Sophia

# Unit 5 Tutorials: Integration

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# Area

*by Sophia*

WHAT'S COVERED

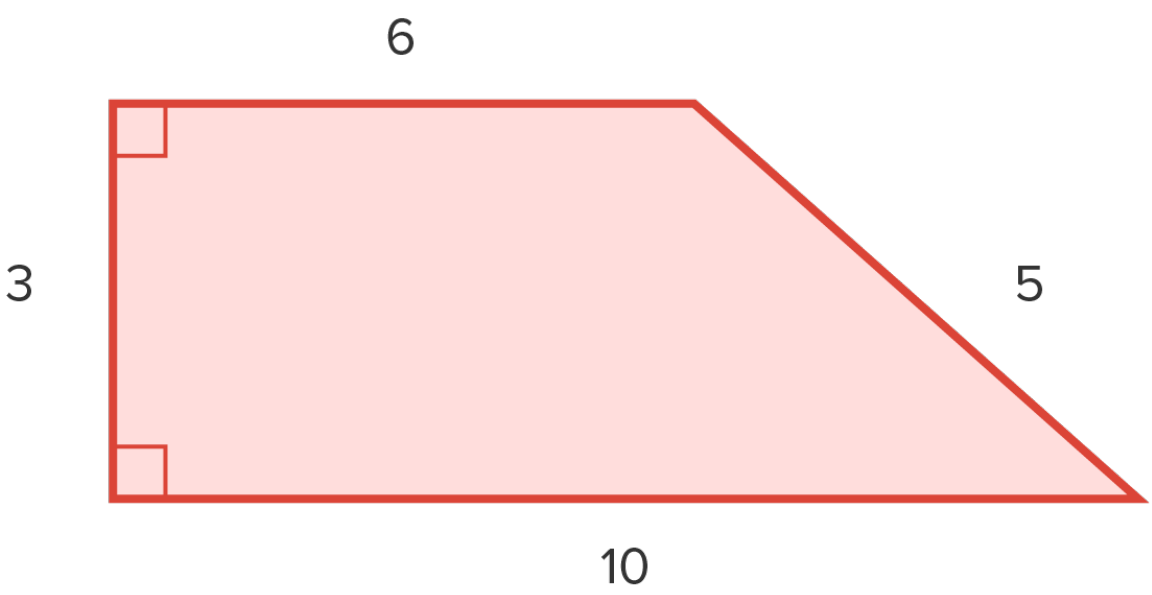
In this lesson, you will find areas by using formulas and approximation methods. So far, we have been using differential calculus to solve problems by finding rates of change. We are now moving into integral calculus, in which areas are used as a visual to solve problems. In this section, we will start by finding areas that involve combining simpler areas. Specifically, this lesson will cover:

1. Finding the Area by Using Geometric Formulas
2. Approximating Areas by Using Rectangles and Graphs

## 1. Finding the Area by Using Geometric Formulas

Before we get into the connection between areas and calculus, let's get some practice finding areas of some shapes using basic geometric formulas.

Consider the figure shown here.




There are three ways to find the area.

- Method I: This is a trapezoid, with area formula $A=\frac{1}{2}h(b_1+b_2)$, where $h$ is the height and $b_1$ and $b_2$ are

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the lengths of the parallel bases.

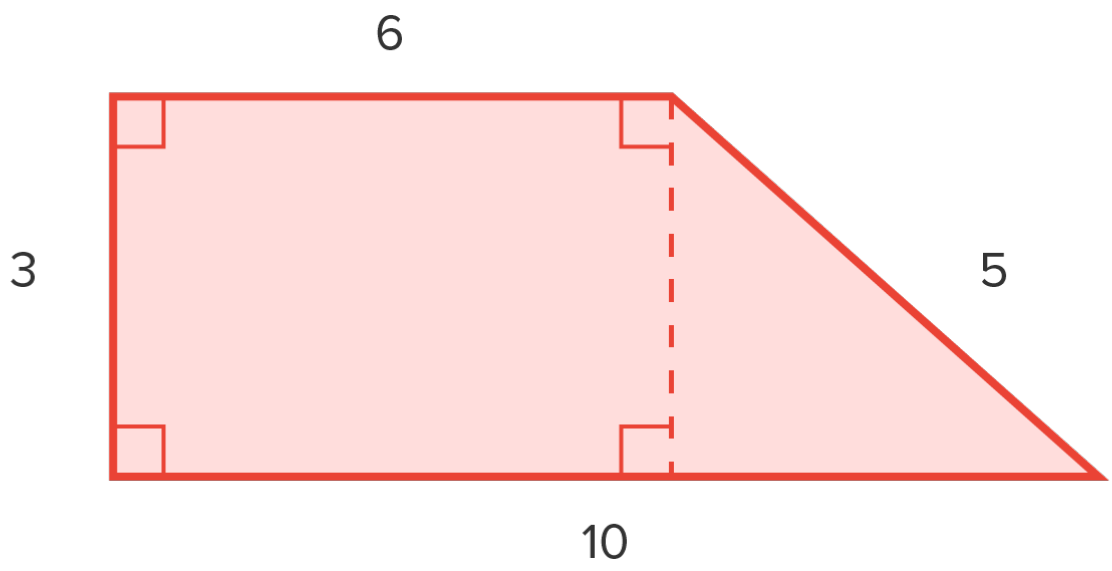
Then, $A = \frac{1}{2}(3)(10+6) = 24 \text{ units}^2$.

- Method II: Split the trapezoid into a rectangle and a triangle, as shown in the figure below.

  The area of the rectangle is $3(6) = 18 \text{ units}^2$.

  The triangle has base 4 (found by 10 – 6) and height 3.

  The area of the triangle is $\frac{1}{2}(4)(3) = 6 \text{ units}^2$.




  The combined area is $18+6=24 \text{ units}^2$.

- Method III: Split the trapezoid into two triangles, as shown in the figure below.

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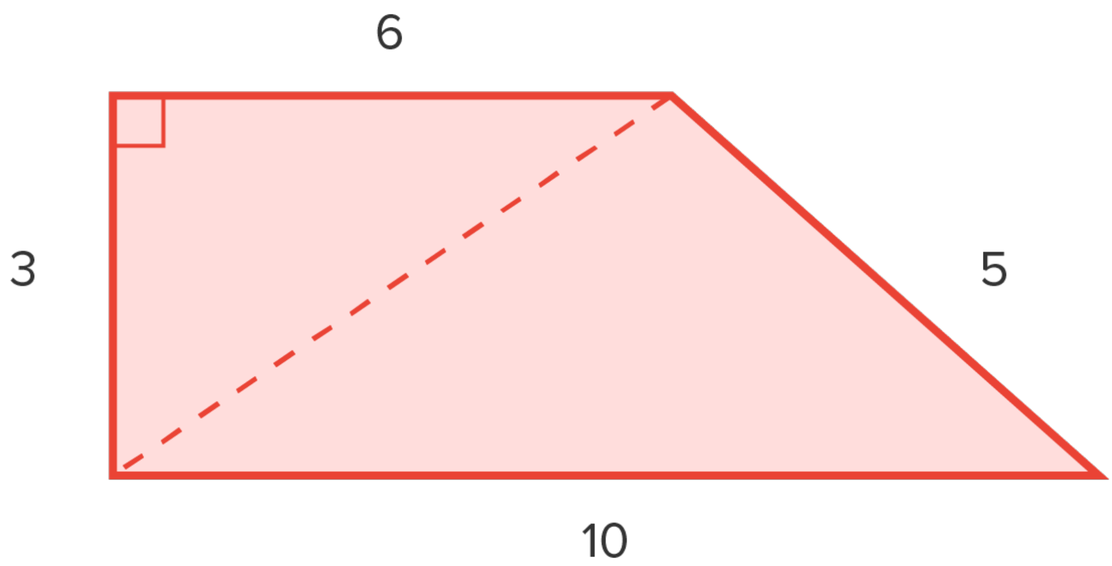


The area of the triangle on the left is $A = \frac{1}{2}(6)(3) = 9 \text{ units}^2$.

The area of the triangle on the right is $A = \frac{1}{2}(10)(3) = 15 \text{ units}^2$.

Once again, the combined area is $9 + 15 = 24 \text{ units}^2$.

HINT

Even though the trapezoid formula was the most straightforward method to find the area, most people are more comfortable with rectangles and triangles.

TRY IT

Consider the shape below:

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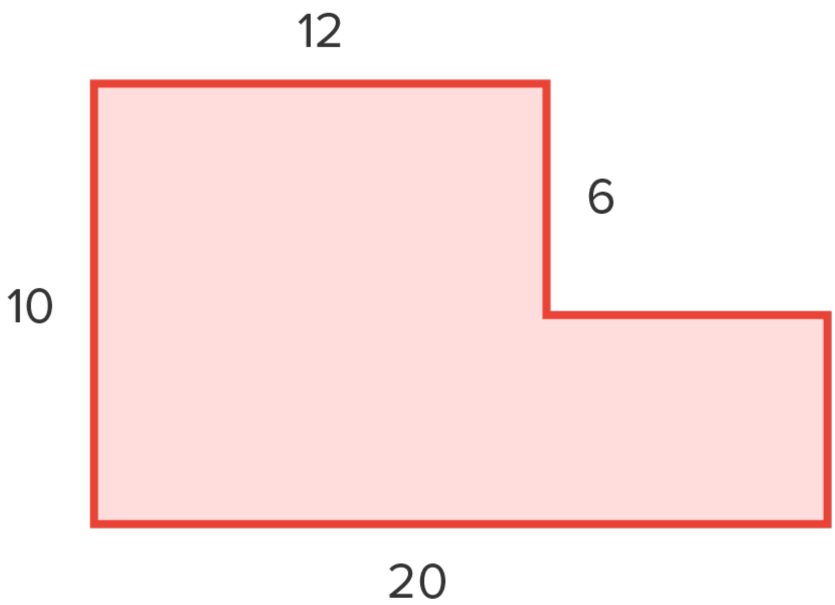


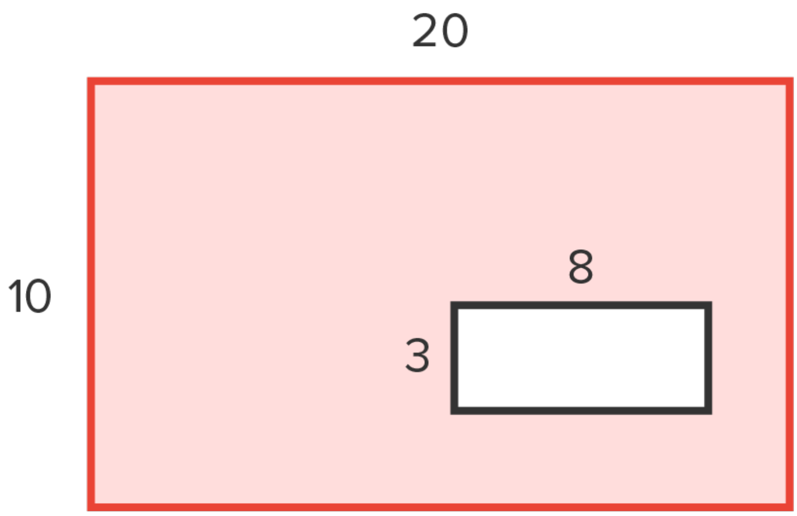

Find the area of this shape.

+

The area is $152 \text{ units}^2$.

Let's look at an example where there is a "hole" in the shape.

↱ EXAMPLE Find the area of the shape (represented by the shaded region), assuming each side is measured in inches.




The outer rectangle has area $10(20) = 200 \text{ in}^2$. The hole, also in the shape of a rectangle, has area $24 \text{ units}^2$. Then, the area of the shape is the difference between the areas: $200 - 24 = 176 \text{ in}^2$.

We can also find areas when certain graphs are used, since they are familiar shapes. Before diving in, here is

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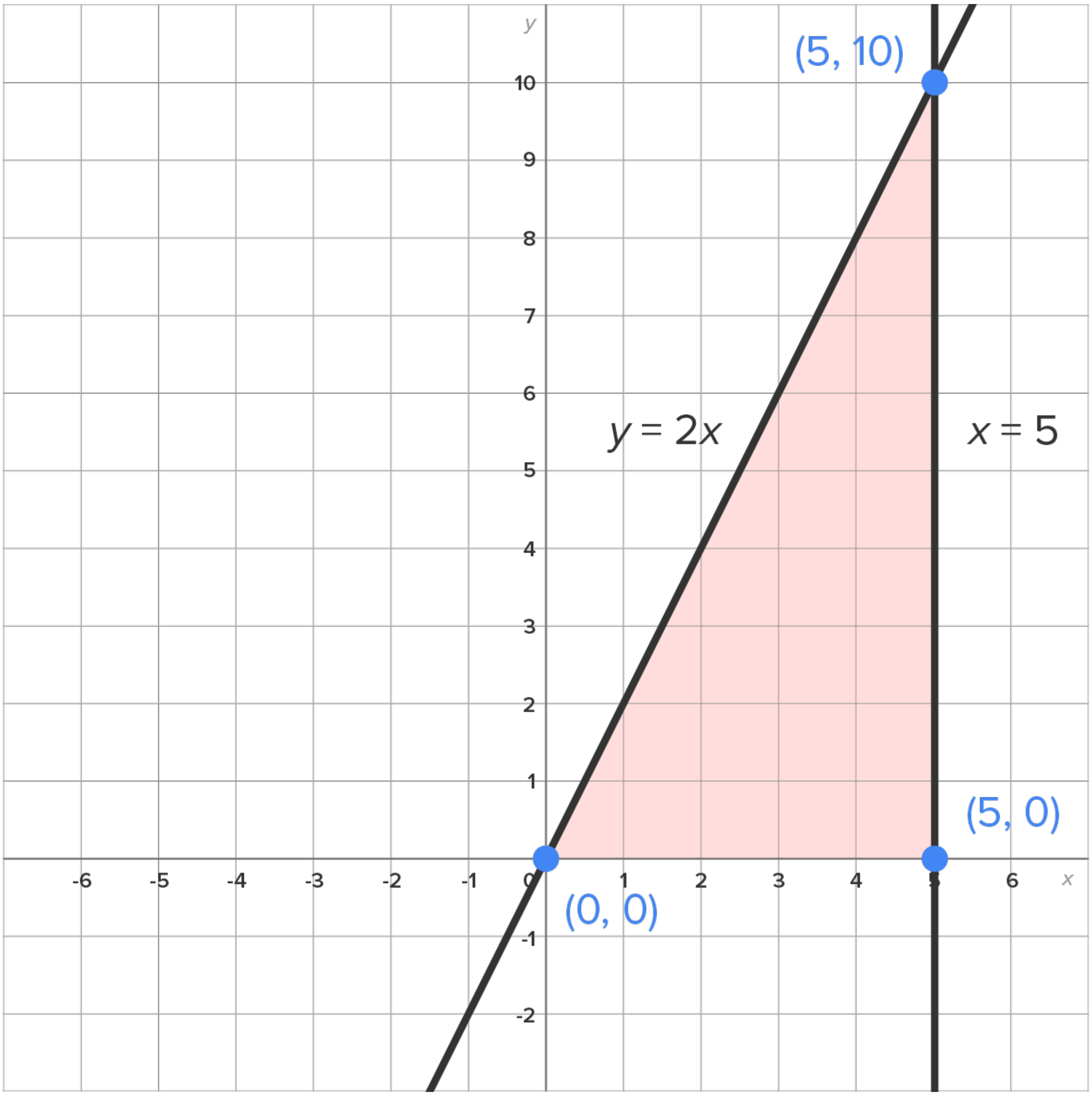
a summary of equations that, when graphed, could form areas we know from formulas:

| Forms | Equation |
|---|---|
| Horizontal line | $y=b$ |
| Slanted line | $y=mx+b$ |
| Circle with radius $r$ | $x^2+y^2=r^2$ |
| Semicircle with radius $r$ | $y=\sqrt{r^2-x^2}$ |

↪ EXAMPLE Find the area of the region bounded by the x-axis, the line $x=5$, and the line $y=2x$. The graph of the region is shown in the figure below.




The region is triangular, with base 5 and height 10.

Thus, the area of the region is $A=\frac{1}{2}(5)(10)=25\text{ units}^2$.

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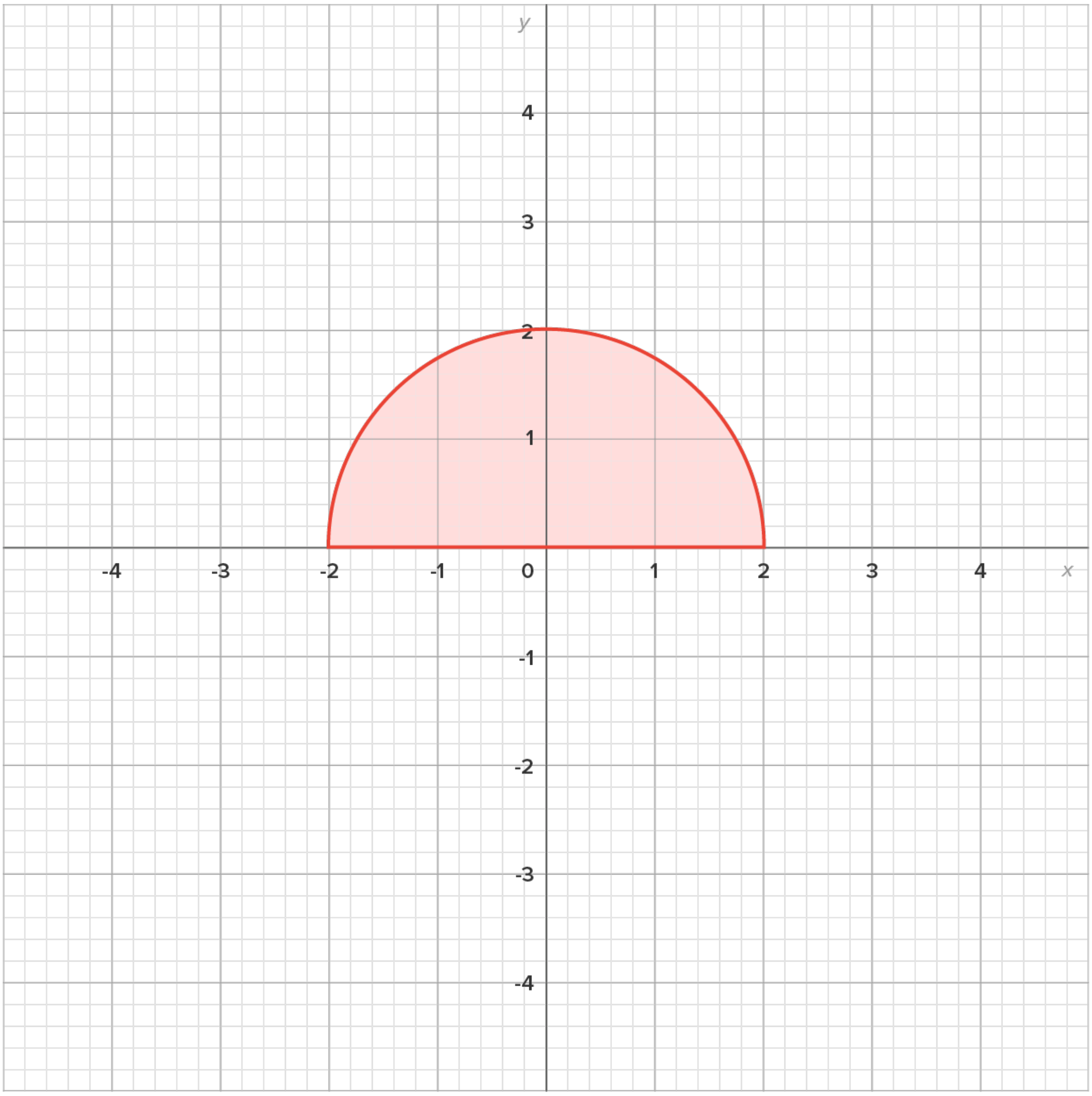
↪ EXAMPLE The figure below shows the graph of the region between $f(x) = \sqrt{4 - x^2}$ and the x-axis. What is the area of this region?




This region is a semicircle whose radius is 2. Recall the area of a circle is $A = \pi r^2$.

The area of the region is $\frac{1}{2} \cdot \pi(2)^2 = 2\pi$ units$^2$.

TRY IT

Consider the graph below:

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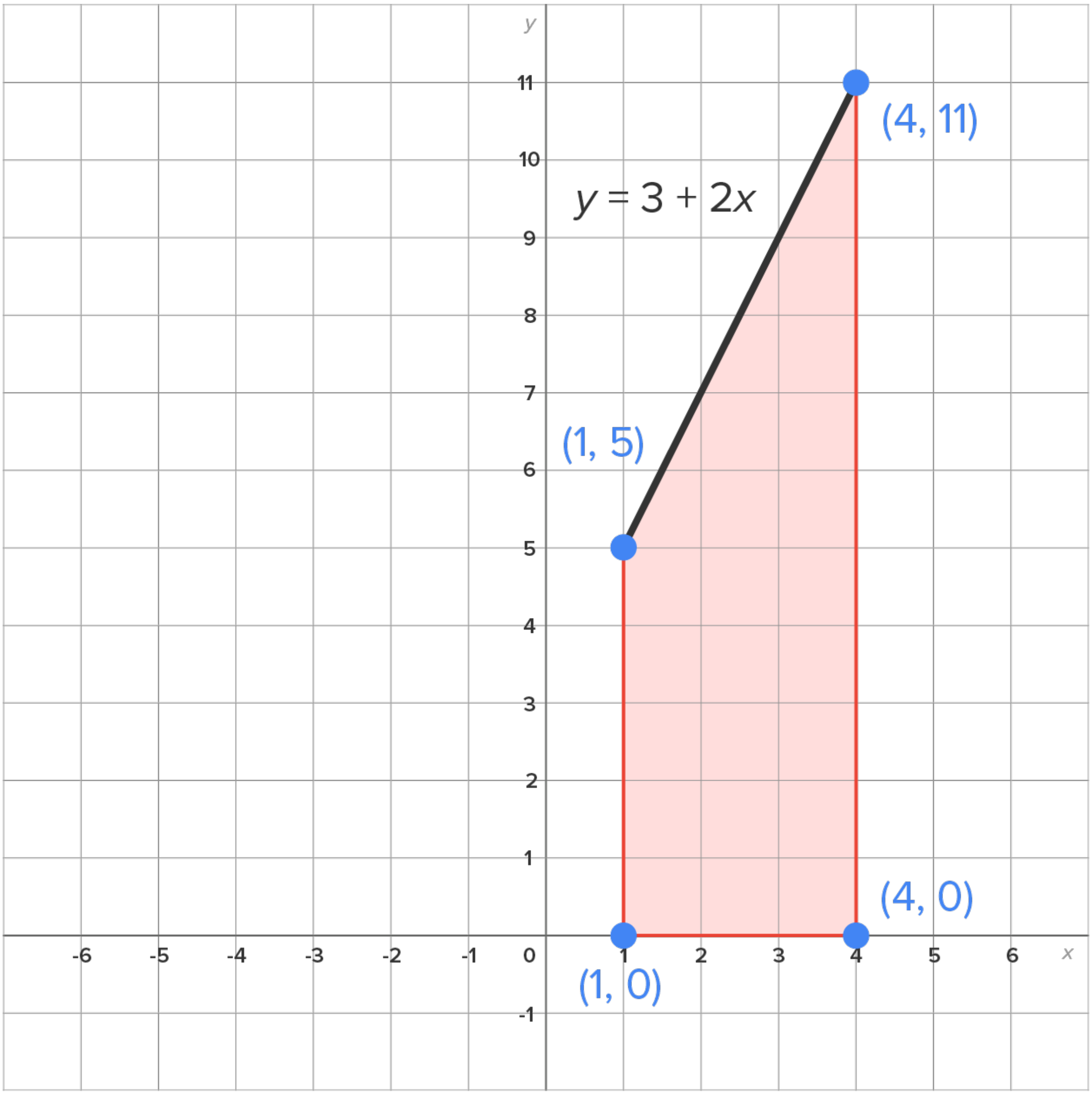



Find the area of the region in the graph.

+

24 $\text{units}^2$

HINT

Since you will be using areas to solve problems in this unit, refer to this sheet which contains formulas for areas of various shapes (circles, trapezoids, etc.). This is also available as a PDF file at the end of this tutorial.

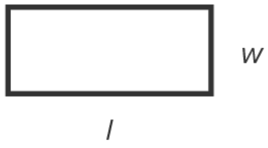
# Geometric Formulas

## Square


$P = 4s$

$A = s^2$

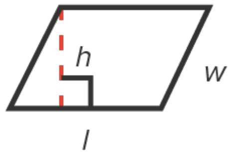
## Rectangle




$P = 2l + 2w$

$A = lw$

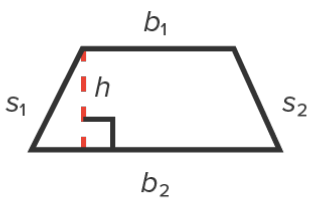
## Parallelogram




$P = 2l + 2w$

$A = lh$

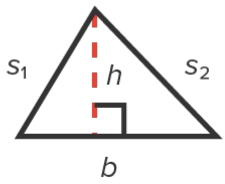
## Trapezoid




$P = s_1 + s_2 + b_1 + b_2$

$A = ½h(b_1 + b_2)$

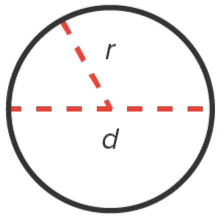
## Triangle




$P = s_1 + s_2 + b$

$A = ½bh$

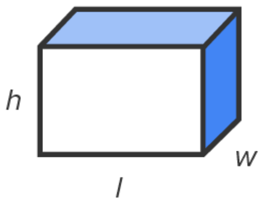
## Circle




$C = 2\pi r$ or $C = \pi d$

$A = \pi r^2$

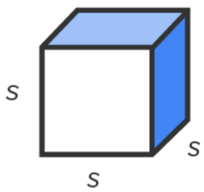
## Rectangular Solid




$S = 2lh + 2wh + 2wl$

$V = lwh$

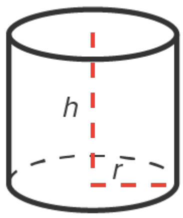
## Cube




$S = 6s^2$

$V = s^3$

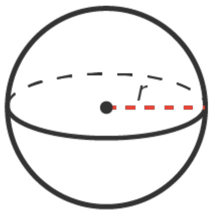
## Right Circular Cylinder




$S = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h$

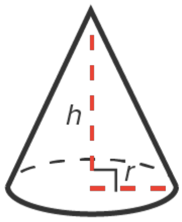
## Sphere




$S = 4\pi r^2$

$V = {}^4/_3\pi r^3$

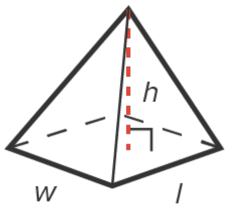
## Right Circular Cone




$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$

$V = ⅓\pi r^2 h$

## Square or Rectangular Pyramid




$V = ⅓lwh$

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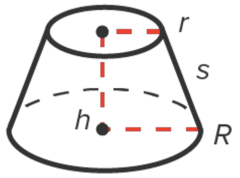
| Right Circular Cone Frustum | Geometric Symbols |
|---|---|
| <br>$S = \pi s(R + r) + \pi r^2 + \pi R^2$<br>$V = \dfrac{\pi(r^2 + rR + R^2)h}{3}$ | A = Area<br>P = Perimeter<br>V = Volume<br>S = Surface Area<br>C = Circumference<br>π = Pi Constant |

## 2. Approximating Areas by Using Rectangles and Graphs

There are some regions, particularly those which come from a graph, which cannot be found using formulas.

For example, consider the region bounded by the graphs of $y = x^2 + 3$, the x-axis, $x = 1$, and $x = 3$. The region is shown below.

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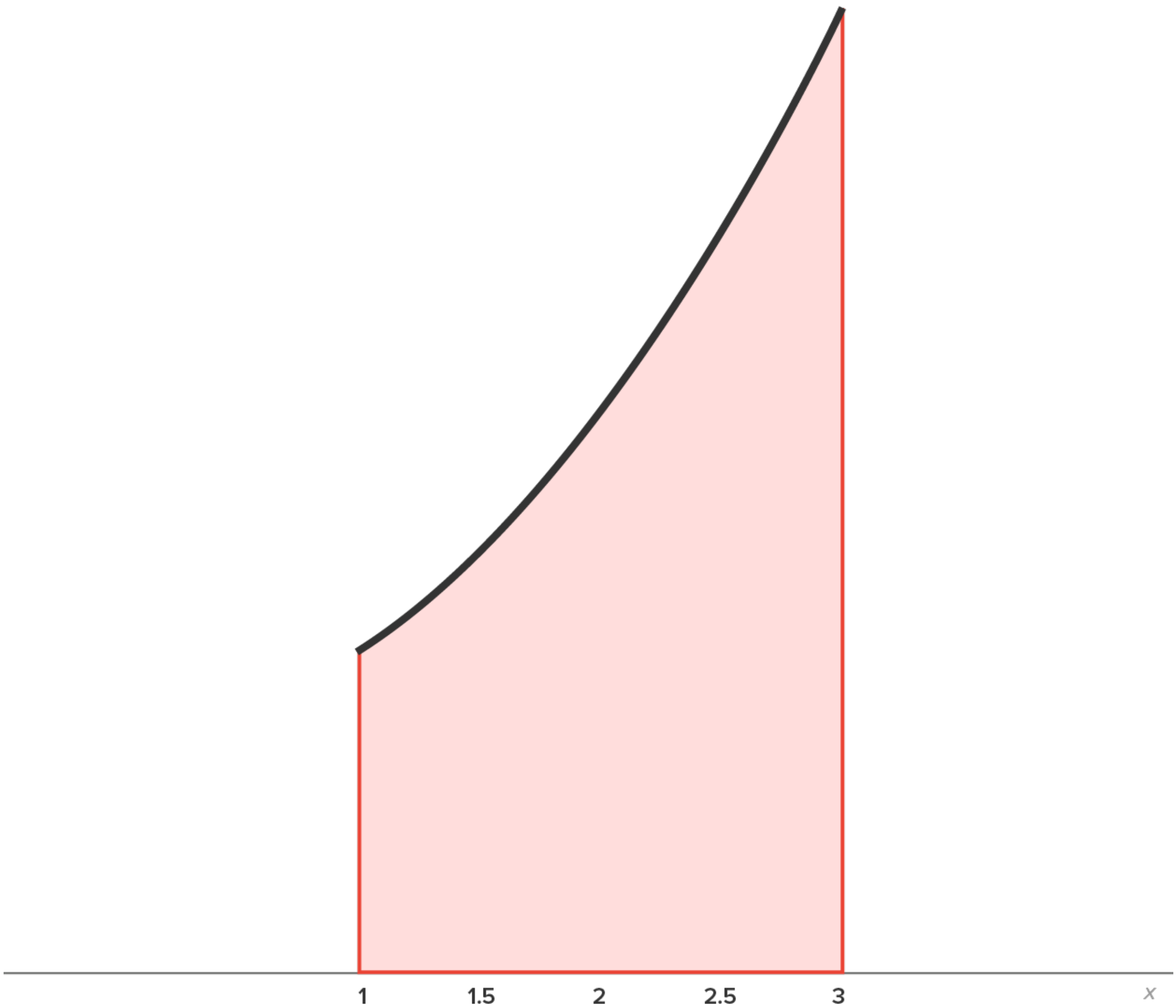
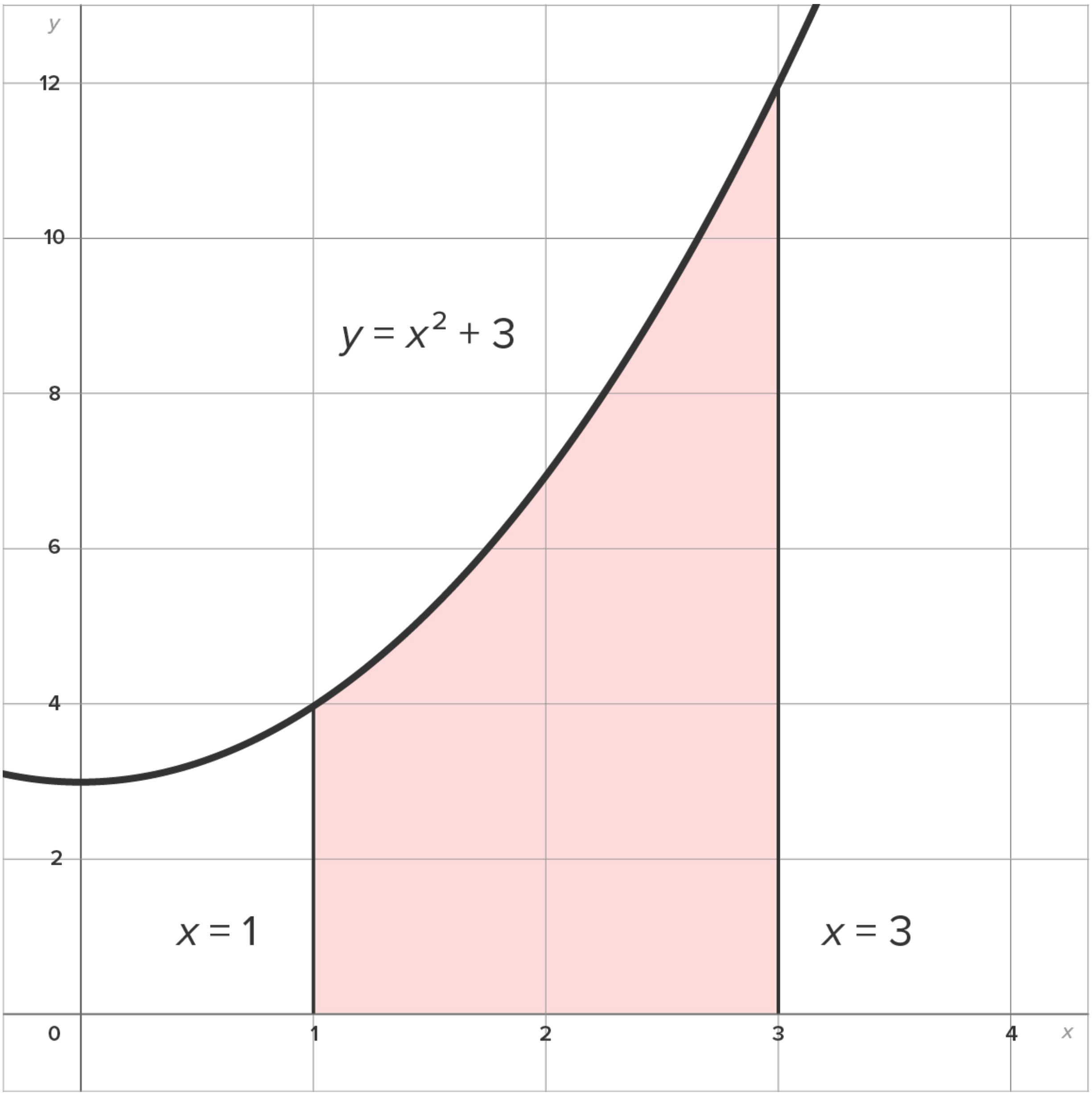


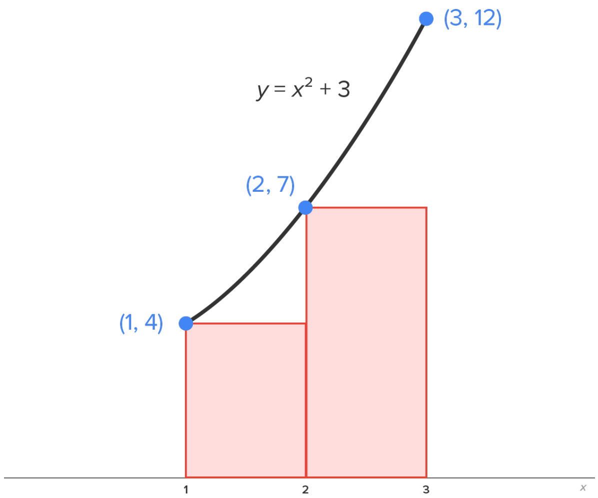

A major focus of integral calculus is being able to find areas of regions like this. For now, we need to come up with a way to estimate the area.

The most convenient way is to use rectangles whose bases are along the x-axis. This is illustrated in the next few examples.

> ↱ EXAMPLE Approximate the area of the region bounded by $y=x^2+3$, the x-axis, $x=1$, and $x=3$, as shown in the graph below:




To find this area, first find the combined area of the rectangles, as shown in each figure.

| Graph | Description |
|---|---|
| <br> | The rectangles used here are **inscribed**, meaning the largest possible rectangle drawn within the region. Notice how one corner of each rectangle is also on the curve.<br><br>• Each rectangle is 1 unit wide.<br>• The rectangle on the left has a height of 4 units.<br>• The area of the first rectangle is $(1)(4) = 4 \text{ units}^2$.<br>• The rectangle on the right has a height of 7 |

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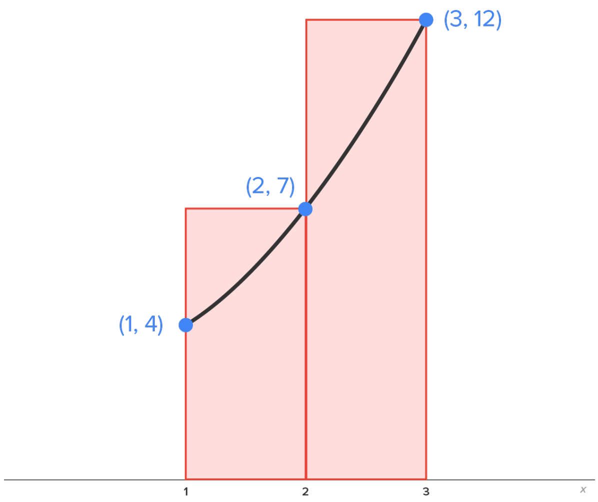
| | |
|---|---|
| | units.<br>• The area of the second rectangle is $(1)(7) = 7 \text{ units}^2$.<br><br>The combined area is $11 \text{ units}^2$. We know this is an underestimate of the actual area since the rectangles are inscribed. |
|  | The rectangles used here are **circumscribed**, meaning drawn in such a way that the rectangle completely encloses the region, but is as small as possible. Notice how one corner of each rectangle is also on the curve.<br><br>• Each rectangle is 1 unit wide.<br>• The rectangle on the left has a height of 7 units.<br>• The area of the first rectangle is $(1)(7) = 7 \text{ units}^2$.<br>• The rectangle on the right has a height of 12 units.<br>• The area of the second rectangle is $(1)(12) = 12 \text{ units}^2$.<br><br>The combined area is $19 \text{ units}^2$. We know this is an overestimate of the actual area since the rectangles are circumscribed. |

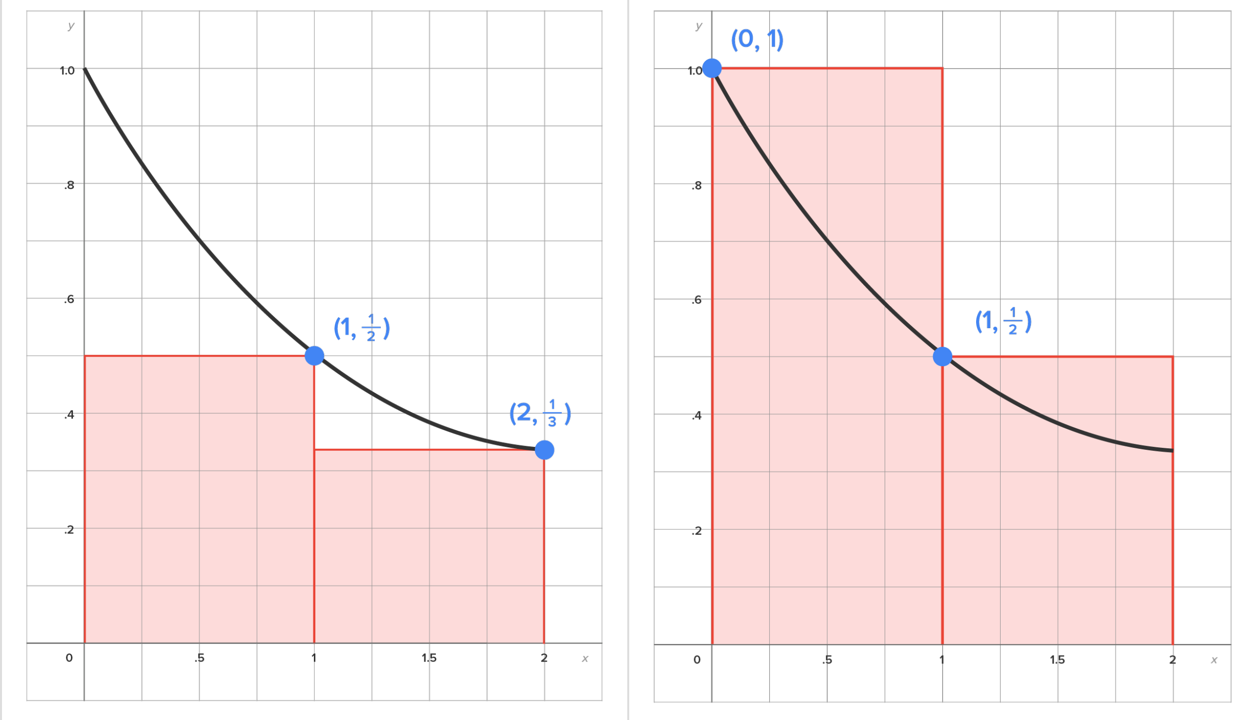
Since one estimate is an underestimate and one is an overestimate, one way to get a better approximation is to average them.

Using this logic, an estimate for the actual area is $\frac{11+19}{2} = 15 \text{ units}^2$.

TRY IT

Approximate the area of the region bounded by $y = \frac{1}{x+1}$, the x-axis, $x = 0$, and $x = 2$ by finding the combined area of the rectangles, as shown in each figure. Then, find the average of the estimates.

| Figure 1 | Figure 2 |
|---|---|
| | |




What is the area of Figure 1?

+

$\text{Area} = \frac{5}{6} \text{ units}^2$

What is the area of Figure 2?

+

$\text{Area} = \frac{3}{2} \text{ units}^2$

What is the average area?

+

$\frac{7}{6} \text{ units}^2$

TERMS TO KNOW

**Inscribed (Rectangles)**

A rectangle is inscribed inside a region if it is the largest rectangle that stays inside the region.

**Circumscribed (Rectangles)**

A rectangle is circumscribed outside a region if it is the smallest rectangle that encompasses the region.

SUMMARY

In this lesson, you learned that **area can be found using basic geometric formulas**, by combining areas, or by subtracting areas. For example, when finding the area of a trapezoid, you could use the trapezoid area formula, or you could split the trapezoid into a rectangle and a triangle and combine their respective areas. You also learned that when there is no area formula available, you can **approximate areas by using rectangles and graphs**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Circumscribed (Rectangles)**
A rectangle is circumscribed outside a region if it is the smallest rectangle that encompasses the region.

**Inscribed (Rectangles)**
A rectangle is inscribed inside a region if it is the largest rectangle that stays inside the region.

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# Some Applications of "Area"

*by Sophia*

WHAT'S COVERED

In this lesson, you will use the connection between area and velocity graphs in order to find the distance traveled on an interval. This will be the key to understanding the connection between differential calculus and integral calculus. Specifically, this lesson will cover:

1. Applications of Area
2. Calculating Distance Using Area

# 1. Applications of Area

Let's first look at an example where an object is moving with constant velocity.

EXAMPLE An object is moving at $v(t) = 40$ miles per hour, where *t* is the number of hours since the object started to move.

If we wanted to find the distance traveled between $t = 1$ and $t = 3$ (time measured in hours), we use the distance formula, $\text{distance} = \text{rate} \cdot \text{time}$.

The time elapsed is 2 hours, and the distance is $\frac{40 \text{ miles}}{1 \text{ hour}} \cdot 2 \text{ hours} = 80 \text{ miles}$. Note how the "hours" are canceled out.

The distance on the interval $[2, 7]$ is found in a similar way. The time elapsed is 5 hours, and the distance is $\frac{40 \text{ miles}}{1 \text{ hour}} \cdot 5 \text{ hours} = 200 \text{ miles}$.

Let's now look at the previous situation, but graphically.

EXAMPLE An object is moving at $v(t) = 40$ miles per hour, where *t* is the number of hours since the object started to move.

Here is a graph of the velocity of the object:

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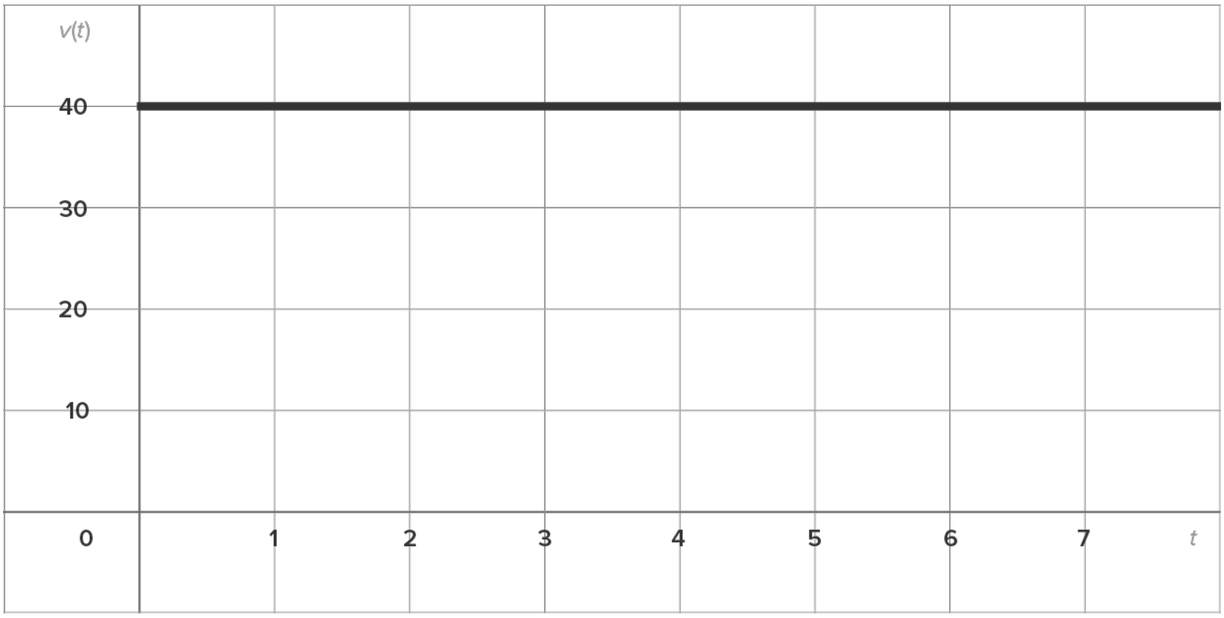
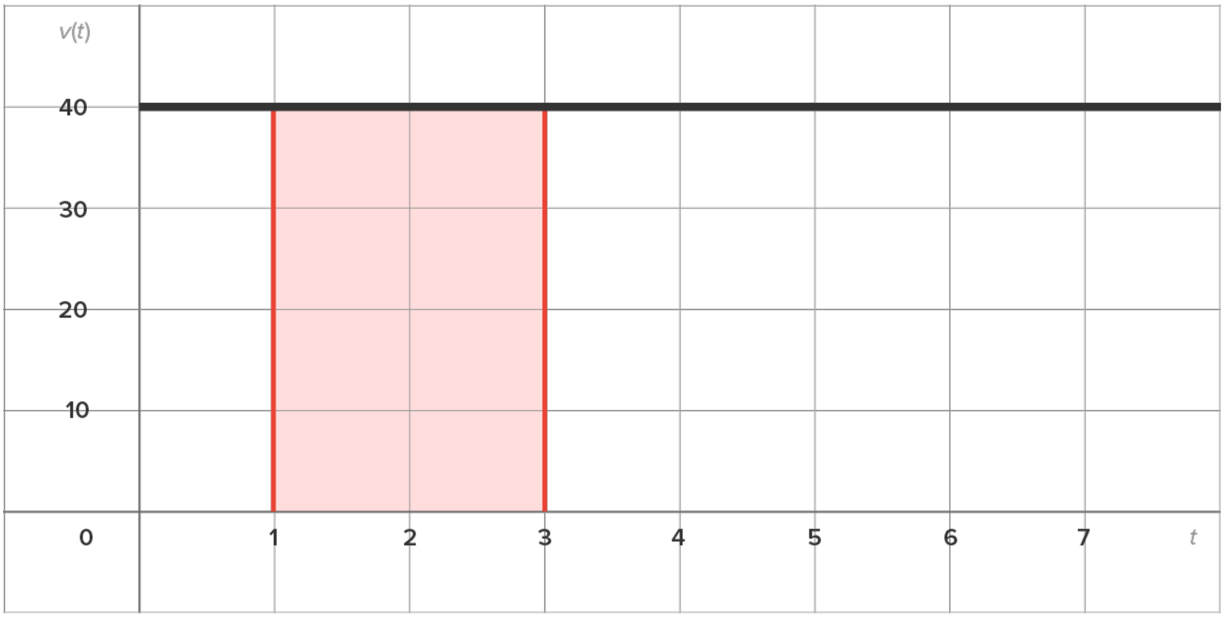



To find the distance traveled between $t = 1$ and $t = 3$, now consider this graph:




Notice that the area of the region is $2(40) = 80$. Since *t* is measured in hours and $v(t)$ is measured in miles per hour, we have the same calculation as in the previous example.

$\text{Area} = (2\text{ hours}) \cdot (40\text{ miles/hour}) = 80\text{ miles}$

Thus, the area between the velocity graph and the t-axis is the distance traveled.

On the interval $[2, 7]$, we see the same thing:

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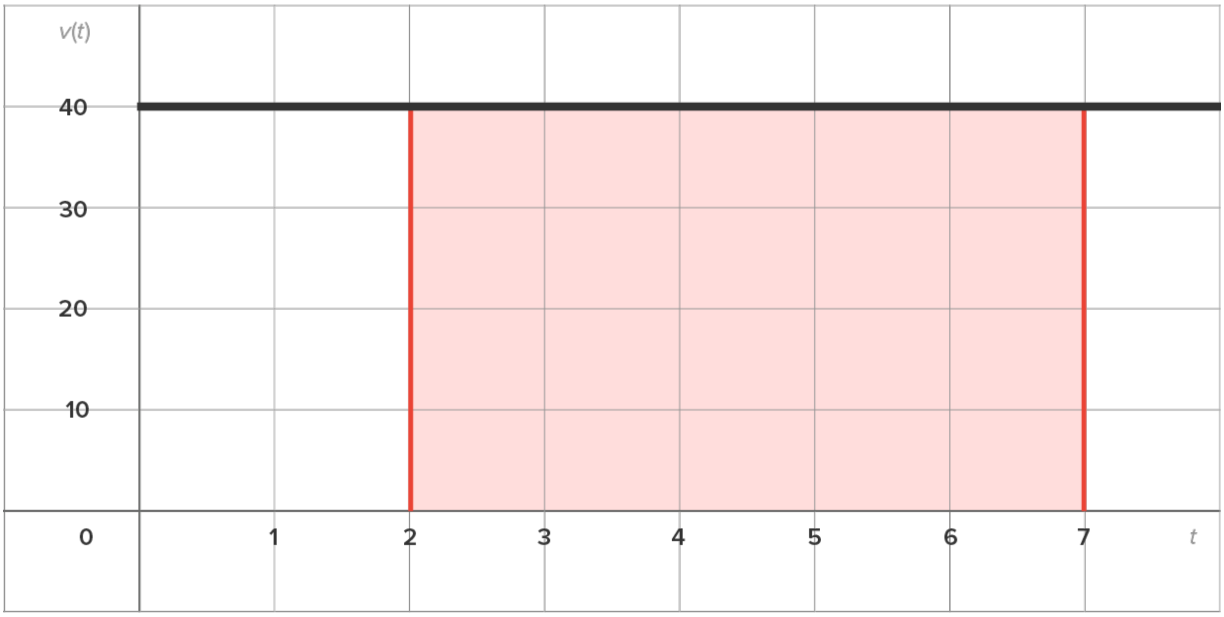



$\text{Area} = (5 \text{ hours}) \cdot (40 \text{ miles/hour}) = 200 \text{ miles}$, which is the same as the answer in the previous exercise.

BIG IDEA

If given $v(t)$, the velocity after $t$ units of time (seconds, minutes, hours), the distance traveled between $t = a$ and $t = b$ is the area between the graph of $v(t)$ and the t-axis on the interval $[a, b]$.

The area concept can be extended to other applications as well. If $c(x) =$ the cost per unit and $x =$ the number of units, the area under $c(x)$ gives the total cost.

If $f(x)$ measures the force (pounds) at a distance of $x$ (feet), then the area under the graph of $f(x)$ has units "pound-feet," which is a measure of work done.

In summary, the area's units are the product of the units of the horizontal axis and the units of the vertical axis. Therefore, the product has to make sense for the area to be useful.

EXAMPLE If both the horizontal and vertical axes were measured in dollars, then the area would be $\text{dollars}^2$, which makes no sense.

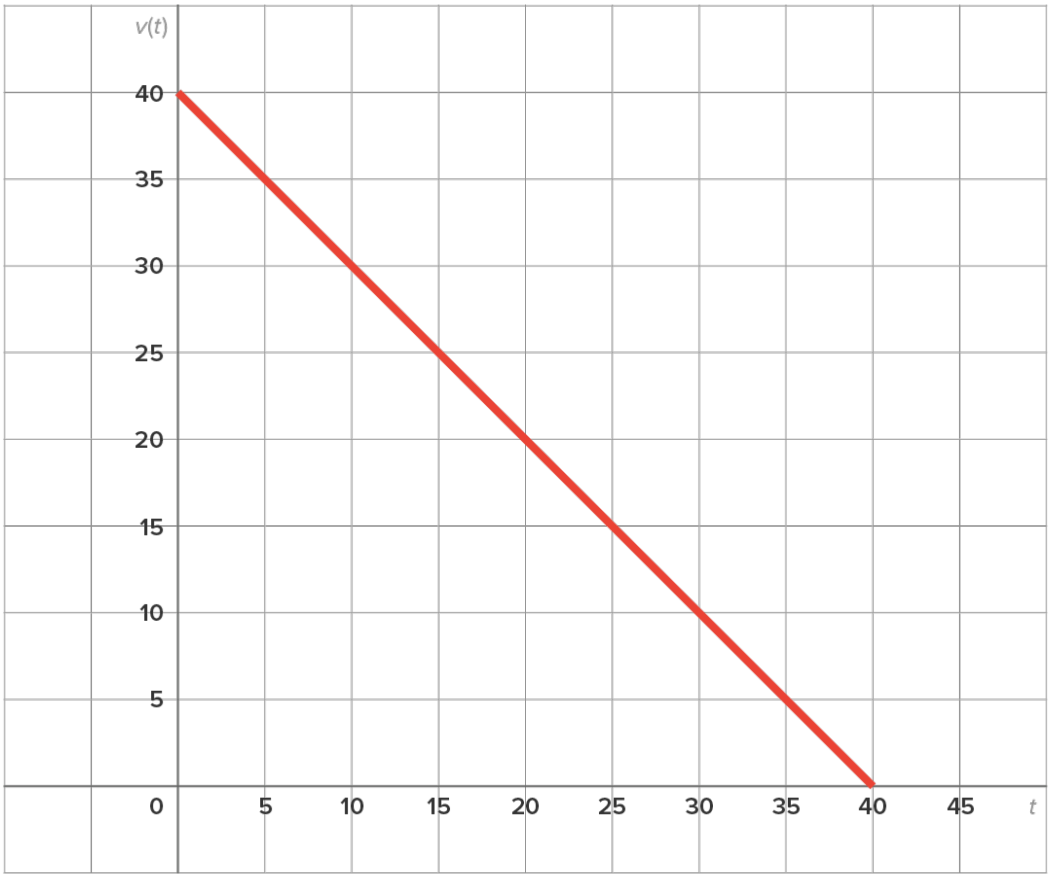
# 2. Calculating Distance Using Area

Now, let's explore the relationship between distance and velocity by looking at some varied situations.

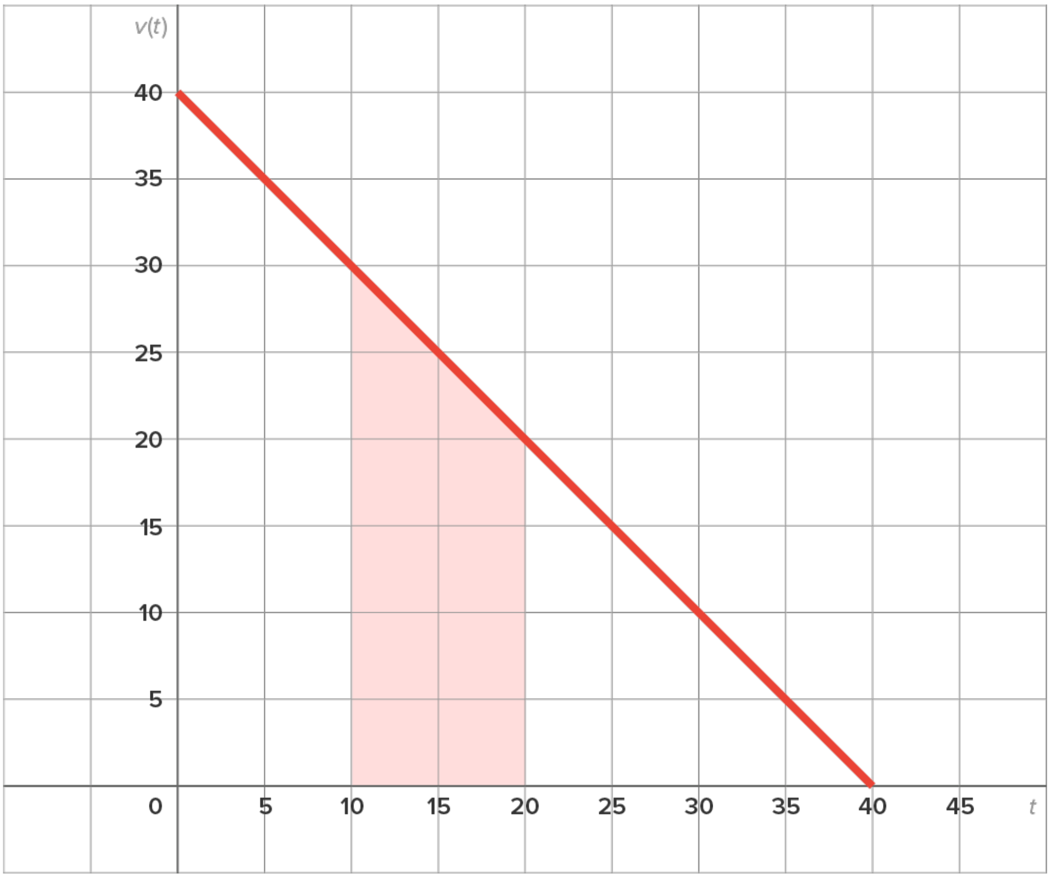
EXAMPLE Suppose the velocity of some object is $v(t) = 40 - t$, where $0 \le t \le 40$.

Here, $t$ is the time in seconds and $v(t)$ is the velocity in meters per second.

The graph of $v(t)$ is shown below:




If we want to know the total distance traveled between 10 and 20 seconds, we want the area between $v(t)$ and the t-axis, between $t = 10$ and $t = 20$, as shown below:




The region is a trapezoid. Note that $v(10) = 30$ and $v(20) = 20$. The parallel bases are 30 and 20 and the

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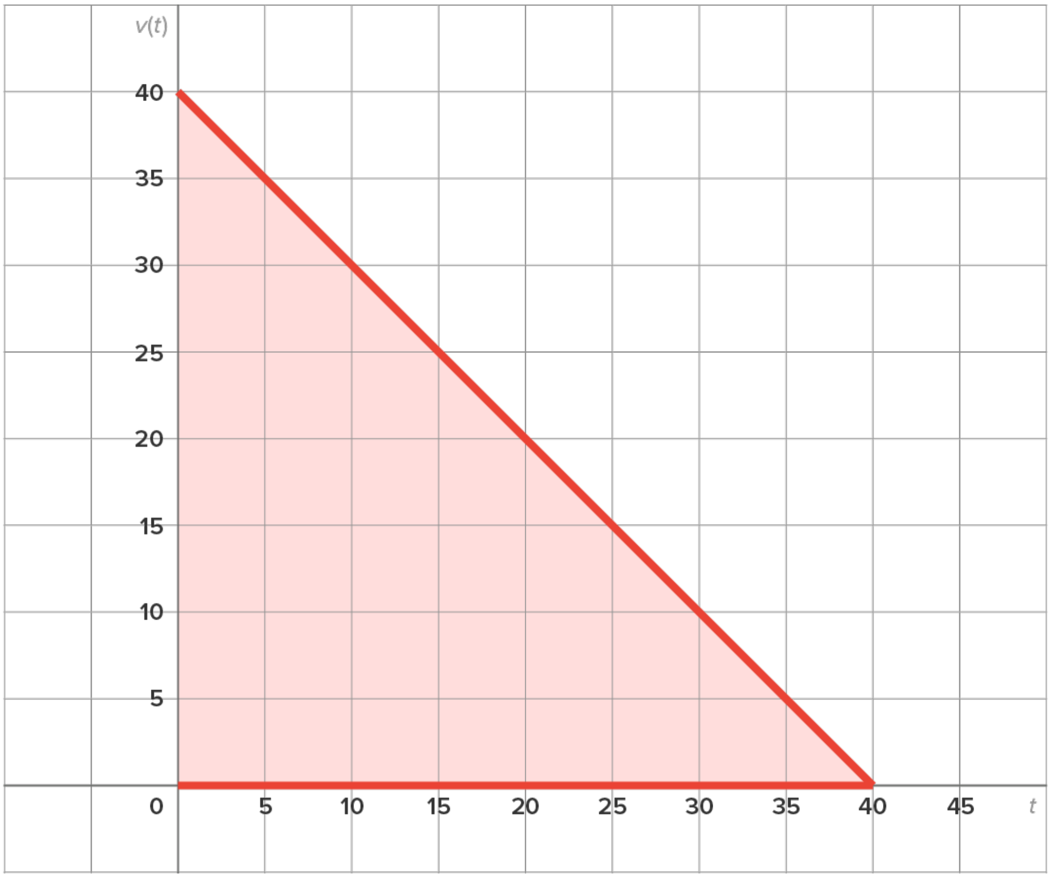
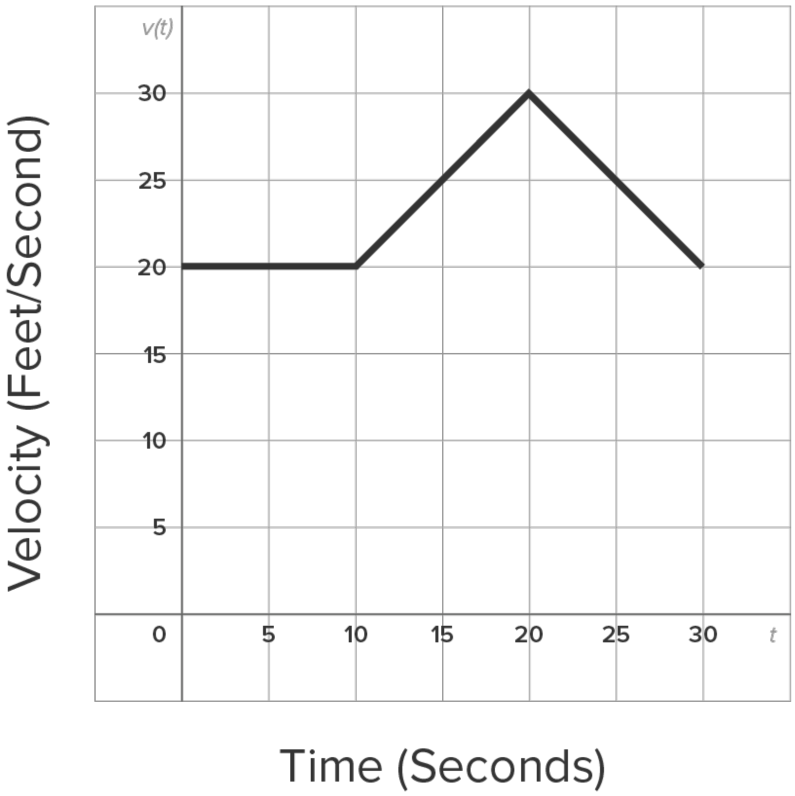
height (along the t-axis) is 10. The area is $\frac{1}{2}(10)(30+20)=250$, which means that the object traveled 250 meters in that time.

Now, if we want to know the total distance traveled between 0 and 40 seconds, we need the area between $v(t)$ and the t-axis, between $t=0$ and $t=40$, as shown below:

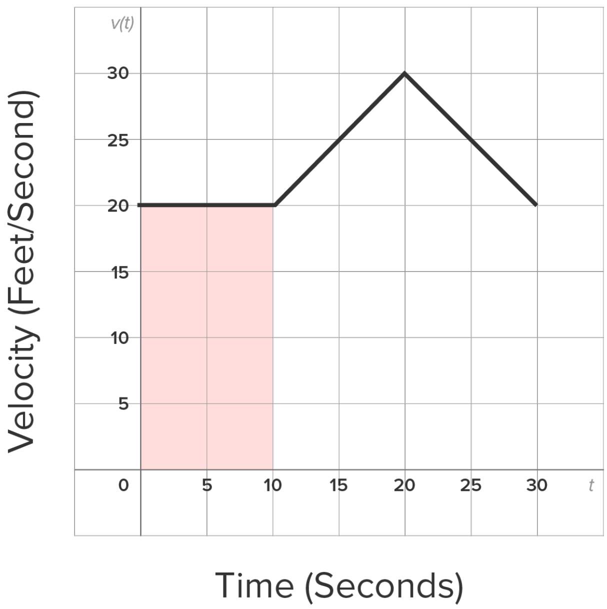



The region is a triangle. Note that $v(0)=40$ and $v(40)=0$. The area is $\frac{1}{2}(40)(40)=800$, which means the object traveled 800 meters in that time.

↪ EXAMPLE The graph shows the velocity of a bicycle over time.




Let's see how the distance traveled changes starting at $t = 0$ and ending at $t = 10$, 20, and 30.

| Time | Graph | Description |
|---|---|---|
| $t = 10$ |  | The region is a rectangle with base 10 and height 20. Thus, the area is $(10)(20) = 200$.<br><br>The distance traveled on $[0, 10]$ is 200 feet. |
| | | Since we already know the area on $[0, 10]$ from above, let's find the area on $[10, 20]$. |

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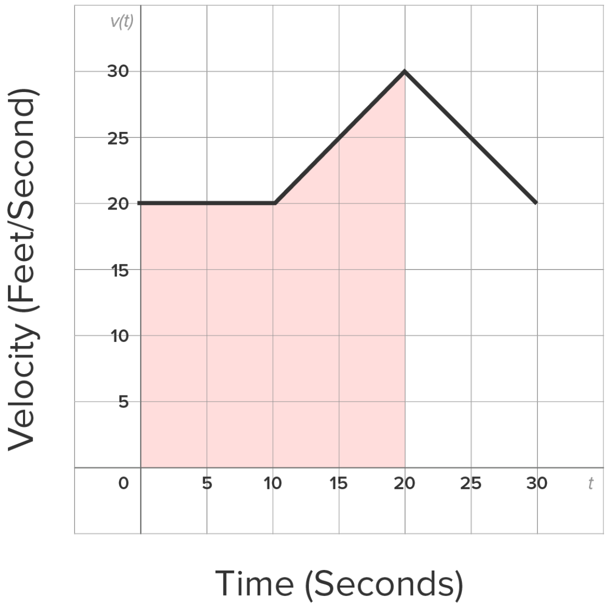
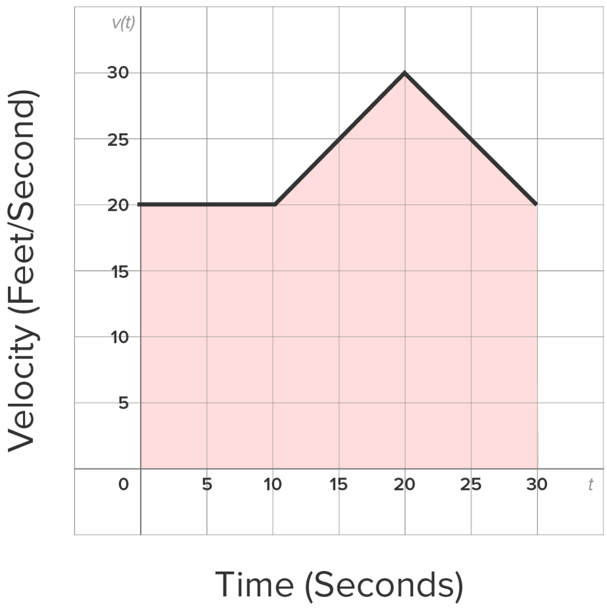
| | | |
|---|---|---|
| $t = 20$ |  | The region is a trapezoid (or triangle on top of a rectangle).<br><br>The rectangle has area $(20)(10) = 200$. The triangle has area $\frac{1}{2}(10)(10) = 50$.<br><br>The distance traveled on $[10, 20]$ is 250 feet.<br><br>Thus, the distance traveled on $[0, 20]$ is $200 + 250 = 450$ feet. |
| $t = 30$ |  | Since we already know the area on $[0, 20]$ from previous work, let's find the area on $[20, 30]$.<br><br>The region is a trapezoid (or triangle on top of a rectangle).<br><br>The rectangle has area $(20)(10) = 200$. The triangle has area $\frac{1}{2}(10)(10) = 50$.<br><br>The distance traveled on $[20, 30]$ is 250 feet.<br><br>Thus, the distance traveled on $[0, 30]$ is $450 + 250 = 700$ feet. |

**TRY IT**

Tom was hiking from some starting point and his velocity is shown in the graph below. Assume the horizontal axis shows the time in hours and the vertical axis shows the velocity in miles per hour.

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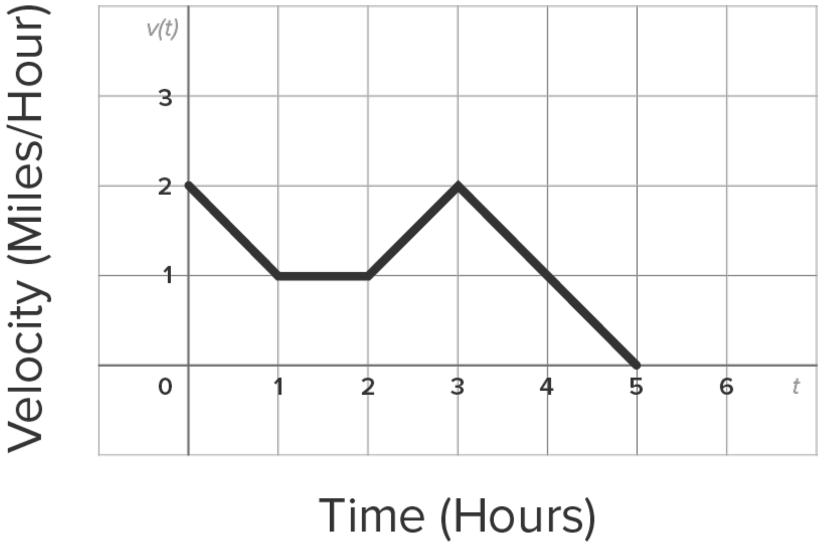


Find Tom's total distance traveled on the interval [0, 1]. +

1.5 miles

Find Tom's total distance traveled on the interval [0, 2]. +

2.5 miles

Find Tom's total distance traveled on the interval [0, 3]. +

4 miles

Find Tom's total distance traveled on the interval [0, 4]. +

5.5 miles

Find Tom's total distance traveled on the interval [0, 5]. +

6 miles

## SUMMARY

In this lesson, you learned about some **applications of area**, notably when units of the product of two quantities are useful, the area between the graph of a function and the x-axis can be used to interpret the result. More specifically, you learned about **calculating distance using area**, understanding that the area between the velocity graph and the t-axis, between $t = a$ and $t = b$, is interpreted as the total

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distance traveled on that interval.

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# Sigma Notation

*by Sophia*

WHAT'S COVERED

In this lesson, you will use sigma notation to evaluate sums, numbers that are a result of some formula. This is a key idea in integral calculus, since we will be adding up several areas. Specifically, this lesson will cover:

1. Introduction to Sigma Notation
2. Using Sigma Notation
3. Special Sums and Summation Properties
   a. Adding Powers of Positive Integers
   b. Summation Properties

## 1. Introduction to Sigma Notation

When there are many numbers to add together, sigma notation $(\Sigma)$ is used to represent this sum (sometimes called a **summation**). Here are some examples of how sums can be written using sigma notation.

EXAMPLE

| Sum | Sigma Notation |
|---|---|
| $1^2+2^2+3^2+4^2+5^2$ | $\sum_{k=1}^{5} k^2$ |
| $2^0+2^1+2^2+2^3$ | $\sum_{k=0}^{3} 2^k$ |
| $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$ | $\sum_{k=3}^{8} \frac{1}{k}$ |

BIG IDEA

In general, sigma notation is written as:

$$\sum_{k=1}^{n} a_k$$

- The symbol $\Sigma$ (capital sigma) is used to represent a sum.
- "$k$" is called the index of summation or the counter.
- The starting value (in this instance) is 1 and the stopping (ending) value is $n$.

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- The notation $a_k$ represents the "formula" for the sequence of numbers being added. In the three problems in the previous example, these were $k^2$, $2^k$, and $\frac{1}{k}$. This is also called the **summand**, in other words, the expression being used to determine the numbers that are added in a sum.

When using many rectangles to represent an area under a curve, sigma notation will be used to represent that sum cohesively. Before discussing the area application, let's see how to use sigma notation.

TERMS TO KNOW

**Summation**
An expression that implies that several numbers are being added together. These are often written using sigma notation.

**Summand**
The expression being used to determine the numbers that are added in a sum.

---

# 2. Using Sigma Notation

We saw earlier how a summation translates to sigma notation; now let's use sigma notation to find sums.

↪ EXAMPLE Evaluate the sum $\sum_{k=1}^{5} k^2$.

To evaluate, substitute the values of $k$ from 1 to 5, then add the results.

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

TRY IT

Consider the sum $\sum_{k=0}^{4} 3^k$.

Evaluate the sum. +

121

↪ EXAMPLE Write out the sum $\sum_{k=1}^{4} f(x_k)$.

Substituting the values of $k$, we have $f(x_1) + f(x_2) + f(x_3) + f(x_4)$.

Now, let's get some practice writing sigma notation.

↪ EXAMPLE Write the sigma notation for 2 + 4 + 6 + ... + 80. Assume the starting value is $k = 1$.

In this example, $k = 1$ corresponds to "2", $k = 2$ corresponds to "4", and $k = 3$ corresponds to "6."

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Deduce that the formula for the summand is $2k$. Then, since the last value is 80, the ending value for the sum is $k = 40$.

The sigma notation is $\sum_{k=1}^{40} 2k$.

TRY IT

Suppose you have 9 + 16 + 25 + ... + 144.

Express this sum using sigma notation. Use a convenient starting value. +

$$\sum_{k=3}^{12} k^2$$

# 3. Special Sums and Summation Properties

## 3a. Adding Powers of Positive Integers

Consider the sum $\sum_{k=1}^{40} 2 = 2 + 2 + 2 + \ldots + 2$ (there are 40 terms; each of them are 2s). We can see that the sum is $40(2) = 80$, but there is a “shortcut” we can take.

FORMULA

**The Summation of a Constant**

If $C$ is a constant, $\sum_{k=1}^{n} C = C \cdot n$

HINT

In this formula, there are $n$ terms, where each term has the value of $C$. Also note that this formula only works when the starting value is 1.

Here are summation formulas for powers of consecutive positive integers.

FORMULA

**Summations of Powers of Consecutive Numbers**

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

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HINT

These formulas only work when the starting value is 1. Note also that the value of these sums depends on the ending value.

EXAMPLE Use a formula to evaluate the sum $\sum_{k=1}^{10} k^2$.

We can use the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. To use the formula, note that $k=1$ and $n=10$. Then, the sum is equal to $\frac{10(10+1)(2(10)+1)}{6} = 385$.

Now, you try one.

TRY IT

Consider the sum $\sum_{k=1}^{6} k^3$.

Use a formula to evaluate this sum. +

441

## 3b. Summation Properties

Consider the summation $\sum_{k=1}^{3} 2k^3 = 2(1)^3 + 2(2)^3 + 2(3)^3$. Notice that each term has a factor of 2, which means this can be written $2(1^3 + 2^3 + 3^3)$. Then, the result can be rewritten using sigma notation: $2 \cdot \sum_{k=1}^{3} k^3$

This leads to the following property of summations:

FORMULA

**Summation of a Constant Multiple**

$$\sum_{k=1}^{n} C \cdot a_k = C \cdot \sum_{k=1}^{n} a_k$$

HINT

In other words, you can move the constant multiple outside of the summation.

Now, consider the summation $\sum_{k=1}^{3} (k^3 + k^2) = (1^3 + 1^2) + (2^3 + 2^2) + (3^3 + 3^2)$. Rearrange the terms so that the common powers are together: $(1^3 + 2^3 + 3^3) + (1^2 + 2^2 + 3^2)$. Then, the result can be written using two summations: $\sum_{k=1}^{3} k^3 + \sum_{k=1}^{3} k^2$

This leads to two more summation formulas:

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FORMULA

**Summation of a Sum or Difference**

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

$$\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$$

BIG IDEA

For the summation of a sum, you can break into a sum of two simpler summations. For the summation of a difference, you can break into a difference of two simpler summations.

EXAMPLE Evaluate the summation $\sum_{k=1}^{8}(3k^2+4k-2)$.

By the sum and difference formulas, this is equivalent to $\sum_{k=1}^{8}3k^2+\sum_{k=1}^{8}4k-\sum_{k=1}^{8}2$.

By the constant multiple formulas, this is equivalent to $3\sum_{k=1}^{8}k^2+4\sum_{k=1}^{8}k-\sum_{k=1}^{8}2$.

Now, use the summation formulas:

$$\sum_{k=1}^{8}k^2=\frac{8(8+1)(2\cdot 8+1)}{6}=204$$

$$\sum_{k=1}^{8}k=\frac{8(8+1)}{2}=36$$

$$\sum_{k=1}^{8}2=8(2)=16$$

When you substitute these values, the value of the sum is $3(204)+4(36)-16=740$.

TRY IT

Consider the summation $\sum_{k=1}^{10}(k^4-2k)$.

Evaluate this summation. +

25,223

## SUMMARY

In this lesson, you learned that when a sum contains many terms, **sigma notation** is a convenient way to express the sum, as long as the terms follow some pattern. Next, you practiced **using sigma notation** to find sums. You also learned about some **special sums and summation properties** that

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enable you to find the sum quickly without having to add a large set of numbers together, such as when **adding powers of positive integers**. In the next section, we'll see why sigma notation is useful when calculating area.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Summand**

The expression being used to determine the numbers that are added in a sum.

**Summation**

An expression that implies that several numbers are being added together. These are often written using sigma notation.

## FORMULAS TO KNOW

**Summation of a Constant Multiple**

$$\sum_{k=1}^{n} C \cdot a_k = C \cdot \sum_{k=1}^{n} a_k$$

**Summation of a Sum or Difference**

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

**Summations of Powers of Consecutive Numbers**

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

**The Summation of a Constant**

If $C$ is a constant, $\sum_{k=1}^{n} C = C \cdot n$

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# Area Under A Curve –– Riemann Sums

*by Sophia*

WHAT'S COVERED

In this lesson, you will form Riemann sums to approximate areas. This idea is very important as it paves the way for some applications in integral calculus. Specifically, this lesson will cover:

1. Definition of Riemann Sum
2. Finding the Riemann Sum
   a. Find the Partition and Subintervals
   b. Find the Width of Each Subinterval
   c. Select x-Values Within Each Partition
   d. Form the Riemann Sum
3. Using Riemann Sums to Calculate Area

# 1. Definition of Riemann Sum

Suppose we want to calculate the area between the graph of a nonnegative function $f(x)$ and the x-axis interval $[a, b]$, as shown in the figure below.

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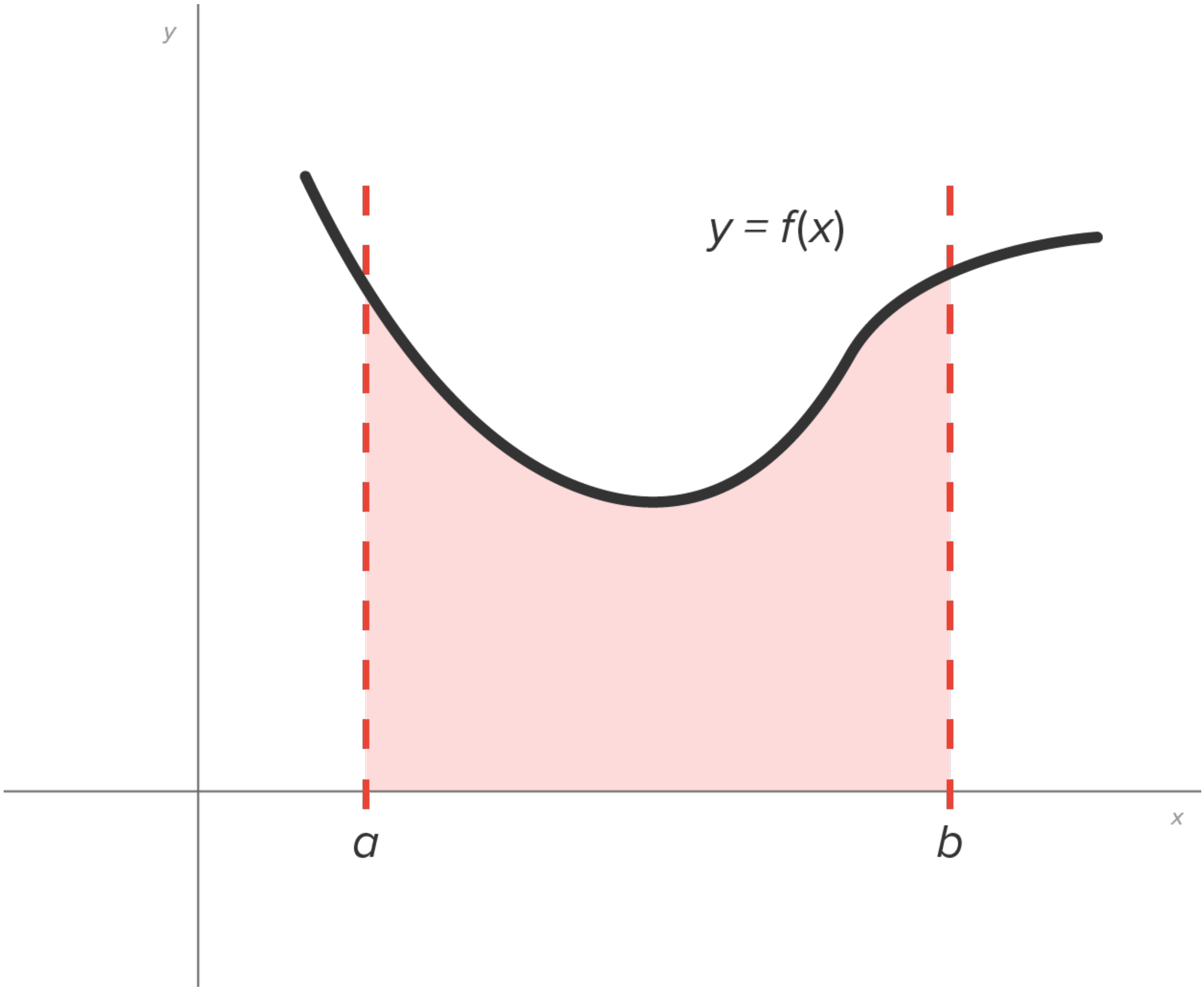


If $f(x)$ is nonnegative, the Riemann sum method is to build several rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of $f(x)$. Then, the areas of the rectangles can be calculated and added together to get a number called a **Riemann sum** of $f(x)$ on $[a, b]$.

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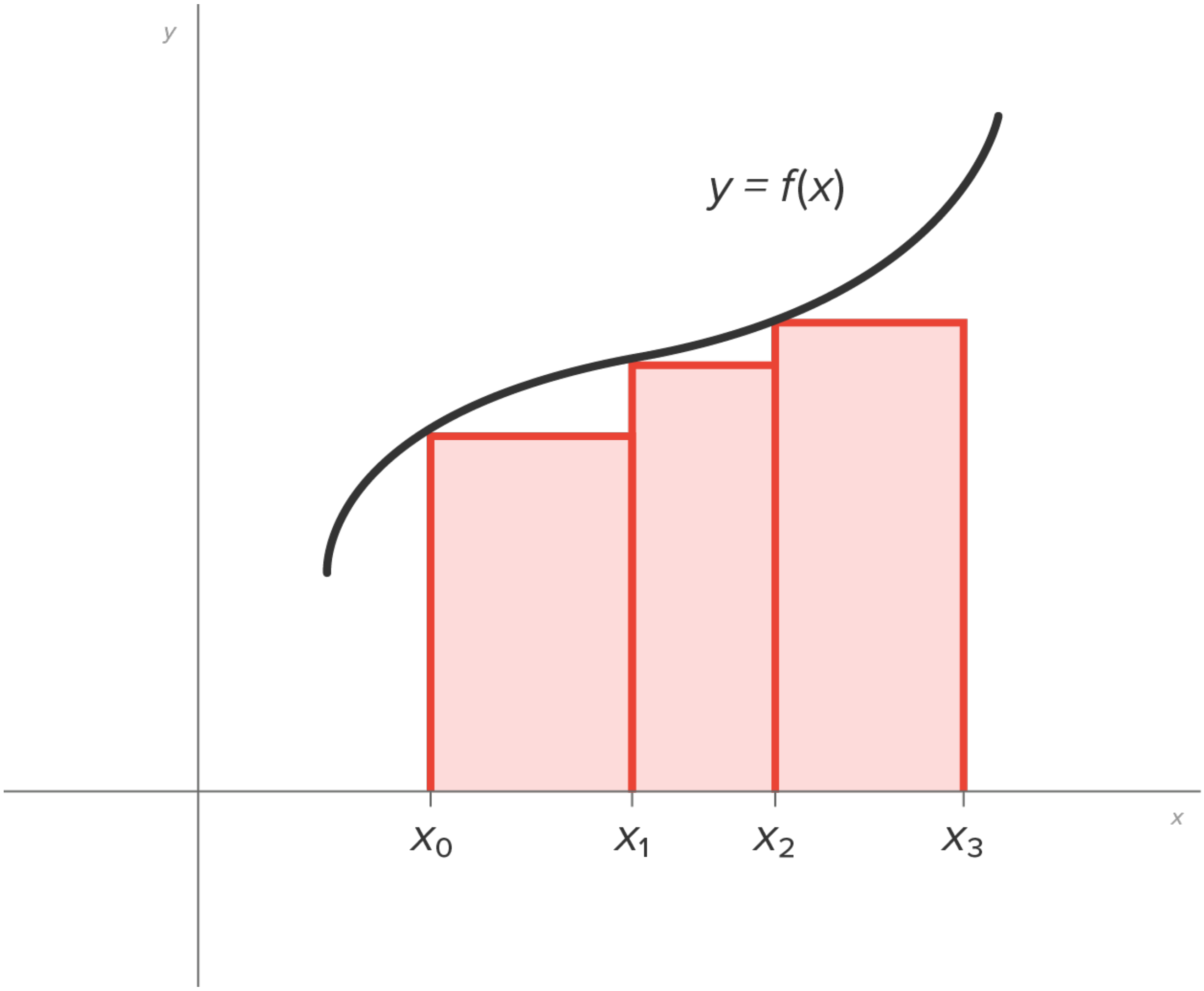


The area of the region formed by the rectangles is an approximation of the area between the graph and the x-axis.

TERM TO KNOW

**Riemann Sum**

The sum obtained from the areas of rectangles that are used to approximate the area between a curve and the x-axis.

# 2. Finding the Riemann Sum

In order to find the Riemann sum, there are several quantities that need to be established first.

1. Find the partition and subintervals.
2. Find the width of each subinterval.
3. Select x-values within each partition.
4. Form the Riemann sum.

Let's take a deeper look at each step.

## 2a. Find the Partition and Subintervals

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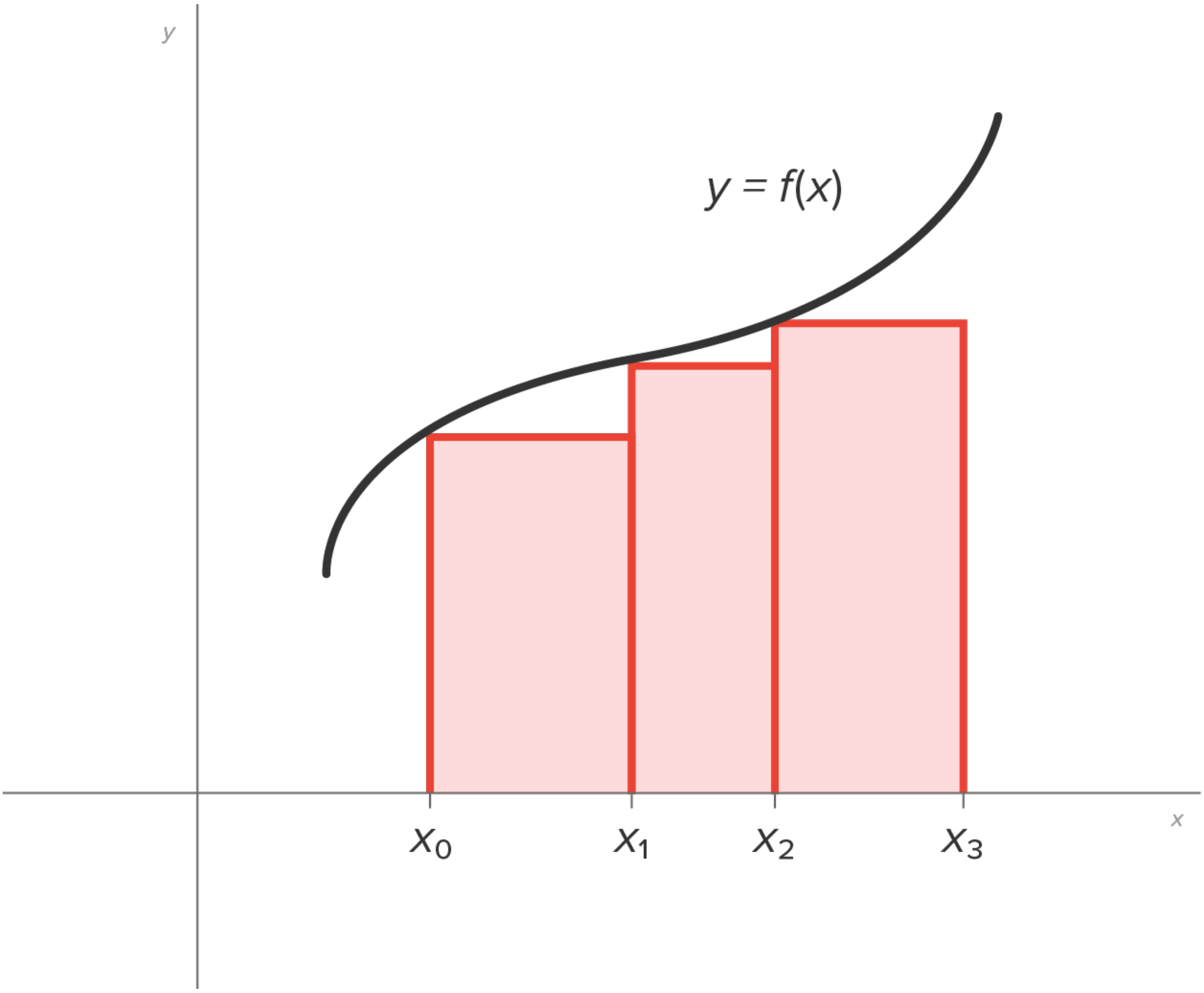
First, a **partition** of the interval $[a, b]$ is needed to establish the bases of the rectangles. Consider the graph in the figure.




The endpoints of the interval are $x_0$ and $x_3$. In order to form three rectangles, two more values ($x_1$ and $x_2$) are added to form a partition of the interval $[x_0, x_3]$.

We label the partition by the x-coordinates, namely $\{x_0, x_1, x_2, x_3\}$. The numbers are listed in increasing order.

Note that there are 4 x-values in the partition for three rectangles. In general, if $n$ rectangles are desired, there would be $n+1$ x-values in the partition. This is why the first one is labeled as $x_0$ (which is the left-hand endpoint of the interval), so the last one can be called $x_n$ (to match the number of rectangles).

The **subintervals** for this partition are $[x_0, x_1]$, $[x_1, x_2]$, and $[x_2, x_3]$.

TERMS TO KNOW

**Partition**

A set of x-values that are used to split the interval $[a, b]$ into smaller intervals.

**Subinterval**

A smaller interval that is part of a larger interval.

## 2b. Find the Width of Each Subinterval

It is most convenient to select a partition where each x-value is the same distance apart from its neighbor, but

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that is not necessary.

Continuing with this partition, we use the notation $\Delta x_k$ to represent the width of the $k^{th}$ subinterval. Recall that the width of an interval is the difference between its endpoints.

| Subinterval | Width |
|---|---|
| $[x_0, x_1]$ | $\Delta x_1 = x_1 - x_0$ |
| $[x_1, x_2]$ | $\Delta x_2 = x_2 - x_1$ |
| $[x_2, x_3]$ | $\Delta x_3 = x_3 - x_2$ |

## 2c. Select x-Values Within Each Partition

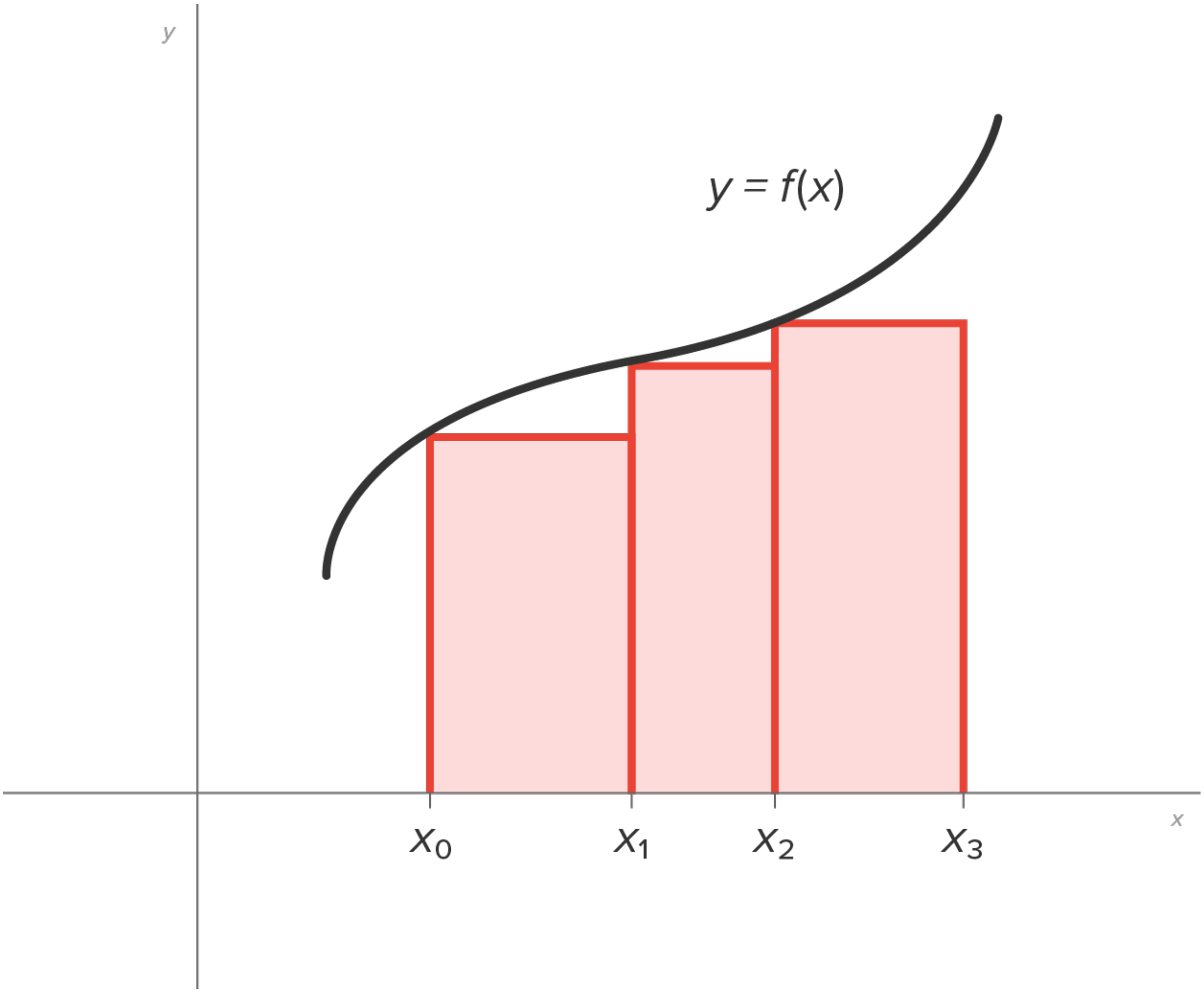
Let $c_k =$ the value of *x* used in the $k^{th}$ subinterval. There are popular choices for $c_k$:

- The left endpoint of each subinterval
- The right endpoint of each subinterval
- The midpoint of each subinterval

Of course, we are not forced to use any one of these, but these are the most convenient.

## 2d. Form the Riemann Sum

Consider the figure shown below:




As the rectangles suggest, the left-hand endpoint was used in each sub-interval to set the height of the rectangle. This means:

| Subinterval | Value Chosen | Width |
|---|---|---|
| $[x_0, x_1]$ | $c_1 = x_0$ | $\Delta x_1 = x_1 - x_0$ |
| $[x_1, x_2]$ | $c_2 = x_1$ | $\Delta x_2 = x_2 - x_1$ |
| $[x_2, x_3]$ | $c_3 = x_2$ | $\Delta x_3 = x_3 - x_2$ |

So, we can say:

- Area of the first rectangle: $f(c_1) \cdot \Delta x_1$
- Area of the second rectangle: $f(c_2) \cdot \Delta x_2$
- Area of the third rectangle: $f(c_3) \cdot \Delta x_3$

Then, the approximation for the area between $f(x)$ and the x-axis is the sum of these areas. Written using sigma notation, the Riemann sum is:

$$\sum_{k=1}^{3} f(c_k) \cdot \Delta x_k$$

In general, here is the definition (formula) for a Riemann sum.

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FORMULA

**Riemann Sum**

When approximating the area between a nonnegative function $y = f(x)$ and the x-axis by using $n$ rectangles, the summation $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called the Riemann sum, where $c_k$ is a value of $x$ in the $k^{th}$ subinterval, and $\Delta x_k$ is the width of the $k^{th}$ subinterval.

---

# 3. Using Riemann Sums to Calculate Area

Now that we have all the definitions, let's compute a few Riemann sums.

EXAMPLE Use a Riemann sum with 4 rectangles of equal width to approximate the area between $y = x^2 + 1$ and the x-axis on the interval $[1, 3]$. Use the left-hand endpoint of each subinterval.

Since each subinterval will have equal width, that width is $\frac{\textit{width of } [1, 3]}{4} = \frac{2}{4} = 0.5$.

Based on the problem, we have the following information:

| Subinterval | Width of Subinterval | Value Chosen in Each Subinterval |
|---|---|---|
| $[1, 1.5]$ | 0.5 | 1 |
| $[1.5, 2]$ | 0.5 | 1.5 |
| $[2, 2.5]$ | 0.5 | 2 |
| $[2.5, 3]$ | 0.5 | 2.5 |

Here's a picture of the graph with the rectangles that were used:

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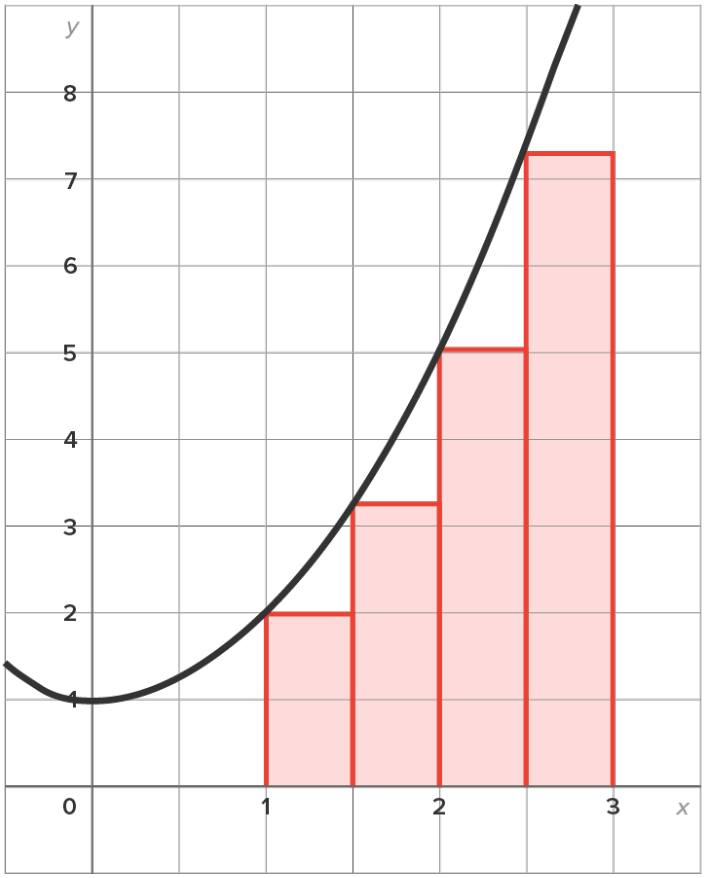


Then, the Riemann sum is:

| | |
|---|---|
| $\sum_{k=1}^{4} f(c_k)\Delta x_k = f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5$ | Use the Riemann sum formula. |
| $= 0.5[f(1)+f(1.5)+f(2)+f(2.5)]$ | Factor out 0.5. |
| $= 0.5(2+3.25+5+7.25)$ | Substitute values: $f(1)=2$, $f(1.5)=3.25$, $f(2)=5$, $f(2.5)=7.25$ |
| $= 8.75$ | Simplify. |

Thus, an approximation of the area is $8.75 \text{ units}^2$.

BIG IDEA

When the width of each subinterval is the same, we call the width of the interval $\Delta x$ since they are all the same, then $\Delta x = \frac{b-a}{n}$.

TRY IT

Use a Riemann sum with 4 rectangles of equal width to approximate the area between $y = x^2 + 1$ and the x-axis on the interval $[1, 3]$. Use the right-hand endpoint of each subinterval. Note, this is the same information as in the last example, except that right-hand endpoints are used.

Approximate the area. +

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$12.75 \text{ units}^2$

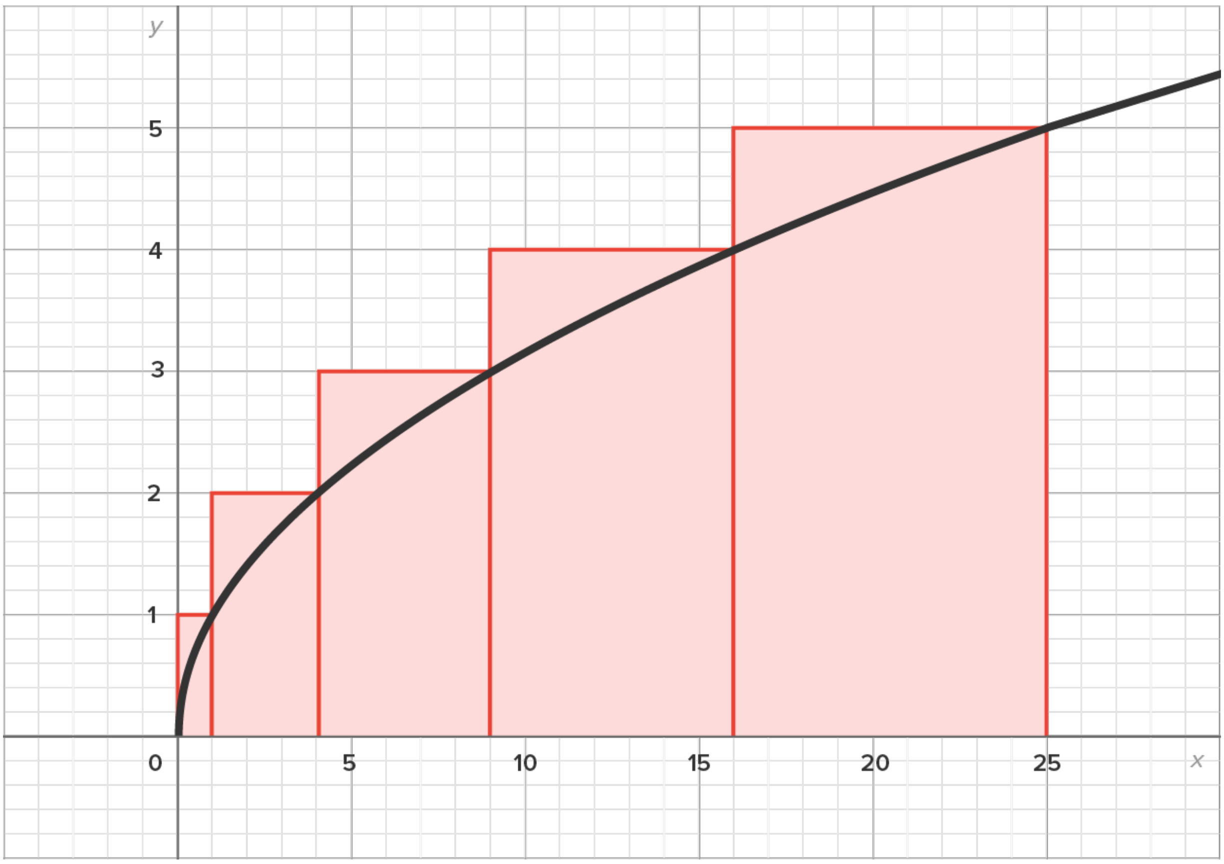
Let's look at an example where the widths of the intervals are not the same.

↱ EXAMPLE Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 25]$. Estimate the area between $f(x)$ and the x-axis by using the partition $\{0, 1, 4, 9, 16, 25\}$. Use the right-hand endpoint of each subinterval.

Since there are 6 numbers in the partition, there are 5 rectangles. The table shows the information we need to set this up:

| Subinterval | Width of Subinterval | Value Chosen in Each Subinterval |
|---|---|---|
| $[0, 1]$ | 1 | 1 |
| $[1, 4]$ | 3 | 4 |
| $[4, 9]$ | 5 | 9 |
| $[9, 16]$ | 7 | 16 |
| $[16, 25]$ | 9 | 25 |

The graph of $f(x)$ along with the rectangles is shown below.

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Then, the Riemann sum is $\sum_{k=1}^{5} f(c_k)\Delta x_k$.

| | |
|---|---|
| $\sum_{k=1}^{5} f(c_k)\Delta x_k = f(1)\cdot 1 + f(4)\cdot 3 + f(9)\cdot 5 + f(16)\cdot 7 + f(25)\cdot 9$ | Use the Riemann sum formula. |
| $= 1(1)+2(3)+3(5)+4(7)+5(9)$ | Substitute values: $f(1)=1$, $f(4)=2$, $f(9)=3$, $f(16)=4$, $f(25)=5$ |
| $=95$ | Simplify. |

Thus, the approximation for the area is $95 \text{ units}^2$.

WATCH

In this video, we will use a Riemann sum to approximate the area below the graph of $f(x)=x^2+2$ on the interval $[0, 4]$ using 4 rectangles of equal width, using the left-hand endpoints of each subinterval.

## Video Transcription

[MUSIC PLAYING] Hi there, and welcome back. What we're going to do is continue our exploration into Riemann sums with an example of a parabola, f of x equals x squared plus 2.

And we're going to find the Riemann sum to estimate the area below that graph and above the x-axis, on the interval 0 to 4, using four equally spaced rectangles along the x-axis. And we're going to use the left-hand endpoint of each subinterval to determine the height of each rectangle.

So here we have a hand-drawn graph of y equals x squared plus 2 or f of x equals x squared plus 2. Each interval is going to have width 1 unit-- or should I say, each subinterval. So that means, for the first subinterval, which is from 0 to 1, this point right here is going to represent the height of our rectangle.

On the interval 1 to 2, we'll use x equals 1 for the height of the rectangle. On the interval 2 to 3, we'll use x equals 2 to determine the height. And on the interval 3 to 4, we'll use x equals 3. So there are four rectangles. And even though it doesn't look like they have equal width, they theoretically have equal width.

So all we need to do is find the area of each rectangle. Now, the height of each rectangle is determined by the value of the function. And x equals 0. This is the point 0, 2. So I'm just going to write the y-coordinate here. So that's 2.

And x equals 1. If we plug-in x equals 1 to the function, f of 1 is equal to 1 squared plus 2, which is 3. So the height of that rectangle is 3. I'm actually going to write the height right where the height actually is. Why not? So then, when x equals 2, the height is 2 squared plus 2, which is 6. So this rectangle here is 6 units high.

And the last one, which is determined by f of 3, 3 squared plus 2, is 11. So-- and just to complete the picture here-- f of 0 is 0 squared plus 2, which is 2. So those are the heights of my rectangles.

Now the total area, which is the Riemann sum-- so remember the Riemann sum is the sum going from k going from 1 to 4 of f of c sub k times delta x sub k, where c sub k are the values that we're substituting, so that would be 0, 1, 2, and 3. And delta x sub k would be the width of each subinterval. In this case, they're all the same. They're all 1.

So that means that this is going to be-- so f of 0 is 2 times 1 plus 3 times 1 plus 6 times 1 plus 11 times 1, which is just the sum of 2, 3, 6, and 11. And that looks to be 22. And that is the estimate for the area between the x-axis and y equals x squared plus 2, between x equals 0 and x equals 4.

Now as we can see, that area is an underestimate for the actual area because all of the rectangles are what we call inscribed. They are all underneath the curve. There is space between the curve and the rectangles, which means that there's actually more area between the x-axis and the curve.

And one way we could close in on that actual area is to use more rectangles. We could use eight, ten, 20, 100, 1,000 rectangles to try to get closer to the estimate for the actual area. So there's our Riemann sum for four rectangles.

[MUSIC PLAYING]

### THINK ABOUT IT

What effect would increasing the number of rectangles (partitions) have on the estimate in terms of the actual area?

## SUMMARY

In this lesson, you learned that a **Riemann sum** provides a systematic way to approximate the area between a curve $y=f(x)$ and the x-axis on the interval $[a, b]$, by obtaining the sum from the areas of rectangles. You learned that when **finding the Riemann sum**, there are several quantities that need to be established first: **find the partition and subintervals**; **find the width of each subinterval**; **select x-values within each partition**; and finally, **form the Riemann sum**. Using this knowledge, you were then able to explore several examples of **using Riemann sums to calculate area**. Many applications we will investigate later in this course are based on Riemann sums, which makes this a very important topic to understand.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Partition**
A set of x-values that are used to split the interval $[a, b]$ into smaller intervals.

**Riemann Sum**
The sum obtained from the areas of rectangles that are used to approximate the area between a curve

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and the x-axis.

**Subinterval**

A smaller interval that is part of a larger interval.

## FORMULAS TO KNOW

**Riemann Sum**

When approximating the area between a nonnegative function $y = f(x)$ and the x-axis by using $n$ rectangles, the summation $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called the Riemann Sum, where $c_k$ is a value of $x$ in the $k^{th}$ subinterval, and $\Delta x_k$ is the width of the $k^{th}$ subinterval.

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# Definition of the Definite Integral

*by Sophia*

## WHAT'S COVERED

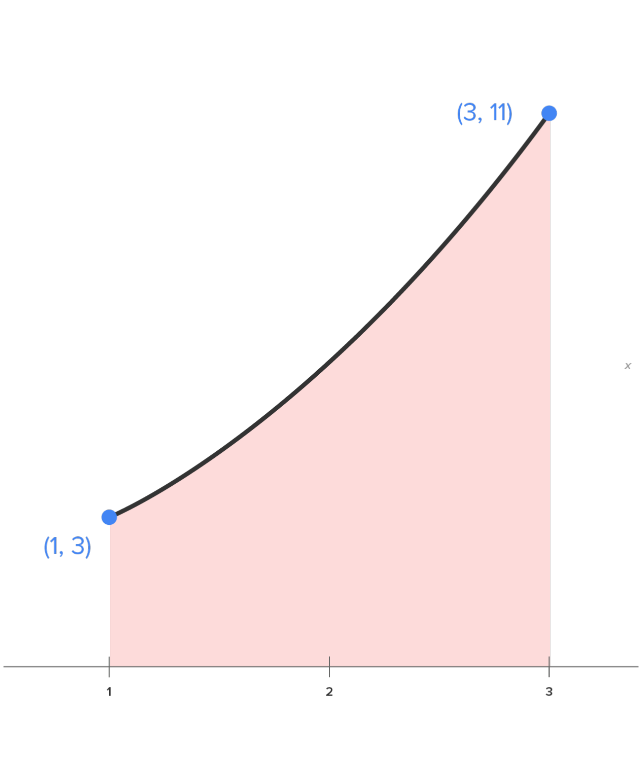
In this lesson, you will connect Riemann sums and the definition of the definite integral. This is the key step in moving into integral calculus. Specifically, this lesson will cover:

1. The Definition of the Definite Integral
2. Definite Integrals and Riemann Sums
3. Using Riemann Sums to Evaluate Definite Integrals
4. Using Area to Evaluate Riemann Sums and Definite Integrals

# 1. The Definition of the Definite Integral

Consider the area of the region bounded by $y=x^2+2$ between the x-axis, $x=1$, and $x=3$.

Shown below is the actual region as well as the region approximated by 4, 8, and 16 rectangles; all have equal width.

<table>
<tr><th>Actual Region</th><th>4 Rectangles</th></tr>
<tr><td>


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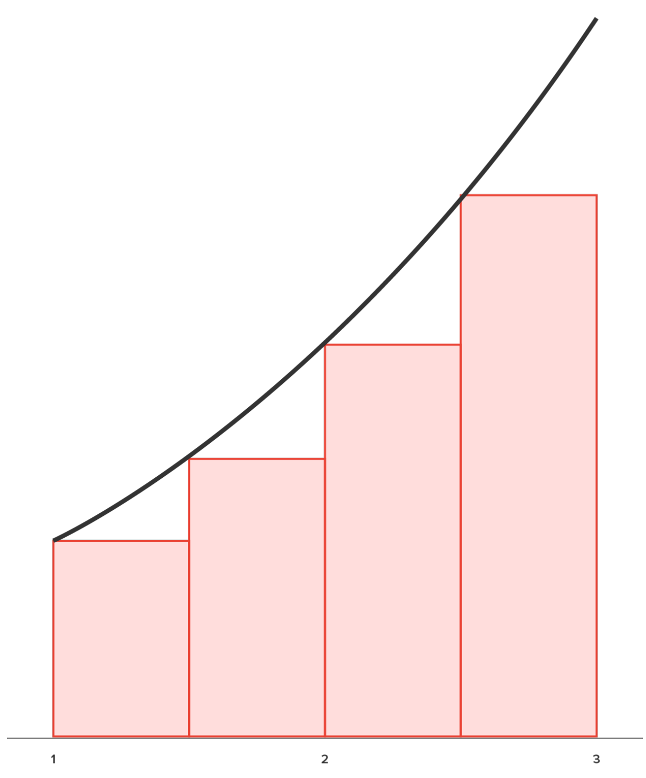

</td></tr>
<tr><th>8 Rectangles</th><th>16 Rectangles</th></tr>
</table>

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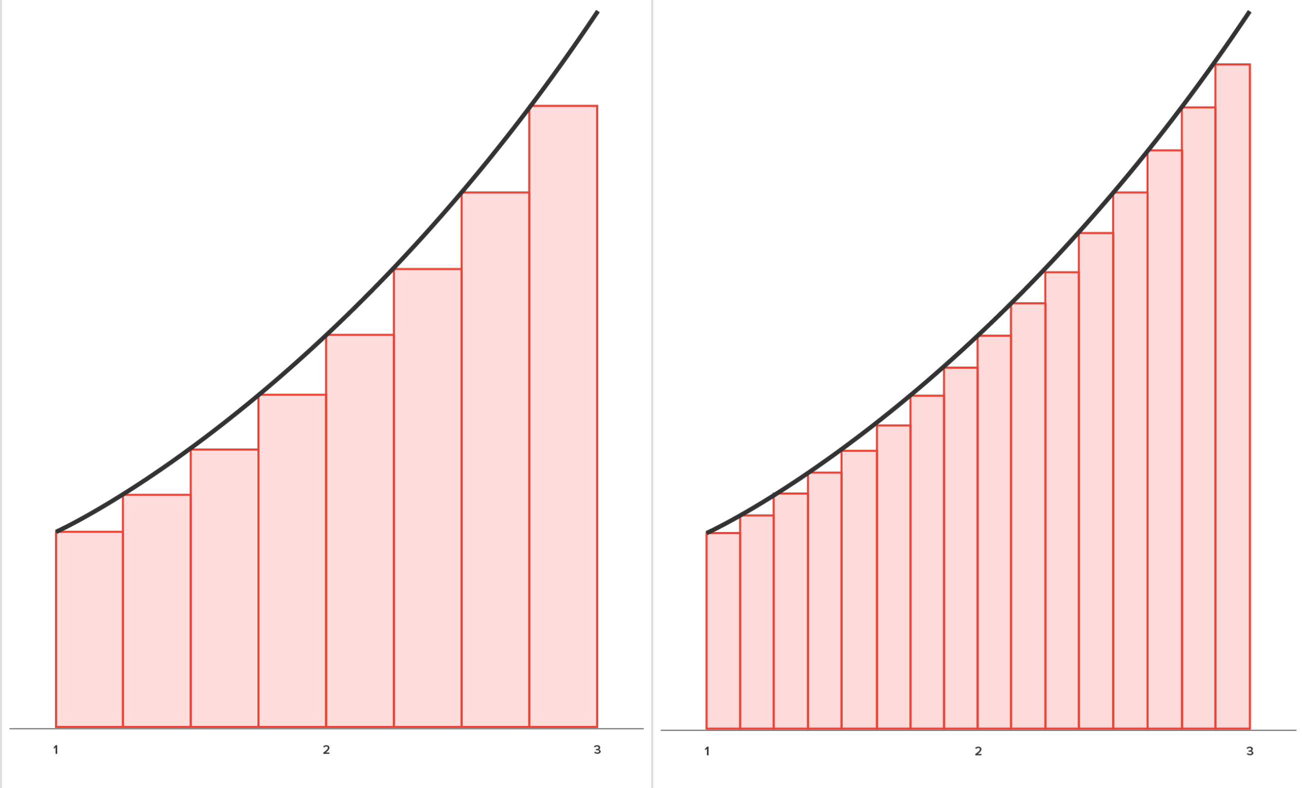


When the subintervals have equal width, we notice the following as the number of rectangles (and subintervals) increases:

1. The width of each rectangle (and subinterval) decreases.
2. The sum of the areas of the rectangles gets closer to the actual area under the curve.

When calculating a Riemann sum for a function $f(x)$ on $[a, b]$, we will only use rectangles that have equal width. That is, when *n* subintervals are used, the width of each subinterval is $\Delta x = \frac{b-a}{n}$. This also means that Riemann sums from this point forward will be written as:

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

Assume (for now) that $f(x)$ is nonnegative. As the number of rectangles gets larger, which means that $n \to \infty$, this quantity will get closer to the actual area as long as $\Delta x \to 0$ for all *k*.

When $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x$ exists and has the same value regardless of the values of $c_k$ used in each subinterval, then $f(x)$ is **integrable** on $[a, b]$.

If we call *A* the definite integral of $f(x)$ on $[a, b]$, then $A = \int_a^b f(x)dx$.

In this notation,

- The numbers *a* and *b* represent the lower and upper limits of integration.
- The function $f(x)$ is called the integrand.
- *x* is called the variable of integration.
- The differential *dx* tells us that the definite integral is computed by letting values of *x* increase from *a* to *b*.

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BIG IDEA

For a non-negative function $f(x)$, the value of the Riemann sum approaches the definite integral as $n \to \infty$.

Then, the quantity $\int_a^b f(x)dx$ is the area between the graph of a nonnegative function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

TERM TO KNOW

**Integrable**

If the value of $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to $A$ regardless of the values of $c_k$ used in each subinterval, then we say that $f(x)$ is integrable on the interval $[a, b]$.

---

# 2. Definite Integrals and Riemann Sums

Using the definition of a definite integral, we can write Riemann sums as definite integrals and vice versa.

EXAMPLE Write $\int_0^4 2xdx$ as a Riemann sum.

Recall the Riemann sum for a function $f(x)$ is $\sum_{k=1}^{n} f(c_k)\Delta x$.

Since $f(x) = 2x$, the definite integral is the value of the Riemann sum as $n \to \infty$: $\lim_{n \to \infty} \sum_{k=1}^{n} 2c_k \Delta x$

Now, let's take a Riemann sum and write it as a definite integral.

EXAMPLE A function $f(x)$ on the interval $[-4, 4]$ has the Riemann sum $\sum_{k=1}^{n} \frac{1}{1+c_k^2} \Delta x$.

The definite integral is the value of the Riemann sum as $n \to \infty$, and is written $\int_{-4}^{4} \frac{1}{1+x^2} dx$.

TRY IT

Given below are (a) the limit of a Riemann sum and (b) a definite integral.

(a) $\lim_{n \to \infty} \sum_{k=1}^{n} \sin(c_k)\Delta x,\ 0 \le x \le \pi$

(b) $\int_0^9 \sqrt{x+16}\, dx$

Write the definite integral that corresponds to the Riemann sum in (a). +

$\int_0^{\pi} \sin x dx$

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Write the limit of the Riemann sum that corresponds to the definite integral in (b). +

$$\lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{c_k + 16}\,\Delta x,\ 0 \le x \le 9$$

---

# 3. Using Riemann Sums to Evaluate Definite Integrals

We learned that $f(x)$ is integrable on $[a, b]$ if $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to the same value for any choice of $c_1, c_2, \ldots, c_n$, each of which is in their respective subintervals.

By using the formulas for sigma notation combined with the limit definition, we can evaluate some definite integrals of functions for which we don't know the area of the corresponding region.

Here is how:

STEP BY STEP

1. Since each subinterval has equal width, we know $\Delta x = \dfrac{b-a}{n}$.
2. Select $c_k$ to be the right-hand endpoint of the interval. Then, $c_1 = a + \Delta x,\ c_2 = a + 2\Delta x,\ c_3 = a + 3\Delta x,$ ... which means $c_k = a + k\Delta x$.
3. Substitute $c_k$ into the function.
4. Evaluate the sum (using formulas), then compute the limit.

EXAMPLE Use a Riemann sum to evaluate $\int_0^4 2x\,dx$.

1. Find the width of each subinterval: $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$
2. Find the right-hand endpoints: $c_k = a + k\Delta x = 0 + k\left(\dfrac{4}{n}\right) = \dfrac{4k}{n}$
3. Evaluate the function at each $c_k$: $f(c_k) = 2\left(\dfrac{4k}{n}\right) = \dfrac{8k}{n}$
4. Then, $\int_0^4 2x\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\dfrac{8k}{n}\right)\left(\dfrac{4}{n}\right)$.

Next, simplify the sum and calculate the limit.

| | |
|---|---|
| $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\dfrac{8k}{n}\right)\left(\dfrac{4}{n}\right)$ | Calculate the limit. |

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$= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{32k}{n^2} \right)$ Simplify.

$= \lim_{n \to \infty} \frac{32}{n^2} \sum_{k=1}^{n} k$ $\frac{32}{n^2}$ is a constant factor since $k$ is the index of summation. Therefore, it can be factored out.

$= \lim_{n \to \infty} \left( \frac{32}{n^2} \cdot \frac{n(n+1)}{2} \right)$ Apply the summation formula: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$= \lim_{n \to \infty} \frac{16n^2 + 16n}{n^2}$ Simplify.

$= \lim_{n \to \infty} \left( 16 + \frac{16}{n} \right)$ Divide each term by $n^2$.

$= 16$ Evaluate the limit.

Regardless of the values of $c_k$ used (left endpoints, etc.), this holds true.

Thus, $\int_0^4 2x\,dx = 16$.

WATCH

In this video, we'll use the Riemann sum to evaluate $\int_1^3 (x^2 + 2)dx$.

## Video Transcription

[MUSIC PLAYING] Hello there and welcome back. What we're going to do in this video is determine the area under a curve, or should I say the, value of a definite integral. In this case, it is the area under the curve because our function is not negative. But we're going to find that area using a Riemann sum and taking the limit as the number of rectangles approaches infinity.

So over to the left, we have the function f of x equals x squared plus 2 on the interval 1 to 3. And you see there we have the region is sketched there. And I do have a rectangle there that is representative of all the other rectangles. We're going to use right-hand end points because it turns out that getting the values of ck is easier with right-hand end points. So the pieces we need are the width of each rectangle, which is the width of the interval divided by the number of rectangles, which is b minus a over n.

That's right here. So delta x is 3 minus 1 over n, which is 2 over n. So that's a familiar quantity that's going to be used. And the values of our c sub k's is 1, the left-hand endpoint, plus k times delta x, which simplifies to 1 plus 2k over n. So coming over to the right, the value of the definite integral is the limit as n approaches infinity of f of ck times delta x, which is right there in the first line.

So now we're going to substitute everything that we know. We have the limit as n approaches infinity of the sum of f of 1 plus 2k over n. That was the value of ck times 2 over n. And just to note that this is all under the summation just to avoid confusion there. So then from here, we substitute into the function, we simplify, and then we use summation formulas where possible to simplify the sum and to, hopefully, be

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able to take the limit as n goes to infinity.

So here we go. So since f of x is x squared plus 2, that means we have 1 plus 2k over n squared plus 2. And then we have the times 2 over n. So then we just simplify squaring 1 plus 2k over n. We have these three terms, and then the plus 2 still all multiplied by 2 over n.

In the next step, all we're doing is combining the like terms. The 1 plus 2 becomes the 3. So that's the only major highlight happening there. And still that whole sum multiplied by 2 over n. And then now distributing the 2 over n through.

So 3 times 2 over n becomes 6 over n, and so on and so on. So now you notice we have these sums with k squared. Now k is the index of summation. So we're going to be able to utilize these summation formulas by another property of sums, which says I can now have three separate sums since I have a sum of a sum. A lot of sums going on there.

So it's the sum k going from 1 to infinity of 6 over n. The 8 over n squared is a constant, so it can come outside the sum, like so. And the 8 over n to the third can be factored outside of the sum as well. So from here, this is where the summation formulas kick in. So this is equivalent to the limit as n approaches infinity.

Now the summation of 6 over n is the summation of a constant. So that means that that's equal to 6 over n times n terms plus 8 over n squared. The summation of k starting from 1 and going to n is n times n plus 1 over 2. And the summation of k squared is k times k plus 1 times-- I'm sorry, not k, n. Fix those there.

So we have n. n plus 1, 2n plus 1 over 6. So now we should be able to take the limit as n approaches infinity of that. Now let's clean some things up here. So we have the limit as n approaches infinity 6 over n times n is 6.

In the second term we have, so if I multiply the numerator through, that's going to be 8n squared plus 8n, and that's all over 2n squared. And then here in the last term, we have a denominator of 6n to the third. And if we were to multiply the numerator through, well, let's see. This part right here is 2n to the third plus 3n squared plus n. So if I multiply that by 8 we have 16n to the third plus 24n squared plus 8n.

And now what I'm going to do in this last step before evaluating the limit is I have the limit as n approaches infinity. We have 6 plus, now if I divide through by n squared, that's 4 plus 4 over n. And if we divide through by 6n to the third, 16 over 6 is 8 over 3. 24n squared over 6n to the third is 4 over n. And 8n over 6n to the third is 4 over 3n squared.

So then as n goes to infinity, anything that has an n in the denominator is going to cancel out and go to 0. So this term goes to 0. This term goes to 0. And this term goes to 0. So we're left with 6 plus 4 plus 8 over 3, which becomes 6 plus 4 is 10.

10 is 30 over 3. So that means that the value of the definite integral is 38 over 3. And we know that a function is integrable as long as any limit that we do of this form, if we use left-hand endpoints, midpoints, random points in each of the subintervals, if we use those, that come out to be the same sum. And it would.

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If we were to use the left-hand points, we would get 38/3. If we were to use midpoints, we would also get 38/3. So the value of the definite integral is 38/3.

[MUSIC PLAYING]

TRY IT

Consider the integral $\int_2^6 (10-x)dx$.

Use the Riemann sum to evaluate this integral. +

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(10-\left(2+\frac{4k}{n}\right)\right)\left(\frac{4}{n}\right)\right]=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{32}{n}-\frac{16k}{n^2}\right)=24$$

THINK ABOUT IT

Fortunately, this is not the only method to evaluate definite integrals. These samples were chosen based on known summation formulas.

For example, consider the definite integral $\int_0^{\pi} \sin x dx$. The corresponding limit of a Riemann sum is $\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sin\left(\frac{k\pi}{n}\right)\cdot\frac{\pi}{n}\right]$, which has no known summation formula. We will learn how to find the value of the definite integral without summations in a future challenge.

# 4. Using Area to Evaluate Riemann Sums and Definite Integrals

→ EXAMPLE Evaluate the definite integral: $\int_0^4 2x dx$

The figure shows the region bounded by the graph of $f(x)=2x$ and the x-axis on the interval $[0, 4]$.

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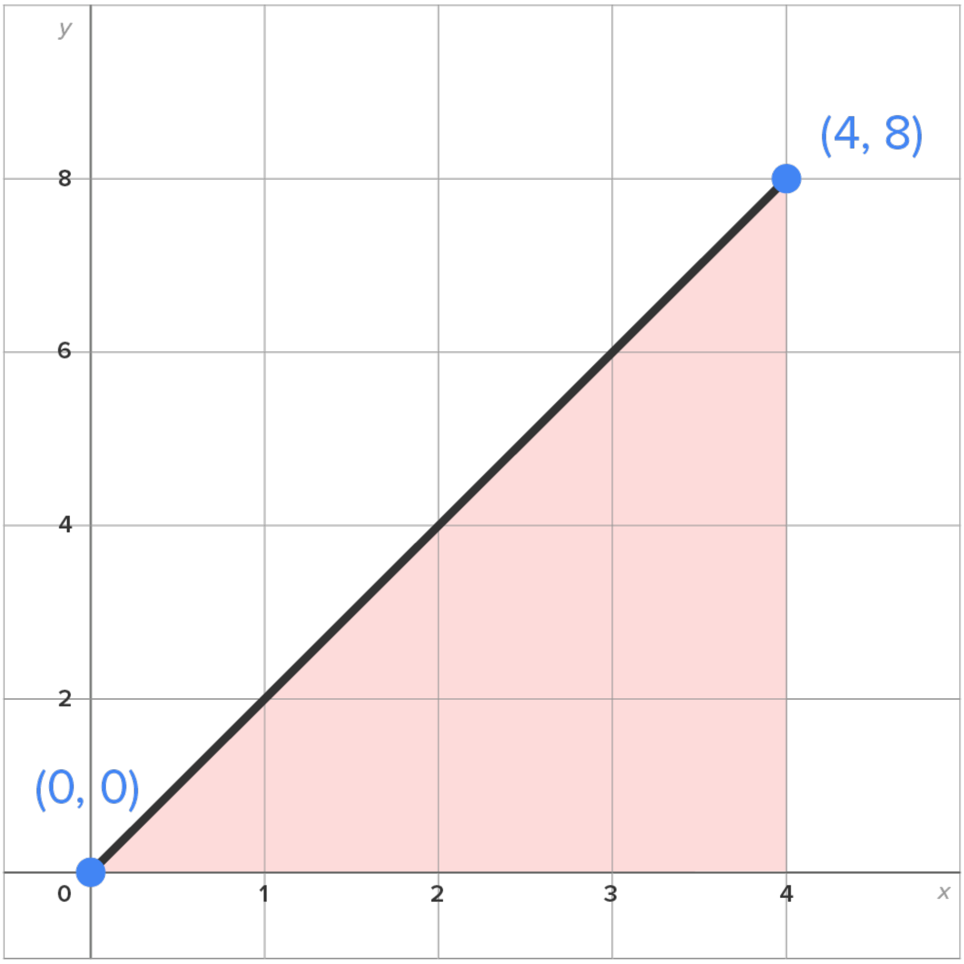


The region is the shape of a triangle with base 4 and height 8, which has area $A=\frac{1}{2}(4)(8)=16$ square units.

Then, $\int_0^4 2x dx = 16$.

Let's look at an example of a continuous piecewise function.

→ EXAMPLE Consider the graph of $f(x)$ shown in the figure. Use it to evaluate $\int_0^6 f(x)dx$.

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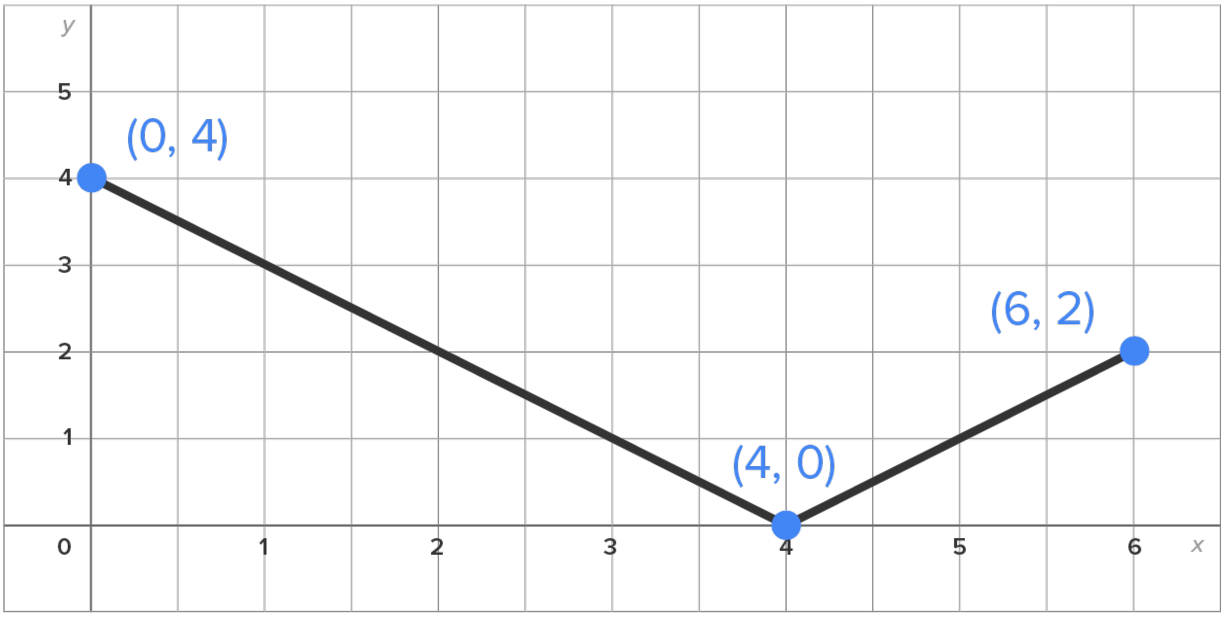
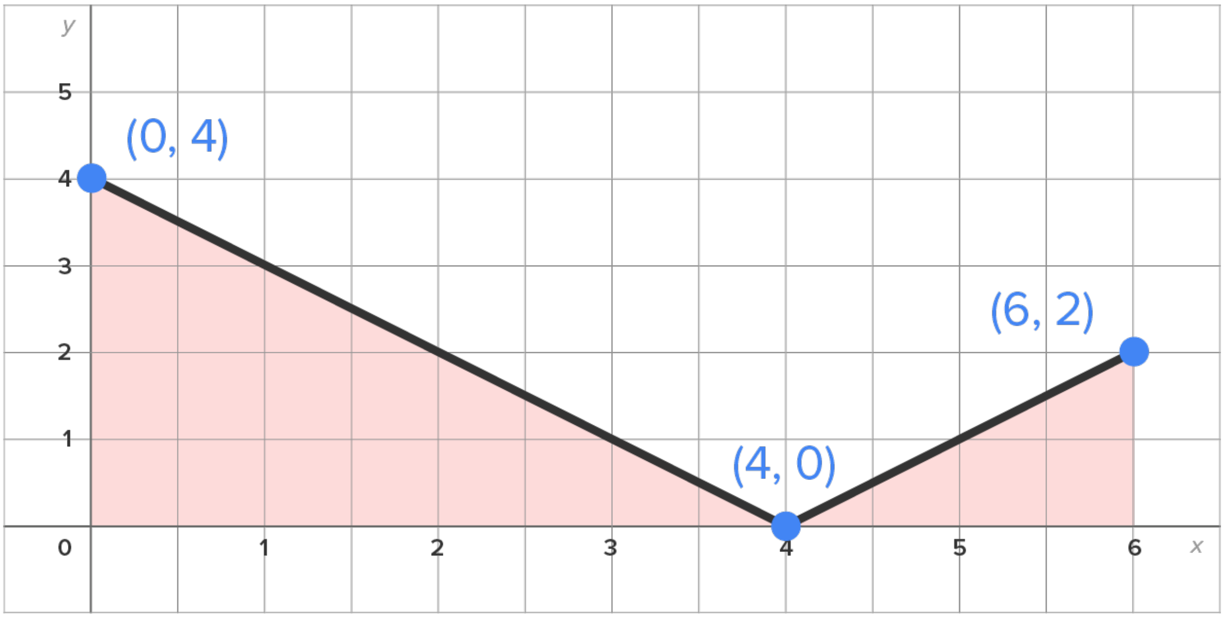



The region is shown in the figure below and is composed of two triangles.




The triangle on $[0, 4]$ has area $\frac{1}{2}(4)(4) = 8 \text{ units}^2$ and the triangle on $[4, 6]$ has area $\frac{1}{2}(2)(2) = 2 \text{ units}^2$.

The total area is $10 \text{ units}^2$, which means that $\int_0^6 f(x)dx = 10$.

TRY IT

Consider the graph of $f(x)$ shown in the figure that can be used to evaluate $\int_{-2}^{2} f(x)dx$.

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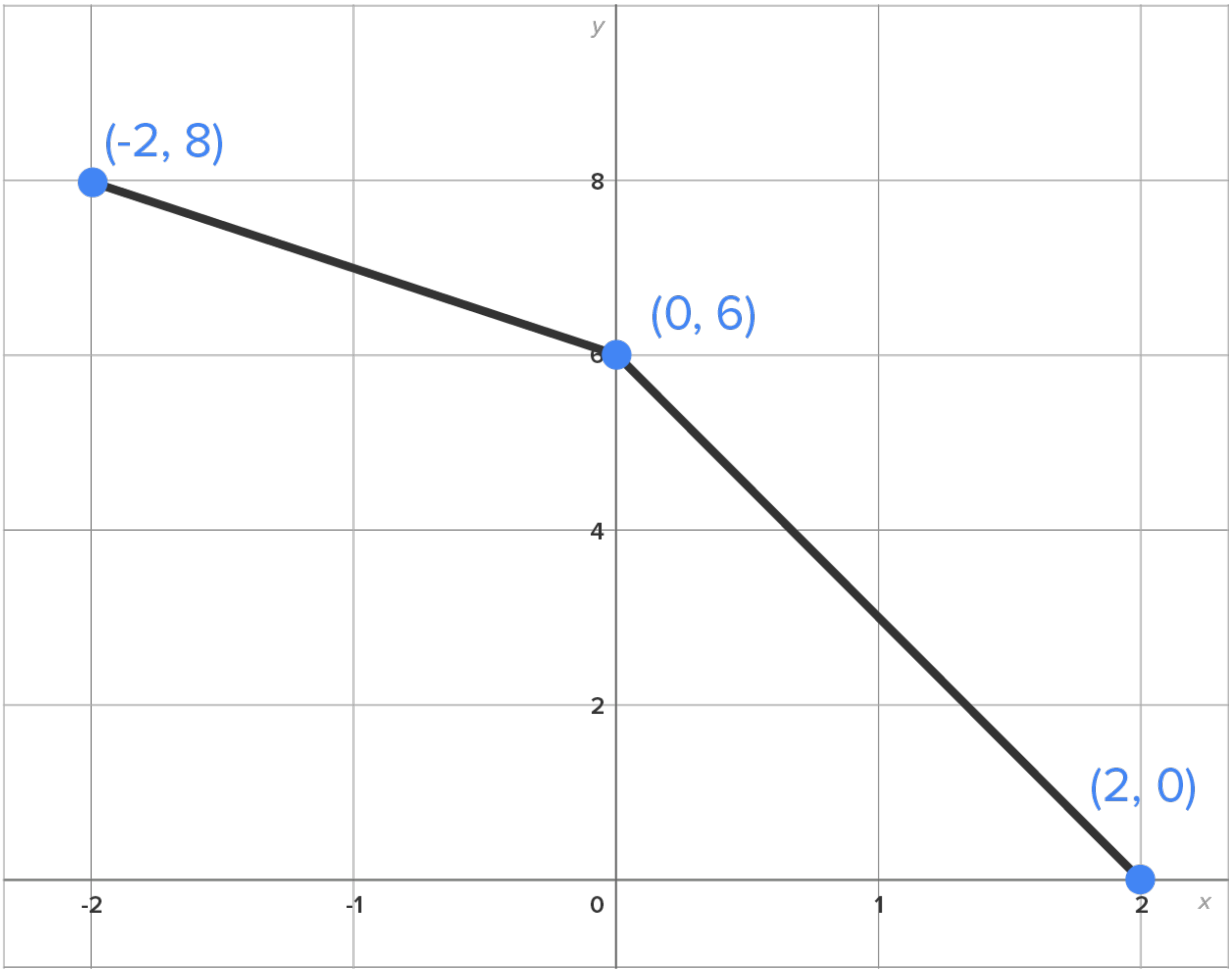


Evaluate this integral. +

$$\int_{-2}^{2} f(x)dx = 20$$

As it turns out, $f(x)$ doesn't have to be continuous in order to be integrable. Here is an example that illustrates this.

WATCH

In this video, we'll use a graph of some function $y = f(x)$ shown to evaluate $\int_{0}^{10} f(x)dx$.

### Video Transcription

[MUSIC PLAYING] Hi there. And welcome back. What we're going to do in this video is evaluate the definite integral over some discontinuous function, f of x, as x varies from 0 to 10. So to understand how we do this, you're probably going to notice very quickly that this is really not that different than things we've done in the past here.

If I go out to, say, x equals 2 first, I know that the area of this region, being above the x-axis, I know that the definite integral is equal to the area. So this piece right here is the integral from 0 to 2 of f of x, the accumulated area from 0 to 2.

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If I keep increasing x, and if I go out to 3, I basically stop there, and that gives me the area of that entire rectangle right there. Now, if I move just a notch pass 3-- in fact, I'm just going to go out to 4-- but as soon as I cross over 3, here is the piece that we pick up.

We pick up the area underneath the line. It looks like the line y equals 2, starting at x equals 3 and going out to x equals 8. So if you're thinking about this correctly at this point, all we have to do is find the areas under each of those lines. And they're all above the x-axis. So again, definite integral translates to area of the region.

So if we go from 3 to 8, that's the area of this rectangle right here. And then, as soon as we cross over x equals 8, we're now talking about that region right here. So those areas represent the areas we pick up as we go from 0 to 10, increasing the upper limit from up to 10.

So the definite integral over the entire interval 0 to 10 is just some of the areas of those regions. So this region has area 3 times 6 because it goes up to a height of 6, and that's 18. This rectangle has a base of 5 units and a height of 2 to make 10. And this rectangle has a base of 2, but a height of 5 for another 10. So that means the integral from 0 to 10 of f of x dx is 18 plus 10 plus 10, which is 38.

And we don't have any units to carry here, but I'm just going to call this units squared, square units. And there is evaluating a definite integral over a discontinuous function. All we do is accumulate area.

[MUSIC PLAYING]

## SUMMARY

In this lesson, you learned **the definition of the definite integral**, understanding that for a non-negative function $f(x)$, the value of the **Riemann sum approaches the definite integral** as $n \to \infty$. You also learned that by using the formulas for sigma notation combined with the limit definition, you can **evaluate definite integrals** of functions for which you don’t know the area of the corresponding region, by **using Riemann sums** to visualize how the area between $f(x)$ and the x-axis on $[a, b]$ is obtained. Finally, through a series of examples **using area to evaluate Riemann sums and definite integrals**, you have seen that $f(x)$ does not have to be continuous to be integrable.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Integrable**

If the value of $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to *A* regardless of the values of $c_k$ used in each subinterval, then we say that $f(x)$ is integrable on the interval $[a, b]$.

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# Definite Integrals of Negative Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will connect the ideas from Riemann sums, integrals, and regions that are below the x-axis. Specifically, this lesson will cover:

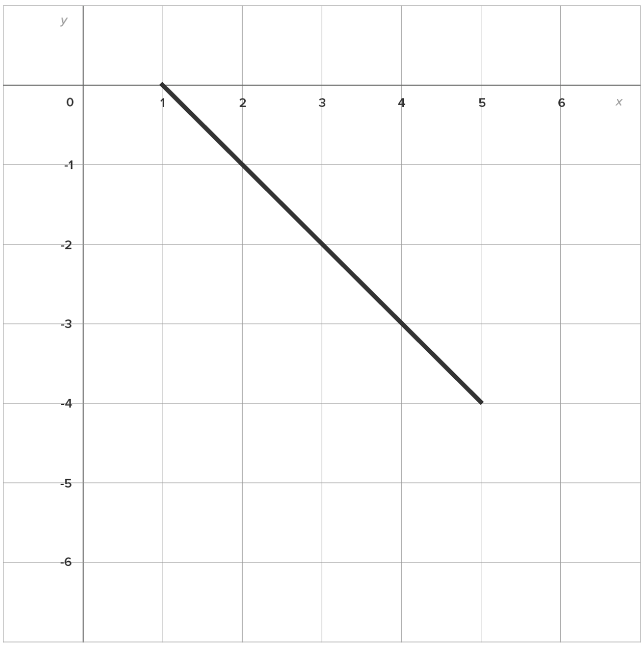
1. Riemann Sums of Functions That Are Below the x-Axis
2. Evaluating Definite Integrals When $f(x) \le 0$ on $[a, b]$
3. Evaluating Definite Integrals When $f(x)$ Is Both Negative and Positive on $[a, b]$

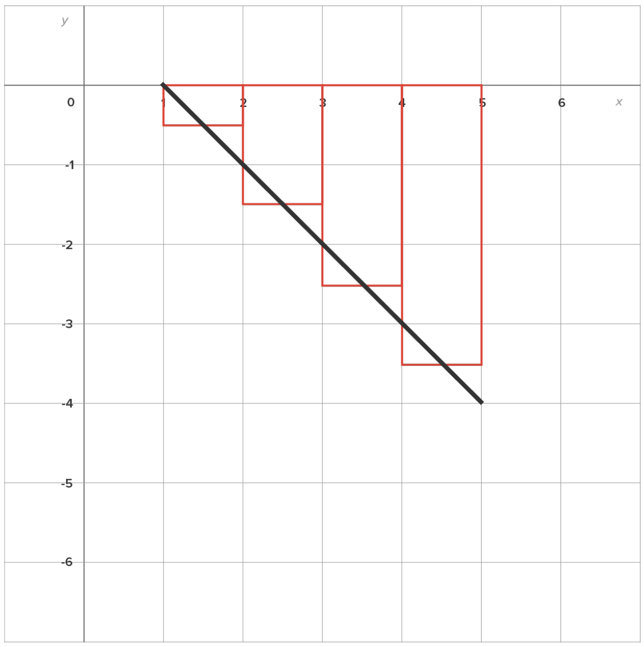
# 1. Riemann Sums of Functions That Are Below the x-Axis

Up until now, we only integrated functions that were above the x-axis on $[a, b]$. We'll use this example to see what happens when that is not the case.

Consider the function $f(x) = 1 - x$ on the interval $[1, 5]$.

The graph on the left is $f(x)$ on the interval $[1, 5]$, and the graph on the right shows the rectangles that could be used in a Riemann sum. Remember, the rectangles have a base on the x-axis and extend out to the graph of $f(x)$.




Now consider a Riemann sum, $\sum_{k=1}^{n} f(c_k)\Delta x$.

- When $f(c_k)$ is positive, we know the quantity $f(c_k) \cdot \Delta x$ is the area of one rectangle.
- As a result, when $f(c_k)$ is negative, the quantity $f(c_k) \cdot \Delta x$ is the negative of the area of that one rectangle.

Now picture adding these quantities, all of which are negative. The Riemann sum would be an estimate for the negative of the area.

Now, consider the definite integral of this function on the interval $[1, 5]$, written $\int_1^5 (1-x)dx$.

We know the value of this integral is the limit of the Riemann sums as the number of rectangles gets larger and larger $(n \to \infty)$.

Note that the area of the region between $f(x)$ and the x-axis is $\frac{1}{2}(4)(4) = 8 \text{ units}^2$.

Then, $\int_1^5 (1-x)dx = -8$, the negative of the area of the region.

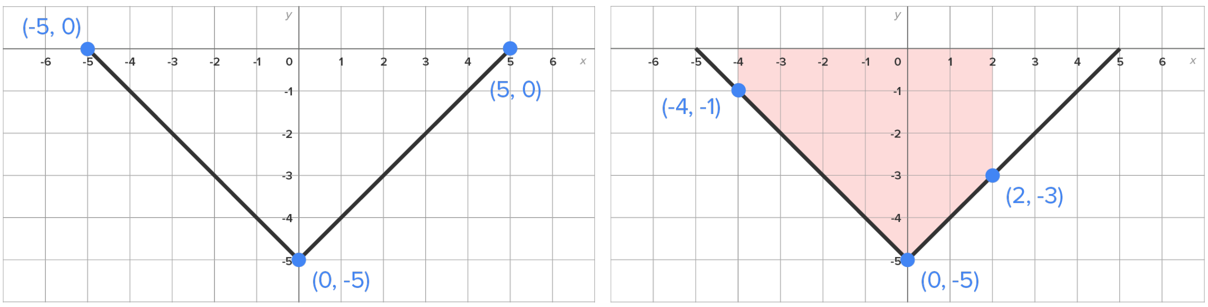
BIG IDEA

If the graph of $f(x)$ is below the x-axis on $[a, b]$, then $\int_a^b f(x)dx$ is the negative of the area of the region between $f(x)$ and the x-axis on $[a, b]$.

## 2. Evaluating Definite Integrals of $f(x)$ when $f(x) \leq 0$ on $[a, b]$

EXAMPLE Evaluate $\int_{-4}^{2} (|x| - 5)dx$.

The graph of $f(x)$ is shown on the left and the region is shown on the right.




Note that the graph of $f(x)$ is below the x-axis on the interval $[-4, 2]$. The region itself is not a standard shape, so let's split the region at $x = 0$.

On $[-4, 0]$, the region is a trapezoid with parallel (vertical) bases 1 and 5, and (horizontal) height 4. The

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area is $\frac{1}{2}(4)(1+5) = 12 \text{ units}^2$.

On $[0, 2]$, the region is a trapezoid with parallel (vertical) bases 5 and 3, and (horizontal) height 2. The area is $\frac{1}{2}(2)(5+3) = 8 \text{ units}^2$.

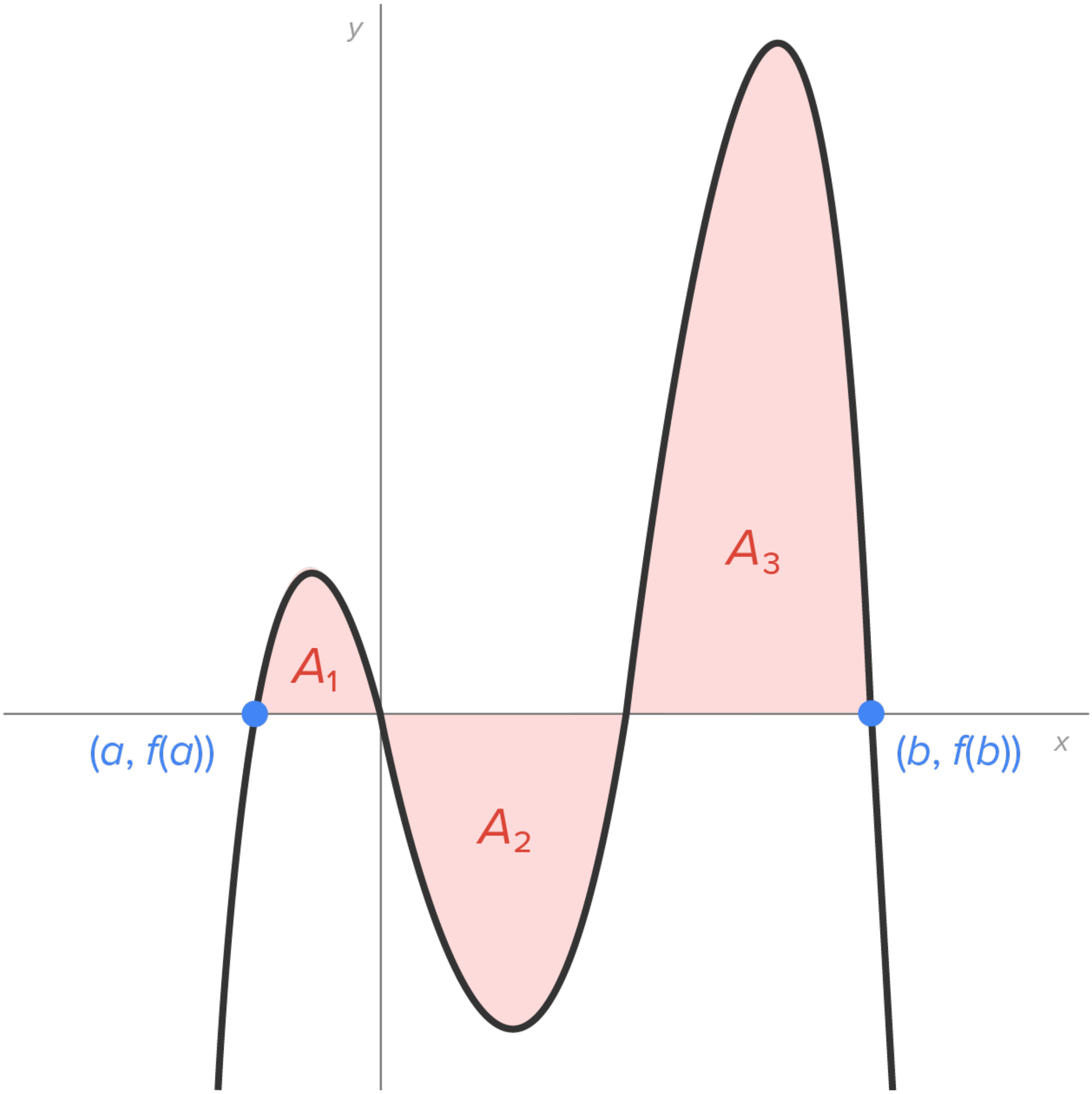
Then, the total area is $20 \text{ units}^2$.

Since the region is completely below the x-axis on $[-4, 2]$, $\int_{-4}^{2}(|x|-5)dx = -20$.

---

# 3. Evaluating Definite Integrals When $f(x)$ Is Both Negative and Positive on $[a, b]$

Suppose we wish to evaluate $\int_{a}^{b} f(x)dx$ for the function whose graph is shown in the figure.




Notice how this region is broken into 3 smaller regions with areas $A_1$, $A_2$, and $A_3$.

Now, consider the definite integral on $[a, b]$.

- For the region with area $A_1$, the definite integral is equal to $A_1$ since the region is above the x-axis.
- For the region with area $A_2$, the definite integral is equal to $-A_2$ since the region is below the x-axis.
- For the region with area $A_3$, the definite integral is equal to $A_3$ since the region is above the x-axis.

Thus, the definite integral over $[a, b]$ is equal to the sum of the three definite integrals, or $A_1 - A_2 + A_3$. In general, we would add any area above the x-axis and subtract any area below the x-axis.

BIG IDEA

If $f(x)$ takes on both positive and negative values on $[a, b]$, then $\int_a^b f(x)dx =$ (sum of all areas above the x-axis) - sum of all areas below the x-axis.

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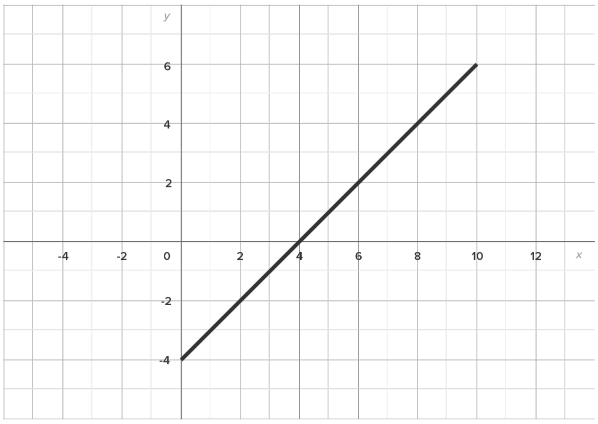
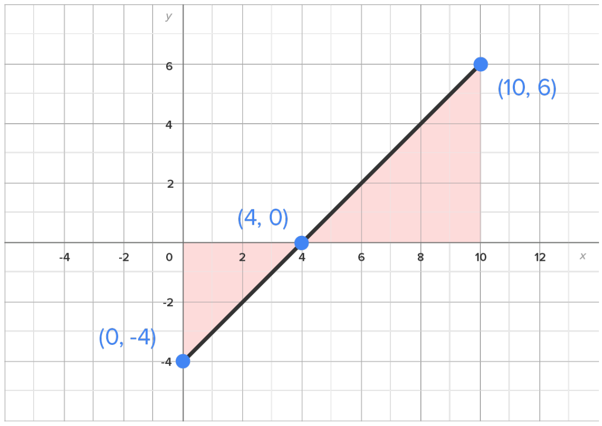
Let's look at an example.

↪ EXAMPLE Evaluate $\int_0^{10} (x-4)dx$.

Consider the graph of $f(x)=x-4$ on the interval $[0, 10]$, as shown in the figure on the left. The figure on the right shows the graph with the relevant regions.




The triangle between $x=0$ and $x=4$ has area $\frac{1}{2}(4)(4)=8$, and is below the x-axis.

The triangle between $x=4$ and $x=10$ has area $\frac{1}{2}(6)(6)=18$, and is above the x-axis.

Then, $\int_0^{10} (x-4)dx = -8+18=10$.

WATCH

Given the graph of $y=f(x)$, we'll find $\int_0^6 f(x)dx$.

## Video Transcription

[MUSIC PLAYING] Hi there, and welcome back. We're going to do, in this video, is, given the graph of a function y equals f of x. We're going to evaluate the definite integral of this function from x equals 0 to x equals 6.

And the key thing to remember is that the value of the definite integral is the accumulated area as we move from x equals 0 to x equals 6. And since part of the graph is below the x-axis, we know that contributes to the area differently.

So the first thing we're going to do is find the area of each region. So this region right here is a trapezoid. So I'm just going to very lightly shade this in here. There's the area of the region.

So remember that the area of a trapezoid is 1/2 times the height times the sum of the parallel bases that we call b1 and b2. The parallel bases, here, being the x-axis and this horizontal line here.

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So the height is 3. And the upper base is 1. And the lower base is 2. So this looks like 9/2. So the area of that first region is 9/2.

Now, the region under the x-axis is a triangle. This might look tricky at first, but, remember, it's just 1/2 base height. So the height of the triangle is 3. And the base of the triangle is 4. So the area of this region is 1/2 base is 4, height is 3, and that is 6.

So how does that all figure in? Well, the definite integral basically tells us that it's the sum of the positive areas. And then, subtracting the sum of the negative areas. And by negative area, I mean any area that is below the x-axis.

So the integral from 0 to 6 of f of x is going to be, well, the positive area is 9/2 the area above the x-axis. And the area below the x-axis is 6. So we're going to subtract 6. And that result is-- well, let's see. 9/2 minus 12/2 is negative 3/2.

So because the result is negative, what that really means is that there's more area below the x-axis than there is above the x-axis. So as a way of interpreting this as net area, the net is negative, which means there's more below than there is above.

[MUSIC PLAYING]

BIG IDEA

In this context, the definite integral can be thought of as a "net area." If $\int_a^b f(x)dx > 0$, there is more area above the x-axis than below the x-axis on $[a, b]$.

If $\int_a^b f(x)dx < 0$, there is more area below the x-axis than above the x-axis on $[a, b]$.

If $\int_a^b f(x)dx = 0$, there is as much area above the x-axis as there is below the x-axis on $[a, b]$.

TRY IT

Consider the graph of $f(x)$ as shown in the figure below that can be used to evaluate $\int_0^{12} f(x)dx$.

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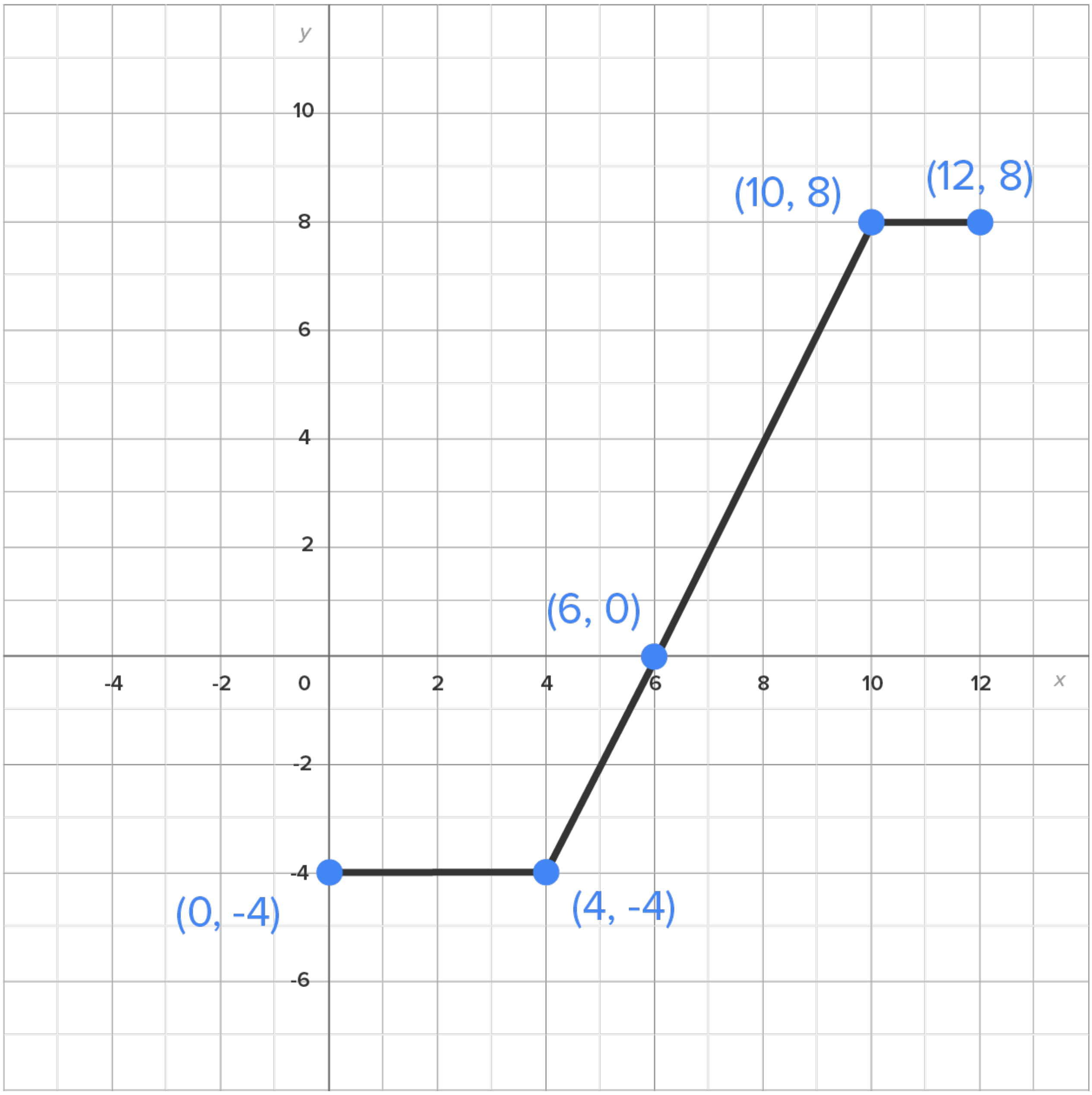


Evaluate this integral. +

12

## SUMMARY

In this lesson, you learned about the **Riemann sums of functions that are below the x-axis**, understanding how to evaluate $\int_a^b f(x)dx$ when the graph of f(x) is below the x-axis. You applied this knowledge in **evaluating definite integrals when $f(x) \leq 0$ on** $[a, b]$. Lastly, you learned that when **evaluating definite integrals when $f(x)$ is both negative and positive on** $[a, b]$, you can interpret the value of the definite integral as "net area," considering regions that are above and below the x-axis. This will be very useful when investigating applications in the next tutorial.

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# Units for the Definite Integral

*by Sophia*

WHAT'S COVERED

In this lesson, you will investigate the various interpretations of the definite integral. Specifically, this lesson will cover:

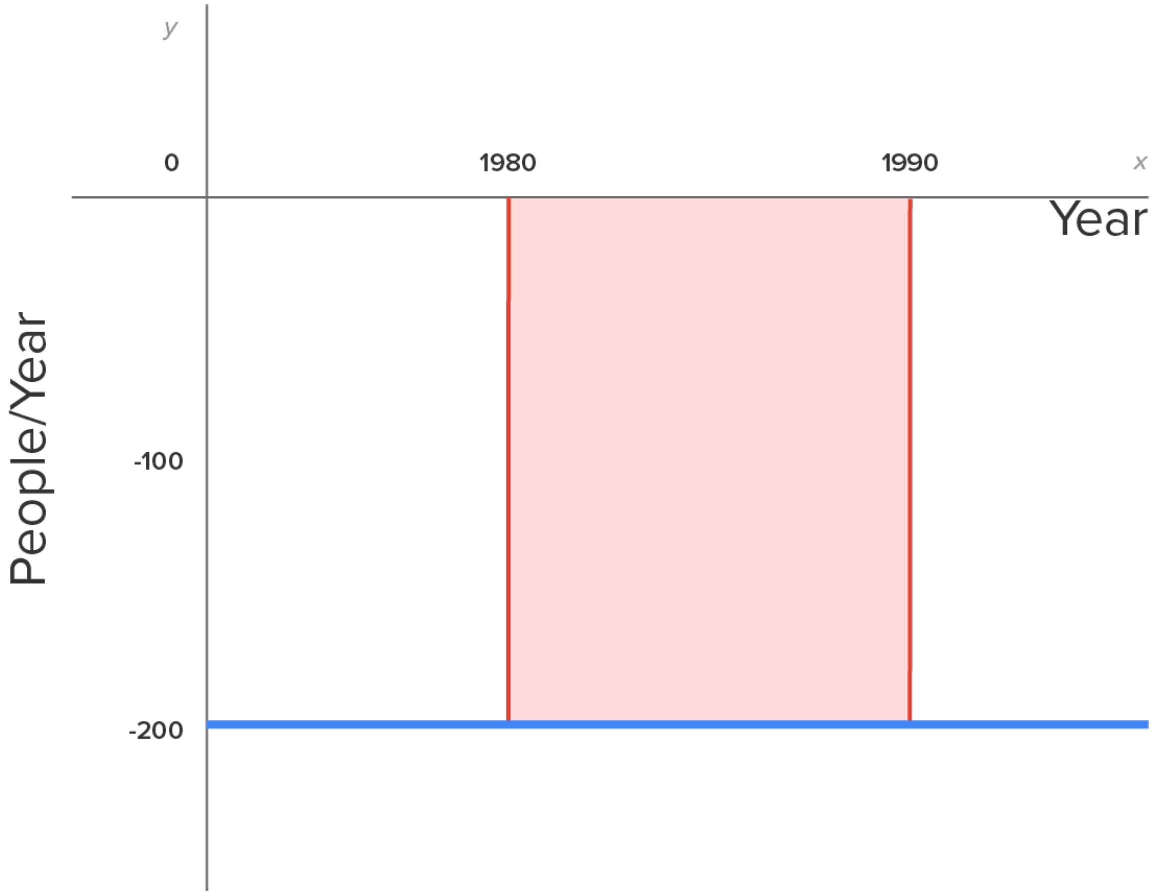
1. The Definite Integral As Net Change
2. The Definite Integral As It Relates to Everyday Situations
   a. Velocity and Distance
   b. Other Situations

# 1. The Definite Integral As Net Change

Let's say that $f(x) =$ the rate at which the population of a small town is changing in year *t*.

- If $f(x) > 0$, this means that the population is increasing.
- If $f(x) < 0$, this means that the population is decreasing.

Suppose that $f(x)$ has this graph between years $x = 1980$ and $x = 1990$.



This means that at any point between 1980 and 1990, the population is decreasing at a rate of 200 people per year.

Note that the area of the region is $10(200) = 2000$, and the region is below the x-axis.

This means that the value of the definite integral is $\int_{1980}^{1990} f(x)dx = \int_{1980}^{1990} -200dx = -2000$.

So, what are the units? Since the horizontal scale is measured in years and the vertical scale is measured in people/year, the area is the product of the units, which is the number of people. Thus, the area can be interpreted as 2,000 people. But how does this fit in, especially since this region is below the x-axis?

What does this result mean? The value of the definite integral means that the town lost 2,000 people between 1980 and 1990. So, if the population of the town in 1980 was 14,250, the population in 1990 was $14{,}250 - 2{,}000 = 12{,}250$ people. Thus, the definite integral $\int_{1980}^{1990} f(x)dx$ is interpreted as the change in population over the interval $[1980, 1990]$.

BIG IDEA

The meaning of the definite integral $\int_a^b f(x)dx$ can be interpreted as the net change in $f(x)$ over the interval $[a, b]$.

The units of the definite integral are (units of $x$)(units of $f(x)$).

# 2. The Definite Integral As It Relates to Everyday Situations

## 2a. Velocity and Distance

Recall that velocity is the rate of change of distance.

- A positive velocity means that the distance from a specific point is increasing (moving further away from the point).
- A negative velocity means that the distance from a specific point is decreasing (getting closer to the point).
- A zero velocity (over an interval) means the object is at rest.

Let's say an object in motion has velocity $v(t)$ on the interval $[a, b]$, where velocity is measured in miles per hour and $t$ is measured in hours.
Then, the units of the definite integral are (miles/hour)(hours) = miles, and the value of the definite integral means the net distance traveled on $[a, b]$.

EXAMPLE Consider the graph below, which represents the velocity $v(t)$ of a car moving east of its starting point at 12 PM $(t = 0)$. A positive velocity indicates eastward motion, while a negative velocity indicates westward motion.

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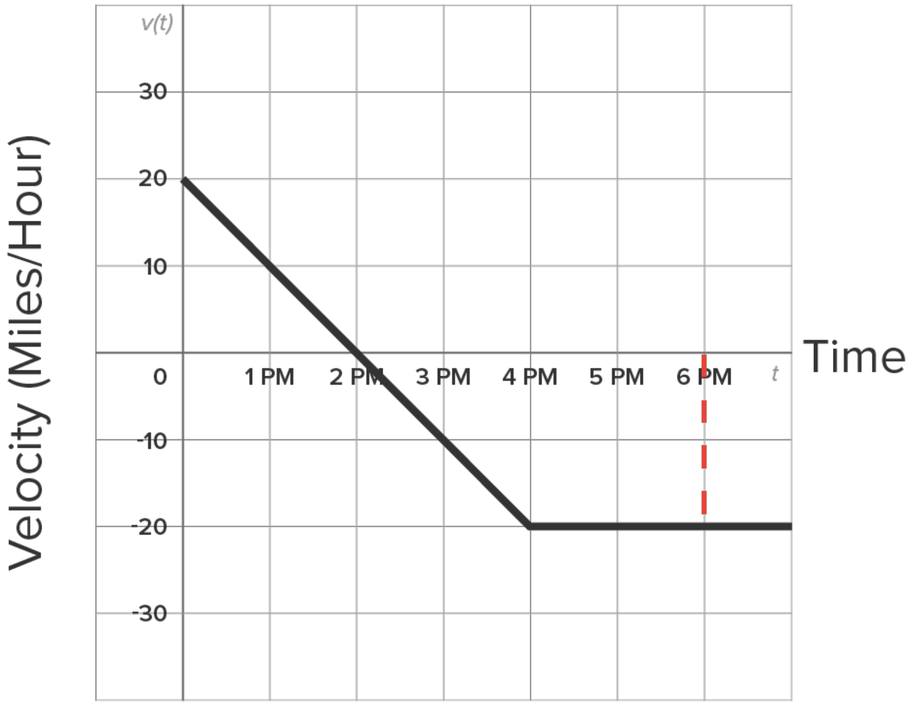
1. What is the distance traveled from 12 PM to 6 PM?
2. What does $\int_0^6 v(t)dt$ mean in this situation?




Note that the vertical scale is measured in miles per hour and the horizontal scale is measured in hours. Therefore, the area of any region is measured in $\frac{\text{miles}}{\text{hour}} \cdot \text{hours} = \text{miles}$, which is distance. To find the total area, add the area of each region.

- On $[0, 2]$, the region is a triangle, which means its area is $\frac{1}{2}(2)(20) = 20$ miles to the east.
- On $[2, 4]$, the region is a triangle, but under the x-axis. Its area is $\frac{1}{2}(2)(20) = 20$ miles to the west.
- On $[4, 6]$, the region is a rectangle, but under the x-axis. Its area is $(2)(20) = 40$ miles to the west.

1. The total area is the distance traveled (in any direction), $20 + 20 + 40 = 80$ miles.

   The change in position is the value of the definite integral, which considers the location of the region (above or below the t-axis). If the region is above, use the area. If the region is below the t-axis, use the negative of the area.

2. $\int_0^6 v(t)dt = 20 - 20 - 40 = -40$

   This result means that the car is 40 miles west of its starting point after 6 hours (since east is positive, west is negative).

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WATCH

In this video, we'll use another graph to find the distance traveled and other interpretations based on the results.

## Video Transcription

[MUSIC PLAYING] Hi there, good to see you again. What we're going to do in this video is, given a velocity graph over the interval 0 to 10, we're going to try to determine the distance traveled on 0 to 10. And we're going to look at two different things. We're going to look at not only what our position is relative to the starting point, but we're also going to look at the total distance traveled and how those two things differ from looking at this graph.

So remember that we know that the area between the axis and the curve is distance. Because area, as you know, is length times width. But length, in this case, is the number of seconds. That's a t interval. And the v of t interval, the height, is measured in feet per second. So when you multiply feet per second times seconds, you get feet, which means it's a distance traveled.

So since we have three different regions, we're going to look at three different intervals here. So first, looking at the interval 0 to 1, I notice that region is above the x-axis-- above the t-axis, in this case. So that means that the area represents the distance traveled in the positive direction. So that means this area being 1/2 times 1 times 5 is 2.5 and measured in feet.

Now, looking at this next region, it is being low the t-axis, which means the value of the definite integral will be negative. But we still calculate it just as area, because we know that the definite integral is related to area. It's the negative of the area if the region is below the axis, and it's equal to the area when the region is above the axis.

So this is a trapezoidal figure. And we know that the area of a trapezoid is 1/2 times the height, times the sum of the parallel bases. Now, our parallel bases are 2 units and 6 units. And our height, being the side perpendicular to that, is 5. So that in total is 20.

And that does represent a distance of 20 feet. It's just that it's moving in a certain direction. So if positive meant we were moving east, a negative would mean that we're moving west. But it is still distance. We can still interpret it that way.

Now, for our final region, we have a triangular region from 7 to 10. And that means that since it's above the axis, the actual distance traveled is equal to the area. But that also means that the definite integral is positive as well. So that area is 1/2. The base is 3 units. The height is 4 units. And that gives us 6. And we know that that is feet.

OK, so the two questions we have-- the total distance. Well, the total distance, no matter which way you go, is just the sum of the areas positive. And we just make them all positive. So that means this is 2.5 plus 20 plus 6, which is 28.5 feet.

So to understand this, think about if you wear a fitness tracker. No matter what direction you're walking in, it still records steps. It doesn't take away steps if you start walking the other way. It's going to keep

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continuing to count in a forward, in accumulating number of steps.

Now, as far as how far are we from our starting point, that's where the positives and negatives have an effect. So from 0 to 1, we've walked 2.5 feet away from the beginning point. But then on the interval 1 to 7, we went 20 feet in the other direction. And then from the interval 7 to 10, we have gone 6 feet back in the positive direction.

So I'm going to call that a change in displacement. And that looks like 2.5 and 6 is 8.5, minus 20 is negative 11.5 feet, which means 11.5 feet in the negative direction. So again, if we said positive direction was east, this would mean that we ended up 11.5 feet west of the original starting point.

And there is how we can use velocity to interpret distance traveled and change in displacement.

[MUSIC PLAYING]

## 2b. Other Situations

TRY IT

Shown below is the graph of the flow rate $f(t)$ of a pipe, in gallons per minute. Here, $t =$ the number of minutes.

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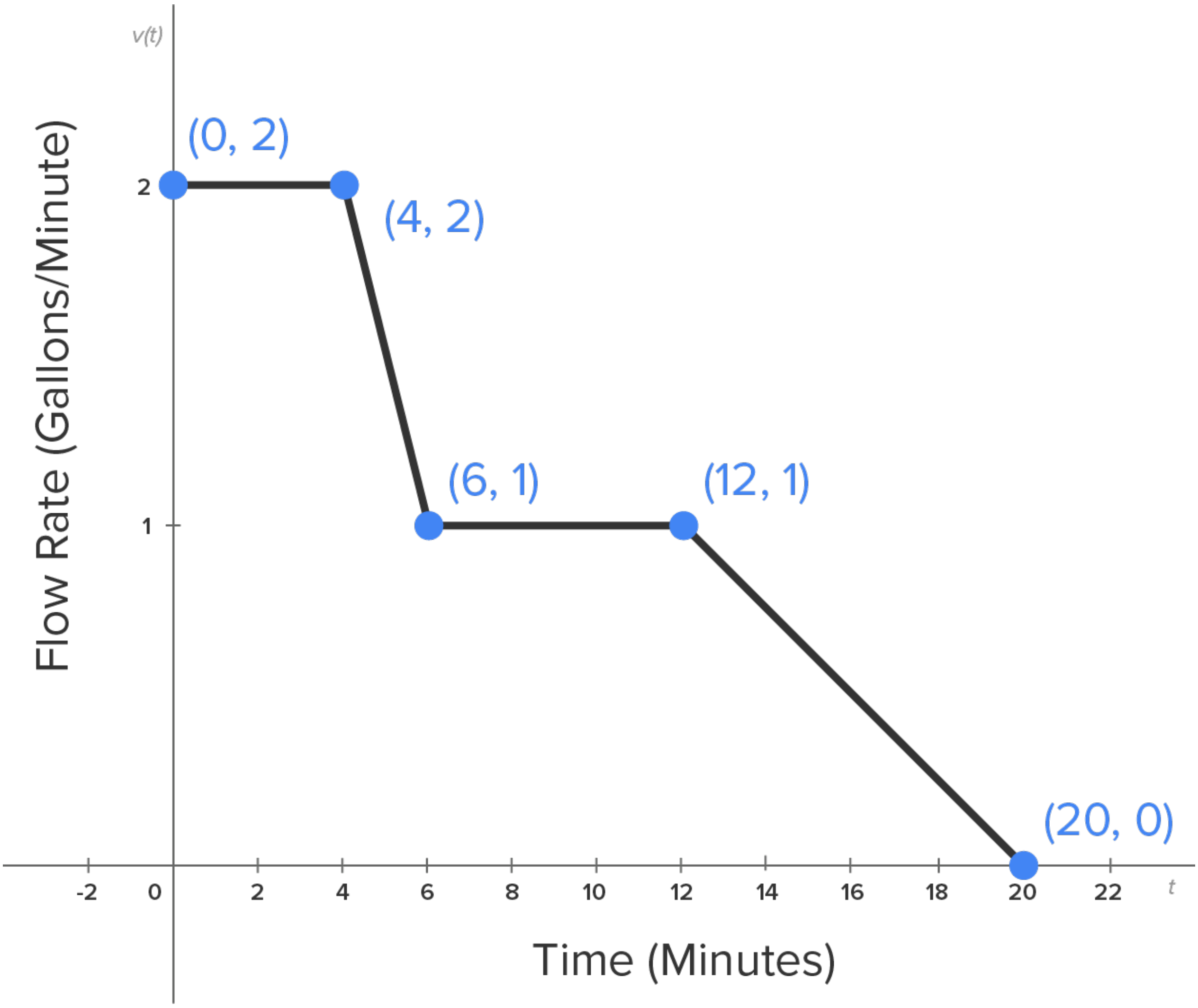


Using this information, we can find the total number of gallons of water that was flowing through this pipe in 20 minutes. To get this, we need to find $\int_0^{20} f(t)dt$.

Calculate the value of the definite integral. +

21 gallons

## SUMMARY

In this lesson, you learned that **the definite integral is interpreted as a net change** over an interval. The units of $\int_a^b f(x)dx$ are (units of *x*)(units of $f(x)$). Next, you explored some examples using **the definite integral as it relates to everyday situations**, involving **velocity and distance** and **other situations**. For example, when given a **velocity** function $v(t)$, the definite integral of $v(t)$ is the net distance traveled on the interval.

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# Properties of the Definite Integral

*by Sophia*

WHAT'S COVERED

In this lesson, you will learn some useful properties of definite integrals. They are mostly quite intuitive and have been hinted at in previous tutorials, but this is where we will establish these properties officially. Specifically, this lesson will cover:

1. Properties of the Definite Integral
2. Properties of Definite Integrals of Combinations of Functions
3. Comparison Properties
   a. Comparing Two Functions
   b. Bounds on the Value of a Definite Integral

# 1. Properties of the Definite Integral

FORMULA

| Property | Integral Formula | In Words |
|---|---|---|
| 1 | **Definite Integral When Lower and Upper Bounds Are Equal** $\int_a^a f(x)dx = 0$ | When the limits of integration are equal, the value of the definite integral is 0. |
| 2 | **Definite Integral When Upper and Lower Bounds Are Interchanged** $\int_b^a f(x)dx = -\int_a^b f(x)dx$ | When the order of the limits of integration are interchanged, the values of the definite integrals are opposites. |
| 3 | **Definite Integral of a Constant Function** $\int_a^b kdx = k(b-a)$ | The definite integral of a constant is equal to the constant multiplied by the width of the interval $(b-a)$. |
| 4 | **Definite Integral of a Constant Multiple of a Function** | The constant $k$ can be moved outside, and the definite integral of $f(x)$ is multiplied by $k$. |

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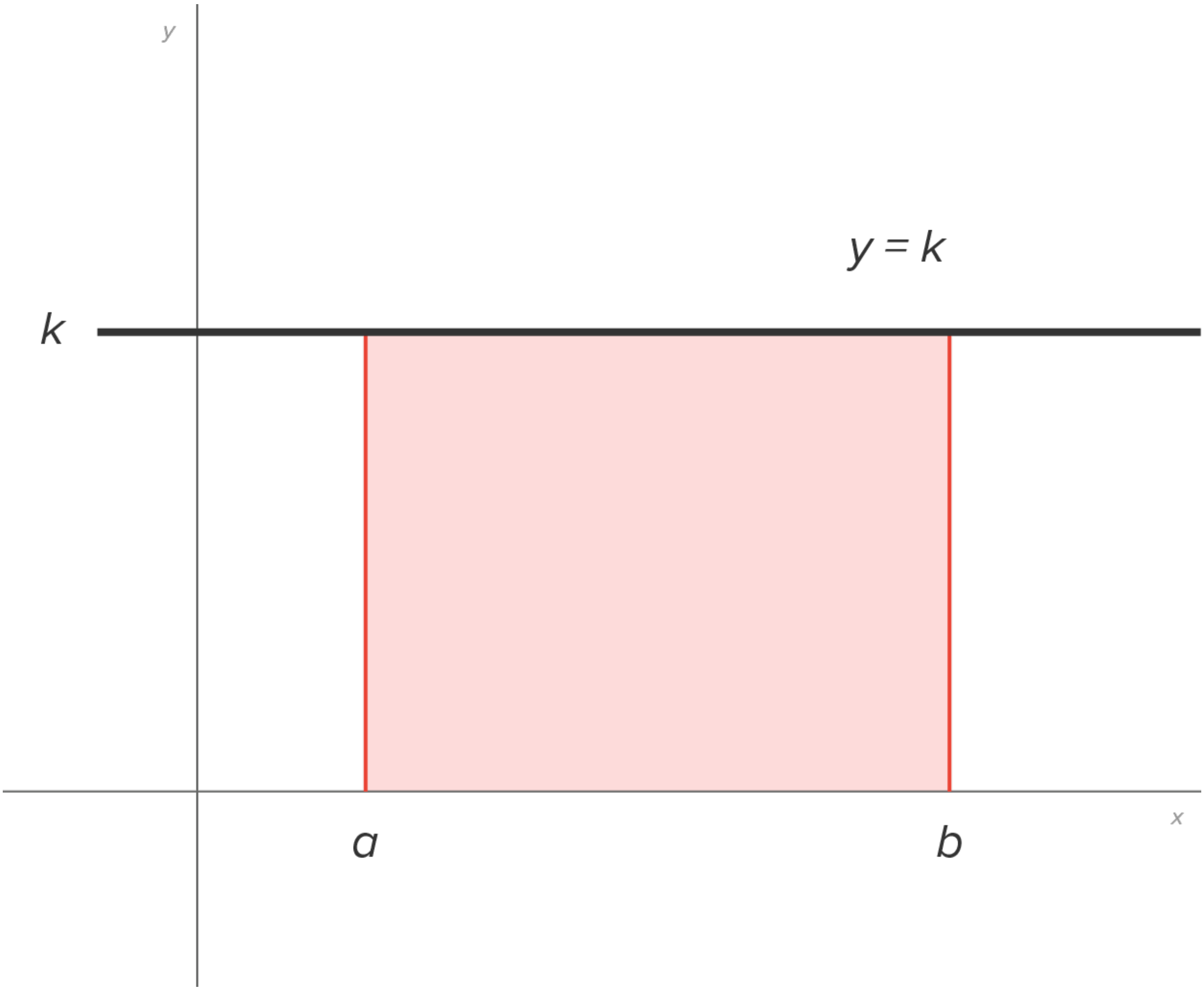
| | | |
|---|---|---|
| | $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$ | |
| 5 | **Definite Integral Over a Partition of an Interval, with a ≤ b ≤ c**<br><br>$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ | Adding areas<br>Note: $a \le b \le c$ |

To understand some of these properties, we'll look at Riemann sums and areas.

**Property 1:** If the lower and upper limits of integration are equal, then the width of the interval is 0, which means there is no accumulated area.

**Property 2:** As we have seen, $\int_a^b f(x)dx$ accumulates area from $x = a$ to $x = b$. Then, moving in the opposite direction from $x = b$ to $x = a$, whatever was added would be subtracted, and vice versa. Thus, the definite integrals have opposite signs.

**Property 3:** Consider the graph of $y = k$ on the interval $[a, b]$.




Assume $k > 0$, as in the picture. The region formed by $y = k$, $x = a$, and $x = b$ is a rectangle with height *k* and width $b - a$. Then, the area is $k(b - a)$.

- When $k > 0$, the definite integral is equal to the area.

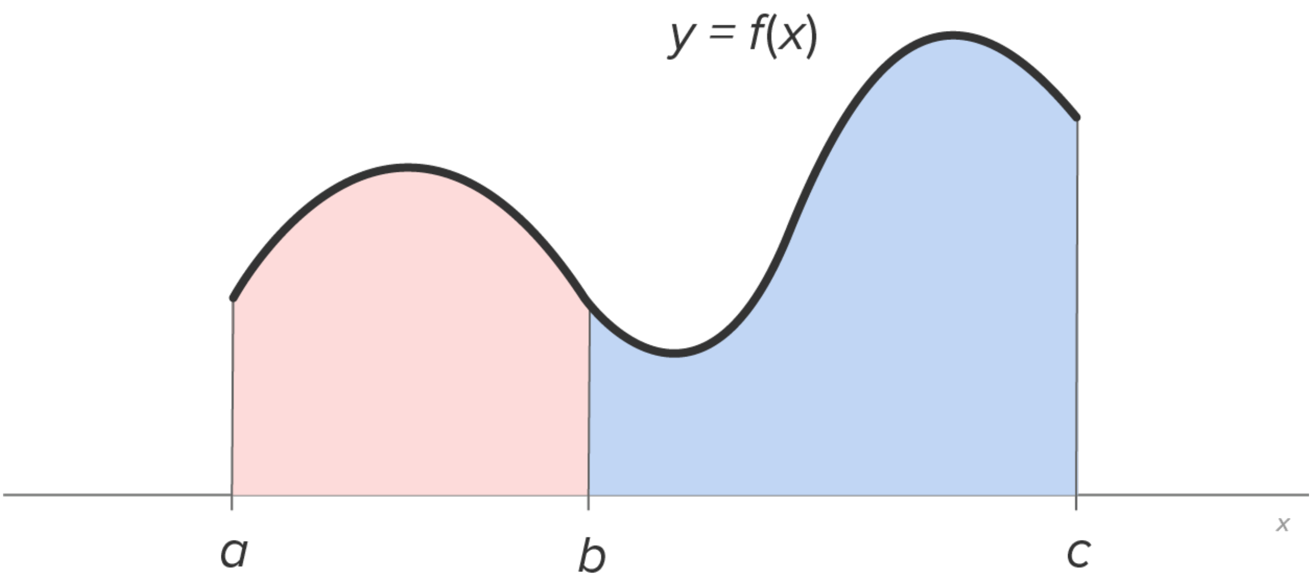
- When $k < 0$, $k(b-a) < 0$, which is also true since then the rectangle is below the x-axis.

**Property 4:** Consider the Riemann sums of each function, where the same partition is used:

| Riemann Sum for $y = f(x)$ | Riemann Sum for $y = k \cdot f(x)$ |
|---|---|
| $\sum_{i=1}^{n} f(c_i)\Delta x$<br>Height of each rectangle: $f(c_i)$<br>Area of each rectangle: $f(c_i) \cdot \Delta x$ | $\sum_{i=1}^{n} k \cdot f(c_i)\Delta x$<br>Height of each rectangle: $k \cdot f(c_i)$<br>Area of each rectangle: $k \cdot f(c_i) \cdot \Delta x$ |

In the Riemann sum, all terms have a common factor of $k$, meaning it can be factored outside the sum, $k \cdot \sum_{i=1}^{n} f(c_i)\Delta x$. Since this is the original Riemann sum multiplied by $k$, this justified the integral version of this property.

**Property 5:** Consider the graph in the figure:




By adding areas, we see that the area on $[a, c]$ is the sum of the areas on $[a, b]$ and $[b, c]$.

↳ EXAMPLE The graph in the figure shows a function $f(x)$ and areas between $f(x)$ and the x-axis.

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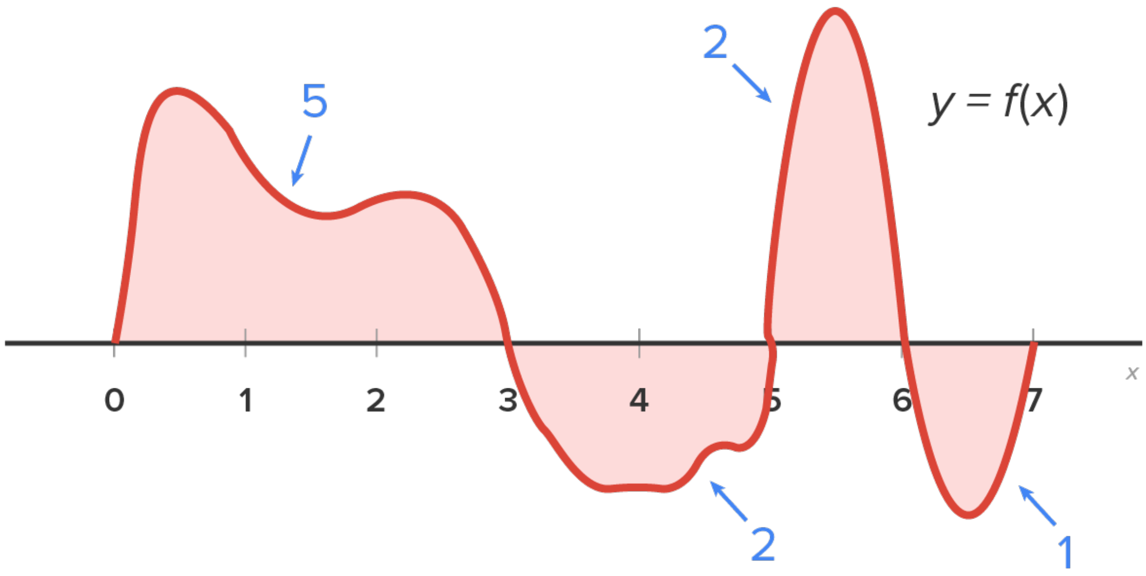


Find the definite integrals of $f$ for each of the following:

a. Find the definite integral: $\int_3^5 f(x)dx$

$\int_3^5 f(x)dx$ Evaluate this definite integral.

$= -2$ The area of the region is 2, but is below the x-axis.

Conclusion: $\int_3^5 f(x)dx = -2$

b. Find the definite integral: $\int_0^5 f(x)dx$

$\int_0^5 f(x)dx$ Evaluate this definite integral.

$= \int_0^3 f(x)dx + \int_3^5 f(x)dx$ Use the following property: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Note, “3” was chosen since at $x = 3$, the first region ends and the second one begins.

$= 5 + (-2)$ Substitute values from the graph: $\int_0^3 f(x)dx = 5,\ \int_3^5 f(x)dx = -2$

$= 3$ Simplify.

Conclusion: $\int_0^5 f(x)dx = 3$

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c. Find the definite integral: $\int_5^3 f(x)dx$

| | |
|---|---|
| $\int_5^3 f(x)dx$ | Evaluate this definite integral. |
| $= -\int_3^5 f(x)dx$ | Use the following property: $\int_b^a f(x)dx = -\int_a^b f(x)dx$ <br> Since the limits of integration are in reverse order, this property is appropriate to use. |
| $= -(-2) = 2$ | Substitute values from the graph: <br> $\int_3^5 f(x)dx = -2$ |
| $= 2$ | Simplify. |

Conclusion: $\int_5^3 f(x)dx = 2$

d. Find the definite integral: $\int_0^5 4f(x)dx$

| | |
|---|---|
| $\int_0^5 4f(x)dx$ | Evaluate this definite integral. |
| $= 4\int_0^5 f(x)dx$ | Use the following property: $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$ |
| $= 4(3)$ | $\int_0^5 f(x)dx$ was evaluated in part b. |
| $= 12$ | Simplify. |

Conclusion: $\int_0^5 4f(x)dx = 12$

TRY IT

Consider the following definite integrals in relation to the same graph as in the previous example.

a. $\int_4^4 [f(x)]^3 dx$

b. $\int_5^7 -3f(x)dx$

c. $\int_3^6 f(x)dx$

Evaluate each definite integral.

a. $\int_4^4 [f(x)]^3 dx = 0$

b. $\int_5^7 -3f(x)dx = -3$

c. $\int_3^6 f(x)dx = 0$

---

# 2. Properties of Definite Integrals of Combinations of Functions

FORMULA

| Formula | In Words |
|---|---|
| **Definite Integral of a Sum of Two Functions** $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ | The definite integral of a sum of two functions is the sum of the definite integrals of the functions. |
| **Definite Integral of a Difference of Two Functions** $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$ | The definite integral of a difference of two functions is the difference of the definite integrals of the functions. |

These properties follow directly from Riemann sums (using properties of summations).

For the sum property:

$$\int_a^b (f(x)+g(x))dx$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(f(c_k)+g(c_k))\Delta x\right]$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(f(c_k)\Delta x+g(c_k)\Delta x)\right]$$

By the property of summations, this is written as $\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(c_k)\Delta x + \sum_{k=1}^{n} g(c_k)\Delta x\right]$, which by the limit property is equal to $\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x + \lim_{n\to\infty}\sum_{k=1}^{n} g(c_k)\Delta x$, which is equal to $\int_a^b f(x)dx + \int_a^b g(x)dx$.

A very similar sequence of steps can be followed for the difference between *f* and *g*.

EXAMPLE Given $\int_2^6 f(x)dx = 10$ and $\int_2^6 g(x)dx = 4$, find each of the following:

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a. Find the definite integral: $\int_2^6 [f(x)+g(x)]dx$

| | |
|---|---|
| $\int_2^6 [f(x)+g(x)]dx$ | Evaluate this definite integral. |
| $=\int_2^6 f(x)dx+\int_2^6 g(x)dx$ | Use the definite integral of a sum of two functions property. |
| $=10+4$ | Substitute values. |
| $=14$ | Simplify. |

Conclusion: $\int_2^6 [f(x)+g(x)]dx=14$

b. Find the definite integral: $\int_2^6 [3+f(x)]dx$

| | |
|---|---|
| $\int_2^6 [3+f(x)]dx$ | Evaluate this definite integral. |
| $=\int_2^6 3dx+\int_2^6 f(x)dx$ | Use the property: $\int_a^b [f(x)+g(x)]dx=\int_a^b f(x)dx+\int_a^b g(x)dx$ |
| $=3(6-2)+10$ | For the first integral, $\int_a^b kdx=k(b-a)$. For the second integral, the value is given: 10 |
| $=22$ | Simplify. |

Conclusion: $\int_2^6 [3+f(x)]dx=22$

c. Find the definite integral: $\int_2^6 [3f(x)-g(x)]dx$

| | |
|---|---|
| $\int_2^6 [3f(x)-g(x)]dx$ | Evaluate this definite integral. |
| $=\int_2^6 3f(x)dx-\int_2^6 g(x)dx$ | Use the property: $\int_a^b [f(x)-g(x)]dx=\int_a^b f(x)dx-\int_a^b g(x)dx$ |
| $=3\int_2^6 f(x)dx-\int_2^6 g(x)dx$ | Use the property: $\int_a^b k\cdot f(x)dx=k\int_a^b f(x)dx$ |

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$$= 3(10) - 4 \quad \text{Substitute given values of integrals.}$$

$$= 26 \quad \text{Simplify.}$$

Conclusion: $\int_2^6 [3f(x) - g(x)]dx = 26$

WATCH

In this video, given $\int_1^4 f(x)dx = 12$, we'll find the value of $\int_1^4 (f(x) + x)dx$.

### Video Transcription

[MUSIC PLAYING] Hello there. Welcome back. What we're going to do is take a look at a known definite integral-- the integral from 1 to 4 of f of x dx. And its value is 12. And we're going to use it to evaluate the integral from 1 to 4 of f of x plus x dx. And that's going to be by using the various properties of definite integrals that we've learned.

So the first property, remember, is that when you have a sum as your integrand, we can break the definite integral into pieces. So this is the integral from 1 to 4 of f of x dx plus the integral from 1 to 4 of just x dx. Now, we already know that the value of this first definite integral is 12. And because it's a known formula-- a known shape, I should say-- we know what the definite integral of f of x equals x is.

So I just sketch a graph of that. And we have our axes here. y equals x is the diagonal line, making a 45-degree angle with the x-axis. And we have x equals 1 here and x equals 4 here. Now since the region is above the x-axis on the interval 1 to 4, we know that the definite integral from 1 to 4 is the area of said region, which is in the shape of a trapezoid.

So the height of this side here is 1. The height of this side here is 4. Those are your parallel bases. And then what we call the height of the trapezoid-- the side that's perpendicular to the parallel bases-- is 3. So that means that this area is 1/2 times 3 times 1 plus 4, which looks to be 15/2. So that means that this second definite integral-- I'm going to have to go to the next line here-- has value 15 over 2, since, again, it's the area of the region because the region is above the x-axis. So that means this is equal to 12 plus 15/2, which is 7.5. So I can write this as either 19.5 or 39/2. and that is using properties to evaluate a definite integral.

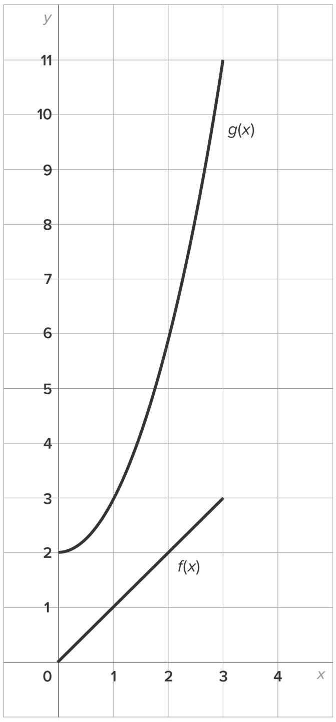
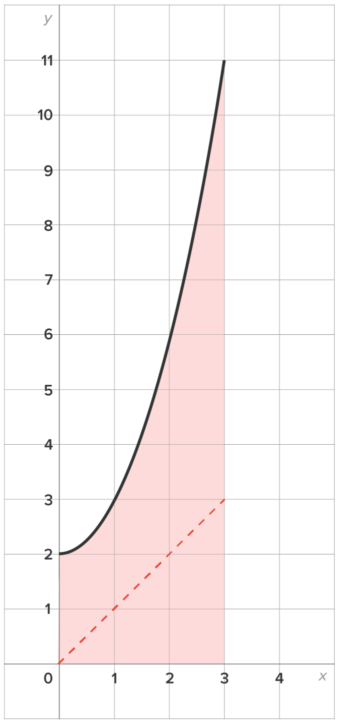
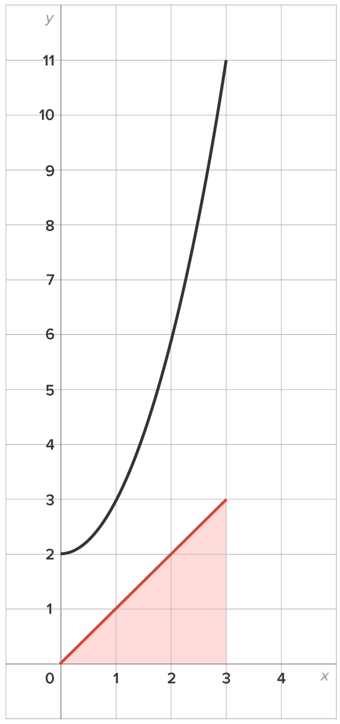
[MUSIC PLAYING]

# 3. Comparison Properties

## 3a. Comparing Two Functions

If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

To visualize this, consider these graphs. Clearly, the area between the graph of $g(x)$ and the x-axis is greater than the area between the graph of $f(x)$ and the x-axis.

| The graphs of $f(x)$ and $g(x)$ together on the same axes. | The region bounded by the graphs of $g(x)$ and the x-axis. Area $= \int_0^3 g(x)dx$ | The region bounded by the graphs of $f(x)$ and the x-axis. Area $= \int_0^3 f(x)dx$ |
|---|---|---|
|  |  |  |



## 3b. Bounds on the Value of a Definite Integral

Let $m$ = the minimum value of $f(x)$ on $[a, b]$.
Let $M$ = the maximum value of $f(x)$ on $[a, b]$.

Given that $m \le f(x) \le M$ on the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

Below is the graphical justification:

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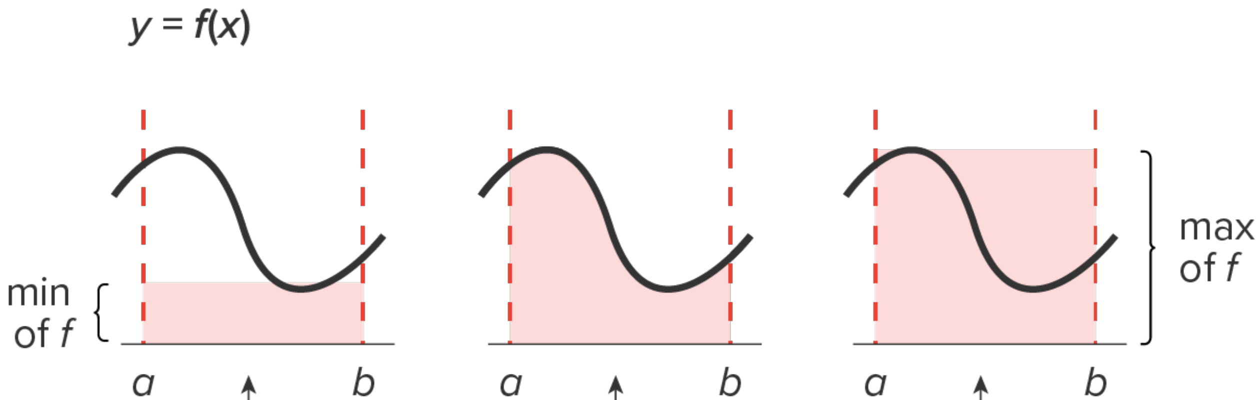


$$\{\text{ min of } f\ \}(b - a) \leq \int_a^b f(x)\,dx \leq \{\text{ max of } f\ \}(b - a)$$

→ EXAMPLE Use the graph to determine the upper and lower bounds of the value of $\int_1^5 f(x)dx$.

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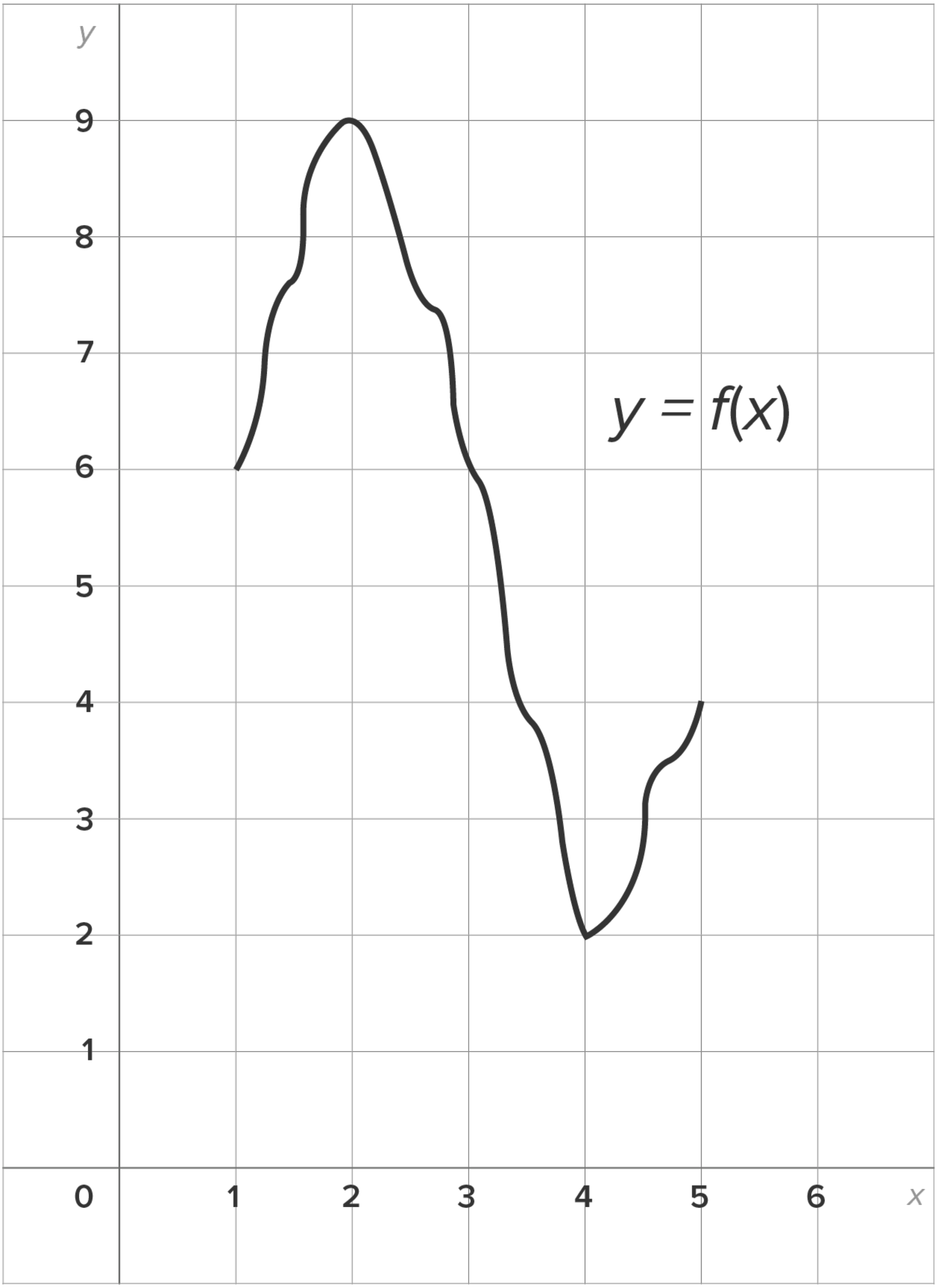


The minimum value of $f(x)$ on $[1, 5]$ is 2; the maximum value is 9.

Thus, the minimum value for the definite integral is $2(5-1)=8$ and the maximum value is $9(5-1)=36$. That is, $8 \le \int_1^5 f(x)dx \le 36$.

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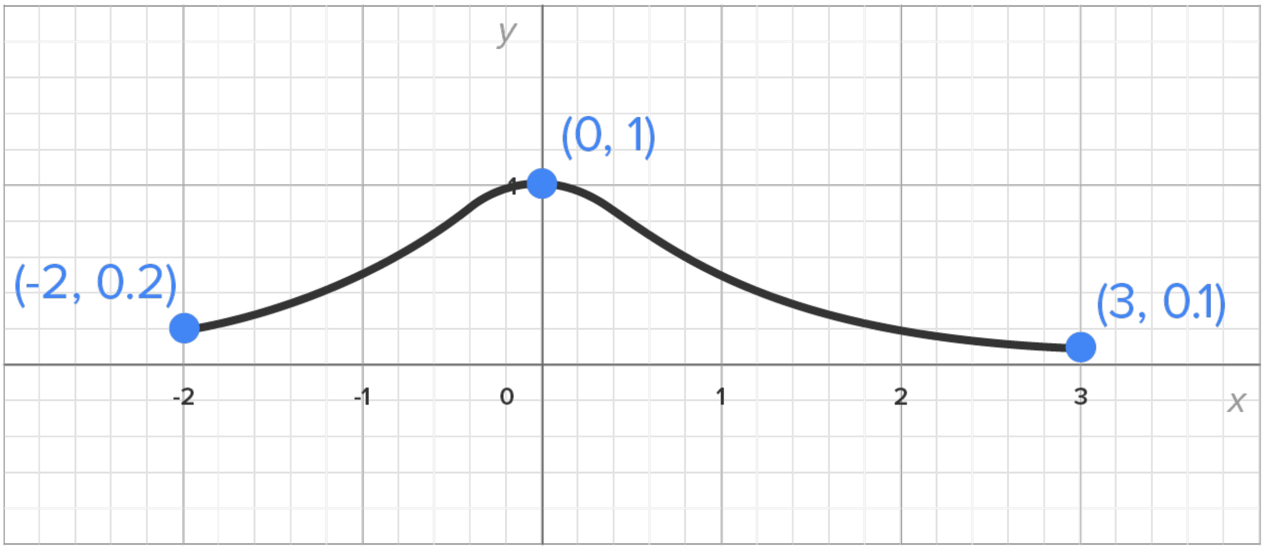
TRY IT

Consider the graph below in relation to the integral $\int_{-2}^{3} f(x)dx$.




Use the graph to determine the upper and lower bounds of the value of this integral. +

The minimum value is 0.5 and the maximum value is 5.

## SUMMARY

In this lesson, you examined some useful **properties of the definite integral** as well as **properties of definite integrals of combinations of functions**. As a result of properties of areas, you are now able to find definite integrals of sums, differences, and constant multiples of functions, as well as utilize **comparison properties** to **compare two functions** and put **bounds on the value of a definite integral**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## FORMULAS TO KNOW

**Definite Integral Over a Partition of an Interval, with a ≤ b ≤ c**

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

**Definite Integral When Lower and Upper Bounds Are Equal**

$$\int_a^a f(x)dx = 0$$

**Definite Integral When Upper and Lower Bounds Are Interchanged**

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$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

**Definite Integral of a Constant Function**

$$\int_a^b k\,dx = k(b-a)$$

**Definite Integral of a Constant Multiple of a Function**

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

**Definite Integral of a Difference of Two Functions**

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

**Definite Integral of a Sum of Two Functions**

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

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# Areas, Integrals, and Antiderivatives

*by Sophia*

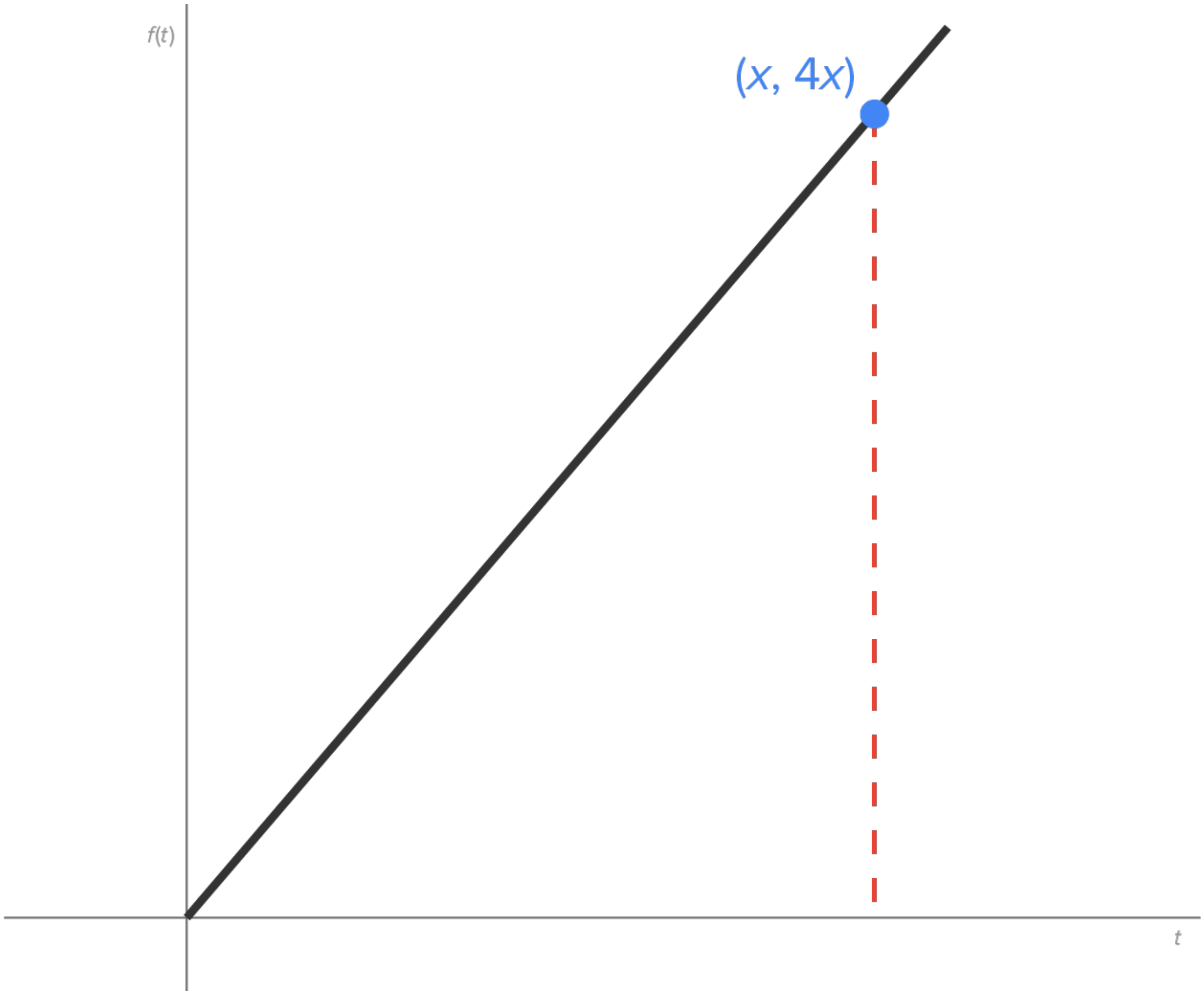
WHAT'S COVERED

In this lesson, you will connect the idea of area, definite integrals, and antiderivatives. Specifically, this lesson will cover:

1. The Area Function, $A(x)$ and Its Relationship to $f(x)$
2. Finding Basic Antiderivatives
3. The First and Second Fundamental Theorems of Calculus
   a. The First Fundamental Theorem of Calculus
   b. The Second Fundamental Theorem of Calculus
4. Using Antiderivatives to Calculate Area

# 1. The Area Function, $A(x)$ and Its Relationship to $f(x)$

Consider the area of the region bounded by the t-axis (horizontal axis), the function $f(t)=4t$, and the vertical line $t=x$. The graph is shown in the figure below.




As the value of *x* changes, the area of the region changes, meaning that the area depends on *x*, meaning the area is a function of *x*.

Let $A(x) =$ the area of the region, which is a triangle.

Since $A(x)$ defines the area of a region between $f(t)$ and the t-axis, we can define $A(x)$ as a definite integral:

$$A(x) = \int_0^x 4t\,dt$$

Assuming that $x > 0$, the area of the region is $A(x) = \frac{1}{2}(x)(4x) = 2x^2$.

TRY IT

Consider the region bounded by $f(t) = 3$ and the t-axis between $t = 0$ and $t = x$.

Write the area function *A*(*x*) as both a definite integral and as a function of *x*. +

$$A(x) = \int_0^x 3dt = 3x$$

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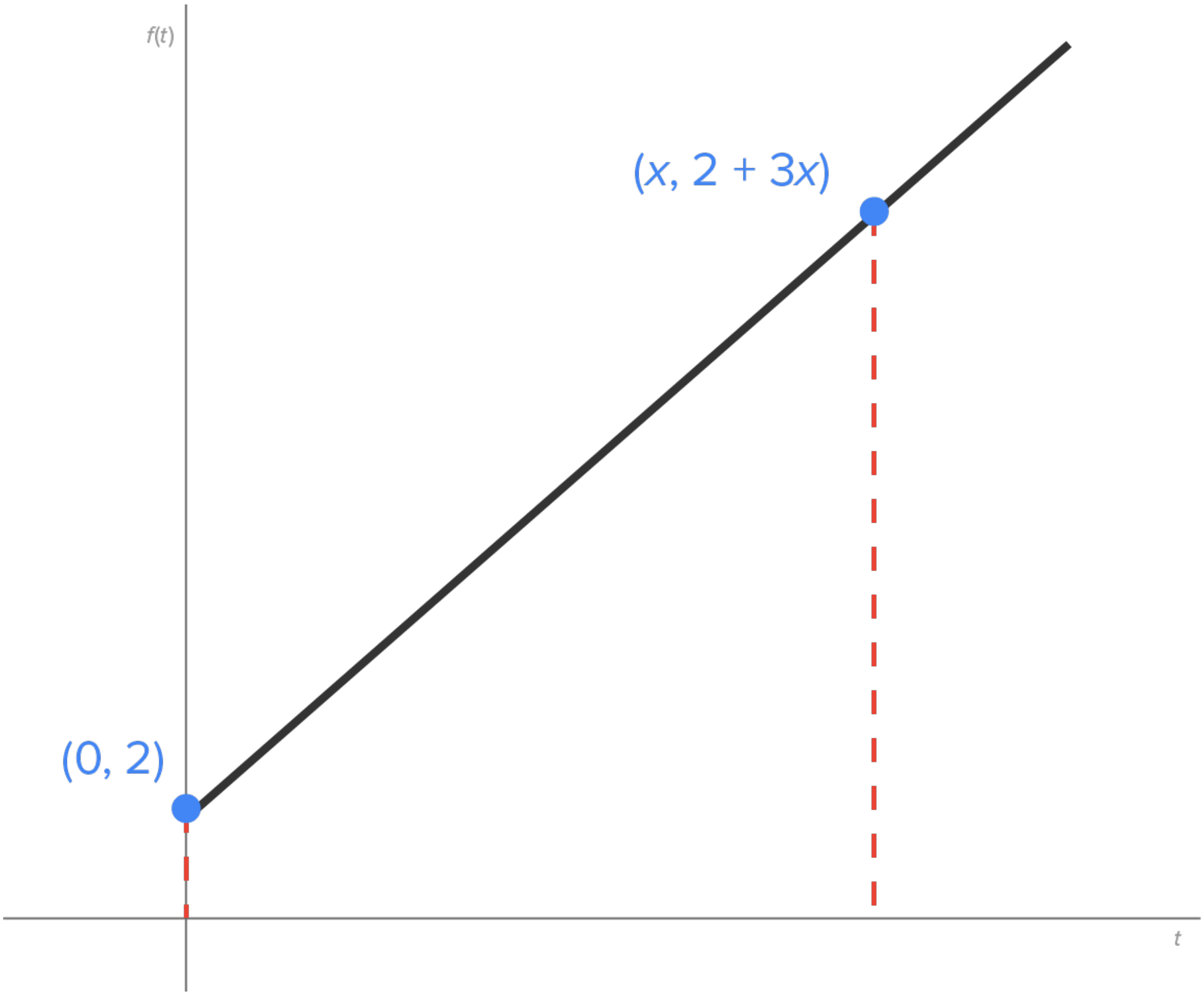
BIG IDEA

If $x$ is one of the limits of integration, it is important that another variable be used inside the integral sign. This is because we can't have variables serving two roles at once (upper limit of integration and the variable inside the integral).

→ EXAMPLE Consider the region bounded by the t-axis and the line $f(t) = 2 + 3t$ between $t = 0$ and $t = x$.




As $x$ increases, the shape remains a trapezoid.

As a definite integral, the area is:

$$A(x) = \int_0^x (2+3t)dt$$

Using the trapezoid area formula, we have:

| | |
|---|---|
| $\frac{1}{2}(x)(2+(2+3x))$ | Use the trapezoid area formula. |
| $\frac{1}{2}x(4+3x)$ | Simplify parentheses. |
| $2x+\frac{3}{2}x^2$ | Distribute. |

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Thus, $A(x)=2x+\frac{3}{2}x^2$.

You might notice that there is a relationship between the area function $A(x)$ and the associated curve $y=f(x)$. We're going to explore this in this next segment.

Consider the last three examples. Here is a summary of the area functions with their associated curves, as well as the derivatives of each area function.

| Regions | Area Function, $A(x)$ | "Height" Function, $f(x)$ | $A'(x)$ |
|---|---|---|---|
| Region bounded by the t-axis (horizontal axis), the function $f(t)=4t$, and the vertical line $t=x$ | $A(x)=2x^2$ | $f(x)=4x$ | $A'(x)=4x$ |
| Region bounded by $f(t)=3$ and the t-axis between $t=0$ and $t=x$ | $A(x)=3x$ | $f(x)=3$ | $A'(x)=3$ |
| Region bounded by the t-axis and the line $f(t)=2+3t$ between $t=0$ and $t=x$ | $A(x)=2x+\frac{3}{2}x^2$ | $f(x)=2+3x$ | $A'(x)=2+3x$ |

Note that in each situation, $A'(x)=f(x)$. It turns out that this is always the case, which is a very useful idea in finding areas of regions that use any choice of $f(x)$. First, we need to learn a bit about antiderivatives.

---

# 2. Finding Basic Antiderivatives

We call $F(x)$ an **antiderivative** of $f(x)$ if $F'(x)=f(x)$. That is, $F(x)$ is the function whose derivative is $f(x)$. For instance, an antiderivative of $f(x)=3x^2$ is $F(x)=x^3$ since $D[x^3]=3x^2$. In fact, we could also say that $F(x)=x^3+4$ is an antiderivative of $f(x)=3x^2$ since $D[x^3+4]=3x^2$.

As it turns out, any function of the form $F(x)=x^3+C$ (where $C$ is constant) is an antiderivative of $f(x)=3x^2$ since $D[x^3+C]=3x^2$.

↪ EXAMPLE Find three antiderivatives of $f(x)=\cos x$.

Since $D[\sin x]=\cos x$, it follows that $F(x)=\sin x$ is an antiderivative of $f(x)=\cos x$. To find others, all you would need to do is add a constant.

Two more antiderivatives are $F(x)=\sin x+2$ and $F(x)=\sin x-3$.

In summary, any function of the form $F(x)=\sin x+C$ is an antiderivative of $f(x)=\cos x$.

TRY IT

Consider the function $f(x)=6x^2$.

Write three antiderivatives of this function. +

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Any function of the form $F(x)=2x^3+C$ is an antiderivative of $f(x)=6x^2$. Here are three examples:

- $F(x)=2x^3$
- $F(x)=2x^3+2$
- $F(x)=2x^3-4$

TERM TO KNOW

**Antiderivative**
$F(x)$ is an antiderivative of $f(x)$ if $F'(x)=f(x)$.

---

# 3. The First and Second Fundamental Theorems of Calculus

## 3a. The First Fundamental Theorem of Calculus

Consider the area function $A(x)=\int_0^x f(t)dt$.

By substituting $x=1$ and $x=2$, we have $A(1)=\int_0^1 f(t)dt$ and $A(2)=\int_0^2 f(t)dt$.

By properties of integrals, $\int_0^2 f(t)dt=\int_0^1 f(t)dt+\int_1^2 f(t)dt$.

Replacing the first two integrals by their values, we have $A(2)=A(1)+\int_1^2 f(t)dt$.

Finally, let's write the definite integral to one side: $A(2)-A(1)=\int_1^2 f(t)dt$

Remember that $A'(x)=f(x)$, meaning that $A(x)$ is an antiderivative of $f(x)$. Therefore, we could write $A(x)=F(x)$.

Thus, we can rewrite as $\int_1^2 f(t)dt=F(2)-F(1)$. This is generalized in **the first fundamental theorem of calculus**, as shown below:

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x)=f(x)$. Then, $\int_a^b f(x)dx=F(b)-F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

To show that we are substituting $a$ and $b$ into $F(x)$, we use the following notation: $F(x)\Big|_a^b$

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Then, it follows that $F(x)\Big|_a^b = F(b) - F(a)$.

↪ EXAMPLE Evaluate $\int_0^2 3x^2 dx$.

Since any function of the form $F(x) = x^3 + C$ is an antiderivative of $f(x) = 3x^2$, we have the following:

| | |
|---|---|
| $\int_0^2 3x^2 dx$ | Start with the original expression. |
| $= (x^3 + C)\Big|_0^2$ | Apply the first fundamental theorem of calculus with $F(x) = x^3 + C$. |
| $= (2^3 + C) - (0^3 + C)$ | Substitute $x = 2$ and $x = 0$ into $F(x)$, then subtract. |
| $= 8 + C - C$ | Evaluate operations in the parentheses. |
| $= 8$ | Simplify. |

Thus, $\int_0^2 3x^2 dx = 8$.

BIG IDEA

Notice that the "+*C*" dropped out when evaluating the definite integral. Intuitively, this will always happen. Thus, when evaluating a definite integral, select $C = 0$. Some students even say "You don't need the *C*."

TERM TO KNOW

**The First Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x) = f(x)$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

## 3b. The Second Fundamental Theorem of Calculus

Recall that $A'(x) = f(x)$. If we replace $A(x)$ with $F(x)$ to correspond with $f(x)$, we have another important theorem in calculus, **the second fundamental theorem of calculus**:

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a \le x \le b$.

Let $F(x) = \int_a^x f(t)dt$. Then, $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

↪ EXAMPLE Let $F(x) = \int_2^x \sqrt{t+1}\,dt$.

Since $f(t) = \sqrt{t+1}$ is continuous on $[-1, \infty)$, which includes 2, then $F'(x) = \sqrt{x+1}$.

TRY IT

Let $F(x)=\int_{\sqrt{\pi}}^{x}\sin(t^2)dt$.

Find $F'(x)$. +

$F'(x)=\sin(x^2)$

Suppose $x$ is replaced by $u$, where $u$ is a function of $x$.

That is, $F(x)=\int_a^u f(t)dt$. Then, by the chain rule, $F'(x)=f(u)\cdot\frac{du}{dx}$.

EXAMPLE Let $F(x)=\int_1^{x^3} e^t dt$.

Then, $F'(x)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$.

TRY IT

Let $F(x)=\int_0^{\sin x}\ln t\,dt$.

Find $F'(x)$. +

$F'(x)=\cos x\cdot\ln(\sin x)$

TERM TO KNOW

**The Second Fundamental Theorem of Calculus**

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a\le x\le b$.

Let $F(x)=\int_a^x f(t)dt$. Then, $F'(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x)$.

# 4. Using Antiderivatives to Calculate Area

As a result of the fundamental theorem of calculus, we have a new way to compute areas with definite integrals. Instead of relying on a sketch of the region, we can use antiderivatives to compute areas.

EXAMPLE Find the area between the graph of $f(x)=\cos x$ and the x-axis between $x=0$ and $x=\frac{\pi}{2}$.

The region is shown in the figure:

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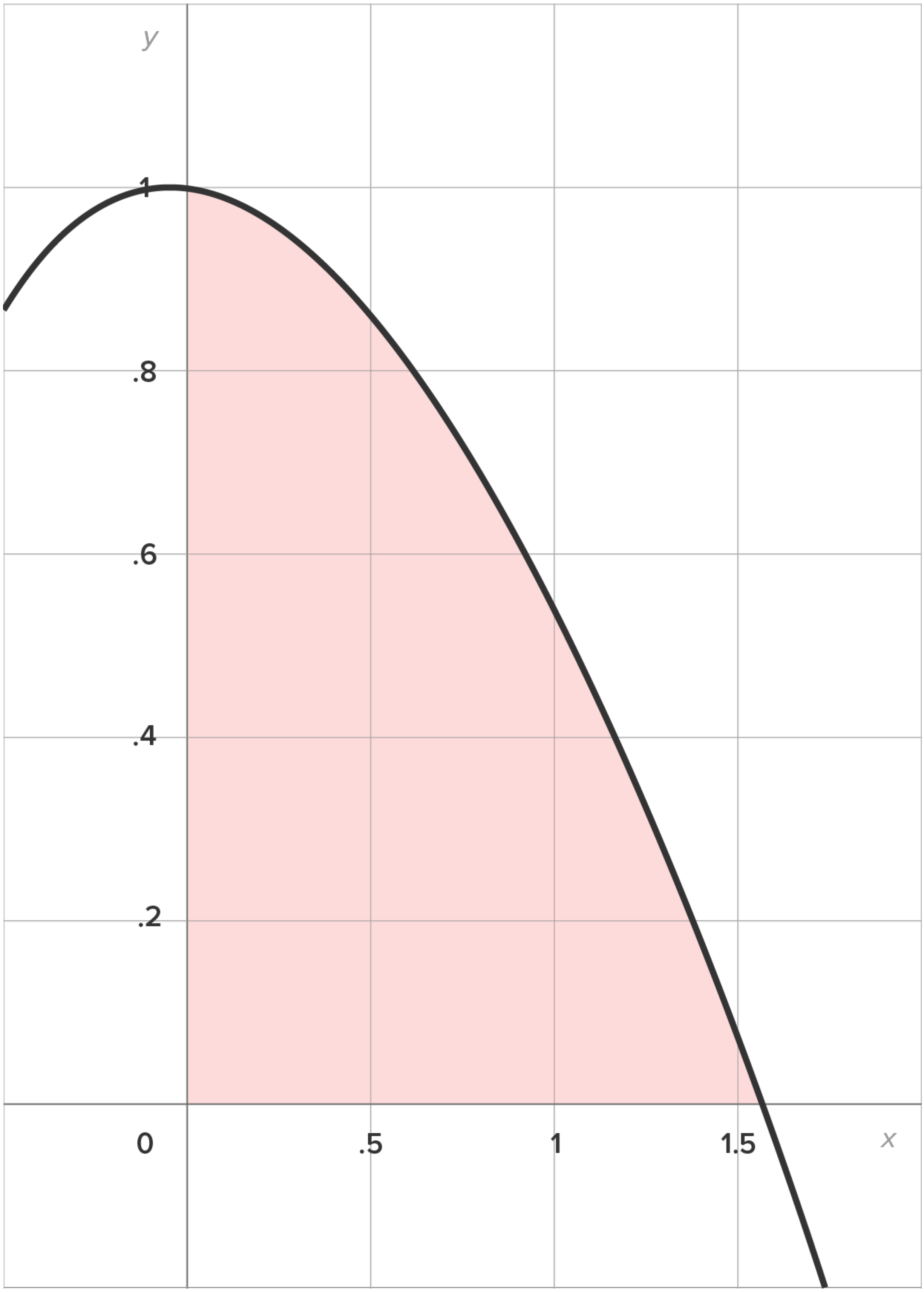


We know the area is given by the definite integral $\int_0^{\pi/2} \cos x dx$.

Earlier, we saw that sin $x$ is an antiderivative of cos $x$. Therefore, the area is as follows:

| | |
|---|---|
| $\int_0^{\pi/2} \cos x dx$ | Start with the original expression. |
| $= \sin x \Big\vert_0^{\pi/2}$ | Use the fundamental theorem of calculus with $F(x) = \sin x$. Remember, we do not need to write "+C," meaning we are choosing $C = 0$. |
| $= \sin\frac{\pi}{2} - \sin 0$ | Substitute $x = \frac{\pi}{2}$ and $x = 0$ into $F(x)$, then subtract. |
| $= 1$ | Simplify. Recall $\sin\frac{\pi}{2} = 1$ and $\sin 0 = 0$. |

Thus, the area of the region is 1 square unit.

TRY IT

Consider the graphs of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = 2$.

Calculate the area between these graphs. +

$e^2 - 1$ square units

## SUMMARY

In this lesson, you learned that with a **link between the area function $A(x)$ and a region with $f(x)$** as the upper boundary, **basic antiderivatives** can be used to **calculate areas** and compute definite integrals by using **the first and second fundamental theorems of calculus**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Antiderivative**

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

**The First Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x) = f(x)$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

**The Second Fundamental Theorem of Calculus**

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a \le x \le b$.

Let $F(x) = \int_a^x f(t)dt$. Then, $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

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# Indefinite Integrals and Antiderivatives of Polynomial Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will begin your journey in finding antiderivatives (or indefinite integrals) by using properties and formulas, similar to how you learned derivatives. Finding an antiderivative is the reverse of the process used to find a derivative, so this will be used to our advantage. We'll start with polynomial functions, then work our way through trigonometric and exponential functions. Specifically, this lesson will cover:

1. The Definition of Indefinite Integrals
2. The Power Rule
3. Properties of Antiderivatives and Antiderivatives of Polynomials

# 1. The Definition of Indefinite Integrals

In the last tutorial, we defined the antiderivative of a function. Another name for an antiderivative is an **indefinite integral**. The indefinite integral of a function $f(x)$, written $\int f(x)dx$, is the collection of functions whose derivatives are equal to $f(x)$. In other words, the indefinite integral of $f(x)$ is the antiderivative of $f(x)$.

If $F'(x) = f(x)$, then we write $\int f(x)dx = F(x) + C$, where $C$ is an arbitrary constant. $C$ is also known as the constant of integration.

TERM TO KNOW

**Indefinite Integral of *f*(*x*)**

The collection of functions whose derivatives are equal to $f(x)$. In other words, the indefinite integral of $f(x)$ is the antiderivative of $f(x)$.

# 2. The Power Rule

To get a better understanding of the antiderivative of $f(x) = x^n$, watch this video.

WATCH

In this video, we'll develop the ideas needed to establish a rule to find $\int x^n dx$.

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## Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do in this video is take a look at antiderivatives, which is the reverse process of derivatives. In order to understand how to find antiderivatives we need to have a working understanding of derivatives. So as you see here, we notice that we have the derivative of x squared, which we know is 2x, and the derivative of x to the third, which we know is 3x squared.

So we need to have a similar notation or a similar kind of relationship to show that we're starting at 2x and ending up at x squared. So that notation happens to be the integral sign that you might remember from definite integrals. So already you're seeing there might be a connection here. So this is an indefinite integral because there is no starting x value or ending x value. So the way this is written, the correspondence is, the antiderivative of 2x dx is equal to-- Well, what function did you start with?

You started with x squared, but remember we're basically answering the question, the derivative of what is 2x? x squared is not the whole story, but it's most of the story. Remember that you can also add any constant you want 2x squared and its derivative will still be 2x. So we write x squared plus c. What that is, it really is a family of solutions rather than just one solution. x squared plus 4 would have worked, x squared minus 1, x squared plus 8, all of those have derivative equal to 2x. So the most general of forms is just to write x squared plus a constant.

So then for 3x squared, as you can probably gather, we would write antiderivative of 3x squared dx is equal 2x to the 1/3 plus a constant. OK. So here's the question, what if I just had a power function without a coefficient? What is the antiderivative there? So let's look at a couple patterns here. We have the antiderivative, let's do x to the fourth.

Now, from what we know about derivatives I'm basically asking the question, what do I differentiate to end up with x to the fourth? Well, let's see. If it's a power function I know when I bring the power down I subtract 1 from the power. So I know that had to start with an x to the fifth and a plus c, but that's not the story because the derivative of x to the fifth is 5x to the fourth. So the question is, what do I need to multiply that by to make that five go away? That number is one fifth.

If you take the derivative of one fifth x to the fifth, you get five times one fifth times x to the fourth, which is 1x to the fourth. There it is. OK. So that means that this is our first antiderivative that we're guessing here. So if we wanted, say, x to the 13th dx, well, again, I know that my power had to start at a 14. What do I need to multiply by x to the 14th to make the 14 go away? Because remember, the derivative of x to the 14th is 14x to the 13th. You might have guessed it, I need to multiply by 1 over 14. So there's our next correspondence. OK.

So it seems as though the antiderivative of x to the n dx is equal to-- Well, let's just trace out what's happening here. When the power in the integrand was four, the result ended up being one fifth x to the fifth. When it was x to the 13th we got 1/14 x to the 14th. So it seems as though we are adding 1 to the power and putting that in the denominator as the coefficient, and then times x to the same power, the n plus 1, and then a constant. Now, you can write that in a shorter form, x to the n plus 1 over n plus 1. Shorter, now it takes up less room anyway.

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So those are both acceptable forms. However, there's a problem. Notice in the denominator we have an n plus 1, which means this rule is not valid if n is negative 1. So you have to write a caveat that n is not equal to negative 1. So then the question is, how do we handle n equals negative 1? We handle it separately. So if n is equal to negative 1 we have the antiderivative of x to the negative 1 dx. Now, that negative 1 is not doing me any good as a power that way so I'm going to rewrite this as 1 over x, the more familiar way.

Now maybe if you rifle through your derivative rules, there's a derivative that you learned whose result is 1 over x. The derivative of some function is 1 over x. It turns out that function is the natural log of x. Great. Does that mean that the antiderivative of 1 over x dx is equal to the natural log of x plus c? Well, not exactly. Turns out, and this is a rule when we're trying to find any antiderivative, same kind of rule that tells us we have to add the plus c, we want the most general answer possible. So it turns out there is a better generalization of this antiderivative that is leaving it at natural log of x.

So let me explain. Let's look at the function natural log of absolute value of x, which is equal to the natural log of just x if x is positive. Remember the definition of absolute value, absolute value of x is equal to itself if x is positive and its opposite, its additive inverse, if x is negative. So we already know the derivative of natural log of x is 1 over x. The derivative of natural log of absolute value of x is equal to 1 over x if x is greater than 0. So what is it if x is less than 0? Well, let's see.

The derivative of natural log of negative x is 1 over negative x times the derivative of negative x. You notice those negatives drop out and we get 1 over x. So that means that both pieces have the same derivative, which means we can form one function that's defined for x not equal to 0, as opposed to just x greater than 0. So that means in our antiderivative up here I'm going to add the absolute value bars. Now, why is that so important?

Well, if you look at the integrand, which is 1 over x, its domain is everything except for 0, all real values of x except 0. So it stands to reason that our antiderivative formula should have the same domain, or at least a similar domain, we don't want to take half of it away. So at least with the absolute value we're able to consider negative values of x as well as positive values of x.

So in summary, here are our two formulas that we just talked about. Antiderivative x to the n dx is 1 over n plus 1, times x to the n plus 1, plus c. Then the antiderivative of 1 over x dx is the natural log of the absolute value of x, plus z. These are the first of many, many, many antiderivative formulas that you will learn as you go through this unit. It'll be a fun journey.

[MUSIC PLAYING]

Here is a summary of the antiderivative formulas for $f(x) = x^n$.

FORMULA

**Power Rule for Antiderivatives**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$$

**Antiderivative of a Constant**

Antiderivative of a Constant

$$\int k\,dx = kx + C$$

**Natural Logarithm Rule**

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

↱ EXAMPLE Find $\int x^4\,dx$.

| | |
|---|---|
| $\int x^4\,dx$ | Start with the original expression. |
| $= \frac{1}{5}x^5 + C$ | Use the power rule with $n = 4$. |
| $= \frac{x^5}{5} + C$ | Rewrite. |

Thus, $\int x^4\,dx = \frac{x^5}{5} + C$.

↱ EXAMPLE Find $\int t^{1/2}\,dt$.

| | |
|---|---|
| $\int t^{1/2}\,dt$ | Start with the integral. |
| $= \frac{1}{3/2}t^{3/2} + C$ | Use the power rule with $n = \frac{1}{2}$: $\frac{1}{2} + 1 = \frac{3}{2}$ |
| $= \frac{2}{3}t^{3/2} + C$ | Rewrite: $\frac{1}{3/2} = \frac{2}{3}$<br>Fractions should always be written in the simplest form. |

Thus, $\int t^{1/2}\,dt = \frac{2}{3}t^{3/2} + C$.

TRY IT

Consider $\int x^{-4}\,dx$.

Find the indefinite integral. +

$\frac{-1}{3}x^{-3} + C$

# 3. Properties of Antiderivatives and Antiderivatives of Polynomials

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Recall the rules for derivatives of combinations of functions (sums, differences, and constant multiples). For example, recall that $D[\sin x + \cos x] = D[\sin x] + D[\cos x]$.

Below are the formulas for the antiderivative of a sum, difference, and constant multiple of a function:

FORMULA

**Antiderivative of a Constant Multiple of a Function**

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

**Antiderivative of a Sum of Functions**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

**Antiderivative of a Difference of Functions**

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

BIG IDEA

In general:

- The antiderivative of a constant times $f(x)$ is the constant times the antiderivative of $f(x)$.
- The antiderivative of a sum is the sum of the antiderivatives.
- The antiderivative of a difference is the difference of the antiderivatives.

Recall that a polynomial function is a function in which all the terms are nonnegative integer powers of some variable. For example, $f(x) = 15x^4 - 3x^2 + 20x - 41$. This means we have all the properties necessary to find the antiderivative of any polynomial function.

EXAMPLE Find the indefinite integral: $\int (15x^4 + 2x^3 - 12x^2)dx$

| | |
|---|---|
| $\int (15x^4 + 2x^3 - 12x^2)dx$ | Start with the original expression. |
| $= \int 15x^4 dx + \int 2x^3 dx - \int 12x^2 dx$ | Use the sum/difference properties. |
| $= 15\int x^4 dx + 2\int x^3 dx - 12\int x^2 dx$ | Apply the constant multiple property. |
| $= 15\left(\frac{1}{5}x^5\right) + 2\left(\frac{1}{4}x^4\right) - 12\left(\frac{1}{3}x^3\right) + C$ | Use the power rule. Note: there is only one "$+C$" needed. If a constant were added to each indefinite integral, they could be merged and written as one constant. |
| $= 3x^5 + \frac{1}{2}x^4 - 4x^3 + C$ | Simplify. |

Thus, $\int (15x^4 + 2x^3 - 12x^2)dx = 3x^5 + \frac{1}{2}x^4 - 4x^3 + C$.

TRY IT

Consider $\int (6x^8 - 14x^6 + 22x + 17)dx$.

Find the indefinite integral. +

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$$\frac{2}{3}x^9 - 2x^7 + 11x^2 + 17x + C$$

WATCH

In this video, we'll find $\int\left(2x - \frac{8}{x}\right)dx$.

## Video Transcription

[MUSIC PLAYING] Hi there, and welcome back. Thanks for joining me. What we're going to do in this video is find the antiderivative using the rules that we know so far. The antiderivative of 2x minus 8 over x with respect to x. So the first thing to notice is that we do have a subtraction inside of the integral sign there, which means we can separate these into two separate integrals like so.

And then because we have constant multiples in each of these integrals, we can remove the constant-- or, I should say move it to the outside. So this is 2 times the antiderivative of x minus 8 times the antiderivative of 1 over x. So by the power rule-- now remember, x is really x to the first. To utilize the power rule, which is this property right here, we have two-- whoops.

We have 2 times-- we go up to the next power, which is x squared, and divide by the new power, and then minus 8 times-- now, the antiderivative of 1 over x. The power rule does not work on that one because that's really an x to the negative 1, which would mean x to the 0 over 0. That's not good news.

But remember, the antiderivative of 1 over x is your natural log function. And it's natural log of the absolute value of x. And we put the plus C on the very end. We don't put a plus C for each antiderivative we did because you'd have a C1 and a C2, and having two constants put together, we can just write that as one constant.

So simplifying here. The 2's are going to cancel, and we have x squared minus 8, natural log, absolute value x plus C. And that is the antiderivative of our function.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that another name for an antiderivative (covered in the last tutorial) is **indefinite integral**. The indefinite integral of $f(x)$ is the antiderivative of $f(x)$. You also learned that by applying **the power rule** for antiderivatives, as well as **properties of antiderivatives and antiderivatives of polynomials**, you can find antiderivatives of sums, differences, and constant multiples of powers of a variable.

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## TERMS TO KNOW

**Indefinite Integral of f(x)**

The collection of functions whose derivatives are equal to $f(x)$. In other words, the indefinite integral of $f(x)$ is the antiderivative of $f(x)$.

## FORMULAS TO KNOW

**Antiderivative of a Constant**

$$\int k dx = kx + C$$

**Antiderivative of a Constant Multiple of a Function**

$$\int k \cdot f(x) dx = k \cdot \int f(x) dx$$

**Antiderivative of a Difference of Functions**

$$\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$$

**Antiderivative of a Sum of Functions**

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

**Natural Logarithm Rule**

$$\int \frac{1}{x} dx = \ln|x| + C$$

**Power Rule for Antiderivatives**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$$

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# Indefinite Integrals of Functions Requiring Rewriting Before Applying Rules

*by Sophia*

WHAT'S COVERED

In this lesson, you will use algebraic manipulation so that the power rule can be used to find the antiderivative. Specifically, this lesson will cover:

1. Radicals and Powers in the Denominator of the Integrand
2. Performing Multiplication and/or Division First
   a. Performing Multiplication First
   b. Performing Division First

# 1. Radicals and Powers in the Denominator of the Integrand

Let's remind ourselves of the following:

- When you want to take the *derivative* of $f(x)=\sqrt[n]{x}$, you first rewrite $f(x)$ as $x^{1/n}$.
- When you want to take the *derivative* of $g(x)=\dfrac{1}{x^n}$, you first rewrite $g(x)$ as $x^{-n}$.

These rewrites are also used when finding antiderivatives!

BIG IDEA

When the integrand of $\int f(x)dx$ contains a radical and/or a power in a monomial denominator, rewrite these expressions in exponential form so that the power rule may be used.

EXAMPLE Find the indefinite integral: $\int \sqrt[3]{t}\,dt$

We need to rewrite the expression as a power, then find the antiderivative.

| | |
|---|---|
| $\int \sqrt[3]{t}\,dt$ | Start with the original expression. |
| $=\int t^{1/3}dt$ | Rewrite $\sqrt[3]{t}$ as $t^{1/3}$. |

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$$= \frac{1}{\left(\frac{4}{3}\right)} t^{4/3} + C$$ Use the power rule with $n = \frac{1}{3}$: $\frac{1}{3} + 1 = \frac{4}{3}$

$$= \frac{3}{4} t^{4/3} + C$$ Simplify.

Thus, $\int \sqrt[3]{t}\, dt = \frac{3}{4} t^{4/3} + C$.

↪ EXAMPLE Find the indefinite integral: $\int \frac{6}{x^3} dx$

We need to rewrite the expression as a power, then find the antiderivative.

$$\int \frac{6}{x^3} dx$$ Start with the original expression.

$$= \int 6x^{-3} dx$$ Rewrite $\frac{1}{x^3}$ as $x^{-3}$.

$$= 6\left(\frac{x^{-2}}{-2}\right) + C$$ Use the power rule with $n = -3$: $-3 + 1 = -2$

$$= -3x^{-2} + C$$ Simplify.

$$= \frac{-3}{x^2} + C$$ You can also write the final answer using positive exponents.

Thus, $\int \frac{6}{x^3} dx = \frac{-3}{x^2} + C$.

TRY IT

Consider $\int (12\sqrt[3]{t} + 12\sqrt[4]{t}) dt$.

Find the indefinite integral. +

$$9t^{4/3} + \frac{48}{5} t^{5/4} + C$$

↪ EXAMPLE Find the indefinite integral: $\int \left(\frac{3}{\sqrt{x}} - \frac{4}{x^2}\right) dx$

We need to rewrite each expression as a power, then find the antiderivative.

$$\int \left(\frac{3}{\sqrt{x}} - \frac{4}{x^2}\right) dx$$ Start with the original expression.

$$= \int (3x^{-1/2} - 4x^{-2}) dx$$ Rewrite each expression as a power of $x$.
Note: $\frac{3}{\sqrt{x}} = \frac{3}{x^{1/2}} = 3x^{-1/2}$

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| | |
|---|---|
| $= 3\int x^{-1/2}dx - 4\int x^{-2}dx$ | Use the sum/difference rules and the constant multiple rules. |
| $= 3\left(\frac{x^{1/2}}{\left(\frac{1}{2}\right)}\right) - 4\left(\frac{x^{-1}}{-1}\right) + C$ | Apply the power rule to both terms $(n = -\frac{1}{2}$ and $n = -2)$. |
| $= 6x^{1/2} + 4x^{-1} + C$ | Simplify. |
| $= 6\sqrt{x} + \frac{4}{x} + C$ | Rewrite in more “common” form. |

Thus, $\int\left(\frac{3}{\sqrt{x}} - \frac{4}{x^2}\right)dx = 6\sqrt{x} + \frac{4}{x} + C$.

TRY IT

Consider $\int\left(\frac{12}{\sqrt[3]{x}} - \frac{8}{x^5}\right)dx$.

Find the indefinite integral. +

$18x^{2/3} + \frac{2}{x^4} + C$

# 2. Performing Multiplication and/or Division First

Now, we’ll look at examples where algebraic manipulation is required before using the power rule. This could include multiplication and division.

## 2a. Performing Multiplication First

EXAMPLE Consider the indefinite integral $\int(2x-3)(x+5)dx$.

The integrand $(2x-3)(x+5)$ is not a power of *x*, nor is it a sum or difference of powers of *x*.

If we perform the multiplication, the integrand becomes $(2x-3)(x+5) = 2x^2 + 7x - 15$, which is a polynomial, which we learned to antidifferentiate in the last tutorial.

This means we now have $\int(2x-3)(x+5)dx = \int(2x^2 + 7x - 15)dx$.

| | |
|---|---|
| $\int(2x-3)(x+5)dx$ | Start with the original expression. |
| $= \int(2x^2 + 7x - 15)dx$ | Multiply to form the expanded expression. |
| $= 2\int x^2dx + 7\int xdx - \int 15dx$ | Use the sum/difference properties followed by the constant multiple rule. |

$= 2\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) - 15x + C$ Apply the power rule.

$= \frac{2}{3}x^3 + \frac{7}{2}x^2 - 15x + C$ Simplify.

Thus, $\int (2x-3)(x+5)dx = \frac{2}{3}x^3 + \frac{7}{2}x^2 - 15x + C$.

TRY IT

Consider $\int 3x^2\left(2x + \frac{2}{x^2}\right)dx$.

Find the indefinite integral. +

$\frac{3}{2}x^4 + 6x + C$

EXAMPLE Find the definite integral: $\int (4x^2+3)^2 dx$

Here, we can write $(4x^2+3)^2 = (4x^2+3)(4x^2+3) = 16x^4 + 24x^2 + 9$, which is a polynomial. Now, use the power rule and properties to find the indefinite integral.

| | |
|---|---|
| $\int (4x^2+3)^2 dx$ | Start with the original expression. |
| $= \int (16x^4 + 24x^2 + 9)dx$ | Multiply out the square of the binomial. |
| $= 16\int x^4 dx + 24\int x^2 dx + \int 9dx$ | Use the sum/difference properties followed by the constant multiple rule. |
| $= 16\left(\frac{x^5}{5}\right) + 24\left(\frac{x^3}{3}\right) + 9x + C$ | Apply the power rule. |
| $= \frac{16}{5}x^5 + 8x^3 + 9x + C$ | Simplify. |

Thus, $\int (4x^2+3)^2 dx = \frac{16}{5}x^5 + 8x^3 + 9x + C$.

HINT

When the integrand is a quantity raised to a power, there are only certain instances in which performing the multiplication is desirable.

When the power is 2, this is fairly straightforward. For instance, $(2x^3 - 7x)^2 = 4x^6 - 28x^4 + 49x^2$.

If the power is 3, this could take some time, but is still reasonable. For instance,

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$(5x^2+4)^3 = 125x^6+300x^4+240x^2+64.$

For higher whole-number powers, the process is much longer and time-consuming. Consider examples like $(2x+1)^4$ or $(x^2-4)^{10}$. These can be done, but there may be faster methods presented later.

## 2b. Performing Division First

If the integrand has a certain form, division can be used to rewrite so that the power rule can be used.

→ EXAMPLE Perform the division in the expression $\dfrac{3x^2+14x}{x}$.

This expression isn't a power of *x*, nor a sum/difference of powers of *x*. Since the denominator is a single term, we can divide each term in the numerator by the denominator (and some simplification should occur).

$$\frac{3x^2+14x}{x} = \frac{3x^2}{x} + \frac{14x}{x} = 3x+14$$

This expression is a polynomial, which means it can be integrated.

→ EXAMPLE Perform the division in the expression $\dfrac{6x^2-20x^5}{4x^3}$.

This expression isn't a power of *x*, nor a sum/difference of powers of *x*. Once again, since the denominator is a single term, we can divide each term in the numerator by the denominator (and some simplification should occur).

$$\frac{6x^2-20x^5}{4x^3} = \frac{6x^2}{4x^3} - \frac{20x^5}{4x^3} = \frac{3}{2x} - 5x^2$$

This can be integrated.

→ EXAMPLE Find the indefinite integral: $\displaystyle\int \frac{3x^2+14x}{x}dx$

As we saw in the previous example, performing the division gives a sum of powers of *x*.

| | |
|---|---|
| $\displaystyle\int \frac{3x^2+14x}{x}dx$ | Start with the original expression. |
| $\displaystyle= \int (3x+14)dx$ | Perform the division, as seen in the previous example. |
| $\displaystyle= 3\int x dx + \int 14 dx$ | Use the sum property and the constant multiple rule. |
| $\displaystyle= 3\left(\frac{x^2}{2}\right) + 14x + C$ | Apply the power rule. |

$$= \frac{3}{2}x^2 + 14x + C \quad \text{Simplify.}$$

Thus, $\int \frac{3x^2+14x}{x}dx = \frac{3}{2}x^2 + 14x + C$.

Now, let's look at the other expression we simplified in the first example.

↪ EXAMPLE Find the definite integral: $\int \frac{6x^2-20x^5}{4x^3}dx$

As we saw in the previous example, performing the division gives a difference of powers of *x*.

| | |
|---|---|
| $\int \frac{6x^2-20x^5}{4x^3}dx$ | Start with the original expression. |
| $= \int \left(\frac{3}{2x} - 5x^2\right)dx$ | Perform the division, as seen in the previous example. |
| $= \frac{3}{2}\int \frac{1}{x}dx - 5\int x^2 dx$ | Use the difference property followed by the constant multiple rule. |
| $= \frac{3}{2}\ln\lvert x\rvert - 5\left(\frac{x^3}{3}\right) + C$ | Apply the natural logarithm rule and power rule. |
| $= \frac{3}{2}\ln\lvert x\rvert - \frac{5}{3}x^3 + C$ | Simplify. |

Thus, $\int \frac{6x^2-20x^5}{4x^3}dx = \frac{3}{2}\ln\lvert x\rvert - \frac{5}{3}x^3 + C$.

TRY IT

Consider $\int \frac{6x^4+1}{3x^3}dx$.

Find the indefinite integral. +

$$x^2 - \frac{1}{6x^2} + C$$

## SUMMARY

In this lesson, you learned that when there are **radicals and powers in the denominator of the integrand** of $\int f(x)dx$, it is sometimes necessary to rewrite these expressions in exponential form so that the power rule may be used. You also explored examples where algebraic manipulation is required before using the power rule, including **performing multiplication first** and **performing division first**.

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# Indefinite Integrals of Trigonometric Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will use the derivative rules for trigonometric functions to establish antiderivative rules for some trigonometric functions, then use these rules in combination with the power rule to find antiderivatives. Specifically, this lesson will cover:

1. Derivative Rules and Associated Antiderivative Rules
2. Indefinite Integrals Involving Trigonometric Functions
3. Using Trigonometric Identities to Find Indefinite Integrals

# 1. Derivative Rules and Associated Antiderivative Rules

Recall the derivative rules for the six trigonometric functions:

- $D[\sin x] = \cos x$
- $D[\cos x] = -\sin x$
- $D[\tan x] = \sec^2 x$
- $D[\csc x] = -\csc x \cot x$
- $D[\sec x] = \sec x \tan x$
- $D[\cot x] = -\csc^2 x$

These lead to the following antiderivative rules:

FORMULA

**Antiderivative of sin*x***

$$\int \sin x\,dx = -\cos x + C$$

**Antiderivative of cos*x***

$$\int \cos x\,dx = \sin x + C$$

**Antiderivative of $\sec^2 x$**

$$\int \sec^2 x\,dx = \tan x + C$$

**Antiderivative of $\csc^2 x$**

$$\int \csc^2 x\,dx = -\cot x + C$$

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**Antiderivative of** $\sec x \tan x$

$$\int \sec x \tan x dx = \sec x + C$$

**Antiderivative of** $\csc x \cot x$

$$\int \csc x \cot x dx = -\csc x + C$$

---

# 2. Indefinite Integrals Involving Trigonometric Functions

Let's use these new rules to find some antiderivatives:

EXAMPLE Find the indefinite integral: $\int (4\cos t - 3\sin t) dt$

| | |
|---|---|
| $\int (4\cos t - 3\sin t) dt$ | Start with the original expression. |
| $= 4\int \cos t dt - 3\int \sin t dt$ | Use the sum/difference and constant multiples rules. |
| $= 4(\sin t) - 3(-\cos t) + C$ | Use antiderivative formulas for $\sin t$ and $\cos t$. |
| $= 4\sin t + 3\cos t + C$ | Simplify. |

In conclusion, $\int (4\cos t - 3\sin t) dt = 4\sin t + 3\cos t + C$.

TRY IT

Consider $\int (2\sec x \tan x + 3\csc^2 x) dx$.

Find the indefinite integral. +

$2\sec x - 3\cot x + C$

Here are some practice problems that also require the rules you've already learned.

TRY IT

Consider $\int (-2\sin x - 3x) dx$.

Find the indefinite integral. +

$2\cos x - \frac{3}{2}x^2 + C$

TRY IT

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Consider $\int\left(\frac{2}{t^3} - \sec t \tan t\right)dt$.

Find the indefinite integral. +

$\frac{-1}{t^2} - \sec t + C$

---

# 3. Using Trigonometric Identities to Find Indefinite Integrals

Consider the indefinite integral $\int(\sin^2\theta + \cos^2\theta)d\theta$. According to rules we know so far, we are unable to find $\int \sin^2\theta d\theta$ or $\int \cos^2\theta d\theta$. However, you may recall the identity $\sin^2\theta + \cos^2\theta = 1$. This means that $\int(\sin^2\theta + \cos^2\theta)d\theta = \int 1 d\theta = \theta + C$.

Thus, sometimes it is possible to rewrite the integrand using a (trigonometric) identity so that an indefinite integral formula can be used.

EXAMPLE Find the indefinite integral: $\int \tan^2 x dx$

While $\tan^2 x$ doesn't have an integration formula, we can apply a trigonometric identity.

| | |
|---|---|
| $\int \tan^2 x dx$ | Start with the original expression. |
| $= \int(\sec^2 x - 1)dx$ | Use the identity $1 + \tan^2 x = \sec^2 x$. |
| $= \int \sec^2 x dx - \int 1 dx$ | Apply the difference property. |
| $= \tan x - x + C$ | Use formulas for $\int \sec^2 x dx$ and $\int 1 dx$. |

Thus, $\int \tan^2 x dx = \tan x - x + C$.

THINK ABOUT IT

Notice that the antiderivatives of $\sin x$ and $\cos x$ are known, but the other four are not. It turns out that the antiderivatives of the other four trigonometric functions require more advanced techniques that are covered later in this challenge. This speaks to a larger idea that the processes for derivatives and antiderivatives can be similar, or they can be quite different.

Here is a derivative to think about for a later challenge.

- $f(x) = \ln(\sec x)$

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- $f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$, using $D[\ln u] = \frac{1}{u} \cdot u'$

The significance is that we now have a function whose derivative is tan*x*. In other words, $\int \tan x dx = \ln|\sec x| + C$. You do not need to know this (yet)—this is just something to think about!

## SUMMARY

In this lesson, you reviewed the **derivative rules** for the six trigonometric functions and learned about their **associated antiderivative rules**. Then, you were able to apply these rules to find**indefinite integrals (antiderivatives) involving trigonometric functions**, expanding on the antiderivatives that you are able to find. You also learned that it is possible to rewrite the integrand **using a trigonometric identity to find indefinite integrals**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## FORMULAS TO KNOW

**Antiderivative of cos x**

$$\int \cos x dx = \sin x + C$$

**Antiderivative of csc x cot x**

$$\int \csc x \cot x dx = -\csc x + C$$

**Antiderivative of csc² x**

$$\int \csc^2 x dx = -\cot x + C$$

**Antiderivative of sec x tan x**

$$\int \sec x \tan x dx = \sec x + C$$

**Antiderivative of sec² x**

$$\int \sec^2 x dx = \tan x + C$$

**Antiderivative of sin x**

$$\int \sin x dx = -\cos x + C$$

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# Indefinite Integrals of Exponential Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will find antiderivatives of exponential functions and incorporate them into the antiderivatives we already know (powers and trigonometric functions). Specifically, this lesson will cover:

1. Antiderivatives of Exponential Functions
2. Antiderivatives of Functions Containing Exponential Functions

# 1. Antiderivatives of Exponential Functions

Recall that $D[e^x] = e^x$ and $D[a^x] = a^x \cdot \ln a$, assuming $a > 0$. This leads to the following antiderivative formulas:

FORMULA

**Antiderivatives of Exponential Functions**

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

---

# 2. Antiderivatives of Functions Containing Exponential Functions

Let's get into some examples.

EXAMPLE Find the indefinite integral: $\int (4e^x - 2x + 1)dx$

Note that the same properties can be used.

| | |
|---|---|
| $\int (4e^x - 2x + 1)dx$ | Start with the original expression. |
| $= 4\int e^x dx - 2\int x dx + \int 1 dx$ | Use the sum/difference properties and the constant multiple rule. |
| $= 4(e^x) - 2\left(\frac{x^2}{2}\right) + x + C$ | Apply exponential and power rules. |
| | Simplify. |

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$$= 4e^x - x^2 + x + C$$

Thus, $\int (4e^x - 2x + 1)dx = 4e^x - x^2 + x + C$.

↱ EXAMPLE Find the indefinite integral: $\int \left(3^x - \frac{2}{3}\sin x\right)dx$

| | |
|---|---|
| $\int \left(3^x - \frac{2}{3}\sin x\right)dx$ | Start with the original expression. |
| $= \int 3^x dx - \frac{2}{3}\int \sin x dx$ | Use the sum/difference properties and the constant multiple rule. |
| $= \frac{3^x}{\ln 3} - \frac{2}{3}(-\cos x) + C$ | Apply formulas for $\int a^x dx$ and $\int \sin x dx$. |
| $= \frac{3^x}{\ln 3} + \frac{2}{3}\cos x + C$ | Simplify. |

Thus, $\int \left(3^x - \frac{2}{3}\sin x\right)dx = \frac{3^x}{\ln 3} + \frac{2}{3}\cos x + C$.

TRY IT

Consider $\int (x^2 - 5e^x)dx$.

Find the indefinite integral. +

$$\frac{1}{3}x^3 - 5e^x + C$$

TRY IT

Consider $\int \left(10^x - \frac{6}{\sqrt{x}}\right)dx$.

Find the indefinite integral. +

$$\frac{10^x}{\ln 10} - 12\sqrt{x} + C$$

## SUMMARY

In this lesson, you learned the formula for **antiderivatives of exponential functions**, expanding your abilities to find derivatives to include finding **antiderivatives of functions containing exponential functions**.

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**Antiderivatives of Exponential Functions**

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

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# Changing the Variable: u-substitution with Power Rule

*by Sophia*

WHAT'S COVERED

In this lesson, you will explore indefinite integrals of composite functions. Specifically, this lesson will cover:

1. Introduction to *u*-Substitution/Review of Chain Rule
2. Using *u*-Substitution With the Power Rule

# 1. Introduction to *u*-Substitution/Review of Chain Rule

One way to view *u*-substitution is "undoing the chain rule." Consider the function $f(x)=(x^2+1)^9$, which is a composite function.

Taking the derivative, we get $f'(x)=9(x^2+1)^8\cdot 2x=18x(x^2+1)^8$.

It follows that $f(x)=(x^2+1)^9+C$ is the antiderivative of $f'(x)=18x(x^2+1)^8$.

In other words, $\int 18x(x^2+1)^8dx=(x^2+1)^9+C$.

The big question: How do we get the antiderivative without having to guess? The answer to this question lies in the very way we find the derivative.

Let's look again at $f(x)=(x^2+1)^9$.

When we first learned the chain rule, we let $u=x^2+1$. Then, $f(u)=u^9$, and $f'(u)=9u^8\cdot\dfrac{du}{dx}$.

As you can see, the derivative is less complicated when the "*u*" is used. You can focus on the "inner" function. While you may have gotten used to not using the "*u*" idea in derivatives, it is very useful for antiderivatives. Let's walk through an example.

> EXAMPLE Find the indefinite integral: $\int\sqrt{4x^2+3}\cdot 8xdx$
>
> In the spirit of the chain rule, let $u=4x^2+3$.

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Since a chain rule derivative also contains $\frac{du}{dx}$, we'll find that as well: $\frac{du}{dx}=8x$

Notice that the integrand is written in differential form (a function multiplied by *dx*).

To align with that, we'll write $\frac{du}{dx}=8x$ in differential form: $du=8xdx$

The goal is to get an integral that has *u* as its only variable. With these substitutions, $4x^2+3$ gets replaced by *u* and $8xdx$ is replaced by *du*.

This means the integral can now be written as $\int\sqrt{u}\,du$, which is much simpler, and it is clearer what to do:

| | |
|---|---|
| $\int\sqrt{u}\,du$ | Start with the original expression. |
| $=\int u^{1/2}du$ | Rewrite as a power so that the power rule can be used. |
| $=\frac{u^{3/2}}{\left(\frac{3}{2}\right)}+C$ | Apply the power rule with $n=\frac{1}{2}$. |
| $=\frac{2}{3}u^{3/2}+C$ | Simplify. |
| $=\frac{2}{3}(4x^2+3)^{3/2}+C$ | Replace *u* with $4x^2+3$. |

The last step is the key step. The original function was in terms of *x*, which means that the final answer should also be in terms of *x*. The *u*-substitution was more or less used to help us to get organized.

Thus, $\int\sqrt{4x^2+3}\cdot 8xdx=\frac{2}{3}(4x^2+3)^{3/2}+C$.

This is the essence with *u*-substitution. When you identify an antiderivative that requires *u*-substitution, here is what you do:

**STEP BY STEP**

1. Identify the "inner" function, which is the function within another function. For now, it will be the function that is raised to a power (or under a radical). If we call the inside function $g(x)$, then let $u=g(x)$.
2. Find the differential $du=g'(x)dx$. This will only work if $g'(x)$ or some multiple of $g'(x)$ is in the integrand before the substitutions are made.
3. Substitute *u* and *du* into the integral so that the integral has *u* as its only variable.
4. Find the antiderivative with respect to *u* (and don't forget +*C*!).
5. Replace $u=g(x)$ in the antiderivative. This is called back-substitution.

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# 2. Using *u*-Substitution With the Power Rule

Now that we have a process, let's look at some examples.

↱ EXAMPLE Find the indefinite integral: $\int 2x(x^2+3)^7 dx$

| | |
|---|---|
| $\int 2x(x^2+3)^7 dx$ | Start with the original expression. |
| $= \int u^7 du$ | Make the substitution: $u = x^2+3$<br>Find the differential: $du = 2xdx$<br>Replace $x^2+3$ with *u* and $2xdx$ with *du*. |
| $= \frac{1}{8}u^8 + C$ | Apply the power rule with $n = 7$. |
| $= \frac{1}{8}(x^2+3)^8 + C$ | Back-substitute $u = x^2+3$. |

Thus, $\int 2x(x^2+3)^7 dx = \frac{1}{8}(x^2+3)^8 + C$.

The next example will illustrate what happens when *du* is not exactly in the integral, but is a constant multiple.

WATCH

In this video, we will find $\int x^2(4x^3+5)^6 dx$.

### Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do in this video is find an antiderivative that requires the use of u substitution. In other words, reversing the chain rule. So we have the antiderivative of x squared times the quantity, 4x to the third plus 5 raised to the sixth power. What signals us to use u substitution for this is that we have a function within a function. In this case, it's a quantity that's being raised to a power. That's what's telling us that it's a composite function.

So we are going to let u equal 4x to the 1/3 plus 5. Then the differential form is du equals 12x squared dx. Now, notice that we do not have a 12x squared dx in the integral. We just have x squared dx. That turns out to be the most important part. We can work with constants, it's just the variable piece does have to be there. Since x squared dx is in the integral, I am going to go ahead and isolate-- Somehow my x got cut off there. There we go. I'm going to isolate x squared dx to one side.

So this means that 1/12 du is equal to x squared dx. So then u equals 4x to the 1/3 plus 5 is going to replace this piece, and x squared dx becomes 1/12 du. We're going to have a whole brand new integral with u as the variable instead of x.

We have the integral of u to the sixth and x squared dx is 1/12 d u. That integral sign is a little bit too far away from my taste. How about that? So now we know that the 1/12 can go outside of the integral by the constant multiple rule. So we have 1/12 integral of u to the sixth du. Now this is where the power rule is

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going to be used. So we have 1/12 times 1/7 u to the seventh plus c, which is 1 over 84 u to the seventh plus c.

Then u is 4x to the 1/3 plus 5. We need to remember to write our answer in terms of the original variable. So we have 1 over 84, 4x to the 1/3 plus 5 to the seventh plus c, which of course, can also be written as 4x to the 1/3 plus 5 to the seventh, all over 84, plus c. So either one of those two forms is acceptable for a final answer. There is an integral using u substitution.

[MUSIC PLAYING]

↪ EXAMPLE Find the indefinite integral: $\int \sqrt{8x+1}\,dx$

| | |
|---|---|
| $\int \sqrt{8x+1}\,dx$ | Start with the original expression. |
| $= \int \frac{1}{8}\sqrt{u}\,du$ | First, make the substitution: $u = 8x+1$<br>Write the differential: $du = 8dx$<br>Solve for *dx*: $dx = \frac{1}{8}du$<br>Replace $8x+1$ with *u* and *dx* with $\frac{1}{8}du$. |
| $= \frac{1}{8}\int \sqrt{u}\,du$ | Move the constant $\frac{1}{8}$ outside the integral sign. |
| $= \frac{1}{8}\int u^{1/2}du$ | Rewrite as a power. |
| $= \frac{1}{8}\frac{u^{3/2}}{\left(\frac{3}{2}\right)}+C$ | Use the power rule with $n = \frac{1}{2}$. |
| $= \frac{1}{12}u^{3/2}+C$ | Simplify. |
| $= \frac{1}{12}(8x+1)^{3/2}+C$ | Back-substitute $u = 8x+1$. |

Thus, $\int \sqrt{8x+1}\,dx = \frac{1}{12}(8x+1)^{3/2}+C$.

In the next example, we'll look at a power in the denominator.

↪ EXAMPLE Find the indefinite integral: $\int \frac{2x+1}{(x^2+x+4)^5}dx$

| | |
|---|---|
| $\int \frac{2x+1}{(x^2+x+4)^5}dx$ | Start with the original expression. |
| $= \int \frac{1}{u^5}du$ | First, make the substitution: $u = x^2+x+4$<br>Write the differential: $du = (2x+1)dx$<br>Replace $x^2+x+4$ with *u* and $(2x+1)dx$ with *du*. |

Note: $dx$ is multiplied by the expression, which means it is multiplied by the numerator.

| | |
|---|---|
| $=\int u^{-5}du$ | Rewrite as a negative power so that the power rule can be used. |
| $=\frac{1}{-4}u^{-4}+C$ | Apply the power rule with $n=-5$. |
| $=\frac{-1}{4}u^{-4}+C$ | Simplify. |
| $=\frac{-1}{4}(x^2+x+4)^{-4}+C$ | Back-substitute $u=x^2+x+4$. |
| $=\frac{-1}{4(x^2+x+4)^4}+C$ | Write in terms of positive exponents if desired or directed. |

Thus, $\int\frac{2x+1}{(x^2+x+4)^5}dx=\frac{-1}{4(x^2+x+4)^4}+C$.

TRY IT

Consider $\int\frac{4x}{(x^2+8)^{3/4}}dx$.

Find the indefinite integral. +

$8(x^2+8)^{1/4}+C$

Finally, here is an example where there doesn't appear to be an inner function.

EXAMPLE Find the indefinite integral: $\int\frac{4x}{8x^2+1}dx$

At first glance, it looks like we could manipulate the integrand, but since the denominator has more than one term, this is not possible. Taking a closer look, notice that the substitution $u=8x^2+1$ has differential $du=16xdx$, which is a constant multiple of $4x$ (which is in the integral). This is the direction we'll go.

| | |
|---|---|
| $\int\frac{4x}{8x^2+1}dx$ | Start with the original expression. |
| $=\int\frac{1}{u}\cdot\frac{1}{4}du$ | First, make the substitution: $u=8x^2+1$<br>Write the differential: $du=16xdx$<br>Solve for $x\cdot dx$: $xdx=\frac{1}{16}du$.<br>Replace $8x^2+1$ with $u$ and $dx$ with $\frac{1}{16}du$.<br>Note: $dx$ is multiplied by the expression, which means it is multiplied by the numerator.<br>Note: $4xdx=4\left(\frac{1}{16}du\right)=\frac{1}{4}du$<br>We isolate $xdx$ since the goal is to rewrite the integrand in terms of $u$ |

and *du*.

$=\frac{1}{4}\int\frac{1}{u}du$ Move the constant $\frac{1}{4}$ outside the integral sign.

$=\frac{1}{4}\ln|u|+C$ Apply the natural logarithm rule.

$=\frac{1}{4}\ln|8x^2+1|+C$ Back-substitute $u=8x^2+1$.

$=\frac{1}{4}\ln(8x^2+1)+C$ It is worth mentioning that since $8x^2+1$ is positive for all real numbers, there is no need to use absolute value. That said, it is not incorrect to use absolute value, but it is not necessary in this case.

Thus, $\int\frac{4x}{8x^2+1}dx=\frac{1}{4}\ln(8x^2+1)+C$.

The substitution method isn't exclusively used for reversing the chain rule. It can also be used to rewrite expressions that could not otherwise be manipulated for antidifferentiation.

→ EXAMPLE Find the indefinite integral: $\int x(x+3)^{15}dx$

Since we certainly don't want to multiply $(x+3)^{15}$ out, we'll try a substitution.

$\int x(x+3)^{15}dx$ Start with the original expression.

$=\int(u-3)\cdot u^{15}du$ Make the substitution: $u=x+3$
Use the differential form: $du=dx$
At this point, there is no replacement for the "*x*" in front. Since $x+3$ will be replaced with *u*, we need a replacement for *x*.
From the substitution $u=x+3$, we can solve for *x* to obtain $x=u-3$.
$x\to u-3,\ x+3\to u,\ dx\to du$

$=\int(u^{16}-3u^{15})du$ Distribute $u^{15}$.

$=\frac{1}{17}u^{17}-\frac{3}{16}u^{16}+C$ Apple the power rule and combine constants.

$=\frac{1}{17}(x+3)^{17}-\frac{3}{16}(x+3)^{16}+C$ Back-substitute $u=x+3$.

Thus, $\int x(x+3)^{15}dx=\frac{1}{17}(x+3)^{17}-\frac{3}{16}(x+3)^{16}+C$.

TRY IT

Consider $\int x(x-9)^{10}dx$.

Find the indefinite integral. +

$\frac{1}{12}(x-9)^{12}+\frac{9}{11}(x-9)^{11}+C$

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WATCH

In this video, we'll find $\int \frac{(2+\ln x)^3}{x}dx$.

## Video Transcription

[MUSIC PLAYING] Hi there. Welcome back. What we're going to do in this video is look at another example of an antiderivative that needs to be found using u-substitution. So, looking at this antiderivative, we have 2 plus the natural log of x quantity raised to the third divided by x. And this is what we're trying to find the antiderivative of. So our convention for u-substitution is we let u equal the quantity that's being raised to the power of the function that's within the function.

So you have 2 plus the natural log of x. Oh, and that's it. So then the differential is 0 plus 1 over x dx. Now, it might not be apparent right away that is in the integral. So I'm going to rewrite this integral another way. I'm going to write 2 plus natural log of x raised to the third times 1 over x dx. And now it's very clear where everything belongs. This right here is going to be du. And this right here is going to be u to the third. So that means our integral successfully converts to an integral with only u as the variable.

So we have u to the third du. And now the power rule can be used. So this means this is 1/4 u to the fourth plus c. And then we're back substituting because, again, we want our answer to be in terms of the original variable, which is 1/4 2 plus natural log x to the fourth plus c. And that is our antiderivative. Remember, to check an antiderivative answer, you can find its derivative. And as long as you do everything correctly, you will end up with the integrand. So there's our antiderivative-- 1/4 quantity 2 plus natural log x raised to the fourth plus an arbitrary constant.

[MUSIC PLAYING]

TRY IT

Consider $\int \frac{\ln x}{x}dx$.

Find the indefinite integral. +

$\frac{1}{2}(\ln x)^2 + C$

SUMMARY

In this lesson, you learned about the ***u*-substitution** method, which is primarily used to reverse **the chain rule**. When the $u$ is used, the derivative is less complicated, allowing you to focus on the “inner” function. You also practiced **using *u*-substitution with the power rule**. As you learned, $u$-substitution isn't exclusively used for reversing the chain rule; it can also be used to “rearrange” the expression so that it can be manipulated for antidifferentiation.

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# Changing the Variable: u-Substitution with Trigonometric Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will continue using substitutions to find antiderivatives, but now we will focus on the trigonometric functions. Specifically, this lesson will cover:

1. The Inner Function Is Trigonometric
2. The Outer Function Is Trigonometric

# 1. The Inner Function Is Trigonometric

Since we have already been through *u*-substitution, there really is nothing too different about these indefinite integrals. The idea is the same, it's just that we have to use rules for trigonometric functions now.

Let's take a look at an example.

EXAMPLE Find the indefinite integral: $\int \sin^2 x \cos x dx$

First, note that the integral can be written $\int (\sin x)^2 \cos x dx$. At this point, notice that sin*x* is the "inner" function since it is raised to a power. Notice also that its derivative, cos*x*, is also in the integral. This means that *u*-substitution should work!

| | |
|---|---|
| $\int \sin^2 x \cos x dx$ | Start with the original expression. |
| $= \int u^2 du$ | First, make the substitution: $u = \sin x$<br>Write the differential: $du = \cos x dx$<br>Replace $\sin x$ with *u* and $\cos x dx$ with *du*. |
| $= \frac{1}{3}u^3 + C$ | Find the antiderivative with respect to *u*. |
| $= \frac{1}{3}\sin^3 x + C$ | Back-substitute $u = \sin x$. |

Thus, $\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + C$.

TRY IT

Consider $\int \frac{\sin x}{\sqrt{\cos x}} dx$.

Find the indefinite integral.

+

$-2\sqrt{\cos x} + C$

We can also use substitution to find antiderivatives of certain trigonometric functions.

WATCH

In this video, we'll find $\int \tan x dx$.

## Video Transcription

[MUSIC PLAYING] Hello, there, and welcome back. What we're going to do in this video is find an antiderivative of tangent of x. As you see here we have the integral of tangent of xdx. So the first thing we'll do since it's not quite clear what the antiderivative of tangent is, we're going to rewrite tangent as sine x over cosine x. While it might be becoming a little bit more clear we're still going to do one more rewrite. I'm going to rewrite this as 1 over cosine x times sine xdx.

Now, if you look at the function the way it's written now the cosine x is actually part of a bigger function. We have 1 over something. So here it makes sense to do a u substitution with cosine, but before doing so we need to make sure that its derivative is inside the integral in some form as well. Remember that the derivative of cosine is negative sine. So we essentially have the sine of x, but we're going to have to adjust for a negative. So u substitution seems to be a good strategy to use here.

So we're going to let u equal cosine of x because then we get 1 over u. Then du is negative sine of xdx, the differential form. Since we don't have a negative inside of the integral sign we're going to get sine of xdx by itself by multiplying both sides by negative 1. So you have negative du equals sine of xdx. Now to make our substitution we have the integral of 1 over u times negative du. Since that negative is really a constant we're going to write it on the outside of the integral sign. It's really a negative 1 getting multiplied.

Now, remember that the antiderivative of 1 over u is the natural log of the absolute value of u plus a constant. Now we back substitute, u is equal to cosine of x. So we have the natural log of the absolute value of cosine of x plus c-- negative natural log of absolute value of cosine of x plus c. Sorry, I misspoke there.

Now, that is a perfectly fine form for the antiderivative, but we can take it one step further. Remember that the negative out front of the natural log is really a negative 1 multiplied, so by property of logarithms we can write the natural log of the absolute value of cosine x to the negative 1 plus c. Remember negative 1 means reciprocal, and remember that the reciprocal of cosine is secant. So we can also write this as the natural log of secant x plus c.

So that means we have two forms for the antiderivative. I'm just going to write the other one here right next to it. Either one of these is an acceptable form for the antiderivative of tangent of x. So there we have it, natural log of absolute value of secant x, that one appeases the crowd that doesn't like

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negatives in expressions. And we have the negative natural log of absolute value of cosine x plus c, that one appeases the crowd that likes to write trig functions in terms of sines and cosine. So there's something for everyone here.

[MUSIC PLAYING]

TRY IT

Consider $\int \cot x dx$.

Find the indefinite integral. +

$\ln|\sin x| + C$

TRY IT

Consider $\int (2+\tan x)^4 \sec^2 x dx$.

Find the indefinite integral. +

$\frac{1}{5}(2+\tan x)^5 + C$

---

# 2. The Outer Function Is Trigonometric

EXAMPLE Find the indefinite integral: $\int \cos(4x)dx$

| | |
|---|---|
| $\int \cos(4x)dx$ | Start with the original expression. |
| $= \int \cos u \cdot \frac{1}{4}du$ | Make the substitution: $u=4x$<br>Find the differential: $du=4dx$<br>Solve for *dx*: $dx=\frac{1}{4}du$<br>Replace $4x$ with *u* and *dx* with $\frac{1}{4}du$. |
| $=\frac{1}{4}\int \cos u du$ | Move the constant $\frac{1}{4}$ outside the integral sign. |
| $=\frac{1}{4}\sin u + C$ | Use the antiderivative rule for $\cos x$. |
| $=\frac{1}{4}\sin(4x)+C$ | Back-substitute $u=4x$. |

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Thus, $\int \cos(4x)dx = \frac{1}{4}\sin(4x) + C$.

↪ EXAMPLE Find the indefinite integral: $\int x\sec(3x^2)\tan(3x^2)dx$

| | |
|---|---|
| $\int x\sec(3x^2)\tan(3x^2)dx$ | Start with the original expression. |
| $= \int \sec u \tan u \cdot \frac{1}{6}du$ | Make the substitution: $u = 3x^2$<br>Find the differential: $du = 6xdx$<br>Solve for *xdx*: $xdx = \frac{1}{6}du$<br>Replace $3x^2$ with *u* and *xdx* with $\frac{1}{6}du$. |
| $= \frac{1}{6}\int \sec u \tan u du$ | Move the constant $\frac{1}{6}$ outside the integral sign. |
| $= \frac{1}{6}\sec u + C$ | Use the antiderivative rule for $\sec x \tan x$. |
| $= \frac{1}{6}\sec(3x^2) + C$ | Back-substitute $u = 3x^2$. |

Thus, $\int x\sec(3x^2)\tan(3x^2)dx = \frac{1}{6}\sec(3x^2) + C$.

TRY IT

Consider $\int \frac{\sin(4\ln x)}{x}dx$.

Find the indefinite integral. +

$\frac{-1}{4}\cos(4\ln x) + C$

THINK ABOUT IT

Consider the antiderivative $\int \cos(x^2)dx$.

A first instinct is to let $u = x^2$, then $du = 2xdx$, which means $xdx = \frac{1}{2}du$. Here is the problem: the original integral only has a "*dx*" term, not an "*x dx*" term. From the substitution, we could also write $x = \sqrt{u}$. Let's see where that takes us.

$xdx = \frac{1}{2}du$ becomes $\sqrt{u}\,dx = \frac{1}{2}du$, so $dx = \frac{1}{2\sqrt{u}}du$.

Making all the substitutions, our integral becomes $\int \frac{\cos u}{\sqrt{u}}du$, which is much more complicated. As it turns

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out, there is no substitution that would solve $\int \cos(x^2)dx$ because there is no antiderivative for $f(x) = \cos(x^2)$. There are several other functions that do not have antiderivatives.

So, when making substitutions, be careful that all the variables are covered in your substitution. If they aren't, you either should check your work, try another substitution, or it is possible that the antiderivative doesn't exist.

## SUMMARY

In this lesson, you learned how to apply your knowledge of $u$-substitution to find indefinite integrals when **the inner function is trigonometric** and **the outer function is trigonometric**. With the addition of trigonometric functions, your abilities to find antiderivatives expands even further. As you saw in one example, however, there are several functions that do not have antiderivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

# Changing the Variable: u-Substitution with Exponential Functions

*by Sophia*

WHAT'S COVERED

In this lesson, you will find antiderivatives of composite functions that involve exponential functions. Specifically, this lesson will cover:

1. Indefinite Integrals Where the Inner Function Is Exponential
2. Indefinite Integrals Where the Outer Function Is Exponential

# 1. Indefinite Integrals Where the Inner Function Is Exponential

Even though we talked about exponential functions with base "*a*" in the past, we will focus on exponential functions with base *e* for the sake of finding antiderivatives.

Recall the derivative formula: $D[e^u] = e^u \cdot u'$

These will be useful in making substitutions where the inside function is exponential.

EXAMPLE Find the indefinite integral: $\int e^x \sin(e^x)dx$

Note that $e^x$ is the inner function, and it is also a factor in the integrand. This means a *u*-substitution will work.

| | |
|---|---|
| $\int e^x \sin(e^x)dx$ | Start with the original expression. |
| $= \int \sin u du$ | Make the substitution: $u = e^x$<br>Find the differential: $du = e^x dx$<br>Replace the inner $e^x$ with *u* and *dx* with $e^u du$. |
| $= -\cos u + C$ | Use the antiderivative rule for $\sin u$. |
| $= -\cos(e^x) + C$ | Back-substitute $u = e^x$. |

Thus, $\int e^x \sin(e^x)dx = -\cos(e^x) + C$.

HINT

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When making the substitution $u = e^x$, it can be difficult to know where the substitutions go in the integral.

Remember, the goal is to make $\int (expression)dx$ look like $\int (expression)du$. Therefore, the term with the $du$ should always be outside the function, and $u$ should always go inside the function.

Here is one to try on your own.

TRY IT

Consider $\int e^x\sqrt{e^x+4}\,dx$.

Find the indefinite integral. +

$\frac{2}{3}(e^x+4)^{3/2}+C$

Here is an example with a more complicated substitution.

EXAMPLE Find the indefinite integral: $\int \frac{e^{4x}}{e^{4x}+20}dx$

| | |
|---|---|
| $\int \frac{e^{4x}}{e^{4x}+20}dx$ | Start with the original expression. |
| $= \int \frac{1}{u}\cdot\frac{1}{4}du$ | Make the substitution: $u = e^{4x}+20$<br>Find the differential: $du = 4e^{4x}dx$<br>Solve for $e^{4x}dx$: $e^{4x}dx = \frac{1}{4}du$<br>Then, make the following replacements: $e^{4x}+20 \to u$, $e^{4x}dx \to \frac{1}{4}du$ |
| $= \frac{1}{4}\int \frac{1}{u}du$ | Move the constant $\frac{1}{4}$ outside the integral sign. |
| $= \frac{1}{4}\ln\|u\|+C$ | Use the antiderivative rule for $\frac{1}{u}$. |
| $= \frac{1}{4}\ln\left\|e^{4x}+20\right\|+C$ | Back-substitute $u = e^{4x}+20$. |

Thus, $\int \frac{e^{4x}}{e^{4x}+20}dx = \frac{1}{4}\ln\left|e^{4x}+20\right|+C$. It's worth noting that since $e^{4x}+20$ is positive for all real numbers $x$, the antiderivative can be written without the use of absolute value. That is, $\int \frac{e^{4x}}{e^{4x}+20}dx = \frac{1}{4}\ln(e^{4x}+20)+C$.

TRY IT

Consider $\int \frac{3e^{8x}}{(e^{8x}+15)^3}dx$.

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Find the indefinite integral. +

$\frac{-3}{16}(e^{8x}+15)^{-2}+C$ or $\frac{-3}{16(e^{8x}+15)^2}+C$

---

# 2. Indefinite Integrals Where the Outer Function Is Exponential

Now we will move on to antiderivatives where the exponential is the outer function. Let's look at an example to see how this is different:

↱ EXAMPLE Find the indefinite integral: $\int e^{3x}dx$

Note: the inner function is "3*x*" since we know the antiderivative of $e^u$. This is where we start:

| | |
|---|---|
| $\int e^{3x}dx$ | Start with the original expression. |
| $=\int e^u \cdot \frac{1}{3}du$ | Make the substitution: $u=3x$<br>Find the differential: $du=3dx$<br>Solve for *dx*: $dx=\frac{1}{3}du$ |
| $=\frac{1}{3}\int e^u du$ | Move the constant $\frac{1}{3}$ outside the integral sign. |
| $=\frac{1}{3}e^u+C$ | Use the antiderivative rule for $e^u$. |
| $=\frac{1}{3}e^{3x}+C$ | Back-substitute $u=3x$. |

Thus, $\int e^{3x}dx=\frac{1}{3}e^{3x}+C$.

This result is used so often that it might be handy to remember this formula:

FORMULA

**Antiderivative of $e^{kx}$, Where *k* is a Constant**

$\int e^{kx}dx=\frac{1}{k}e^{kx}+C$

TRY IT

Consider $\int e^{-x}dx$.

Find the indefinite integral. +

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$-e^{-x}+C$

Here is another more complicated example.

↪ EXAMPLE Find the indefinite integral: $\int 12x^2e^{-6x^3}dx$

Note: the inner function is "$-6x^3$" since we know the antiderivative of $e^u$. This is where we start:

| | |
|---|---|
| $\int 12x^2e^{-6x^3}dx$ | Start with the original expression. |
| $=\int e^u\cdot\left(-\frac{2}{3}\right)du$ | Make the substitution: $u=-6x^3$<br>Find the differential: $du=-18x^2dx$<br>Solve for $x^2dx$: $x^2dx=\frac{-1}{18}du$<br>Then, make the following replacements: $-6x^3\rightarrow u,\ x^2dx\rightarrow\frac{-1}{18}du$<br>Note: $12\left(\frac{-1}{18}du\right)=\frac{-2}{3}du$ |
| $=-\frac{2}{3}\int e^u du$ | Move the constant $-\frac{2}{3}$ outside the integral sign. |
| $=\frac{-2}{3}e^u+C$ | Use the antiderivative rule for $e^u$. |
| $=\frac{-2}{3}e^{-6x^3}+C$ | Back-substitute $u=-6x^3$. |

Thus, $\int 12x^2e^{-6x^3}dx=\frac{-2}{3}e^{-6x^3}+C$.

TRY IT

Consider $\int e^{-2\cos x}(\sin x)dx$.

Find the indefinite integral. +

$\frac{1}{2}e^{-2\cos x}+C$

## SUMMARY

In this lesson, you learned how to use *u*-substitution to find **indefinite integrals where the inner function is exponential** and **the outer function is exponential**. With exponential functions added, this expands your capabilities for finding more antiderivatives. You now have a sizable toolbox from which to apply antiderivatives, which is what we'll do in the next tutorial and in Challenge 5.4.

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## FORMULAS TO KNOW

**Antiderivative of $e^{kx}$, Where k is a Constant**

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$$

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# Solving y' = f(x)

*by Sophia*

WHAT'S COVERED

In this lesson, you will solve differential equations of the form $y' = f(x)$. Specifically, this lesson will cover:

1. What Is a Differential Equation?
2. Solving a Differential Equation
   a. Finding the General Solution
   b. Finding a Particular Solution Based on Initial Conditions
3. Applications
   a. Finding Height Given Velocity and Initial Position
   b. Finding Velocity and Position Given Acceleration

# 1. What Is a Differential Equation?

A **differential equation** is an equation that contains derivatives of some function $y$. The solution of the differential equation is the function $y$ that satisfies the equation.

Examples of differential equations include:

- $y' = 2x + 5$
- $y'' + 4y' + 4y = 5\sin t$

In this course, we will solve differential equations of the form $y' = f(x)$.

TERM TO KNOW

**Differential Equation**
An equation that contains derivatives of some function $y$.

---

# 2. Solving a Differential Equation

## 2a. Finding the General Solution

Consider the differential equation $y' = f(x)$. The solution can be written $y = \int f(x)dx$. This tells us to take the antiderivative of $f(x)$ to solve this type of differential equation.

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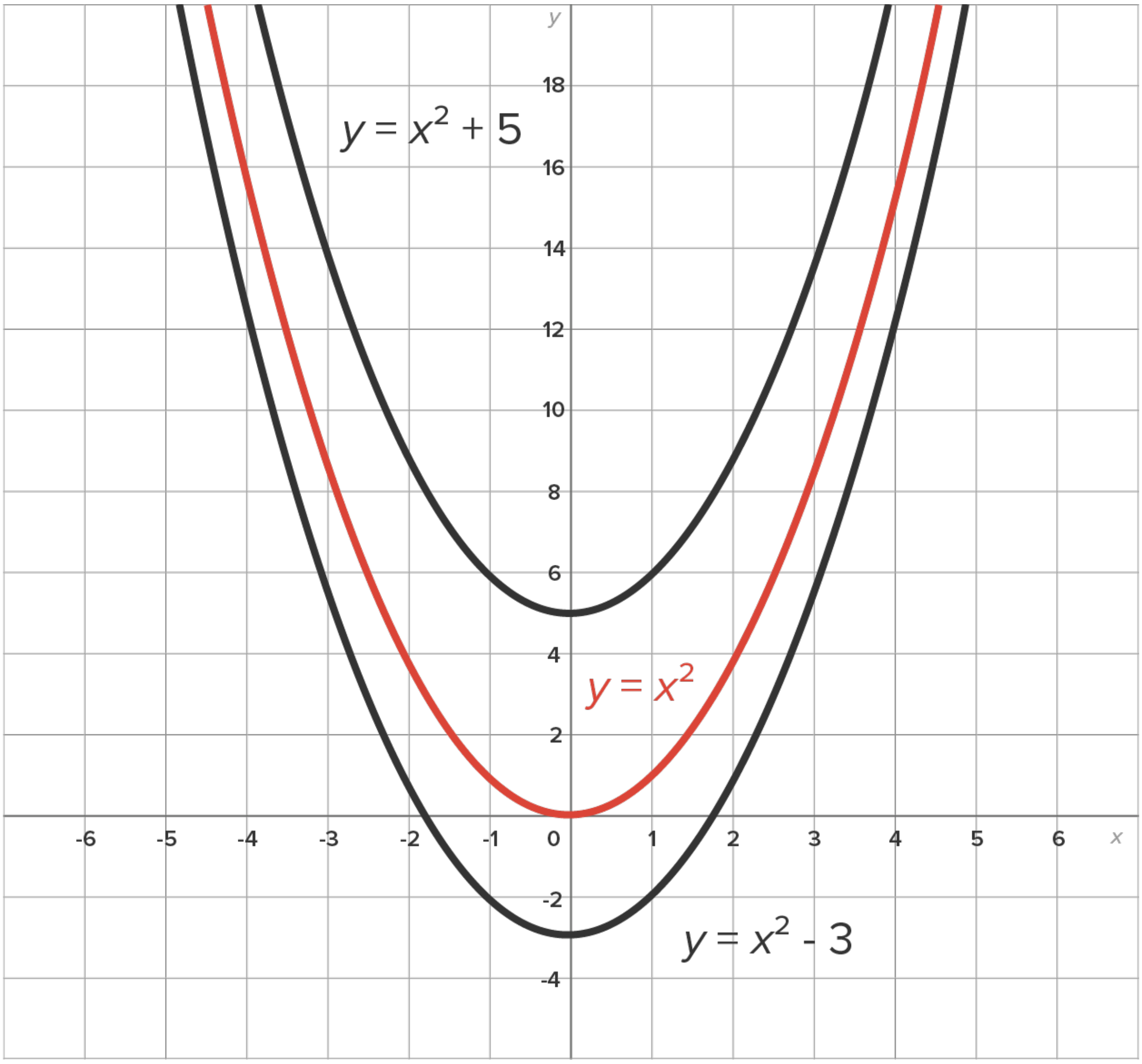
Now, recall that we often use $F(x)$ to represent the antiderivative of $f(x)$. That is, $F(x) = \int f(x)dx$.

This all considered, the function $y = F(x) + C$ is called the **general solution** of the differential equation $y' = f(x)$. The general solution is actually a family of solutions since the equation represents a set of functions that differ only by the value of $C$, the arbitrary constant.

↱ EXAMPLE Consider the differential equation $y' = 2x$, which we know has solution $y = x^2 + C$.

This means that $y = x^2$, $y = x^2 - 3$, and $y = x^2 + 5$ are solutions to the differential equation, to name a few.

In fact, if you were to graph each solution, they only differ by a vertical shift.




☆ BIG IDEA

To solve a differential equation of the form $y' = f(x)$, find the antiderivative of $f(x)$, which is called $F(x) = \int f(x)dx$. Then, the general solution is $y = F(x) + C$.

↱ EXAMPLE Solve $y' = \cos(2x)$.

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This means that $y$ is the antiderivative of $\cos(2x)$, written $y=\int \cos(2x)dx$.

Let's go through the antiderivative process.

| | |
|---|---|
| $\int \cos(2x)dx$ | Start with the original expression. |
| $=\int \cos u \cdot \frac{1}{2}du$ | Let $u=2x$.<br>Then, the differential is $du=2dx$.<br>Then, $dx=\frac{1}{2}du$. |
| $=\frac{1}{2}\int \cos u du$ | Move the constant $\frac{1}{2}$ outside the integral sign. |
| $=\frac{1}{2}\sin u+C$ | Use the antiderivative rule for $\cos u$. |
| $=\frac{1}{2}\sin(2x)+C$ | Back-substitute $u=2x$. |

Thus, the solution to the differential equation $y'=\cos(2x)$ is $y=\frac{1}{2}\sin(2x)+C$.

TRY IT

Consider the differential equation $y'=x^2-4x+7$.

Find the solution. +

$$y=\frac{1}{3}x^3-2x^2+7x+C$$

TRY IT

Consider the differential equation $y'=\cos(3x)+3e^x-9x^2$.

Find the solution. +

$$y=\frac{1}{3}\sin(3x)+3e^x-3x^3+C$$

→ EXAMPLE Solve $y'=\frac{3}{2x+1}$.

This means $y=\int \frac{3}{2x+1}dx$.

Again, let's go through the antiderivative process.

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| | |
|---|---|
| $y = \int \frac{3}{2x+1} dx$ | Start with the original expression. |
| $= \int \frac{1}{u} \cdot \frac{3}{2} du$ | Let $u = 2x + 1$.<br>Then, the differential is $du = 2dx$.<br>Then, $dx = \frac{1}{2} du$. |
| $= \frac{3}{2} \int \frac{1}{u} du$ | Move the constant $\frac{3}{2}$ outside the integral sign. |
| $= \frac{3}{2} \ln\|u\| + C$ | Use the antiderivative rule for $\frac{1}{u}$. |
| $= \frac{3}{2} \ln\|2x+1\| + C$ | Back-substitute $u = 2x + 1$. |

Thus, the solution to the differential equation $y' = \frac{3}{2x+1}$ is $y = \frac{3}{2} \ln|2x+1| + C$.

TERM TO KNOW

**General Solution**

The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

## 2b. Finding a Particular Solution Based on Initial Conditions

When obtaining the solution to the differential equation, sometimes we are given a value of *y* when *x* is some number in the domain. This is called an **initial condition**.

For example, if the graph is to pass through the point $(1, 5)$, then the initial condition is written $y(1) = 5$.

↪ EXAMPLE Solve $y' = 2x$, given that the solution passes through the point $(3, 20)$.

We know $\int 2x dx = x^2 + C$. Thus, the solution to the differential equation is $y = x^2 + C$. Use this equation to find the solution that passes through $(3, 20)$.

| | |
|---|---|
| $y = x^2 + C$ | Use this equation to find the solution. |
| $20 = 3^2 + C$ | Replace *x* with 3 and *y* with 20. |
| $11 = C$ | Simplify. |

Substituting this answer back into $y = x^2 + C$, the solution to the differential equation is $y = x^2 + 11$.

BIG IDEA

The initial condition is used to find the value of *C*, the constant of integration.

A **particular solution** is the solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

↪ EXAMPLE Solve $y' = e^{-3x} + x + \sin x$, given that the solution passes through the point $(0, 4)$.

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First, find the family of solutions.

| | |
|---|---|
| $y' = e^{-3x} + x + \sin x$ | Start with the original expression. |
| $y = \int (e^{-3x} + x + \sin x) dx$ | If $y' = f(x)$, then $y = \int f(x) dx$. |
| $y = \int e^{-3x} dx + \int x dx + \int \sin x dx$ | Use the sum of antiderivatives property. |
| $y = \frac{-1}{3} e^{-3x} + \frac{1}{2} x^2 - \cos x + C$ | Apply antiderivative formulas. |
| $4 = \frac{-1}{3} e^{(3 \cdot 0)} + \frac{1}{2} (0)^2 - \cos 0 + C$ | Apply the initial condition and replace $x$ with 0 and $y$ with 4. |
| $C = \frac{16}{3}$ | Solve for $C$. |

Thus, the particular solution is $y = \frac{-1}{3} e^{-3x} + \frac{1}{2} x^2 - \cos x + \frac{16}{3}$.

TRY IT

Consider the differential equation $y' = 2\cos(4x) - 12\sin(6x)$ with $y(0) = -1$.

Solve the differential equation. +

$$y = \frac{1}{2}\sin(4x) + 2\cos(6x) - 3$$

WATCH

In this video, we'll solve a differential equation: $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$, $y(3) = 1$

## Video Transcription

[MUSIC PLAYING] Well, hello there. Thank you for joining me today. What we have here is a differential equation, dy dx is equal to x divided by the square root of x squared plus 16, where we know that the y value when x equals 3 is equal to 1. So the goal is to solve this differential equation, which means we're trying to find the function y that satisfies all these conditions. So if dy dx is equal to this expression, that means that y is equal to the antiderivative of this expression.

This is what's going to get us the indefinite answer, the general answer, the family of curves. Then we apply the initial condition to find the value of c. So that's our framework here. The first thing to think about here is, remember that whenever we see a square root in calculus we always write it as a one half power. Since this one is in a denominator, it's really a negative one half power.

That's going to be where I start here. Then it looks like we have a complicated expression to the negative one half power. I'm going to call that u equals x squared plus 16 and that means that differential form is 2x dx. Since I don't have the 2, I'm going to isolate x dx. So that means this piece. This means

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that x is equal to one half du, just divide both sides by 2. OK.

So looking at that now, x squared plus 16 is u. So u to the negative one half, and the x dx, that all becomes one half du. So I'm going to put the one half here and the du here. OK. Remember that constants can come outside, so I'm going to put the one half outside, u to the negative 1/2 du. Then we use our power rule.

So this is one half times, going up 1 power is one half. I divide by a half right away, plus c. A couple of ways to handle this, I could either bring the one half up as a reciprocal, make it a 2, and then the 2 and the one half would cancel each other out or if I just multiply my fraction straight across that's a one. So either way, I just end up with u to the one half plus c, which is-- Well, we could write that as the square root of u plus c just to make it more familiar, which is the square root of x squared plus 16 plus c. Remember, that's all y.

So the family of curves that solves the functional part of the differential equation is y equals square root of x squared plus 16 plus c. So c could be any constant at this point. We want to know the particular one where 3,1 is a point on the curve. So that means when x equals 3, y is equal to 1. So we substitute 1 for y, we substitute 3 for x, and we see what happens.

So 1 is equal to square root. 3 squared is 9, plus 16, plus c. So you have 1 equals 9 plus 16 is 25 and the square root of 25 is 5, plus c. So that means that c is negative 4. So that we're almost at the final answer. Taking this, this is where we started from, the final answer is y equals square root, x squared plus 16 minus 4. I will share this with you, a lot of times there is confusion about where the square root starts and stops so it is often customary to write the constant first, followed by the square root. That way you know that 16 is the last thing intended to be under the square root, otherwise somebody might think that the minus 4 is also under the square root.

So that's just a little tidbit of information there for you, but there's our solution to our differential equation. The function that satisfies both the condition about y of 3 equals 1 and dy dx equals, as we saw up here, x divided by square root x squared plus 16.

[MUSIC PLAYING]

TERMS TO KNOW

**Initial Condition**
From a differential equation, a point that the solution’s graph passes through.

**Particular Solution**
The solution to a differential equation that doesn’t contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

# 3. Applications

## 3a. Finding Height Given Velocity and Initial Position

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↪ EXAMPLE Suppose the velocity of an object that is falling from a height of 400 feet is given by $v(t) = 120 - 120e^{-0.4t}$, where $t$ is measured in seconds, and $v(t)$ is measured in meters per second.

What is the object's height above the ground after $t$ seconds, given that the object's initial position was 400 feet above the ground?

We know that the height, $h(t)$, is the antiderivative of the velocity $v(t)$. Let's start there.

| | |
|---|---|
| $v(t) = 120 - 120e^{-0.4t}$ | Start with the original expression. |
| $h(t) = \int (120 - 120e^{-0.4t})dt$ | We can say $h(t) = \int v(t)dt$. |
| $h(t) = 120t + 300e^{-0.4t} + C$ | Integrate the right-hand side. |
| $400 = 120(0) + 300e^{(-0.4 \cdot 0)} + C$ | We were also told that the initial position was 400 feet away in the positive direction. This means $h(0) = 400$. |
| $400 = 300 + C$ | Simplify. |
| $100 = C$ | Solve for $C$. |

Thus, the object's height function is

## 3b. Finding Velocity and Position Given Acceleration

If $s(t)$ represents a function's position at time $t$ (usually measured in seconds), recall the following:

- If $v(t)$ represents its velocity at time $t$, then $v(t) = s'(t)$.
- If $a(t)$ represents its acceleration at time $t$, then $a(t) = v'(t) = s''(t)$.
- It follows that $v(t) = \int a(t)dt$ and $s(t) = \int v(t)dt$.

↪ EXAMPLE A tennis ball is dropped (no starting velocity) from a height of 400 feet at time $t = 0$, where $t$ is measured in seconds. This means that $v(0) = 0$ and $s(0) = 400$.

If the tennis ball's acceleration (due to gravity) is $-32 \text{ ft/s}^2$, find its velocity and position as functions of time.

Since we are given acceleration, we find the velocity function, $v(t)$, first.

| | |
|---|---|
| $v(t) = \int a(t)dt$ | Given the acceleration, we can find the velocity first. |
| $v(t) = \int (-32)dt$ | Plug in the known value of acceleration, -32. |
| $v(t) = -32t + C$ | Evaluate. |
| $0 = -32(0) + C$ | Knowing that $v(0) = 0$, substitute to find $C$. |
| $0 = C$ | Simplify. |

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Thus, $v(t) = -32t$.

Now repeat the process to find the position function, $s(t)$.

| | |
|---|---|
| $s(t) = \int v(t)dt$ | Given the velocity, we can find the position. |
| $s(t) = \int(-32t)dt$ | Plug in the known velocity function, $v(t) = -32t$. |
| $s(t) = -32\left(\frac{1}{2}\right)t^2 + C$ | Evaluate. |
| $s(t) = -16t^2 + C$ | Simplify. |
| $400 = -16(0)^2 + C$ | Knowing that $s(0) = 400$, substitute to find *C*. |
| $400 = C$ | Simplify. |

Then, $s(t) = -16t^2 + 400$.

TRY IT

At time $t = 0$, a tomato is launched with an upward velocity of 20 feet per second from a height of 300 feet, where *t* is measured in seconds. This means that $v(0) = 20$ and $s(0) = 200$. Assume the tomato's acceleration due to gravity is $-32\ \text{ft}/\text{s}^2$.

Find the velocity of the tomato as a function of time. +

$v(t) = -32t + 20$

Find the position of the tomato as a function of time. +

$s(t) = -16t^2 + 20t + 200$

DID YOU KNOW

In the metric system, the basic unit of distance is meters. The acceleration due to gravity is approximately $-9.8\ \text{m}/\text{s}^2$. Therefore, we can find $v(t)$ and $s(t)$ for a moving object when the distance is measured in meters rather than feet.

BIG IDEA

If an object moves with acceleration *a* (constant) with initial velocity $v_0$ and has initial position $s_0$, we know the following:

- Its velocity after *t* seconds is $v(t) = at + v_0$.
- Its position at time *t* is $s(t) = \frac{1}{2}at^2 + v_0t + s_0$.

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## SUMMARY

In this lesson, you began by defining **a differential equation**, which is an equation that contains derivatives of some function *y*. You learned that when **solving a differential equation** of the form $y' = f(x)$, there are two ways to express the solution. When there is no initial condition, **finding the general solution** of the differential equation has the form $y = F(x) + C$, where *C* is any constant. This produces a family of solutions that are vertical shifts of one another. When **finding a particular solution based on an initial condition** that is given, you can find the specific curve from the same family of solutions that satisfied the initial condition.

Extending this idea to other **applications**, you are able to **find the height** of an object in motion **given velocity and initial position** as well as **find the velocity and position** of an object in motion, **given its acceleration** at any time *t*.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 6 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Differential Equation**
An equation that contains derivatives of some function *y*.

**General Solution**
The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

**Initial Condition**
From a differential equation, a point that the solution's graph passes through.

**Particular Solution**
The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

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# The Fundamental Theorem of Calculus

*by Sophia*

WHAT'S COVERED

In this lesson, you will apply the fundamental theorem of calculus to definite integrals. Specifically, this lesson will cover:

1. The Fundamental Theorem of Calculus
2. Using the Fundamental Theorem of Calculus
3. Fundamental Theorem of Calculus with *u*-Substitutions
   a. Finding the Indefinite Integral First
   b. Changing the Limits of Integration to Match the Substitution

# 1. The Fundamental Theorem of Calculus

Recall the fundamental theorem of calculus:

FORMULA

**Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of a continuous function $f(x)$ on the interval $[a, b]$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$.

To show that we are evaluating $F(x)$ at $x = a$ and $x = b$ and then subtracting, we use the notation:

$$F(x)\Big|_a^b$$

Note that $F(x)$ is any antiderivative of $f(x)$. Recall that the general antiderivative is $F(x) + C$, where $C$ is an arbitrary constant.
Then, evaluating the definite integral, we obtain:

$$\begin{aligned}(F(x)+C)\Big|_a^b &= (F(b)+C)-(F(a)+C)\\ &= F(b)+C-F(a)-C\\ &= F(b)-F(a)\end{aligned}$$

What this means is that when evaluating a definite integral, the result is the same regardless of the value of $C$ used. Therefore, to keep it simple, use $C = 0$.

BIG IDEA

When evaluating a definite integral, use $C = 0$ (which means not to write the integration constant).

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## 2. Using the Fundamental Theorem of Calculus

We can now use this powerful theorem to evaluate definite integrals, which also includes finding exact areas of regions we could only approximate before.

→ EXAMPLE Evaluate the definite integral: $\int_1^2 4x^2 dx$

| | |
|---|---|
| $\int_1^2 4x^2 dx$ | Start with the original expression. |
| $= \frac{4}{3}x^3 \Big\|_1^2$ | Find the antiderivative.<br>Remember: no "+*C*" is required. |
| $= \frac{4}{3}(2)^3 - \frac{4}{3}(1)^3$ | Evaluate at the upper and lower endpoints. |
| $= \frac{32}{3} - \frac{4}{3}$ | Evaluate the exponents. |
| $= \frac{28}{3}$ | Simplify. |

Thus, $\int_1^2 4x^2 dx = \frac{28}{3}$.

→ EXAMPLE Evaluate $\int_0^1 (e^{2x} - 2e^x) dx$.

| | |
|---|---|
| $\int_0^1 (e^{2x} - 2e^x) dx$ | Start with the original expression. |
| $= \left(\frac{1}{2}e^{2x} - 2e^x\right)\Big\|_0^1$ | Find the antiderivative.<br>Remember: no "+*C*" is required. |
| $= \left(\frac{1}{2}e^{2(1)} - 2e^1\right) - \left(\frac{1}{2}e^{2(0)} - 2e^0\right)$ | Evaluate at the upper and lower endpoints. |
| $= \frac{1}{2}e^2 - 2e - \frac{1}{2} + 2$ | Evaluate each parentheses. |
| $= \frac{1}{2}e^2 - 2e + \frac{3}{2}$ | Simplify. |

Thus, $\int_0^1 (e^{2x} - 2e^x) dx = \frac{1}{2}e^2 - 2e + \frac{3}{2}$.

TRY IT

Consider the definite integral $\int_4^9 (x - \sqrt{x}) dx$.

Evaluate the definite integral, giving the exact answer. +

$\frac{119}{6}$

WATCH

Check out this video to see the example $\int_1^4 \frac{3}{x^4}dx$.

## Video Transcription

[MUSIC PLAYING] Hello there, and welcome to today's video on evaluating a definite integral from 1 to 4 of 3 over x to the fourth dx. Now, remember, to evaluate a definite integral, we can use the fundamental theorem of calculus if our function is continuous on the closed interval.

And if I'm evaluating the definite integral from a to b of f of x dx, the fundamental theorem says if you find an antiderivative for your integrand, then you can find the definite integral value by evaluating that antiderivative at the upper limit of integration minus that antiderivative evaluated at the lower limit of integration.

So the first thing we want to do is to find our capital F of x, our antiderivative that we're going to use. And to find that, we find the indefinite integral of 3 over x to the fourth dx. Remember, to find this indefinite integral, we need to first write it so that I don't have the division of the x to the fourth. So we're going to write this as the integral 3 times x to the negative 4 dx.

Using the constant multiple rule, we will bring the constant factor of 3 out in front of the integral. And then we notice that we have letter base to number exponent, and the number exponent is not negative 1. So I'll use the general power rule to integrate that expression. So I have 3 times-- general power rule is keep your base of x, add 1 to the exponent. Negative 4 plus 1 is negative 3. And then divide by that number, that negative 3. And remember to add your arbitrary constant.

Simplifying this gives me a negative x to the negative 3 plus C. And then rewriting it without negative exponents gives me a negative 1 over x to the 3 plus C. And because, with the fundamental theorem of calculus, I can use an antiderivative of small case f of x, it can be any antiderivative of it, I can choose the antiderivative where C is equal to 0. So antiderivative we're going to use is f of x is equal to negative 1 over x to the third.

Bringing that over to our work, then, with our definite integral from 1 to 4 of 3 over x to the fourth dx, we are going to have our f of x evaluated at our upper limit of integration and then subtract from that our antiderivative evaluated at the lower limit of integration.

So we start with our negative 1 over, plug in the upper limit of integration, and subtract from that-- from the fundamental theorem of calculus-- a negative 1 over, now plug in the lower limit of integration. So this gives us negative 1 over 64, and then minus negative is plus, and then 1 cubed is 1, and 1 divided by 1 is 1. And taking negative 1 over 64 plus 1 gives me 63 over 64 as the value of my definite integral.

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# 3. Fundamental Theorem of Calculus with *u*-Substitutions

When the antiderivative requires *u*-substitution, there are two ways to evaluate the definite integral:

- Finding the indefinite integral first
- Changing the limits of integration to match the substitution

## 3a. Finding the Indefinite Integral First

↪ EXAMPLE Evaluate the definite integral: $\int_0^3 x^2\sqrt{x^3+9}\,dx$

First, find the indefinite integral:

| | |
|---|---|
| $\int x^2\sqrt{x^3+9}\,dx$ | Start with the original expression. |
| $=\int x^2(x^3+9)^{1/2}dx$ | Rewrite $\sqrt{x^3+9}$ as $(x^3+9)^{1/2}$. |
| $=\frac{1}{3}\int u^{1/2}du$ | Let $u=x^3+9$. Then, $du=3x^2dx$, or $\frac{1}{3}du=x^2dx$. |
| $=\frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C$ | Use the power rule. Dividing by $\frac{3}{2}$ is equivalent to multiplying by $\frac{2}{3}$. |
| $=\frac{2}{9}u^{3/2}+C$ | Simplify. |
| $=\frac{2}{9}(x^3+9)^{3/2}+C$ | Back-substitute $u=x^3+9$. |

Now, evaluate the definite integral:

| | |
|---|---|
| $=\frac{2}{9}(x^3+9)^{3/2}\Big\vert_0^3$ | Apply the fundamental theorem of calculus.<br>Note, "+*C*" is omitted since we are evaluating a definite integral. |
| $=\frac{2}{9}(3^3+9)^{3/2}-\frac{2}{9}(0^3+9)^{3/2}$ | Substitute the upper and lower endpoints. |
| $=\frac{2}{9}(36)^{3/2}-\frac{2}{9}(9)^{3/2}$ | Evaluate parentheses. |
| $=48-6$ | Evaluate. |
| $=42$ | Simplify. |

In conclusion, $\int_0^3 x^2\sqrt{x^3+9}\,dx=42$.

TRY IT

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Consider the definite integral $\int_1^2 \frac{1}{(2x+1)^3}dx$.

Evaluate the definite integral. Give an exact answer. +

$\frac{4}{225}$

↪ EXAMPLE Evaluate the definite integral: $\int_1^{e^2} \frac{(\ln x)^3}{x}dx$

First, find the indefinite integral:

| | |
|---|---|
| $\int \frac{(\ln x)^3}{x}dx$ | Start with the original expression. |
| $= \int u^3 du$ | First, write $\frac{(\ln x)^3}{x} = (\ln x)^3 \cdot \frac{1}{x}$. Let $u = \ln x$. Then, $du = \frac{1}{x}dx$. |
| $= \frac{1}{4}u^4 + C$ | Use the power rule. |
| $= \frac{1}{4}(\ln x)^4 + C$ | Back-substitute $u = \ln x$. |

Now, evaluate the definite integral:

| | |
|---|---|
| $= \left.\frac{1}{4}(\ln x)^4\right\vert_1^{e^2}$ | Apply the fundamental theorem of calculus. The "+$C$" is omitted since we are evaluating a definite integral. |
| $= \frac{1}{4}(\ln e^2)^4 - \frac{1}{4}(\ln 1)^4$ | Substitute the upper and lower endpoints. |
| $= \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4$ | Evaluate parentheses. Recall that $\ln(e^k) = k$. |
| $= 4 - 0$ | Evaluate. |
| $= 4$ | Simplify. |

In conclusion, $\int_1^{e^2} \frac{(\ln x)^3}{x}dx = 4$.

WATCH

In this video, the example $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x dx$ is presented.

## Video Transcription

[MUSIC PLAYING] Hello there, and welcome to the video on how to evaluate the definite integral from 0 to pi/4 of tangent cubed x secant squared x dx. Now, when working with this definite integral, we're

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going to use the fundamental theorem of calculus.

And the fundamental theorem of calculus says, if you have a continuous function on a closed interval, then the definite integral from a to b of f of x dx is equal to finding an antiderivative for f of x. And we'll use the notation of capital F for the antiderivative. Evaluate that antiderivative at the upper limit of integration b minus evaluate that antiderivative at the lower limit of integration a.

So here, what we want to do is to look at our work to find our antiderivative F of x. Now, looking at that we can do the indefinite integral of tangent cubed x. And we're going to write that as tangent x to the third power and secant squared x dx.

And the reason that I chose to rewrite the equivalent notation for the tangent cubed x as tangent x to the third power was that it helps us to see our u substitution a little bit better. Now, notice when we're doing our u substitution, what we let u represent, we also need to have the differential du, at least within a constant multiple.

And if you notice that if I would have chosen secant x as my u, its derivative is secant x tangent x. But choosing tangent x as my u, its derivative is secant squared x, which we have in the [INAUDIBLE] So we are going to let u represent tangent to x-- that amount that's being acted on-- and also what we have the derivative of in the [INAUDIBLE]

And then our du-- well, the derivative of tangent x is secant squared x. And it's the differential, so we'll say secant squared x dx. Now, when we make our substitution, our u is tangent x. So we're going to put u where tangent x is.

And our du is secant squared dx. So we are going to replace the secant squared x dx with the du. So now, we have our integral of u to the third. And remember, the secant squared x dx all goes out with a du replacement.

Now, integrating that, it's a letter base to a number power. And the number power is not negative 1. So using the general power rule, we'll have u to the power 4 divided by 4 and then plus our arbitrary constant C. And I can write that as 1/4 u to the power of 4 plus C. And then going back to the original variable that we had, u, remember, is the tangent x. So we are going to write this as 1/4 tangent to the power 4 of x and then plus C.

Now, from the fundamental theorem of calculus, we just need to have an antiderivative of f of x. So we can choose the antiderivative to be the one whose arbitrary constant is 0. And we'll get that our antiderivative for our f of x is capital F of x is equal to 1/4 tangent to the fourth of x.

Now, let's come over and do our definite integral. So we have the definite integral from 0 to pi/4 is of our tangent cubed x secant squared x dx. We are going to evaluate our antiderivative at the upper limit of integration. So putting the pi over 4 in first, minus at the lower limit of integration, putting in the 0.

So we have 1/4 tangent to the power of 4 of pi/4 minus 1/4 tangent to the 4 power of 0. Now, again, remember that power 4 means we have 1/4 times the tangent of pi/4 to the fourth power minus 1/4 the tangent of 0 to the power of 4.

And then simplifying this, the tangent of pi/4-- well, the tangent of pi/4 is 1-- and I'm raising that to the fourth power-- minus 1/4 times the tangent of 0 is 0. And 1 to the fourth is 1 times 1/4 is 1/4, minus 0 to the power of 4 is 0 times 1/4 fourth is 0. And 1/4 minus 0 is 1/4. And that gives us our value of the definite integral from 0 to pi/4 of tangent cubed x secant squared x dx.

[MUSIC PLAYING]

## 3b. Changing the Limits of Integration to Match the Substitution

Now we will look at a method where we change the limits of integration. This method allows us to avoid having to back-substitute.

We'll look at the same examples as before to help make the connection.

↪ EXAMPLE Evaluate the definite integral: $\int_0^3 x^2\sqrt{x^3+9}\,dx$

First, make the $u$-substitution:

| | |
|---|---|
| $\int_0^3 x^2\sqrt{x^3+9}\,dx$ | Start with the original expression. |
| $=\frac{1}{3}\int u^{1/2}du$ | First, write $\sqrt{x^3+9}=u^{1/2}$. Let $u=x^3+9$. Then, $du=3x^2dx$, or $\frac{1}{3}du=x^2dx$. |

At this point, we write the original definite integral as a new definite integral, this time with $u$-values as the limits of integration.

The original definite integral is taken from $x=0$ to $x=3$. What are the corresponding values of $u$? We use the substitution $u=x^3+9$:

- When $x=0$, $u=0^3+9=9$.
- When $x=3$, $u=3^3+9=36$.

Thus, through our substitution, $\int_0^3 x^2\sqrt{x^3+9}\,dx=\frac{1}{3}\int_9^{36}u^{1/2}du$. We can now continue.

| | |
|---|---|
| $=\frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big\vert_9^{36}$ | Apply the fundamental theorem of calculus. The "+$C$" is omitted since we are evaluating a definite integral. |
| $=\frac{2}{9}u^{3/2}\Big\vert_9^{36}$ | Simplify before evaluating. |
| $=\frac{2}{9}(36)^{3/2}-\frac{2}{9}(9)^{3/2}$ | Substitute the upper and lower endpoints. |

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$$= 48 - 6 \quad \text{Evaluate.}$$

$$= 42 \quad \text{Simplify.}$$

Thus, $\int_0^3 x^2\sqrt{x^3+9}\,dx = 42$, which is the same as the result from the last method.

Having done the same problem two different ways now, let's compare the methods.

By using the indefinite integral, we had these steps:

$$\int_0^3 x^2(x^3+9)^{1/2}dx$$

$$= \frac{2}{9}(x^3+9)^{3/2}\Big|_0^3$$

$$= \frac{2}{9}(36)^{3/2} - \frac{2}{9}(9)^{3/2}$$

$$= 42$$

After making the *u*-substitution and replacing the *x*-values with *u*-values, we had these steps:

$$\int_0^3 x^2(x^3+9)^{1/2}dx$$

$$= \frac{1}{3}\int_9^{36} u^{1/2}du$$

$$= \frac{2}{9}u^{3/2}\Big|_9^{36}$$

$$= \frac{2}{9}(36)^{3/2} - \frac{2}{9}(9)^{3/2}$$

$$= 42$$

REFLECT

Now that you have seen two methods to evaluate definite integrals with *u*-substitution, which method do you think is easier, and why? If you can't answer this yet, wait to answer this question until you have gone through more examples.

TRY IT

Consider the definite integral $\int_0^3 \frac{1}{\sqrt{5x+1}}dx$.

Write the definite integral with values of *u* as limits of integration. +

$$\frac{1}{5}\int_1^{16} u^{-1/2}du$$

Evaluate the integral in the answer from the first question. +

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$\frac{6}{5}$

Now, let's take another look at a previous example so we can compare.

↳ EXAMPLE Evaluate the definite integral: $\int_1^{e^2} \frac{(\ln x)^3}{x} dx$

| | |
|---|---|
| $\int_1^{e^2} \frac{(\ln x)^3}{x} dx$ | Start with the original expression. |
| $= \int_0^2 u^3 du$ | First, write $\frac{(\ln x)^3}{x} = (\ln x)^3 \cdot \frac{1}{x}$. Let $u = \ln x$, then $du = \frac{1}{x} dx$. When $x = 1$, $u = \ln 1 = 0$. When $x = e^2$, $u = \ln e^2 = 2$. |
| $= \frac{1}{4} u^4 \Big\vert_0^2$ | Apply the fundamental theorem of calculus. The "$+C$" is omitted since we are evaluating a definite integral. |
| $= \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4$ | Substitute the upper and lower endpoints. |
| $= 4 - 0$ | Evaluate. |
| $= 4$ | Simplify. |

Just as before, $\int_1^{e^2} \frac{(\ln x)^3}{x} dx = 4$.

WATCH

Here is a video showing an example writing the limits of integration with values of $u$ for the definite integral $\int_0^{\frac{5\pi}{6}} \sin^4 x \cos x dx$.

## Video Transcription

[MUSIC PLAYING] Hi, there. It's nice to see you again. This video is on evaluating the definite integral from 0 to 5 pi over 6 of sine of the fourth x cosine xdx. In this one, we're actually going to also do our substitutions for our endpoints of our integral, our limits of our integration, as well as with the rest of the integral.

So remember to look at evaluating a definite integral by the fundamental theorem of calculus. If we have a continuous function on a closed interval, then the definite integral from a to b of f of x dx can be found by finding an antiderivative of small case, f of x and calling it capital F of x, and evaluating that antiderivative at the upper limit of integration minus evaluating that antiderivative at the lower limit of integration. Now, what we have in this specific situation is our definite integral from 0 to 5 pi over 6, and I have sine to the fourth x cosine x dx.

So I can rewrite the meaning of sine to the fourth x. Remember, that means that your sine of x is being

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raised to the fourth power. Then we have times cosine x dx. So with this, we need to do a u substitution in order to integrate it. We are going to let u represent what is being acted on. So sine x is being acted on by the power of 4, sine x is going to be what we represent u with. Then remember we also need to figure out the differential du.

Well, the differential du, we first think, well, what's the derivative of sine x? The derivative of sine x is cosine x and then dx. Then when we look back we also have our end points of integration. Well, if u is equal to the sine of x, then when x is equal to zero that gives us that u will be equal to the sine of 0. In this case, we get u is equal to 0. When x is equal to 5 pi over 6, that gives us that u is equal to the sine of 5 pi over 6. We get that u is equal to, well, the sign of 5 pi over 6 is positive one half.

So we will make all of those substitutions. We now have the integral from 0 to one half of u to the power 4, and then remember the cosine x dx both go and get replaced by the du. Now, integrating this we're going to use our power rule because we have a letter base to a number power and the number power is not negative 1. The general power rule for integration, we add 1 to the exponent, 4 plus 1 is 5, and divide by that. So divide by 5, and we don't put the arbitrary constant when we have a definite integral we just show our vertical line for evaluated at, and our lower limit within the u values is 0 up to the upper limit of integration, the one half.

I can write this u to the fifth over 5 as one fifth times u to the fifth. So it's one fifth times, put in the one half for u and raise that to the power of 5, minus one fifth times, take out the u and put in the 0 and raise it to the power 5. So this next gives me one fifth times one half to the power 5 is 1 over 32, and then minus one fifth time 0. So I get 1 over 160 minus 0 is 1 over 160. That is the value of our original definite integral from 0 to 5 pi over 6 sine fourth x, cosine x dx is equal to 1 over 160.

[MUSIC PLAYING]

REFLECT

How do you feel about this method for evaluating definite integrals with $u$-substitution versus the indefinite integral with back-substitution?

## SUMMARY

In this lesson, you learned that **the fundamental theorem of calculus** uses antiderivatives to evaluate definite integrals exactly rather than with a Riemann sum. When $u$-substitution is not required, this is a rather straightforward process. When applying the **fundamental theorem of calculus with $u$-substitution**, you learned that there are two methods that could be used to evaluate the integral: 1) **finding the indefinite integral first**, and 2) **changing the limits of integration to match the substitution**. In later sections, we will revisit applications that utilized definite integrals, but in a different light now that we know how to evaluate definite integrals more generally rather than relying on using geometric formulas.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

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## FORMULAS TO KNOW

**Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of a continuous function $f(x)$ on the interval $[a, b]$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$.

# Antiderivative Applications

*by Sophia*

WHAT'S COVERED

In this lesson, you will revisit the ideas of area and distance traveled now that we have a more general way to evaluate definite integrals (the fundamental theorem of calculus). Specifically, this lesson will cover:

1. Calculating Areas of Regions
2. Calculating Distance Traveled and Net Change in Distance

# 1. Calculating Areas of Regions

Recall the following about areas and definite integrals:

1. When $f(x)$ is nonnegative on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area of the region between the graph of $y=f(x)$ and the x-axis on $[a, b]$.

   That is, if the area of the region is *A* (a positive number), then $\int_a^b f(x)dx = A$.

2. When $f(x)$ is negative on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the negative of the area of the region between the graph of $y=f(x)$ and the x-axis on $[a, b]$.

   That is, if the area of the region is *A* (a positive number), then $\int_a^b f(x)dx = -A$.

We use these ideas to find areas of regions that are above the x-axis, below the x-axis, or a combination of the two.

↳ EXAMPLE Find the area of the region bounded by $f(x)=\sqrt{x}$ and the x-axis between $x=1$ and $x=9$. The region is shown in the figure below.

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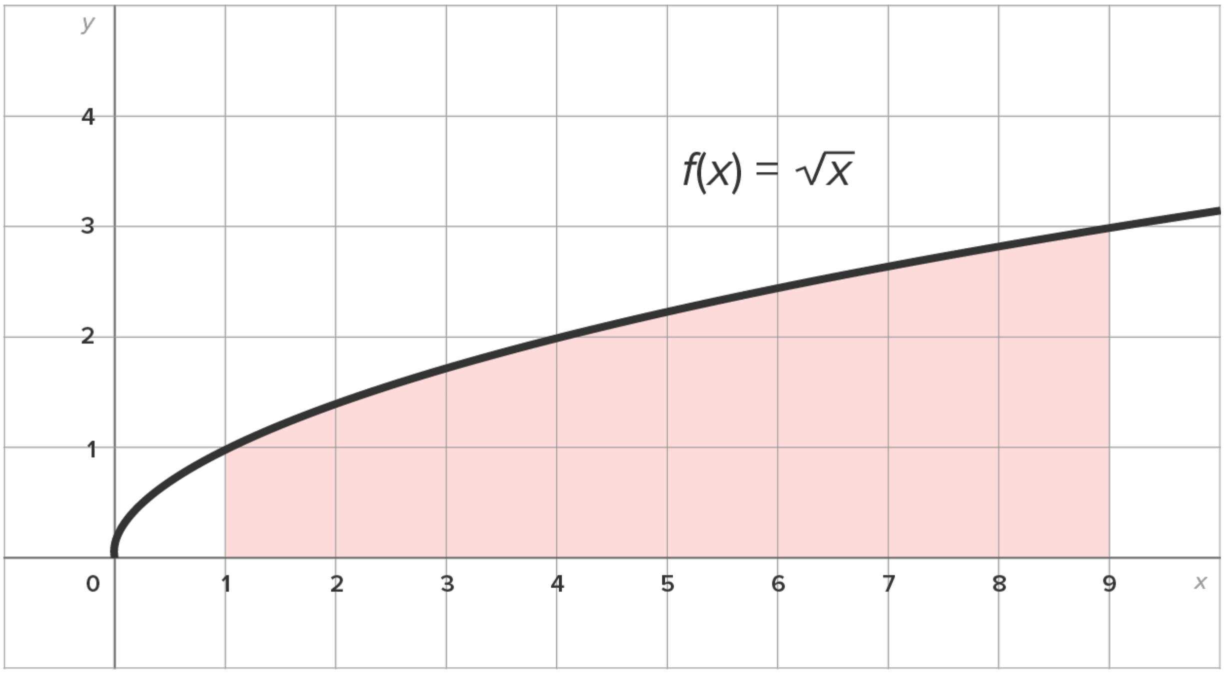


Since the region is above the x-axis, the value of the definite integral is equal to the area of the region.

The definite integral that describes this area is $\int_1^9 \sqrt{x}\,dx$.

Now we evaluate:

| | |
|---|---|
| $\int_1^9 \sqrt{x}\,dx$ | Start with the original expression. |
| $= \int_1^9 x^{1/2}dx$ | Rewrite as a power. |
| $= \frac{2}{3}x^{3/2}\Big\|_1^9$ | Apply the fundamental theorem of calculus and the power rule for antiderivatives. |
| $= \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2}$ | Substitute the upper and lower endpoints. |
| $= \frac{2}{3}(27) - \frac{2}{3}$ | Evaluate. |
| $= \frac{52}{3}$ | Simplify. |

In conclusion, the area of the region bounded by $f(x) = \sqrt{x}$ and the x-axis between $x = 1$ and $x = 9$ is equal to $\frac{52}{3}\text{units}^2$.

Now, let's look at a region that is below the x-axis.

↱ EXAMPLE Find the area of the region between the x-axis and the curve $f(x) = x^2 - 6x$ on the interval between $x = 1$ and $x = 3$. The region is shown in the figure below.

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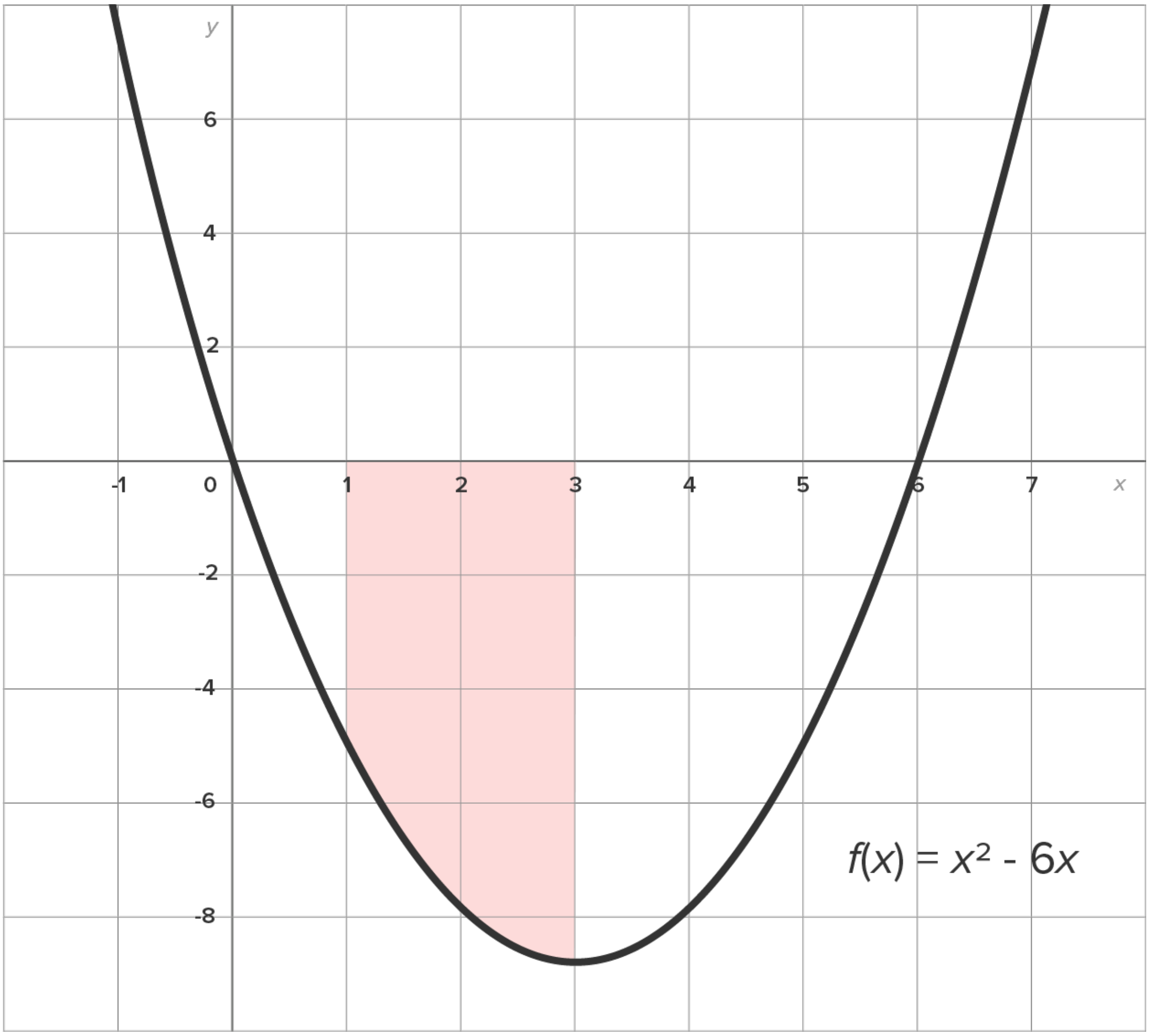


Since the region is entirely below the x-axis, we know that the definite integral will be negative. Thus, we'll evaluate $\int_a^b f(x)dx$ as usual, but remember that its value is the negative of the area.

| | |
|---|---|
| $\int_1^3 (x^2-6x)dx$ | Start with the definite integral that is tied to the area of the region. |
| $=\left(\frac{1}{3}x^3-3x^2\right)\Big\|_1^3$ | Apply the fundamental theorem of calculus. |
| $=\left[\frac{1}{3}(3)^3-3(3)^2\right]-\left[\frac{1}{3}(1)^3-3(1)^2\right]$ | Substitute the limits of integration and subtract. Grouping symbols are used to make the subtraction more clear. |
| $=-18-\left(\frac{-8}{3}\right)$ | Evaluate each bracket. |
| $=\frac{-46}{3}$ | Simplify. |

The value of the definite integral is $\frac{-46}{3}$. Then, the area of the region is $\frac{46}{3}\text{units}^2$.

Let's look at a region that contains parts above and below the x-axis.

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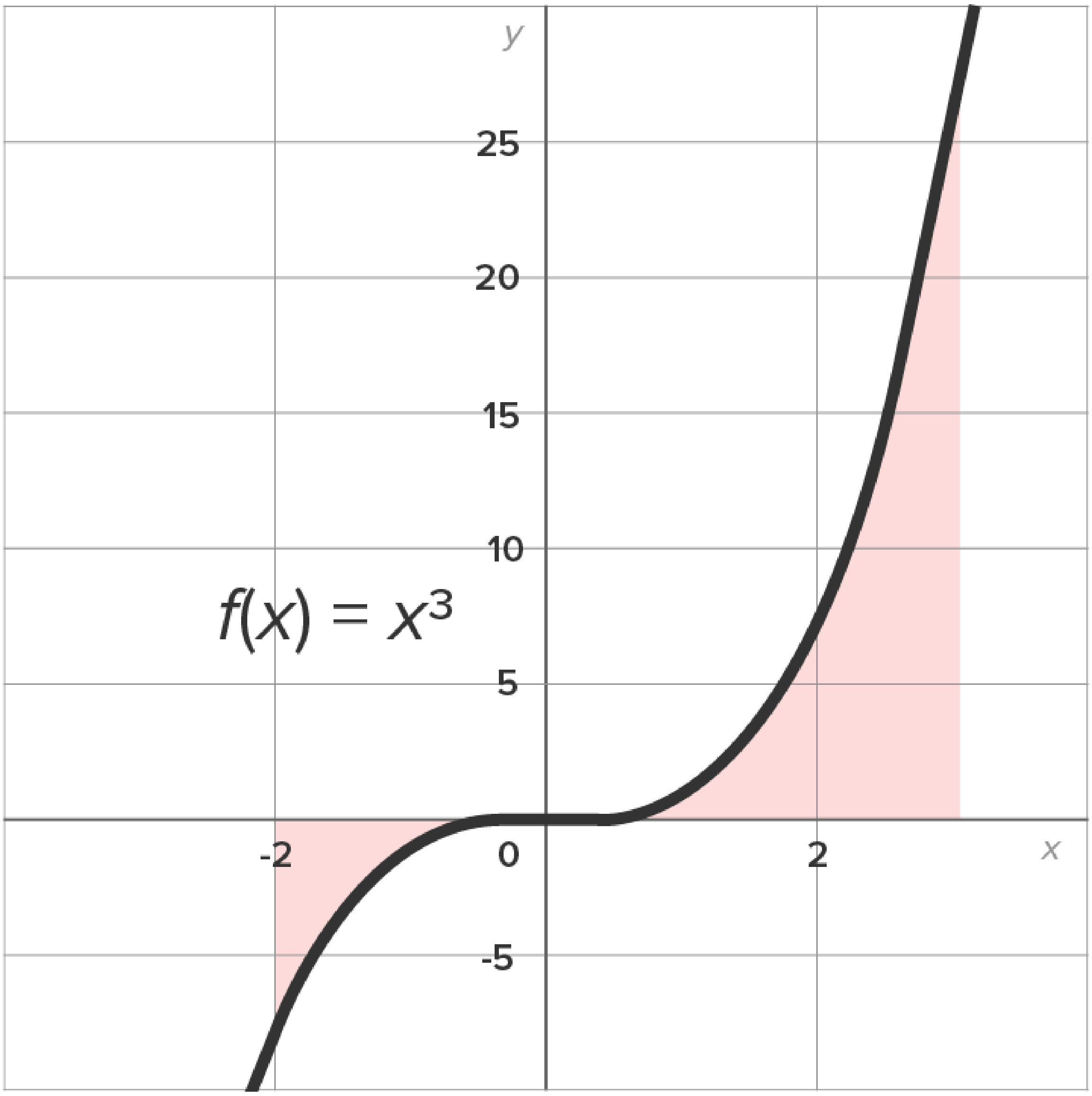
↪ EXAMPLE Find the total area between the x-axis and the curve $f(x) = x^3$ between $x = -2$ and $x = 3$. The region is shown in the figure below.




Notice that part of the region is below the x-axis and part of it is above the x-axis.

- On the interval $[-2, 0]$, the region is below the x-axis.
- On the interval $[0, 3]$, the region is above the x-axis.

This means $\int_{-2}^{0} x^3 dx$ will give the negative of the area and $\int_{0}^{3} x^3 dx$ will give the area.

- The region on $[-2, 0]$:
  $\int_{-2}^{0} x^3 dx = \frac{1}{4}x^4 \Big|_{-2}^{0} = \frac{1}{4}(0)^4 - \frac{1}{4}(-2)^4 = -4$
- The region on $[0, 3]$:
  $\int_{0}^{3} x^3 dx = \frac{1}{4}x^4 \Big|_{0}^{3} = \frac{1}{4}(3)^4 - \frac{1}{4}(0)^4 = \frac{81}{4}$

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Then, the total area of the region is $4+\frac{81}{4}=\frac{97}{4}\text{units}^2$.

WATCH

Check out this video where substitution is required, that shows finding the area bounded by $f(x)=x\sqrt{25-x^2}$, the x-axis, $x=-4$, and $x=3$.

## Video Transcription

[MUSIC PLAYING] Hello. Welcome to today's video. Here, I'm going to show you how to calculate the area of a region bounded by a function that is negative for some of the values over the interval and positive for others and also that requires u-substitution when integrating it. So here I've drawn the graph of f of x equals x times the square root of 25 minus x squared. So I'll outline it here. The x-axis, x equal negative 4-- that vertical line-- and x equal 3-- the vertical line there.

Now, if you notice from the picture, over the interval from an x value of negative 4 to an x value of 3, the region caught between those graphs is below the x-axis. So the definite integral there will come out negative and we'll need to take the negative of that negative number to get our positive value that will represent area. Over the interval from 0 to 3, the region is above the x-axis. So there the definite integral will be the same as the area.

Now, let's look then at our first definite integral from -4 to 0 of our x times the square root of 25 minus x squared dx. Here, we look at the expression 25 minus x squared as being acted on by the square root. So that's what we will let our u represent, that expression 25 minus x squared.

Now we want to do take the differential. The differential is du is equal to, and the derivative of 25 minus x squared is-- well, the derivative of 25 is 0. The derivative of negative x squared is a negative 2x and then our differential x

Now, notice here, we have our expression 25 minus x squared is the u. And then, as far as the x and the dx, that is part of our differential. However, that factor of negative 2 isn't in in integrand. So since it's a constant factor, I can divide that over to the other side and have negative 1/2 du will be the replacement for that x dx expression as we go.

So now, when we look at this, we also need to see that it's a definite integral. So we need to find out our replacements for our limits of integration. For the lower limit, when x is equal to negative 4, we'll get our u value is equal to-- we'll look at what u represent, 25 minus x squared. So it's 25 minus that negative 4 squared. And that gives me 25 minus 16, or u is equal to 9.

When x is equal to 0 for the upper limit of integration, our u comes out to be 25 minus 0 squared, which is 25. So this gives us the integral from replacing the negative 4x value with the 9 u value, to replacing the 0 x value to a 25 for my u. Then I have the square root of 25 minus x squared replacement is u. And then the x dx replacement is a negative 1/2 du.

Now factor out the negative 1/2 numerical factor, and rewrite the radical as a fractional exponent. So I have the negative 1/2 times the integral from 9 to 25 of u to the 1/2 du. Now, as we're integrating that, we notice that it's a letter base to a number exponent, and the number exponent is not negative 1. So we will

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use the power rule.

Keep the letter base u. Add 1 to the exponent. 1/2 plus 1 is 3/2. Divide by 3/2, which is multiplying by 2/3. And we are going to evaluate that from 9 to 25.

Common factors of 2's remove. I have a negative 1/3 times 25 to the 3/2 minus a negative 1/3 times 9 to the 3/2. Now remember, when you're working with fractional exponents, the denominator of the fractional exponent is the root, and the numerator is the exponent. So we'll think the denominator of my fractional exponent is 2, so that's square root. Square root of 25 is 5, and 5 cubed is 125. So negative 1/3 times 125 is a negative 125/3.

And then I have minus a negative or plus. Square root of 9 is 3. 3 cubed is 27. So I have 27/3. And I know that simplifies, but I want to add these fractions, so we need to have those common denominators.

Negative 125/3 plus 27/3 is a negative 98/3. It came out to be a negative definite integral like we anticipated. So remember, when we go to get the total area, we're actually going to use the opposite of negative 98/3 or 98/3 in with our area work. OK.

Now, let's look at the interval from 0 to 3. On our interval from 0 to 3, the substitution work that we did in integrating is going to be the same, it's just our limits of integration that we're going to need to work on. So remember, this is the integral from 0 to 3 of our x times the square root of 25 minus x squared dx.

Our u substitution, our du differential, and dividing the negative 1/2 over is all going to be the same. But for our limits of integration, a lower limit of integration of x equals 0, that's going to be u equal to my 25 minus 0 squared, or 25. And for the x value of 3, you will be 25 minus 3 squared. That's 25 minus 9, which is 16. So here I have the integral from 0's x value goes to a u value of 25, and 3's x value goes to a u value of 16.

And then I have the square root of u times that negative 1/2 du. And then integrating that, remember we're going to pull out the negative 1/2, have that as u to the 1/2 power, and we're going to add 1 to the exponent, divide by that like before. And you've got negative 1/3 u to the 3/2 evaluated from 25 to 16.

Plug in the upper limit of integration. I have 1/3 times 16 to the 3/2's power minus then a negative 1/3 times 25 to the 3/2 power. And that gives me-- well, the square root is 16 is 4, 4 cubed is 64-- I have negative 64/3 plus-- and then the square root of 25 is 5, 5 cubes is 125. So that gives me 125/3. And then my negative 64/3 plus 125/3 gives me 61/3.

So then my total area is positive 98/3 plus the 61/3, which gives me 159/3, which simplifies to 53 square units. And that gives our total area of the regions bounded by the function f of x equals x times the square root of 25 minus x squared. The x-axis, x equals negative 4 and x equal 3.

[MUSIC PLAYING]

Now that you've seen a few examples, here are some examples for you to try.

TRY IT

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Consider the region bounded by $f(x)=e^{2x}-x$, the x-axis, $x=0$, and $x=2$.

Find the exact area of the region. +

The region is completely above the x-axis, so the area is equal to $\int_0^2 (e^{2x}-x)dx = \left(\frac{1}{2}e^4 - \frac{5}{2}\right)$ units$^2$.

TRY IT

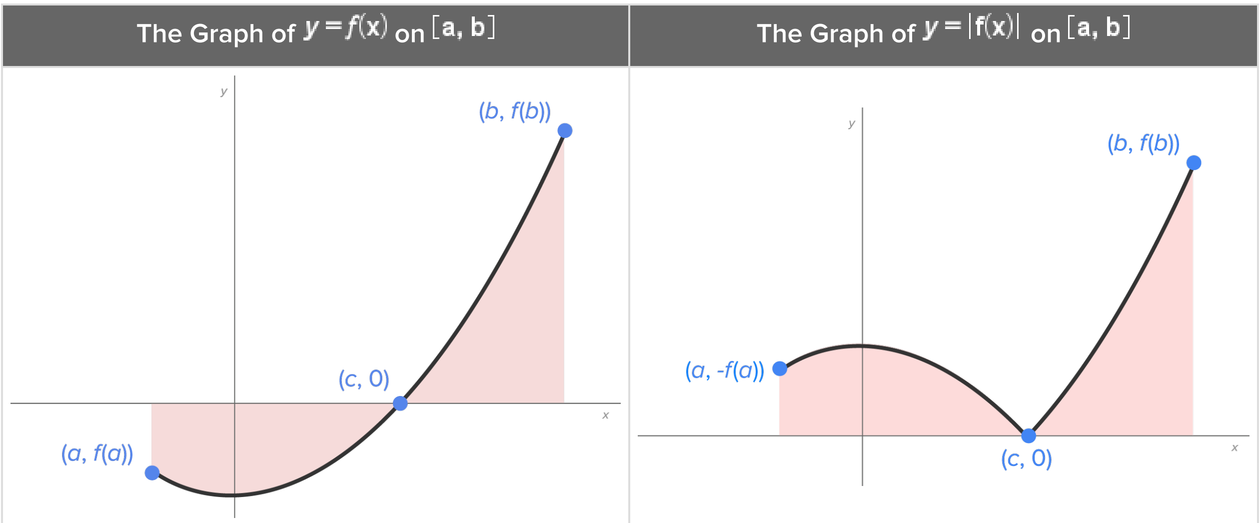
Consider the region bounded by $f(x)=\sin x$, the x-axis, $x=0$, and $x=\frac{3\pi}{2}$.

Find the exact area of the region. +

The region is partially above the x-axis and partially below the x-axis, so two integrals are required. Since $\int_0^{\pi} \sin x dx = 2$ and $\int_{\pi}^{3\pi/2} \sin x dx = -1$, the total area is $2+1=3$ units$^2$.

BIG IDEA

Consider the graphs of $y=f(x)$ and $y=|f(x)|$ shown below.




The regions on $[c, b]$ are identical. The regions on $[a, c]$ have the same area; one is above the x-axis, and the other is below the x-axis.

Since the graph of $y=|f(x)|$ is nonnegative on $[a, b]$, the definite integral $\int_a^b |f(x)|dx$ gives the area of the region between the graph of $y=|f(x)|$ and the x-axis between $x=a$ and $x=b$.

The drawback, however, is that $\int_a^b |f(x)|dx$ can be difficult to compute since finding antiderivatives with absolute value can be difficult if $f(x)$ changes sign over the interval $[a, b]$. However, if using technology, using $\int_a^b |f(x)|dx$ to calculate area is a nice way to find area, since it doesn't require a graph to calculate the area.

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↱ EXAMPLE Consider the region bounded by $f(x) = \sin x$, the x-axis, $x = 0$, and $x = \frac{3\pi}{2}$.

In a previous “TRY IT,” you calculated the total area to be 3, but that was by using two integrals since part of the region is below the x-axis.

Using technology, $\int_0^{3\pi/2} |\sin x| dx = 3$.

As it turns out, $\int_a^b |f(x)| dx$ can be extended to represent distance, as you’ll see in the next portion of this tutorial.

---

# 2. Calculating Distance Traveled and Net Change in Distance

Let $v(t)$ equal the velocity of an object at time *t*.

- If $v(t) > 0$, the object is moving in a forward direction.
- If $v(t) < 0$, the object is moving in a negative direction.

So, if $v(t)$ is the velocity of an object at time *t*, then $\int_a^b v(t) dt$ is the change in position between $t = a$ and $t = b$.

- If $\int_a^b v(t) dt$ is positive, then the object’s final position is ahead of its starting point.

  (Example: if $v(t)$ represents upward velocity, then the object finishes above its starting position at $t = a$).
- If $\int_a^b v(t) dt$ is negative, then the object’s final position is behind its starting point.

  (Example: if $v(t)$ represents upward velocity, then the object finishes below its starting position at $t = a$).
- If $\int_a^b v(t) dt = 0$, then the object’s final position is the same as its starting point.

It follows that $\int_a^b |v(t)| dt$ gives the total distance traveled (in either direction) between $t = a$ and $t = b$. We will still compute this by examining regions.

↱ EXAMPLE An object has velocity $v(t) = 60 - 12\sqrt{t}$ feet per second, where *t* is the number of seconds.

a. What is the object’s change in position after its first 100 seconds of travel?
b. What is the total distance traveled after the first 100 seconds?

Let’s first find the object’s change in position.

a. What is the object’s change in position after its first 100 seconds of travel?

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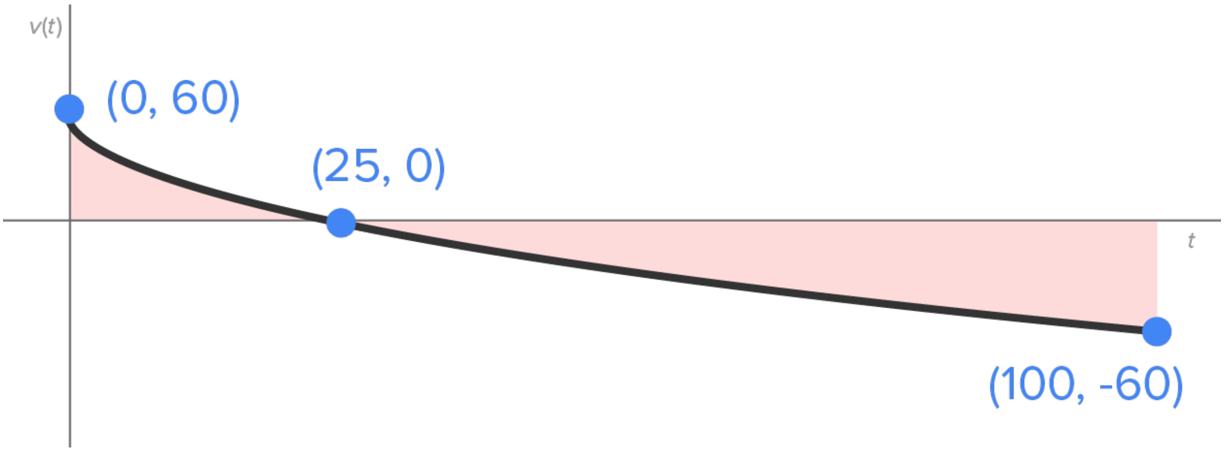
Note: since $v(t)$ is measured in feet per second and $t$ is measured in seconds, distance is measured in feet. If we are looking for a change in position, this is found by evaluating $\int_0^{100} (60 - 12\sqrt{t})dt$.

| | |
|---|---|
| $\int_0^{100} (60 - 12\sqrt{t})dt$ | Start with the original expression. |
| $= \int_0^{100} (60 - 12t^{1/2})dt$ | Rewrite the square root as a power so that the power rule can be used. |
| $= \left(60t - 8t^{3/2}\right)\Big\vert_0^{100}$ | Apply the fundamental theorem of calculus. Note: $\int 12t^{1/2}dt = 12\left(\frac{2}{3}\right)t^{3/2} = 8t^{3/2}$ |
| $= [60(100) - 8(100)^{3/2}] - [60(0) - 8(0)^{3/2}]$ | Substitute the upper and lower endpoints. |
| $= 6000 - 8(1000)$ | Evaluate. |
| $= -2000$ | Simplify. |

Since the result is negative, this means that the object's final position is 2000 ft behind its starting point at $t = 0$.

b. What is the total distance traveled after the first 100 seconds?

This requires us to look at the graph of $v(t)$ and the t-axis over the interval $[0, 100]$. The graph along with the region between $v(t)$ and the t-axis is shown in the figure below.




On the interval $[0, 25]$, the region is above the t-axis, and on the interval $[25, 100]$, the region is below the t-axis.

Remember also that you can find the t-intercept using algebra:

$$60 - 12\sqrt{t} = 0$$

$$60 = 12\sqrt{t}$$

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$$5 = \sqrt{t}$$
$$25 = t$$

To find the total distance traveled, we'll need to compute two integrals. Luckily, from part (a), we already know the antiderivative.

| Interval | Calculation | Explanation | Distance Traveled |
|---|---|---|---|
| Distance traveled on $[0, 25]$ | $\int_0^{25} (60 - 12t^{1/2})dt$<br>$= (60t - 8t^{3/2})\Big\vert_0^{25}$<br>$= [60(25) - 8(25)^{3/2}] - [60(0) - 8(0)^{3/2}]$<br>$= 1500 - 8(125)$<br>$= 500$ | This result means that the object traveled 500 feet in the positive direction on the interval $[0, 25]$. | 500 feet |
| Distance traveled on $[25, 100]$ | $\int_{25}^{100} (60 - 12t^{1/2})dt$<br>$= (60t - 8t^{3/2})\Big\vert_{25}^{100}$<br>$= [60(100) - 8(100)^{3/2}] - [60(25) - 8(25)^{3/2}]$<br>$= [6000 - 8(1000)] - [1500 - 8(125)]$<br>$= -2000 - 500$<br>$= -2500$ | This result means that the object traveled 2500 feet in the negative direction on the interval $[25, 100]$. | 2500 feet |

Thus, the total distance traveled on $[0, 100] = 500 + 2500 = 3000$ feet.

WATCH

This video walks you through an example of finding an object's change in position and total distance traveled using a definite integral.

## Video Transcription

[MUSIC PLAYING] Hi, there. Welcome to the video on antiderivative applications. This application is going to focus on having a velocity function, and then answering questions about changing position and total distance traveled. So let's get started. Here we have of the velocity of an object in motion after t seconds is given by the function v of t is equal to 48 minus 24 times the cubed root of t in feet per second.

And the first part says, what is the object's change in position after the first 64 seconds of travel? Well, that is just found by the definite integral from 0 the start to 64 for our-- after the first 64 seconds of our velocity function, 48 minus 24. And we're going to write the cubed root of t as t to the 1/3 power to get it ready to integrate. And then our differential is dt.

Now, here, I can just integrate term by term. So this is 48t minus 24 times. And 4t to the 1/3 power we'll use the general power rule. So we we'll add 1 to the exponent. 1/3 plus 1 is 4/3. And then we'll divide by that 4/3, and evaluate that from 0 to 64.

Simplifying this a little bit before we plug in, we have 48t minus-- dividing by 4/3 is multiplying by 3/4. And when we multiply 24 by 3/4, we get 18. And that's t to the 4/3 power, and evaluated from 0 to 64.

So we are going to plug in the 64 for the t's first. Remember, plug in that upper limit of integration to start, and then minus 18 times that 64 to the 4/3 power, and then minus plugging in the lower limit of integration, our 0, so 48 times 0 minus 18 times 0 to the 4/3.

And as we go through and evaluate that, you will find that you get negative 1,536. So that is telling us that our object is 1,536 feet behind its original starting point when it ends, or after its 64 seconds of travel. So we will write that. It's 1,536 feet behind the starting point. And that's after it's first 64 seconds.

Now, part B asks, what is the total distance traveled after the first 64 seconds? So here what we need to do is make sure that we keep track of the distance traveled, no matter which direction it's going. I have graphed the velocity function down here underneath part B.

And you'll notice that the graph has some portion of it that's shaded above the t axis, and a larger region that's shaded below the t axis. And that's actually in line with what we got for part A. We ended up with a negative value overall. And that's because more of the shading was below.

For us to get the total distance, well remember, the distance will be the same as the area when our area is above the t axis. And then when our region is below the t axis, it'll be the negative of it. So we'll have to take the opposite of that negative to add those together.

So the first thing we need to do is to find this point where the graph crosses the t axis. And that we find by just taking our original function, 48 minus 24 times the cubed root of t, and setting that equal to 0, because we want to find the t value where our function value comes out 0.

Adding the 24 times the cubed root of t to both sides, we get 48 equals 24 times the cubed root of t. Divide both sides by 24, we get 2 is equal to the cubed root of t. And cubing both sides, we get t equals 8. So what we want to do for our distance is, for our interval 0 to 8, we notice that's where it's above the t axis.

So we'll just calculate that. And that's the integral 0 to 8 of 48 minus 24 times t to the 1/3 dt. We already found the antiderivative for this in the previous line, in the previous work. So we have-- that's equal to 48t minus 18 t to the 4/3. And we're evaluating that from 0 to 8.

When you plug in the 8, then minus plug in the 0, you get a value of 96. Now on the interval from 8 to 64, when we evaluate that definite integral of our velocity function, and go through, find-- we have our antiderivative that we're using. We're going to evaluate that from 8 to 64, plugging in the 64 minus plugging in the 8. For that value, we get a negative 1,632.

And so then our total area, which would be our total distance, is going to be our 96, and then plus we're going to take the opposite of negative 1,632, which is a 1,632. And we get our total distance is 1,728. And that is in feet. And that is an application of our antiderivative.

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TRY IT

The velocity of an object in motion after $t$ minutes is given by the function $v(t) = 20 - 10e^{-t}$ feet per minute on the interval $[0, 5]$.

Find the distance traveled on the interval [0, 5]. Give both the exact answer and rounded to the nearest whole foot. +

Exact: $90 + 10e^{-5}$ feet. Approximate: 90 feet.

Is the distance traveled on the interval [0, 5] equal to the change in position after 5 minutes? Why or why not? +

They are equal since the graph of $v(t)$ is above the t-axis on the interval $[0, 5]$, indicating that $v(t)$ is positive on $[0, 5]$.

## SUMMARY

In this lesson, you learned that by applying the fundamental theorem of calculus, you are now able to **calculate areas of regions** as well as **calculate distance traveled and net change in distance** exactly rather than using approximation techniques.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

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# The Area Between Two Curves that Do Not Intertwine

*by Sophia*

WHAT'S COVERED

In this lesson, you will use the fundamental theorem of calculus to find the areas of regions bounded by two curves (for now, the regions will not intertwine). Specifically, this lesson will cover:

1. The Idea Behind the Area Between Two Curves That Do Not Intertwine
2. Finding the Area Between Two Curves That Do Not Intertwine

# 1. The Idea Behind the Area Between Two Curves That Do Not Intertwine

In addition to the regions we have worked with up to now, we also need to consider regions that do not have an axis as an edge.

Consider the areas in the figure below where:

- $R_1$ is the area of the region between the graphs of $y=f(x)$ and $y=g(x)$ on the interval $[a, b]$.
- $R_2$ is the area of the region between the x-axis and $y=g(x)$ on the interval $[a, b]$.
- $R_1+R_2$ is the area of the region between the x-axis and the graph of $y=f(x)$ on $[a, b]$.

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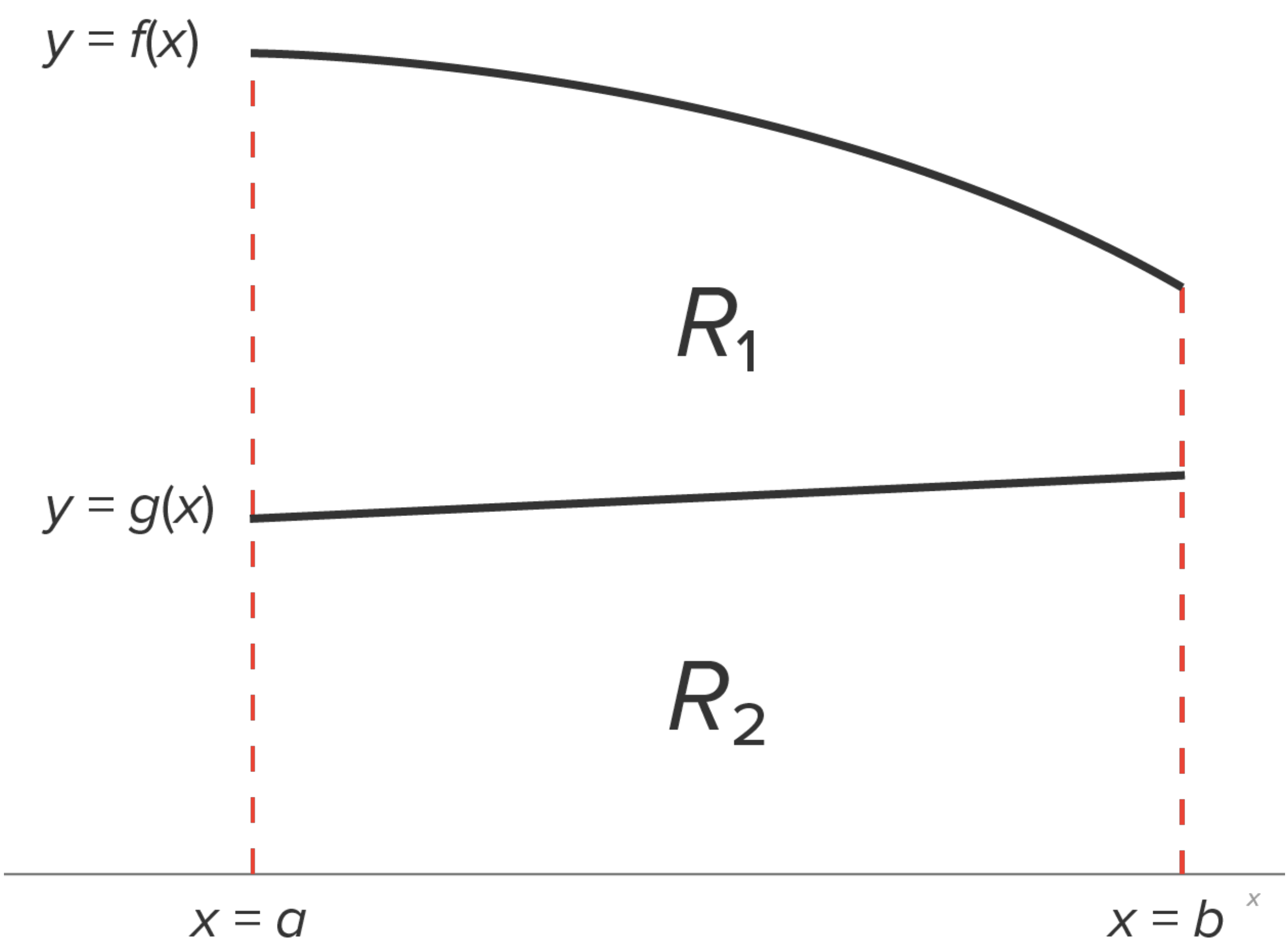


From the picture, we know that $\int_a^b f(x)dx = R_1 + R_2$. We also see that $R_2 = \int_a^b g(x)dx$.

Replacing $R_2$ with $\int_a^b g(x)dx$ into the first equation, we have $\int_a^b f(x)dx = R_1 + \int_a^b g(x)dx$.

Solving for $R_1$, we have $R_1 = \int_a^b f(x)dx - \int_a^b g(x)dx$.

By properties of definite integrals, we know this is equal to $R_1 = \int_a^b [f(x) - g(x)]dx$.

Thus, assuming that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $[a, b]$, we have the following formula to find the area of the region between the graphs.

FORMULA

**Area Between Two Curves, $y = f(x)$ and $y = g(x)$, Assuming $f(x) \geq g(x)$ on $[a, b]$**

$$Area = \int_a^b [f(x) - g(x)]dx$$

We'll use this idea to find the areas of some regions bounded between two curves.

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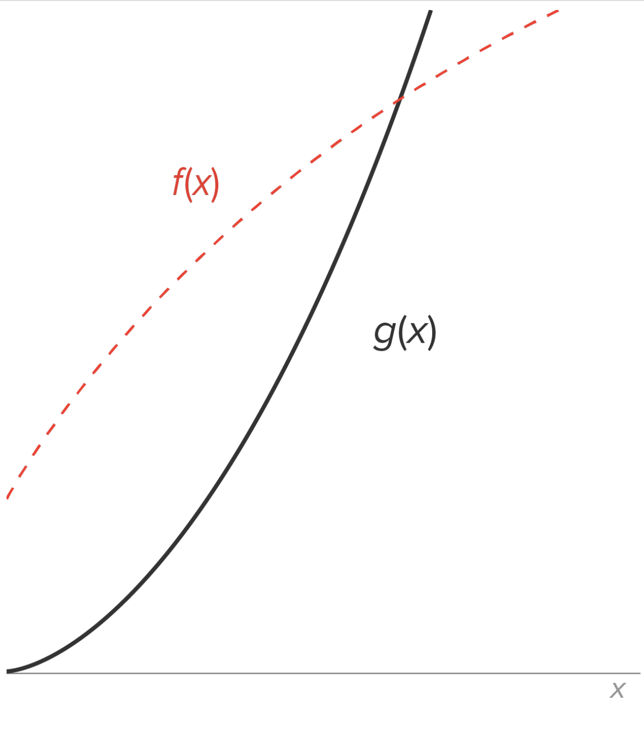
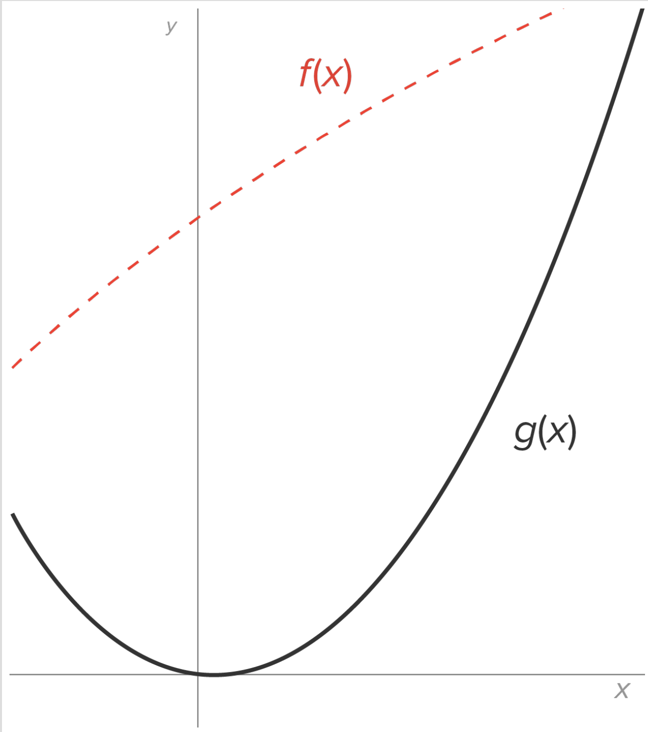
# 2. Finding the Area Between Two Curves That Do Not Intertwine

In order to find the area of this type of region, first verify that the graphs do not intertwine.

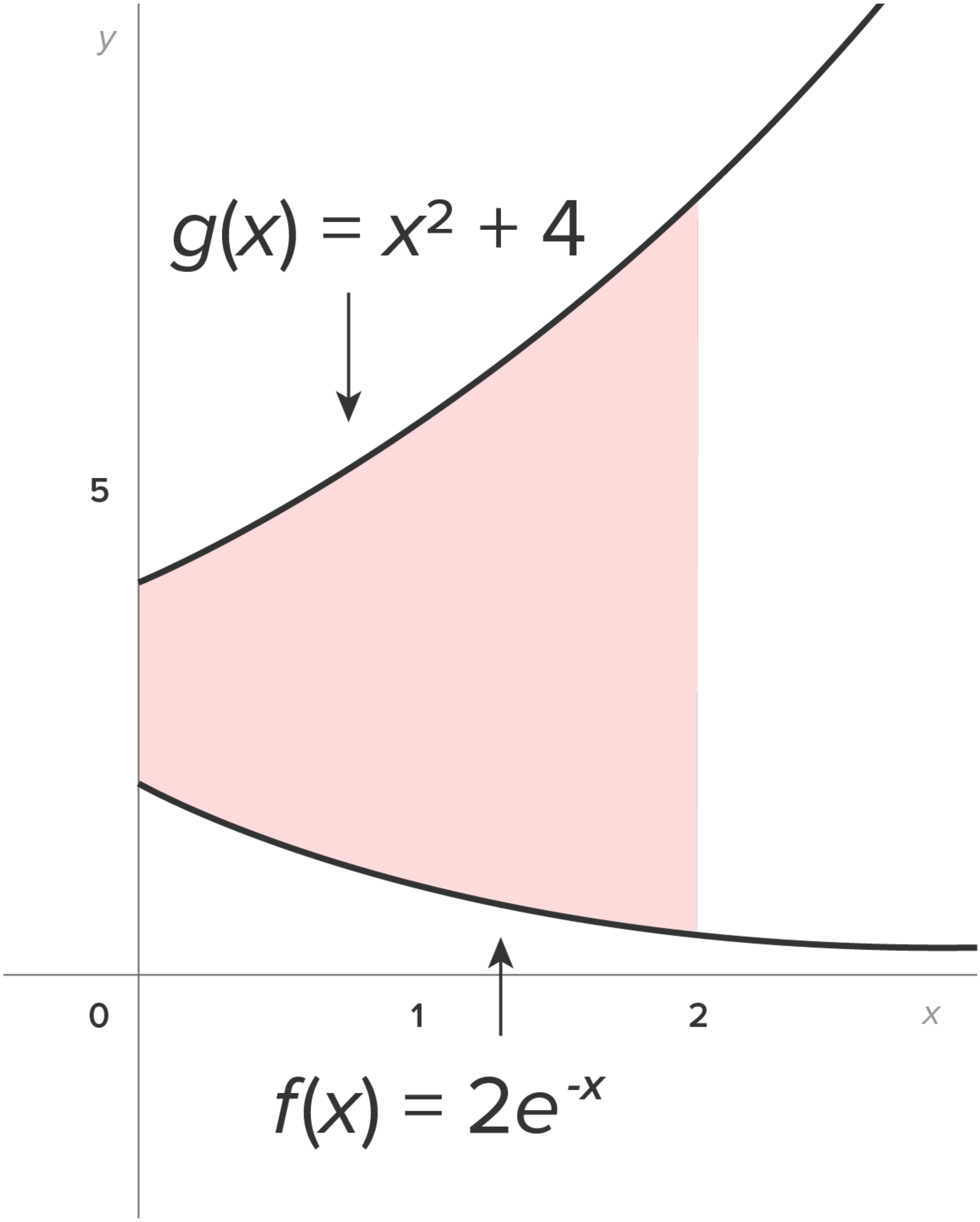
| When $f(x)$ and $g(x)$ Intertwine | When $f(x)$ and $g(x)$ Do NOT Intertwine |
|---|---|
|   |   |

HINT

If the graphs meet at either endpoint, they are not considered intertwining. We will discuss intertwining graphs in the next tutorial.

EXAMPLE Find the exact area between the graphs of $f(x) = 2e^{-x}$ and $g(x) = x^2 + 4$ between $x = 0$ and $x = 2$.

The graph of the two curves on $[0, 2]$ is shown in the figure below.




The figure shows that $g(x)=x^2+4$ is higher than $f(x)=2e^{-x}$ on the entire interval. Then, the area of the region is $\int_0^2 (x^2+4-2e^{-x})dx$.

Now we evaluate the integral:

$\int_0^2 (x^2+4-2e^{-x})dx$ Start with the original expression.

Apply the fundamental theorem of calculus.

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$$= \left(\frac{1}{3}x^3 + 4x + 2e^{-x}\right)\Big|_0^2$$ Note: $\int 2e^{-x}dx = \frac{2}{-1}e^{-x} + C = -2e^{-x} + C$

$$= \left[\frac{1}{3}(2)^3 + 4(2) + 2e^{-2}\right] - \left[\frac{1}{3}(0)^3 + 4(0) + 2e^{-0}\right]$$ Substitute the upper and lower endpoints.

$$= \frac{8}{3} + 8 + 2e^{-2} - 2e^0$$ Evaluate the brackets.

$$= \frac{8}{3} + 8 + \frac{2}{e^2} - 2$$ Note: $e^0 = 1$

$$= \frac{26}{3} + \frac{2}{e^2}$$ Simplify.

Then, the area of the region is $\frac{26}{3} + \frac{2}{e^2}$ units$^2$.

Here is an example where the interval is not given.

→ EXAMPLE Find the exact area of the region in the first quadrant that is bounded by the graphs of $y = x$, $y = \frac{4}{x}$, and $x = 4$.

The graph of the region in the first quadrant is shown in the figure.

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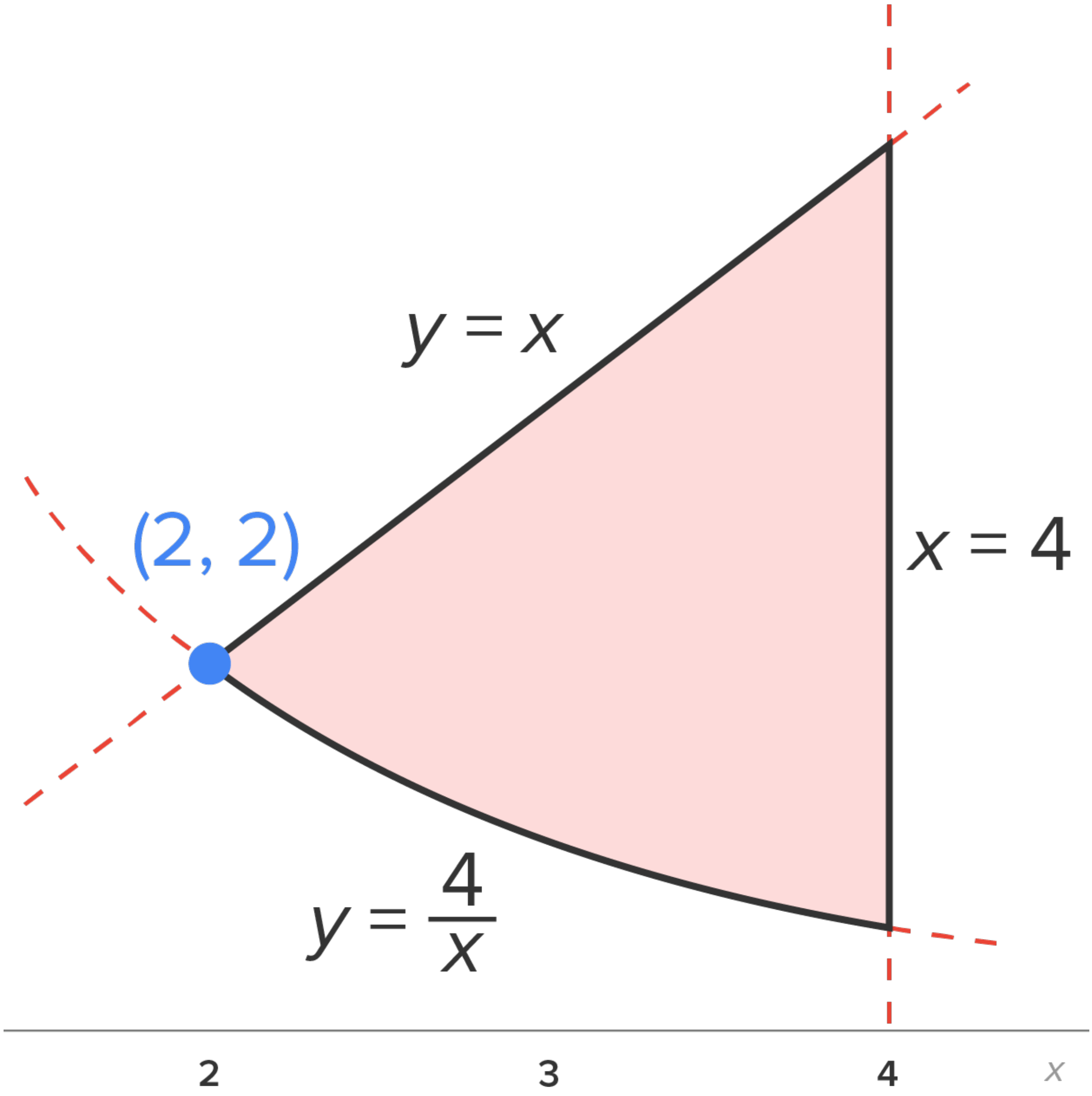


To find the intersection point, set $x = \frac{4}{x}$ and solve:

$$x = \frac{4}{x}$$

$$x^2 = 4$$

$$x = \pm 2$$

Since the region is in the first quadrant, only $x = 2$ is considered.

Also, since the graph of $y = x$ is above the graph of $y = \frac{4}{x}$ on the interval $[2, 4]$, the definite integral that gives the area is $\int_2^4 \left(x - \frac{4}{x}\right)dx$.

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Now evaluate the definite integral.

| | |
|---|---|
| $\int_2^4 \left(x - \frac{4}{x}\right)dx$ | Start with the original expression. |
| $= \left(\frac{1}{2}x^2 - 4\ln\lvert x\rvert\right)\Big\rvert_2^4$ | Apply the fundamental theorem of calculus. |
| $= \left[\frac{1}{2}(4)^2 - 4\ln\lvert 4\rvert\right] - \left[\frac{1}{2}(2)^2 - 4\ln\lvert 2\rvert\right]$ | Substitute the upper and lower endpoints. |
| $= [8 - 4\ln\lvert 4\rvert] - [2 - 4\ln\lvert 2\rvert]$ | Evaluate the parentheses. |
| $= 6 - 4\ln 4 + 4\ln 2$ | Simplify. |
| $= 6 - 4(\ln 4 - \ln 2)$ | Factor. |
| $= 6 - 4\ln\left(\frac{4}{2}\right)$ | Use the property $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$. |
| $= 6 - 4\ln 2$ | Simplify. |

Thus, the area of the region is equal to $6 - 4\ln 2 \text{ units}^2$.

WATCH

Check out this video to learn how to find the exact area between $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

## Video Transcription

[MUSIC PLAYING] Welcome to the video on how to find the exact area between the graphs of f of x equals sine x, and g of x equal cosine x, and the interval from pi over 4 to 5 pi over 4. Now, I've graphed these two functions over on this graph on the right side of the screen. It's important for us to identify which function is which. So let's look at the function f of x is equal to sine x while remembering that sign of 0 is 0, and sine of pi over 2 is 1, et cetera, the curve that is represented with this curve graph is your f of x is equal to sine x.

Then for our g of x is equal to cosine x we have that it is cosine of 0 is 1 and cosine of pi over 2 is 0. So we get this curve for g of x is equal to cosine of x. So now, when we are looking at finding the exact area between the graphs over a specific interval we want to think about where that interval is and to graph vertical lines at x equal the start of your interval and then x equal the end of your interval your, 5 pi over 4.

Now, pi over 4, when we think about that value pi over 4 is a value where the sine of pi over 4 is the square root of 2 over 2. The cosine of pi over 4 is also the square root of 2 over 2. So that's actually right underneath that point of intersection. Then when we look at 5 pi over 4, the sine of 5 pi over 4 is negative the square root of 2 over 2 and the cosine of 5 pi over 4 is also negative the square root of 2 over 2. So we have right in line with this point of intersection is my 5 pi over 4.

Now, we could go through and actually find those using our methods of solving a trig equation to find

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the point where they intersect. We have that the sine of x would have to equal the cosine of x and as long as the cosine of x is not zero I can divide both sides by cosine of x. Then sine x divided by cosine x with a trig identity is tangent x and cosine x divided by cosine x when cosine x isn't 0 is 1. The angles that have a tangent ratio of 1, still keeping in mind this interval from pi over 4 to 5 pi over 4, well, it's true that the tangent of pi over 4 is equal to 1 and it's also true that the tangent of 5 pi over 4 is equal to 1.

So I actually have the points of intersection for these two curves are happening at the end points of my interval, but my interval is a closed interval so I can use those values. So when I think about what region is enclosed by those two curves over that interval, it's this region in here. So that's what we want to find the area of.

Now, as I look from left to right the blue curve is consistently higher on the graph than the yellow curve. Remember when we're finding the area between the two curves we want to take the definite integral over the interval where we have that consistency of one curve either higher than the other or if I'm going to do my differential dy one curve right of the other throughout, but here we have it higher than the other. We take our curve that is further up on the coordinate plane, that's the sine of x, minus the curve that's lower on the coordinate plane, that's the cosine of x.

Now, to evaluate this definite integral we'll find an antiderivative of this function and it's term by term that we can use. The antiderivative of sine x is negative cosine x, minus antiderivative of cosine x is sine x, and I'm going to evaluate that from pi over 4 to 5 pi over 4. Recall with the fundamental theorem of calculus, you're definite integral from a to b of f of x is equal to your antiderivative evaluated at the upper limit of integration minus antiderivative evaluated at the lower limit of integration.

So we are going to put in the 5 pi over 4 first for the x throughout the expression, minus, keeping track of your negative signs, putting in now the pi over 4 for the x. This gives me now, simplifying negative-- Well, the cosine of 5 pi over 4 is negative the square root of 2 over 2, then minus the sine of 5 pi over 4 is also negative the square root of 2 over 2. Then I have my subtraction from the fundamental theorem of calculus. Then I have minus the cosine of pi over 4 is square root of 2 over 2, subtract, then the sine of pi over 4 is square root of 2 over 2.

So in this first set of parentheses I have minus a negative, so plus the square root of 2 over 2. Again, plus the square root of 2 over 2 is two square roots of 2 over 2. Then minus, in the second set of parentheses, that's going to be negative two square root of 2 over 2. While the common factors of twos will remove and I have the square root of 2, minus negative, so plus the square root of 2. I get my definite integral value, which will then represent the area in between there is two square roots of 2. So it's two square roots of 2, square units for the exact area between those two graphs over that interval.

[MUSIC PLAYING]

TRY IT

Consider the region bounded by the graphs of $y=4$ and $y=\cos x$ on the interval $[0, 2\pi]$.

Find the exact area of the region. +

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$8\pi$ units$^2$

TRY IT

Consider the region bounded by the graphs of $y = 4x$ and $y = x^2 + 3$.

Find the exact area of the region. +

$\frac{4}{3}$ units$^2$

WATCH

There are situations where it is advantageous to use horizontal subrectangles instead of vertical ones. Check out this video to learn how to find the exact area between $x = y^2 - 4$ and $x = -2y - 1$.

## Video Transcription

[MUSIC PLAYING] Well, hi there. What I have to show you today is how to find the exact area between the graphs of two different curves that aren't necessarily functions. Now, to kind of set it up, Let's look back at what we've already done when we were finding the exact area between graphs of functions.

So over on the right, down below, I've written up, just graphed a picture of what we have seen before when we're finding the area between two curves that are functions, and they don't intertwine, meaning they don't loop back and forth across each other. So here in this picture, when we see our graph, and we are looking at, if we spanned from left to right and had vertical subrectangles as we go along, then we would use the differential dx.

And the way we would find the area caught between those two curves is we would identify the curve that is above the other one over the region-- here, that's the line, which is g of x-- and then subtract off of that the curve that is below throughout the region. Here, that is the parabola, which I've called f of x

And then we would find our endpoints of integration by the x-coordinate of the leftmost point that they intersect at and the upper limit of integration to be the x-coordinate of the rightmost point that they intersect at if they didn't give me an interval. Certainly, if they gave me an interval, then I would do the endpoints of the interval for my integration.

But if I look at the problem that we're asked to do in this segment, find the exact area between the graphs of x equal y squared minus 4, well, that y squared minus 4 is the parabola that's opening to the right. And x equal negative 2y minus 1-- well, that's just the line that we've graphed here.

And if I try to look at using my subrectangles, vertical like before, and going across the region from left to right, I see, as I start, it's the top of the parabola is the top of my vertical line, or my vertical subrectangle. The bottom of the parabola is the bottom. And then I get to this point where then it's the line on top and the lower branch of the parabola on the bottom.

And while that's not impossible to do, it's really not very advantageous for us. And it's harder to work with

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that way. So what we want to do is, if we recognize that it's just not to our advantage to think of it in that way, what if we ran our subrectangles horizontally and scanned from the bottom of the region to the top of the region?

So starting at the bottom of this region, running them horizontally, I see that as I draw these, there is consistency of one of the curves is always on the right edge of the region. Here on the right edge of the region is that line. And then the other curve is always to the left edge of that region. And I wanted to change that color as we go.

So I want to run my subrectangles horizontally. I get more of a unified idea there. And if I run them horizontally, that's using the differential dy. Also, when I use the differential dy, the endpoints of my integration have to be the lower coordinate. That's the y-coordinate. And then the upper is the upper y-coordinate of your point of intersection.

And then we do our subtraction of the right curve minus the left curve. Some questions can actually be done either way. You'll still get the same answer. Some, you'll find one way is just much more beneficial than the other, as is this case.

Now, when we do this with the subtraction, and it's the differential of dy-- we've used the y-coordinate-- I need to also make sure that these are expression in y's. Your variable expressions have to have the same variable as your differential when you're doing these problems.

So now, when I look at setting this up, one of the first things I need to do is find my points of intersection. And I want the y-coordinate of this lower point of intersection and this higher point of intersection. So with our equations, since x is equal to y squared minus 4, and x is equal to negative 2y minus 1, I'll just take out the x, and put in the negative 2y minus 1, and set that equal to y squared minus 4.

And it's quadratic. So we're going to take everything on one side set equal to 0. Then factoring y squared plus 2y minus 3, that's y plus 3, y minus 1. And then setting each factor equal to 0, I get y is equal to negative 3, or y is equal to 1. So my lower limit of integration is that negative 3, to my upper limit of integration is my y value of 1. And I don't need to go find the x's because, again, we're using the differential dy.

Now, the rightmost curve is the line, remember. So that is this equation. And if it weren't solved with the x alone, and the expression y is on the other side, I would need to do that first. But it is solved that way. So I have my negative 2y minus 1, and then minus the parentheses. And for the blue, that's the parabola. And it has x equal. So let's solve for x in terms of y-- y squared minus 4. And remember, this is dy.

So we are going to remove the parentheses and collect like terms. And we'll have a negative y squared minus 2y. And then remember, that negative 1 has to be distributed through that parentheses. My constant will be negative 1 plus 4, which is a positive 3. And that's dy.

Integrating, that's negative 1/3 y cubed minus-- well, I'm going to have y squared, and I'm going to divide by 2. And that'll remove that 2 that we have there. And then plus-- the integral of 3dy is 3y. And we are evaluating that from -3 to 1.

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Plug in the 1 first. That's negative 1/3 times 1 to the power 3 minus 1 squared plus 3 times 1. Subtract from it, plugging in the negative 3-- negative 1/3 times negative 3 cubed minus negative 3 squared plus 3 times negative 3.

So that's negative 1/3 minus 1 plus 3, and then minus, in the parentheses, negative 3 cubed is -27. Negative 1/3 times -27 is a positive 9. Negative 3 squared is positive 9. So then I have minus 9. And then 3 times negative 3 is minus 9. So we have negative 1/3 plus 2 and minus a negative 9. And that gives me negative 1/3 plus 2 plus 9. And that gives me 32/3.

And remember, this is area. So it's 32/3 square units. And that is how you find the area between graphs where it's better to do horizontal subrectangles instead of vertical subrectangles. So we are working with the differential dy and our expressions in y instead of dx and expressions in x.

[MUSIC PLAYING]

To summarize the previous video, we have the following formula to calculate the area between two curves using horizontal subrectangles:

FORMULA

**Area Between Two Curves, $x = h(y)$ and $x = k(y)$, Assuming $h(y) \geq k(y)$ on $[c, d]$ (Horizontal Subrectangles)**

$$Area = \int_c^d [h(y) - k(y)]dy$$

## SUMMARY

In this lesson, you learned how to apply the fundamental theorem of calculus to compute the area of a region that does not have the x-axis as a boundary, **finding the area between two curves that do not intertwine**. Note that if the graphs meet at either endpoint, they are not considered intertwining. In the next tutorial, we will apply what we’ve learned in this tutorial to tackle areas where the boundary curves do intertwine.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## FORMULAS TO KNOW

**Area Between Two Curves, x = h(y) and x = k(y), Assuming h(y) ≥ k(y) on [c, d] (Horizontal Subrectangles)**

$$Area = \int_c^d [h(y) - k(y)]dy$$

**Area Between Two Curves, y = f(x) and y = g(x), Assuming f(x) ≥ g(x) on [a, b]**

$$Area = \int_a^b [f(x) - g(x)]dx$$

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# The Area Between Two Curves that Intertwine

*by Sophia*

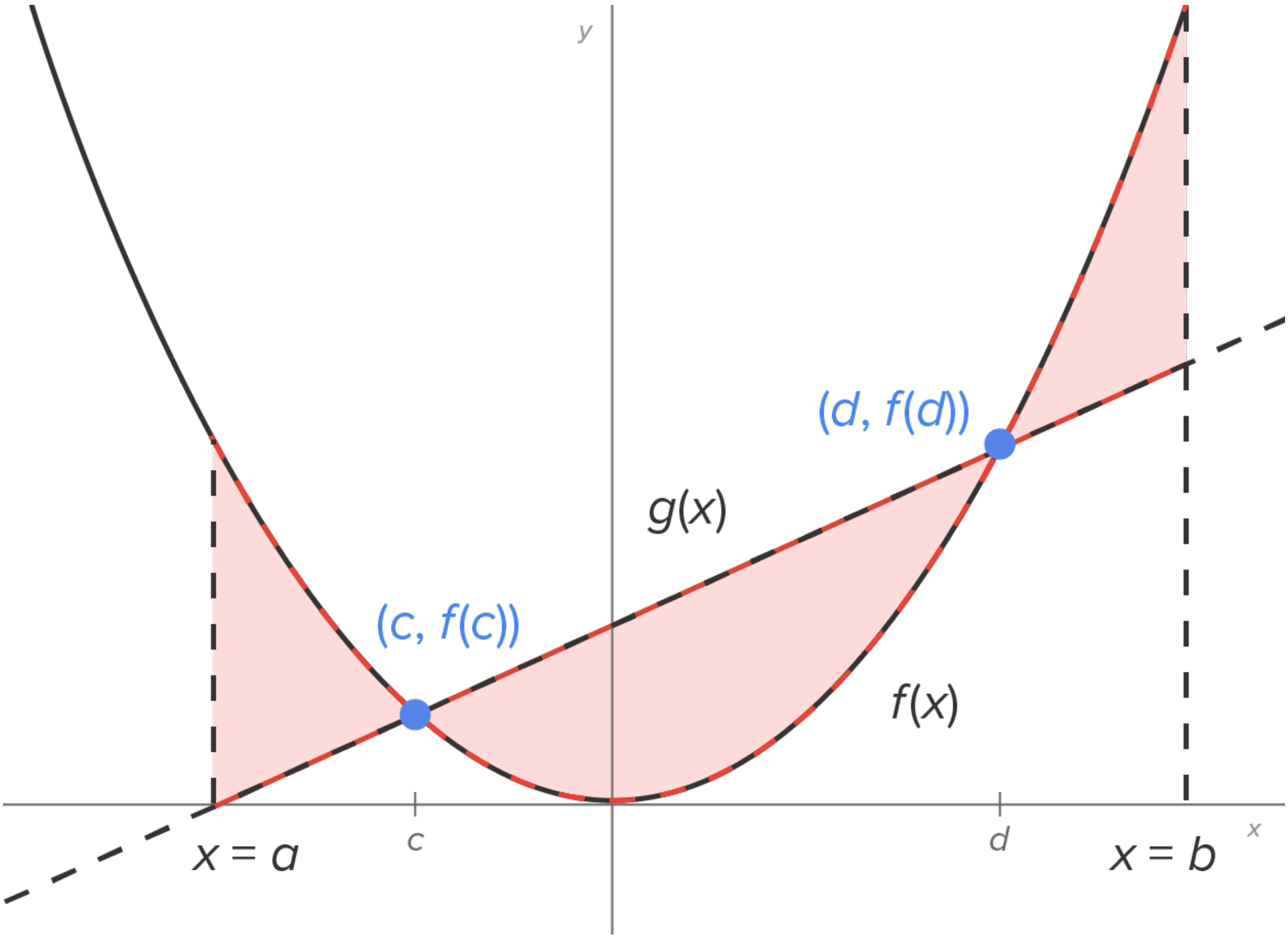
WHAT'S COVERED

In this lesson, you will learn how to find areas of regions when the curves intertwine. Specifically, this lesson will cover:

1. Introduction: A Strategy
2. Finding Areas of Regions Between Two Curves That Intertwine

## 1. Introduction: A Strategy

Consider the region shown in the figure below.




The graph of $f(x)$ has a parabolic shape, and the graph of $g(x)$ is linear. Observe the following:

- On the interval $[a, c]$, the graph of $f(x)$ is above the graph of $g(x)$.
- On the interval $[c, d]$, the graph of $g(x)$ is above the graph of $f(x)$.

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- On the interval $[d, b]$, the graph of $f(x)$ is above the graph of $g(x)$.

Recall from the last tutorial that $\int_a^b (f(x) - g(x))dx$ is the area of the region between two graphs as long as $f(x) \geq g(x)$, meaning that the graph of $f(x)$ is at least as high as the graph of $g(x)$ on $[a, b]$.
Considering the region again, it stands to reason that the integral expression for the total area is:

$$\int_a^c (f(x) - g(x))dx + \int_c^d (g(x) - f(x))dx + \int_d^b (f(x) - g(x))dx$$

This is our strategy for calculating the area of a region between two curves when the curves intertwine.

BIG IDEA

When finding the area of the region between two curves $y = f(x)$ and $y = g(x)$, keep the following in mind:

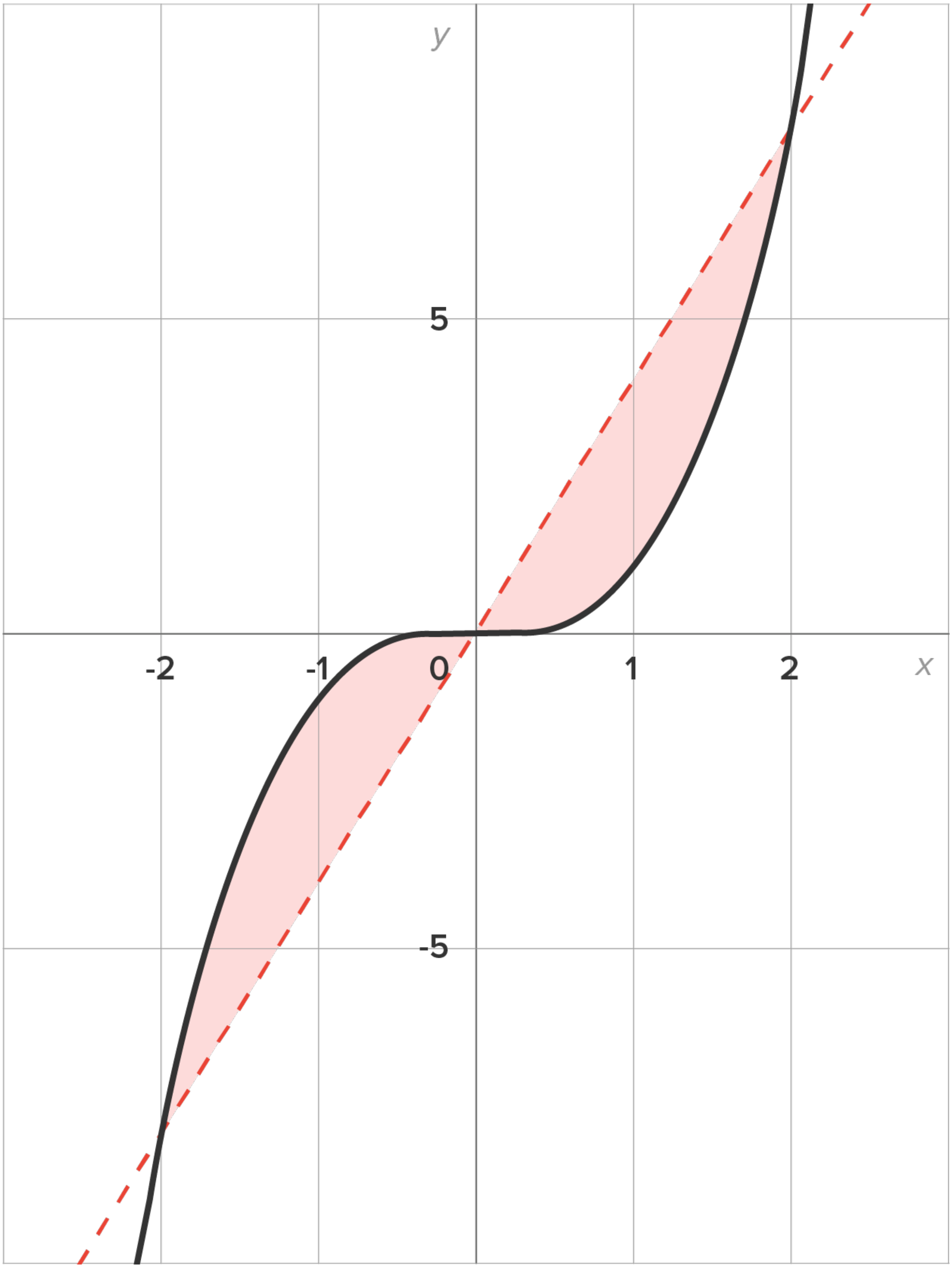
- If $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [f(x) - g(x)]dx$.
- If $f(x) \leq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [g(x) - f(x)]dx$.

That is, the integrand is “upper – lower.”

---

## 2. Finding Areas of Regions Between Two Curves That Intertwine

→ EXAMPLE Find the total area bounded by the graphs of $y = x^3$ and $y = 4x$. The graph of the region is shown in the figure below.




First, find the points where the graphs intersect:

$$x^3 = 4x$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x+2)(x-2) = 0$$

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$x = 0, x = -2, x = 2$

- On the interval $[-2, 0]$, the graph of $y = x^3$ is above the graph of $y = 4x$.
- On the interval $[2, 0]$, the graph of $y = 4x$ is above the graph of $y = x^3$.

Thus, the total area is $\int_{-2}^{0}(x^3 - 4x)dx + \int_{0}^{2}(4x - x^3)dx$.

Now evaluate each integral, starting with $\int_{-2}^{0}(x^3 - 4x)dx$.

| | |
|---|---|
| $\int_{-2}^{0}(x^3 - 4x)dx$ | Start with the first integral. |
| $= \left(\frac{1}{4}x^4 - 2x^2\right)\Big\|_{-2}^{0}$ | Apply the fundamental theorem of calculus.<br>Note: $\int 4x dx = 4\left(\frac{1}{2}x^2\right) = 2x^2$ |
| $= \left[\frac{1}{4}(0)^4 - 2(0)^2\right] - \left[\frac{1}{4}(-2)^4 - 2(-2)^2\right]$ | Substitute the upper and lower endpoints. |
| $= 0 - [-4]$ | Evaluate the parentheses. |
| $= 4$ | Simplify. |

The area of this part of the region is $4 \text{ units}^2$.

Now evaluate the second integral, $\int_{0}^{2}(4x - x^3)dx$.

| | |
|---|---|
| $\int_{0}^{2}(4x - x^3)dx$ | Start with the second integral. |
| $= \left(2x^2 - \frac{1}{4}x^4\right)\Big\|_{0}^{2}$ | Apply the fundamental theorem of calculus.<br>Note: $\int 4x dx = 4\left(\frac{1}{2}x^2\right) = 2x^2$ |
| $= \left[2(2)^2 - \frac{1}{4}(2)^4\right] - \left[2(0)^2 - \frac{1}{4}(0)^4\right]$ | Substitute the upper and lower endpoints. |
| $= 4 - 0$ | Evaluate the parentheses. |
| $= 4$ | Simplify. |

The area of this part of the region is $4 \text{ units}^2$.

Thus, the total area bounded by the graphs of $y = x^3$ and $y = 4x$ is $4 + 4 = 8 \text{ units}^2$.

WATCH

Here is another example of intertwining curves. In the video, we will find the total area between $f(x)=\frac{9}{x}$ and $g(x)=x$ over the interval $[1, 4]$.

## Video Transcription

[MUSIC PLAYING] Hi, there. In this video, I'd like to show you how to find the total area between the graphs of f of x is equal to 9 over x, and g of x is equal to x on the interval from 1 to 4. So here we have an example where we actually have the curves crossing over each other in the interval. And that's called intertwining curves.

Let's look at what we have so far. So f of x is 9 over x. And if we take and just highlight that a little bit, that is this curve right here. And it actually has another piece of it on the other side of the y-axis, but our interval is from 1 to 4. So I just graphed the quadrant 1 that applies to our problem.

Now, g of x equaling is the line. So we have the line going through the origin with the slope of 1. And on the interval from 1 to 4, we're going to graph vertical lines at the endpoints of that interval between the curves. So I have an x equal 1, we will draw this vertical line, and then at x equal 4.

So here when we look from left to right, we see that the blue curve is higher in the plane than the red line from x equal 1 until you get to the point of intersection. And it looks like that's happening at x equal 3. And then from x equal 3 to x equal 4, the red line is higher on the plane than the blue curve. And we can actually verify that by finding the point of intersection of these two curves in that interval.

So to do that, we have that y is equal to 9 over x, and y is equal to x. To find the point of intersection, we'll just substitute x in for the y. And we have x is equal to 9 over x. Multiplying both sides by x, we get x squared is equal to 9, taking the square root of both sides, remembering our plus or minus, we have x is equal to plus or minus 3.

However, we need to make sure that we stay within the given interval. So our value that is applicable to this problem is x equal 3, which we could tell from the way it was graphed. So we are going to set up our definite integrals, and do the first definite integral to find this area, add it to the definite integral to find that area.

So we have our integral from 1 to 3 of-- there the blue curve, the 9 over x is higher in the plane. So we'll write that first and subtract from it the red line, the x dx, and add to it the definite integral from 3 to 4 of-- and the next part, the red line is higher in the plane. So we'll write the x first minus the 9 over x dx.

So now we just need to evaluate these two definite integrals and add them together. So let's first look at the integral from 1 to 3 of our-- and we're going to write the 9 over x as 9 times x to the negative 1 to get ready to integrate it. So that's equal to-- well, the integral of 9 times x to the negative 1, well, x to the negative 1 is a letter base with an x number exponent, but the number exponent's negative 1.

And, remember, that's the case that goes back to the natural log. So I have 9 times the natural log of the absolute value of x, and then minus-- for my x to the first power, I can use the general power rule and get 1/2 x squared.

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And we are going to evaluate that from 1 to 3. Plugging in 3, I have 9 times the natural log of-- the absolute value of 3 is 3-- minus 1/2 times 3 squared. And then subtract off of that 9 times the natural log of 1 minus 1/2 times 1 squared.

So I have 9 times the natural log of 3, minus 9/2. And then natural log of 1, remember, that's 0. So that's 9 times 0 is 0. And then 0 minus 1/2 is negative 1/2. And minus negative 1/2 is plus 1/2. And I get 9 times the natural log of 3.

Minus 9/2 plus 1/2 is minus 8/2, so that's minus 4. Now, remember, that's just the first part. We still are going to take the definite integral from 3 to 4 of our x minus 9 times x to the negative 1 dx. And that's 1/2 x squared minus 9 times the natural log of the absolute value of x, evaluated from 3 to 4.

And we have 1/2 times 4 squared minus 9 times the natural log of 4, and then minus 1/2 times 3 squared minus 9 times the natural log of 3. So 4 squared 16, half of 16 is 8, minus 9 times a natural log of 4. And then I get a negative 9/2, and plus 9 times the natural log of 3.

Here I'm going to take and look at the 8 as 16 halves, and 16 halves minus 9/2 is 7/2. So I have 7/2 minus 9 times a natural log of 4, plus 9 times the natural log of 3. Now, what we need to do is add this to our other area region.

And when we add these together, we take our 9 natural log of 3 minus 4, and add it to 7/2 minus 9 natural log of 4, plus 9 natural log of 3. We get a value of 18 times a natural log of 3, minus 9 times the natural log of 4, and then minus 1/2

And that is in square units. And that is our area, total area caught between those graphs of f of x equals 9 over x, and g of x equal x, on the interval from 1 to 4. And it's an example of when you have intertwining curves.

[MUSIC PLAYING]

TRY IT

Consider the region bounded by the graphs of $y=x^2$ and $y=4$ between $x=1$ and $x=4$.

Find the exact area of the region. +

$\frac{37}{3}$ units$^2$

## SUMMARY

In this lesson, you learned the **strategy** for **finding the area of regions between two curves that intertwine**, which is a natural extension of the area computations that you investigated in the last tutorial. The main idea is to make sure that the integrand is “upper function – lower function” to ensure that the definite integral yields a positive result (area).

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SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

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# The Average Value of a Continuous Function on a Closed Interval

*by Sophia*

WHAT'S COVERED

In this lesson, you will learn about the average value of a continuous function over an interval $[a, b]$. Recall that the average of a set of numbers is the sum of the numbers, divided by the number of numbers. This takes on a different meaning for continuous functions. Specifically, this lesson will cover:

1. The Idea Behind Average Value
2. Computing the Average Value of a Continuous Function

# 1. The Idea Behind Average Value

When finding the average of a set of numbers, you add up all the numbers, then divide by how many numbers there are.

EXAMPLE Given the numbers 81, 85, 89, and 71, the average of these four numbers is $\frac{81+85+89+71}{4}=81.5.$

Now consider a function $y=f(x)$ on some interval $[a, b]$. Break up the interval $[a, b]$ into *n* equal subintervals. Then, select a value of *x* from each subinterval. Call these values $x_1, x_2, \ldots, x_n$.

Then, the average of these values is $\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{1}{n}\right].$

The summation resembles a Riemann sum, but the $\Delta x$ term is missing inside the summation. Recall that $\Delta x=\frac{b-a}{n}$.

We can multiply the summation by $\frac{b-a}{b-a}$ as follows:

$$\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{b-a}\,\frac{1}{n}\right]$$
$$=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{n}\,\frac{1}{b-a}\right]$$

We replace $\frac{b-a}{n}$ with $\Delta x$:

$$= \sum_{k=1}^{n} \left[ f(x_k) \cdot \Delta x \cdot \frac{1}{b-a} \right]$$

Since $\frac{1}{b-a}$ is a constant, it can be factored out and written in front of the summation:

$$\frac{1}{b-a} \sum_{k=1}^{n} [f(x_k) \cdot \Delta x]$$

Recall that the summation $\sum_{k=1}^{n} [f(x_k) \cdot \Delta x]$ approaches the value of $\int_a^b f(x)dx$ as $n \to \infty$ as long as $f(x)$ is integrable on $[a, b]$. Since we are assuming $f(x)$ is continuous on $[a, b]$, $f(x)$ is also integrable on $[a, b]$. Note that the summation for the average value is the Riemann sum for $f(x)$ but multiplied by $\frac{1}{b-a}$.

This leads to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$.

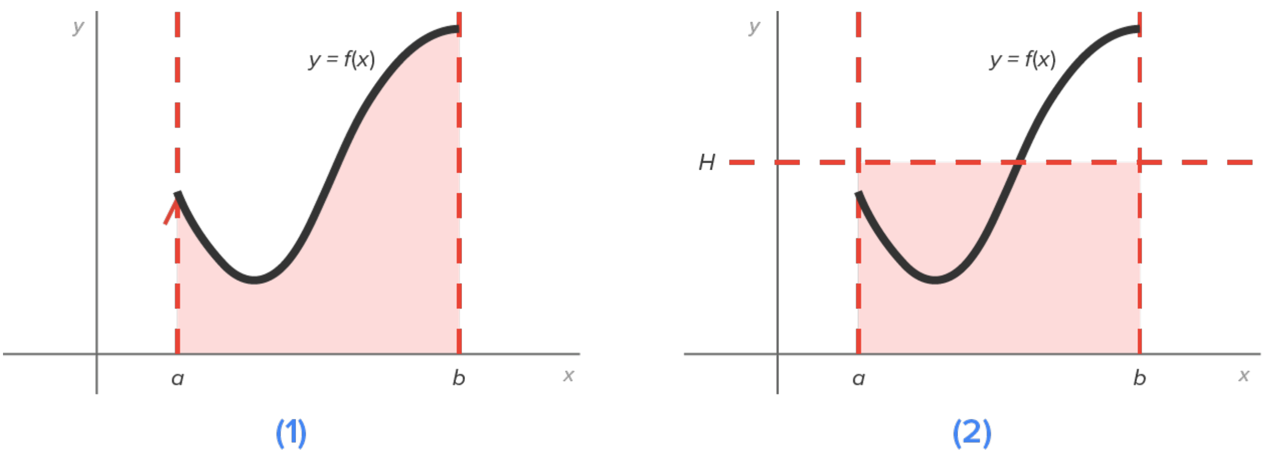
FORMULA

**Average Value of a Function**

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x)dx$.

BIG IDEA

For a geometric interpretation of average value, let $H =$ the average value of a nonnegative function $f(x)$ on $[a, b]$. The figure below shows an illustration of this.




- The graph in (1) is the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.
- The graph in (2) is the rectangle with an area equal to $\int_a^b f(x)dx$. Note that the base is $b-a$, and its height is $H$, where $H$ is the average value of $f(x)$ on $[a, b]$.

The area of the rectangle with height $H$ and width $b-a$ is equal to the area of the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.

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# 2. Computing the Average Value of a Continuous Function

Now that we have a formula for average value, let's compute and interpret average values.

↪ EXAMPLE Find the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

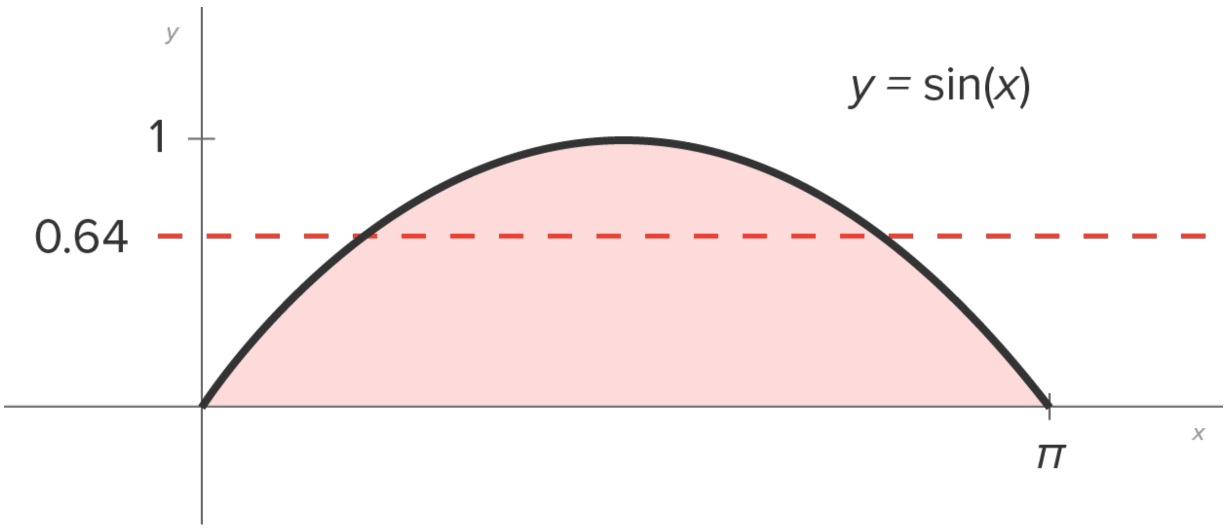
From the formula, this is equal to $\frac{1}{\pi - 0}\int_0^{\pi} \sin x dx = \frac{1}{\pi}\int_0^{\pi} \sin x dx$.

Now, we evaluate the definite integral:

| | |
|---|---|
| $\frac{1}{\pi}\int_0^{\pi} \sin x dx$ | Start with the original expression. |
| $= \frac{1}{\pi}(-\cos x)\Big\|_0^{\pi}$ | Apply the fundamental theorem of calculus. |
| $= \frac{1}{\pi}(-\cos \pi) - \frac{1}{\pi}(-\cos 0)$ | Substitute the upper and lower endpoints. |
| $= \frac{1}{\pi} + \frac{1}{\pi}$ | Evaluate the parentheses. |
| $= \frac{2}{\pi}$ | Simplify. |

The average value of $f(x) = \sin x$ on the interval $[0, \pi]$ is equal to $\frac{2}{\pi}$.

To see the geometric interpretation, here is the graph of the region bounded by $f(x) = \sin x$ and the x-axis on the interval $[0, \pi]$ and the rectangle whose height is the average value and whose width is $\pi$.
Note: $\frac{2}{\pi} \approx 0.64$

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WATCH

Find the average value of $f(x)=\frac{15x}{x^2+1}$ on the interval $[0, 5]$.

## Video Transcription

[MUSIC PLAYING] Hi, there. Welcome. What I'd like to show you in this video is how to find the average value of a continuous function on a closed interval. Here, the function we're going to find the average value of is f of x equals 15x over x squared plus 1. And the closed interval is the closed interval 0 to 5.

Now, remember to do this. We're looking at finding our average value of our function. f sub a vg. And we're just using that subscript with that abbreviation to make sure that we identify what we're finding. And then, that's equal to 1 over b minus a times the definite integral from a to b of f of xdx

And we have our function is 15x over x squared plus 1. And then, for a closed interval, where a is the left endpoint of the closed interval. And b is the right endpoint of the closed interval.

So here, we have f sub average is equal to 1 over 5 minus 0 times the definite integral from 0 to 5 of my 15x over x squared plus 1dx.

Now, when I look at that integrand, remember we want to find an antiderivative of 15x over x squared plus 1. And then, use the antiderivative where c is equal to 0. So we don't need to add this c when we're evaluating the definite integral.

And then, plug in the upper limit of integration through the antiderivative minus plugging in the lower limit. And when I look at this integrand, I see that my denominator has two different terms. And I need to use substitution with this.

Now, Let's rewrite this, first, before we identify what we will have for our use substitution. So I have 1 over-- well, 5 minus 0 is 5. And the definite integral from 0 to 5 of--

Well, I can bring this 15 factor in the numerator out as a constant factor from the constant multiple rule. And it's in the numerator. So I'm going to write it times 15 in the numerator out in front of the integral.

And then, this x squared plus 1 is in the denominator. So we're going to bring that up out of the denominator by going x squared plus 1 to the negative 1 power. And then, our factor, x, that's in a numerator. We have that times xdx.

And I rewrite it in that form because that helps me identify what my u designation should be. Remember, u is that expression that's being acted on. So the action of negative 1 power is on that expression x squared plus 1.

Now, if you represents x squared plus 1, we also need to look at the differential du. And taking the derivative of the right hand side, I get the derivative of x squared plus 1 is 2x. And then, times a differential dx.

And I look, and I don't have the factor of 2. But I do have the factor of x and the dx. So we're going to

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divide the 2 over to the other side. And we have 1/2 du is the replacement for xdx.

Now, we also have a definite integral. So I need to also find the values for the limits of integration. When x is equal to 0, that gives me that u will be equal to 0 squared plus 1. Or u is equal to 1.

When x is equal to 5, The upper limit of integration I get u is equal to 5 squared plus 1. Or u is equal to 26. So now, back over with our information for the average value of the function. Well, 1 times 15 divided by 5 is 3.

And I have times the definite integral from my u lower limit of integration is 1, To my u upper limit of integration is 26. And x squared plus 1 is u. So I view to the negative 1. And xdx goes out. And 1/2du goes in its place.

So now, we have our factor of 1/2 half can be brought out with a constant multiple rule. And 3 times 1/2 is 3/2. And the indefinite integral of u to the negative is the natural log of the absolute value of u.

And we are evaluating that from 1 to 26. And then, we plug in our upper limit of integration first. So it's 3/2 times the natural log. Well, the absolute value of 26 is 26 minus the natural log and the absolute value of 1 is 1.

And then, simplifying one more step, I get 3/2 times natural log of 26 minus-- with a natural log of 1, remember, the natural log of 1 is 0. And so our final result is our average value of the function f of x equals 15x over x squared plus 1 on the closed interval from 0 to 5 is 3/2 times the natural log of 26.

And there you go-- an example of finding the average value of a continuous function on a closed interval.

[MUSIC PLAYING]

TRY IT

Consider the following table.

| Function on a Given Interval | Average Value |
|---|---|
| $f(x) = x^2 + 2$ on the interval $[0, 4]$ | ? |
| $f(x) = \frac{4}{x^2}$ on the interval $[1, 2]$ | ? |

Find the average value of each function on the given interval.

+

| Function on a Given Interval | Average Value |
|---|---|
| $f(x) = x^2 + 2$ on the interval $[0, 4]$ | $\frac{22}{3}$ |
| | |

| | |
|---|---|
| $f(x)=\frac{4}{x^2}$ on the interval $[1, 2]$ | 2 |

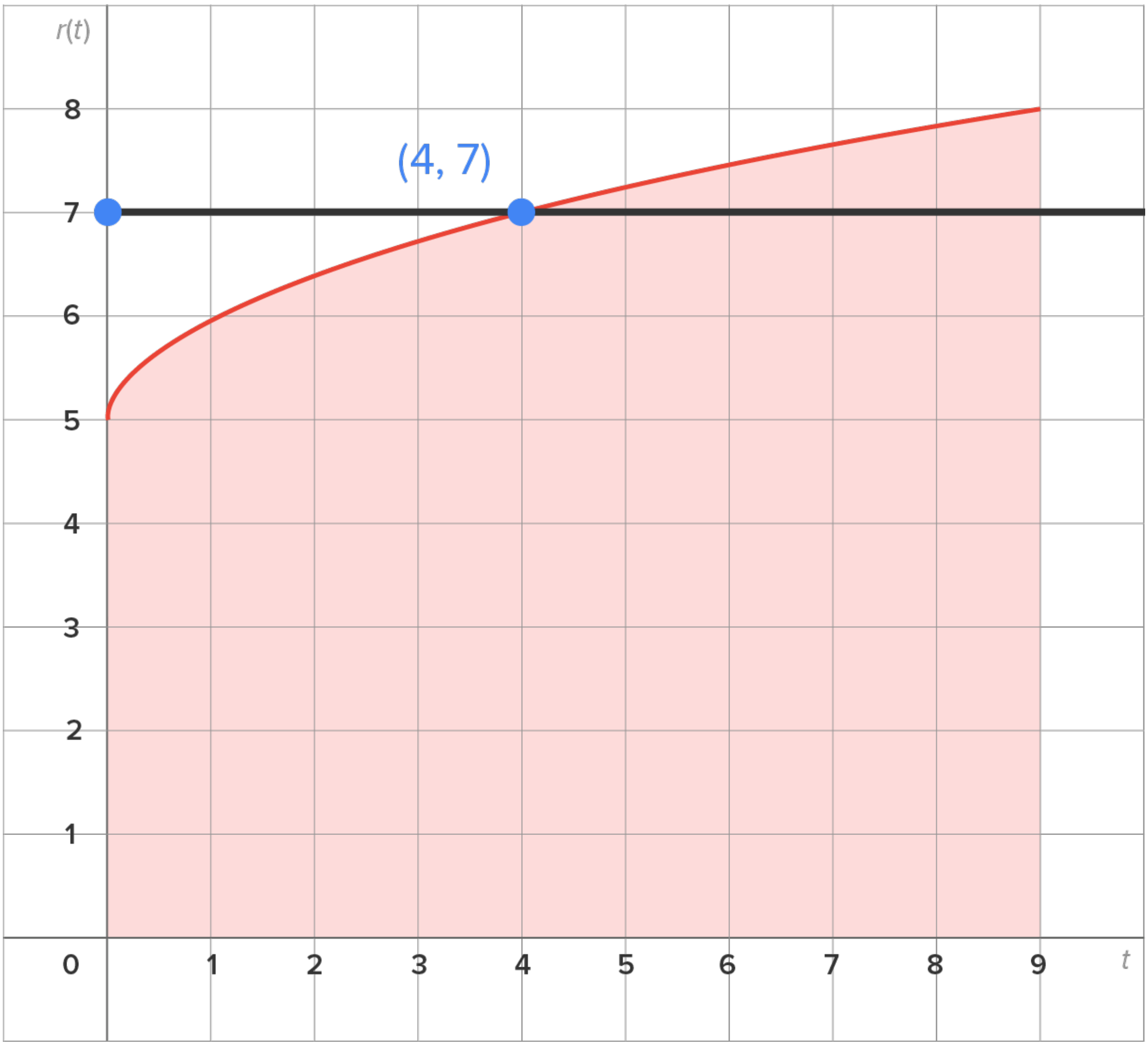
↪ EXAMPLE During a 9-hour workday, the production rate at time $t$ hours is $r(t)=5+\sqrt{t}$ cars per hour. What is the average hourly production rate?

We seek the average value of $r(t)$ over the interval $[0, 9]$.

| | |
|---|---|
| $Average\ Value=\frac{1}{9-0}\int_0^9 (5+t^{1/2})dt$ | Start with the original expression. Rewrite $\sqrt{t}=t^{1/2}$ to be able to use the power rule. |
| $=\frac{1}{9}\left(5t+\frac{2}{3}t^{3/2}\right)\Big\vert_0^9$ | Apply the fundamental theorem of calculus. |
| $=\frac{1}{9}\left[5(9)+\frac{2}{3}(9)^{3/2}\right]-\frac{1}{9}\left[5(0)+\frac{2}{3}(0)^{3/2}\right]$ | Substitute the upper and lower endpoints. |
| $=\frac{1}{9}(45+18)-\frac{1}{9}(0)$ | Evaluate. |
| $=7$ | Simplify. |

The average rate of production is 7 cars per hour.

Shown in the figure is the region between $r(t)=5+\sqrt{t}$ and the t-axis, as well as the horizontal line $r(t)=7$. Note that the area between $r(t)$ and the t-axis is equal to the area of the rectangle with the same base (9) and height 7 (the average value).




## SUMMARY


In this lesson, you began by understanding **the idea behind average value**, following the path from the formula to find the average of a set of numbers to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$. You also learned how the fundamental theorem of calculus can be used to **compute the average value of a continuous function** $f(x)$ on an interval $[a, b]$.


SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## FORMULAS TO KNOW

**Average Value of a Function**

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

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# The Mean Value Theorem for Integrals

*by Sophia*

WHAT'S COVERED

In this lesson, you will connect the mean value theorem to integrals. Specifically, this lesson will cover:

1. The Mean Value Theorem for Integrals
2. Finding the Value of *c* Guaranteed by the Mean Value Theorem for Integrals

# 1. The Mean Value Theorem for Integrals

Similar to the mean value theorem for derivatives, we can establish a theorem for integrals. If $f(x)$ is continuous on $[a, b]$, then at some point $c$ in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

In other words, there is at least one value of $c$ in the interval $[a, b]$ such that $f(c) =$ the average value of $f(x)$ on $[a, b]$.

TERM TO KNOW

**The Mean Value Theorem for Integrals**
If $f(x)$ is continuous on $[a, b]$, then at some point $c$ in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

---

# 2. Finding the Value of *c* Guaranteed by the Mean Value Theorem for Integrals

Let's look at a few examples to help illustrate the mean value theorem for integrals.

EXAMPLE Consider the function $f(x) = x^2$ on the interval $[0, 3]$.

The average value of $f(x)$ on $[0, 3]$ is $\frac{1}{3}\int_0^3 x^2dx$. Evaluating, we have:

$$\frac{1}{3}\int_0^3 x^2dx = \frac{1}{3}\cdot\left.\frac{1}{3}x^3\right|_0^3 = \frac{1}{9}(3)^3 - \frac{1}{9}(0)^3 = 3$$

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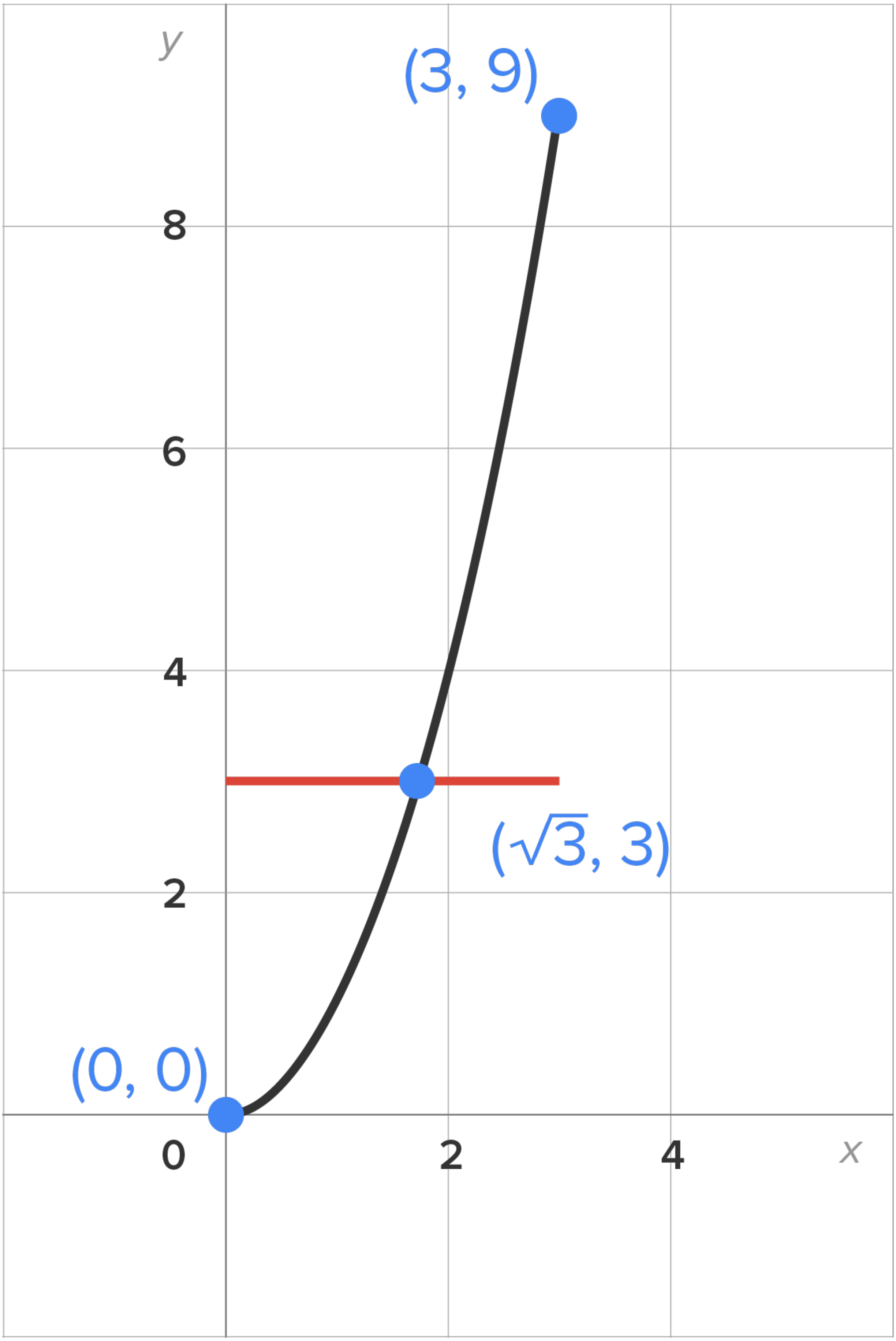
To find the value of *c*, set $f(c) = 3$. This means $c^2 = 3$, which means $c = \pm\sqrt{3}$. Since $-\sqrt{3}$ is not in the interval $[0, 3]$, the value of *c* guaranteed by the theorem is $c = \sqrt{3}$.

Here is the graph of $f(x) = x^2$ on the interval $[0, 3]$ along with the line $y = 3$ (the average value). Note that they intersect at the point $(\sqrt{3}, 3)$.




WATCH

Check out this video to see the example to find the average value and the value of $c$ guaranteed by the

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mean value theorem for $f(x) = 2x^2 - x$ on $[-1, 3]$.

## Video Transcription

[MUSIC PLAYING] Hi, there. What I'd like to show you today is how to find the average value and then also find the value of c guaranteed by the mean value theorem for integrals with our continuous function f of x is equal to 2x squared minus x, and we're on the closed interval from negative 1 to 3.

Now, when we do this type of problem, we want to first find the average value of the function. So our f sub-- and I'm just going to use this notation of subscripting with the abbreviation for average, avg. And that, recall, is equal to 1 over b minus a times the definite integral from a to b of f of x dx. And here, our f of x is x squared-- excuse me. Our f of x is 2x squared minus x. Our a is the beginning of the closed interval, negative 1. And our b is the end of the closed interval 3.

So we have our f sub avg is equal to 1 over 3 minus negative 1 times a definite integral from negative 1 to 3 of 2x squared minus x dx. Now, this integrand can be integrated term by term. And remember, to evaluate a definite integral, we find an antiderivative and then evaluate that antiderivative at the upper limit of integration minus evaluated at the lower limit.

So putting that into play, we have our average value of our function is 1/4 times-- well, the integral of 2x squared is-- keep your base x. Add 1 to the exponent, so 2 plus 1 is 3. And we take our constant factor and divide that by that 3 exponent. And then minus the an antiderivative of x is 1/2 x squared doing the same process.

And then, this is evaluated from x equal negative 1 to x equal 3. Plugging in the upper limit of integration minus the lower limit, we have our 1/4 times 2/3 times 3 cubed minus 1/2 times 3 squared and then that in parentheses minus open parentheses plugging in the negative 1.

And when you work through the arithmetic of what's in the brackets, you'll get that that's 44/3 in the brackets. Excuse me, there. And 1/4 of 44/3, we get our average value of our function over that interval to be 11/3. So that's the first part of the process, the average value.

Now we want to find the value of c guaranteed by the mean value theorem. And to do that, we have that our f of c will attain that average value for some-- at least one c in the closed interval. So we have f of c is going to be that 11/3 for some value in our interval. Well, f of c tells you exactly how to set up that left-hand side. Go to your function, f, take out the x's and put in c's instead. So I have 2c squared minus c is equal to 11/3.

Now, this is quadratic in c. So this will-- we'll go ahead and multiply both sides by 3 to remove the fractions and make it just a little bit easier to work with. That gives us 6c squared minus 3c is equal to 11. And then, quadratic, we're going to bring everything on one side set equal to 0. And since this quadratic, the equation isn't factorable, we'll use the quadratic formula.

So c is equal to the opposite of the coefficient in front of the first-degree term, so the opposite of negative 3, then plus or minus the square root of that coefficient squared minus 4 times the coefficient in front of the square term, so 6, times the constant. And remember to take that negative with the 11, so negative 11. And that's all over 2 times that coefficient of the squared term.

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And when we simplify all of that, we get 3 plus or minus the square root of 273, and that's all over 12. Now, that gives us two different values. We have c is equal to 3 minus the square root of 273 over 12, and c is equal to 3 plus the square root of 273 over 12.

And we want to look at the decimal approximation to only report the values of c that are in the interval. We want to report the answer that's exact. But we have to look at the decimal approximations to see which ones are in the interval. So for the 3 minus the square root of 273 over 12, the decimal approximation is approximately negative 1.12689. And for the c is equal to 3 plus the square root of 273 over 12, that's approximately 1.62689.

And then, looking at our interval from negative 1 to 3 right here, and seeing which one or both could be, but, in this case, the negative 1.126, that's not in the interval. But the 1.62689 is. So we will report c is equal to 3 plus the square root of 273 all over 12. So there you have it, our average value of our function and the c value guaranteed by the mean value theorem.

Well, thank you for watching today, and I hope you have found this video helpful.

[MUSIC PLAYING]

TRY IT

Consider the function $f(x) = \dfrac{16}{x^3}$ on the interval $[1, 2]$.

Find the average value and the value of *c* guaranteed by the mean value theorem for integrals. +

Average value = 6, $c = \sqrt[3]{\dfrac{8}{3}} \approx 1.39$

## SUMMARY

In this lesson, you learned that through **the mean value theorem for integrals**, you are able to guarantee that there is some input value (*c*) of a function $f(x)$ on $[a, b]$ in which $f(x)$ is equal to its average value on $[a, b]$. Next, you practiced **finding the value of *c* guaranteed by the mean value theorem for integrals**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**The Mean Value Theorem for Integrals**

If $f(x)$ is continuous on $[a, b]$, then at some point *c* in $[a, b]$:

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$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

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# Using Tables to Find Antiderivatives

*by Sophia*

WHAT'S COVERED

In this lesson, you will use a table of integrals to find antiderivatives and solve problems that cannot be solved only using the formulas and techniques learned thus far. Specifically, this lesson will cover:

1. Using a Table to Find Antiderivatives
2. Using a Table to Solve Applications Involving Definite Integrals

# 1. Using a Table to Find Antiderivatives

Given any function, we have the necessary tools to find its derivative.

But to find antiderivatives of many functions, new techniques are required. Since these techniques are not covered in this course, we will make use of the table of integrals as referenced below.

↪ EXAMPLE Assuming $a$ is a constant, use formula #44, which is $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$, to find $\int \frac{dx}{x^2+81}$.

Using the formula as a model, we see that $a=9$. Then, $\int \frac{dx}{x^2+81} = \frac{1}{9}\arctan\left(\frac{x}{9}\right)+C$.

Here is another example, this time using logarithmic functions.

↪ EXAMPLE Find the indefinite integral: $\int x^4 \ln x dx$

According to formula #38 in the integral table, $\int x^n \ln x dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1}+C,\ n \neq -1$.

Therefore, use this formula with $n=4$.

Then, $\int x^4 \ln x dx = \frac{x^{4+1}}{(4+1)^2}\{(4+1)\ln x - 1\}+C = \frac{x^5}{25}\{5\ln x - 1\}+C$.

TRY IT

Consider $\int \frac{dt}{100-t^2}$.

Find the antiderivative by using formula #46 in the table. +

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$\frac{1}{20}\ln\left|\frac{t+10}{t-10}\right|+C$

↱ EXAMPLE Use an appropriate formula to find the indefinite integral: $\int \sin^3(2x)dx$

According to formula #24, $\int \sin^3(ax)dx = \frac{-\sin^2(ax)\cos(ax)}{3a} - \frac{2}{3a}\cos(ax)+C$.

With $a=2$, we have:

$\int \sin^3(2x)dx = \frac{-\sin^2(2x)\cos(2x)}{3(2)} - \frac{2}{3(2)}\cos(2x)+C = \frac{-\sin^2(2x)\cos(2x)}{6} - \frac{1}{3}\cos(2x)+C$

## 2. Using a Table to Solve Applications Involving Definite Integrals

Since the key step in evaluating $\int_a^b f(x)dx$ is finding the antiderivative of $f(x)$, we can solve area and distance problems using the tables of integrals when necessary.

↱ EXAMPLE Evaluate the definite integral: $\int_0^{\pi} 4\cos^2(5x)dx$

According to formula #19 in the table, $\int \cos^2(ax)dx = \frac{1}{2}x + \frac{1}{2a}\sin(ax)\cos(ax)+C$.

In our integral, $a=5$, which means $\int \cos^2(5x)dx = \frac{1}{2}x + \frac{1}{10}\sin(5x)\cos(5x)+C$.

Then, $\int 4\cos^2(5x)dx = 4\left[\frac{1}{2}x + \frac{1}{10}\sin(5x)\cos(5x)\right]+C = 2x + \frac{2}{5}\sin(5x)\cos(5x)+C$.

To use the fundamental theorem of calculus, use $C=0$. This leads to:

$$\int_0^{\pi} 4\cos^2(5x)dx = \left(2x+\frac{2}{5}\sin(5x)\cos(5x)\right)\Big|_0^{\pi}$$
$$= 2\pi + \frac{2}{5}\sin(5\pi)\cos(5\pi) - 2(0) + \frac{2}{5}\sin(5\cdot 0)\cos(5\cdot 0)$$
$$= 2\pi$$

Thus, $\int_0^{\pi} 4\cos^2(5x)dx = 2\pi$.

↱ EXAMPLE Find the exact area of the region between the graphs of $y=\ln x$ and the x-axis between $x=2$ and $x=6$. The region is in the figure below.

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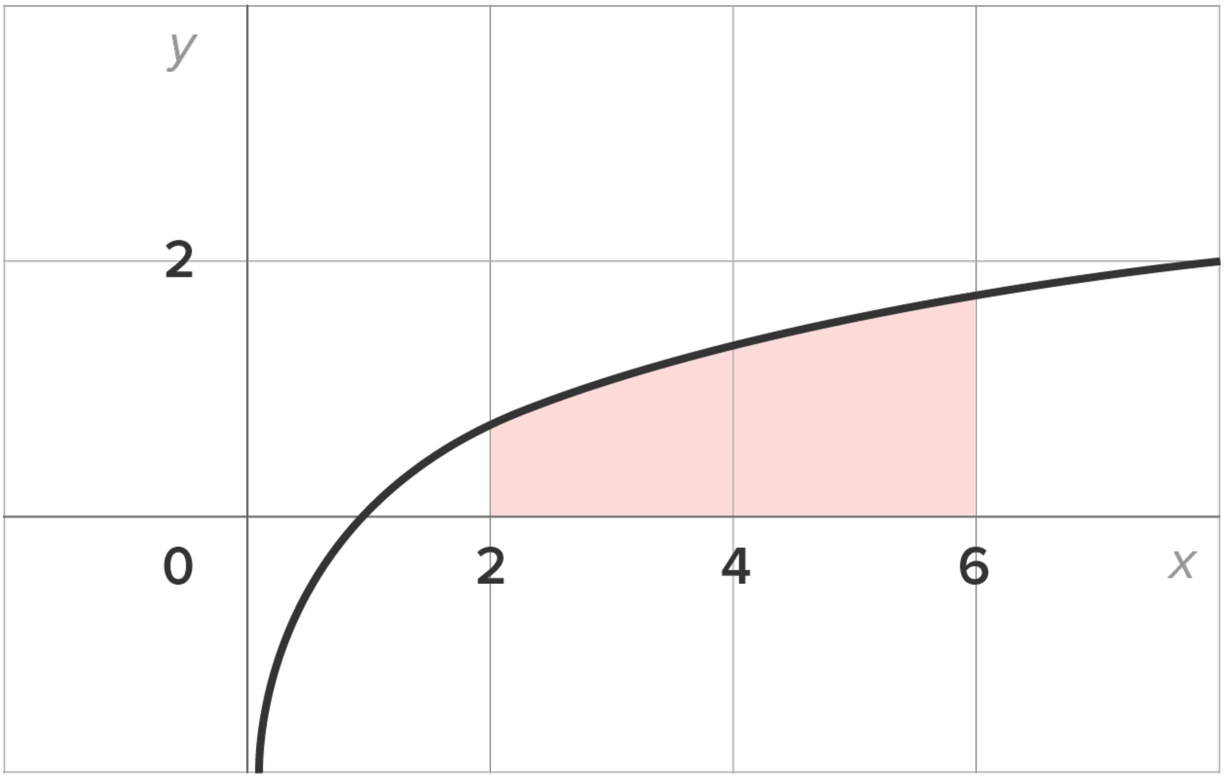


Since the region is entirely above the x-axis on the interval $[2, 6]$, the area is given by the definite integral $\int_2^6 \ln x dx$.

Formula #37 in the table states that $\int \ln x dx = x\ln x - x + C$. It follows that:

$$\int_2^6 \ln x dx = (x\ln x - x)\Big|_2^6$$
$$= 6\ln 6 - 6 - (2\ln 2 - 2)$$
$$= 6\ln 6 - 2\ln 2 - 4$$

TRY IT

Consider the region bounded by the graphs of $f(x) = x(2x+1)^4$ and the x-axis between $x = 0$ and $x = 3$.

Set up and evaluate a definite integral that gives the area of the described region. +

The definite integral is $\int_0^3 x(2x+1)^4 dx$. Since the graph of $f(x)$ is above the x-axis on $[0, 3]$, the value of the definite integral is equal to the area of the region. Using formula #9 in the table, the area of the region is $4061.7 \text{ units}^2$. Note, this integral could have also been evaluated using u-substitution.

SUMMARY

In this lesson, you learned how to **use a table to find antiderivatives**, as well as **to solve applications**

**involving definite integrals**. Having a table of integrals is a great tool when solving problems in which antiderivatives are required, but as you can tell, these would be very difficult (and not realistic) to memorize. Through more study of antiderivatives, you can learn the techniques that are required to arrive at these antiderivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

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# Approximating Definite Integrals

*by Sophia*

WHAT'S COVERED

In this lesson, you will use approximation methods for evaluating definite integrals by using other shapes to approximate said areas. Specifically, this lesson will cover:

1. Approximating a Definite Integral Using Rectangles
2. Approximating a Definite Integral Using Trapezoids (Trapezoidal Rule)
3. Approximating a Definite Integral Using Parabolas (Simpson's Rule)
4. Application: Approximating Area

# 1. Approximating a Definite Integral Using Rectangles

When you combine our knowledge of antiderivatives with the tables of integrals from the last tutorial, there are still functions for which we do not know antiderivatives.

> ↱ EXAMPLE $\int_0^2 e^{-x^2} dx$ cannot be evaluated exactly since the antiderivative of $f(x) = e^{-x^2}$ can't be written in terms of the functions we are familiar with in calculus.

Recall in challenge 5.2, we approximated definite integrals by using rectangles of equal width along the x-axis.

Consider a nonnegative function $y = f(x)$ on the interval $[a, b]$. Break the interval $[a, b]$ into *n* equal subintervals; then, each subinterval has width $\Delta x = \frac{b-a}{n}$.

Note, these approximation methods can be used whether $f(x)$ is positive, negative, or both on the interval $[a, b]$. We are considering only nonnegative functions for now to make the visual connection with the area between $y = f(x)$ and the x-axis on $[a, b]$.

The graph below shows the interval $[a, b]$ broken into 4 subintervals, where the left-hand endpoints are used to determine the height of each rectangle.

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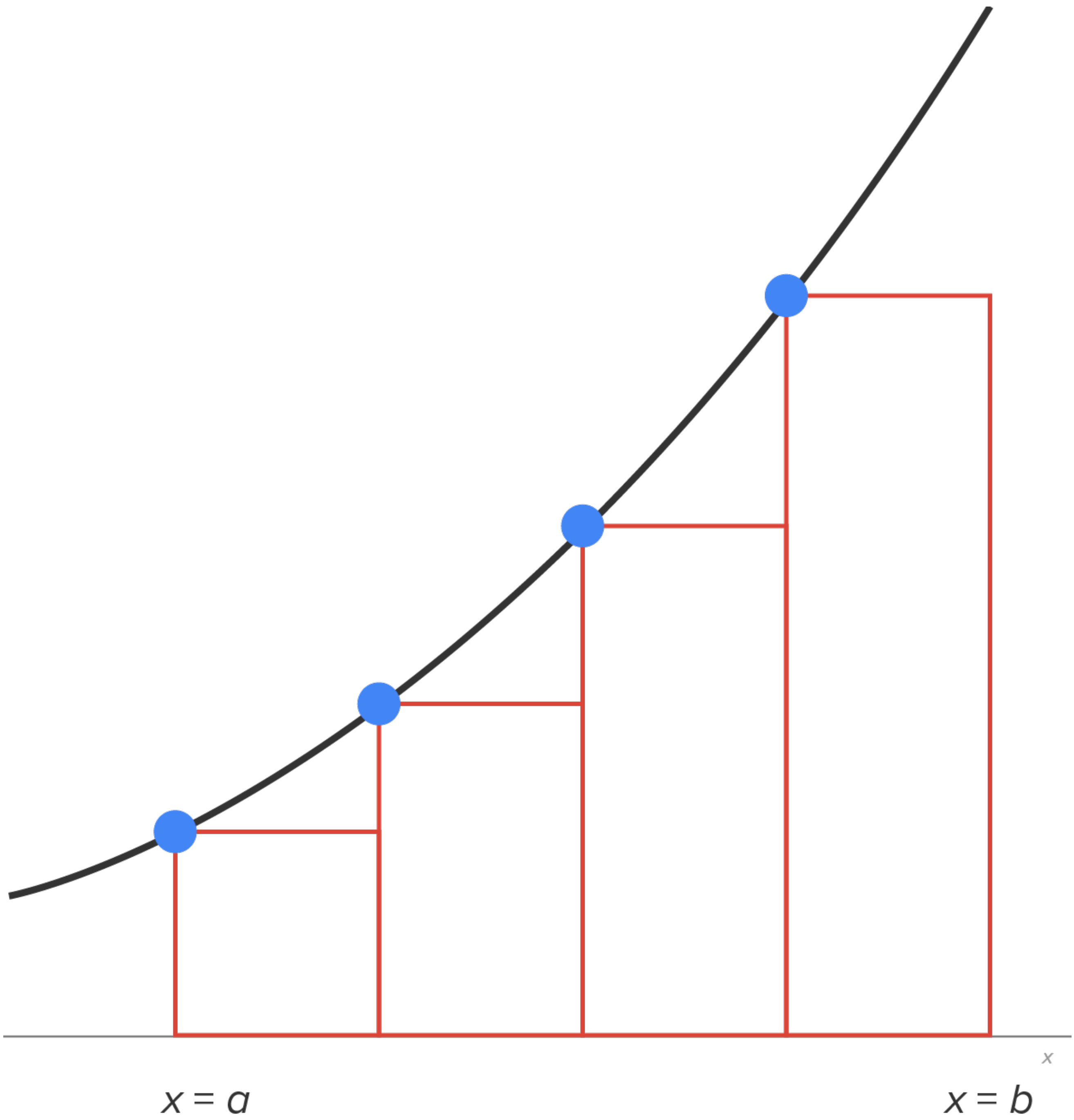


Naturally, as the number of rectangles (subintervals), *n*, gets larger, the approximation gets closer to the actual area (in this case, also the definite integral).

Now that we have our framework, let's look at an example where we use all three approximation methods.

> ↪ EXAMPLE Estimate the value of $\int_0^2 e^{-x^2}dx$ by using $n=4$ subintervals of equal width using (a) left-hand endpoints, (b) right-hand endpoints, and (c) midpoints of each subinterval.
>
> The graph of the region is shown in the figure below. Note: $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$ This means the subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$.

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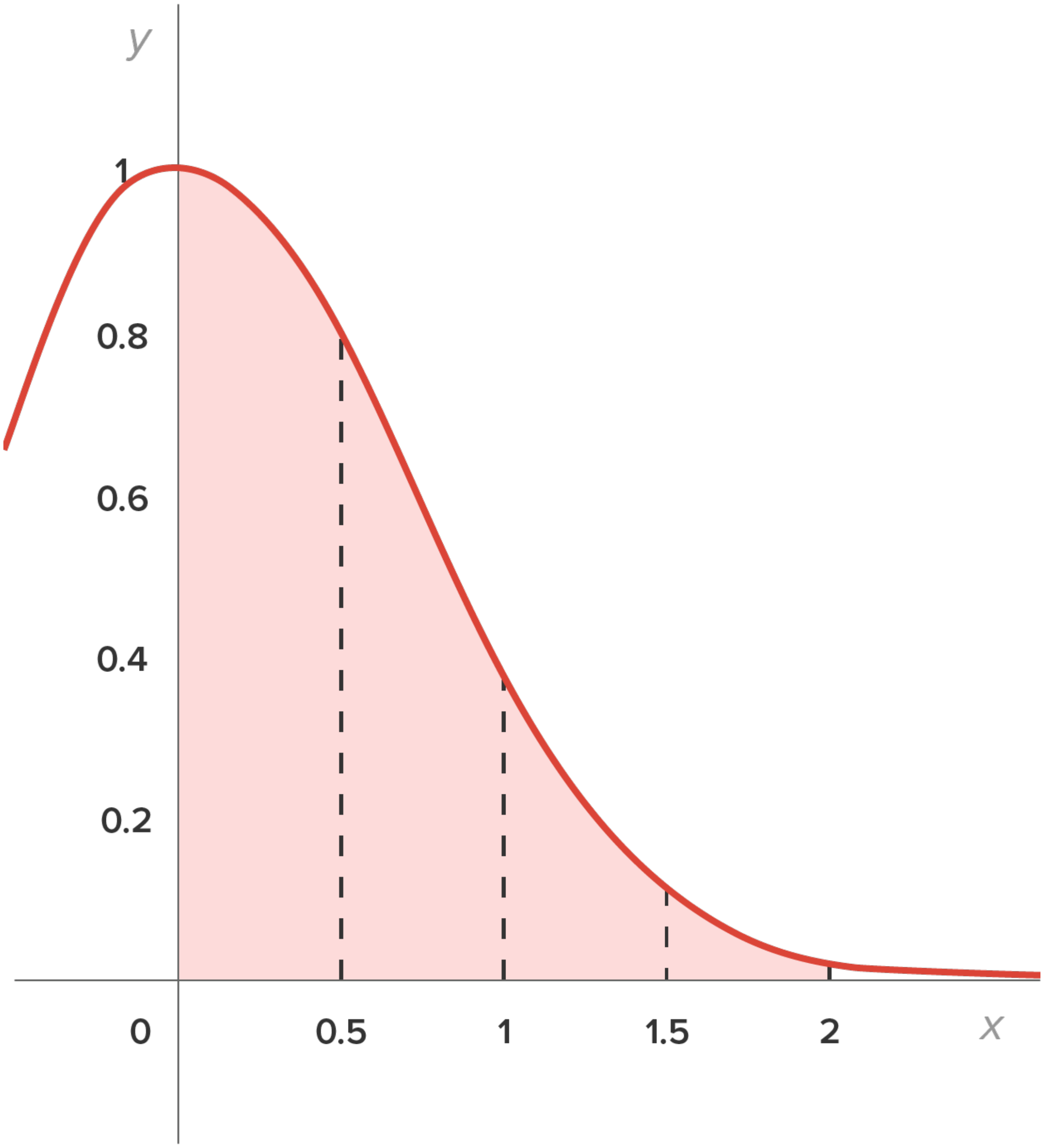


| Points | Value |
|---|---|
| Left-hand endpoints | Using the left-hand endpoint of each interval, this means the x-values are 0, 0.5, 1, and 1.5.<br><br>Then, the estimate for the definite integral is $0.5f(0)+0.5f(0.5)+0.5f(1)+0.5f(1.5)$.<br><br>Note that we can factor out 0.5:<br><br>$0.5[f(0)+f(0.5)+f(1)+f(1.5)]$<br>$0.5\left[e^{-0^2}+e^{-0.5^2}+e^{-1^2}+e^{-1.5^2}\right]$<br><br>After using a calculator, this is approximately 1.12604 (to five decimal places). |

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| | |
|---|---|
| Right-hand endpoints | Using the right-hand endpoint of each interval, this means the x-values are 0.5, 1, 1.5, and 2.<br><br>Then, the estimate for the definite integral is $0.5f(0.5)+0.5f(1)+0.5f(1.5)+0.5f(2)$.<br><br>Note that we can factor out 0.5:<br><br>$0.5[f(0.5)+f(1)+f(1.5)+f(2)]$<br>$0.5\left[e^{-0.5^2}+e^{-1^2}+e^{-1.5^2}+e^{-2^2}\right]$<br><br>After using a calculator, this is approximately 0.63520 (to five decimal places). |
| Midpoints | Using the midpoint of each interval, this means the x-values are 0.25, 0.75, 1.25, and 1.75.<br><br>Then, the estimate for the definite integral is $0.5f(0.25)+0.5f(0.75)+0.5f(1.25)+0.5f(1.75)$.<br><br>Note that we can factor out 0.5:<br><br>$0.5[f(0.25)+f(0.75)+f(1.25)+f(1.75)]$<br>$0.5\left[e^{-0.25^2}+e^{-0.75^2}+e^{-1.25^2}+e^{-1.75^2}\right]$<br><br>After using a calculator, this is approximately 0.88279 (to five decimal places). |

Looking at the region, the left-hand endpoints produce an overestimate and the right-hand endpoints produce an underestimate. It is more difficult to tell if the midpoint estimate is higher or lower than the actual value, but one thing is for sure: it is closer to the actual than the others.

For your reference, $\int_0^2 e^{-x^2}dx \approx 0.8820813908$, so the midpoint estimate was very close!

To account for the left-hand and right-hand endpoints providing overestimates or underestimates, sometimes the average of the left-hand and right-hand estimates is used.

In this case, this average is $\frac{1.12604+0.63520}{2}=0.88062$, which is indeed closer.

TRY IT

Consider $\int_1^3 \sqrt{x^3+1}\,dx$.

Using $n=4$ subintervals, estimate the definite integral using left-hand endpoints, right-hand endpoints, and midpoints of each subinterval. Round each estimate to 5 decimal places.

Estimate using left-hand endpoints. +

5.29162

Estimate using right-hand endpoints. +

7.23026

Estimate using midpoints. +

6.21450

---

# 2. Approximating a Definite Integral Using Trapezoids (Trapezoidal Rule)

Consider the region shown in the figure, this time connecting consecutive points with a line.

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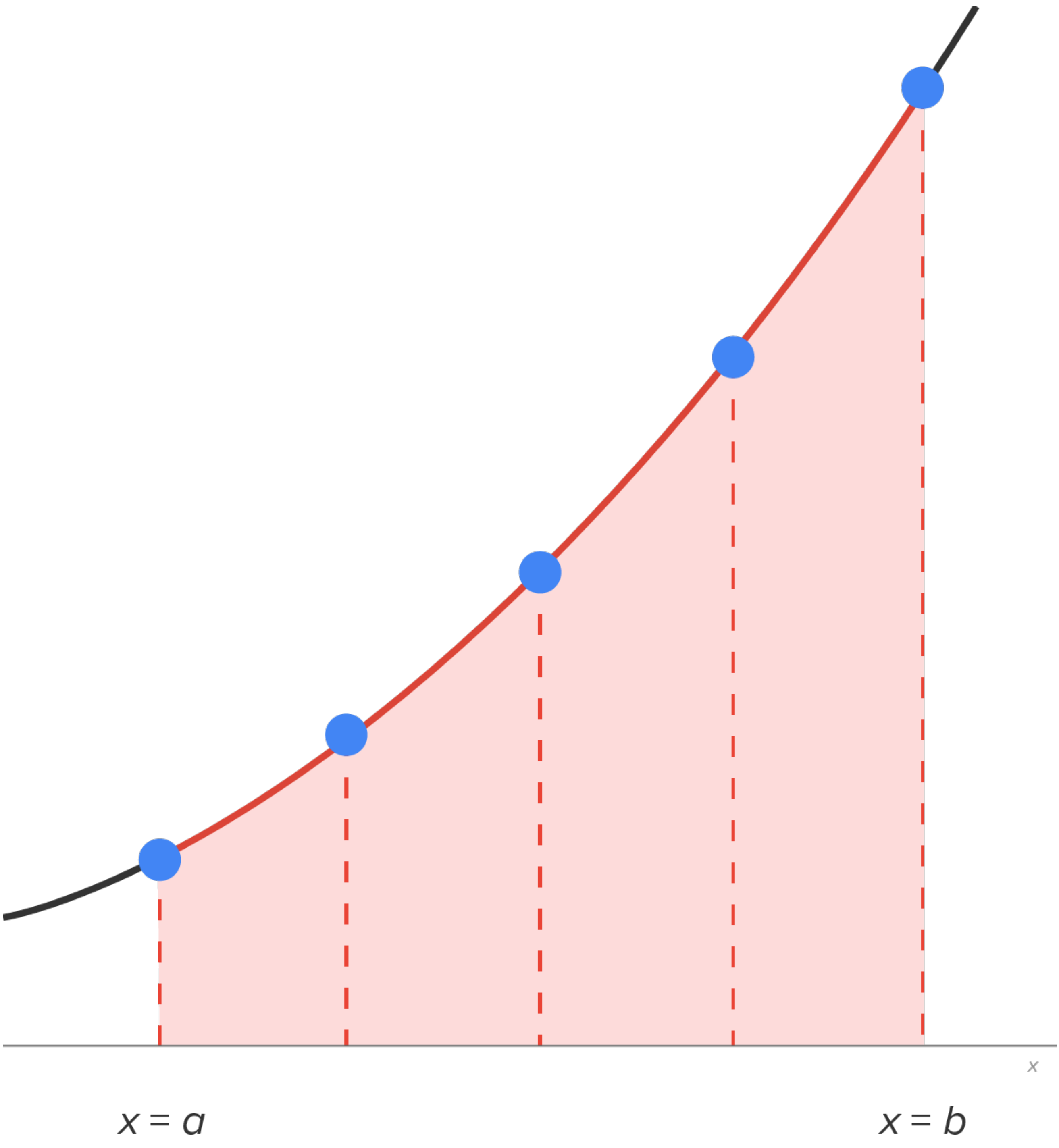


This means that trapezoids are used to approximate the area of the region in each subinterval. Recall that the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where $h$ is the height (the distance between the parallel sides) and $b_1$ and $b_2$ are parallel bases.

In this case, the height is the width of each trapezoid $(\Delta x)$, and the bases are the function values on both sides.

Let $x_0 = a$, $x_k =$ the right-hand endpoint of each subinterval, and $x_n = b$.

| Trapezoid | Bases | Height | Area |
|---|---|---|---|
| | | | |

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| | | | |
|---|---|---|---|
| First trapezoid on the interval $[x_0, x_1]$ | $f(x_0)$ and $f(x_1)$ | $\Delta x$ | $\frac{1}{2}(f(x_0)+f(x_1))\Delta x$ |
| Second trapezoid on the interval $[x_1, x_2]$ | $f(x_1)$ and $f(x_2)$ | $\Delta x$ | $\frac{1}{2}(f(x_1)+f(x_2))\Delta x$ |
| Third trapezoid on the interval $[x_2, x_3]$ | $f(x_2)$ and $f(x_3)$ | $\Delta x$ | $\frac{1}{2}(f(x_2)+f(x_3))\Delta x$ |
| Last trapezoid on the interval $[x_{n-1}, x_n]$ | $f(x_{n-1})$ and $f(x_n)$ | $\Delta x$ | $\frac{1}{2}(f(x_{n-1})+f(x_n))\Delta x$ |

Notice that the endpoints (*a* and *b*) are only used in one trapezoid each. All the interior values of $x$ are used in two trapezoids.

To estimate $\int_a^b f(x)dx$ (assuming $f(x)$ is nonnegative), add all the areas together:

$$\frac{1}{2}(f(x_0)+f(x_1))\Delta x+\frac{1}{2}(f(x_1)+f(x_2))\Delta x+\frac{1}{2}(f(x_2)+f(x_3))\Delta x+\ldots+\frac{1}{2}(f(x_{n-1})+f(x_n))\Delta x$$

Notice that each quantity has a factor of $\frac{1}{2}\Delta x$. We'll factor this out:

$$\frac{1}{2}\Delta x[f(x_0)+f(x_1)+f(x_1)+f(x_2)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})+f(x_n)]$$

There are some like terms within the brackets. Combine them:

$$\frac{1}{2}\Delta x[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(x_n)]$$

Note: we know that the $f(x_{n-1})$ term will show up twice since it is a base in each of the last two trapezoids.

This leads to a formula for estimating $\int_a^b f(x)dx$ using trapezoids. This method is aptly named the trapezoidal rule.

FORMULA

**Trapezoidal Rule**

$\int_a^b f(x)dx$ can be approximated by the sum

$\frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(x_n)]$, where $\Delta x=\frac{b-a}{n}$ and

$a=x_0, x_1, x_2, \ldots, x_{n-1}, x_n=b$ are the endpoints of each equally spaced subinterval.

BIG IDEA

The estimate obtained from the trapezoidal rule is actually the average of the left- and right-hand endpoint rules. Thus, if you have the left- and right-hand estimates already calculated, the trapezoidal estimate follows quickly.

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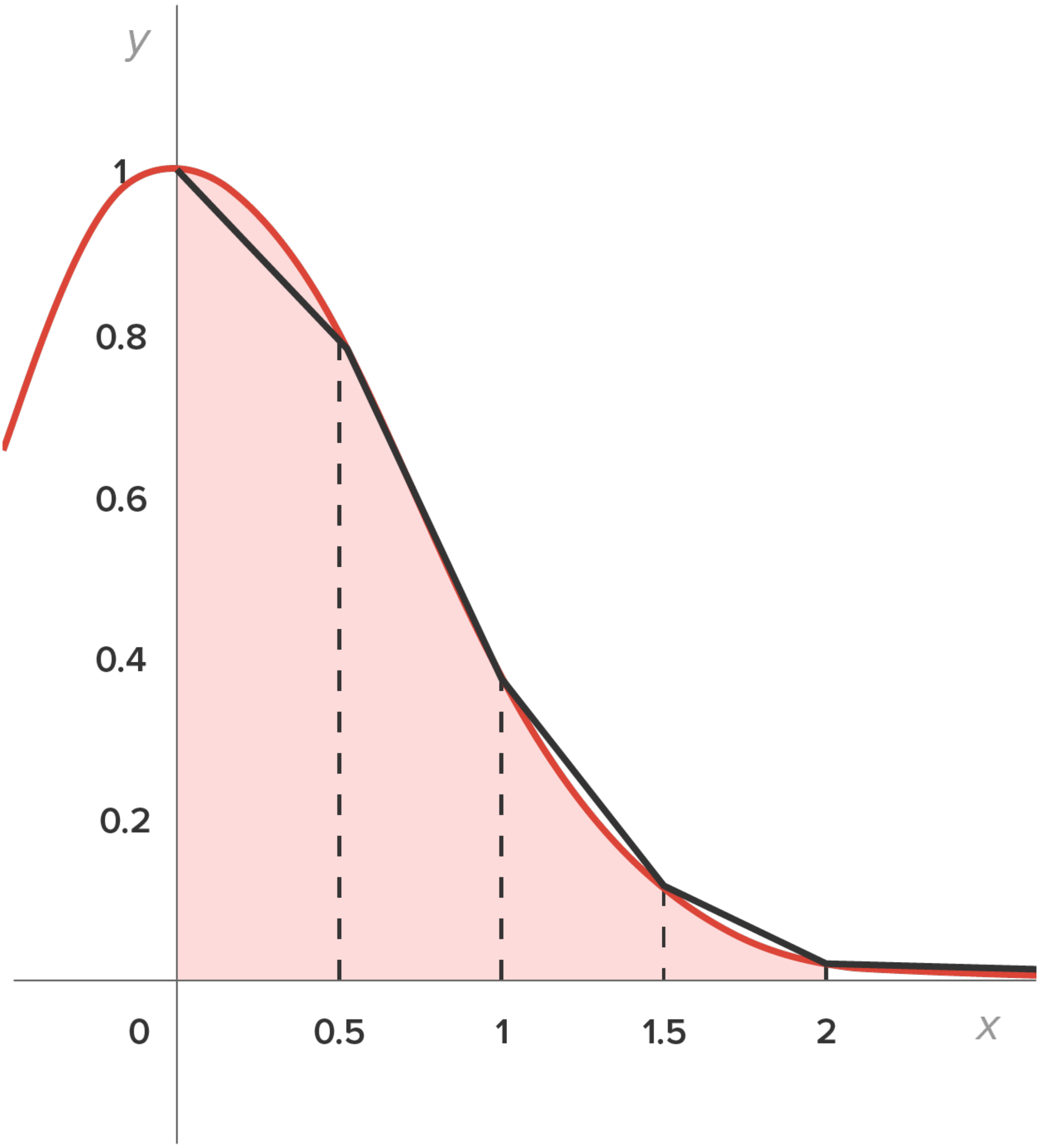
↪ EXAMPLE Estimate the value of $\int_0^2 e^{-x^2}dx$ by using $n=4$ subintervals of equal width using the trapezoidal rule. Check that this is equal to the average of the left- and right-hand endpoint estimates.

The graph of the region (with trapezoids drawn) is shown in the figure below. Note: $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. This means the subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$.




The x-values used are $x = 0, 0.5, 1, 1.5$, and 2.

Then, the estimate for the definite integral is:

$$\frac{0.5}{2}[f(0)+2f(0.5)+2f(1)+2f(1.5)+f(2)]$$

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$$= 0.25\left[e^{-0^2} + 2e^{-0.5^2} + 2e^{-1^2} + 2e^{-1.5^2} + e^{-2^2}\right]$$

After using a calculator, this is approximately 0.88062 (to five decimal places).
In an earlier example, we also computed the average of the left- and right-hand estimates, and this was also approximately 0.88062. (If you look beyond the third decimal place in each, the estimates should be identical.)

TRY IT

Consider $\int_1^3 \sqrt{x^3+1}\,dx$.

Using $n=4$ subintervals, estimate the definite integral using the trapezoidal rule. Round your answer to 5 decimal places.

+

6.26094

---

# 3. Approximating a Definite Integral Using Parabolas (Simpson's Rule)

While the methods discussed so far seem to do a fair job of approximating $\int_0^2 f(x)dx$, it makes more sense to approximate $f(x)$ with a curve rather than a straight line (if $f(x)$ is a straight-line function, there is no need to approximate the definite integral). To that end, we can also use parabolas, but not before we establish a rule for the definite integral of a parabola.

If $f(x) = Ax^2 + Bx + C$, then $\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$.

Now, consider this region.

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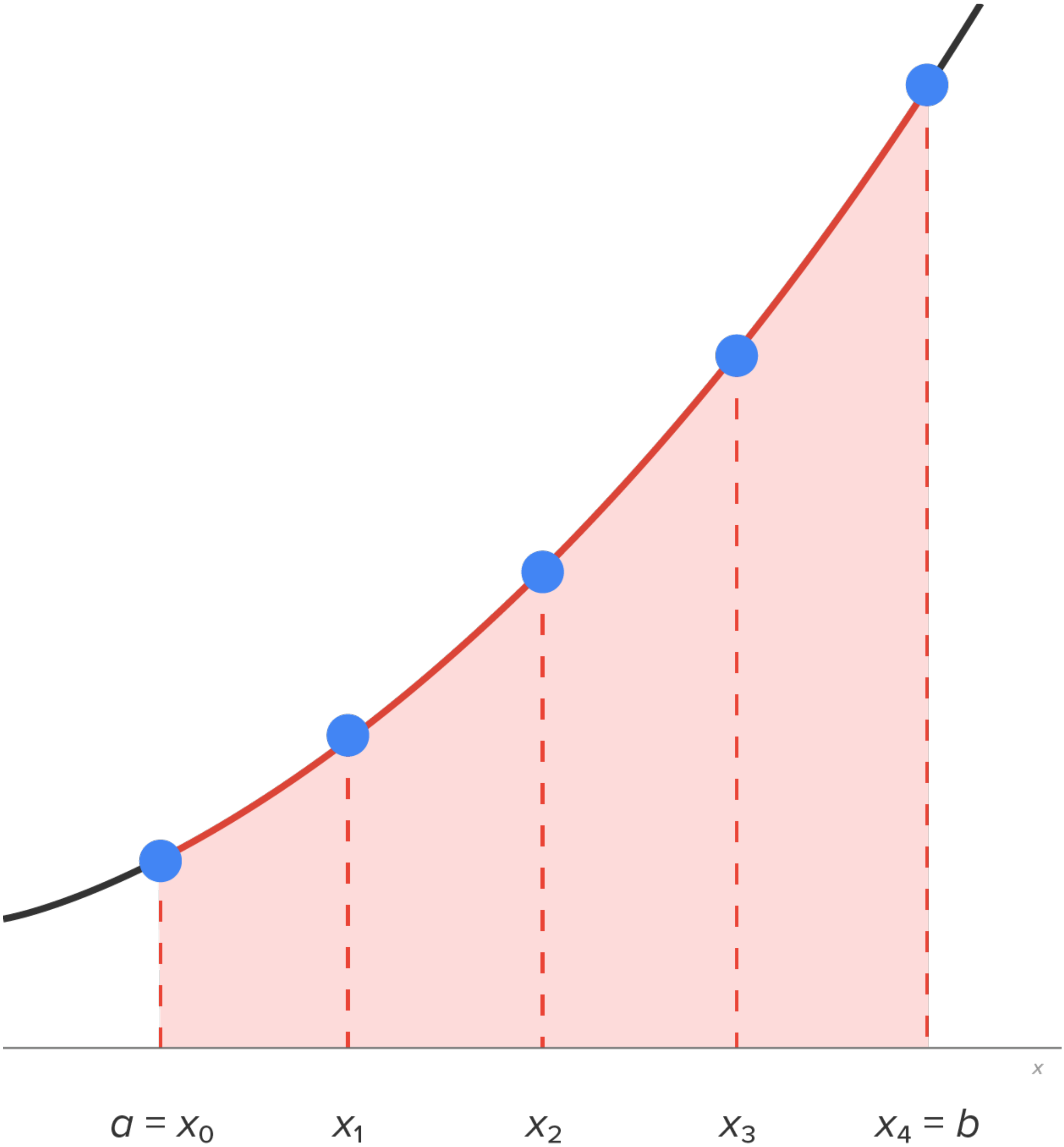


If we want to approximate $f(x)$ by using a parabola, we would have $\int_{x_0}^{x_2} f(x)dx \approx \frac{x_2 - x_0}{6}\left[f(x_0) + 4f\left(\frac{x_0 + x_2}{2}\right) + f(x_2)\right]$.

Note: $\frac{x_0 + x_2}{2} = x_1$ (that is, since the x-values are evenly spaced, $x_1$ is the average of $x_0$ and $x_2$).

Also, consider the quantity $\frac{x_2 - x_0}{6}$. Since $\Delta x$ is the distance between two consecutive x-values, it follows that $x_2 - x_0 = 2\Delta x$, which means $\frac{x_2 - x_0}{6} = \frac{2\Delta x}{6} = \frac{\Delta x}{3}$.

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Thus, we can write $\int_{x_0}^{x_2} f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)]$.

Then, considering the interval $[x_2, x_4]$, we have $\int_{x_2}^{x_4} f(x)dx \approx \frac{\Delta x}{3}[f(x_2)+4f(x_3)+f(x_4)]$.

Then, the approximation for $\int_{x_0}^{x_4} f(x)dx$ is the sum of the previous two approximations:

That is, $\int_{x_0}^{x_4} f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)]+\frac{\Delta x}{3}[f(x_2)+4f(x_3)+f(x_4)]$.

Factoring out $\frac{\Delta x}{3}$, this becomes $\frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)]$.

Combining like terms, this becomes $\frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$.

Note: since each parabola uses two subintervals, the number of subintervals must be even.

If this process were to continue, notice the following:

- $f(x_0)$ and $f(x_n)$ would have coefficients of 1.
- $f(x_1), f(x_3),$ ... would have coefficients of 4 (odd-numbered subscripts).
- $f(x_2), f(x_4),$ ... would have coefficients of 2 (even-numbered subscripts, but not the endpoints).

This means that the pattern in the coefficients is 1, 4, 2, 4, 2, 4, ..., 2, 4, 1.
This approximation method is known as Simpson's rule.

FORMULA

**Simpson's Rule**

The value of $\int_a^b f(x)dx$ is approximated by

$\frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+...+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$ where $\Delta x = \frac{b-a}{n}$ and $a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b$ are the endpoints of each equally spaced subinterval. Note: $n$ must be even.

EXAMPLE Estimate the value of $\int_0^2 e^{-x^2}dx$ by using $n=4$ subintervals of equal width using Simpson's rule.

The graph of the region is shown in the figure. Note: $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. This means the subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$.

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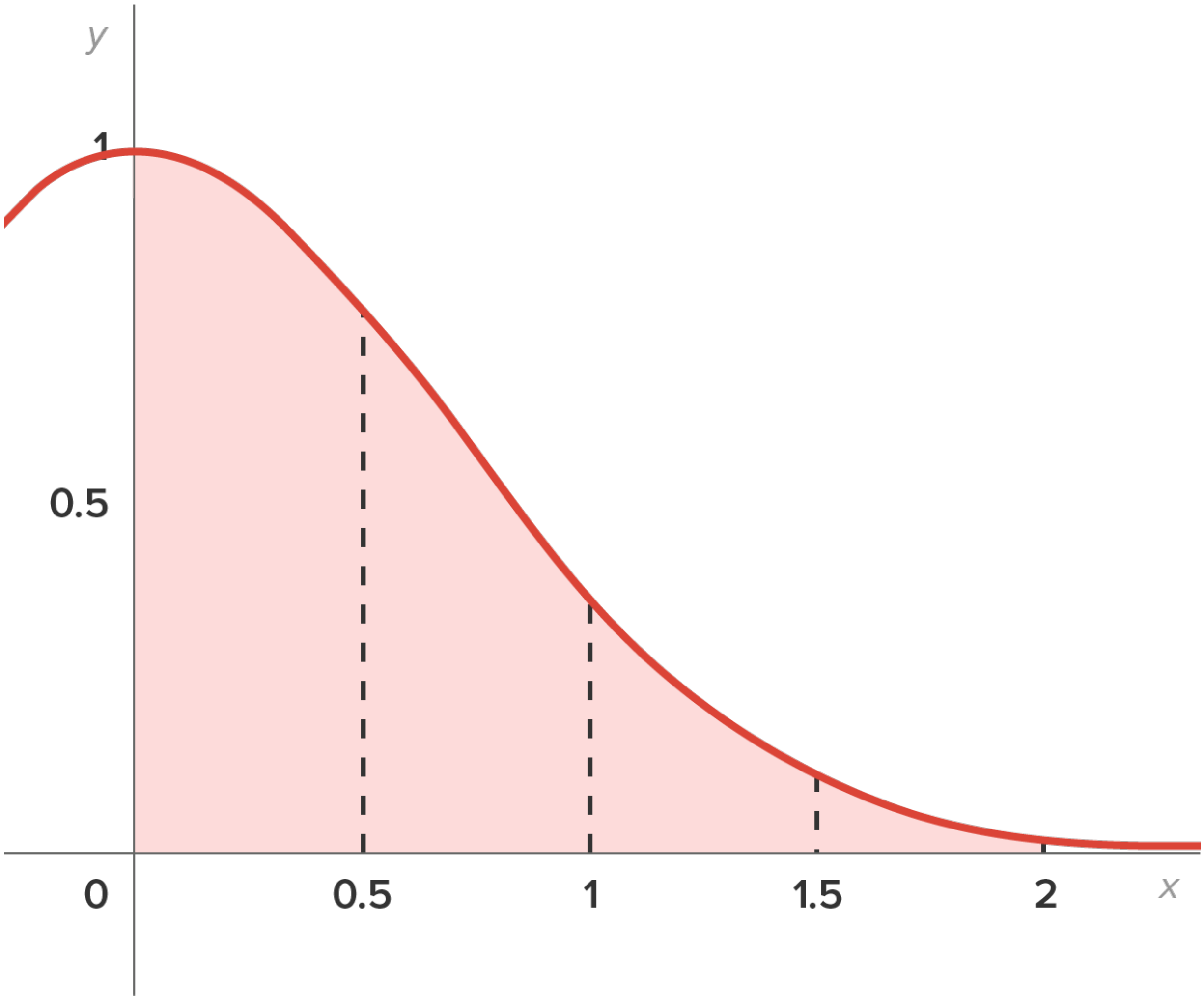


The x-values used are $x = 0, 0.5, 1, 1.5$, and 2.

Then, the estimate for the definite integral is:

$$\frac{0.5}{3}[f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)] = \frac{1}{6}\left[e^{-0^2}+4e^{-0.5^2}+2e^{-1^2}+4e^{-1.5^2}+e^{-2^2}\right]$$

After using a calculator, this is approximately 0.88181 (to five decimal places).

Comparing this to the left-hand, right-hand, midpoint, and trapezoidal estimates, these are all very close, but it turns out that Simpson's rule gives the closest approximation for this function on this interval. In general, Simpson's rule usually gives the best estimate out of the five we discussed, but there are exceptions.

TRY IT

Consider $\int_1^3 \sqrt{x^3+1}\,dx$.

Using $n = 4$ subintervals, estimate the definite integral using Simpson's rule. +

6.23030

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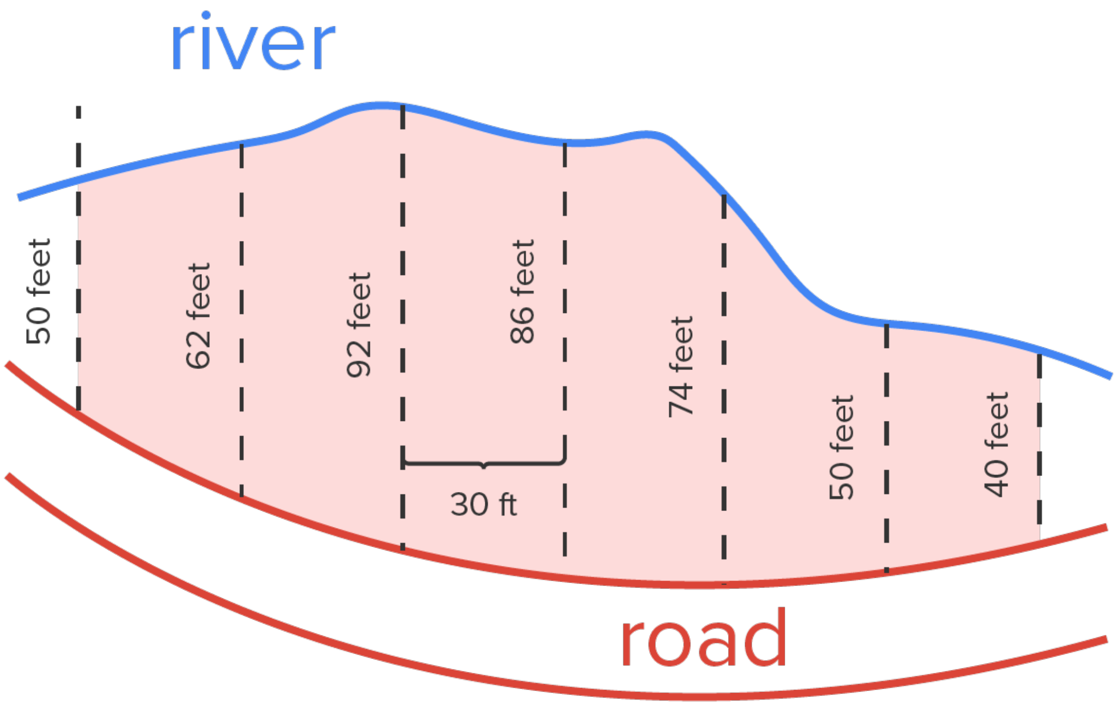
The approximation methods shown in this challenge are all used to approximate the value of a definite integral. Using that connection, these methods can be used to approximate the area of an irregularly shaped region, as we'll see in the next part.

# 4. Application: Approximating Area

↱ EXAMPLE Suppose the measurements of a park are shown below. What is the approximate area of the park? As the figure suggests, the measurements were taken every 30 feet.




Let $f(x) =$ the length of the park at a distance of *x* feet from the left-hand side (as shown). Then, the area of the park is $\int_0^{180} f(x)dx$ (there are 6 widths of 30 feet, for a total of 180 feet).

Since we don't have an expression for $f(x)$, we'll need to use the approximation techniques learned in this section to estimate the area.

Using $\Delta x = 30$ with $n = 6$ subintervals, we can easily use the left-hand and right-hand endpoints, the trapezoidal rule and Simpson's rule to approximate the area. (Since we don't know the values of $f(x)$ in the middle of each interval, the midpoint method cannot be used.)

Here is a table of values for all the values of *x*.

| | $a = x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6 = b$ |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 |
| Length | 50 | 62 | 92 | 86 | 74 | 50 | 40 |

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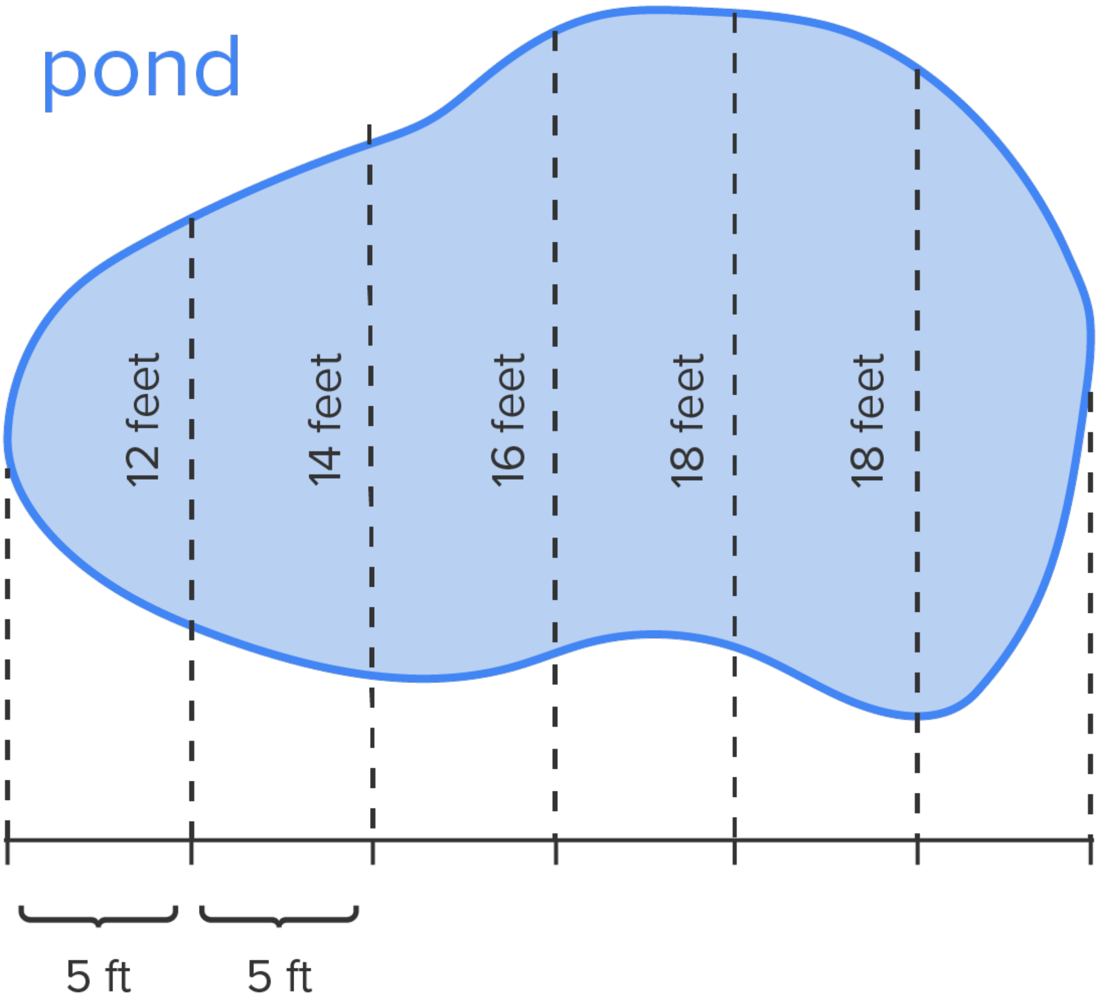
Below is a table that shows the various approximations for the area. Notice that the estimates are fairly close to each other.

| Method | Estimation for the Area |
|---|---|
| Left-Hand Endpoints | $30(50+62+92+86+74+50)=12{,}420\ ft^2$ |
| Right-Hand Endpoints | $30(62+92+86+74+50+40)=12{,}120\ ft^2$ |
| Trapezoids | $\frac{30}{2}[50+2(62)+2(92)+2(86)+2(74)+2(50)+40]=12{,}270\ ft^2$ |
| Simpson's Rule | $\frac{30}{3}[50+4(62)+2(92)+4(86)+2(74)+4(50)+40]=12{,}140\ ft^2$ |

TRY IT

Consider the pond shown in the figure below. Assume each subinterval is 5 feet wide and that the distance across at the endpoints is 0 feet.




Use left- and right-hand endpoints, the trapezoidal rule, and Simpson’s rule to estimate the surface area of the pond.

+

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The left- and right-hand endpoints, and trapezoidal rule all give a surface area of $390\text{ ft}^2$, while Simpson's rule gives an estimate of $413.33\text{ ft}^2$ (rounded).

## SUMMARY

In this lesson, you learned that when **approximating definite integrals**, there are several methods that can be used: **using rectangles**, **using trapezoids (trapezoidal rule)**, and **using parabolas (Simpson's rule)**. The key takeaway here is the progression. Using rectangles is simpler, but doesn't give as good an estimate as trapezoids, which still has flaws since a line is used to approximate a curve. Simpson's rule, using parabolas, seems to be the best match for a curve $y=f(x)$, but since not all curves are parabolic, there is still room for some error. In all cases, the estimate is improved by increasing the number of subintervals. Finally, you were able to **apply** the techniques learned in this section to **approximate the area** in several examples.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## FORMULAS TO KNOW

**Simpson's Rule**

The value of $\int_a^b f(x)dx$ is approximated by

$\frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$ where $\Delta x=\frac{b-a}{n}$ and $a=x_0, x_1, x_2, \ldots, x_{n-1}, x_n=b$ are the endpoints of each equally spaced subinterval. Note: *n* must be even.

**Trapezoidal Rule**

$\int_a^b f(x)dx$ can be approximated by the sum $\frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(x_n)]$, where $\Delta x=\frac{b-a}{n}$ and $a=x_0, x_1, x_2, \ldots, x_{n-1}, x_n=b$ are the endpoints of each equally spaced subinterval.

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# Terms to Know

**Antiderivative**

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

**Circumscribed (Rectangles)**

A rectangle is circumscribed outside a region if it is the smallest rectangle that encompasses the region.

**Differential Equation**

An equation that contains derivatives of some function *y*.

**General Solution**

The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

**Indefinite Integral of f(x)**

The collection of functions whose derivatives are equal to $f(x)$. In other words, the indefinite integral of $f(x)$ is the antiderivative of $f(x)$.

**Initial Condition**

From a differential equation, a point that the solution's graph passes through.

**Inscribed (Rectangles)**

A rectangle is inscribed inside a region if it is the largest rectangle that stays inside the region.

**Integrable**

If the value of $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to *A* regardless of the values of $c_k$ used in each subinterval, then we say that $f(x)$ is integrable on the interval $[a, b]$.

**Particular Solution**

The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

**Partition**

A set of x-values that are used to split the interval $[a, b]$ into smaller intervals.

**Riemann Sum**

The sum obtained from the areas of rectangles that are used to approximate the area between a curve and the x-axis.

**Subinterval**

A smaller interval that is part of a larger interval.

**Summand**

The expression being used to determine the numbers that are added in a sum.

**Summation**

An expression that implies that several numbers are being added together. These are often written using sigma notation.

**The First Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x)=f(x)$.

Then, $\int_a^b f(x)dx = F(b)-F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

**The Mean Value Theorem for Integrals**

If $f(x)$ is continuous on $[a, b]$, then at some point *c* in $[a, b]$:

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx$$

**The Second Fundamental Theorem of Calculus**

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a \le x \le b$.

Let $F(x)=\int_a^x f(t)dt$. Then, $F'(x)=\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

# Formulas to Know

**Antiderivative of a Constant**

$$\int k dx = kx + C$$

**Antiderivative of a Constant Multiple of a Function**

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

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**Antiderivative of a Difference of Functions**

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

**Antiderivative of a Sum of Functions**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

**Antiderivative of cos x**

$$\int \cos x dx = \sin x + C$$

**Antiderivative of csc x cot x**

$$\int \csc x \cot x dx = -\csc x + C$$

**Antiderivative of csc$^2$ x**

$$\int \csc^2 x dx = -\cot x + C$$

**Antiderivative of e$^{kx}$, Where k is a Constant**

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$$

**Antiderivative of sec x tan x**

$$\int \sec x \tan x dx = \sec x + C$$

**Antiderivative of sec$^2$ x**

$$\int \sec^2 x dx = \tan x + C$$

**Antiderivative of sin x**

$$\int \sin x dx = -\cos x + C$$

**Antiderivatives of Exponential Functions**

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

**Area Between Two Curves, x = h(y) and x = k(y), Assuming h(y) ≥ k(y) on [c, d] (Horizontal Subrectangles)**

$$Area = \int_c^d [h(y) - k(y)]dy$$

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**Area Between Two Curves, y = f(x) and y = g(x), Assuming f(x) ≥ g(x) on [a, b]**

$$Area = \int_a^b [f(x) - g(x)]dx$$

**Average Value of a Function**

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

**Definite Integral Over a Partition of an Interval, with a ≤ b ≤ c**

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

**Definite Integral When Lower and Upper Bounds Are Equal**

$$\int_a^a f(x)dx = 0$$

**Definite Integral When Upper and Lower Bounds Are Interchanged**

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

**Definite Integral of a Constant Function**

$$\int_a^b k dx = k(b-a)$$

**Definite Integral of a Constant Multiple of a Function**

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

**Definite Integral of a Difference of Two Functions**

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

**Definite Integral of a Sum of Two Functions**

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

**Fundamental Theorem of Calculus**

Let $F(x)$ be an antiderivative of a continuous function $f(x)$ on the interval $[a, b]$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$.

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### Natural Logarithm Rule

$\int \frac{1}{x} dx = \ln|x| + C$

### Power Rule for Antiderivatives

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$

### Riemann Sum

When approximating the area between a nonnegative function $y = f(x)$ and the x-axis by using *n* rectangles, the summation $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called the Riemann Sum, where $c_k$ is a value of *x* in the $k^{th}$ subinterval, and $\Delta x_k$ is the width of the $k^{th}$ subinterval.

### Simpson's Rule

The value of $\int_a^b f(x)dx$ is approximated by

$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ where $\Delta x = \frac{b-a}{n}$ and $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ are the endpoints of each equally spaced subinterval. Note: *n* must be even.

### Summation of a Constant Multiple

$$\sum_{k=1}^{n} C \cdot a_k = C \cdot \sum_{k=1}^{n} a_k$$

### Summation of a Sum or Difference

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

### Summations of Powers of Consecutive Numbers

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

### The Summation of a Constant

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If $C$ is a constant, $\sum_{k=1}^{n} C = C \cdot n$

**Trapezoidal Rule**

$\int_a^b f(x)dx$ can be approximated by the sum $\frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+...+2f(x_{n-1})+f(x_n)]$, where $\Delta x = \frac{b-a}{n}$ and $a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b$ are the endpoints of each equally spaced subinterval.