and move on. In the next term, I have, actually, a multiplication of 2x, which is an expression with the independent variable, and then y, which is my dependent variable. When I'm differentiating a product of variable expressions by the derivative rules, I have to use the product rule.

So we're going to take the derivative of the first factor-- and the derivative of 2x is 2-- times-- keep the second factor y alone-- and then plus-- this time, we're going to leave the 2x factor the way it is, and we are going to multiply that by the derivative of that second factor, y. But y is the dependent variable. So once I apply the derivative rule, I'm going to also multiply by dy, dx.

Well, the derivative rule for y is 1. But it's the y letter, so I multiply by dy, dx. And then plus-- now my last term, this 4y squared. That is a term that involves the dependent variable, y. So again, we apply the derivative rule. 4 times my variable base to the number 2 power is the general power rule, so I have 8y.

But it was the y letter that we were working the derivative on, so once I do that by the chain rule, I need to multiply by dy, dx. And that's equal to. And the derivative of 12 on the right-hand side-- well, 12 is a number. It's a constant. And the derivative of constant term is 0.

OK. So this gives us our implicit differentiation. Next, what we want to do is solve for dy, dx. So I'm going to write this in a little bit friendlier manner, by within the multiplications, write that with whatever multiplications I can do out. And then we want all the terms with dy, dx on one side of the equal sign. So my equal sign is there, and my 2x dy, dx plus my 8y dy, dx both have dy, dxx in them.

And any term that doesn't have the dy, dx, we are going to move over to the other side. So we are going to subtract a 2x and subtract a 2y from both sides. And we will get 2x dy, dx plus 8y dy, dx is equal to negative 2x minus 2y. Factor out the dy, dx. I have 2x plus 8y.

Quantity times dy, dx is equal to negative 2x minus 2y. And then divide both sides by what's ever multiplied to your dy, dx. So dy, dx is equal to negative 2x minus 2y over 2x plus 8y.

Now, that's the formula that gives me the slope of that mathematical relation at any point on the curve. I specifically want to know the slope at the .21, so we are going to take that derivative dy, dx, and we are going to evaluate it-- so we do this vertical line to show we're going to evaluate it-- at the ordered pair 2,1. And remember, in the ordered pair, your first coordinate is your x-coordinate, and your second coordinate is your y-coordinate. So we're going to come over to our formula. Wherever we see an x, we're going to take it out and put in 2. Wherever we see a y, we're going to take it out and put in 1.

So I have negative 2 times 2 minus 2 times 1 over 2 times 2 plus 8 times 1. So that gives me negative 4 minus 2, which is negative 6, over 4 plus 8, which is 12. And so my dy, dx evaluated at 2,1 is negative one-half. And that is my slope of my line.

Now, the point is actually given to me with both coordinates. So I have my point is 2,1. So now, remember that's your x sub 1, y sub 1. And so our equation is y minus the y-coordinate of the point 1 is equal to negative one-half times x minus the x-coordinate of the point 2. Removing the parentheses, I get y minus 1 equals negative one-half x plus 1. Adding 1 to both sides, I get y is equal to negative one half x plus 2. And that is the equation of the tangent line to x squared plus 2xy plus 4y squared equals 12 at the point 2,1.

SUMMARY

In this lesson, you learned that through **implicit differentiation**, it is possible to find the derivative of a mathematical relation that is not explicitly solved for y. You also learned that in an equation where y is defined implicitly, when asked to write the equation of a tangent line, you will be given a point on the curve; therefore, you can use the **point-slope form to write the equation of the tangent line**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Logarithmic Differentiation

by Sophia

WHAT'S COVERED

In this lesson, you will find derivatives of combinations of products, quotients, and powers by using a technique called logarithmic differentiation. Specifically, this lesson will cover:

1. Defining Logarithmic Differentiation
2. Finding Derivatives Using Logarithmic Differentiation

1. Defining Logarithmic Differentiation

While we have a vast set of rules we could use to find the derivative of most any function, there are still functions beyond our reach.

For example, consider the function $y = x^x$. While we can find the derivative of $y = x^n$ (exponent is constant) and $y = a^x$ (base is constant), there is no rule that handles the case when both the base and exponent are variables.

Also, consider the function $y = \frac{x \cdot \cos x}{2x + 1}$. While this could be differentiated using the quotient and product rules, it would be very lengthy and cumbersome.

These are examples of functions that could benefit from logarithmic differentiation. Here is how it works:

STEP BY STEP

Given $y = f(x)$:

1. Take the natural logarithm of both sides: $\ln y = \ln f(x)$.
2. Rewrite the right-hand side using properties of logarithms.
3. Take the derivative of both sides with respect to x (implicitly).
4. Solve for $\frac{dy}{dx}$.

HINT

This technique is only useful when $f(x)$ is some combination of powers, products, or quotients.

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2. Finding Derivatives Using Logarithmic Differentiation

↪ EXAMPLE Using logarithmic differentiation, find the derivative of $y = x^x$. Since both the base and power are variables, logarithmic differentiation will be useful.

$y = x^x$	Start with the original function.
$\ln y = \ln x^x$	Apply the natural logarithm to both sides.
$\ln y = x\ln x$	Use the property of logarithms: $\ln a^b = b\ln a$
$\frac{1}{y}\cdot\frac{dy}{dx} = (1)\ln x + x\left(\frac{1}{x}\right)$	Take the derivative of both sides: $D[\ln y] = \frac{1}{y}\cdot\frac{dy}{dx}$ Apply the product rule for $D[x\cdot\ln x] = D[x]\cdot\ln x + x\cdot D[\ln x]$.
$\frac{1}{y}\cdot\frac{dy}{dx} = \ln x + 1$	Simplify the right-hand side.
$\frac{dy}{dx} = y(\ln x + 1)$	Solve for $\frac{dy}{dx}$ by multiplying both sides by y.
$\frac{dy}{dx} = x^x(1 + \ln x)$	Substitute $y = x^x$ so that the derivative is a function of x alone.

Thus, the derivative is $\frac{dy}{dx} = x^x(1 + \ln x)$.

TRY IT

Consider the function $y = (\sin x)^x$.

Using logarithmic differentiation, find the derivative. +

$$\frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x\cot x]$$

Now, let's see how logarithmic differentiation can help with a combination of products, quotients, and/or powers.

↪ EXAMPLE Using logarithmic differentiation, find the derivative of $y = \frac{x\cdot\cos x}{2x+1}$. Since this function is a product within a quotient, logarithmic differentiation will be very useful to compute the derivative.

$y = \frac{x\cdot\cos x}{2x+1}$	Start with the original function.
$\ln y = \ln\left(\frac{x\cdot\cos x}{2x+1}\right)$	Apply the natural logarithm to both sides.
$\ln y = \ln x + \ln\cos x - \ln(2x+1)$	Use the properties of logarithms on the right-hand side.
$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos x}\cdot(-\sin x) + \frac{1}{2x+1}\cdot 2$	Find the derivatives of each term.
$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} - \tan x + \frac{2}{2x+1}$	Simplify the right-hand side.
$\frac{dy}{dx} = y\left(\frac{1}{x} - \tan x + \frac{2}{2x+1}\right)$	Solve for $\frac{dy}{dx}$ by multiplying both sides by y.

$\frac{dy}{dx}=\frac{x\cdot\cos x}{2x+1}\left(\frac{1}{x}-\tan x+\frac{2}{2x+1}\right)$ Substitute $y=\frac{x\cdot\cos x}{2x+1}$ so that the derivative is a function of x alone.

Thus, $\frac{dy}{dx}=\frac{x\cdot\cos x}{2x+1}\left(\frac{1}{x}-\tan x+\frac{2}{2x+1}\right)$.

TRY IT

Consider the function $y=x^2e^{-4x}\sin x$.

Using logarithmic differentiation, find the derivative. +

$$\frac{dy}{dx}=x^2e^{-4x}\sin x\left(\frac{2}{x}-4+\cot x\right)$$

WATCH

Use logarithmic differentiation to find the derivative of $y=\sqrt[7]{\frac{x^2(2x-7)^3}{(4x+5)^5}}$.

Video Transcription

Hello, and welcome to another example of how to use logarithmic differentiation to find dy dx for a function that involves products, powers, quotients, and also if in another situation you have a variable base to a variable power. In this particular problem, we want to find the derivative of y equals the seventh root of x squared, times the quantity 2x minus 7 to the third power, divided by the quantity 4x plus 5 to the fifth power.

So this has a lot of powers, products, and a quotient. And it's perfectly suited to work using our logarithmic differentiation. Now notice when I have this function, I can rewrite my expression to the 1/7 power to write it without using the radicals, but instead using my fractional exponents.

Then next step, we see that with all the products and powers, there's not a logarithm originally in this question. But I know that with the way this function is, with all those products, powers, and quotients, having a log would actually help immensely. So I'm going to incorporate the natural log. But since I'm doing it as a new process, I have to incorporate the natural log to both sides of the equation.

Now what does that do for me? What it does-- it allows me to use the properties of logarithms to expand out that right-hand side. And it helps me enormously not having to do as in-depth with as many layers of the chain rule.

So what we're going to do is we're going to use the properties of logarithms to expand that right-hand side. So we have the natural log of y is equal to, well, this is the natural log of a quantity to a power. So that 1/7 overall exponent can come down in front.

Next, the natural log of a quotient is the natural log of the numerator minus the natural log of the denominator. And since I'm turning it into two terms, I need to put brackets around the expression.

Next, the natural log of a product-- and I have that in my first term-- is the natural log of the first factor plus the natural log of the second factor, so natural log of x squared plus the natural log of 2x minus 7 to the third power, minus the natural log of 4x plus 5 to the fifth power. And then I can bring the exponents in the individual terms of the natural logs acting on quantities to powers as coefficients just in front of those terms. So the natural log of y is equal to 1/7 times-- so this is 2 times the natural log of x, plus 3 times the natural log of 2x minus 7, and then minus 5 times the natural log of 4x plus 5.

Now again, these are all properties of logarithms that we're using. We haven't differentiated yet. Then one more step before we differentiate-- we're going to distribute that 1/7 through. So I have the natural log of y is equal to 2/7 times the natural log of x, plus 3/7 times the natural log of 2x minus 7, minus 5/7 times the natural log of 4x plus 5.

Now that I have this out as term by term and I've brought all the exponents down, I've simplified it as much as I can with the properties of logarithms, we are going to go through and take the derivative. And because there's an action on the dependent variable y, we need to use our implicit differentiation on that left-hand side when we go through and do that.

So remember, the natural log of something is 1 over the something times the derivative of the something. And the derivative of y is 1 dy dx. So that's where your implicit differentiation comes in.

Now on the right, we're going to differentiate this, term by term. So I have 2/7 times the derivative of the natural log of x is 1 over x. And then plus 3/7 times the derivative of the natural log of 2x minus 7 is 1 over 2x minus 7 times the derivative of x minus 7. And the derivative of x minus 7 is 2, and then minus 5 over 7 times the derivative of the natural log of 4x plus 5, and that derivative is 1 over 4x plus 5, times the derivative of 4x plus 5, which is 4.

So now simplifying this, we're going to move it up a little bit to get some more space, I have 1 over y, dy dx, is equal to 2 over 7x plus 3 over 7 times 1 over 2x minus 7 times 2 is 6 over 14x minus 49, and then minus, and I have a 5 over 7 times 1 over 4x plus 5 times 4 over 1. So that is a 20 over 28x plus 35.

And then we have just actually two more steps to do. The next thing-- we want to solve for dy dx. So we are going to multiply both sides by y over 1. And it's the entire side that gets multiplied by y over 1.

So for that, our left side, the y's cancel. And we get dy dx is equal to. And then on the right side, I do have that y over 1 as a factor in front but we know what y is worth. Y is worth the seventh root of x squared times 2x minus 7 cubed over 4x plus 5 to the fifth. So we want to replace our y with what it's worth.

So we have the seventh root of x squared times 2x minus 7 cubed over 4x plus 5 to the fifth. That's from the y. And then open your brackets, and you've got 2 over 7x plus 6 over 14x minus 49 minus 20 over 28x plus 35.

And that is the derivative of our original function using logarithmic differentiation.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that through **logarithmic differentiation**, it is now possible to find the derivative of functions with variable bases and powers (such as $y = x^x$). It is also much simpler to **find derivatives** of combinations of products, quotients, and powers **using logarithmic differentiation**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

The Inverse Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn about the inverse trigonometric functions and how they are evaluated.

1. The Inverse Trigonometric Functions
2. Evaluating the Inverse Sine, Cosine, and Tangent Functions for Known Ratios
3. Evaluating the Inverse Cosecant, Secant, and Cotangent Functions for Known Ratios

1. The Inverse Trigonometric Functions

Recall the six basic trigonometric functions: sin*x*, cos*x*, tan*x*, sec*x*, csc*x*, and cot*x*.

For each of them, the input is some angle and the output is a real number.

The **inverse trigonometric functions** do just the reverse. The input is the real number, while the output is the angle that produces the ratio.

For example, we define the inverse sine function as $y = \sin^{-1}x$, which means $x = \sin y$. Looking at the equation $x = \sin y$, it's clear that *x* must be between -1 and 1 (inclusive) since the sine function only returns ratios between -1 and 1.

The six inverse trigonometric functions, with their domains and ranges, are summarized in the table below.

Function	Domain	Range
$y = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x$	All real numbers	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \csc^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \cot^{-1}x$	All real numbers	$(0, \pi)$

Note: when you use your calculator to evaluate an inverse trigonometric function, it will return the correct value.

HINT

The inverse trigonometric functions often go by other names. For example, $\sin^{-1}x$ can also be written as $\arcsin x$. This is sometimes more convenient since the "-1" in $\sin^{-1}x$ is often mistaken for an exponent of -1. Naturally, the other trigonometric functions follow suit. For example, $\tan^{-1}x$ is also known as $\arctan x$, etc.

TERM TO KNOW

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

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2. Evaluating the Inverse Sine, Cosine, and Tangent Functions for Known Ratios

Recall that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Then, we can say $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ or $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

When evaluating inverse trigonometric functions, we need to keep the range in mind.

EXAMPLE $\sin^{-1}(1) = \frac{\pi}{2}$ since $\frac{\pi}{2}$ is inside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ since $\frac{2\pi}{3}$ is inside the interval $[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is inside the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

---

3. Evaluating the Inverse Cosecant, Secant, and Cotangent Functions for Known Ratios

Most calculators do not have dedicated buttons for $\csc^{-1}x$, $\sec^{-1}x$, or $\cot^{-1}x$; as you might suspect, these are related to their corresponding reciprocal functions.

For example, let's say we wish to find $\csc^{-1}(2)$.

This means that we want to find y so that $\csc y = 2$.

Since $\csc y$ and $\sin y$ are reciprocals, this is equivalent to writing $\sin y = \frac{1}{2}$, which means $y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Through all this, something to notice is that $\csc^{-1}(2) = \sin^{-1}\left(\frac{1}{2}\right)$. This leads to some important identities.

FORMULA

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

EXAMPLE Find $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$.

$$\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Note: $\frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{2\cdot 3} = \frac{\sqrt{3}}{2}$

EXAMPLE Find $\cot^{-1}(-1)$.

$$\cot^{-1}(-1) = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

EXAMPLE Find $\csc^{-1}\left(\frac{1}{2}\right)$.

$$\csc^{-1}\left(\frac{1}{2}\right) = \sin^{-1}(2)$$

Since $\sin^{-1}(2)$ is undefined, $\csc^{-1}\left(\frac{1}{2}\right)$ is undefined as well.

TRY IT

Consider the following inverse trigonometric function:

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	?
$\sec^{-1}(\sqrt{2})$	?
$\tan^{-1}(-\sqrt{3})$	?

Find the exact value of each inverse trigonometric function. +

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	$\frac{\pi}{4}$
$\sec^{-1}(\sqrt{2})$	$\frac{\pi}{4}$
$\tan^{-1}(-\sqrt{3})$	$-\frac{\pi}{3}$

SUMMARY

In this lesson, you learned that **the inverse trigonometric functions** provide a way to express the angle as a function of the trigonometric ratio. You also learned how to **evaluate the inverse trigonometric functions for known ratios**, noting that while not all of these inverse functions are available on most calculators, there are identities that can be used to relate to other more common functions.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 7 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

FORMULAS TO KNOW

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

Derivatives of Inverse Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn and use rules to differentiate the inverse trigonometric functions. Specifically, this lesson will cover:

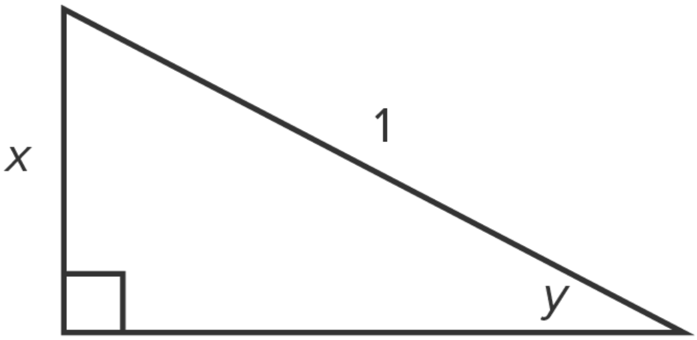
1. Derivatives of the Inverse Trigonometric Functions
2. Derivatives of Functions That Involve Inverse Trigonometric Functions

1. Derivatives of the Inverse Trigonometric Functions

Consider the function $y = \sin^{-1} x$, which is also written $x = \sin y$. To find $\frac{dy}{dx}$, we will use the equation $x = \sin y$ and find the derivative implicitly.

$x = \sin y$	Start with the original equation.
$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$	Set up the derivative on each side.
$1 = \cos y \frac{dy}{dx}$	Take the derivative of each side.
$\frac{dy}{dx} = \frac{1}{\cos y}$	Solve for $\frac{dy}{dx}$.

At this point, it would appear that we are done, but the goal is to get an expression in terms of x alone, instead of a function of y.



To do so, let's use a right triangle with angle y. Since $x = \sin y$, this means the side opposite y is x and the hypotenuse is 1.

By using the Pythagorean theorem, the length of the adjacent side is $\sqrt{1-x^2}$.

Then, $\cos y = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Thus, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$.

In summary, $D[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$.

Through similar reasoning, the derivatives of all six inverse trigonometric functions are shown below. Note that each formula has the basic version (with x as the variable) and the chain rule version (with u as the variable, where u represents a function of x.)

FORMULA

Derivative of the Inverse Sine Function

$$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\sin^{-1}u] = \frac{u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Cosine Function

$$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cos^{-1}u] = \frac{-u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Tangent Function

$$\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{u'}{1+u^2}$$

Derivative of the Inverse Cotangent Function

$$\frac{d}{dx}\left[\cot^{-1}x\right] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left[\cot^{-1}u\right] = \frac{-u'}{1+u^2}$$

Derivative of the Inverse Secant Function

$$\frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\sec^{-1}u\right] = \frac{u'}{|u|\sqrt{u^2-1}}$$

Derivative of the Inverse Cosecant Function

$$\frac{d}{dx}\left[\csc^{-1}x\right] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\csc^{-1}u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

---

2. Derivatives of Functions That Involve Inverse Trigonometric Functions

With our new derivative rules, we can now find derivatives of functions that contain inverse trigonometric functions.

→ EXAMPLE Find the derivative of $y = \tan^{-1}(2x)$.

$y = \tan^{-1}(2x)$	Start with the original equation.
$\frac{dy}{dx} = \frac{2}{1+(2x)^2}$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$, $u = 2x$, $u' = 2$
$\frac{dy}{dx} = \frac{2}{1+4x^2}$	Simplify.

Thus, $\frac{dy}{dx} = \frac{2}{1+4x^2}$.

→ EXAMPLE Consider the function $f(x) = x^2 \cdot \sin^{-1}x$. Find its derivative.

$f(x) = x^2 \sin^{-1}x$	Start with the original equation.
$f'(x) = 2x\sin^{-1}x + x^2\frac{1}{\sqrt{1-x^2}}$	Use the product rule with x^2 and $\sin^{-1}x$.

$$f'(x) = 2x\sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$$ Simplify.

Thus, $f'(x) = 2x\sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$.

TRY IT

Consider the function $f(x) = \cos^{-1}(x^3)$.

Find the derivative. +

$$f'(x) = \frac{-3x^2}{\sqrt{1-x^6}}$$

WATCH

Find the derivative of $y = 2x^3\arctan(5x^2+3)$.

Video Transcription

[MUSIC PLAYING] Welcome to the video on how to find a derivative dy/dx of a function that involves an inverse trig function. Now, remember, we can use the notation for an inverse trig function with the inverse notation or with arc in front of the abbreviation for the trig function.

So involved in this, I see arctan of a ratio. So recalling that the derivative of inverse tan of u is equal to u prime over 1 plus u squared, then that also is the derivative of arctan of u. Just a different notation for the exact same concept. So I have that involved in my function as a second factor. And as a first factor, I have to execute.

So now to find the derivative, we see that the dependent variable y is on a side by itself. So we'll look at this derivative. But we also notice that the 2x cubed is multiplied to the arctan of 5x squared plus 3. So we are going to have to use the product rule because of that, the derivative we just learned of our tan because of the second factor, and then we're using the chain rule form of it because arctan is acting on an expression 5x squared plus 3.

Let's get started. So we have dy/dx. Well, that's going to be the derivative of the first factor, 2x cubed, times the second factor, arctan of 5x squared plus 3, and then plus the first factor, 2x cubed, times the derivative of the second factor, arctan of 5x squared plus 3.

So now, doing those-- applying those derivative rules, the derivative of 2x cubed is 6x squared. And then leave that arctan the way it is. And then plus-- keep the 2x cubed. And this time multiply by the derivative of arctan of the 5x squared plus 3.

So u, here, is 5x squared plus 3. So we have our fraction in the numerator is the derivative of what's being acted on. The derivative of 5x squared plus 3 is 10x. That's in the numerator. And in the denominator, it's 1 plus that expression that arctan is acting on, that 5x squared plus 3 squared. So just following our derivative rule for that.

Now, there's some simplification that we can do with this. We have our dy/dx is equal to our 6x squared arctan of 5x squared plus 3. And then plus-- that 2x cubed is over 1, so that gives me 20x to the fourth in the numerator. And in the denominator, we can square out the 5x squared plus 3. So that gives us 25x to the fourth plus 30x squared plus 9.

And then, lastly, we can collect like terms in the denominator of that fraction. So dy/dx is equal to 6x squared arctan of 5x squared plus 3 and then plus 20x to the fourth over 25x to the fourth plus 30x squared plus 10. And that is our derivative of our function y equal 2x cubed times the arctan of 5x squared plus 3.

Naturally, we can apply what we know about inverse trigonometric functions to applications such as finding the slope of the tangent line.

➦ EXAMPLE Compute the slope of the line tangent to the function $y = \sec^{-1}(x^2+1)$ when $x = -1$.

First, find the derivative of $y = \sec^{-1}(x^2+1)$.

$y = \sec^{-1}(x^2+1)$	Start with the original equation.
$\frac{dy}{dx} = \frac{2x}{\lvert x^2+1\rvert\sqrt{(x^2+1)^2-1}}$	$\frac{dy}{dx} = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$, $u = x^2+1$, $u' = 2x$
$\frac{dy}{dx} = \frac{2x}{\lvert x^2+1\rvert\sqrt{x^4+2x^2}}$	Simplify $(x^2+1)^2-1 = x^4+2x^2+1-1 = x^4+2x^2$.
$m_{\tan} = -\frac{\sqrt{3}}{3}$	Substitute -1 for x to get $-\frac{1}{\sqrt{3}}$, then rationalize the denominator.

Thus, the slope of the tangent line is $-\frac{1}{\sqrt{3}}$, which after rationalizing the denominator, is $-\frac{\sqrt{3}}{3}$.

SUMMARY

In this lesson, you learned that by knowing the **derivative rules for the inverse trigonometric functions**, you can now find **derivatives of functions that involve inverse trigonometric functions**, thus expanding on the types of functions you are able to analyze for slope and rates of change, etc.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 7 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of the Inverse Cosecant Function

$$\frac{d}{dx}\left[\csc^{-1}x\right] = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left[\csc^{-1}u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Derivative of the Inverse Cosine Function

$$\frac{d}{dx}\left[\cos^{-1}x\right] = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left[\cos^{-1}u\right] = \frac{-u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Cotangent Function

$$\frac{d}{dx}\left[\cot^{-1}x\right] = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}\left[\cot^{-1}u\right] = \frac{-u'}{1+u^2}$$

Derivative of the Inverse Secant Function

$$\frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left[\sec^{-1}u\right] = \frac{u'}{|u|\sqrt{u^2-1}}$$

Derivative of the Inverse Sine Function

$$\frac{d}{dx}\left[\sin^{-1}x\right] = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left[\sin^{-1}u\right] = \frac{u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Tangent Function

$$\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{u'}{1+u^2}$$

Related Rates Problems Using Geometric Formulas

by Sophia

WHAT'S COVERED

In this lesson, you will apply rates of change in a geometric setting. When a solid expands or contracts, there are several rates of change that are related. We can use implicit differentiation to establish those relationships and also find instantaneous rates of change. Specifically, this lesson will cover:

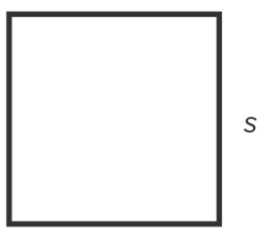
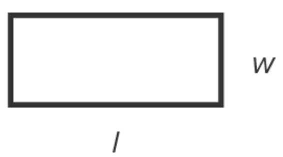
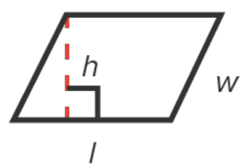
1. Geometric Formulas
2. What Exactly Are Related Rates?
3. Related Rates Applied to Geometric Situations

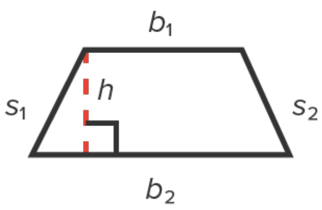
1. Geometric Formulas

In this part of the challenge, we will focus on related rates that come from geometric formulas.

Refer to the geometric formula sheet below for a complete list of the formulas you should know. You can also download this formula sheet as a PDF file at the end of the tutorial.

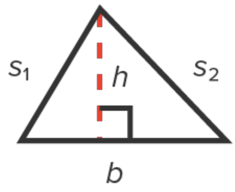
Geometric Formulas

Square	Rectangle	Parallelogram
		
$P = 4s$ $A = s^2$	$P = 2l + 2w$ $A = lw$	$P = 2l + 2w$ $A = lh$
Trapezoid	**Triangle**	**Circle**



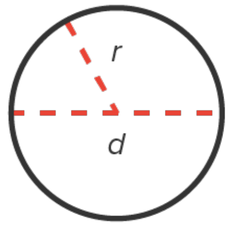
$P = s_1 + s_2 + b_1 + b_2$

$A = \frac{1}{2}h(b_1 + b_2)$



$P = s_1 + s_2 + b$

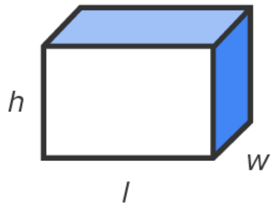
$A = \frac{1}{2}bh$



$C = 2\pi r$ or $C = \pi d$

$A = \pi r^2$

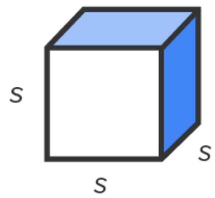
Rectangular Solid



$S = 2lh + 2wh + 2wl$

$V = lwh$

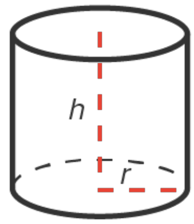
Cube



$S = 6s^2$

$V = s^3$

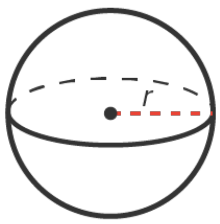
Right Circular Cylinder



$S = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h$

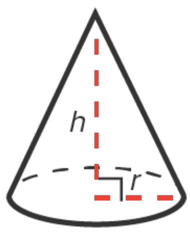
Sphere



$S = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

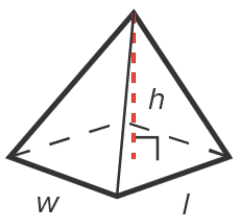
Right Circular Cone



$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$

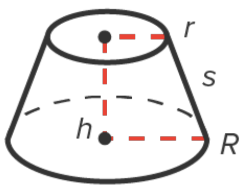
$V = \frac{1}{3}\pi r^2 h$

Square or Rectangular Pyramid



$V = \frac{1}{3}lwh$

Right Circular Cone Frustum



$S = \pi s(R + r) + \pi r^2 + \pi R^2$

$V = \frac{\pi(r^2 + rR + R^2)h}{3}$

Geometric Symbols

A = Area	S = Surface Area
P = Perimeter	C = Circumference
V = Volume	π = Pi Constant

3	

2. What Exactly Are Related Rates?

A stone is dropped in a lake. The ripple that is formed is circular, and the radius is increasing at a rate of 2 inches per second. Let's see how this impacts the area enclosed by the ripple.

Recall that the area enclosed by a circle is $A = \pi r^2$. The table below shows the time elapsed (in seconds), along with the radius and the area inside the ripple.

Time	Radius	Area
1	2	4π
2	4	16π
3	6	36π
4	8	64π

The table suggests that the radius and the area are functions of time, meaning that as time changes, so do the values of the radius and the area. In addition, it is pretty clear that even though the radius is changing at the same rate each second, the rate at which the area is changing is at different rates (in fact, it is increasing).

This also means that the rate at which the area is changing is affected by the rate of change in the radius and the radius itself.

Formally, let's call $\frac{dA}{dt}$ the rate of change in the area with respect to time and let $\frac{dr}{dt}$ represent the rate of change in the radius with respect to time.

Then, $\frac{dA}{dt}$ and $\frac{dr}{dt}$ are related by some equation. Here is how we get there.

We know that the area of the circle is $A = \pi r^2$. Since A and r are functions of t, we will differentiate both sides with respect to t:

$A = \pi r^2$	Start with the area of a circle formula.
$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$	Apply the derivative with respect to t to both sides.
$\frac{dA}{dt} = \pi(2r) \cdot \frac{dr}{dt}$	Take the derivative of each side. Multiply $\pi(2r)$ by $\frac{dr}{dt}$ since we are differentiating r implicitly.
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$	Simplify.

Thus, the equation $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ shows us how the rates are related.

Furthermore, we also assumed that the radius was increasing by 2 inches per second, which means $\frac{dr}{dt} = 2$

in/sec.

This means that $\frac{dA}{dt} = 2\pi r(2) = 4\pi r \text{ in}^2/\text{sec}$.

With this relationship, we can answer two questions:

- What is the radius when the area is changing by a certain amount?
- What is the rate of change in the area for a specific radius?

---

3. Related Rates Applied to Geometric Situations

In this part of the challenge, we will focus on related rates that come from the geometric formula sheet.

> ↱ EXAMPLE A spherical balloon is being inflated at a rate of $20\pi \text{ cm}^3/\text{sec}$. At what rate is the radius increasing when the radius is 10 cm?
>
> First, identify the geometric formula to use. Since the balloon is spherical, use the formula for the volume of a sphere: $V = \frac{4}{3}\pi r^3$
>
> Now, identify all quantities, and what we are looking for.
>
> - Given : $\frac{dV}{dt} = 20\pi$
> - Want to know: $\frac{dr}{dt}$ when $r = 10$
>
> Since the information we have (and need) involves rates, we need to use implicit differentiation to take the derivative:
>
> | | |
> |---|---|
> | $V = \frac{4}{3}\pi r^3$ | Start with the original formula. |
> | $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$ | Take the derivative of both sides, remembering that each variable is being differentiated implicitly. |
> | $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ | Simplify. |
> | $20\pi = 4\pi(10)^2 \cdot \frac{dr}{dt}$ | Given $\frac{dV}{dt} = 20\pi$, we want to know $\frac{dr}{dt}$ when $r = 10$. |
> | $20\pi = 400\pi \cdot \frac{dr}{dt}$ | Simplify. |
> | $\frac{dr}{dt} = \frac{1}{20} \text{ in}/\text{sec}$ | Divide both sides by 400π and affix appropriate units. |
>
> The radius is increasing at a rate of $\frac{1}{20}$ in/sec at the exact moment the radius is 10cm.

↪ EXAMPLE A large ice cube is melting in such a way that its volume is decreasing at a rate of $40\ \text{in}^3/\text{hour}$. At what rate is one of its sides changing when it measures 45 inches on each side?

First, identify the geometric formula to use. Since the ice cube is in the shape of a cube, use the formula for the volume of a cube, $V = s^3$.

Note that both quantities (*V* and *s*) are changing with respect to times, so they remain as variables in the equation. Now, identify all quantities, and what we are looking for.

- Given: $\frac{dV}{dt} = -40$ (Since the ice cube is melting, the volume is decreasing, making $\frac{dV}{dt}$ negative.)
- Want to know: $\frac{ds}{dt}$ when $s = 45$

Since the information we have (and need) involves rates, we need to take the derivative:

$V = s^3$	Start with the original formula.
$\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$-40 = 3(45)^2 \cdot \frac{ds}{dt}$	Given $\frac{dV}{dt} = -40$, we want to know $\frac{ds}{dt}$ when $s = 45$.
$-40 = 6075 \cdot \frac{ds}{dt}$	Simplify.
$\frac{ds}{dt} = -\frac{8}{1215}$ in/min	Divide both sides by 6,075 and reduce; affix appropriate units.

The length of one side is decreasing at a rate of $-\frac{8}{1215}$ in/min, or approximately -0.00658 in/min at the exact moment the radius is 45 in.

TRY IT

A graphic designer takes an image of a square and enlarges it in such a way that each side increases by 0.25 inch every second.

When the length of each side is 6 inches, at what rate is the area changing? +

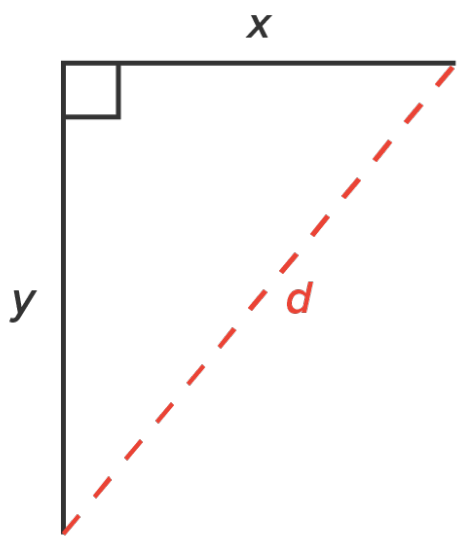
$3\text{in}^2/\text{sec}$

We can also use related rates to examine how the distance between two objects is changing.

↪ EXAMPLE Two joggers start from the same point in the middle of a large field, one heading east and one heading south. If the eastbound jogger runs at a rate of 5 mph and the southbound jogger runs at a rate of 6 mph, at what rate is the distance between them changing after 2 hours?

- Let x = the distance traveled by the eastbound jogger.

- Let y = the distance traveled by the southbound jogger.
- Let d = the distance between the joggers.



Then, the variables are related by the Pythagorean theorem, $x^2 + y^2 = d^2$.

Since all three quantities are changing with respect to time, t, in this problem, they will remain as variables when we find the related rates.

- Given: The rates of change in the distances (the speeds), $\frac{dx}{dt} = 5$ mi/hr, $\frac{dy}{dt} = 6$ mi/hr
- Want to know: $\frac{dd}{dt}$ after 2 hours

Since $\text{distance} = \text{rate} \cdot \text{time}$, this means $x = 10$ and $y = 12$.

Because the information we have (and need) involves rates, we need to take the derivative:

$x^2 + y^2 = d^2$	Start with the original formula.
$2d \cdot \frac{dd}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$d^2 = x^2 + y^2$ $d^2 = 10^2 + 12^2$ $d^2 = 244$ $d = \sqrt{244}$	The value of d wasn't given, but we can use the original equation to find it.
$2\sqrt{244} \cdot \frac{dd}{dt} = 2(10)(5) + 2(12)(6)$	Given $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = 6$, we want to know $\frac{dd}{dt}$ when $x = 10$, $y = 12$, and $d = \sqrt{244}$.

$\frac{dd}{dt} = \frac{244}{2\sqrt{244}} \approx 7.81$ mi/hr   Solve for $\frac{dd}{dt}$.

The distance between the joggers is increasing at a rate of 7.81 mph after 2 hours.

SUMMARY

In this lesson, you learned how to **apply related rates to geometric situations**, using a comprehensive list of **geometric formulas**. You learned that derivatives can be applied to geometric formulas to produce **related rates**, which are equations that establish a relationship between two (or more) rates of change.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Related Rates Problems Using Proportional Reasoning and Trigonometry

by Sophia

WHAT'S COVERED

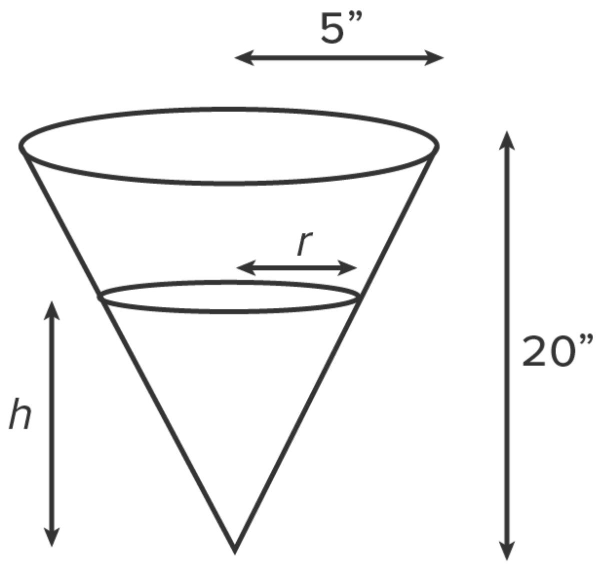
In this lesson, you will continue to explore related rates problems that involve proportional reason (such as similar triangles) and trigonometry. For example, we may want to determine how an angle of inclination changes for a camera that is following a rocket that is taking off. Specifically, this lesson will cover:

1. Related Rates Problems Involving Proportions
2. Related Rates Problems Involving Trigonometry

1. Related Rates Problems Involving Proportions

We will now look at problems where we need to use proportional reasoning (similar triangles, etc.) to get a relationship between the variables.

> EXAMPLE Water is filling a cone-shaped container at a rate of $30\pi \text{ in}^3/\text{min}$. The container is 10" wide at the top and 20" deep. At what rate is the height of the water changing when the water is 10 inches deep?
>
> This information means we are given $\frac{dV}{dt} = 30\pi$, and we want $\frac{dh}{dt}$ when $h = 10$.
>
> From the geometry formulas, we know that $V = \frac{\pi}{3}r^2h$, where:
>
> - r = the radius of the water in the cone
> - h = the height of the water in the cone
>
> Notice that there isn't any information given about the radius of the conical shape. However, we do have the information we need to solve this problem.



Since the cone is 10” wide across at the top, its radius is 5”.

Notice that the water in the cone forms a smaller version of the conical container, which means that the height and radius are in proportion to each other. In the full cone, the height is 4 times the radius. To figure out the relationship, we know that $\frac{h}{r}=\frac{20}{5}$.

Now we have to decide whether to solve for *h* or *r*.

Since there is no information given about the radius in this problem, we want to replace *r* in the formula. This means we want to solve for *r*. Solving $\frac{h}{r}=\frac{20}{5}$ gives $5h=20r$, or $r=\frac{1}{4}h$.

Now, we can write a volume formula in terms of only *h*.

$$V=\frac{\pi}{3}r^2h=\frac{\pi}{3}\left(\frac{h}{4}\right)^2h=\frac{\pi}{3}\cdot\frac{h^2}{16}\cdot h=\frac{\pi}{48}h^3$$
$$V=\frac{\pi}{48}h^3$$

Since rates are involved, find the derivative with respect to time:

$V=\frac{\pi}{48}h^3$	Start with the original equation.
$\frac{dV}{dt}=\frac{\pi}{48}(3h^2)\cdot\frac{dh}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$\frac{dV}{dt}=\frac{\pi}{16}h^2\cdot\frac{dh}{dt}$	Simplify.
$30\pi=\frac{\pi}{16}(10)^2\cdot\frac{dh}{dt}$	$\frac{dV}{dt}=30\pi$, h = 10, $\frac{dh}{dt}=?$

$$30\pi = \frac{25\pi}{4} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = 30\pi \cdot \frac{4}{25\pi} = \frac{24}{5} \text{ in/min}$$

In conclusion, the height is increasing at a rate of 4.8 inches per minute when the water is 10 inches deep.

WATCH

This is an example of related rates of distance from a flagpole.

Video Transcription

Welcome. In this video, we're going to look at an example of a related rate question. Here, a woman walks away from a flagpole at the rate of 4 feet per second. If the flagpole is 25 feet tall, at what rate is she walking away from the top of the flagpole when she is 15 feet from its base.

So what I've done is drawn a picture to give this scenario, and we need to put some variables in here. We are assuming that it's on level ground and that the flagpole is set perpendicular to the ground, and so that gives us the right triangle. The flagpole is 25 feet tall. So I've done the vertical part of our right triangle as 25 feet.

And so the ground is the horizontal component of the right triangle, and she is walking away from the flagpole. So her distance from the flagpole we're going to call x. And we're measuring this distance that she's away from the ground to the top of the flagpole. So from her feet up is the distance that we're talking about.

Now, we also need a variable for the hypotenuse. We'll just call this s. So when we look at the information, it's important, with the related rates, to identify the information that's given and identify the information that you want to find, and also notice any sort of geometrical or other sorts of formulas that might help you.

So it says that she is walking away from the flagpole at the rate of 4 feet per second. And so, since that's her distance from the flagpole and it's a rate-- it's the change of distance with respect to the change in time, we have that dx dt-- the change in her X distance away from the flagpole with respect to the change in time-- is that 4 feet per second.

Now, we have that the flagpole is 25 feet tall, and it says, at what rate is she walking away from the top of the flagpole? So it's asking me for the rate that she's walking away from the top of the flagpole. So that is from the top of the flagpole down to her feet. So that is the rate of change of s.

So ds dt is what we are looking for. And we're looking for that when she is 15 feet from the base. So that's when x is equal to 15 feet.

So now, what geometrical formula can we use here? Well, it's a right triangle. And when we are talking about the relationships between the sides of the right triangle, that's Pythagorean theorem. So we have legs squared-- 25 squared plus x squared, the other leg squared, is equal to the hypotenuse, s squared.

Now, 25 squared is 625, plus x squared is equal 2x squared. And we can go through and find our derivative now with implicit differentiation, and we are looking at it where x and s are both dependent variables on the independent variable of time t. And we know that because of the way our derivatives are set up. The dx dt-- x is dependent on t, and the dt s is dependent on t.

So the derivative of the constant term, 625, is 0, plus the derivative of x squared is 2x for the derivative rule, but x is dependent on t. So I have to multiply by dx dt by the chain rule-- is equal to the derivative of s squared is 2s but s is dependent on dt so by the chain rule, I have to multiply by ds dt.

Now, when we look at what we are given and what we want to know, I have a value for x. It's 15. I have a value for dx dt. That's that 4 feet per second. I don't yet have a value that I've calculated for s, and I want to know ds dt.

So at first glance, it looks like maybe we don't have enough information because we don't have our s, but we have to remember that when x is 15, we have that right triangle where I have a 25-foot vertical on my right triangle. When x is 15, the horizontal leg of the right triangle is 15. So I can use the Pythagorean theorem-- 25 squared-- plus 15 squared is equal to s squared-- to find the value of s at that time.

So I have 625 plus 225 is equal to s squared. That gives me 850 is equal to s squared. So s is equal to the square root of 850 when x is 15. And your flagpole stays at its 25 feet, which-- as a simplified radical, s is equal to 5 times the square root of 34.

So now we have all but one thing missing. So we can plug in our values and solve for that. So I have 2 times-- take out the x and put in my 15 feet-- times-- take out the dx dt and put in 4 feet per second-- is equal to 2 times s. Take out the s, and put in 5 square roots of 34 feet, and then times that ds dt.

So simplifying the left-hand side, I have 2 times my 15 times-- feet times my 4. That gives me a value of 120 square feet per second, just bringing our units along, and we will see how they work out as we go through the rest of the calculation. And then I have 2 times 5 times the square root of 34 feet times ds dt. So that's 10 times the square root of 34 feet times ds dt.

And then solving for ds dt, we want to divide both sides by the 10 times the square root of 34 feet. So I have 1 over 10 times the square root of 34 feet times my 120 feet squared over seconds is equal to the ds dt, and my feet-- no unit in the denominator. We'll remove one of those in the numerator. And then, simplifying the values numerically, I have 12 over the square root of 34 feet per second is what we get for our ds dt. And so the units work out to be what they should be because it's in the change in distance with respect to the change in time.

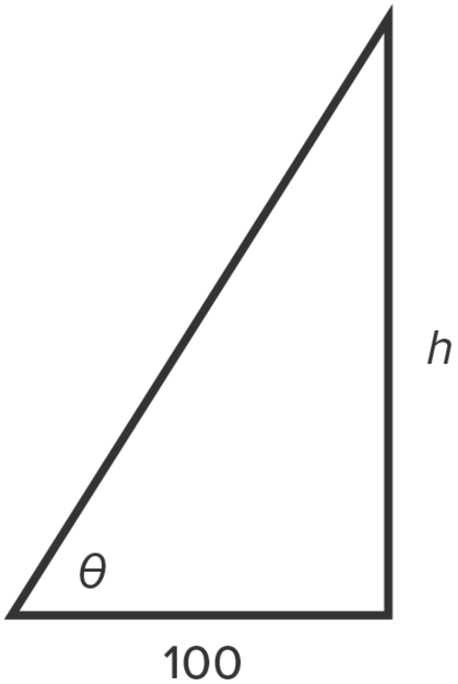
Now, there's a couple other things we can do with this exact answer. First thing we can do is rationalize. And if you rationalize and simplify it, you get ds dt is equal to 6 times the square root of 34 over 17. And if you approximate that, it's approximately 2.058 feet per second. And that is the rate that she is walking away from the top of the flagpole when she is 15 feet from its base.

2. Related Rates Problems Involving

Trigonometry

Related rates can also help answer questions about angles of inclination.

A camera is on ground level 100 feet from a rocket's launchpad and inclines upward as a rocket takes off vertically at a rate of 250 ft/s. At what rate is the angle of inclination changing when the rocket is 1000 feet off the ground?



Notice that the base is always 100 feet. This is not changing. Since the height of the rocket changes, the vertical side is a variable. Since the angle is changing, it is labeled with a variable
(θ) as well.

To relate all the relevant quantities, we need to use a trigonometric function. Since the opposite and adjacent sides to angle θ are labeled, tangent is the best choice.

The equation is $\tan\theta = \frac{h}{100}$. Solving for θ, we have $\theta = \tan^{-1}\left(\frac{h}{100}\right)$.

We were given $\frac{dh}{dt} = 250$ and we want to find $\frac{d\theta}{dt}$ when $h = 1000$ feet.

Now, we take the derivative:

$\theta = \tan^{-1}\left(\frac{h}{100}\right)$	Start with the original equation.
	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{d}{dt}\left[\frac{h}{100}\right]$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{1}{100} \cdot \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1000}{100}\right)^2} \cdot \frac{1}{100} \cdot 250$$

Substitute quantities.

$$\frac{d\theta}{dt} = \frac{1}{101} \cdot \frac{1}{100} \cdot 250 \approx 0.02475 \text{ radians/sec}$$

Simplify.

The angle is increasing at a rate of 0.02475 radians per second. For reference, this is about 1.42°/second.

SUMMARY

In this lesson, you learned how to solve **related rates problems involving proportions and trigonometry** (which can help answer questions about angles of inclination). In these problems, equations were derived using mathematical facts more so than a standard formula. Once the equation is determined, the procedure for finding the related rates is exactly the same: take the derivative, substitute what it is known, then solve for the unknown.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Terms to Know

Acceleration

An object's change in velocity with respect to time.

Cusp

A pointed end where two parts of a curve meet at a vertical tangent.

Derivative

The slope of the tangent line to the graph of a function at a point is also known as the derivative of the function at that point.

Differentiable

A function $y=f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

Fixed Cost (or Overhead)

The costs that are incurred before any items are produced. Mathematically, it is the total cost of producing 0 items.

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

Jerk

An object's change in acceleration with respect to time.

Linear Approximation of f(x) at x = a

The tangent line to the graph of $f(x)$ at $x=a$.

Marginal Cost Function

The derivative (rate of change) of the cost function. Given a production level x, it approximates the cost of the next item.

Velocity

An object's change in distance with respect to time.

Formulas to Know

Average Cost Function

$AC(x) = \frac{C(x)}{x}$

Chain Rule

Suppose $y = f(u)$, a composite function, where u is a function of x.

Then, $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

Using “prime” notation, we can write $\frac{dy}{dx} = f'(u) \cdot u'$.

Using “D” notation, we can write $\frac{dy}{dx} = f'(u) \cdot D[u]$.

Derivative of Cosecant

$D[\csc x] = -\csc x \cot x$

Derivative of Cosine

$\frac{d}{dx}[\cos x] = -\sin x$

Derivative of Cotangent

$D[\cot x] = -\csc^2 x$

Derivative of Secant

$D[\sec x] = \sec x \tan x$

Derivative of Sine

$\frac{d}{dx}[\sin x] = \cos x$

Derivative of Tangent

$D[\tan x] = \sec^2 x$

Derivative of a Composite Logarithm Function, Base a

$D\left[\log_a u\right] = \frac{1}{u \cdot \ln a} \cdot u' = \frac{u'}{u \cdot \ln a}$

Derivative of a Constant Multiple

$D[k \cdot f(x)] = k \cdot D[f(x)]$

Derivative of a Difference

$D[f(x) - g(x)] = D[f(x)] - D[g(x)]$

Derivative of a Logarithm Function, Base a

$$D[\log_a x] = \frac{1}{x \cdot \ln a}$$

Derivative of a Sum

$$D[f(x) + g(x)] = D[f(x)] + D[g(x)]$$

Derivative of lnu, Where u Is a Function of x

$$D[\ln u] = \frac{1}{u} \cdot u'$$

Derivative of the Inverse Cosecant Function

$$\frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}[\csc^{-1} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$$

Derivative of the Inverse Cosine Function

$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\cos^{-1} u] = \frac{-u'}{\sqrt{1 - u^2}}$$

Derivative of the Inverse Cotangent Function

$$\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1 + x^2}$$

$$\frac{d}{dx}[\cot^{-1} u] = \frac{-u'}{1 + u^2}$$

Derivative of the Inverse Secant Function

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$$

Derivative of the Inverse Sine Function

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\sin^{-1} u] = \frac{u'}{\sqrt{1 - u^2}}$$

Derivative of the Inverse Tangent Function

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{u'}{1 + u^2}$$

Derivative of the Natural Logarithmic Function

$D[\ln x] = \frac{1}{x}$

Differential of f

$df = f'(x)dx$ for any choice of *x* and any real number *dx*.

When $y = f(x)$, we can also write $dy = f'(x)dx$.

Equation of a Tangent Line to y = f(x) at x = a

$y = f(a) + f'(a)(x - a)$

Evaluating Inverse Cosecant

$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

Evaluating Inverse Cotangent

$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

Evaluating Inverse Secant

$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D\left[[f(x)]^n\right] = n \cdot [f(x)]^{n-1} \cdot f'(x)$.

Limit Definition of Derivative

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Linear Approximation Error

$\text{Error} = |f(x) - L(x)|$

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Power Property

$\ln(a^b) = b \cdot \ln a$

Power Rule

$$\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$$

Product Property

$$\ln(ab) = \ln a + \ln b$$

Product Rule for Derivatives

$$D[f(x) \cdot g(x)] = D[f(x)] \cdot g(x) + f(x) \cdot D[g(x)]$$

Using alternate notation: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient Property

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

Quotient Rule for Derivatives

Using "Prime" Notation: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Using "D" Notation: $D\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot D[f(x)] - f(x) \cdot D[g(x)]}{[g(x)]^2}$

"High and Low" Version: $D\left[\frac{high}{low}\right] = \frac{low\ dee\ high - high\ dee\ low}{low\ low}$

Slope of the Line Passing Through the Points (x_1, y_1) and (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The Derivative of a^x

$$D\left[a^x\right] = a^x \cdot \ln a$$

The Derivative of a^u, Where u Is a Function of x

$$D\left[a^u\right] = (a^u \cdot \ln a) \cdot u'$$

The Derivative of e^x

$$D\left[e^x\right] = e^x$$

The Derivative of e^u, Where u Is a Function of x

$$D\left[e^u\right] = e^u \cdot u'$$