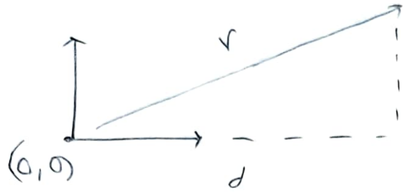
HW2 | Spencer Freeman | AOE 5784 | 10/2/24

2-11)



$r = \sqrt{d^2 + y^2}$

$z = r + w$

$w \sim N(0, \sigma^2)$

① likelihood function of Y

$$\Lambda_{Z^k}(y) = p(Z^k \mid y), \qquad Z^k = \{z_1, z_2, \dots z_k\}$$

$$z = \sqrt{d^2+y^2} + w, \qquad w = f(z) = z - \sqrt{d^2+y^2}$$

$$P_z(\eta) = p_w(f(\eta))\,\Big|\det\Big[\underbrace{\frac{df}{dz}\Big|_\eta}_{1}\Big]\Big| = p_w(z - \sqrt{d^2+y^2})$$

$$\boxed{p(Z^k \mid y) = \frac{1}{(2\pi)^{k/2}\sigma^k} \exp\Big\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{k} \Big(z_j - \sqrt{d_j^2 + y^2}\Big)^2 \Big\}}$$

② find the CRLB for estimating Y
(Cramer-Rao Lower Bound)

$$CRLB = J^{-1}, \quad J: \text{Fisher Information matrix}$$

$$J \triangleq -E\Big[\frac{d^2 \ln(p(Z^k \mid y))}{dy^2}\Big|_{y=y_{true}} \Big]$$

$$\ln(p(Z^k \mid y)) = \ln\Big(\frac{1}{(2\pi)^{k/2}\sigma^k}\Big) + \Big(\frac{-1}{2\sigma^2}\Big)\sum_{j=1}^{k}\Big(z_j - \sqrt{d_j^2+y^2}\Big)^2$$

$$\frac{d^2}{dy^2}\left[\ln(p(z^k|y))\right] = \frac{d}{dy}\left[\frac{-1}{2\sigma^2}\frac{d}{dy}\left[\sum_{j=1}^{k}\left(z_j - \sqrt{d_j^2+y^2}\right)^2\right]\right]$$

$$= \frac{d}{dy}\left[\frac{-1}{2\sigma^2}\sum_{j=1}^{k} 2y\left(1 - \frac{z_j}{\sqrt{d_j^2+y^2}}\right)\right]$$

$$= \frac{1}{\sigma^2}\sum_{j=1}^{k}\left(\left(\frac{z_j}{\sqrt{d_j^2+y^2}} - 1\right) + y\left(\frac{-2y z_j (1/2)}{(d_j^2+y^2)^{3/2}}\right)\right)$$

$$= \frac{1}{\sigma^2}\sum_{j=1}^{K}\left(\frac{d_j^2 z_j}{(d_j^2+y^2)^{3/2}} - 1\right)$$

$$J = -E\left[\frac{1}{\sigma^2}\sum_{j=1}^{K}\left(\frac{d_j^2 z_j}{(d_j^2+y^2)^{3/2}} - 1\right)\right] = -\frac{1}{\sigma^2}\sum_{j=1}^{K}\left(\frac{d_j^2}{(d_j^2+y^2)^{3/2}}\overset{\sqrt{d_j^2+y^2}}{E[z_j]} - \overset{1}{E[1]}\right)$$

$$J = \frac{-1}{\sigma^2}\sum_{j=1}^{K}\left(\frac{d_j^2(d_j^2+y^2)^{1/2}}{(d_j^2+y^2)^{3/2}} - 1\right) = \frac{-1}{\sigma^2}\sum_{j=1}^{K}\left(\frac{d_j^2}{d_j^2+y^2} - 1\right)$$

$$\boxed{CRLB = J^{-1} = \frac{-\sigma^2}{\sum_{j=1}^{K}\left(\frac{d_j^2}{d_j^2+y^2} - 1\right)}}$$

(3) $\sigma = 10^2$, $d = 10^5$, $y = 10^3$ $\rightarrow$ $\sqrt{J^{-1}} = 10^4$

$(k=1)$

CRLB 1-sigma is larger than the actual altitude, not very useful!

④ find the expression of the MLE of y in terms of z & d

$$\text{MLE:} \quad 0 = \frac{d}{dy}\left[p(z^k|y) \right]\Big|_{y=\hat{y}}$$

$$\frac{d}{dy}\left[p(z^k|y) \right]\Big|_{y=\hat{y}} = \frac{d}{dy}\left[\frac{1}{(2\pi)^{k/2}\sigma^k} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{k} \left(z_j - \sqrt{d_j^2 + y^2} \right)^2 \right\} \right]\Big|_{y=\hat{y}}$$

$$0 = \underbrace{p(z^k|y)}_{>0\ \forall y} \sum_{j=1}^{k} \left(\frac{\hat{y}}{\sigma^2} \left(\frac{z_j}{\sqrt{d_j^2 + \hat{y}^2}} - 1 \right) \right)$$

$$0 = \frac{\hat{y}}{\sigma^2} \sum_{j=1}^{k} \left(\frac{z_j}{\sqrt{d_j^2 + \hat{y}^2}} - 1 \right)$$

$$0 = \sum_{j=1}^{k} \left(\frac{z_j}{\sqrt{d_j^2 + \hat{y}^2}} \right) - k$$

ASSUME $d_j = c$

$$0 = \frac{1}{\sqrt{d^2 + \hat{y}^2}} \sum_{j=1}^{k} z_j - k$$

$$\boxed{\hat{y} = \sqrt{\left(\frac{1}{k} \sum_{j=1}^{k} z_j \right)^2 - d^2} = \sqrt{\bar{z}^2 - d^2}}$$