Unit 1 Tutorials: Precalculus Review in Context

INSIDE UNIT 1

Distance Between Two Points

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to find the distance between two points on a number line and also on the xy-plane. Specifically, this lesson will cover:

1. The Distance Between Two Numbers on a Number Line
2. The Distance Between Two Points in the xy-Plane

1. The Distance Between Two Numbers on a Number Line

Suppose you want to calculate the **distance** between two locations on a number line, as shown below.

The distance between these two points is $b - a$, but that is assuming that b is larger than a.

In general, just so we don't have to worry about which number is larger, the distance between two numbers a and b is $dist(a, b) = |b - a|$. The absolute value is used to ensure that the result is not negative.

FORMULA

Distance on a Number Line

$dist(a, b) = |b - a|$

TRY IT

Find the distance between a and b in each example below.

What is the distance when $a = 13$ and $b = 5$? +

The distance between 13 and 5 is 8.

$dist(13,5) = |5 - 13| = |-8| = 8$

What is the distance when $a = -21$ and $b = 9$? +

The distance between -21 and 9 is 30.

$dist(-21,9) = |9 - (-21)| = |9 + 21| = |30| = 30$

TERM TO KNOW

Distance

The length of a line segment between two points.

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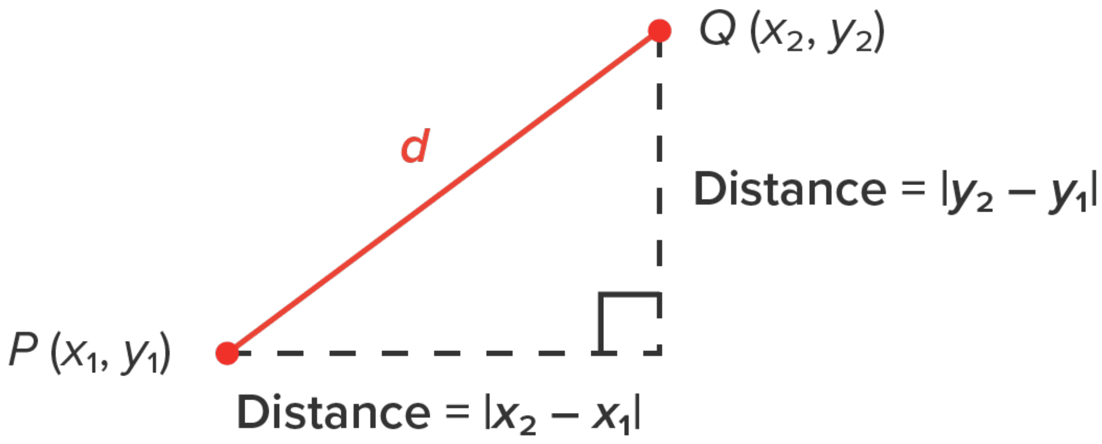
2. The Distance Between Two Points in the xy-Plane

The following image shows two points, P and Q, and the distance between them in the xy-plane, d. Let's find a formula for the distance between these two points.



In the image above:

- The vertical side is the distance between the y-coordinates, which is $|y_2 - y_1|$.
- The horizontal side is the distance between the x-coordinates, which is $|x_2 - x_1|$.
- The distance between the points is labeled as d.

Notice that we have three sides of a right triangle. This means that the Pythagorean theorem can be used to relate the sides to each other. Recall that the Pythagorean theorem states that $(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$, where a leg is defined as a side that makes up the right angle and the hypotenuse is the side opposite the right angle (the longest side).

Applying the Pythagorean theorem to our image, we have $|x_2 - x_1|^2 + |y_2 - y_1|^2 = d^2$.

HINT

Notice that the first two terms are squares of absolute values. Since squaring also guarantees a nonnegative result, there is no need to include the absolute value. Thus, the relationship actually can be rewritten as $(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$.

To write an expression for the distance, d, take the square root of both sides to get the following formula:

FORMULA

Distance in the xy-Plane

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

HINT

You might remember from algebra that taking the square root of both sides results in a positive solution and a negative solution. Since distance is always nonnegative, only the positive square root is considered.

EXAMPLE Calculate the exact distance between the points (4, 5) and (8, 1).

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance Formula
$d=\sqrt{(8-4)^2+(1-5)^2}$	Substitute known quantities: $x_1=4$, $y_1=5$, $x_2=8$, $y_2=1$.
$d=\sqrt{4^2+(-4)^2}$	Evaluate subtraction inside parentheses.
$d=\sqrt{16+16}$	Square values.
$d=\sqrt{32}$	Add values under the square root.
$d=\sqrt{16\cdot 2}$	Rewrite the square root with any perfect square factors.
$d=\sqrt{16}\sqrt{2}$	Apply the product property of square roots.
$d=4\sqrt{2}$	Simplify the radical.

The distance between the points (4, 5) and (8, 1) is $4\sqrt{2}$, or about 5.66 units.

WATCH

The following video further illustrates the use of the distance formula.

Video Transcription

Welcome. And what we're going to look at right now is the distance formula and using it to find the distance between two points and the x-y plane.

So we have labeled here are two points, negative 2, 3 and 4, 5. And what we're curious about is how far apart are they. So you might recall the distance formula over to the right is d equals the square root of x2 minus x1 quantity squared plus y2 minus y1 one the quantity squared. Remember that really represents the difference in the x's squared, which we call delta x, and the difference between the y-coordinates squared.

So the first thing we should do whenever we use a formula is to label the inputs. And our first point I'm just going to call x sub 1, y sub 1. And 4, 5 I'm going to call x sub 2, y sub 2. And that makes it very clear what we are going to substitute into this formula.

So we have d is equal to the square root of-- so x2 is 4. x1 is negative 2. And we're squaring that, plus y2 is 5 and y1 is 3. So that's 5 minus 3, the quantity squared. And now we just use our order of operations to simplify the expression.

So we have the square root of 4 minus negative 2 is 6, and 6 squared is 36, plus 5 minus 3 is 2, and 2 squared is 4. And now we combine under the radical. So we have the square root of 40. And the exact form of the answer is what we're going to look for.

So remember that some square roots can be simplified, in that we remove the perfect square factors from the radicand, which is 40. So thinking about a perfect square that divides into 40 evenly, we could think of 4. So 40 can be rewritten as 4 times 10. Remember property of radicals, this is rewritten as the square root of 4 times the square root of 10, which is 2 square roots of 10.

And this is the exact form of the distance. If you want to approximate it, naturally you can take a calculator. And you can either type in the square root of 40 or 2 times the square root of 10, and they should both give you the same result. And we know that 2 times square root of 10 is simplified because there is no other perfect square factor that divides into 10 evenly. So there is a demonstration of the distance formula.

SUMMARY

In this lesson, you learned how to calculate **the distance between two numbers on a number line** by calculating the absolute value of their difference. Next, you applied this idea, along with the Pythagorean theorem, to arrive at the distance formula to calculate **the distance between two points in the xy-plane**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Distance
The length of a line segment between two points.

FORMULAS TO KNOW

Distance in the xy-Plane
$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Distance on a Number Line
$dist(a,b)=|b-a|$

Equations of Circles

by Sophia

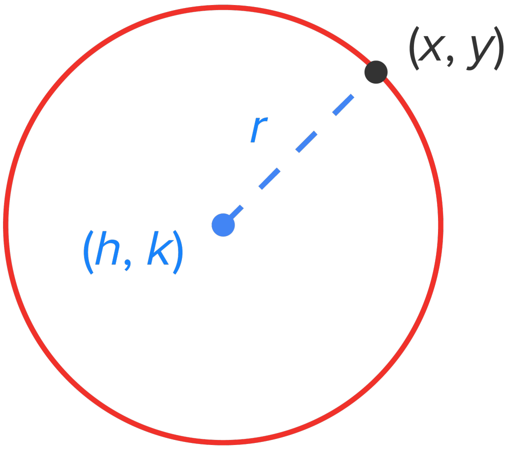
WHAT'S COVERED

In this lesson, you will learn how to write the equation of a circle. Specifically, this lesson will cover:

1. Parts of a Circle
2. Standard Form Equation of a Circle

1. Parts of a Circle

Consider the circle shown below.



In the image above:

- The center of the circle is labeled (h, k).
- (x, y) represents any point on the circle.
- The radius of the circle is *r*, which is the distance from (h, k) to (x, y).

So, how can we calculate this distance?
Using the distance formula from the previous lesson, we can set up a relationship. The distance between (h, k) and (x, y) is the radius, *r*, and can be found with the following formula:

FORMULA

Radius of a Circle

$r=\sqrt{(x-h)^2+(y-k)^2}$

Where:
(h, k) is the center and (x, y) is a point on the circle.

2. Standard Form Equation of a Circle

If we take the radius formula from the above section and square it, we get the following equation:

$(x-h)^2+(y-k)^2=r^2$

This actually is the standard form of the equation of a circle. So, if a circle has center (h, k) and radius *r*, the equation for all points on the circle is $(x-h)^2+(y-k)^2=r^2$.

FORMULA

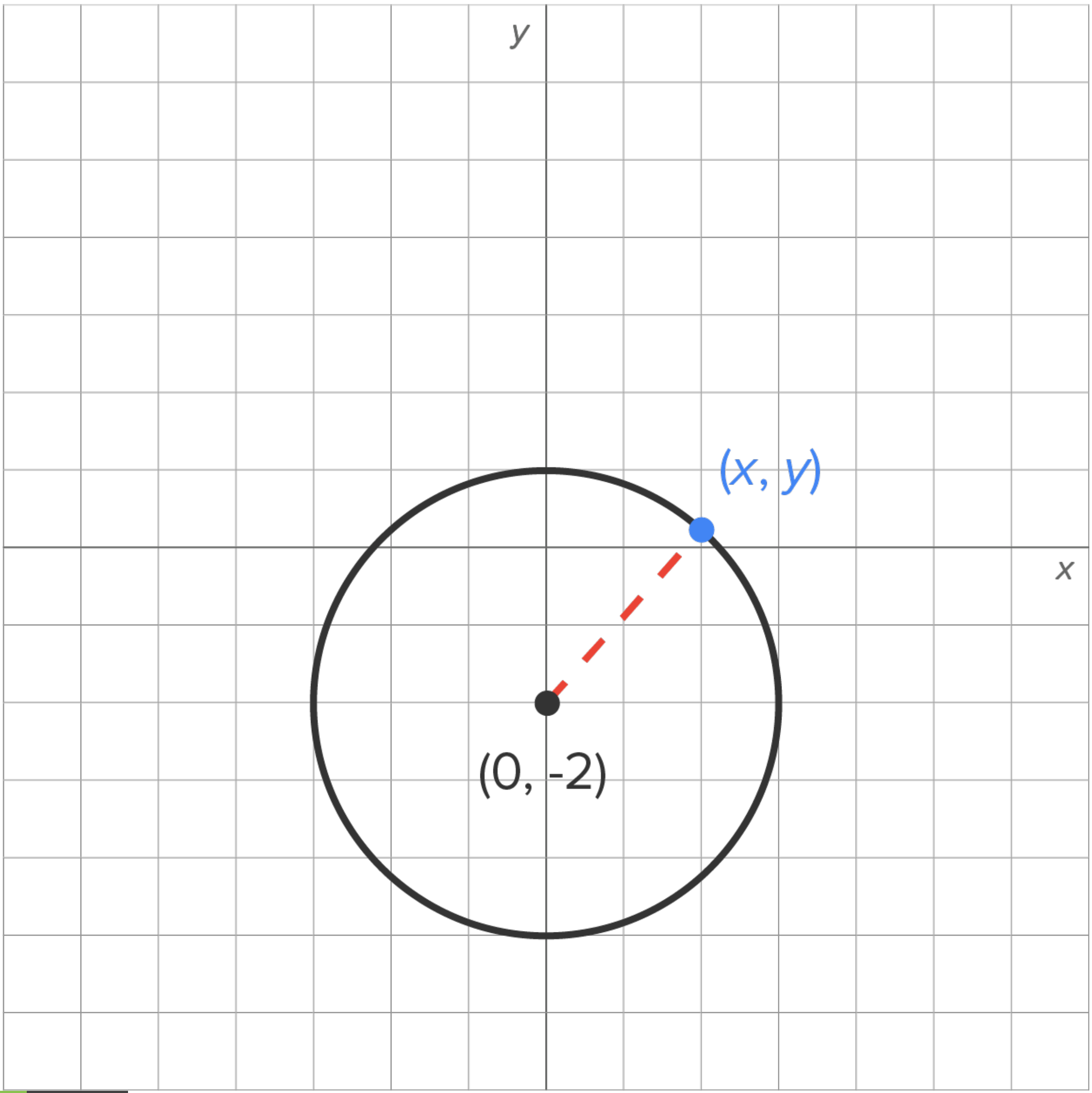
Standard Form Equation of a Circle
$(x-h)^2+(y-k)^2=r^2$

Where:
(h, k) is the center and *r* is the radius.

EXAMPLE Write the equation of a circle whose center is (0, -2) and which has a radius of 3.

$(x-h)^2+(y-k)^2=r^2$	Standard Form Equation of a Circle
$(x-0)^2+(y-(-2))^2=3^2$	Substitute known values: *h* = 0, *k* = -2, and *r* = 3.
$x^2+(y+2)^2=9$	Simplify the equation.

The equation of the circle whose center is (0, -2) and which has a radius of 3 is $x^2+(y+2)^2=9$ and has the following graph:



TRY IT

A circle has the equation $(x-1)^2+(y+2)^2=16$.

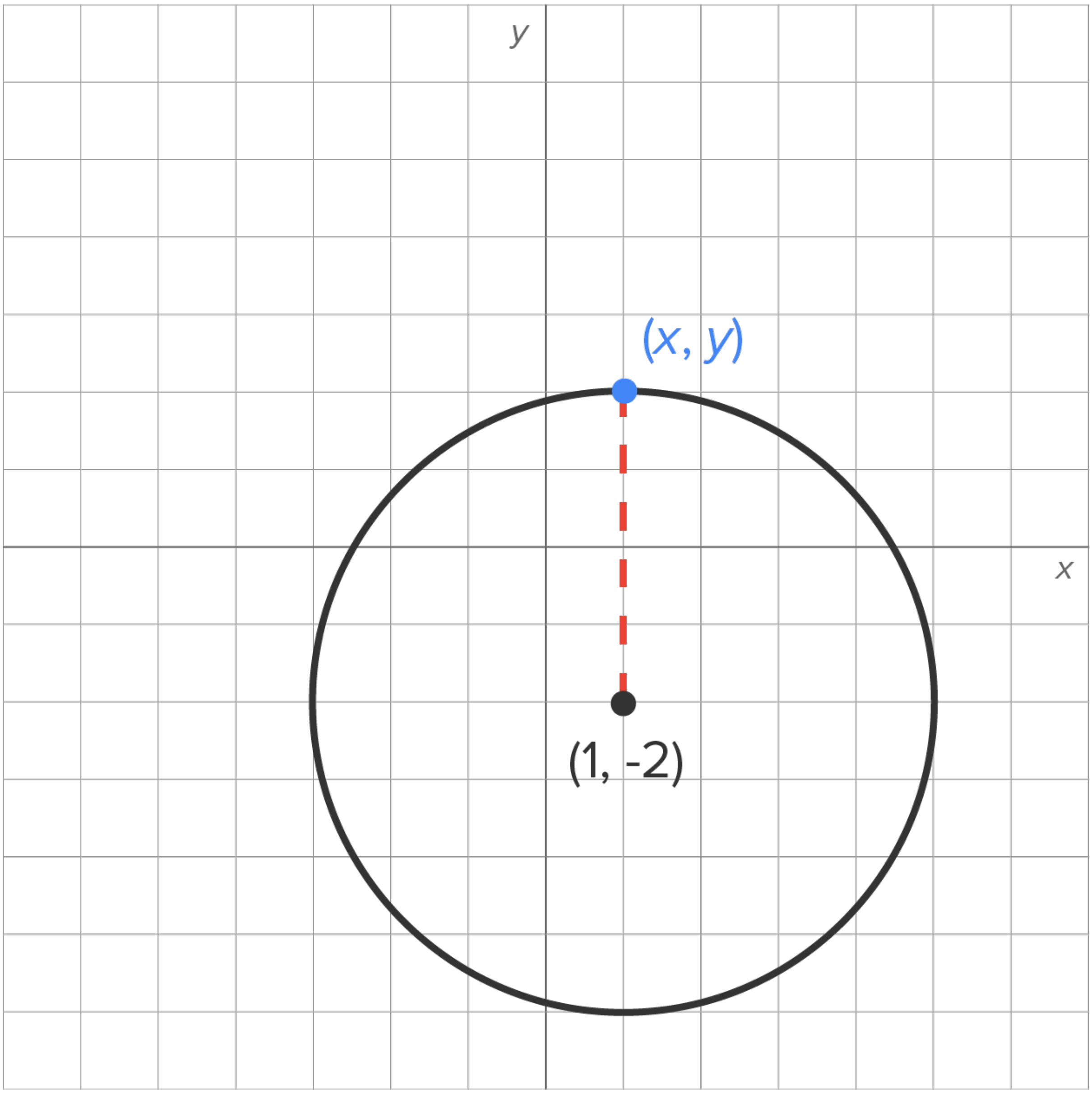
What is the center? +

Matching the equation to the standard form, we see that $h=1$ and $k=-2$, so the center of the circle is (1, -2).

What is the radius? +

Also from the equation, we see that the right side of the equation, 16, corresponds with r^2, which means the radius, r, is 4.

What does this graph look like? +



SUMMARY

In this lesson, you learned about the **parts of a circle**, including the center of the circle, the representation of any point on the circle, and the radius, which is the distance between the center and a given point on the circle. You learned how to calculate this distance using the formula for the radius of a circle. You also learned how to write the equation of a circle using the **standard form equation of a circle**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Radius of a Circle

$r = \sqrt{(x-h)^2 + (y-k)^2}$

Where: (h, k) is the center and (x, y) is a point on the circle.

Standard Form Equation of a Circle

$(x-h)^2 + (y-k)^2 = r^2$

Where: (h, k) is the center and r is the radius.

Slope of a Line Between Two Points

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the slope between two points. Specifically, this lesson will cover:

1. Slope of a Line
2. Slope of a Line Between Two Points on a Curve
3. Applications of Slope to Real-Life Situations
   a. Population
   b. Temperature

1. Slope of a Line

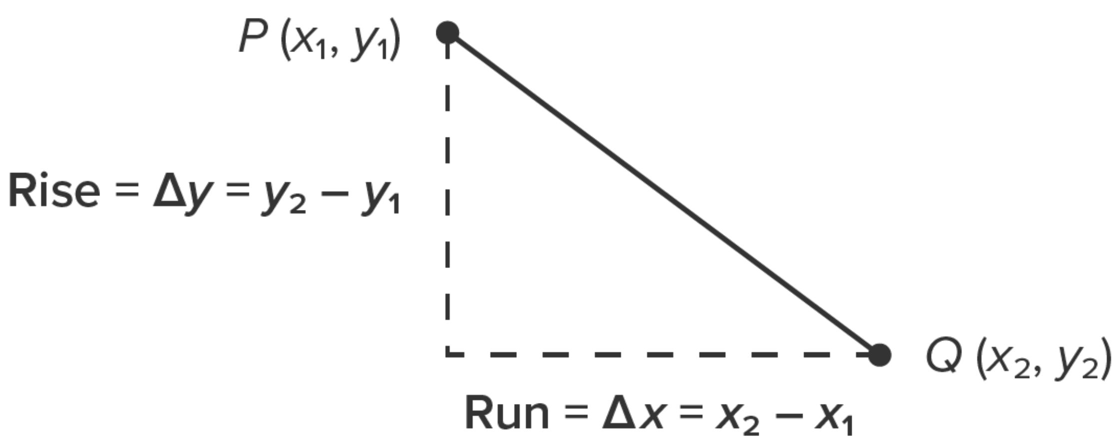
Slope is a very important concept in calculus, as it opens the door to explore rates of change.

When given two points, the slope between them is represented by the letter m, and is given by the following formula:

FORMULA

Slope

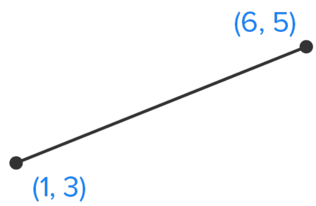
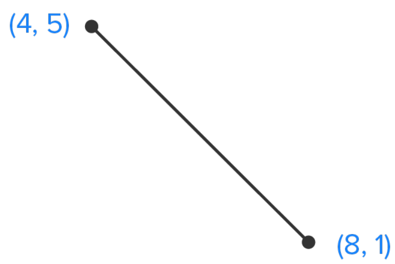
$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



HINT

The easiest way to look at slope is from left to right. When the line falls, or decreases, from left to right, the rise is negative. When the line increases from left to right, the rise is positive.

The table below shows examples of different types of slopes, along with how to compute and graph the slope.

Type of Slope	Points	Calculation	Graph
Positive Slope	$(1, 3)$ and $(6, 5)$	Slope = $m=\frac{5-3}{6-1}=\frac{2}{5}$	
Negative Slope	$(4, 5)$ and $(8, 1)$	Slope = $m=\frac{1-5}{8-4}=\frac{-4}{4}=-1$	
Zero Slope (Horizontal Line)	$(2, 5)$ and $(8, 5)$	Slope = $m=\frac{5-5}{8-2}=\frac{0}{6}=0$	
Undefined Slope (Vertical Line)	$(1, 4)$ and $(1, 6)$	Slope = $m=\frac{6-4}{1-1}=\frac{2}{0}$ This is undefined.	

HINT

Notice that the slope for vertical lines is undefined. This is because you cannot divide by zero.

TERM TO KNOW

Slope

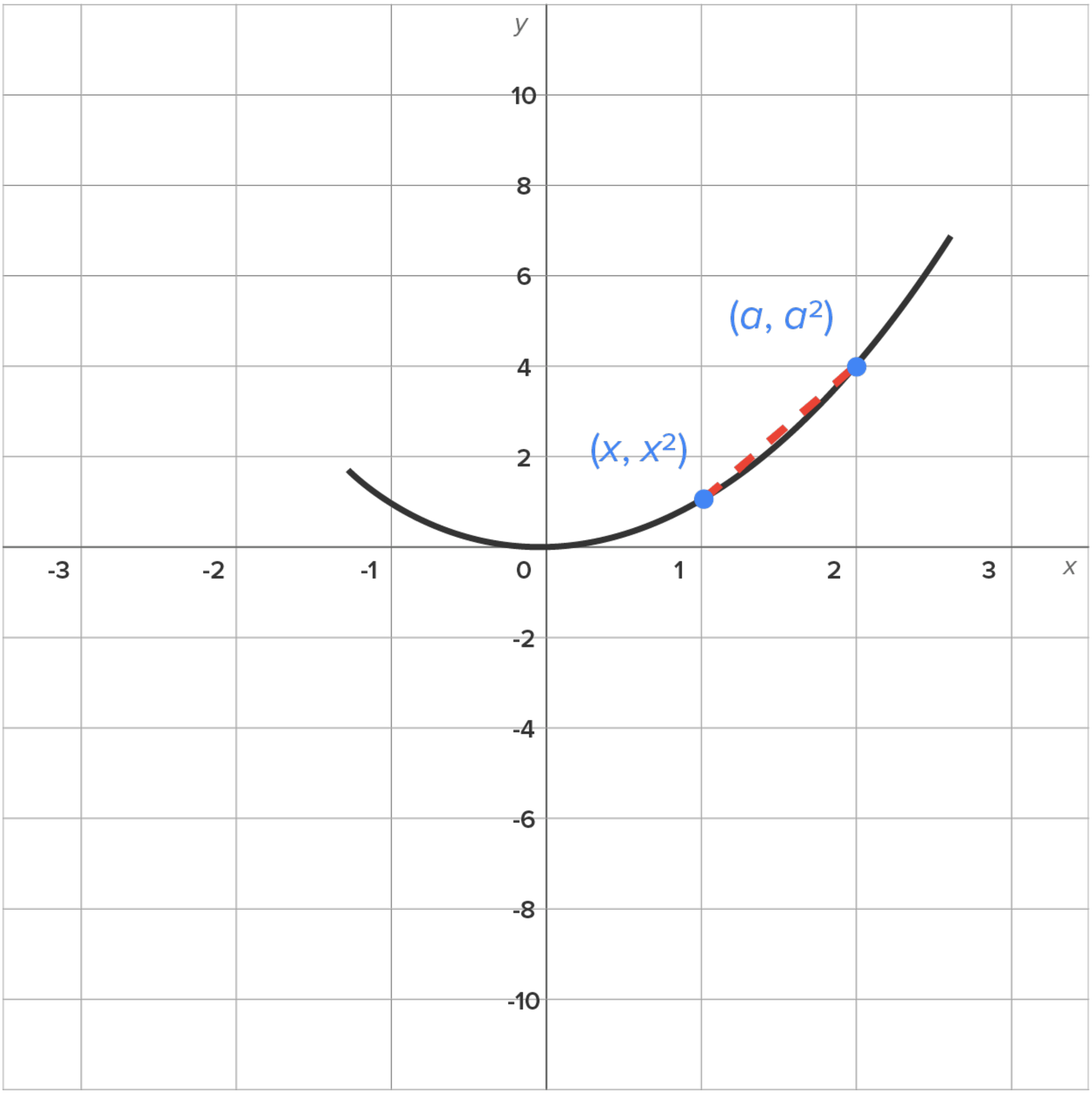
The ratio of the change in y to the change in x; measure of the steepness of a line.

2. Slope of a Line Between Two Points on a

Curve

In this course, we will frequently be examining the slope between two points that are on the same curve.

For example, let's look at the graph of $y = x^2$, which contains the points (x, x^2) and (a, a^2), where $a \neq x$ (meaning they are two different points).



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope Formula

$$m = \frac{a^2 - x^2}{a - x}$$ Substitute the two points.

$$m = \frac{(a - x)(a + x)}{a - x}$$ Factor the numerator.

$$m = a + x$$ Remove the common factors.

This means that for any values of a and x, the slope of the line between these points is $a+x$.

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3. Applications of Slope to Real-Life Situations

3a. Population

Suppose that in 1980, the population of a small town was 3,192 people. Then, in 1990, a count was taken again and the population was found to be 5,362. On average, at what rate did the population grow each year?

If we represent the information with two ordered pairs, (1980, 3192) and (1990, 5362), the slope between these points gives the average growth per year. Here is how:

$$\text{Slope} = m = \frac{\Delta\text{people}}{\Delta\text{time}}$$

$$\text{Slope} = \frac{5362\text{ people} - 3192\text{ people}}{\text{year }1990 - \text{year }1980} = \frac{2170\text{ people}}{10\text{ years}}$$

$$\text{Slope} = 217\text{ people}/\text{year}$$

BIG IDEA

In general, the units of the slope are $\frac{\Delta y\text{ units}}{\Delta x\text{ units}}$.

3b. Temperature

Did you know that the Fahrenheit and Celsius temperature scales are linearly related? Let's investigate.

The temperature at which water freezes is 0° C and 32° F.
The temperature at which water boils is 100° C and 212° F.

We can represent this information as two ordered pairs: (0, 32) and (100, 212).

Then, the slope of the line formed by these two points is $m = \frac{212-32\ (°\text{F})}{100-0\ (°\text{C})} = \frac{1.8\ (°\text{F})}{1\ (°\text{C})}$.

The slope is 1.8, and as we see in the computation, the slope means that the Fahrenheit scale goes up by 1.8° for every 1° increase in the Celsius scale.

SUMMARY

In this lesson, you learned about an important concept in calculus, the **slope of a line**, which is a measure of the steepness of a line. The slope between two points also represents the average rate of change between the points. You learned that the slope can be found using the formula of rise over run, which is the difference in y divided by the difference in x. There are four types of slope: positive, negative, zero (horizontal line), and undefined (vertical line). You also learned how to use the slope formula to calculate the **slope of a line between two points on a curve** Lastly, you explored several **applications of slope to real-life situations**, using slope to calculate the rate of **population** growth and the linear relationship between Fahrenheit and Celsius **temperature** scales.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Slope

The ratio of the change in *y* to the change in *x*; measure of the steepness of a line.

FORMULAS TO KNOW

Slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Equations of Lines

by Sophia

WHAT'S COVERED

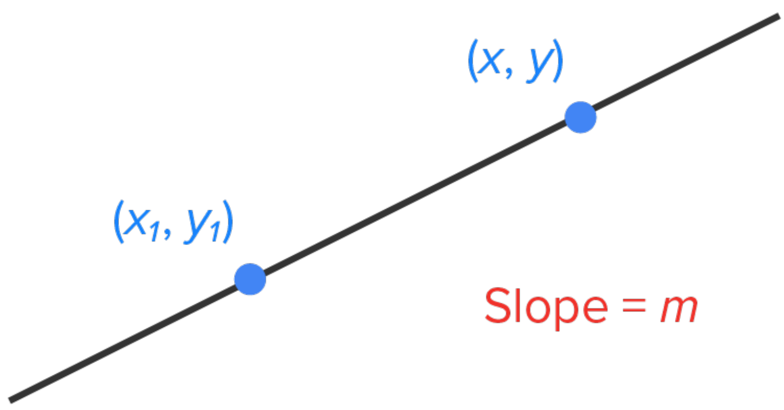
In this lesson, you will be able to write the equation of a line when given the appropriate information. Specifically, this lesson will cover:

1. Point-Slope Form
2. Slope-Intercept Form

1. Point-Slope Form

A line has the property that the slope between any two points on that line is always the same (we call it m).

Let (x, y) represent any point on a line and (x_1, y_1) a specific point on the line.



Using the slope formula, we know $m = \frac{y - y_1}{x - x_1}$. If we multiply both sides by $x - x_1$ this gives us the point-slope form of a linear equation:

FORMULA

Point-Slope Form
$y - y_1 = m(x - x_1)$

HINT

Typically, the point (x_1, y_1) and the slope *m* are substituted into this equation, then the final answer is solved for *y*.

EXAMPLE Use point-slope form to write the equation of the line that contains the point (-1, 4) and has slope 3.

$y - y_1 = m(x - x_1)$	Point-Slope Form
$y - 4 = 3(x - (-1))$	Substitute the value for *m* and the known point for x_1 and y_1.
$y - 4 = 3(x + 1)$	Simplify the subtraction inside the parentheses.
$y - 4 = 3x + 3$	Use the distributive property to simplify the right-hand side.
$y = 3x + 7$	Add 4 to both sides.

The equation of the line is $y = 3x + 7$.

2. Slope-Intercept Form

Another form of a line you may be familiar with is $y = mx + b$, which is the slope-intercept form of a line. The variable *m* is the slope, where the variable *b* is the y-coordinate of the y-intercept. Thus, another way to think about the line in the previous section is that it has slope 3 and y-intercept (0, 7).

FORMULA

Slope-Intercept Form
$y = mx + b$

EXAMPLE Write the equation of the line that contains the points (1, 5) and (4, 7) in slope-intercept form.

First, label the variables: $x_1 = 1$, $y_1 = 5$, $x_2 = 4$, $y_2 = 7$.

Then, the slope of the line is $m = \frac{7-5}{4-1} = \frac{2}{3}$.

You can then use point-slope form, along with either given point and the slope you just found. In this example, the point (1, 5) is used.

$y - y_1 = m(x - x_1)$	Point-Slope Form
$y - 5 = \frac{2}{3}(x - 1)$	Substitute the value for *m* and the known point for x_1 and y_1.
$y - 5 = \frac{2}{3}x - \frac{2}{3}$	Distribute the right-hand side.
$y = \frac{2}{3}x + \frac{13}{3}$	Add 5 to both sides.

Thus, the equation of the line is $y = \frac{2}{3}x + \frac{13}{3}$. This tells us that the line has a slope of $\frac{2}{3}$ and a y-intercept $\left(0, \frac{13}{3}\right)$.

WATCH

The following video illustrates how to write the equation of a line.

Video Transcription

[MUSIC PLAYING] Hi, and welcome back. What we're going to do in this video is take an actual, real life situation and form a linear model that describes the situation. So what we have here, the profit after selling 20 units is $100, while the profit after selling 50 units is $850. So profit is increasing as we sell more units. That actually makes sense. And what we're going to do is write the equation, assuming that it's a linear equation, that gives the profit after x units are sold.

So we have x being the number of units, and we're going to call y the profit. And the information was actually given to us as ordered pairs, without explicitly saying that. We know that when x is 20, the prophet is $100, and when x is 50, the prophet is $850. So we have two ordered pairs. I'm just going to pretend like there's a grid here. We have the point 20 comma 100, and we have the point 50 comma 850. And we want to write the equation of the line that connects those two points.

So remember, when we're writing the equation of a line, we use the point slope form when we're not given the y-intercept. So this is just to remind us that we're going to need this equation right here, y minus y1 equals m times x minus x1. So that's going to be where we actually do the work of writing the equation of the line.

First thing we need is the slope. Now, we know that the slope is the difference between the y-coordinates. So I'm going to say 850 minus 100 divided by the difference in the x-coordinates. And remember, 50 goes with 850, and 20 goes with 100, and then we simplify. So it's 750 divided by 30, which is 25. So 25 is the slope, and that's going to get substituted into our equation here.

Now, remember, it doesn't matter which point, you use when you're going to write the equation of your line, because the slope is what pulls them together. So we're going to use the easier numbers. I like 20 and 100 for x1 and y1. So we're going to say y minus 100 equals 25 times x minus 20, which means y minus 100 is equal to 25x minus 500. And remember, we usually like these solved for y, so I'm going to add 100 to both sides. So it's going to give us, as a final equation, y equals 25x minus 400. And that is the equation of the profit after x units are sold.

And just to kind of double-check reality, here, if x is 0-- so if you sell no units-- this tells us that the profit is negative $400, which seems a little weird. But remember, when you're in the business of selling items, you usually start off at a deficit, and your goal is to at least make up for that deficit. That's called break-even. So this equation actually does model reality pretty well. So there is writing the equation of a linear model, given the information.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that a line has the property that the slope between any two points on that line is always the same (m). You learned that given the slope and a point on the line (or two points contained on the line), you can use the **point-slope form** to write its equation. You also learned how to write the equation of a line using the **slope-intercept form**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Slope-Intercept Form

$$y = mx + b$$

What Is a Function?

by Sophia

WHAT'S COVERED

In this lesson, you will determine which relations are functions and which are not. Specifically, this lesson will cover:

1. What Is a Function?
2. Functions Defined by Sets of Ordered Pairs
3. Functions Defined by Equations
   a. Identifying Which Equations Are Functions
   b. Common Functions From Algebra

1. What Is a Function?

In mathematics, a **function** is a relationship that is designed so each input is assigned to only one output. In mathematics, we say that the output is a function of its input(s).

Here are some examples of functions you have seen:

Situation	Formula	Inputs/Outputs
Area of a Rectangle	$A = LW$	Inputs: L = Length, W = Width Output: A = Area A is a function of L and W.
Perimeter of a Square	$P = 4S$	Input: S = Side Length Output: P = Perimeter P is a function of S.
Volume of a Right Circular Cylinder	$V = \pi r^2 h$	Inputs: r = radius, h = height Output: V = Volume V is a function of r and h.

To get another perspective of what a function is, think about your calculator. When you press the square root key, the calculator returns the square root of the number you input (if your input is 16, the output is 4). The calculator never returns more than one value; this is the essence of a function.

TERM TO KNOW

Function

A correspondence between a set of inputs (x) and a set of outputs (y) such that each input corresponds to at most one output.

2. Functions Defined by Sets of Ordered Pairs

A relationship can be thought of as a collection of ordered pairs (*x, y*), where *x* is the input and *y* is the output.

↱ EXAMPLE Let *x* = the temperature reading in Celsius, and let *y* = the corresponding temperature reading in Fahrenheit. A few ordered pairs represented by the situation would be (0, 32), (100, 212), (37, 98.6), and (-40, -40).

This relationship is a function since there is no Celsius temperature (*x*) that corresponds to more than one Fahrenheit temperature (*y*). In other words, if a Celsius temperature is given, there is one definitive Fahrenheit temperature.

↱ EXAMPLE A teacher gave 5 homework assignments in preparation for a 20-point quiz.

- Let *x* = the number of assignments completed.
- Let *y* = the number of points earned on the quiz.

Ordered pairs in the form (*x, y*) are recorded for several students. Here is a list of several results: {(4, 15), (3, 12), (5, 18), (4, 10), (3, 16), (2, 10), (5, 14), (4, 16)}

This relationship is not a function since there are several inputs that correspond to multiple outputs. For example, the input value *x* = 4 corresponds to outputs 15, 10, and 16.

In a real-life sense, this means that there is no way to definitely predict a student's score by knowing the number of assignments completed, since there are several results for a given input.

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3. Functions Defined by Equations

3a. Identifying Which Equations Are Functions

In this course, we will mostly be sticking with equations with two variables. In general, we will use the variables *x* and *y*, where *x* is the input variable and *y* is the output variable. When the equation is determined to be a function, we say that *y* is a function of *x*.

Our first goal is to determine which equations are functions (in other words, which equations produce one y-value for each choice of x-value). Here are a few examples to help you contrast which relations are functions vs. which are *not* functions.

Equation	Function or Not a Function?
$y=2x+3$	This is a function since each input (*x*) corresponds to only one output (*y*). In other words, whatever is chosen for *x* will result in just one value of *y*. Some examples of ordered pairs are (0, 3), (1, 5), (2, 7), and (3, 9).
$x^2+y^2=25$	This is not a function because there are points that have the same x-coordinate but different y-coordinates. For example, (0, 5) and (0, -5) both satisfy the equation.

$y=x^2-4x-2$	This is a function since each input (x) corresponds to only one output (y). In other words, whatever is chosen for x will result in just one value of y. Some examples of ordered pairs are (1, -5), (2, -6), (3, -5), and (4, -2).

BIG IDEA

Generally speaking, any equation that can be written in "$y=$" form without using "$\pm$" is a function.

EXAMPLE The equation $2x-3y=6$ is a linear function since the slope-intercept form is $y=\frac{2}{3}x-2$.

3b. Common Functions From Algebra

In your mathematical career so far, you have been exposed to many types of equations that were actually functions. Here is a list of some of the most notable (and there are more to come in this unit as well).

Type of Function	Equation
Constant Function	$y=b$ (b is a constant)
Linear Function	$y=mx+b$
Quadratic Function	$y=ax^2+bx+c$
Cubic Function	$y=ax^3+bx^2+cx+d$
Polynomial Function (degree n)	$y=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$
Radical Function (nth Root)	$y=\sqrt[n]{x}$

SUMMARY

In this lesson, you learned **what a function is**, which is a relationship (an equation or formula) that is designed so that each input has at most one output. You learned that there are **functions defined by sets of ordered pairs**—a relationship which is a non-empty collection of ordered pairs (x, y), where x is the input and y is the output—and there are **functions defined by equations**, where x is the input variable and y is the output variable, and we say that y is a function of x. You also learned how to **identify which equations are functions**, or in other words, which equations produce one y-value for each choice of x-value. Finally, you explored a list of **common functions from algebra**, noting that many equations used in the past are actually functions—which is where we will begin our exploration using a calculus perspective.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Function

A correspondence between a set of inputs (x) and a set of outputs (y) such that each input corresponds to at most one output.

Evaluate Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn function notation and use it to evaluate functions. Specifically, this lesson will cover:

1. Evaluate Functions Using Function Notation
2. Why Use Function Notation?

1. Evaluate Functions Using Function Notation

When a relationship meets the requirements of being a function, it can be written using function notation. For instance, $y = x^2$ is a function. In function notation, this is written $f(x) = x^2$.

When written this way, *x* is considered the *input variable* and $f(x)$ is the *output variable*. In the example above, x^2 is the rule for computing the output.

Note, the letter *f* is the name of the function. This could have been called $g(x) = x^2$ or $k(x) = x^2$, for instance.

EXAMPLE Consider the function $f(x) = x^2$. Use it to find $f(2)$, $f(-4)$, and $f(a+3)$.

$f(x) = x^2$	Solution
$f(2) = (2)^2$ $= 4$	$f(2) = 4$
$f(-4) = (-4)^2$ $= 16$	$f(-4) = 16$
$f(a+3) = (a+3)^2$ $= (a+3)(a+3)$ $= a^2 + 3a + 3a + 9$ $= a^2 + 6a + 9$	$f(a+3) = a^2 + 6a + 9$

HINT

For this function, remember to square the whole input! When you square (*a* + 3), you multiply it by itself.

TRY IT

Use the function $f(x) = x^2 - 4x + 2$ to answer the following problems.

Find *f* (1).

+

$$\begin{aligned} f(1) &= (1)^2 - 4(1) + 2 \\ &= 1 - 4 + 2 \\ &= -1 \end{aligned}$$

Find *f*(-3). +

$$\begin{aligned} f(-3) &= (-3)^2 - 4(-3) + 2 \\ &= 9 + 12 + 2 \\ &= 23 \end{aligned}$$

Find *f*(a + 1). +

$$\begin{aligned} f(a+1) &= (a+1)^2 - 4(a+1) + 2 \\ &= a^2 + 2a + 1 - 4a - 4 + 2 \\ &= a^2 - 2a - 1 \end{aligned}$$

WATCH

Depending on which function is used, applying function notation can get very technical. Here is a video that helps guide you through a more complicated function.

Video Transcription

[MUSIC PLAYING] Hello, and thank you for visiting. What we're going to do is a little check in with function notation. We're given the function f of x equals 2x squared minus 5x plus 8, and we're going to find various values of f of x. So remember that with a function, the x is the input. So if we take a look here, everywhere we have an x, it's going to be replaced by a number.

So f of 3, for example, we have 2 times something squared minus 5 times something plus 8. So since it's 3 we are substituting, 3 goes here and here. And then we are left to perform the order of operations. So remember, order of operation sets.

We do the exponents first. We don't have anything inside parentheses, so we jump to the exponents. 2 times 3 squared would be 2 times 9 minus 5 times 3 plus 8, and then we just go from left to right. 2 times 9 is 18. 5 times 3 is 15. Plus 8. 18 -15 is 3. Plus 8. And 3 plus 8 is 11. So there we have our function evaluated at 3. So in summary, we would say f of 3 is equal to 11.

So we do the same thing for f of negative 2. So we have 2 times negative 2 squared minus 5 times negative 2 plus 8. Now here's where we have to be really careful, and this is why the parentheses are crucial. We have negative 2 quantity squared here. Many people think that that's negative 4. It might be tempting to write negative 4. Remember, negative 2 squared is negative 2 times negative 2, which is positive 4. And I'm just going to go ahead and simplify down the line here.

So we have minus 5 times negative 2, which ends up being a plus 10. And then plus 8, which is then 8 plus 10 plus 8. And if we add all three of those numbers together, 8 plus 10 is 18, plus 8 is 26. So there again just to pull that together, f of negative 2 is 26. When you substitute negative 2, you get 26 as an output. So now to substitute a more complicated expression. This is where, again, the grouping symbols and whatnot are very crucial.

So again, our function was 2 times something squared minus 5 times the something plus 8. And inside the parentheses, we're going to put the x plus 4. So now we have to perform x plus 4 the quantity squared. So let's just take a little side step for that. X plus 4 the quantity squared is x plus 4 times x plus 4.

So then if we FOIL that, first times the first is x squared. Outers give us 4x. The inners also give us 4x. And the last times the last gives us 16. And the simplest form of that is x squared plus 8x plus 16. So pulling this into our function, we have 2 times x squared plus 8x plus 16. I'll just leave the rest of it as it was.

And now we can go ahead and do the algebra. So here, the two is going to get distributed to everything. So we have 2x squared plus 16x plus 32. We can go ahead and distribute the minus 5, because that other term is behind us now. So you have minus 5x -20, and then plus 8. And then we'll combine like terms. There is no other squared term, so we have 2x squared.

It looks like we have a plus 16x and a minus 5x. So that's going to be plus 11x. And then for our numbers, we have plus 32 minus 20 plus 8. 32 plus 8 is 40. Minus 20 is 20. And there we have it. So f of x plus 4 is 2x squared plus 11x plus 20. And there we have some examples of function notation.

[MUSIC PLAYING]

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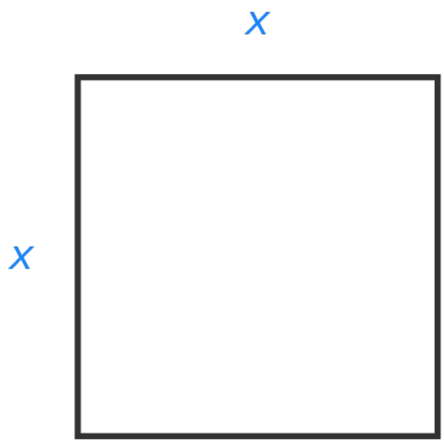
2. Why Use Function Notation?

There are situations in which we want to compute two (or more) different values given a single input. This is where naming functions can be convenient.

↱ EXAMPLE Suppose a square has sides with length x.



- The area of the square is x^2. Using function notation, we could write $A(x) = x^2$.
- The perimeter of the square is $4x$. Using function notation, we could write $P(x) = 4x$.
- The length of the diagonal is $x\sqrt{2}$. Using function notation, we could write $D(x) = x\sqrt{2}$.

Let's now say that the length of the square is 5 inches.

- The area is $A(5) = 5^2 = 25$ square inches.
- The perimeter of the square is $P(5) = 4(5) = 20$ inches.
- The length of the diagonal is $D(5) = (5)\sqrt{2} = 5\sqrt{2}$ inches.

The function names (*A, P,* and *D*) are important here since *A* is used for area, *P* is used for perimeter, and *D* is used for the length of the diagonal. Therefore, we can provide meaningful names for the output of a function rather than using *y* all the time.

HINT

The input variable doesn't necessarily have to be *x*. For example, let's say you are tracking the height of a projectile after *t* seconds. You might name the function $h(t)$.

SUMMARY

In this lesson, you learned that when a relationship is a function, function notation is used to emphasize the roles of the input and output values, and their relationship to each other. In this context, you can **evaluate functions using function notation**. You also learned **why we use function notation**, noting that naming functions can be convenient in situations in which we want to compute the outputs of two (or more) different functions given input values, such as determining the area, perimeter, and length of the diagonal of a square.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Set Up and Simplify a Difference Quotient

by Sophia

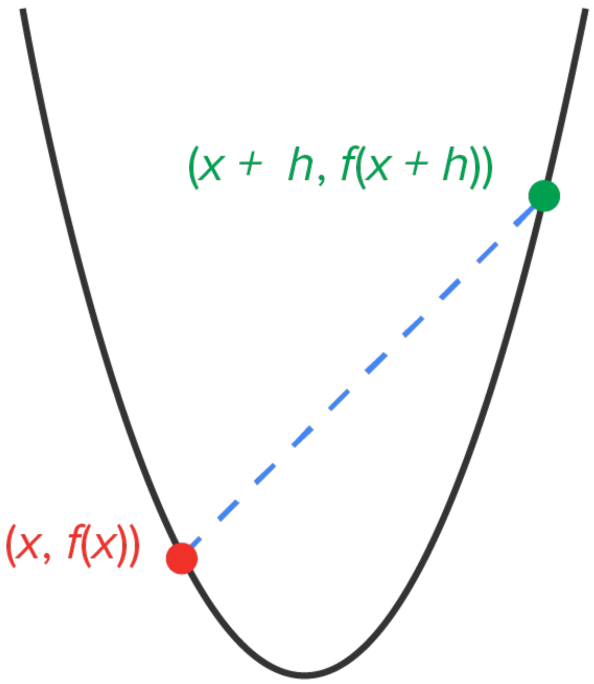
WHAT'S COVERED

In this lesson, you will learn what a difference quotient is, then set up and simplify a difference quotient for specified functions. Specifically, this lesson will cover:

1. What a Difference Quotient Represents
2. Evaluating a Difference Quotient
   a. Linear Functions
   b. Quadratic Functions
   c. Higher-Power Polynomial Functions (Degree 3 or Higher)
   d. Rational Functions
   e. Radical Functions

1. What a Difference Quotient Represents

Consider the graph of a function $y = f(x)$.



In this picture, $(x, f(x))$ is any point on the graph. $(x+h, f(x+h))$ is the resulting point when x is increased by h.

The slope of the line between these points is as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Therefore, the quantity $\frac{f(x+h) - f(x)}{h}$ gives the slope of the line between $(x, f(x))$ and $(x+h, f(x+h))$. This quantity is called the **difference quotient** for $f(x)$ and can be found with the following formula.

FORMULA

Difference Quotient

$$\frac{f(x+h) - f(x)}{h}$$

HINT

Since the slope between two points is also known as the average rate of change between the points, we also say that the difference quotient is the average rate of change between $(x, f(x))$ and $(x+h, f(x+h))$.

TERM TO KNOW

Difference Quotient

An expression that represents the average rate of change between two points on a curve between input values x and $x + h$.

2. Evaluating a Difference Quotient

Evaluating a difference quotient is an algebraic process. Let's take a look at how to evaluate the difference quotient for a few different types of functions.

2a. Linear Functions

Evaluate the difference quotient for $f(x) = 3x - 7$.

$\frac{f(x+h)-f(x)}{h}$	Difference Quotient Formula
$\frac{[3(x+h)-7]-[3x-7]}{h}$	Substitute the function.
$\frac{[3x+3h-7]-[3x-7]}{h}$	Simplify the first quantity in brackets.
$\frac{3x+3h-7-3x+7}{h}$	Distribute the subtraction sign.
$\frac{3h}{h}$	Simplify the numerator.
3	Divide out the common factor.

The difference quotient for the function $f(x) = 3x - 7$ is 3.

THINK ABOUT IT

This answer makes sense since $f(x) = 3x - 7$ is a linear function with slope 3. This means that the slope of this line through any two points is 3.

2b. Quadratic Functions

Evaluate the difference quotient for $f(x) = x^2 + 3x - 2$.

$\frac{f(x+h)-f(x)}{h}$	Difference Quotient Formula
$\frac{[(x+h)^2+3(x+h)-2]-[x^2+3x-2]}{h}$	Substitute the function.
$\frac{[x^2+2xh+h^2+3x+3h-2]-[x^2+3x-2]}{h}$	Simplify the first quantity in brackets.
$\frac{x^2+2xh+h^2+3x+3h-2-x^2-3x+2}{h}$	Distribute the subtraction sign.
$\frac{2xh+h^2+3h}{h}$	Simplify the numerator.
$\frac{h(2x+h+3)}{h}$	Factor the numerator.

$2x+h+3$ Divide out the common factor.

The difference quotient for the function $f(x)=x^2+3x-2$ is $2x+h+3$.

HINT

You may also remember from your background that $\frac{2xh+h^2+3h}{h}$ can be written as $\frac{2xh}{h}+\frac{h^2}{h}+\frac{3h}{h}$, which also yields $2x+h+3$ after performing the division.

2c. Higher-Power Polynomial Functions (Degree 3 or Higher)

The algebra can get quite complicated when evaluating difference quotients, especially when the highest power is more than 2.

WATCH

Here is a video that walks you through a difference quotient for a cubic function. The function used here is $f(x)=x^3-4x+2$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome back. And what we're going to look at is finding a difference quotient for the function f of x equals x to the third minus 4x plus 2. So remember that finding and simplifying a difference quotient means we have to deal with a few pieces here, the first piece being f of x plus h, and the second piece being f of x, and then simplifying from there.

So the first thing we'll do is set up a big fraction here, where the first piece will be f of x plus h. Now, remember, when we're dealing with a function f, the x plus h is what is being input to the function f. So what that means, in this case, is x plus h to the third minus 4 times x plus h plus 2. Now, that part is going to be highlighted in blue, just to show which part we're dealing with there. So there we have that. And then what we're going to do is subtract. Now f of x by itself is x to the third minus 4x plus 2. But since we're subtracting a more complicated quantity, we have to make sure that it's in parentheses.

Now, moving off to the right here, you'll notice that we have to compute x plus h to the third, which is a pretty complicated thing to compute. So you notice that off to the side here, we have x plus h to the third equals. So this just walks through how to find and simplify the cube of x plus h. And you notice that this expression right here, the x to the third plus 3hx squared plus 3x squared h plus h to the third, is the expanded version of that. So I'm just going to go ahead and replace that.

So you have x to the third plus 3hx squared plus 3h squared x plus h to the third. That all replaces x plus h to the third. Now we have to distribute the negative 4. So we have minus 4x minus 4h and then plus 2. And now we have minus a quantity, which is going to switch all the signs in the quantity if we want to write it without parentheses. Just enough room there to the left of the line. And that is all over h. So now just like with past difference quotients, we basically know that all the terms that don't have an h should cross out. So let's take a look at this here.

So we have an x to the third and a minus x to the third, so those are going to cancel. We have a minus 4x and a plus 4x, so those are going to cancel. And we have a plus 2 and a minus 2, so those are going to cancel. So we are left with 3hx squared plus 3h squared x plus h to the third minus 4h, all over h.

And at this point, you could either remove a common factor of h or we could just divide each term by h. When you're dividing a series of terms by the same thing, we can divide each term by that thing. So we're just going to take each term and put it over h, and now we can simplify. So this is 3x squared plus-- now the h squared over h will simplify to h. That's 3hx. h to the third divided by h is h squared, and 4h divided by h is 4. And that is as simplified as our difference quotient can get. So there is our final answer.

[MUSIC PLAYING]

2d. Rational Functions

When $f(x)$ is either a rational function or a radical function, more complex algebraic techniques are needed. These are illustrated in the next two videos.

WATCH

Here is a video that shows the required simplification techniques when evaluating a difference quotient for a rational function. The function used here is $f(x)=\frac{2}{x+4}$.

Video Transcription

Hi, there. Good to see you again. We're going to continue our journey through difference quotients by looking at a rational function f of x equals 2 over x plus 4. So without further ado, we will just get this started.

So remember that we have the two pieces here, the f of x plus h, and the f of x. So f of x plus h just means I'm taking the x and replacing it with an h. So we're going to have 2 over x plus h plus 4. And this is what's replacing f of x plus h. And f of x is just what's written, 2 over x plus 4. So we have all of this all over h.

So you're probably sitting there thinking, wow, that looks kind of simplified, but it really isn't. If you remember through your algebra experience, this is an example of what we call a complex fraction, which means a fraction within a fraction. So the strategy is to multiply by something that's going to clear the fractions.

So looking at this very complicated numerator, we need to multiplied by s plus h plus 4 because that's going to cancel the first denominator, but it's not going to cancel the second denominator, so we also need to multiply by x plus 4. But now I need to multiply something else by x plus h plus 4 times x plus 4 because I don't want to change the value of the expression. So since this is a fraction, I'm going to multiply the denominator as well by x plus h plus 4 times x plus 4.

So it's basically just a big, fancy, complicated version of 1. So when we do the multiplication here, so let's just see what happens. This whole expression gets distributed to the first fraction and the second fraction. And when we multiply that to the first fraction, the x plus h plus 4 goes away, and we're left with 2 times x plus 4 minus-- when we multiply that to the second fraction, the x plus 4s go away, and we're left with 2 times x plus h plus 4.

And that's going to be all over-- now it might be tempting to multiply out the denominator, but we're not going to, because remember that the goal is essentially to get that single h to disappear, to cancel out. So I'm going to leave it in factored form so that I can easily see if that happens. So now at this point, the numerator does need some help because it's not completely expanded and it's not completely factored.

So I'm going to distribute the 2 into the x plus 4, and the minus 2 to the x plus h plus 4. So you have minus 2x, minus 2h, minus 8, all over h times x plus h plus 4, x plus 4. And you notice that several like terms cross out. We have 2x minus 2x that fizzles out, and we have plus 8 minus 8 that fizzles out. So this is equal to negative 2h all over h times x plus h plus 4 x plus 4.

And as luck would have it, now that common factor of h goes away. So we have, as our final answer, negative 2 all over x plus h plus 4, times x plus 4. And that is the simplest version of our difference quotient.

2e. Radical Functions

WATCH

Here is a video that shows the required simplification techniques when evaluating a difference quotient for a radical function. The function used here is $f(x)=\sqrt{x+3}$.

Video Transcription

[MUSIC PLAYING] Hello, and welcome back. And what we have here is another difference quotient, this time for a radical function. So as usual, all we're going to do is substitute what we need to here. So in our difference quotient, we have f of x plus h minus f of x all over h. So we just need expressions for f of x plus h and f of x.

Well, we were given f of x equals the square root of x plus 3, so that part just goes there. And for f of x plus h, remember that that means that we are replacing x with x plus h, which gives us in this case x plus h plus 3. And that's all over h. So at this point, what do we do to simplify? Because it looks like it actually is pretty simplified. But remember, with a difference quotient, the goal is to get the h out of the denominator. We want to rewrite this expression so that the h's cross out.

So this one involves a technique that you may have learned in your algebra experience. We're still going to multiply it by a fancy one, but when we deal with radicals, we're going to multiply by what's called the conjugate. And the conjugate of this radical expression here that I'm going to put in brackets is basically the same terms, but with a plus sign in between.

So we're going to have square root x plus h plus 3 plus square root x plus 3. And I'm going to multiply that to the numerator, because that's where I need it. But whatever I multiply to the numerator I have to multiply to the denominator as well. So when we make that multiplication, let's see what happens.

The first thing to note, is since I want the h to cross out, I'm going to leave the denominator in factored form. I'm not going to multiply it out. So I'm going to have h times the square root of x plus h plus 3 plus square root x plus 3. Now, in the numerator, we basically have a binomial times a binomial. So if we take the first times the first, remember that a square root-- I'm just going to write over to the side here-- a square root times itself just gives you the radicand, or the number inside the radical.

So this is just going to be x plus h plus 3. And if you notice, if we multiply the outers and the inners, those are going to be the same expression, but one's going to be positive and one's going to be negative. So that means, when we combine them, they're going to cancel out. So I'm not even going to bother writing that. And then, for the last times the last, we have the negative square root of x plus 3 times the positive square root of x plus 3. So that's going to be minus x plus 3, because again, when you multiply the radicals together, you just get the inside.

So now it looks like we're getting somewhere. So we're going to simplify the numerator. And up above it, I'm just going to write x plus h plus 3 minus x minus 3. And what you notice is everything cancels out except for the h. So we have equals h divided by h parentheses square root x plus h plus 3 plus square root x plus 3. And the h's cancel out.

And it looks like our final answer is 1 over square root x plus h plus 3 plus square root x plus 3. Now, our mathematical training tells us that we shouldn't leave radicals in the denominator, but this is going to be one case where it's allowed, because it's more useful. And when we get to the next unit, we're going to see exactly why.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned **what a difference quotient represents**, understanding that it is used to find an expression for the average rate of change between two points. You learned that **evaluating a difference quotient** is an algebraic process, and depending on the function, basic simplification or more advanced methods such as simplifying a complex fraction or rationalizing the numerator may be required. You explored how to evaluate the difference quotient for several different types of functions, including **linear functions**, **quadratic functions**, **higher-power polynomial functions (degree 3 or higher)**, **rational functions**, and **radical functions**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Difference Quotient

An expression that represents the average rate of change between two points on a curve between input values x and $x + h$.

FORMULAS TO KNOW

Difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

Functions Defined by Graphs and Tables of Values

by Sophia

WHAT'S COVERED

In this lesson, you will see how functions can be represented with graphs and tables (not only by equations). Specifically, this lesson will cover:

1. Functions Represented by Tables
2. Functions Represented by Graphs
   a. From Tables of Values
   b. From Equations
3. Using a Graph to Determine if It Represents a Function

1. Functions Represented by Tables

Consider the table shown below, which shows the revenue (in thousands of dollars) earned by a particular company after each year for their first 8 years of business.

Year	1	2	3	4	5	6	7	8
Revenue	61	70.5	82	91	112.5	107.5	118.5	134.5

Thinking of the input-output relationship, it makes the most sense to label the year as input and the revenue as output. Since each input (1, 2, ..., 8) corresponds to one output, this table of values represents a function where the input is "Year" and the output is "Revenue." We say that this defines revenue as a function of the year.

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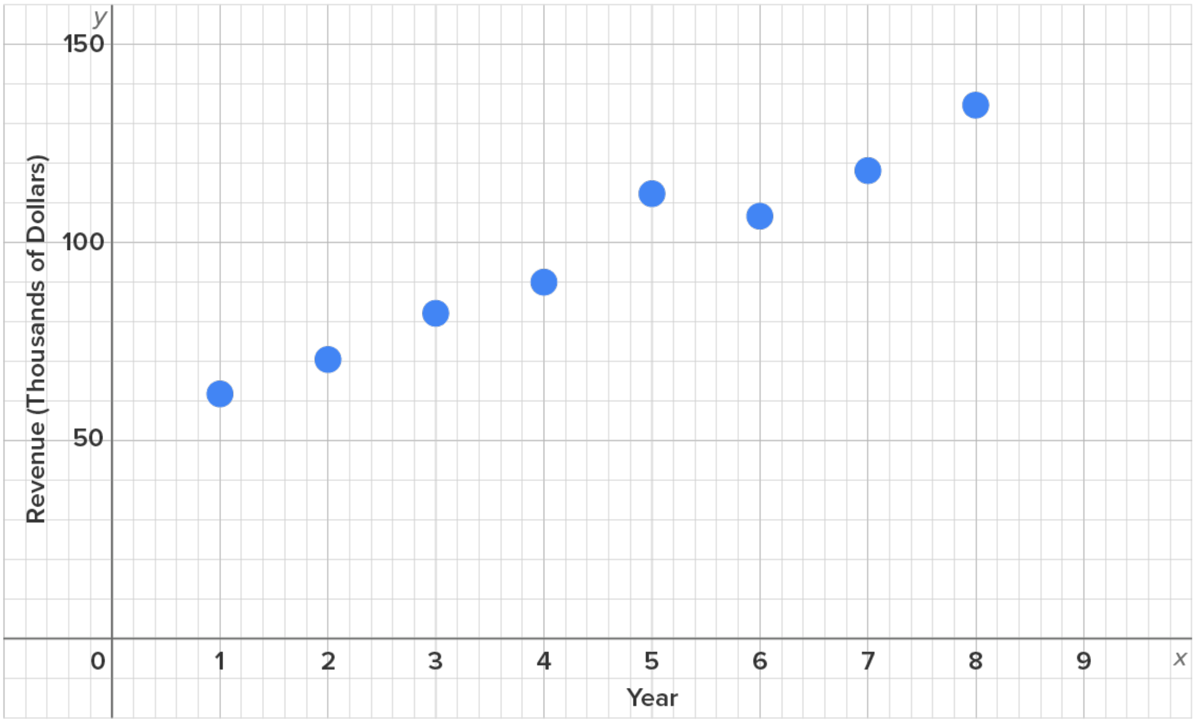
2. Functions Represented by Graphs

2a. From Tables of Values

Let's take the table of values from the first section and graph the points:



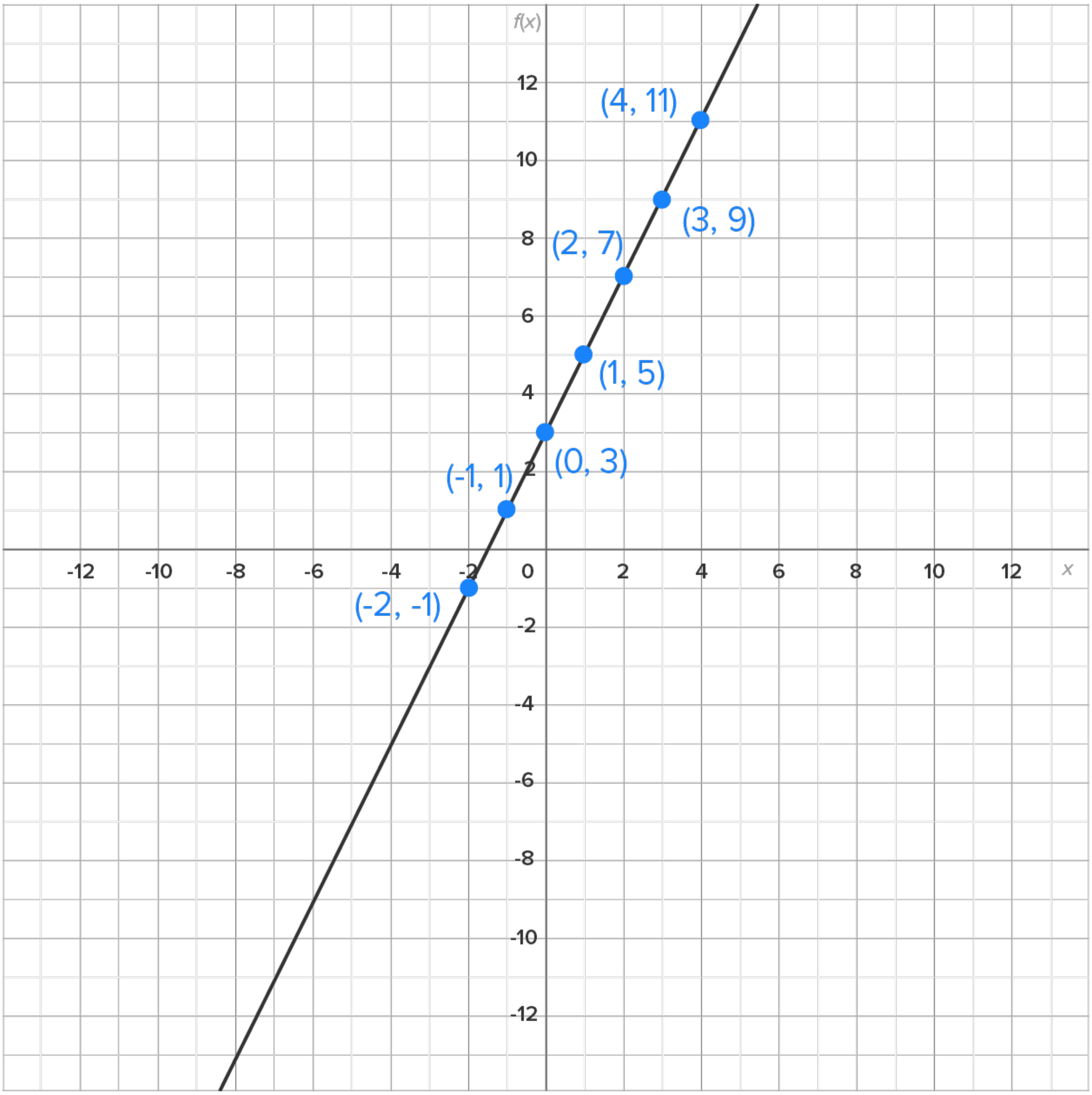
One could say that the graph looks *almost* linear or *roughly* linear, but not perfectly linear.

2b. From Equations

Consider the function $f(x) = 2x + 3$. The following table of values shows some input-output pairs for this function:

x	-2	-1	0	1	2	3	4
$f(x)$	-1	1	3	5	7	9	11

Now we plot the ordered pairs to form the graph (remember, we connect the dots since there are many other points on the graph; we just chose easy ones).



HINT

Note that this looks exactly like the graph of $y = 2x + 3$. In a previous lesson, you learned that $f(x)$ is simply a replacement for *y*.

Thus, graphing a function in the form "$f(x) = \ldots$" is identical to graphing an equation of the form "$y = \ldots$".

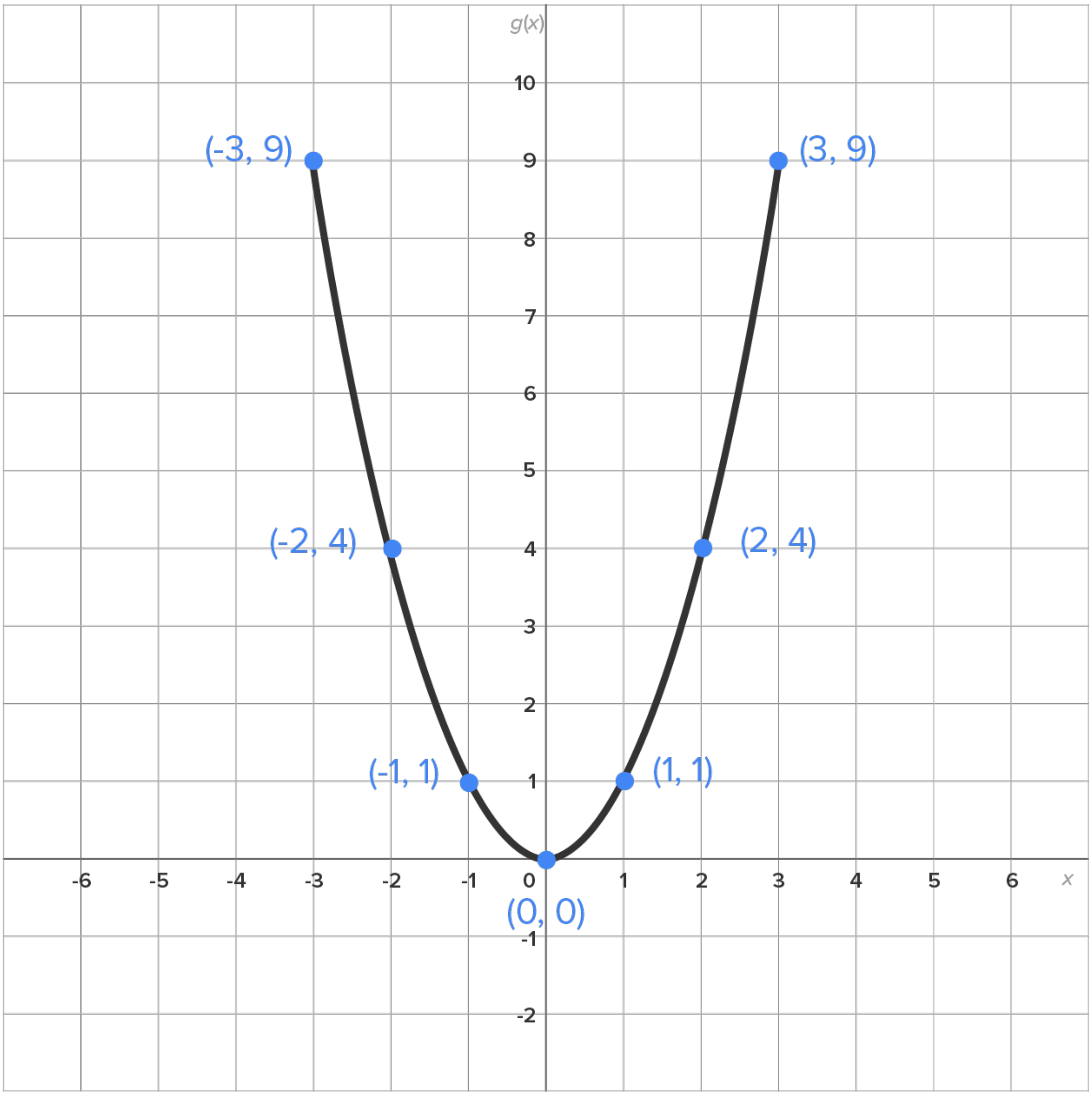
TRY IT

Consider the function $g(x) = x^2$.

Define input-output pairs for this function. +

x	-3	-2	-1	0	1	2	3
g(x)	9	4	1	0	1	4	9

Plot the input-output pairs to create a graph. +



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3. Using a Graph to Determine if It Represents a Function

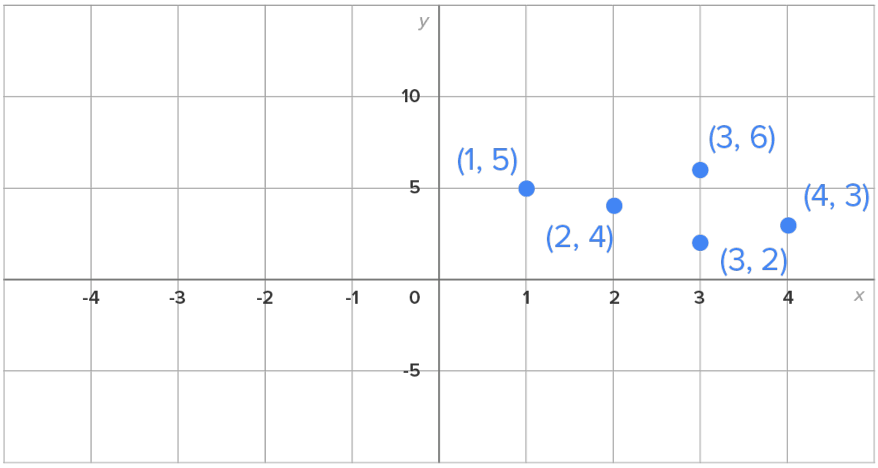
Something to think about: what would a graph look like if it *wasn't* a function?

Remember that if there is an input that corresponds to two or more outputs, then the relationship is not a function.

The equations $f(x)=2x+3$ and $g(x)=x^2$ are both functions, as shown in the last two examples.

Now consider this table of values, and the graph of the ordered pairs to its right:

x	y
1	5
2	4
3	6
3	2
4	3



From the table of values, we can see that this is not a function, since $x = 3$ corresponds to two different outputs, $y = 6$ and $y = 2$.

On the graph, let's pay attention to the points (3, 2) and (3, 6). Notice that these points could be connected by a vertical line. This only happens when there are two points with the same x-coordinate. Thus, we have a simple test to determine whether or not a graph defines y as a function of x.

CONCEPT TO KNOW

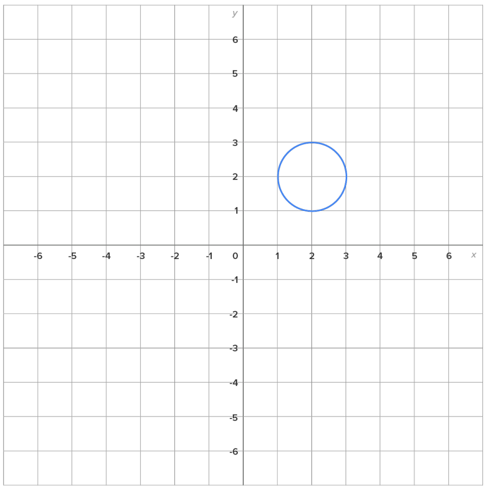
The Vertical Line Test

Given a graph, if a vertical line can be drawn and intersects more than once with the graph, then the graph does not define y as a function of x.

TRY IT

Consider the following graphs and determine which of these graphs defines y as a function of x.

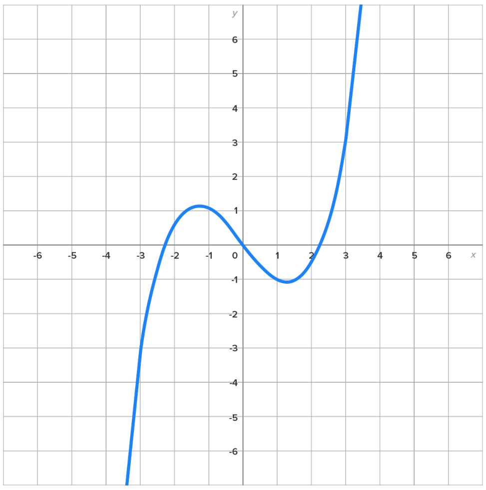
Graph 1



Graph 1: Function or Not a Function? +

This is not a function. There exists a vertical line that passes through two points when drawn through the graph.

Graph 2

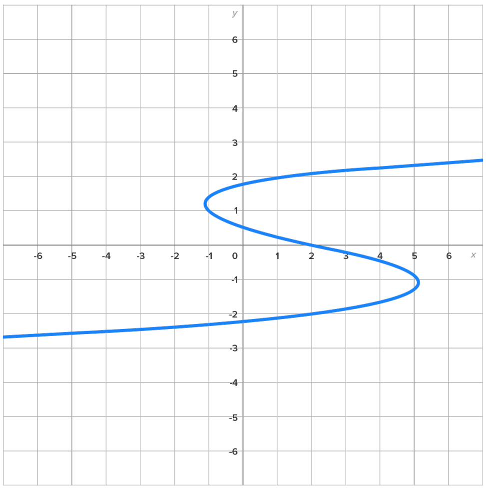


Graph 2: Function or Not a Function?

+

This is a function. Any vertical line will pass through one point on this graph.

Graph 3



Graph 3: Function or Not a Function?

+

This is not a function. There are some places where a vertical line will pass through three points on the graph.

SUMMARY

In this lesson, you learned about the various ways that functions can be represented, including **functions represented by tables** and **functions represented by graphs**, including **from tables of values** and **from equations**. Understanding that if there is an input that corresponds to two or more outputs, then the relationship is not a function, you learned how to **use a graph to determine if it represents a function**, by utilizing the vertical line test: given a graph, if a vertical line can be drawn and intersects more than once with the graph, then the graph does not define *y* as a function of *x*.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Reading Graphs (Carefully)

by Sophia

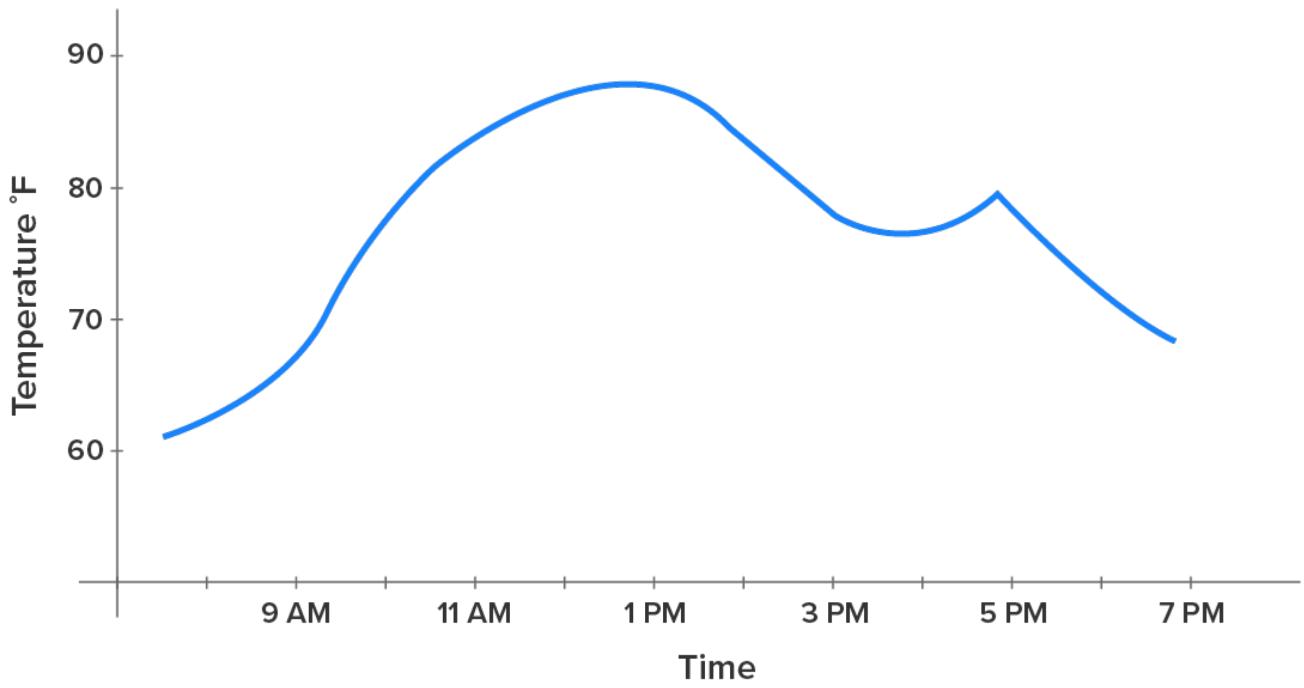
WHAT'S COVERED

In this lesson, you will investigate the different kinds of information that can be extracted from graphs. Specifically, this lesson will cover:

1. Information Related to a Graph
   a. Using Specific Points on a Graph
   b. Using the Shape of a Graph
2. Applications of Graphs to Real-Life Situations

1. Information Related to a Graph

Consider the graph below, which shows the temperature throughout the day.



1a. Using Specific Points on a Graph

Using the graph above, we can extract the following information by examining points on the graph:

- At 9 AM, the temperature is roughly 65°F.
- The high temperature of the day was recorded at 12:30 PM and was about 87°F.

1b. Using the Shape of a Graph

Using the same graph, we can extract the following information by observing the shape of the graph.

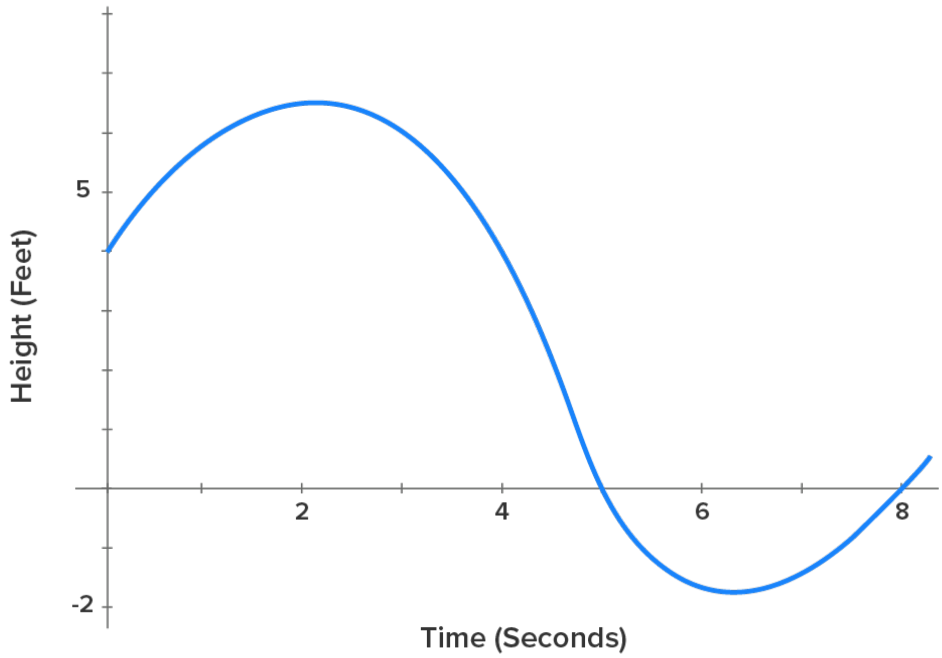
- The temperature appears to have risen most sharply at 9 AM.
- The temperature dropped after 1 PM, but then rose slightly around 3:30 PM, then started falling again at 5 PM.
- At 5 PM, the graph started falling sharply and suddenly. It's possible that a cold front came through, it started raining, or a storm came through.

---

2. Applications of Graphs to Real-Life Situations

TRY IT

Consider this graph, which shows the height of a diver after jumping off the diving board.



How high was the diving board? +

4 feet (the starting point)

How long after jumping did the diver strike the water? +

5 seconds (the first x-intercept)

How far underwater did the diver go? +

About 1.5 feet (the lowest y-value)

When did the diver resurface? +

8 seconds

After jumping off the diving board, at what times was the diver descending? +

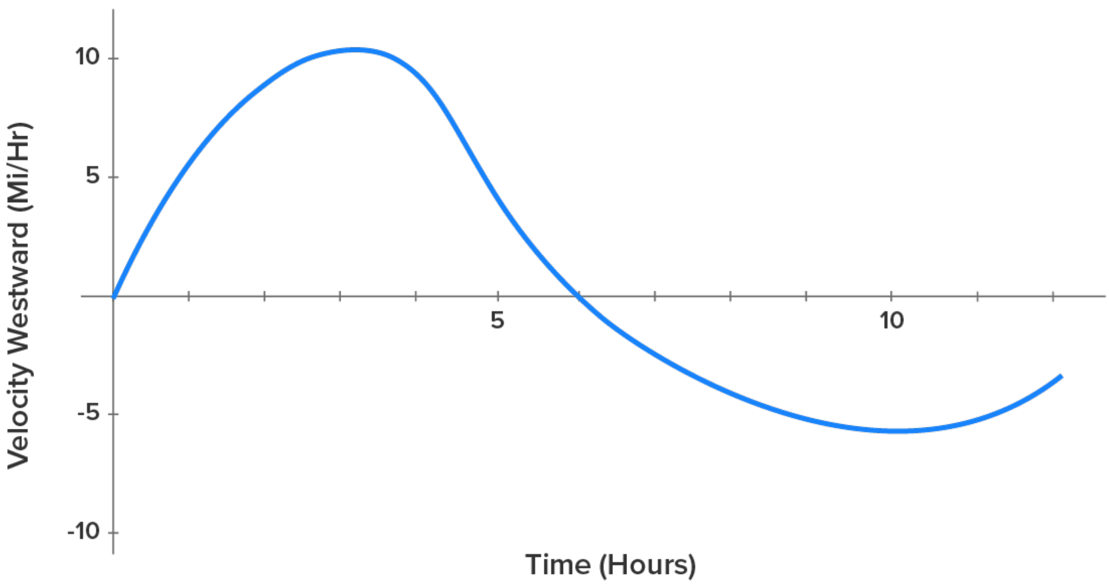
Between 2 and 6.5 seconds

How long was the diver underwater? +

The diver entered the water after 5 seconds and surfaced again after 8 seconds, so this means that the diver was underwater for 3 seconds.

TRY IT

The graph below shows the velocity (heading west) of a boat heading away from St. Thomas (to the west).



When is the boat travelling fastest? +

Around 3 hours, when the graph is at its peak. It looks like the velocity is about 10 mi/hr.

What does a negative velocity mean? +

In this case, since a positive velocity means the boat is heading west, a negative velocity means the boat is heading east.

When is the boat furthest from St. Thomas? +

After 6 hours, the velocity transitions from positive to negative, which means that the boat was at its furthest point west before heading east again.

SUMMARY

In this lesson, you learned that graphs are very useful since they are a visual representation of a situation. They can be created quite easily using technology, so it is important to consider many aspects of the graph. You learned how to extract **information related to a graph** by **using specific points on a graph** and **using the shape of a graph**. You also explored several **applications of graphs to real-life situations** to apply your knowledge of the different kinds of information that can be extracted from graphs.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Evaluate Piecewise Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn how a piecewise function is defined and how it is evaluated. Specifically, this lesson will cover:

1. Uses of Piecewise Functions in the Real World
2. Defining Piecewise Functions
3. Evaluating Piecewise Functions

1. Uses of Piecewise Functions in the Real World

A job pays $24 per hour as long as an employee works at most 40 hours in a week. If an employee works more than 40 hours, they still get $24 for each of the first 40 hours, but they also get $36 for every extra hour beyond 40.

How would we calculate an employee's earnings?

There are two rules, depending on the number of hours worked:

40 hours or less	Earnings = 24 $\cdot$ (Hours)
More than 40 hours	Earnings = 24 $\cdot$ 40 + 36 $\cdot$ (Extra Hours)

If the goal is to calculate several employees' earnings, a piecewise function is needed since there are two rules to calculate the same output (earnings).

2. Defining Piecewise Functions

Let's take this situation and try to represent it mathematically. Let x = the number of hours an employee works in a week.

Each entry in the table above can be translated into a mathematical statement involving the variable x.

Mathematical Statement	Statement in Words
$x \leq 40$	"40 hours or less"
$x > 40$	"More than 40 hours"
Earnings = 24x	Earnings = 24 $\cdot$

	(Hours)
Earnings = 24(40) + 36(x – 40) Note: "x – 40" is the number of extra hours. If someone works more than 40 hours, then you subtract 40 from the hours they worked to get the number of extra hours.	Earnings = 24*40 + 36 $\cdot$ (Extra Hours)

In this situation, notice also that the weekly earnings depend on the number of hours worked. That is, the earnings is a function of the number of hours worked. Using function notation, let $E(x)$ represent the earnings after x hours.

Since there are two rules to find $E(x)$, we can express $E(x)$ as a **piecewise function**. Here is how it would be written:

$$E(x) = \begin{cases} 24x & \text{if } x \le 40 \\ 24(40) + 36(x-40) & \text{if } x > 40 \end{cases}$$

This function isn't quite accurate for the following reasons:

- x cannot be negative (the number of hours worked cannot be negative).
- x cannot be more than 168 (# hours in a week).

The expression 24(40) + 36(x – 40) is not in its final form because it can be simplified.
After addressing these things, the function could be written as:

$$E(x) = \begin{cases} 24x & \text{if } 0 \le x \le 40 \\ 36x - 480 & \text{if } 40 < x \le 168 \end{cases}$$

TERM TO KNOW

Piecewise Function

Assigns an input to an output, but the rule used to determine the output depends on the value of the input.

3. Evaluating Piecewise Functions

The purpose of the function we built in the previous section is to calculate earnings for employees. In order to do so, remember that the input determines which rule is used:

If $0 \le x \le 40$, then use 24x to compute $E(x)$.
If $40 < x \le 168$, then use 36x - 480 to compute $E(x)$.

EXAMPLE Let's use the function to compute earnings for several employees:

Employee, Hours	Hours written in terms of x	Which Rule Should We Use?	Calculate $E(x)$
Holly, 42 hours	$x = 42$	Since 42 satisfies $40 < x \le 168$, use $36x - 480$.	$E(42) = 36(42) - 480 = 1032$
George, 36	$x = 36$	Since 36 satisfies $0 \le x \le 40$, use $24x$.	$E(36) = 24(36) = 864$

hours			
Israel, 40 hours	$x=40$	Since 40 satisfies $0 \le x \le 40$, use $24x$.	$E(40)=24(40)=960$
Savannah, 50 hours	$x=50$	Since 50 satisfies $40 < x \le 168$, use $36x-480$.	$E(50)=36(50)-480=1320$

Conclusion: Holly earned $1032, George earned $864, Israel earned $960, and Savannah earned $1320.

TRY IT

Let $f(x)=\begin{cases} 4x+5 & \text{if } x<2 \\ x^2+x+3 & \text{if } x \ge 2 \end{cases}$

Evaluate $f(-3)$, $f(5)$, and $f(2)$.

+

$f(-3)=4(-3)+5=-7$	Use the first rule since $-3<2$.
$f(5)=5^2+5+3=33$	Use the second rule since $5 \ge 2$.
$f(2)=2^2+2+3=9$	Use the second rule since $2 \ge 2$.

SUMMARY

In this lesson, you learned that whenever a situation involves more than one rule for computing an output, it can be represented mathematically by a piecewise function. A piecewise function assigns an input to an output, where the rule used to determine the output depends on the value of the input. You began the lesson by exploring **uses of piecewise functions in the real world**, then learned how to **define piecewise functions** by taking entries in a table and translating them into a mathematical statement involving the variable x. Lastly, you learned how to **evaluate piecewise functions** by calculating earnings for employees, applying the appropriate rule, determined by the input, to calculate the output (earnings).

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Piecewise Function

Assigns an input to an output, but the rule used to determine the output depends on the value of the input.

Graph Piecewise Functions

by Sophia

WHAT'S COVERED

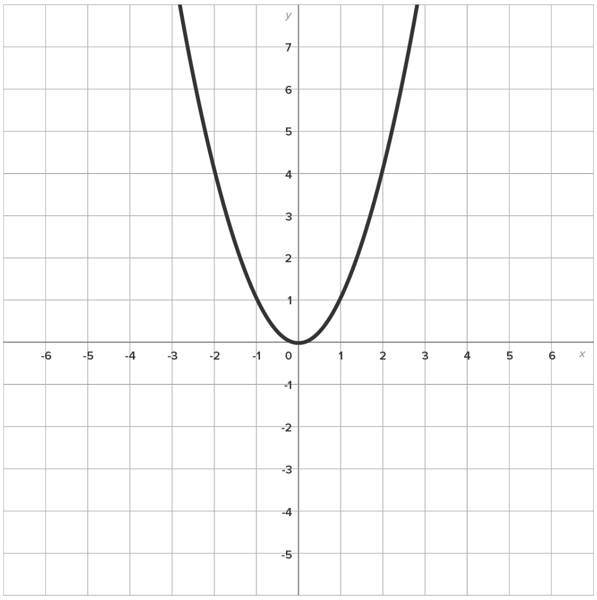
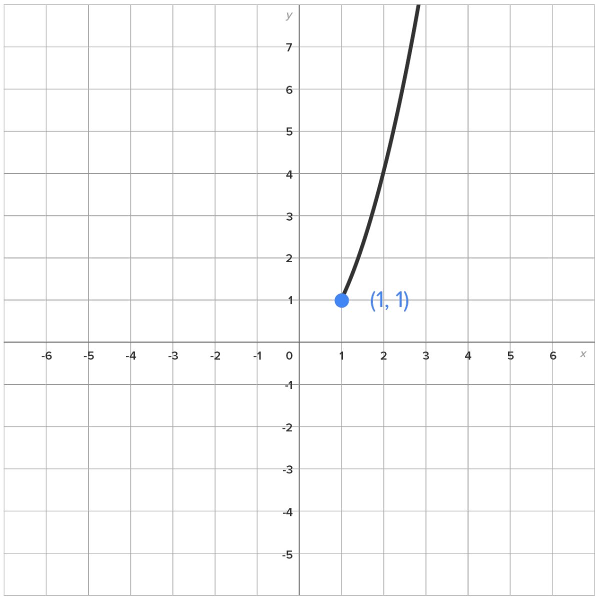
In this lesson, you will graph piecewise functions. Specifically, this lesson will cover:

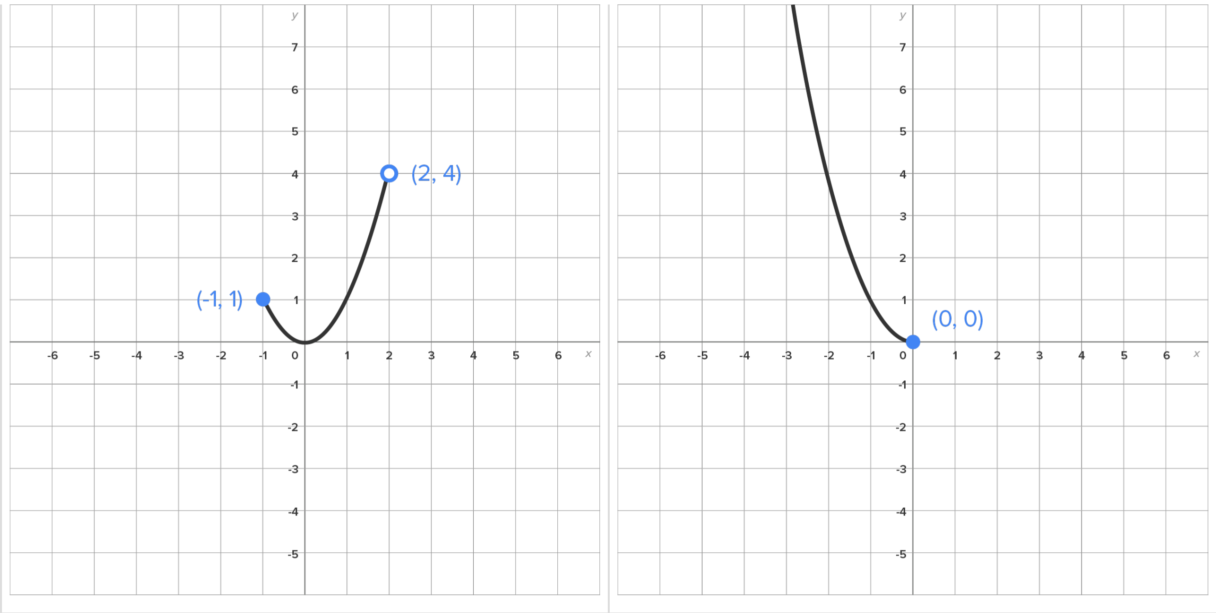
1. Graphing a Function on a Restricted Domain
2. Graphing Piecewise Functions

1. Graphing a Function on a Restricted Domain

When we graph a function, we are considering the entire function. What if we only wanted part of the graph?

EXAMPLE For example, consider the function $f(x) = x^2$, and several "pieces" of the graph, as shown below:

Graph 1: $f(x) = x^2$ (entire graph)	Graph 2: $f(x) = x^2$, $x \geq 1$
	
Graph 3: $f(x) = x^2$, $-1 \leq x < 2$	**Graph 4: $f(x) = x^2$, $x \leq 0$**



To sketch a portion of the graph, a **restricted domain** is used. Recall that the domain of a function is the set of all possible inputs for a function.

For example, in Graph 3 above, the "$-1 \leq x < 2$" is the domain restriction since it is not the entire domain of $f(x) = x^2$ (which is all real numbers).

HINT

When an endpoint is included, we represent it by using a closed circle. See Graphs 2, 3, and 4.

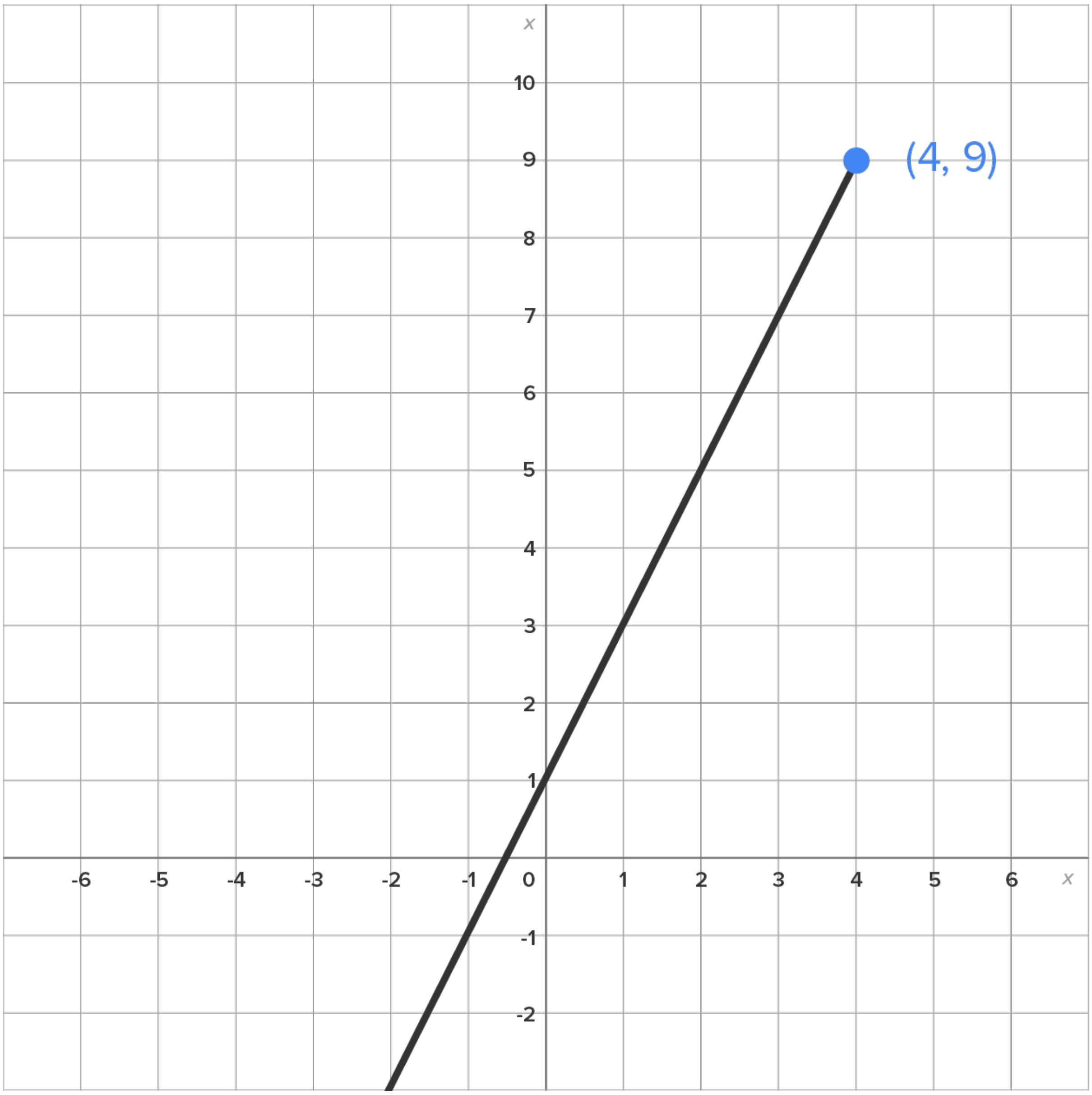
When an endpoint is not included, we represent it by using an open circle. See Graph 3.

TRY IT

Consider the following function: $f(x) = 2x + 1,\ x \leq 4$.

Graph this function. +

Remembering that $y = 2x + 1$ is a line with slope 2 and y-intercept 1, we graph the line but only for values of x up to and including 4.



TERM TO KNOW

Restricted Domain

Part of, but not the entire, domain of a function.

---

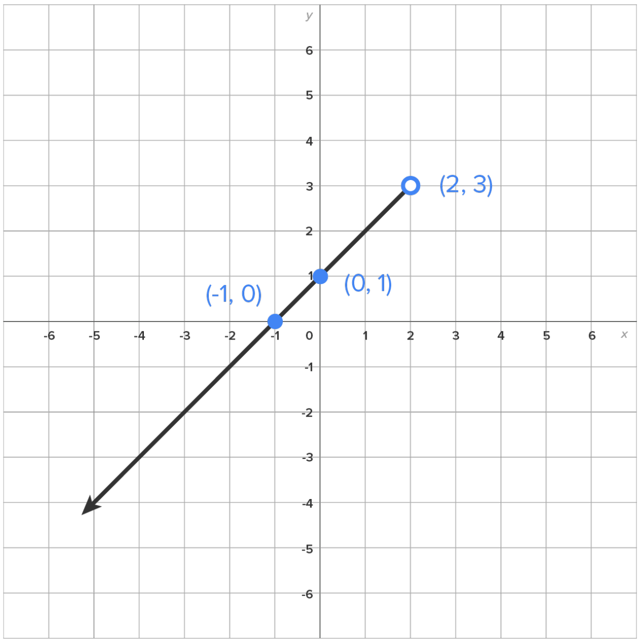
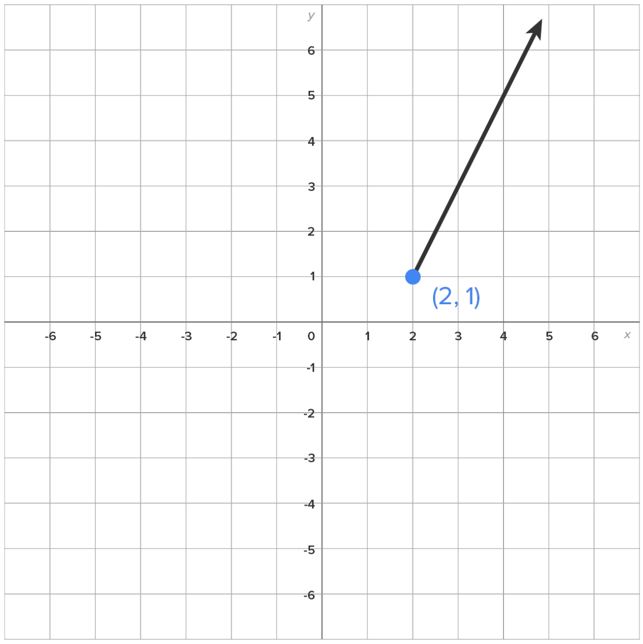
2. Graphing Piecewise Functions

A piecewise function is made up of other functions that are on restricted domains. For example, consider the function:

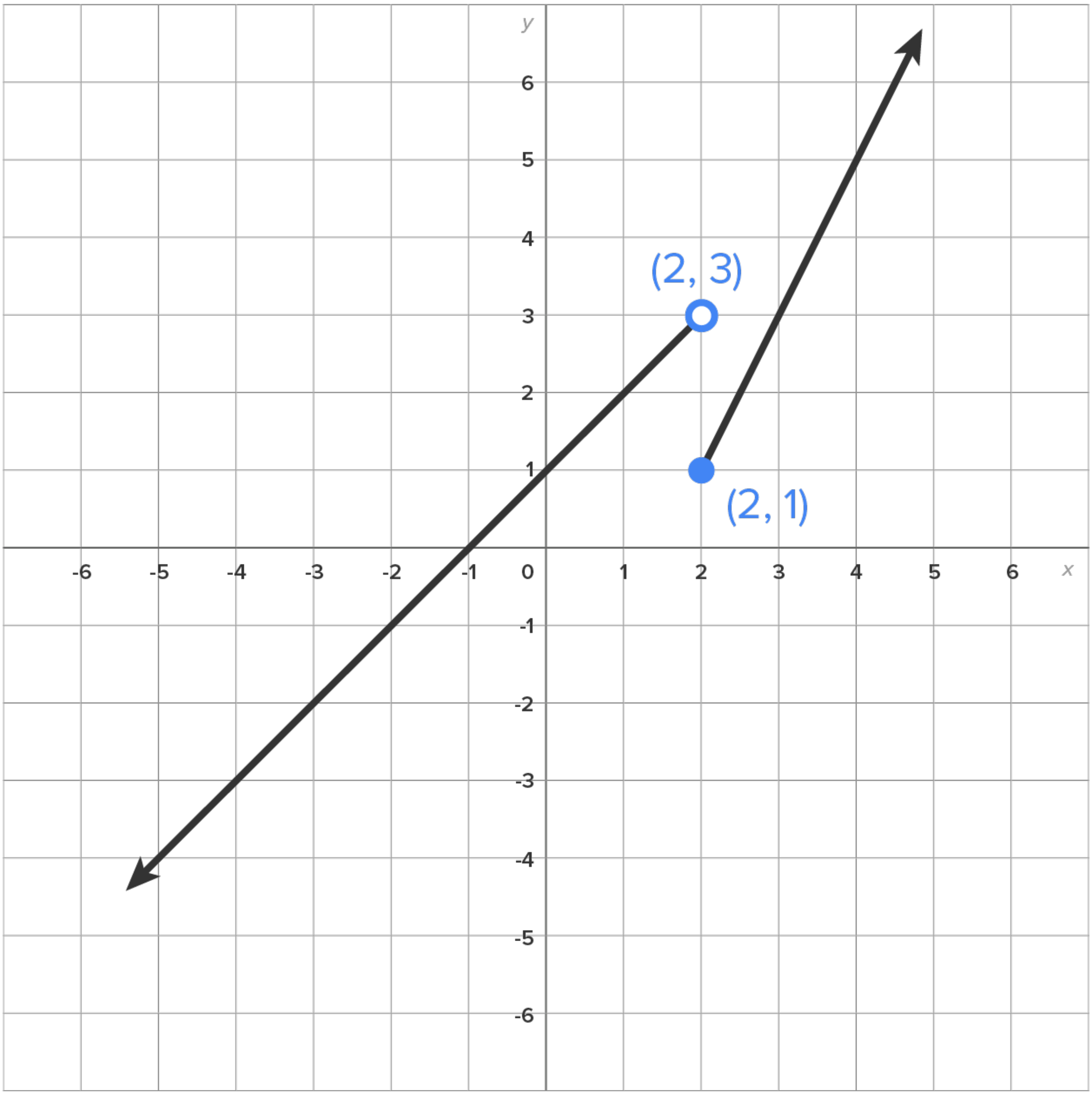
$$f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ 2x-3 & \text{if } x \geq 2 \end{cases}$$

The function tells us to use "$x + 1$", but only if the input is less than 2; and to use "$2x - 3$" if the input is at least 2.

This means that the graph of the function will be "part of" the graph of $y = x + 1$ along with "part of" the graph of $2x - 3$. Here is how we put this together:

$y = x + 1,\ x < 2$	$y = 2x - 3,\ x \geq 2$
	

The graph of $f(x)$ is these pieces put together on one graph as follows:



WATCH

The following video walks you through the process of graphing a piecewise function.

Video Transcription

[MUSIC PLAYING] Hello. I hope you're learning experiences going well so far. What we're going to look at here is how to graph a piecewise function, and we are going to focus on the function f of x equals 3x minus 4 if x is less than 2 and x squared if x is greater than or equal to 2.

So you'll notice that there's two graphs written right below the function, and these are the complete graphs of each piece. This is the graph of y equals 3x minus 4, and this is the graph of y equals x squared. So the thing to understand about a piecewise function is that we just want a piece of each of those graphs, and those two pieces will be put together to form the graph of the piecewise function.

So we're looking at y equals 3x minus 4, but only if x is less than 2. So I'm going to find the point where x is equal to 2, and that's right here. And you notice I'm going to put an open circle there, because we're not including 2. So we want the piece where x is less than 2, which means we are going to only take this piece of the graph.

Now, when we go to put our graphs together, it's going to be important to have some key points labeled. This point right here is 2 comma 2. And we've also managed to include the y-intercept, which is 0, negative 4. So that's going to help us when we go to graph the piecewise function.

Now, for y equals x squared, we're only taking the piece where x is greater than or equal to 2. So if we look where x is equal to 2, this point right here is 2 comma 4, because 2 squared is 4. And we're only taking the piece where x is greater than or equal to 2. So we're only taking that piece that's drawn in black there.

So to put these pieces together, we're just going to make a rough sketch here. I'm just going to zoom out just a little. So we have the point 2, 2 as a key point, because that's where the first graph ends. And it's also passing through 0, negative 4. So our first piece looks like that. And then our second piece starts at 2, 4, so just two units above the open circle. And it curves upward from there, just like y equals x squared. And that is the graph of our piecewise function.

[MUSIC PLAYING]

SUMMARY

In this lesson, you recalled that when you graph a function, you consider the entire function. However, if you only want part of the graph, you learned how to **graph a function on a restricted domain**, which is part of, but not the entire, domain of a function. You learned how to apply this knowledge to **graphing piecewise functions**—which are made up of other functions that are on restricted domains—which requires you to graph each piece on their respective restricted domains of the function.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Restricted Domain
Part of, but not the entire, domain of a function.

Composition of Functions

by Sophia

WHAT'S COVERED

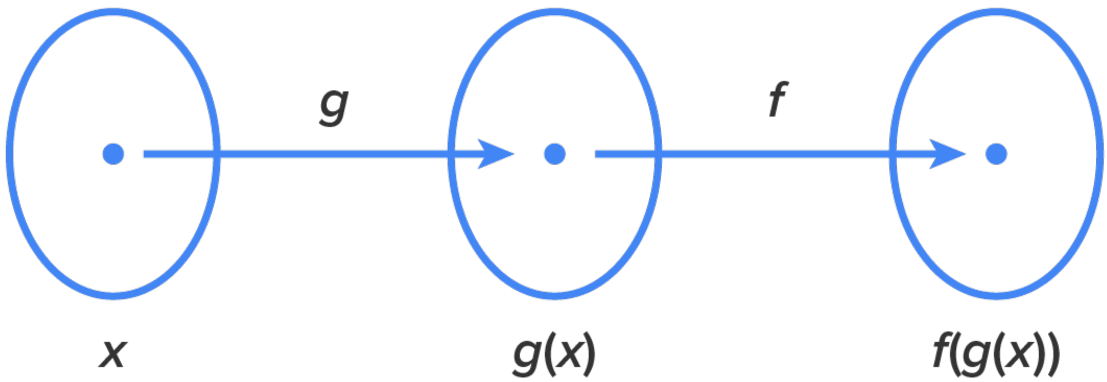
In this lesson, you will find compositions of functions. Specifically, this lesson will cover:

1. The Definition of a Composition of Functions
2. Computing Compositions of Functions
   a. Computing Compositions for Specific Values
   b. Computing the Expression for a Composition of Functions
3. Decomposing Composite Functions
4. Applications of Compositions

1. The Definition of a Composition of Functions

Sometimes it is useful to use one function to get a result, then use that result in another function. This idea is called **composition of functions**.

Here is a picture to show how this works:



The original input is *x*, which is then substituted into $g(x)$. Then, $g(x)$ is input into function *f*, giving the result $f(g(x))$.

The notation used to represent a composition of functions is $(f \circ g)(x)$, which means $f(g(x))$. The expression $(f \circ g)$ means "*f* composed with *g*."

Using this notation, f is considered the outer function and g is the inner function. Notice that g is used first, then f is applied to the result. Therefore, the outer function is what is applied last. We will see how this works more closely in the next section when we evaluate compositions of functions.

TERM TO KNOW

Composition of Functions
Written $(f \circ g)(x)$, it is a function that is obtained by substituting one function into another function.

2. Computing Compositions of Functions

2a. Computing Compositions for Specific Values

EXAMPLE Let $f(x)=2x+3$ and $g(x)=x^2+1$. Find and simplify $(f \circ g)(2)$.

$(f \circ g)(2)=f(g(2))$	Rewrite using the definition of composition.
$=f(5)$	Since $g(2)$ is the innermost expression, find that first: $g(2)=2^2+1=5$
$=2(5)+3=13$	Evaluate $f(5)$.

TRY IT

Let $f(x)=2x+3$ and $g(x)=x^2+1$.

Find and simplify $(g \circ f)(4)$.

+

$(g \circ f)(4)=g(f(4))$	Rewrite using the definition of composition.
$=g(11)$	Since $f(4)$ is the innermost expression, find that first: $f(4)=2(4)+3=11$
$=11^2+1=121+1=122$	Evaluate $g(11)$.

It is also possible to substitute a function into itself. If you keep the definition in mind, this follows the same format.

EXAMPLE Let $f(x)=2x+3$. Find and simplify $(f \circ f)(3)$.

$(f \circ f)(3)=f(f(3))$	Rewrite using the definition of composition.
$=f(9)$	Since $f(3)$ is the innermost expression, find that first: $f(3)=2(3)+3=9$
$=2(9)+3=21$	Evaluate $f(9)$.

2b. Computing the Expression for a Composition of Functions

The process for finding an expression for a composition is very similar to what we just did in the previous section, but this time there is no value to substitute first.

EXAMPLE Let $f(x)=2x+3$ and $g(x)=x^2+1$. Find and simplify $(f \circ g)(x)$.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Rewrite using the definition of composition.} \\
&= f(x^2+1) && \text{Substitute } g(x) = x^2+1. \\
&= 2(x^2+1)+3 && \text{Evaluate the function and simplify.} \\
&= 2x^2+2+3 \\
&= 2x^2+5
\end{aligned}
$$

TRY IT

Let $f(x) = 2x+3$ and $g(x) = x^2+1$.

Find and simplify $(g \circ f)(x)$. +

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) && \text{Rewrite using the definition of composition.} \\
&= g(2x+3) && \text{Substitute } f(x) = 2x+3. \\
&= (2x+3)^2+1 \\
&= (2x+3)(2x+3)+1 \\
&= 4x^2+12x+9+1 && \text{Evaluate the function and simplify.} \\
&= 4x^2+12x+10
\end{aligned}
$$

HINT

Notice that $(f \circ g)(x)$ and $(g \circ f)(x)$ are not equal. In general, we can assume that $(f \circ g)(x) \neq (g \circ f)(x)$.

TRY IT

Consider the same function as above: $f(x) = 2x+3$.

Find and simplify $(f \circ f)(x)$. +

$$
\begin{aligned}
(f \circ f)(x) &= f(f(x)) && \text{Rewrite using the definition of composition.} \\
&= f(2x+3) && \text{Substitute } f(x) = 2x+3. \\
&= 2(2x+3)+3 \\
&= 4x+6+3 && \text{Evaluate the function and simplify.} \\
&= 4x+9
\end{aligned}
$$

There are situations in which a composition can't be simplified.

↪ EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = 4x+7$, find an expression for $(f \circ g)(x)$.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Rewrite using the definition of composition.} \\
&= f(4x+7) && \text{Substitute } g(x) = 4x+7.
\end{aligned}
$$

$= \sqrt{4x+7}$ Evaluate the function.

There is no algebraic way to simplify $\sqrt{4x+7}$, so this is the final answer.

↪ EXAMPLE If $f(x) = x^3$ and $g(x) = 2x - 1$, find an expression for $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$	Rewrite using the definition of composition.
$= f(2x-1)$	Substitute $g(x) = 2x - 1$.
$= (2x-1)^3$	Evaluate the function.

At this point, we could use multiplication to rewrite this expression, but this would be very time-consuming. It is actually more useful to leave the answer as $(2x-1)^3$.

TRY IT

Suppose $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 + x + 5$.

Find an expression for $(f \circ g)(x)$. +

$(f \circ g)(x) = f(g(x))$	Rewrite using the definition of composition.
$= f(x^2+x+5)$	Substitute $g(x) = x^2 + x + 5$.
$= \sqrt[3]{x^2+x+5}$	Evaluate the function.

3. Decomposing Composite Functions

Given a composition of functions, it is important to be able to identify the inner and outer functions.

For example, each of these functions are compositions of other functions:

$$h(x) = (3x-1)^2 \qquad j(x) = \sqrt{5x+6} \qquad m(x) = \frac{3}{(x+1)^2}$$

To decompose composite functions, identify the "inner" function first, then the "outer" function is apparent.

↪ EXAMPLE Consider the expression $(3x+8)^2$, which is the result of a composition of functions. How can we find functions $f(x)$ and $g(x)$ so that $f(g(x)) = f(3x+8) = (3x+8)^2$?

To answer this question, start with the expression inside the grouping symbols. Since $g(x)$ is the inside function, let $g(x) = 3x + 8$. Then, we have $f(g(x)) = f(3x+8) = (3x+8)^2$.

Now, replace $3x+8$ with a symbol, say "?". We can write $f(?) = (?)^2$. As you can see, this tells us that

$f(x) = x^2$.

Conclusion: Given $f(g(x)) = (3x+8)^2$, $f(x) = x^2$ and $g(x) = 3x + 8$.

TRY IT

Suppose $f(g(x)) = \sqrt[3]{x^2+4}$.

Find functions $f(x)$ and $g(x)$ to make the above composition of functions true. +

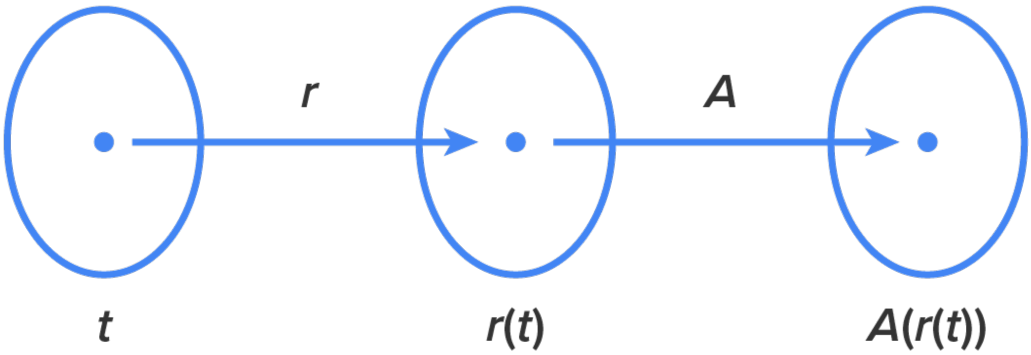
The inside function is $g(x) = x^2 + 4$ and the outside function is $f(x) = \sqrt[3]{x}$.

---

4. Applications of Compositions

When a stone is dropped into a lake, a circular ripple forms and continues to get larger until it dissipates. After *t* seconds, the radius (in inches) of the ripple is $r(t) = 4t$. Recall also that the area of a circle with radius *r* is $A(r) = \pi r^2$.

EXAMPLE Suppose we want to find a function for the area inside the ripple, but as a function of time, *t*. Here is how the functions work together:



Therefore, the composition $(A \circ r)(t)$ will give the area enclosed by the ripple after *t* seconds.

$(A \circ r)(t) = A(r(t))$	Use the definition of composition.
$= A(4t)$	Replace $r(t)$ with *4t*.
$= \pi(4t)^2$	Evaluate the function.
$= 16\pi t^2$	Simplify.

Notice that A is the outer function, which means that the result is an area. Notice also that using this function allows us to bypass knowing the radius in order to get the area.

↱ EXAMPLE The radius of a circle is given by the function $r(C) = \frac{C}{2\pi}$, where C is the circumference of the circle. Recall also that the area of a circle is $A(r) = \pi r^2$. Using this information, we can find $(A \circ r)(C)$.

$A(r(C))$	Rewrite using the definition.
$= A\left(\frac{C}{2\pi}\right)$	Replace $r(C)$ with $\frac{C}{2\pi}$.
$= \pi\left(\frac{C}{2\pi}\right)^2$	Evaluate the function.
$= \pi\left(\frac{C^2}{4\pi^2}\right)$	Apply the exponent.
$= \frac{C^2}{4\pi}$	Remove the common factor of π.

This function gives the area of a circle when its circumference is known. This could be very useful since it is easier to measure the circumference of a circle than it is its radius.

SUMMARY

In this lesson, you learned **the definition of a composition of functions**, which is a function that is obtained by substituting one function into another function. Written $(f \circ g)(x)$, or $f(g(x))$, it means to substitute $g(x)$ into $f(x)$. You learned how to **compute compositions of functions**, including both **computing compositions for specific values** and **computing the expression for a composition of functions**. You learned how to **decompose composite functions** by identifying the “inner” function first, after which the “outer” function is apparent. Finally, you explored **applications of compositions** in real-world situations.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Composition of Functions

Written $(f \circ g)(x)$, it is a function that is obtained by substituting one function into another function.

Shifting and Stretching Graphs

by Sophia

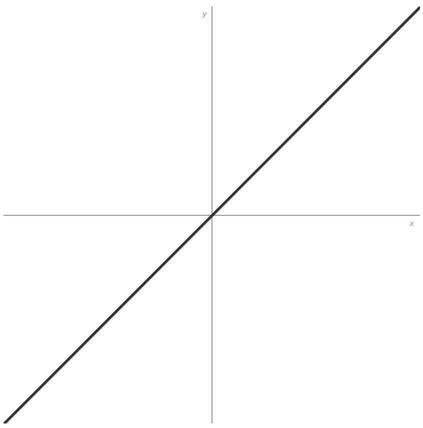
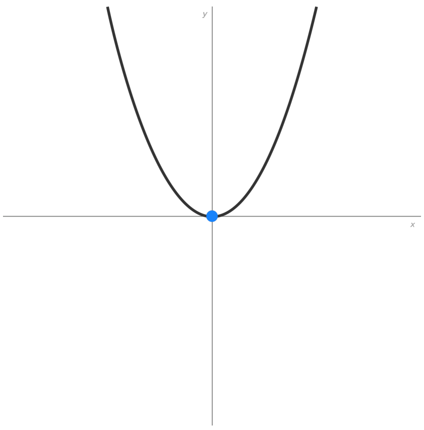
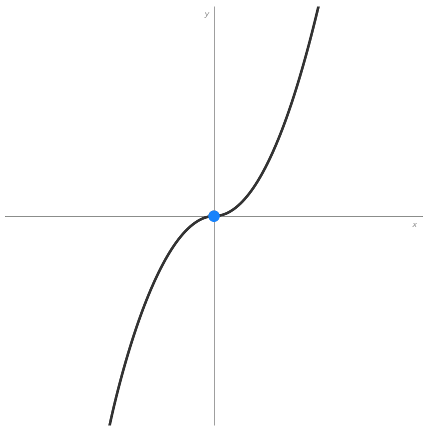
WHAT'S COVERED

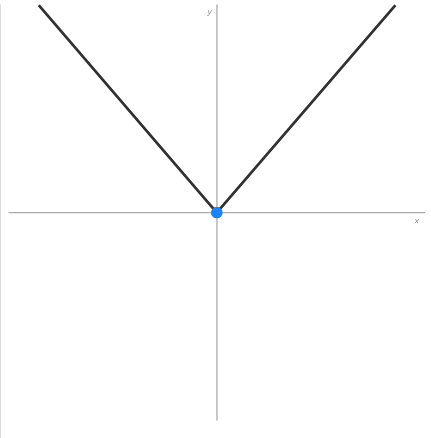
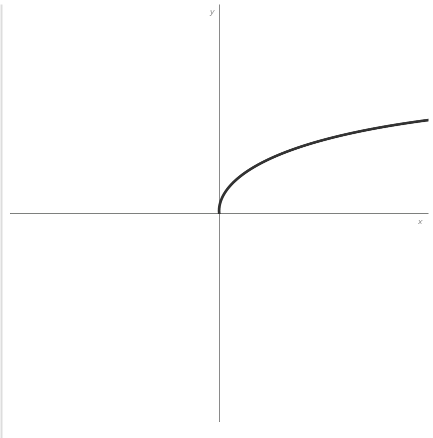
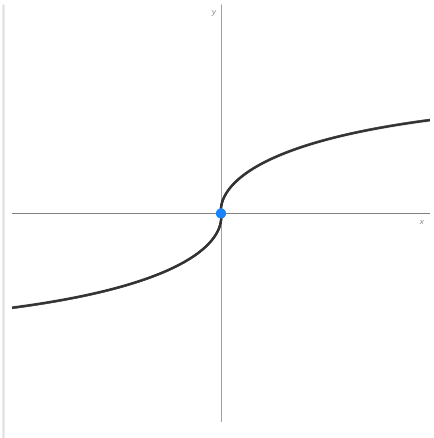
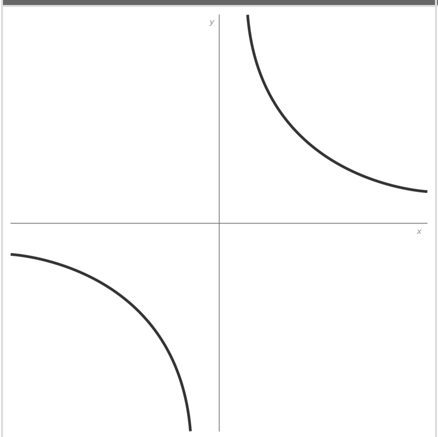
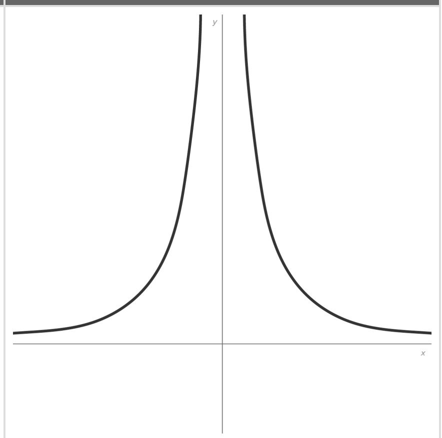
In this lesson, you will investigate how translations affect the graph of a function. Specifically, this lesson will cover:

1. Commonly Used Basic Functions and Their Graphs
2. Applying Basic Translations to $y = f(x)$
3. Applying Several Translations to $y = f(x)$

1. Commonly Used Basic Functions and Their Graphs

Here are the most commonly used graphs that are encountered in a typical algebra course. There are others as well, which will be investigated in future challenges.

Linear Function $f(x) = x$	Quadratic Function $f(x) = x^2$	Cubic Function $f(x) = x^3$
		
Absolute Value Function $f(x) = \|x\|$	**Square Root Function** $f(x) = \sqrt{x}$	**Cube Root Function** $f(x) = \sqrt[3]{x}$

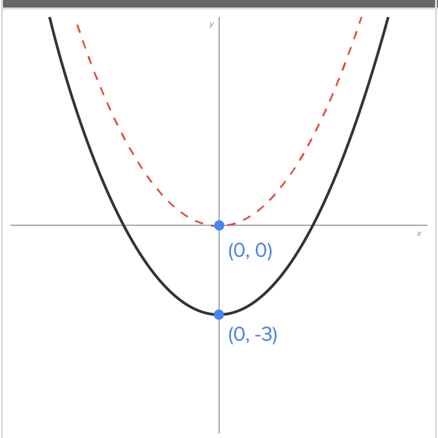
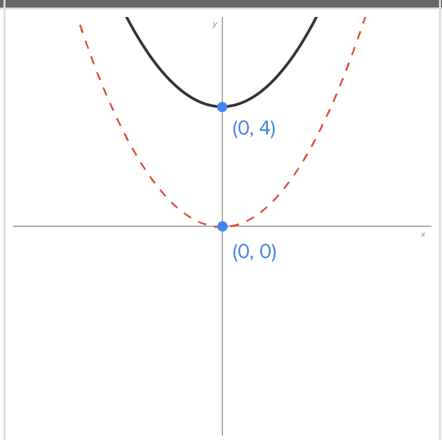
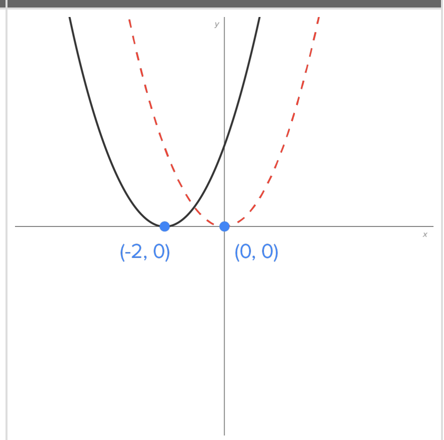
<table>
<tr><td></td><td></td><td></td></tr>
<tr><td>Reciprocal Function
$f(x) = \dfrac{1}{x}$</td><td>Reciprocal Square Function
$f(x) = \dfrac{1}{x^2}$</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
</table>

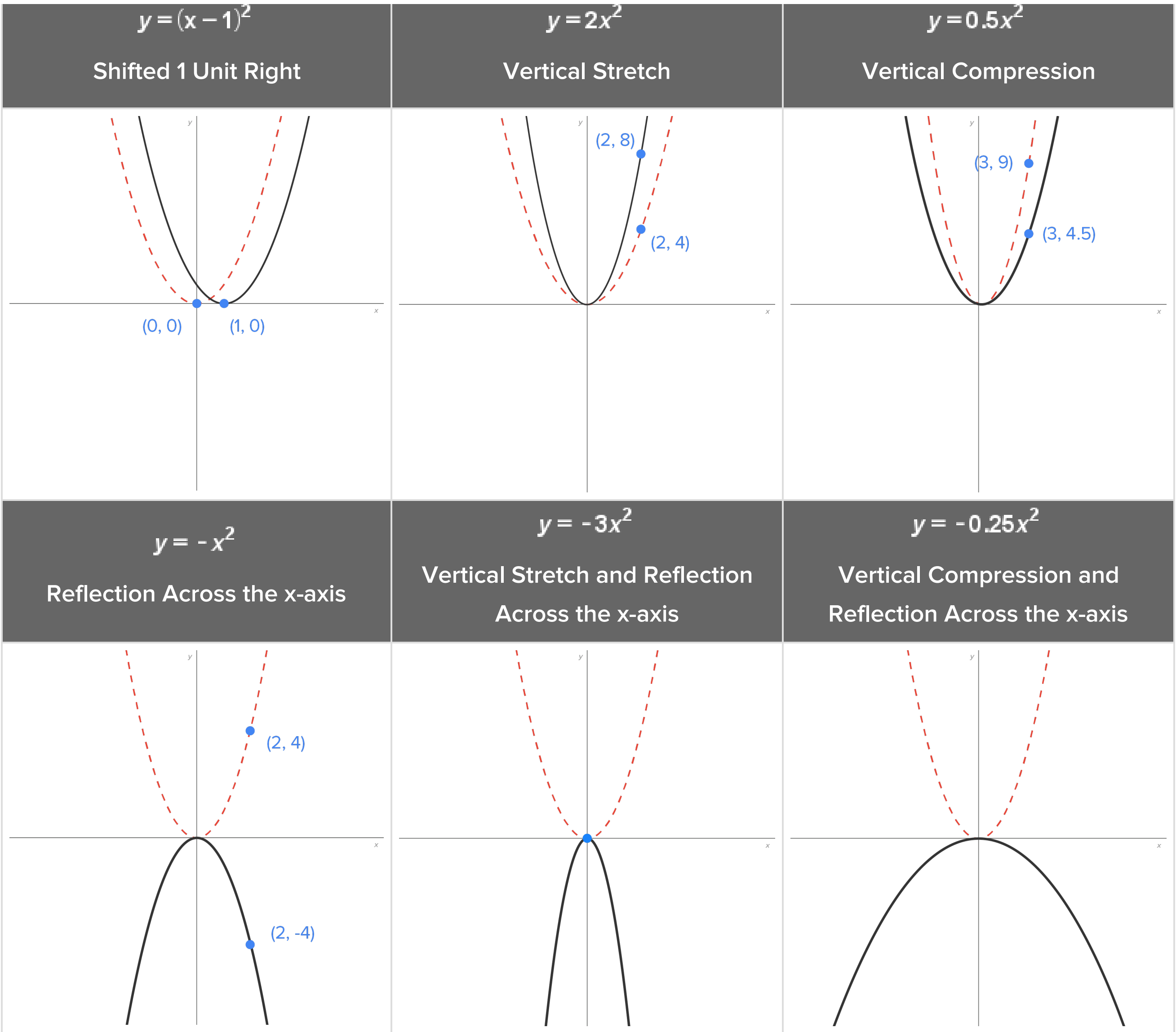
In this challenge, we will investigate translations to an equation and how they affect the graph of said equation.

---

2. Applying Basic Translations to $y = f(x)$

Shown below are several functions which are translations of $f(x) = x^2$. In each graph, the graph of $f(x) = x^2$ is shown with a dashed line.

<table>
<tr><td>$y = x^2 - 3$
Shifted Down 3 Units</td><td>$y = x^2 + 4$
Shifted Up 4 Units</td><td>$y = (x+2)^2$
Shifted Left 2 Units</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>



BIG IDEA

Given the graph of $y=f(x)$ and a positive constant *k*:

- The graph of $y=f(x)+k$ shifts the graph up *k* units.
- The graph of $y=f(x)-k$ shifts the graph down *k* units.
- The graph of $y=f(x-k)$ shifts the graph right *k* units.
- The graph of $y=f(x+k)$ shifts the graph left *k* units.
- The graph of $y=a\cdot f(x)$ is a **vertical stretch** if $|a|>1$ and a **vertical compression** if $|a|<1$. Also, if a is negative, the graph also reflects over the x-axis.

TERMS TO KNOW

Vertical Compression

A translation that makes all y-values of a graph smaller in magnitude, pulling a graph toward the x-axis. This is represented by $y=a\cdot f(x)$, where $|a|<1$.

Vertical Stretch

A translation that makes all y-values of a graph larger in magnitude, pulling a graph toward the y-axis. This is represented by $y=a\cdot f(x)$, where $|a|>1$.

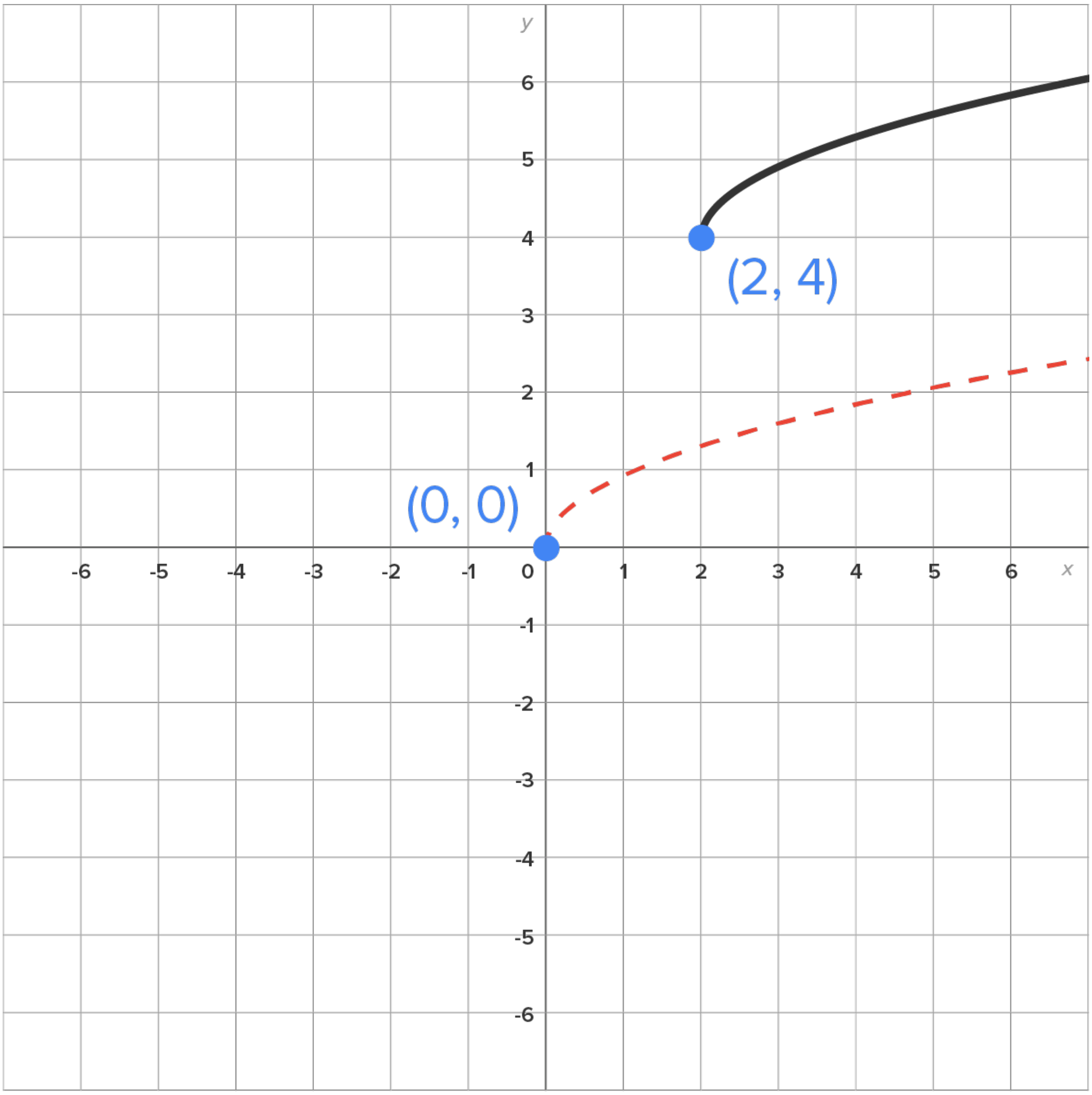
3. Applying Several Translations to $y = f(x)$

Given the translations discussed in the previous section, it is possible to apply several to one function.

↷ EXAMPLE Consider the function $g(x) = \sqrt{x-2} + 4$, which is related to the function $f(x) = \sqrt{x}$.

- There is an "$x - 2$" under the radical where the "x" is in the base function, indicating that the graph is moved to the right two units.
- The "+ 4" outside of the radical indicates that the graph is shifted up 4 units.

The graphs of $f(x)$ and $g(x)$ are shown on the same axes below. The graph of $f(x)$ is dashed to show its relationship to $g(x)$.

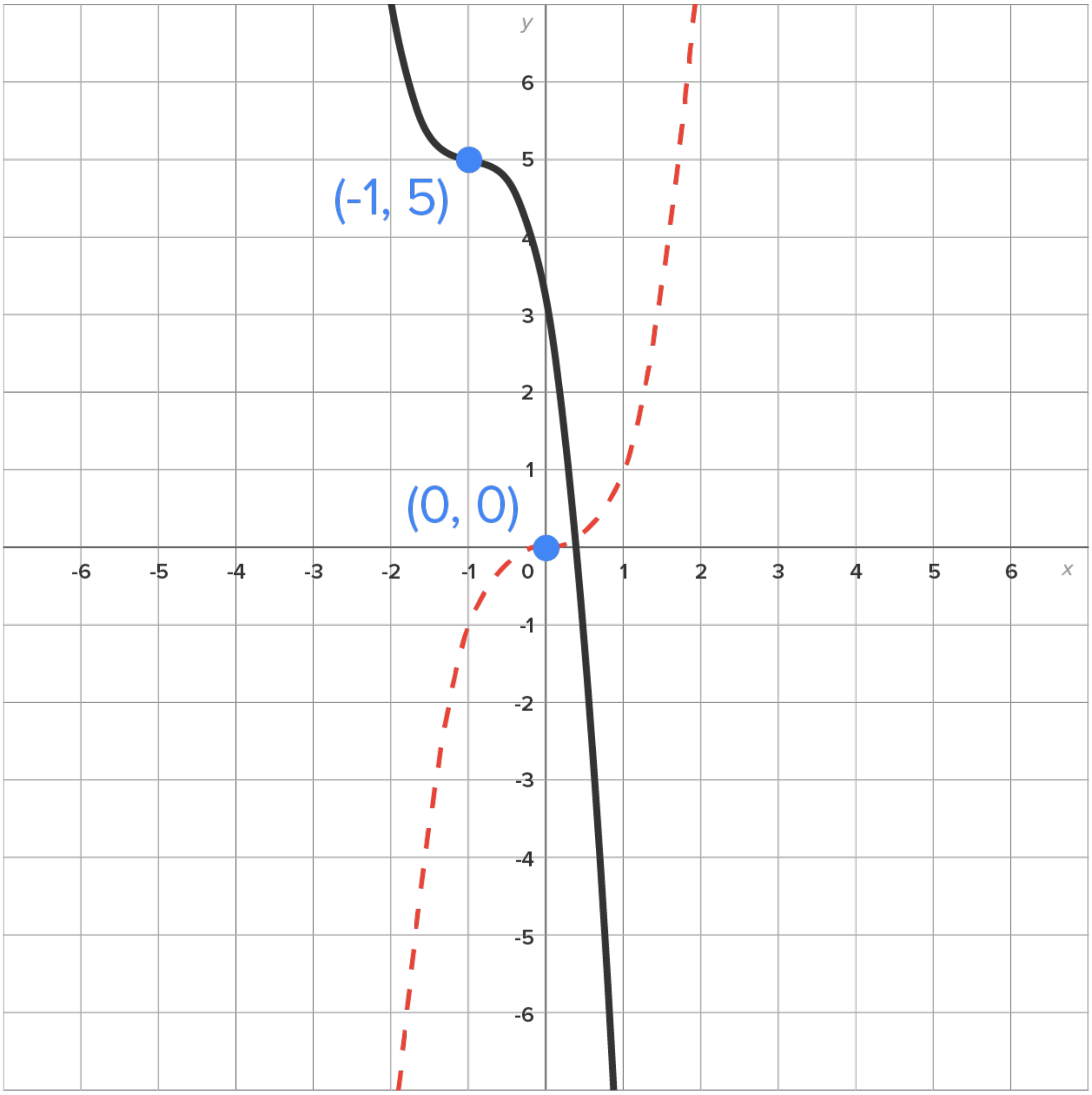


The graph of $g(x)$ is obtained by moving the graph of $f(x)$ to the right 2 units and up 4 units.

↱ EXAMPLE Describe the sequence of transformations that are required to graph $g(x) = -2(x+1)^3 + 5$ based on $f(x) = x^3$.

- The "$x + 1$" tells us that the graph is shifted to the left by 1 unit.
- The "-2" multiplied to the cubed term tells us that the graph is reflected around the x-axis and stretched vertically (since 2 > 1).
- The "+ 5" tells us that the graph is then shifted up five units.

The graph is shown here, with $f(x) = x^3$ dashed.



BIG IDEA

The order in which translations are applied only matters when there is a vertical shift. Here is the order in which translations should be applied:

1. Horizontal Translations
2. Vertical Compressions/Stretches/Reflections
3. Vertical Translations

TRY IT

Consider the equation $g(x) = 0.75(x+4)^2 - 3$.

Identify the basic function. +

The basic function is $f(x) = x^2$.

List the sequence of translations required to graph $g(x)$ based on the basic function. +

The graph is shifted to the left by 4 units, vertically compressed by a factor of 0.75, and shifted down 3 units.

SUMMARY

In this lesson, you began by exploring **commonly used basic functions and their graphs**. You investigated several types of translations to an equation and how they affect the graph of said equation, **applying basic translations to $y = f(x)$** to shift, stretch, compress, and reflect its graph. You also learned how to graph a function $g(x)$ by **applying several translations to $y = f(x)$**, noting the order in which translations should be applied.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Vertical Compression

A translation that makes all y-values of a graph smaller in magnitude, pulling a graph toward the x-axis. This is represented by $y = a \cdot f(x)$, where $|a| < 1$.

Vertical Stretch

A translation that makes all y-values of a graph larger in magnitude, pulling a graph toward the y-axis. This is represented by $y = a \cdot f(x)$, where $|a| > 1$.

Absolute Value Functions

by Sophia

WHAT'S COVERED

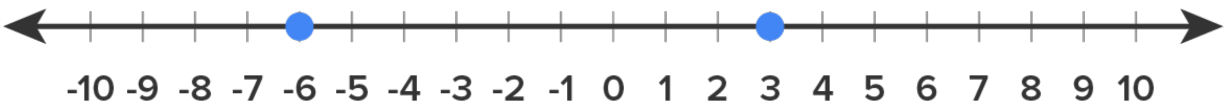
In this lesson, you will learn about absolute value functions. Specifically, this lesson will cover:

1. The Absolute Value Function

1a. The Piecewise Definition of Absolute Value

Recall that $|x|$ means "the **absolute value** of *x*", which represents the distance that a number *x* is from 0 (on the number line).

Consider the number line shown below, with the numbers 3 and -6 marked.



Since the number 3 is a distance of 3 units from 0, we say that $|3| = 3$.

Since the number -6 is a distance of 6 units from 0, we say that $|-6| = 6$.

In general, evaluating $|x|$ requires two different rules, depending on what *x* is.

- If *x* is nonnegative, then $|x|$ and *x* are the same.
- If *x* is negative, then $|x|$ is the opposite of *x* (turning a negative into a positive).

This leads to the piecewise definition for $|x|$.

$$|x| = \begin{cases} -x & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$$

TERM TO KNOW

Absolute Value

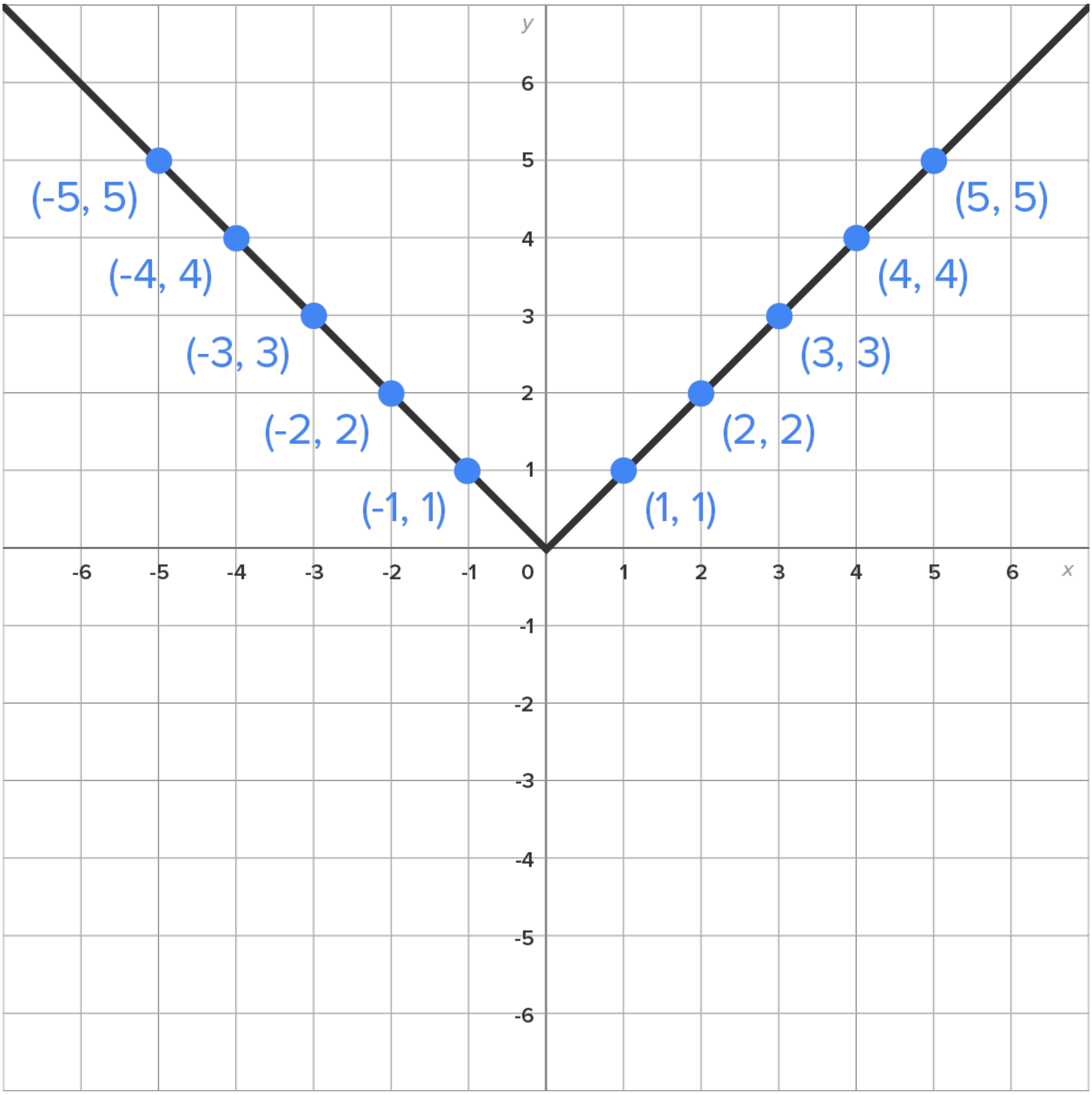
The distance that a number is from 0 on the number line.

1b. The Graph of the Basic Absolute Value Function

In the table below, you see several input-output pairs for $f(x) = |x|$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x) = \lvert x \rvert$	5	4	3	2	1	0	1	2	3	4	5

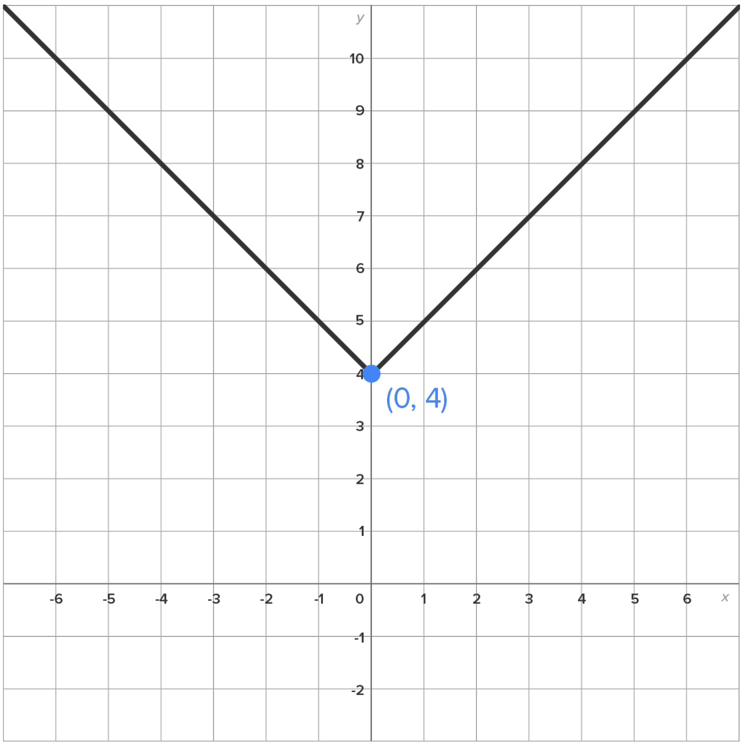
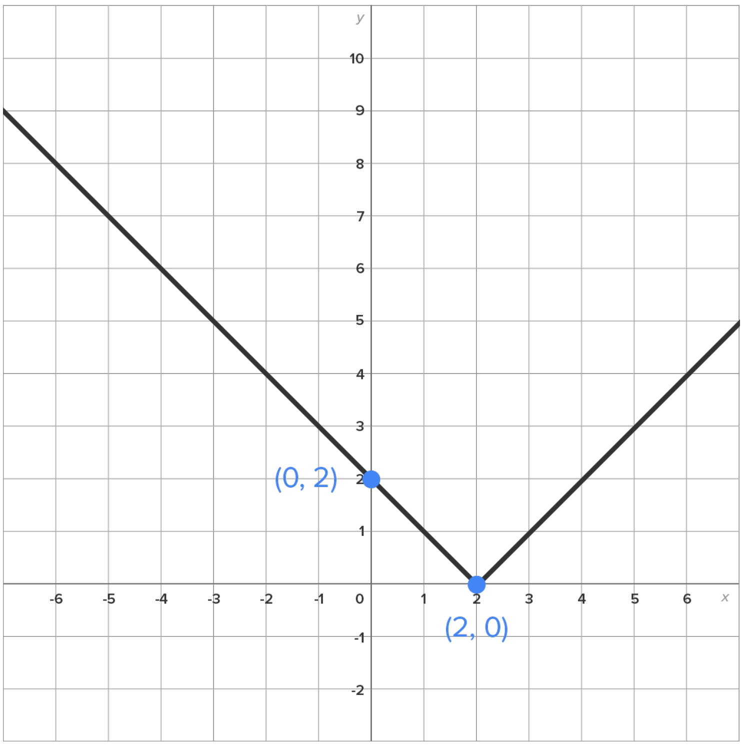
Here is the resulting graph:

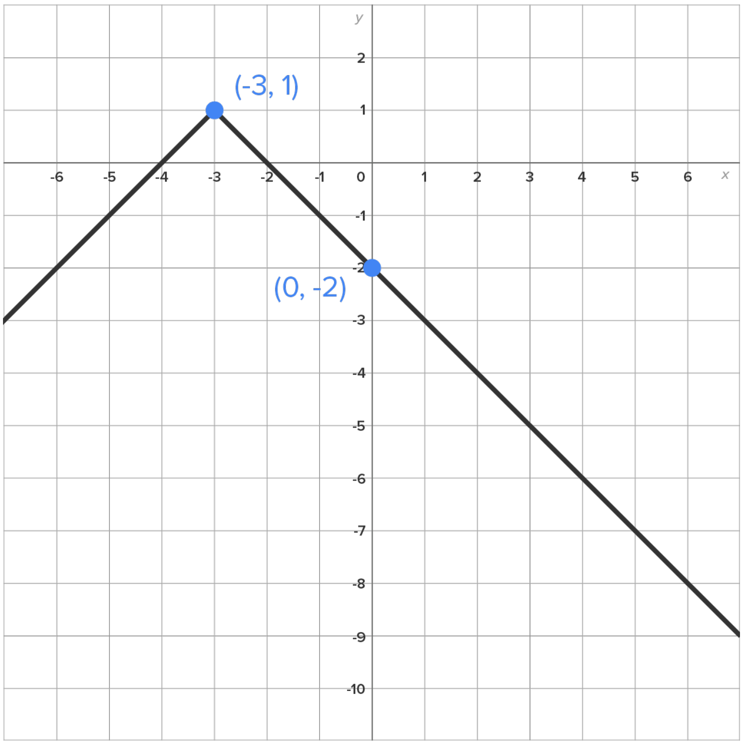


2. Graphing Absolute Value Functions

2a. Shifting, Stretching, and Reflecting the Basic Absolute Value Function

From what you learned in the “Shifting and Stretching Graphs” section, you can apply these rules to the absolute value function.

Function	Graph	Shifts and/or Stretches from $f(x) = \lvert x \rvert$
$g(x) = \lvert x \rvert + 4$		Up 4 units
$h(x) = \lvert x - 2 \rvert$		Right 2 units

$j(x) = -\lvert x+3 \rvert + 1$		Left 3 units, reflected over the x-axis, then moved up 1 unit

2b. Other Absolute Value Graphs

The function $\lvert f(x) \rvert$ can be written in piecewise form by replacing "*x*" with "$f(x)$."

$$\lvert x \rvert = \begin{cases} -x & if\ x < 0 \\ x & if\ x \geq 0 \end{cases} \quad \rightarrow \quad \lvert f(x) \rvert = \begin{cases} -f(x) & if\ f(x) < 0 \\ f(x) & if\ f(x) \geq 0 \end{cases}$$

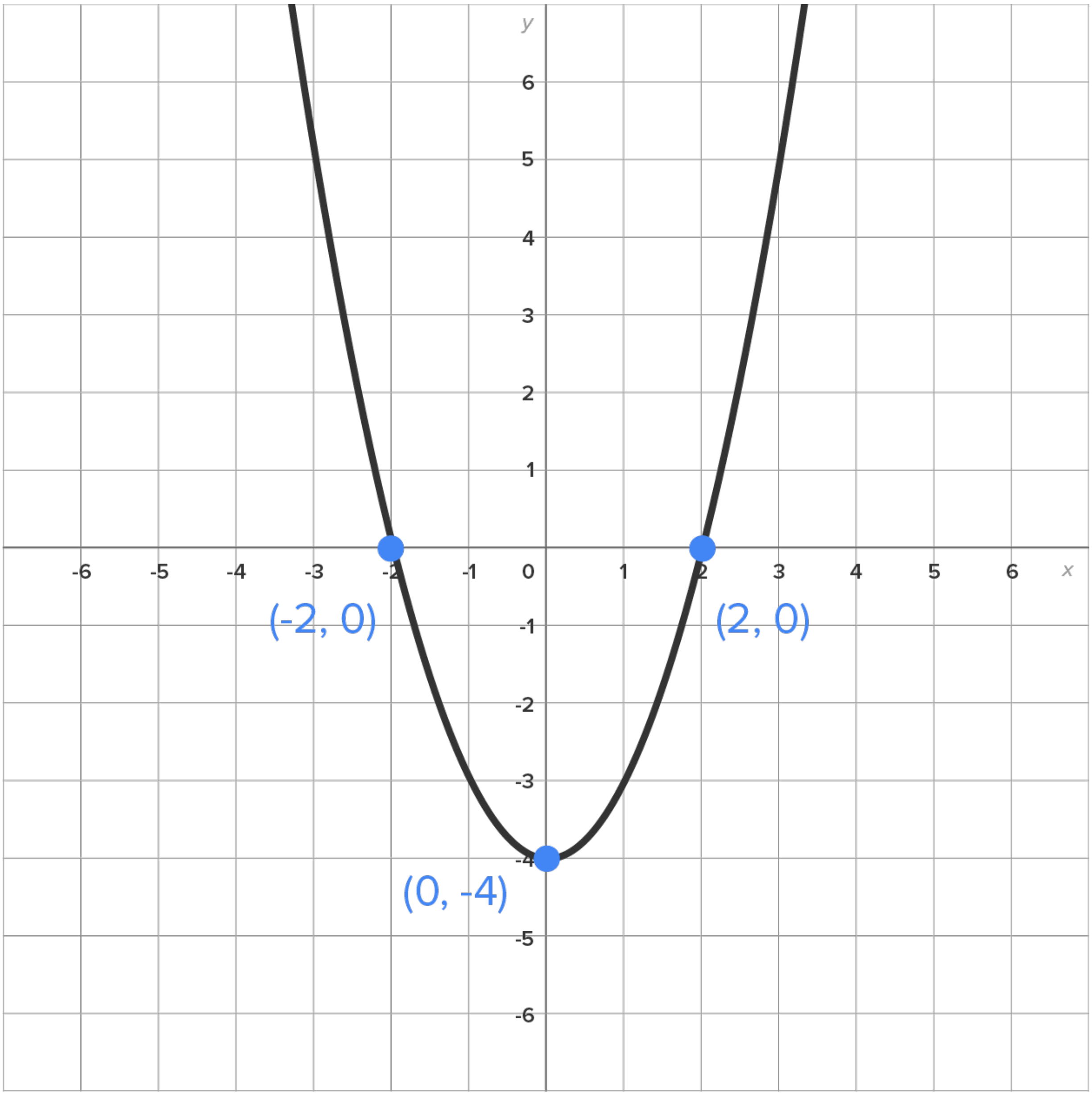
We can adapt this idea to graph a function of the form $y = \lvert f(x) \rvert$. In order to do this, think about what it means when we say $f(x) \geq 0$ and $f(x) < 0$.

If $f(x) < 0$, this really means $y < 0$, indicating that the corresponding point on the graph is below the x-axis.

If $f(x) \geq 0$, this really means $y \geq 0$, indicating that the corresponding point on the graph is on or above the x-axis.

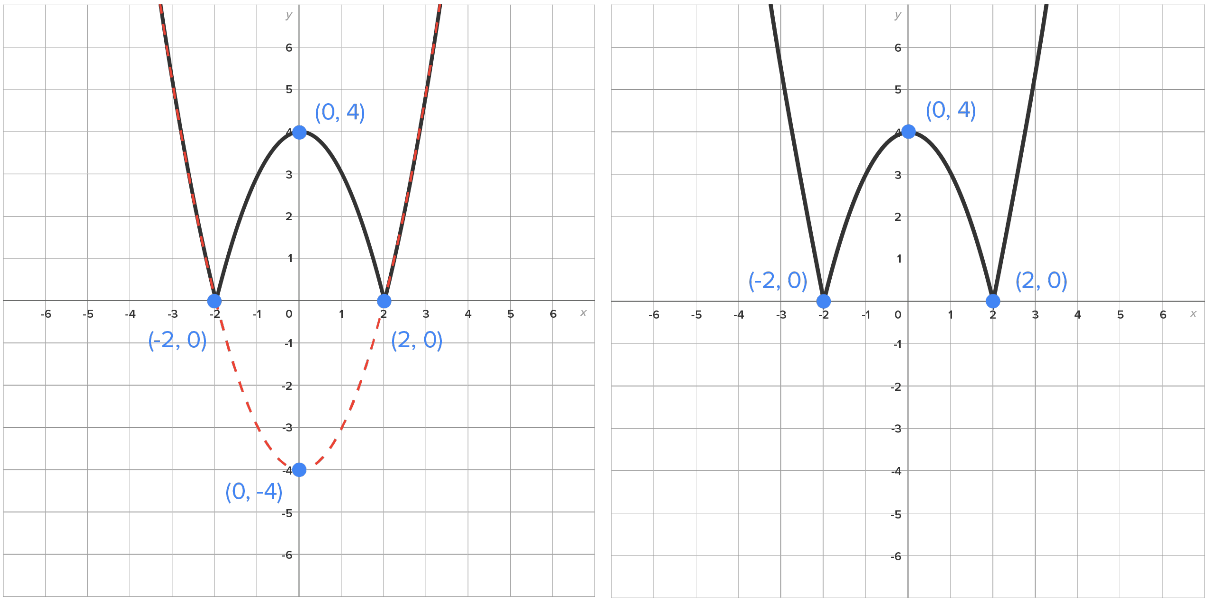
Recall from challenge 1.3.4 "Shifting and Stretching Graphs" that the graph of $y = -f(x)$ reflects the graph of $y = f(x)$ across the x-axis. Thus, if $f(x) < 0$, then the graph of $y = \lvert f(x) \rvert$ reflects over the x-axis (to the positive side). Otherwise, the graphs of $f(x)$ and $y = \lvert f(x) \rvert$ are the same.

↪ EXAMPLE The graph of $f(x) = x^2 - 4$ is shown below:



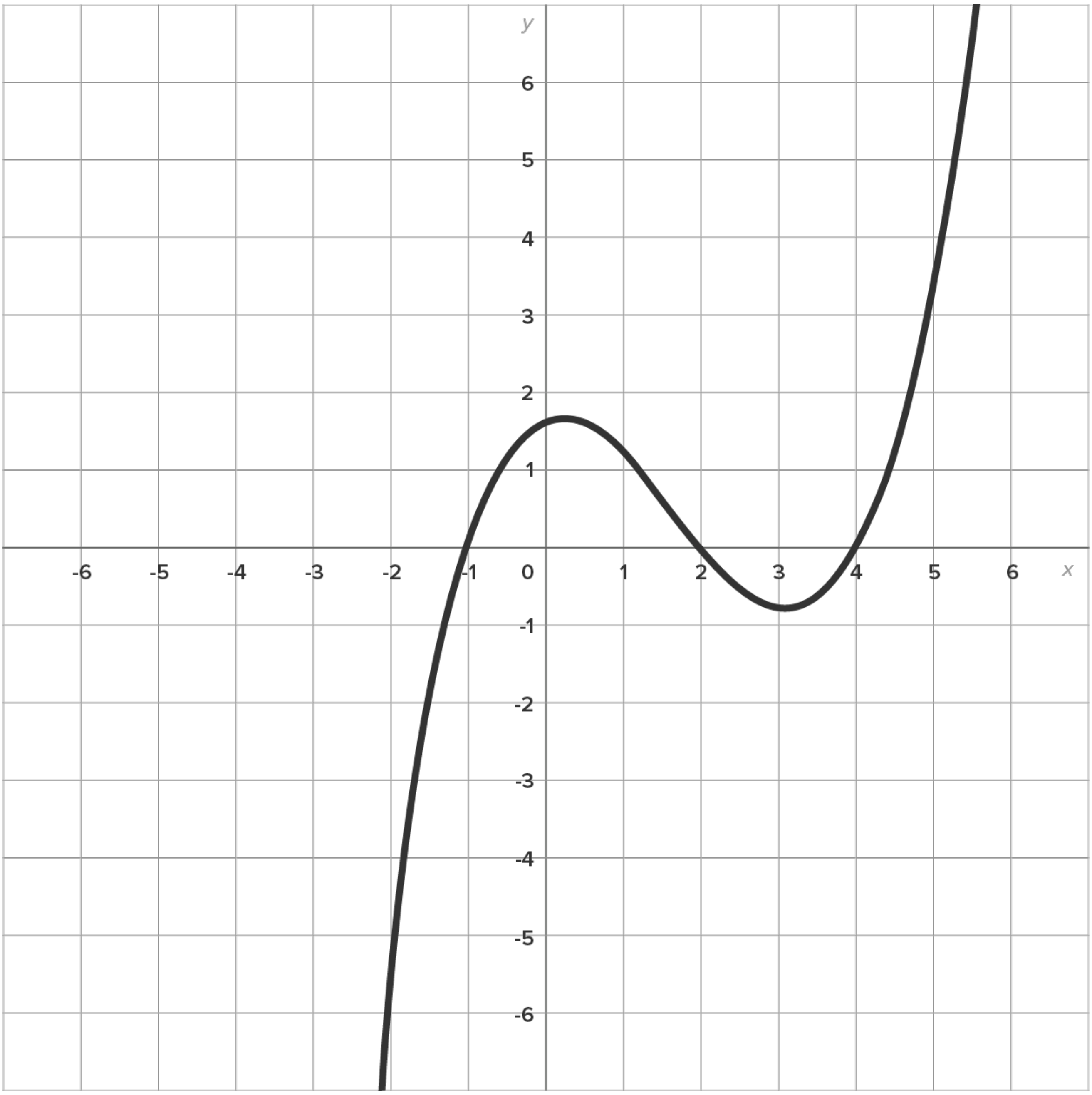
To graph $g(x) = |f(x)| = |x^2 - 4|$, notice that the graph of $f(x) = x^2 - 4$ is below the x-axis between $x = -2$ and $x = 2$. This part reflects over the x-axis, while the rest of the graph remains the same.

On the left is the graph of $g(x) = |f(x)| = |x^2 - 4|$ with the graph of $f(x)$ shown as a dashed line for comparison. On the right is the graph of $g(x) = |x^2 - 4|$.



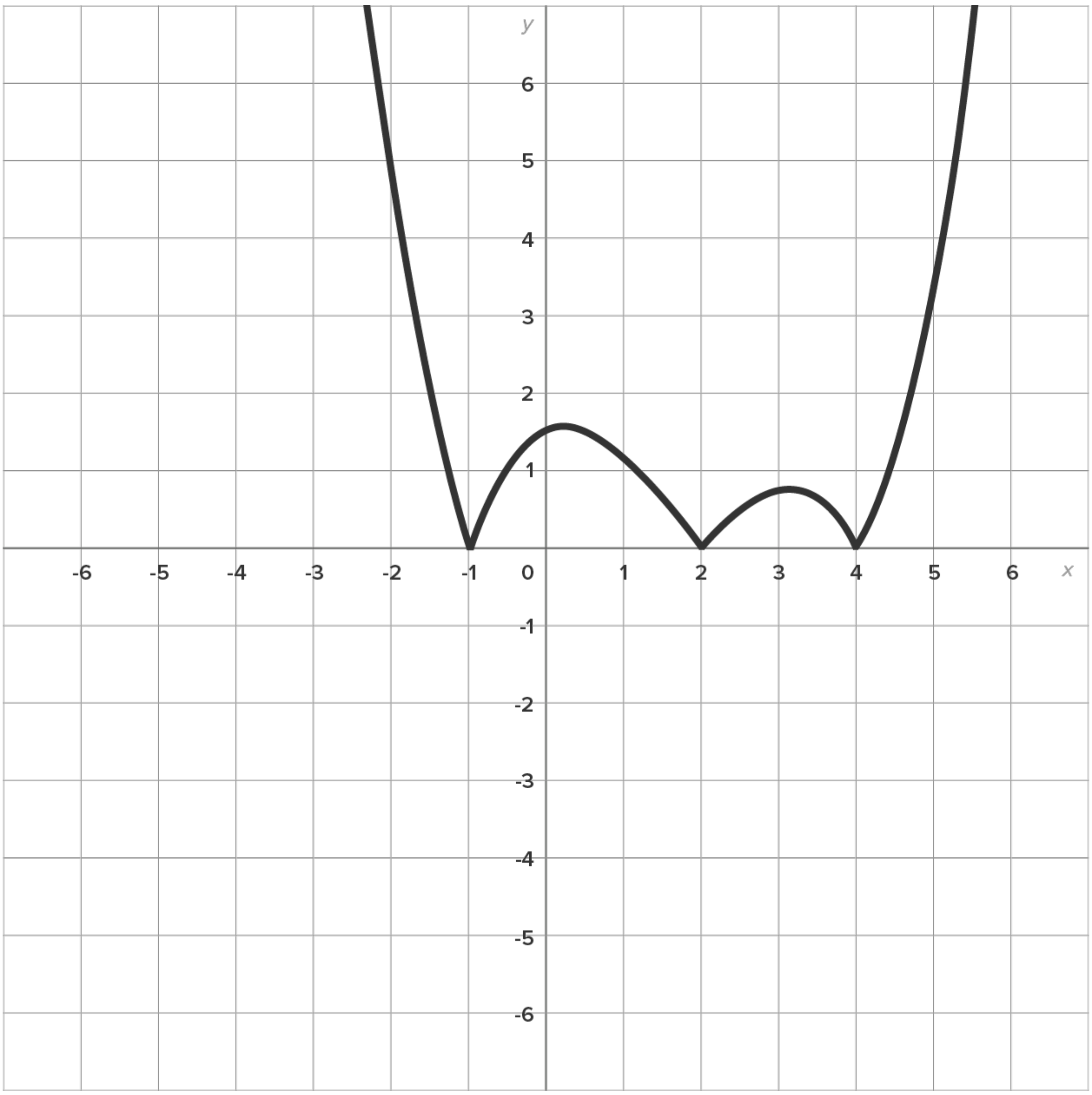
TRY IT

Consider the graph of $y = f(x)$ as shown below.



Sketch the graph of $y = |f(x)|$.

+



SUMMARY

In this lesson, you learned about **the absolute value function** of *x*, which represents the distance that a number *x* is from 0 on the number line. You learned about **the piecewise definition of absolute value**, given that in general, evaluating $|x|$ requires two different rules, depending on what *x* is. It's important to remember that the absolute value function may look simple on the surface, but it has a more complicated definition beyond "turning things nonnegative." You explored **the graph of the basic absolute value function**, applying rules you learned in a previous lesson to create graphs that illustrate the **shifting, stretching, and reflecting of the basic absolute value function**. Lastly, you learned about **other absolute value graphs**, noting that while the basic absolute function is simply a "V" shape, graphing $y = |f(x)|$ requires more thought and care.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Absolute Value

The distance that a number is from 0 on the number line.

Greatest Integer Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to graph the greatest integer function. Specifically, this lesson will cover:

1. The Greatest Integer Function
   a. Motivation: Why Do We Need This?
   b. The Basic Greatest Integer Function
   c. The Graph of the Basic Greatest Integer Function
2. Compositions That Involve the Greatest Integer Function

1. The Greatest Integer Function

1a. Motivation: Why Do We Need This?

You have 60 pieces of candy to give a group of *x* people, and you want to distribute them evenly.

- If there are 15 people, then you would give each person 60/15 = 4 pieces of candy.
- If there are 12 people, then you would give each person 60/12 = 5 pieces of candy.

What happens when you can't divide evenly?
If there are 13 people in the room, then each person would get 60/13 = 4.615... pieces of candy. Since this is impossible and you want to be fair to each person, you would give each person 4 pieces of candy, with some left over.

So, what would be a mathematical rule for this situation?

Let $p(x)$ = the number of pieces of candy each person receives when there are *x* people.

Then, the value of $p(x)$ is found as follows:

- If $\frac{60}{x}$ is a whole number, then use $\frac{60}{x}$.
- If $\frac{60}{x}$ is not a whole number, then use the next whole number less than $\frac{60}{x}$.

1b. The Basic Greatest Integer Function

This leads us to a need to define another function, called the **greatest integer function**, which is denoted $\lfloor x \rfloor$. The piecewise definition of $\lfloor x \rfloor$ is as follows:

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{the closest integer less than } x & \text{if } x \text{ is NOT an integer} \end{cases}$$

HINT

There are other commonly used notations for the greatest integer function:

$INT(x)$ $[x]$ $floor(x)$

(In computer science, this function is often called the floor function, since it usually produces a lower value).

Here are a few function values for $f(x) = \lfloor x \rfloor$:

Function Statement	Reasoning
$f(3.4) = 3$	Since 3.4 is not an integer, the function returns the greatest integer below 3.4, which is 3.
$f(5) = 5$	Since 5 is an integer, the function returns 5.
$f(-2.1) = -3$	Since -2.1 is not an integer, the function returns the greatest integer that is below -2.1, which is -3.
$f(10.8) = 10$	Since 10.8 is not an integer, the function returns the greatest integer that is below 10.8, which is 10.

TRY IT

Consider the greatest integer function $f(x) = \lfloor x \rfloor$.

What does the function return when $x = 2.9$? +

$f(2.9) = 2$

What does the function return when $x = -4$? +

$f(-4) = -4$

What does the function return when $x = 0.8$? +

$f(0.8) = 0$

FORMULA

The Piecewise Greatest Integer Function

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{the closest integer less than } x & \text{if } x \text{ is NOT an integer} \end{cases}$$

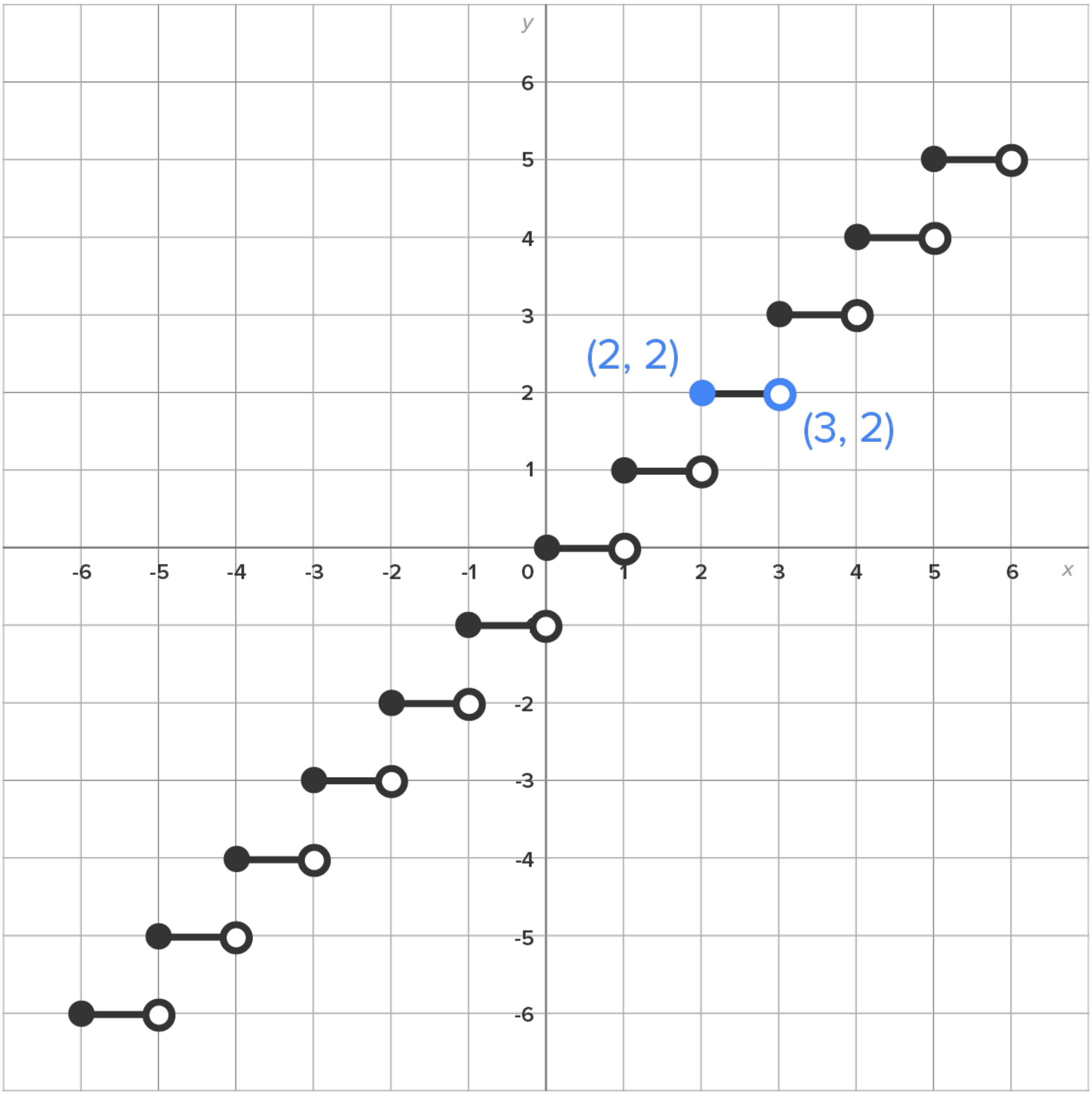
TERM TO KNOW

Greatest Integer Function

Returns the greatest integer that is less than or equal to the input value.

1c. The Graph of the Basic Greatest Integer Function

The graph of the basic greatest integer function is shown below.



Note, the “stair step” pattern continues indefinitely in both directions.

To understand how this graph works, consider the two points that are labelled (2, 2) and (3, 2). If $2 \le x < 3$, the greatest integer function returns the value of 2. We can see this by using some input-output pairs:

x	2	2.2	2.5	2.8	2.9	2.99	2.999	2.9999
$\lfloor x \rfloor$	2	2	2	2	2	2	2	2
(x, y)	(2, 2)	(2.2, 2)	(2.5, 2)	(2.8, 2)	(2.9, 2)	(2.99, 2)	(2.999, 2)	(2.9999, 2)

As soon as the value of *x* jumps to exactly 3, then the greatest integer function returns a value of 3, which means the graph moves to the next “stair step.”

2. Compositions That Involve the Greatest Integer Function

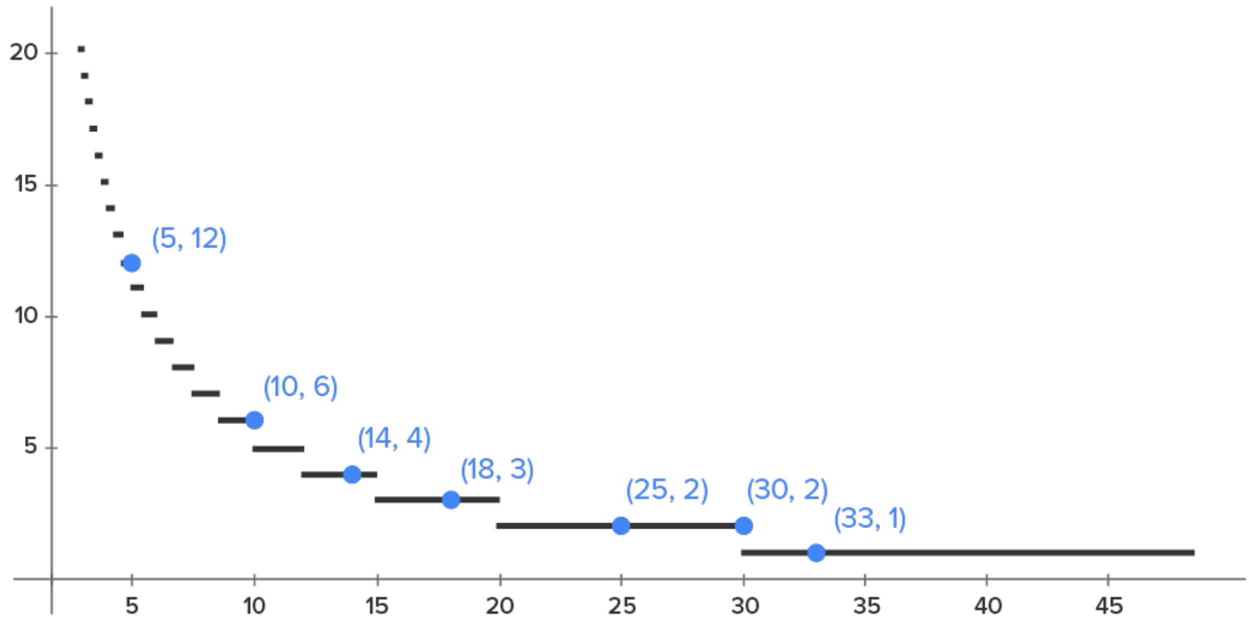
Let's now return to the situation where you were handing out candy to a group of x people.

To find the number of pieces, we calculate $\frac{60}{x}$, then round down if necessary. This is the essence of a greatest integer function!

Using the greatest integer function, the number of pieces received by each person is:

$$p(x) = \left\lfloor \frac{60}{x} \right\rfloor$$

Here is the graph of $p(x)$, with selected points labelled:



Note, the piece of the graph where $y = 1$ would extend out to the point (60, 1) until falling to 0 for $x > 60$. Why is this? If you have more than 60 people in your group and you only have 60 pieces of candy to give out, there is no way to give the same amount of candy to each of them (so each person would receive no candy). On the bright side, that means you get to keep it all.

SUMMARY

In this lesson, you were introduced to **the greatest integer function**, which returns the greatest integer that is less than or equal to the input value. You investigated a real-life situation in which finding a composition involving a greatest integer function was useful, in order to understand the **motivation behind why we need this** function (everyone wants their fair share of candy!). In learning about **the basic greatest integer function**, you explored what **the graph of the basic greatest integer function** looks like, noting its "stair step" pattern which continues indefinitely in both directions. Lastly,

you circled back to the original real-life situation in order to apply your knowledge to **compositions that involve the greatest integer function.**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Greatest Integer Function

Returns the greatest integer that is less than or equal to the input value.

FORMULAS TO KNOW

The Piecewise Greatest Integer Function

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{the closest integer less than } x & \text{if } x \text{ is NOT an integer} \end{cases}$$

Trigonometric Functions

by Sophia

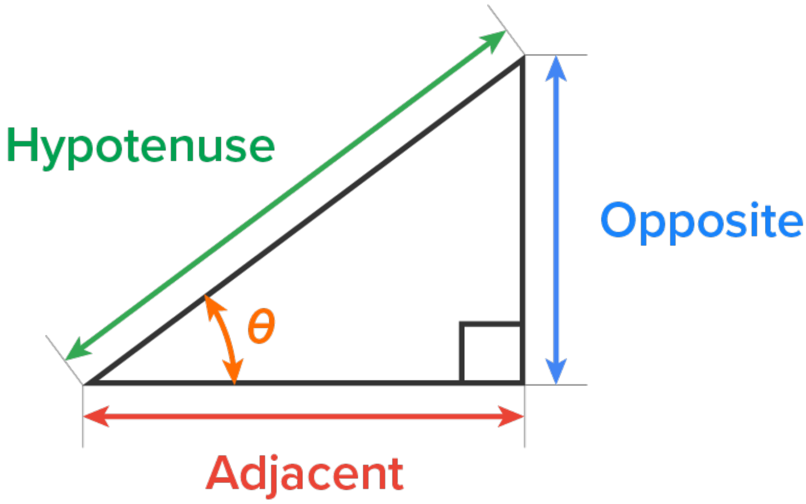
WHAT'S COVERED

In this lesson, you will get a brief overview of the trigonometric functions, some properties, and graphs. Specifically, this lesson will cover:

1. The Three Basic Trigonometric Functions
2. Evaluating Trigonometric Functions
    a. Using Right Triangles to Evaluate Trigonometric Functions
    b. Evaluating Trigonometric Functions for Any Acute Angle
    c. Evaluating Trigonometric Functions for 30°, 45°, and 60°
    d. Defining Non-Acute Angles
    e. The Unit Circle and Evaluating Trigonometric Functions for Any Angle
3. Radian Measure
    a. Converting Between Degrees and Radians
    b. Evaluating Trigonometric Functions Using Radian Measure
4. Finding an Input for a Known Output (Solving Trigonometric Equations)
5. Basic Trigonometric Graphs
    a. The Graph of $y = \sin x$
    b. The Graphs of $y = \cos x$ and $y = \tan x$
6. Frequently Used Trigonometric Identities

1. The Three Basic Trigonometric Functions

Consider the right triangle.



The sides “Opposite” and “Adjacent” are named this way relative to the angle θ (theta) marked in the triangle.

Going forward, “Opposite” means the length of the opposite side and “Adjacent” means the length of the adjacent side.

The hypotenuse is the side opposite the right angle, and is consequently the longest side of the right triangle.

Using this convention, there are six possible ratios that can be computed between two distinct sides of a right triangle. We will focus on three of these for now.

A **trigonometric function** assigns an angle to a ratio. That is, the input is the angle and the output is the ratio. The three basic trigonometric functions we will discuss for now are called the sine function (sin), the cosine function (cos), and the tangent function (tan).

To compute these functions, an angle must be used. Here are their definitions:

FORMULA

Definitions of the Sine, Cosine, and Tangent Functions

$$\sin\theta = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos\theta = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan\theta = \frac{\textit{opposite}}{\textit{adjacent}}$$

TERM TO KNOW

Trigonometric Function

Uses an angle as an input and returns a ratio as the output.

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2. Evaluating Trigonometric Functions

At first, we are going to only consider situations where θ is an **acute angle**, meaning its measure is more than 0° and less than 90°. When using trigonometric functions, it is important to note that an angle must accompany the name of the function. For example, we write $\sin\theta = \frac{5}{7}$, not $\sin = \frac{5}{7}$. (Think about the square root function: You had to write $\sqrt{16} = 4$ as opposed to $\sqrt{\ } = 4$.)

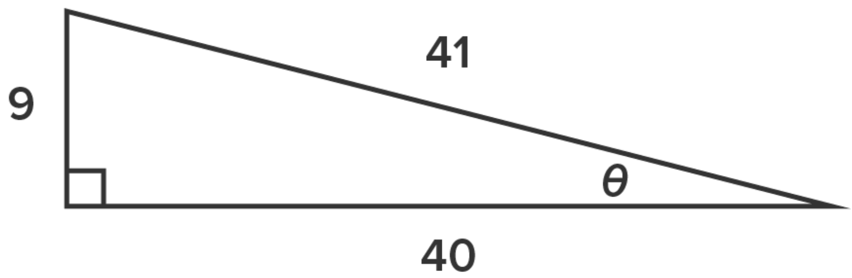
TERM TO KNOW

Acute Angle

An angle whose measure is more than 0° and less than 90°.

2a. Using Right Triangles to Evaluate Trigonometric Functions

↪ EXAMPLE Using the right triangle, find the sine, cosine, and tangent of angle θ.

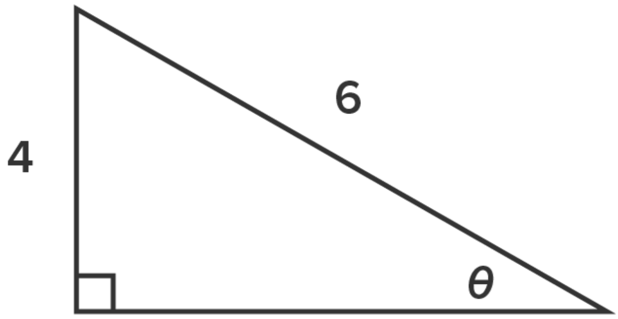


$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{9}{41} \qquad \cos\theta = \frac{adjacent}{hypotenuse} = \frac{40}{41} \qquad \tan\theta = \frac{opposite}{adjacent} = \frac{9}{40}$$

HINT

All right triangles satisfy the Pythagorean theorem. Therefore, if two sides are given, the Pythagorean theorem can be used to find the unknown side.

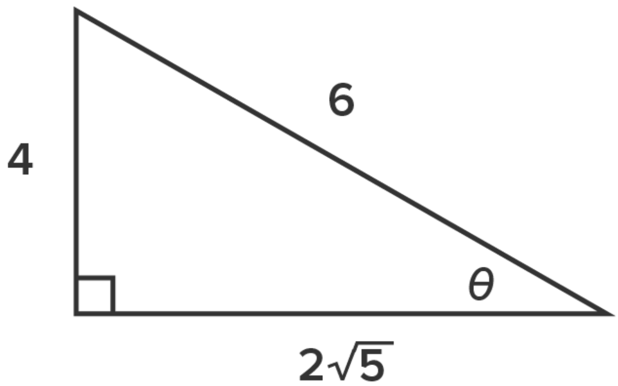
↪ EXAMPLE Using the right triangle, find the sine, cosine, and tangent of angle θ.



Let x = the adjacent side. To find x, the Pythagorean theorem is needed:

$$4^2 + x^2 = 6^2$$
$$16 + x^2 = 36$$
$$x^2 = 20$$
$$x = \sqrt{20} = 2\sqrt{5}$$

Now, here is the triangle with the unknown side filled in:



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{4}{6} = \frac{2}{3}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

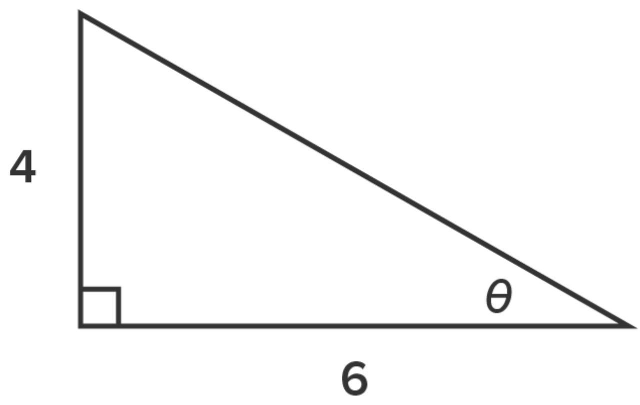
$$\tan\theta = \frac{opposite}{adjacent} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

HINT

In the result for tan θ in the above example, the expression $\frac{2}{\sqrt{5}}$ is not considered simplified since there is a radical in the denominator. The process by which an expression like this is simplified is called rationalizing the denominator. In short, to rationalize a fraction with $\sqrt{b}$ in the denominator, multiply by $\frac{\sqrt{b}}{\sqrt{b}}$.

TRY IT

Consider the right triangle below.



What is the length of the hypotenuse?

+

The length of the hypotenuse is $\sqrt{52} = 2\sqrt{13}$.

What is the sine of angle θ?

+

$$\sin\theta = \frac{4}{2\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

What is the cosine of angle θ?

+

$$\cos\theta = \frac{6}{2\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

What is the tangent of angle θ?

+

$$\tan\theta = \frac{4}{6} = \frac{2}{3}$$

2b. Evaluating Trigonometric Functions for Any Acute Angle

Most trigonometric functions require calculator use. On a typical scientific calculator, you will notice the “sin,” "cos," and "tan" buttons.

Before practicing this, be sure that your calculator is in "degree" mode, which should be the default setting.

> EXAMPLE Suppose we want the sine of the angle 40°. This is written $\sin(40^\circ)$.
>
> Using your calculator, press the "sin" key, then type in the number 40, then close the parenthesis. The result is a long decimal. Rounded to 4 places, we can say $\sin(40^\circ) \approx 0.6428$.

TRY IT

Use your calculator to answer the following questions.

What is the value of cos40°? +

$\cos 40^\circ \approx 0.7660$

What is the value of tan40°? +

$\tan 40^\circ \approx 0.8391$

2c. Evaluating Trigonometric Functions for 30°, 45°, and 60°

Most trigonometric functions produce values that are long decimals, which are irrational numbers. It turns out that the ratios corresponding to 30°, 45°, and 60° are concise enough that they are worth noting (and remembering!). This is why we sometimes refer to these angles as special angles.

WATCH

This video reviews how these values are derived.

Video Transcription

[MUSIC PLAYING] Greetings, and welcome back. What we're going to look at in this video is the trigonometric functions, but applied to specific angles, because they have such nice ratios that they're worth remembering and noting. So if you take a look at the equilateral triangle on the left, we notice that all sides are length 2. And remember that with an equilateral triangle, each angle is also the same. So all three of those angles are 60 even though the one at the top is not marked.

Because here's what we're going to do. Remember that we can only find trig ratios when a right triangle is involved. So I'm going to drop a height here in the middle, and what that's going to produce at the bottom is two right angles. And that means that each of these sides is 1, and that also means that 60-degree angle at the top is now two 30-degree angles.

And focusing on the triangle on the left-- I'm going to highlight that for you here-- if we look at the height, the hypotenuse, and the side whose length is 1, that height, by the Pythagorean theorem, is the square root of 3. So now we have three sides of a right triangle, and we can establish the ratios that are involved with 30 degrees and 60 degrees.

So starting with 30, the sign of 30 is the opposite, which is 1, over the hypotenuse, which is 2. So there we have that. Cosine of 30 is the adjacent side, which is the square root of 3, and hypotenuse is 2. Tangent of 30, that one's a little bit more tricky. Tangent of 30 is opposite over adjacent. So that's going to be 1 over square root of 3, which, in rationalized form, is square root of 3 divided by 3.

Now we could do the same thing for 60 degrees. Opposite the 60 is the square root of 3, hypotenuse is still 2. Cosine of 60, the adjacent side to the 60 degree angle, is 1, and the hypotenuse is still 2. And tangent of 60 is the opposite, which is the square root of 3 divided by the adjacent, which is 1. So tangent of 60 actually simplifies to the square root of 3.

Now, if we come over to the 45, 45, 90 triangle, the isosceles-- and just full point of reference, you've probably heard this word before. Isosceles means that two angles are the same, which, in turn, means that two sides are the same. And since this is a right triangle, that means the other two angles have to split the difference between 180 and 90. So that means they're each 45 degrees.

Now, if we apply our trig ratios to this triangle, if I take the sine of 45 degrees-- which, I'll use this one, the incredible disappearing triangle there. I'll use this one. So sine of 45 is 1 over root 2, opposite over hypotenuse, which, again, can be written in rationalized form. So that's the square root of 2 over 2. Cosine of 45, same thing. And then tangent of 45, this is probably the most surprising of them all. But tangent of 45 is opposite over adjacent, which is 1 over 1, which is 1. And that is how we establish the trig ratios for 30 degrees, 60 degrees, and 45 degrees.

[MUSIC PLAYING]

The exact values for all the special angles are as follows:

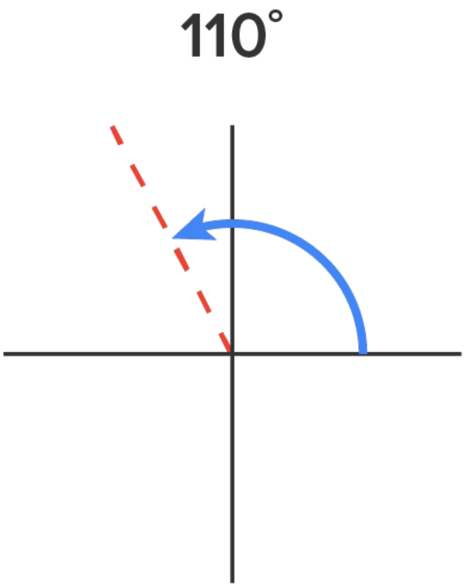
$\sin 30° = \frac{1}{2}$	$\sin 45° = \frac{\sqrt{2}}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 30° = \frac{\sqrt{3}}{2}$	$\cos 45° = \frac{\sqrt{2}}{2}$	$\cos 60° = \frac{1}{2}$
$\tan 30° = \frac{\sqrt{3}}{3}$	$\tan 45° = 1$	$\tan 60° = \sqrt{3}$

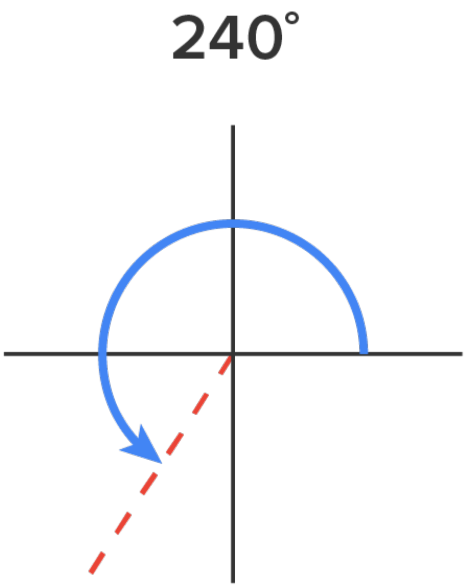
2d. Defining Non-Acute Angles

As it turns out, the work we have done with the trigonometric functions can be applied to any angle, not just acute angles. But how are non-acute angles handled?

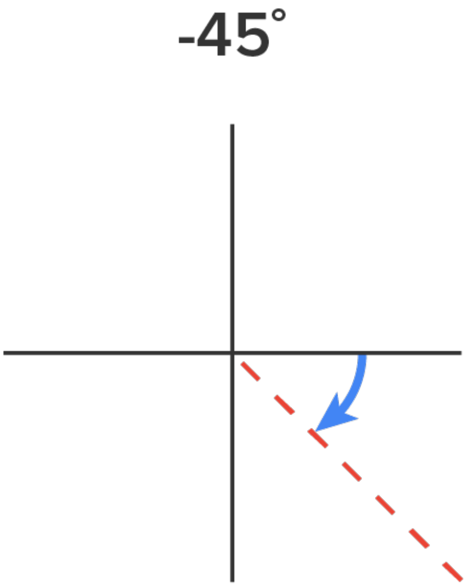
For reference, we draw the x-axis and y-axis. The angle is represented by starting at the positive x-axis (which we call the initial side of an angle), then drawing counterclockwise. To show where the angle stops, a side is created (dashed). This is called the terminal side of the angle.

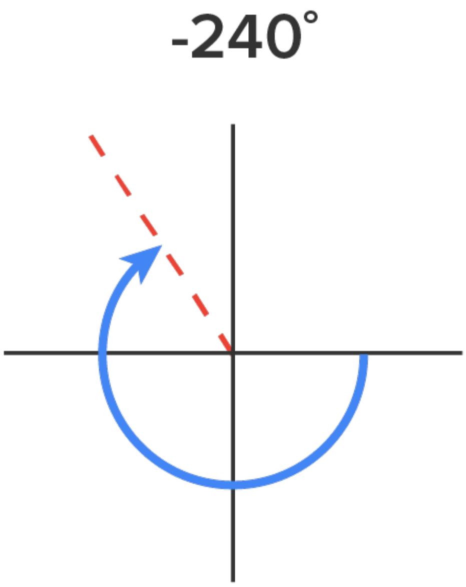
For example, here are what $110°$ and $240°$ look like:



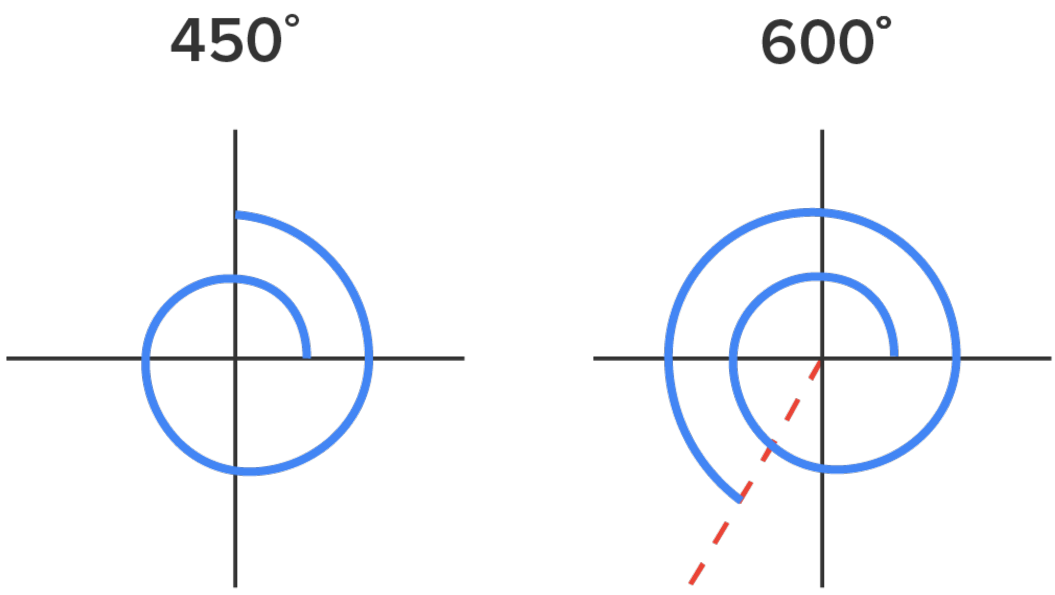


It follows that angles can also be measured clockwise. These angles are negative. Some examples:





Lastly, it is also possible to talk about angles larger than $360°$. These angles go through more than one full revolution.



Notice that the $450°$ angle above terminates at the same place that a $90°$ angle terminates. We call these angles coterminal. In general, coterminal angles have measures that differ by some multiple of $360°$.

TRY IT

Suppose you have the angle $110°$.

Give measures of two angles that are coterminal to this angle.

+

There are many answers, but the most straightforward are $470°$ and $830°$.

2e. The Unit Circle and Evaluating Trigonometric Functions for Any Angle

WATCH

This video shows how a unit circle, which is a circle with radius 1, can be connected with the trigonometric functions.

Video Transcription

[MUSIC PLAYING] Hi, and welcome back. What we're going to look at right now in what we're talking about, trigonometric functions, is its relationship to this thing called the unit circle. And the unit circle is just a circle with a radius of 1. So everything coming out from the center to any point that's on the circle has a length of 1.

So the big question is, what is the relationship between all of these points that are labeled on the circle and the angles that they correspond to? So just for example, we see that the point root 2 over 2, root 2 over 2 corresponds to 45 degrees. Why is that? So let's see. Just to get a snapshot, let's say we're in the first quadrant. So I'm going to take the point 0, 0 and go out to a point that's on the unit circle and call it x comma y. And I'm going to make a triangle out of it.

Now, the major thing to notice about these angles is they seem to be measured from the positive x-axis and going counterclockwise. So that's how we measure angles. We start at the positive x-axis. That's the initial side, and the terminal side is where we end up moving counterclockwise. And that's how we measure the angle. So this angle is theta.

Now, that means, in this case, that this side has length x and this side has length y. So looking at the three basic trigonometric functions, we know that the sine of theta is y over-- remember, this side is 1-- 1. So that's just y. So the y-coordinate of the point is the sine of the angle that corresponds to it.

Cosine of theta is the adjacent side divided by the hypotenuse, so that's just x. So that means, right there, that the actual coordinates tell us the cosine of the corresponding angle and the sine of the corresponding angle. Cool, OK, now let's look at tangent of theta. Tangent is opposite over adjacent, which is y over x. So that means that we could also easily find the tangent of the corresponding angle by just finding the ratio of the coordinates.

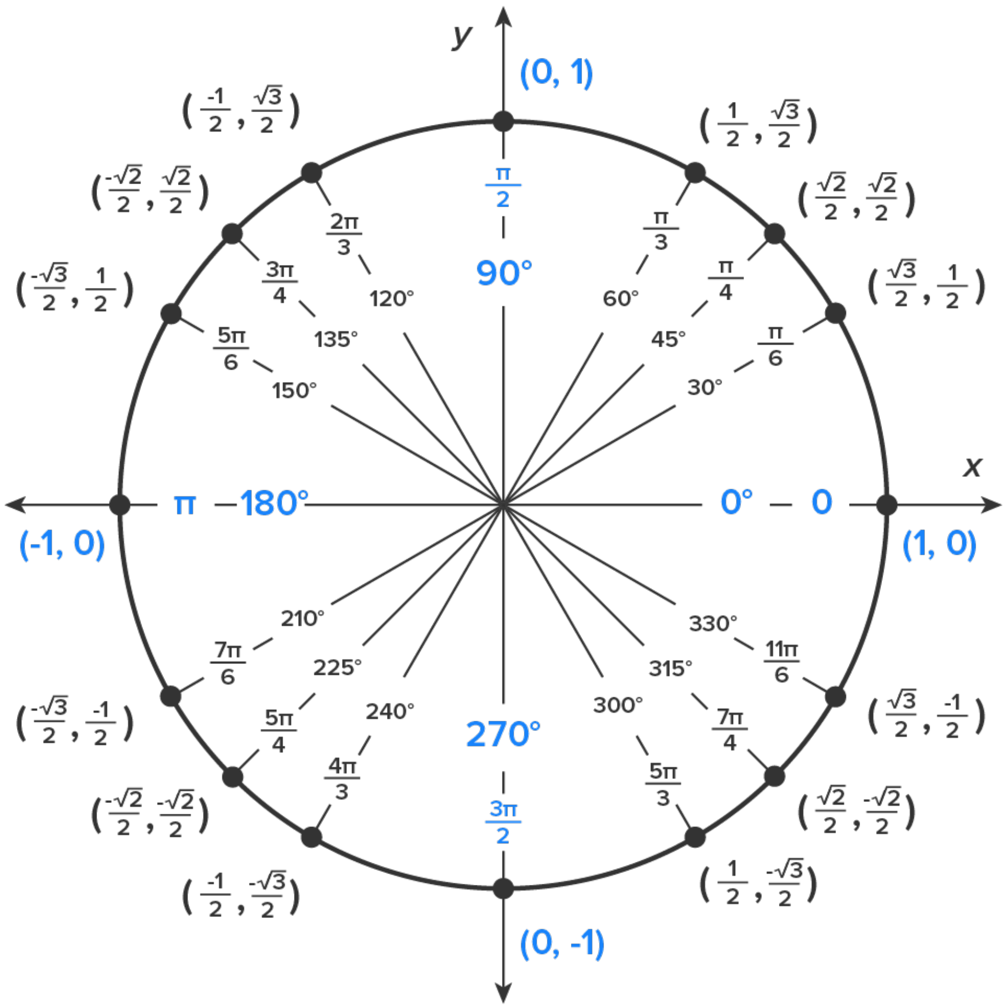
So let's just take a look at how we can apply this. So sine of 120 degrees. 120 degrees terminates at the point 1/2 comma root 3 over 2. Sine of 120 would be the y-coordinate of the 120-degree angle. So that is square root 3 over 2. And notice that when we get into bigger angles, that opens the door for the ratios to end up being negative. This is our way of defining sine, cosine, and tangent moving outside of a right triangle, OK?

So cosine of 135, the 135-degree angle terminates at negative root 2 over 2 comma root 2 over 2. Cosine is the x-coordinate, so that means that cosine of 135 is negative square root of 2 divided by 2. Tangent of 240 terminates here, at the point negative 1/2, negative square root 3 over 2, so we're going to have y over x. So negative square root 3 over 2 divided by negative 1 over 2 is, when we simplify it, the square root of 3. And sine of 90, remember, sine is the y-coordinate of the corresponding point. That is the point 0, 1. So sine of 90 degrees is equal to 1.

So this unit circle can be used to find any trig function, sine, cosine, tangent, or the other three that we haven't really spoken too much about so far, of any of those special angles. And how do you know it's a special angle? Basically, it's any multiple of 30 or any multiple of 45. That's when you know it's going to have a special ratio. And now with this added [INAUDIBLE] with this unit circle, we can also say it's either going to be positive or negative. So there is a complete picture, with sine cosine, and tangent of the special angles.

[MUSIC PLAYING]

For your reference, here is the unit circle. It will also be accessible on the course dashboard.



HINT

From the unit circle, you can see that $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$. This is a very useful identity that can be used to compute θ.

EXAMPLE Find the exact value of sin $180°$ by using the unit circle.

On the unit circle, $180°$ corresponds to the point (-1, 0). Since the y-coordinate of the point is 0, it follows that sin $180° = 0$.

EXAMPLE Find the exact value of tan $315°$ by using the unit circle.

On the unit circle, $315°$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Since $\tan\theta = \frac{y}{x}$, it follows that

$$\tan 315° = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

For negative angles and angles larger than $360°$, use a coterminal angle between $0°$ and $360°$ to evaluate the trigonometric function.

> ↪ EXAMPLE Find the exact value of $\cos 420°$.
>
> On the unit circle, $420°$ has the same terminal side as $60°$. This means that $\cos 420° = \cos 60°$. Since $60°$ corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\cos 420° = \cos 60° = \frac{1}{2}$.

> ↪ EXAMPLE Find the exact value of $\tan(-150°)$.
>
> First, realize that $-150°$ is coterminal with $-150° + 360° = 210°$. Thus, $\tan(-150°) = \tan 210°$. Since $210°$ corresponds to the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, it follows that $\tan 210° = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$.

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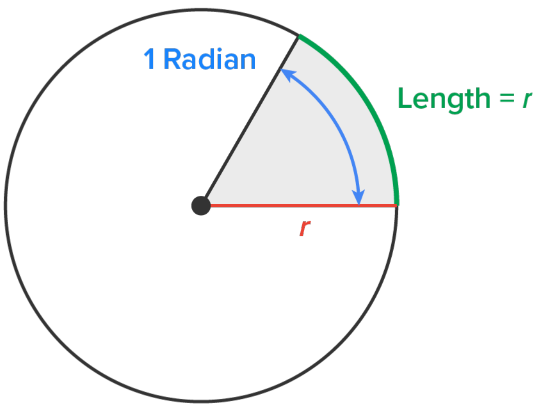
3. Radian Measure

Another way to measure angles is to use radians. One **radian** is defined as the central angle in the circle (see figure) so that the length of the circular arc is equal to the radius of the circle. Radians are used as a way of measuring angles because they represent a quantity, while degrees represent a scale.



TERM TO KNOW

Radian

The angle required to produce a circular arc whose length is equal to the radius. One radian is

$\frac{180}{\pi}$ degrees.

3a. Converting Between Degrees and Radians

Thinking about the previous figure, consider making one trip around the entire circle, which has length $2\pi r$ (circumference). Remembering that each radian contributes a length of *r* to the circle, it follows that one full trip around the circle is $\frac{2\pi r}{r} = 2\pi$ radians.

Also recall that one full trip around the circle is $360°$.

As a result, we see that $360° = 2\pi$ radians. Dividing both sides by 2, we can also say that $180° = \pi$ radians. Dividing both sides by π and 180 respectively, we have two rules to use when converting between radians and degrees:

	Formula	When To Use It
Divide by π:	$\left(\frac{180}{\pi}\right)° = 1 \textit{ radian}$	When given an angle measured in radians, multiply by $\left(\frac{180}{\pi}\right)°$ to get the angle measurement in degrees.
Divide by 180:	$1° = \frac{\pi}{180} \textit{ radians}$	When given an angle measured in degrees, multiply by $\frac{\pi}{180}$ to get the angle measurement in radians.

FORMULA

Conversions Between Degrees and Radians

$$1 \textit{ degree} = \frac{\pi}{180} \textit{ radians}$$

$$1 \textit{ radian} = \frac{180}{\pi} \textit{ degrees}$$

EXAMPLE Convert $45°$ to radians.

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ radians}$$

EXAMPLE Convert 2.1 radians to degrees.

$$2.1 \cdot \frac{180}{\pi} \approx 120.32°$$

TRY IT

Convert the following from degrees to radians or vice versa.

Convert 240 degrees to radians. Leave your answer in terms of pi. +

$$240 \cdot \frac{\pi}{180} = \frac{240\pi}{180} = \frac{4\pi}{3} \text{ radians}$$

Convert 5.7 radians to degrees. Round your final answer to the nearest tenth.

+

$$5.7 \cdot \frac{180}{\pi} \approx 326.6°$$

HINT

When expressing angles in radians, it is customary to leave the angle in terms of π rather than approximate it. These angles are usually multiples of $15°$.

Here is a list of some common angles you will encounter.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

3b. Evaluating Trigonometric Functions Using Radian Measure

Remember that radian measure is another way to represent an angle, so it may be helpful to convert the angle to degrees first before evaluating. (Recall the special values you were given earlier and the values from the unit circle).

EXAMPLE

$$\cos\left(\frac{\pi}{4}\right) = \cos(45°) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{2\pi}{3}\right) = \sin(120°) = \frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{5\pi}{6}\right) = \tan(150°) = -\frac{\sqrt{3}}{3}$$

$$\cos\left(\frac{\pi}{2}\right) = \cos(90°) = 0$$

4. Finding an Input for a Known Output (Solving Trigonometric Equations)

An equation where the angle is unknown and the ratio is known is called a**trigonometric equation**.

Examples:

$$\cos\theta = \frac{1}{2} \qquad \tan x = 3 \qquad \sin A = 0.4$$

Based on what we know from the unit circle, there are infinite solutions to a trigonometric equation if we include all coterminal angles. In most situations, we want to find all solutions in the first revolution of the circle, namely the interval $[0, 360°)$ or in radians, $[0, 2\pi)$.

For now, we will focus on known ratios that correspond to special angles. Other angles/ratios will be investigated in Unit 3.

EXAMPLE Find all solutions to $\sin x = 0$ on the interval $[0, 2\pi)$.

From the unit circle, the y-coordinate is the sine of the angle. The two points with a y-coordinate of 0

are $0°$ and $180°$, or in radians, 0 and π.

Thus, the solutions to the equation are $x = 0$ and $x = \pi$.

↱ EXAMPLE Solve the equation $\tan x + 1 = 0$ on the interval $[0, 2\pi)$.

First, isolate the tan *x* term on one side by subtracting 1 from both sides: $\tan x = -1$.

We seek all angles *x* such that the ratio is -1. Consulting the unit circle, this means that $x = 135°$ and $315°$. In radians, that means $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

TRY IT

Solve $2\sin x + 1 = 0$ on the interval $[0, 2\pi)$ by completing the following steps.

First, isolate sin *x* on one side. +

$\sin x = -\frac{1}{2}$

Now consult the unit circle. +

$x = 210°, x = 330°$

Convert to radians. +

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

TERM TO KNOW

Trigonometric Equation

An equation in which trigonometric functions are involved and the angle is unknown.

5. Basic Trigonometric Graphs

5a. The Graph of *y* = sin *x*

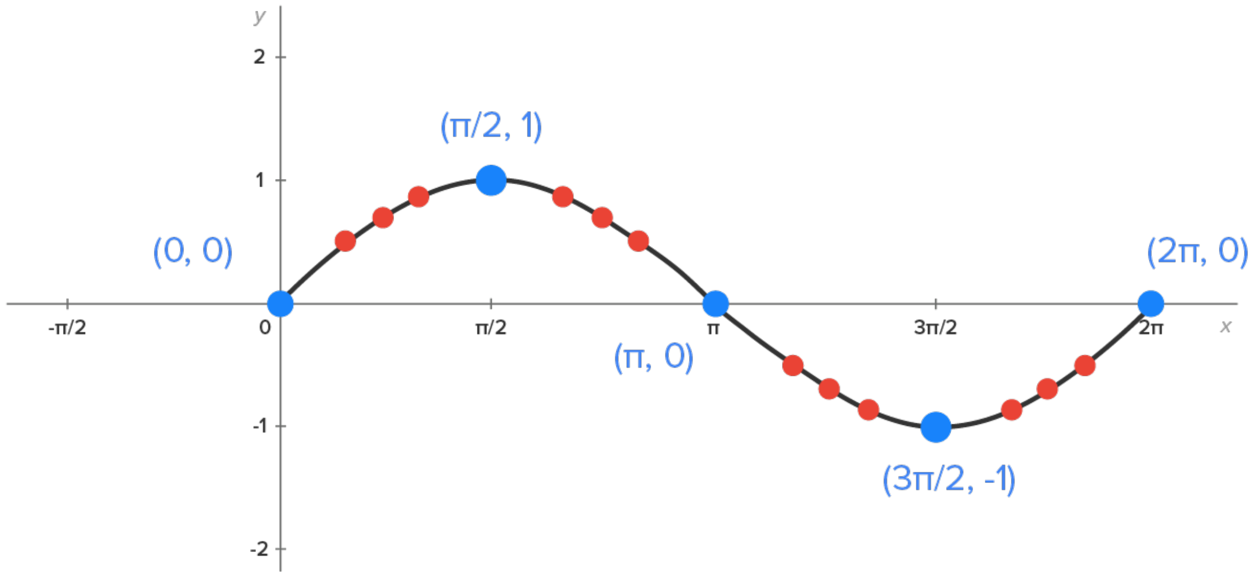
Consider the function *y* = sin *x*.

From the unit circle, here is a table of values that shows how the angles and the ratios are related.

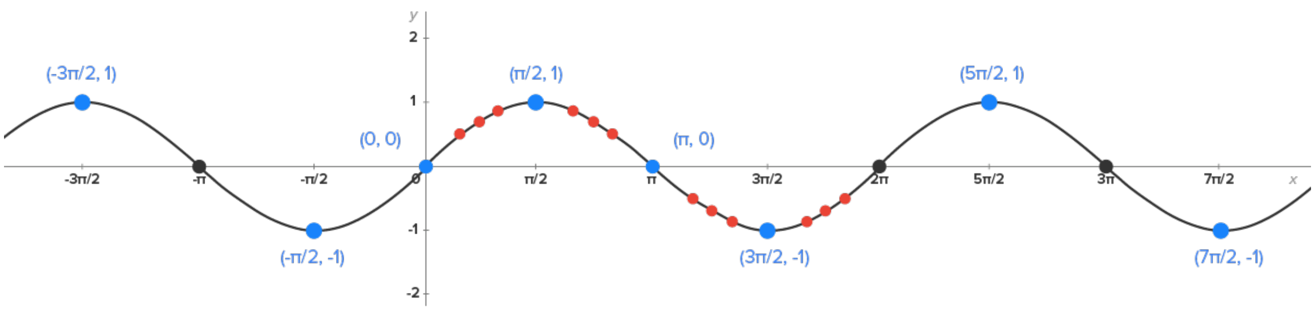
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
***y* = sinx**	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

x	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Here is the graph, limited to the points that are listed above.



Remember also that **coterminal angles** produce the same trigonometric values. Thus, we can say that $\sin(x \pm 2\pi) = \sin x$. Because of this relationship, we say that the sine function has a period of 2π, meaning that the graph repeats itself every 2π units. As a result, the complete graph of $y = \sin x$ is as follows:



The graph continues in this pattern indefinitely. Since there are no breaks or holes in the graph, the domain of $f(x) = \sin x$ is the set of all real numbers, also written in interval notation as $(-\infty, \infty)$. The range of this function is $[-1, 1]$ since the graph goes no higher than $y = 1$ and no lower than $y = -1$.

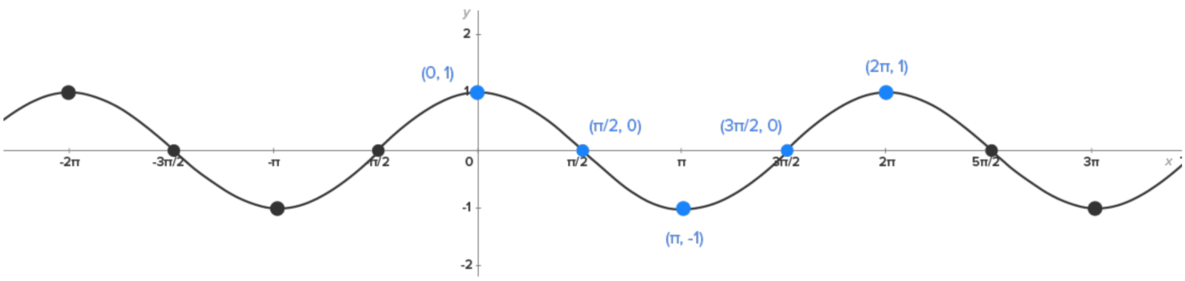
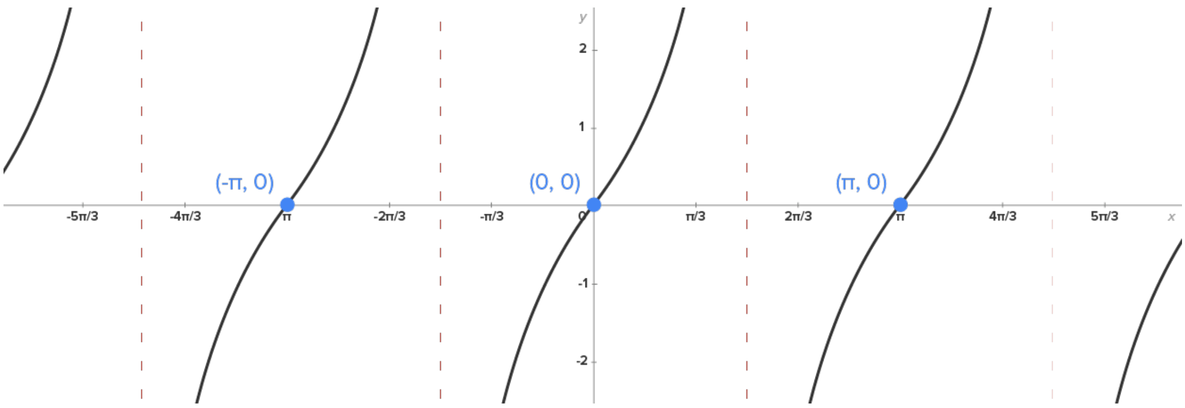
TERM TO KNOW

Coterminal Angles

Angles that have the same terminal side.

5b. The Graphs of $y = \cos x$ and $y = \tan x$

By following a similar process as above, we can obtain the graphs of $y = \cos x$ and $y = \tan x$:

$y = \cos x$	
Graph	
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$
Period	2π
$y = \tan x$	
Graph	
Domain	All reals except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$
Range	$(-\infty, \infty)$
Period	π

HINT

Remember that $\tan x = \frac{\sin x}{\cos x}$. Therefore, $\tan x$ is undefined whenever $\cos x = 0$, which is when x is an odd multiple of $\frac{\pi}{2}$.

6. Frequently Used Trigonometric Identities

Recall that an identity is an equation that is true for all possible values of the variable.

> ↪ EXAMPLE $2(x+3)=2x+6$ is an identity. No matter what is substituted for x, both sides of the equation will have the same value.

The following are the most commonly used trigonometric identities.

Reciprocal Identities	Tangent/Cotangent Identities
Secant: $\sec\theta=\dfrac{1}{\cos\theta}$ Cosecant: $\csc\theta=\dfrac{1}{\sin\theta}$ Cotangent: $\cot\theta=\dfrac{1}{\tan\theta}$	$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$
Cofunction Identities	**Pythagorean Identities**
$\sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta$ $\cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta$ $\tan\left(\dfrac{\pi}{2}-\theta\right)=\cot\theta$ $\cot\left(\dfrac{\pi}{2}-\theta\right)=\tan\theta$ $\sec\left(\dfrac{\pi}{2}-\theta\right)=\csc\theta$ $\csc\left(\dfrac{\pi}{2}-\theta\right)=\sec\theta$ (Note: If θ is measured in degrees, replace $\dfrac{\pi}{2}$ with $90°$.)	$\sin^2\theta+\cos^2\theta=1$ $1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\csc^2\theta$ (Note: The notation $\sin^2\theta$ means $(\sin\theta)^2$.)
Sum of Angles Identities	**Double-Angle Identities**
$\sin(A\pm B)=\sin A\cos B\pm\sin B\cos A$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$ $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$	$\sin 2x=2\sin x\cos x$ $\cos 2x=\cos^2 x-\sin^2 x$ $\cos 2x=2\cos^2 x-1$ $\cos 2x=1-2\sin^2 x$ $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$
Power-Reducing Identities	
$\cos^2 x=\dfrac{1+\cos 2x}{2}$ $\sin^2 x=\dfrac{1-\cos 2x}{2}$	

SUMMARY

In this lesson, you learned about trigonometric functions, which assign an angle to a real number; in other words, they take an angle as input and return a real number as output. These real numbers stem from the ratio of sides of right triangles. You learned about **the three basic trigonometric functions**: the sine function (sinθ), the cosine function (cosθ), and the tangent function (tanθ).

You learned that there are many representations that are useful in **evaluating trigonometric functions**, such as **using right triangles**. You also explored **evaluating trigonometric functions for any acute angle** (an angle whose measure is more than 0° and less than 90°), **evaluating trigonometric functions for 30°, 45°, and 60°** (special angles with concise corresponding ratios),**defining non-acute angles, the unit circle**, and **evaluating trigonometric functions for any angle**.

You learned about another way to measure angles by using **radian measure**, noting that one radian is defined as the central angle in the circle so that the length of the circular arc is equal to the radius of the circle. Radians are used as a way of measuring angles because they represent a quantity, while degrees represent a scale. Using this knowledge, you explored **converting between degrees and radians** and **evaluating trigonometric functions using radian measure**.

You investigated **finding an input for a known output, or solving trigonometric equations**, which are equations where the angle is unknown and the ratio is known, and explored **basic trigonometric graphs**, including **the graph of** $y = \sin x$, $y = \cos x$ **and** $y = \tan x$. Lastly, you covered a selection of **frequently used trigonometric identities** that will be utilized later in this course.

TERMS TO KNOW

Acute Angle

An angle whose measure is more than 0° and less than 90°.

Coterminal Angles

Angles that have the same terminal side.

Radian

The angle required to produce a circular arc whose length is equal to the radius. One radian is $\frac{180}{\pi}$ degrees.

Trigonometric Equation

An equation in which trigonometric functions are involved and the angle is unknown.

Trigonometric Function

Uses an angle as an input and returns a ratio as the output.

FORMULAS TO KNOW

Conversions Between Degrees and Radians

$$1 \; degree = \frac{\pi}{180} \; radians$$

$$1 \; radian = \frac{180}{\pi} \; degrees$$

Definitions of the Sine, Cosine, and Tangent Functions

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

Exponential and Logarithmic Functions

by Sophia

WHAT'S COVERED

In this lesson, you will review the basics of exponential and logarithmic functions and their properties. Specifically, this lesson will cover:

1. Exponential Functions
2. Logarithmic Functions
   a. Evaluating Logarithms
   b. Graphs of Logarithmic Functions
   c. Properties of Logarithms
   d. Expanding Logarithmic Expressions
   e. Condensing a Logarithmic Expression Into a Single Logarithm

1. Exponential Functions

Consider the function $f(x) = 2^x$ with some input-output pairs:

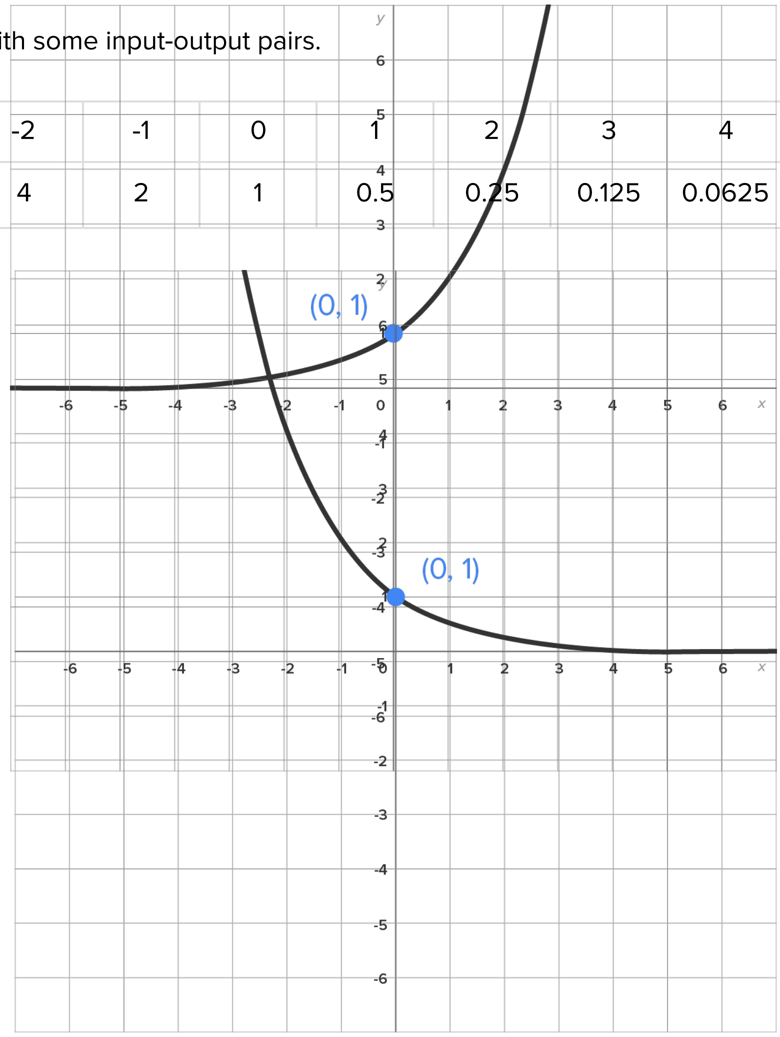
x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	0.0625	0.125	0.25	0.5	1	2	4	8	16

This leads us to the graph on the right:

- The portion of the graph to the right of the y-axis increases sharply.
- The portion of the graph to the left of the y-axis decreases gradually toward $y = 0$, but it never quite gets there.
- This is because there is no value of x for which $2^x = 0$.

Let's now look at the graph of $f(x) = (0.5)^x$ with some input-output pairs.

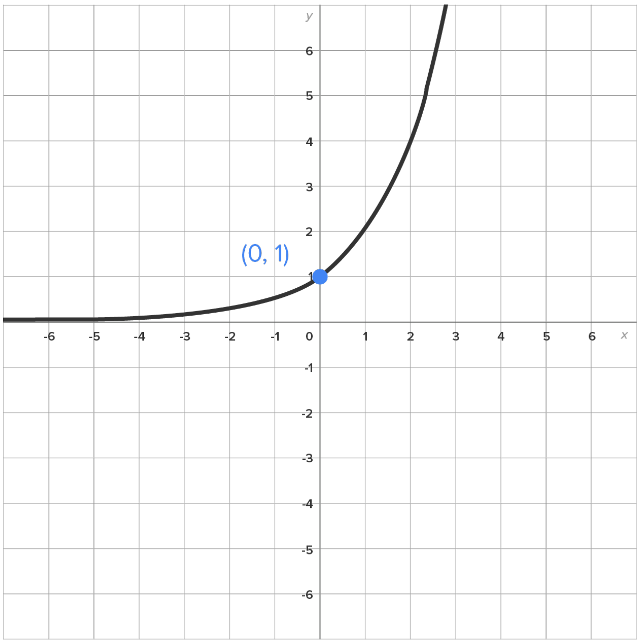
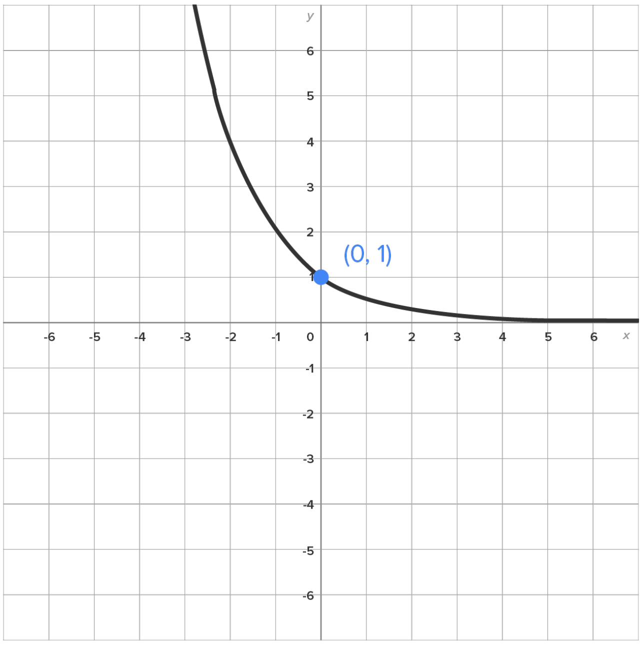
x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = (0.5)^x$	16	8	4	2	1	0.5	0.25	0.125	0.0625



This leads us to the graph on the right:

- The portion of the graph to the left of the y-axis increases sharply.
- The portion of the graph to the right of the y-axis decreases gradually toward $y = 0$, but it never quite gets there.
- This is because there is no value of x for which $(0.5)^x = 0$.

In general, define the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

$f(x) = a^x$, *where* $a > 1$	$f(x) = a^x$, *where* $0 < a < 1$
 The graph is increasing at every point. The domain is $(-\infty, \infty)$. The range is $(0, \infty)$. The graph contains the point (0, 1).	 The graph is decreasing at every point. The domain is $(-\infty, \infty)$. The range is $(0, \infty)$. The graph contains the point (0, 1).

There is a horizontal asymptote at $y = 0$.	There is a horizontal asymptote at $y = 0$.

HINT

Exponential functions can only be defined for $a > 0$ and $a \neq 1$ for the following reasons:

- If $a < 0$, there would be infinite values that are undefined due to fractional exponents. This would not be a useful function.
- If $a = 0$, the function is undefined when $x \leq 0$ and equal to 0 when $x > 0$, which is not an exponential function.
- If $a = 1$, then $f(x) = 1$ for all values of *x*, which is simply a horizontal line, which is not an exponential function.

A commonly used base is the number e, which is called the natural base, where $e \approx 2.718281828...$ (this pattern does not repeat). Since $e > 1$, its graph is the increasing exponential graph as seen above.

2. Logarithmic Functions

2a. Evaluating Logarithms

Recall that the input of an exponential function is the exponent. The output of the exponential function is called the **power**, the result of raising a number to an exponent.

With a **logarithmic function**, the input is the power and the output is the exponent. In other words, a logarithm is the exponent *y* needed to complete the equation $a^y = x$ for given values of *a* and *x*.

That said, to find the value of *y*, we can write $f(x) = \log_a x$ (logarithm with “base *a*” of *x*).

FORMULA

Logarithm Definition

$y = \log_a x$ if $a^y = x$ where $a > 0$ and $a \neq 1$

EXAMPLE

Find the value of $\log_2 8$.

$y = \log_2 8$	Start with the original logarithmic function.
$2^y = 8$	Rewrite in exponential form.
$8 = 2^3$	Write 8 as a power of 2.
$2^y = 2^3$	Equate the exponential expressions.
$y = 3$	Solve for *y*.

Thus, $\log_2 8 = 3$.

EXAMPLE

Find the value of $\log_{10} 0.01$.

$y = \log_{10} 0.01$	Start with the original logarithmic function.
$10^y = 0.01$	Rewrite in exponential form.
$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$	Write 0.01 as a power of 10.
$10^y = 10^{-2}$	Equate the two expressions.
$y = -2$	Solve for *y*.

Thus, $\log_{10} 0.01 = -2$.

HINT

There are two special logarithms that will be handy to know:

- $\log_a 1 = 0$ (We know this because $a^0 = 1$ for any value of *a*.)
- $\log_a a = 1$ (We know this because $a^1 = a$ for any value of *a*.)

Notation Used for Logarithms of Special Bases	
Base 10	$\log_{10} x$ is written $\log x$. No base written means the base is 10.
Base *e*	$\log_e x$ is written $\ln x$, which means the natural logarithm of *x*. You may remember that e is called the natural base, where $e \approx 2.718281828...$ (this pattern does not repeat).

TERMS TO KNOW

A Power

The result of raising a number to an exponent. For example, $2^5 = 32$, and we say that 32 is the 5th power of 2.

Logarithmic Function

$f(x) = \log_a x$ uses the power as its input and returns the exponent required to produce that power when the base is a.

2b. Graphs of Logarithmic Functions

Earlier, we graphed the function $y = 2^x$ by using the following table.

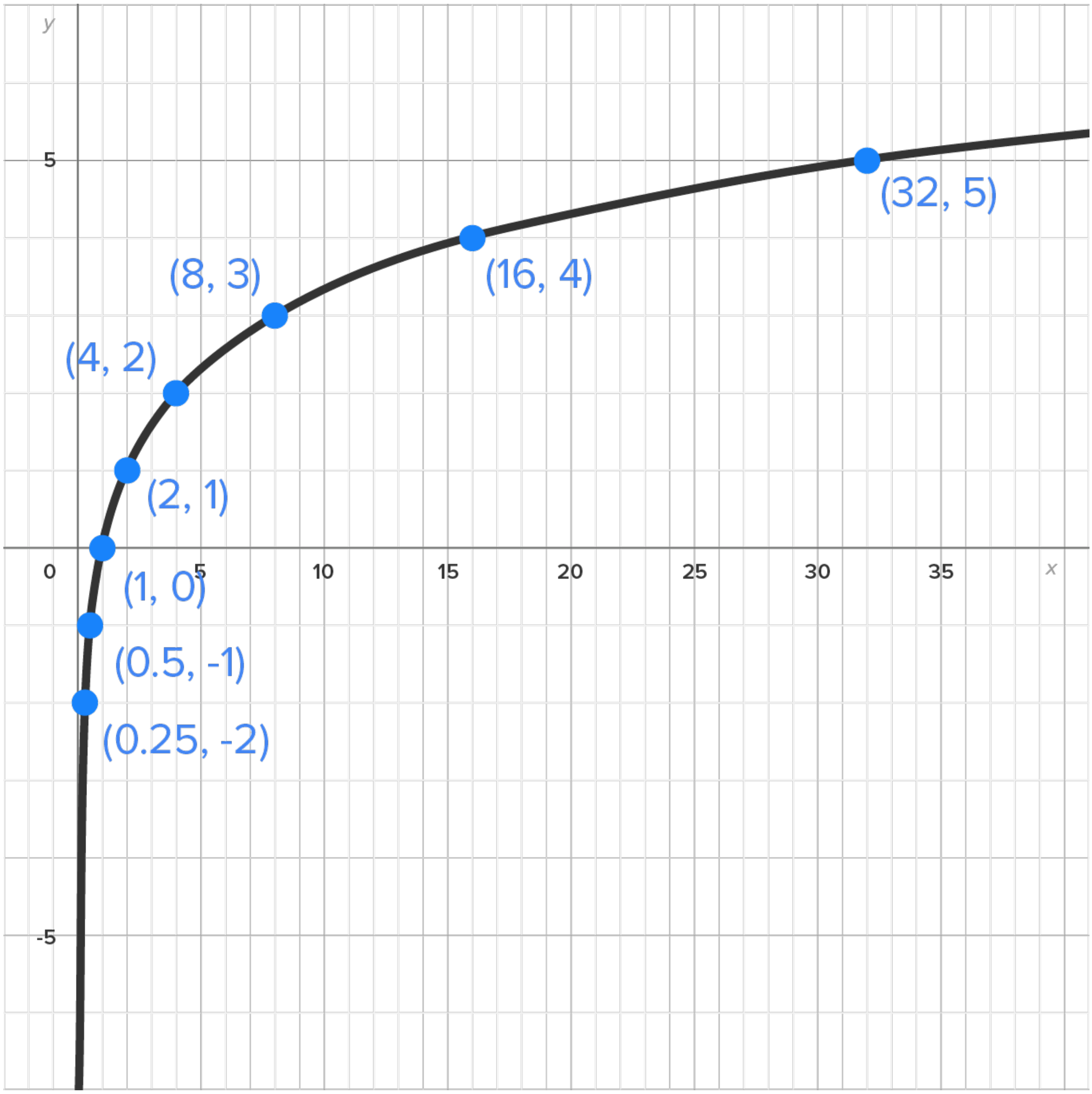
x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	0.0625	0.125	0.25	0.5	1	2	4	8	16

The logarithmic function $y = \log_2 x$ would interchange these values:

x	0.0625	0.125	0.25	0.5	1	2	4	8	16
$y = \log_2 x$	-4	-3	-2	-1	0	1	2	3	4

For example, $\log_2 16 = 4$ since $2^4 = 16$ and $\log_2 1 = 0$ since $2^0 = 1$.

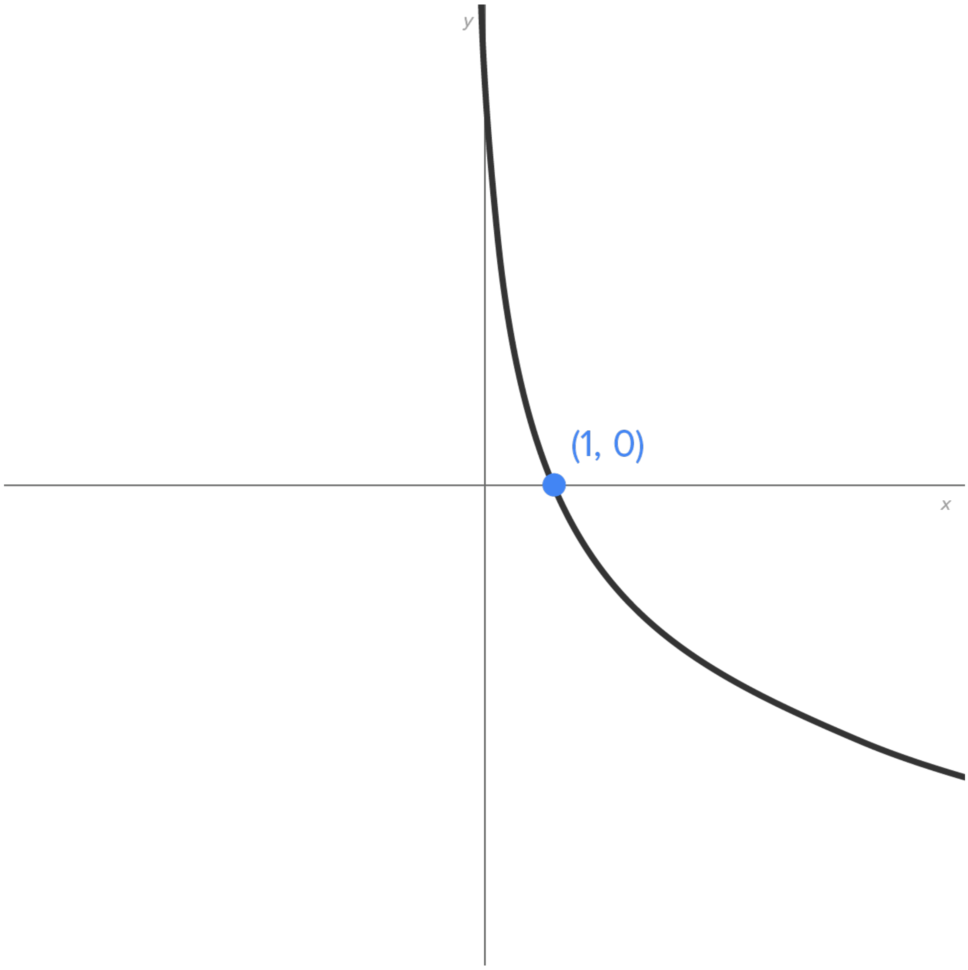
Here is the graph of $y = \log_2 x$ based on these points:



The graph has a vertical asymptote at $x = 0$.

In general, this is what the graph of $y = \log_a x$ looks like when $a > 1$.

When $0 < a < 1$, the graph has this general shape:



Properties of the graph of $y = \log_a x$:

- The domain is $x > 0$.
- The range is all real numbers.
- There is a vertical asymptote at $x = 0$.
- If $a > 1$, the graph is increasing, and if $0 < a < 1$, the graph is decreasing.

2c. Properties of Logarithms

You may recall the following properties of exponents:

$a^x \cdot a^y = a^{x+y}$	Multiply Exponential Expressions, Add Exponents
$\frac{a^x}{a^y} = a^{x-y}$	Divide Exponential Expressions, Subtract Exponents
$(a^x)^y = a^{xy}$	Raise an Exponential Expression to a Power, Multiply the Exponents

Now, remember that a logarithm is an exponent. Thus, the logarithm properties tell us what happens to the exponents when expressions are multiplied, divided, and raised to a power.

FORMULA

Product Property
$\log_a(xy) = \log_a x + \log_a y$

Quotient Property

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Power Property

$$\log_a(x^y) = y \cdot \log_a x$$

These properties are used to rewrite logarithmic expressions in two ways:

- Expand a single logarithm as a sum, difference, or multiple of logarithms.
- Write an expanded logarithmic expression as a single logarithm.

2d. Expanding Logarithmic Expressions

There is a process that you can follow to expand logarithmic expressions:

1. Apply product/quotient property first to “break up” the expression into a sum/difference.
2. Apply power property where relevant.

↪ EXAMPLE Use logarithm properties to expand the expression $\ln\left(\frac{2x}{y}\right)$.

$\ln\left(\frac{2x}{y}\right)$	Start with the original expression.
$\ln(2x) - \ln y$	$\frac{2x}{y}$ is a quotient; apply the quotient property.
$\ln 2 + \ln x - \ln y$	$2x$ is a product; apply the product property.

The expanded form of $\ln\left(\frac{2x}{y}\right)$ is $\ln 2 + \ln x - \ln y$.

↪ EXAMPLE Use logarithm properties to expand the expression $\log(x^2y^4)$.

$\log(x^2y^4)$	Start with the original expression.
$\log(x^2) + \log(y^4)$	x^2y^4 is a product; apply the product property.
$2\log x + 4\log y$	Apply the power property on each logarithm.

The expanded form of $\log(x^2y^4)$ is $2\log x + 4\log y$.

TRY IT

Consider the expression $\log_4\left(\frac{2x}{y^3}\right)$.

Use logarithm properties to expand this expression. +

$\log_4\left(\frac{2x}{y^3}\right)$	Start with the original expression.

$\log_4(2x) - \log_4(y^3)$	$\frac{2x}{y^3}$ is a quotient; apply the quotient property.
$\log_4 2 + \log_4 x - \log_4(y^3)$	$2x$ is a product; apply the product property.
$\log_4 2 + \log_4 x - 3\log_4 y$	Apply the power property.

The expanded form of $\log_4\left(\frac{2x}{y^3}\right)$ is $\log_4 2 + \log_4 x - 3\log_4 y$.

2e. Condensing a Logarithmic Expression Into a Single Logarithm

To condense a logarithmic expression into a single logarithm, apply the properties as we did when expanding an expression, but in reverse. This means:

1. Reverse the power property first for any expressions: $y \cdot \log_a x = \log_a(x^y)$
2. Reverse the sum/difference properties: $\log_a x + \log_a y = \log_a(xy)$ or $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

EXAMPLE Use logarithm properties to write $3\log_4 x + \log_4 5 - 2\log_4 z$ as a single logarithm.

$3\log_4 x + \log_4 5 - 2\log_4 z$	Start with the original expression.
$\log_4 x^3 + \log_4 5 - \log_4 z^2$	Reverse the power property.
$\log_4(5x^3) - \log_4 z^2$	Reverse the product property.
$\log_4\left(\frac{5x^3}{z^2}\right)$	Reverse the quotient property.

The condensed form of $3\log_4 x + \log_4 5 - 2\log_4 z$ is $\log_4\left(\frac{5x^3}{z^2}\right)$.

TRY IT

Consider the expression $2\ln x - 3\ln y + 4\ln(z+1)$.

Write this expression as a single logarithm. +

$\ln\left[\frac{x^2(z+1)^4}{y^3}\right]$

SUMMARY

In this lesson, to add to the library of functions, you explored **exponential functions** and **logarithmic functions** and their **properties**. You learned how to **evaluate logarithms** by rewriting logarithmic functions in exponential form and also explored **graphs of logarithmic functions**. You also learned how to use properties of logarithms to **expand logarithmic expressions**. Lastly, you learned that to **condense a logarithmic expression into a single logarithm**, you need to apply the properties as you did when expanding an expression, but in reverse.

TERMS TO KNOW

A Power

The result of raising a number to an exponent. For example, $2^5 = 32$, and we say that 32 is the 5th power of 2.

Logarithmic Function

$f(x) = \log_a x$ uses the power as its input and returns the exponent required to produce that power when the base is *a*.

FORMULAS TO KNOW

Logarithm Definition

$y = \log_a x$ if $a^y = x$ where $a > 0$ and $a \neq 1$.

Power Property

$\log_a(x^y) = y \cdot \log_a x$

Product Property

$\log_a(xy) = \log_a x + \log_a y$

Quotient Property

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Terms to Know

A Power

The result of raising a number to an exponent. For example, $2^5 = 32$, and we say that 32 is the 5th power of 2.

Absolute Value

The distance that a number is from 0 on the number line.

Acute Angle

An angle whose measure is more than $0°$ and less than $90°$.

Composition of Functions

Written $(f \circ g)(x)$, it is a function that is obtained by substituting one function into another function.

Coterminal Angles

Angles that have the same terminal side.

Difference Quotient

An expression that represents the average rate of change between two points on a curve between input values x and $x + h$.

Distance

The length of a line segment between two points.

Function

A correspondence between a set of inputs (x) and a set of outputs (y) such that each input corresponds to at most one output.

Greatest Integer Function

Returns the greatest integer that is less than or equal to the input value.

Logarithmic Function

$f(x) = \log_a x$ uses the power as its input and returns the exponent required to produce that power when the base is a.

Piecewise Function

Assigns an input to an output, but the rule used to determine the output depends on the value of the input.

Radian

The angle required to produce a circular arc whose length is equal to the radius. One radian is $\frac{180}{\pi}$ degrees.

Restricted Domain

Part of, but not the entire, domain of a function.

Slope

The ratio of the change in *y* to the change in *x*; measure of the steepness of a line.

Trigonometric Equation

An equation in which trigonometric functions are involved and the angle is unknown.

Trigonometric Function

Uses an angle as an input and returns a ratio as the output.

Vertical Compression

A translation that makes all y-values of a graph smaller in magnitude, pulling a graph toward the x-axis. This is represented by $y = a \cdot f(x)$, where $|a| < 1$.

Vertical Stretch

A translation that makes all y-values of a graph larger in magnitude, pulling a graph toward the y-axis. This is represented by $y = a \cdot f(x)$, where $|a| > 1$.

Formulas to Know

Conversions Between Degrees and Radians

$$1\ \textit{degree} = \frac{\pi}{180}\ \textit{radians}$$

$$1\ \textit{radian} = \frac{180}{\pi}\ \textit{degrees}$$

Definitions of the Sine, Cosine, and Tangent Functions

$$\sin\theta = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos\theta = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan\theta = \frac{\textit{opposite}}{\textit{adjacent}}$$

Difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

Distance in the xy-Plane

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Distance on a Number Line

$dist(a,b)=|b-a|$

Logarithm Definition

$y=\log_a x$ if $a^y=x$ where $a>0$ and $a\neq 1$.

Point-Slope Form

$y-y_1=m(x-x_1)$

Power Property

$\log_a(x^y)=y\cdot\log_a x$

Product Property

$\log_a(xy)=\log_a x+\log_a y$

Quotient Property

$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$

Radius of a Circle

$r=\sqrt{(x-h)^2+(y-k)^2}$

Where: (h, k) is the center and (x, y) is a point on the circle.

Slope

$m=\frac{\text{rise}}{\text{run}}=\frac{\text{change in } y}{\text{change in } x}=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$

Slope-Intercept Form

$y=mx+b$

Standard Form Equation of a Circle

$(x-h)^2+(y-k)^2=r^2$

Where: (h, k) is the center and r is the radius.

The Piecewise Greatest Integer Function

$$\lfloor x\rfloor=\begin{cases} x & \textit{if } x \textit{ is an integer} \\ \textit{the closest integer less than } x & \textit{if } x \textit{ is NOT an integer}\end{cases}$$