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AOE 5784, Estimation and Filtering

10/21/2024

Midterm 1

P9) $p(\underline{x}) = \frac{1}{(2\pi)^{n/2}|P|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x}-\bar{\underline{x}})^T P^{-1}(\underline{x}-\bar{\underline{x}})\right\}$

$E[\underline{x}] = \bar{\underline{x}}, \quad E[(\underline{x}-\bar{\underline{x}})(\underline{x}-\bar{\underline{x}})^T] = P, \quad \int p(\underline{x})\,d\underline{x} = 1$

$\underline{z} = V^T\underline{x}, \quad V^{-1} = V^T, \quad P = V\,\mathrm{diag}\{\sigma_{z_1}^2 \cdots \sigma_{z_n}^2\}V^T$

$\underline{x} = V\underline{z} \qquad P^{-1} = V\,\mathrm{diag}\{\frac{1}{\sigma_{z_1}^2}, \cdots \frac{1}{\sigma_{z_n}^2}\}V^T$

$\bar{\underline{x}} = V\bar{\underline{z}}$

$P = VP_zV^T, \quad P^{-1} = VP_z^{-1}V^T$

$p(V\underline{z}) = \frac{1}{(2\pi)^{n/2}|VP_zV^T|^{1/2}} \exp\left\{-\frac{1}{2}(V(\underline{z}-\bar{\underline{z}}))^T VP_z^{-1}V^T(V(\underline{z}-\bar{\underline{z}}))\right\}$

$p(V\underline{z}) = \frac{1}{(2\pi)^{n/2}|P_z|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\underline{z}-\bar{\underline{z}})^T P_z^{-1}(\underline{z}-\bar{\underline{z}})\right\}$

$p(V\underline{z}) = \prod_{j=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma_{z_j}} \exp\left\{-\frac{1}{2\sigma_{z_j}}(z_j-\bar{z}_j)^2\right\}$

$E[\underline{x}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\underline{x}}\,p(\tilde{\underline{x}})\,d\tilde{x}_n \ldots d\tilde{x}_2\,d\tilde{x}_1 \qquad \underline{z} = V^T\underline{x}$

$E[\underline{x}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{V\tilde{\underline{z}}\,p(V\tilde{\underline{z}})}{|\det(V^T)|}\,d\tilde{z} \ldots d\tilde{z}_2\,d\tilde{z}_1$

$$\boxed{E[\underline{x}] = V\int_{\tilde{\underline{z}}} \tilde{\underline{z}}\,p(V\tilde{\underline{z}})\,d\tilde{\underline{z}} = V\bar{\underline{z}} = \bar{\underline{x}}}$$

$\mathrm{cov}(\underline{x}) = \int_{\tilde{\underline{x}}} (\tilde{\underline{x}}-\bar{\underline{x}})(\tilde{\underline{x}}-\bar{\underline{x}})^T p(\tilde{\underline{x}})\,d\tilde{\underline{x}}$

$* \; P_z = \mathrm{diag}\{\sigma_{z_1}^2 \cdots \sigma_{z_n}^2\}$

$$\boxed{\mathrm{cov}(\underline{x}) = \int_{\tilde{\underline{x}}} \frac{V(\tilde{\underline{z}}-\bar{\underline{z}})(\tilde{\underline{z}}-\bar{\underline{z}})^T V^T p(\tilde{\underline{z}})\,d\tilde{\underline{z}}}{|\det(V^T)|} = VP_zV^T = P}$$

$\int_{\tilde{x}} p(\tilde{\underline{x}})\,d\tilde{\underline{x}} = \int_{\tilde{z}} \frac{p(V\tilde{\underline{z}})\,d\tilde{\underline{z}}}{|\det(V^T)|} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1)\,dz_2\,dz_1 \cdots = 1$

HW3 P7 Midterm

Test statistic and it's PDF:

$$\underline{P_e} = P^{-1}\underline{e}$$

$$\beta = \underline{z}^T \underline{P_e}$$

$$P.D.F: \quad y = \frac{1}{\sqrt{2\pi}\underline{P_e}^T P \underline{P_e}} e^{-\frac{1}{2}\left(\frac{\beta - \theta \underline{e}^T \underline{P_e}}{\underline{P_e}^T P \underline{P_e}}\right)^2}$$

Code:

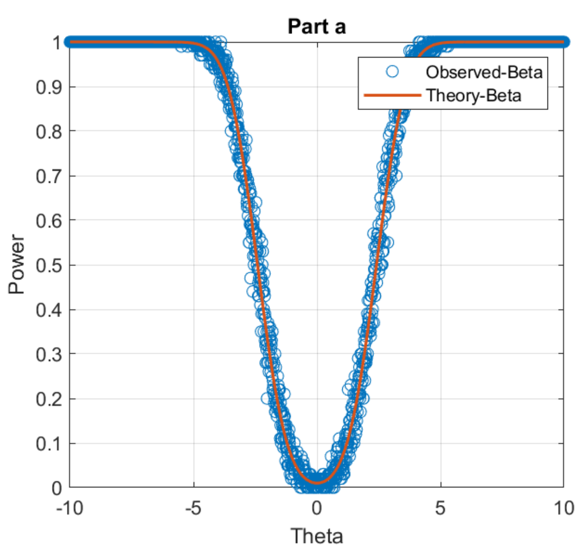
```
%% Optimal Neyman-Pearson 2-sided hypothesis test
% Spencer Freeman, 10/21/2024
% AOE 5784, Estimation and Filtering
%
% This script solves number 7 of problem set 3 which is highy related to
% number 1-9 (Bar Shalom) of problem set 1.
% --------------------------------------------------------------------------
clear;clc;close all

disp('HW3-P7_midterm')


%% a

alpha = .01;
P = [1 .5; ...
    .5 2];

Pinv = inv(P);

e = [1; 1];

Pe = Pinv * e;

sig_beta = sqrt(Pe' * P * Pe); % variance of beta
mu_beta  = 0;              % mean of beta

beta0 = -norminv(alpha/2, mu_beta, sig_beta); % threshold value


% create sample measurements and assess the test
thetas =  -10:.01:10;
for i = 1:length(thetas)
    theta = thetas(i);        % signal
    m = 100;%100e3;              % number of samples
    w = mvnrnd([0; 0], P, m)'; % random draw noise terms
    z = theta * e + w;        % noisy samples
    b = z' * Pinv * e;        % test statistic for each sample

    accept_H1 = abs(b) >= beta0; % test hypothesis

    pw_beta(i) = sum(accept_H1) / m; % detection rate (power)
    Power_beta(i) = ...
        normcdf(-beta0,   (theta * e)' * Pinv * e, sig_beta) + ...
        1-normcdf( beta0, (theta * e)' * Pinv * e, sig_beta);

end



%% b

bs = linspace(-5, 10, 500); % beta's to evaluate

sig_beta = sqrt(Pe' * P * Pe); % variance of beta
```

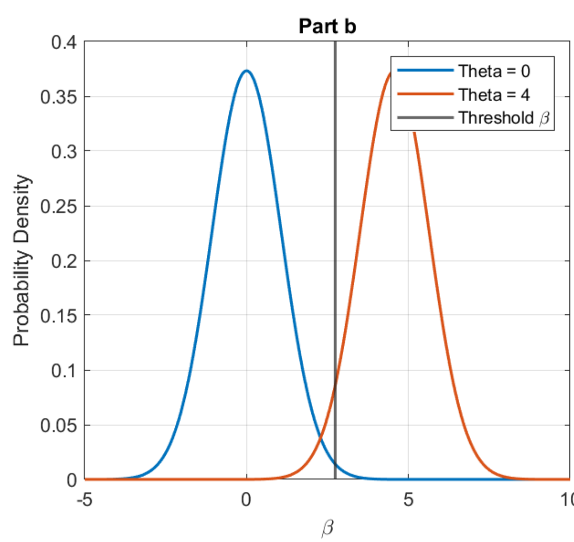
```
mu_beta  = 0;              % mean of beta

y0 = normpdf(bs, mu_beta, sig_beta);

theta1 = 4;
mu_beta  = theta1*e'*Pe;              % mean of beta

y1 = normpdf(bs, mu_beta, sig_beta);



%% plotting
close all

% Be sure to hand in your acquisition test statistic's formula,
% its threshold value, and its probability density functions,
% all with numerical values included where appropriate.

% CDF's of beta and eta
h = figure;
h.WindowStyle = 'Docked';
plot(thetas, pw_beta, 'o', 'Color', "#0072BD"); hold on
plot(thetas, Power_beta, 'LineWidth', 1.5, 'Color', "#D95319")
grid on
title('Part a')
ylabel('Power')
xlabel('Theta')
legend('Observed-Beta', 'Theory-Beta')

% PDF's for beta
h = figure;
h.WindowStyle = 'Docked';
plot(bs, y0, bs, y1)
grid on
title('Part b')

xline(beta0)
ylabel('Probability Density')
xlabel('\beta')
legend('Theta = 0', 'Theta = 4', 'Threshold \beta')

fprintf('\n\tThreshold Beta0: %f\n\t1-Sigma Beta: %f\n', beta0, sig_beta)
```

Output:

HW3-P7_midterm

Threshold Beta0: 2.753677
1-Sigma Beta: 1.069045





HW2 P4 Midterm

Code:

```
%% Write a Matlab function to solve the weighted least-squares problem
% Spencer Freeman, 10/21/2024
% AOE 5784, Estimation and Filtering
%
% This script solves number 4 of problem set 2
% -------------------------------------------------------------------------
clear;clc;close all

disp('HW2-P4_midterm')


z = [ -45.1800;...
   1.7900;...
   -31.3800;...
   26.7700;...
   27.6400] + .25;

H = [ -4.9300, -1.3100, -1.5900;...
   13.2600, 9.7100, 30.7000;...
   -17.0800, -11.9100, -12.1300;...
   -24.0300, -2.9900, -26.9500;...
   -2.4000, -8.7000, 9.3900];

R = [ 5.9700, -0.9200, -1.1800, -7.0600, -1.7900;...
   -0.9200, 3.4500, 1.7100, -0.6000, -4.0500;...
   -1.1800, 1.7100, 1.1900, 0.5600, -1.6700;...
   -7.0600, -0.6000, 0.5600, 9.9200, 4.8500;...
   -1.7900, -4.0500, -1.6700, 4.8500, 6.8700] + 1.2*eye(5);

Ra = chol(R);
Rainv = inv(Ra);
za = Rainv'*z;
Ha = Rainv'*H;
[Qb, Rb0] = qr(Ha);
ind = find(Rb0(:, end) ~= 0, 1, 'last');

zb = Qb'*za;
zbc = zb(1:ind);
Rb = Rb0(1:ind, :);
xhat = inv(Rb)*zbc;

Rinv = inv(R);
sol = norm(-H'*Rinv*(z - H*xhat)) / norm(-H'*Rinv*z);

fprintf('\n\txhat: %1.4f %1.4f %1.4f\n\ttol: %e\n', xhat, sol)
```

Output:

```
HW2-P4_midterm

        xhat: 1.3847 0.5495 -0.2974

        tol: 1.497484e-16
```

BS, P2-7)

Show Bayesian estimation that minimizes the expected value of the cost function

$$C(x-\hat{x}) \triangleq |x-\hat{x}|$$

yields $\hat{x} = x_m$, the median of x:

$$\int_{-\infty}^{x_m} p(x|z)\, dx = \frac{1}{2}$$

$$\hat{x}_{mmse}(z) = \min_{\hat{x}} E\big[(|x-\hat{x}| \mid z)\big] = \min_{\hat{x}} \int_{-\infty}^{\infty} |x-\hat{x}|\, p(x|z)\, dx$$

$$0 = \frac{d}{d\hat{x}}\left[\int_{-\infty}^{\infty} |x-\hat{x}|\, p(x|z)\, dx\right], \quad |x-\hat{x}| = \begin{cases} -(x-\hat{x}), & x < \hat{x} \\ x-\hat{x}, & x \geq \hat{x} \end{cases}$$

$$0 = \frac{d}{dx}\left[\int_{-\infty}^{\hat{x}} -(x-\hat{x})\, p(x|z)\, dx + \int_{\hat{x}}^{\infty} (x-\hat{x})\, p(x|z)\, dx\right]$$

Leibniz rule:

$$0 = (\hat{x}-\infty)\, p(x|z) \overset{0}{\frac{d}{d\hat{x}}(\infty)} - \overset{0}{(\hat{x}-\hat{x})}\, p(x|z) \frac{d}{d\hat{x}}(\hat{x}) + \int_{-\infty}^{\hat{x}} \overset{1}{\frac{d}{d\hat{x}}(\hat{x}-x)}\, p(x|z)\, dx + \ldots$$

$$\overset{0}{(\hat{x}-\hat{x})}\, p(x|z) \frac{d}{d\hat{x}}(\hat{x}) - (\infty-\hat{x})\, p(x|z) \overset{0}{\frac{d}{d\hat{x}}(\infty)} - \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

$$0 = \int_{-\infty}^{\hat{x}} p(x|z)\, dx - \int_{\hat{x}}^{\infty} p(x|z)\, dx, \qquad 1 = \int_{-\infty}^{\hat{x}} p(x|z)\, dx + \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

PDF

$$1 = \overset{2}{(1+1)} \int_{-\infty}^{\hat{x}} p(x|z)\, dx + \overset{0}{(-1+1)} \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

$$\boxed{\frac{1}{2} = \int_{-\infty}^{\hat{x}} p(x|z)\, dx}$$

P6)

$$J(\underline{x},k)=\sum_{j=1}^{k}(\underline{z}_j-H_j\underline{x})^T R_j^{-1}(\underline{z}_j-H_j\underline{x})$$

$$J(\underline{x},k)=\sum_{j=1}^{k}\underline{z}_j^T R_j^{-1}\underline{z}_j-\underline{z}_j^T R_j^{-1}H_j\underline{x}-\underline{x}^T H_j^T R_j^{-1}\underline{z}_j+\underline{x}^T H_j^T R_j^{-1}H_j\underline{x}$$

$$J(\underline{x},k)=\sum_{j=1}^{k}\underline{x}^T H_j^T R_j^{-1}H_j\underline{x}-2\underline{z}_j^T R_j^{-1}H_j\underline{x}+\underline{z}_j^T R_j^{-1}\underline{z}_j$$

define: $R=\begin{bmatrix}R_1 & \cdots & 0\\ \vdots & R_2 & \vdots \\ 0 & \cdots & R_k\end{bmatrix}$, $H=\begin{bmatrix}H_1\\H_2\\ \vdots\\H_k\end{bmatrix}$, $\underline{z}=\begin{bmatrix}\underline{z}_1\\ \underline{z}_2\\ \vdots\\ \underline{z}_k\end{bmatrix}$

$$\sum_{j=1}^{k}\underline{z}_j^T R_j^{-1}\underline{z}_j=\underline{z}^T R^{-1}\underline{z},\quad \sum_{j=1}^{k}-2\underline{z}_j^T R_j^{-1}H_j\underline{x}=-2\underline{z}^T R^{-1}H\underline{x}$$

$$\sum_{j=1}^{k}\underline{x}^T H_j^T R_j^{-1}H_j\underline{x}=\underline{x}^T H^T R^{-1}H\underline{x}$$

redefined: $$J(\underline{x},k)=\underline{x}^T(H^T R^{-1}H)\underline{x}-2(\underline{z}^T R^{-1}H)\underline{x}+(\underline{z}^T R^{-1}\underline{z}) \quad 1$$

$\updownarrow$ equal

prove equal: $$J_p(\underline{x},k)=(\underline{x}-\hat{\underline{x}}(k,\underline{z}^k))^T\hat{P}^{-1}(k,\underline{z}^k)(\underline{x}-\hat{\underline{x}}(k,\underline{z}^k))+J(\hat{\underline{x}}(k,\underline{z}^k),k)$$

$$J_p(\underline{x},k)=\underline{x}^T\hat{P}^{-1}\underline{x}-\underline{x}^T\hat{P}^{-1}\hat{\underline{x}}-\hat{\underline{x}}^T\hat{P}^{-1}\underline{x}+\hat{\underline{x}}^T\hat{P}^{-1}\hat{\underline{x}}+J(\hat{\underline{x}},k)$$

$$J_p(\underline{x},k)=\underline{x}^T\hat{P}^{-1}\underline{x}-2\hat{\underline{x}}^T\hat{P}^{-1}\underline{x}+\hat{\underline{x}}^T\hat{P}^{-1}\hat{\underline{x}}+J(\hat{\underline{x}},k)$$

$$\hat{\underline{x}}=(H^T R^{-1}H)^{-1}H^T R^{-1}\underline{z}=\hat{P}H^T R^{-1}\underline{z},\quad \hat{P}=(H^T R^{-1}H)^{-1}$$

$$\hat{\underline{x}}^T \hat{P}^{-1} \hat{\underline{x}} = (\hat{P} H^T R^{-1} \underline{z})^T \overbrace{\hat{P}^{-1} \hat{P}}^{I} H^T R^{-1} \underline{z}$$

$$= \underline{z}^T R^{-1T} H \hat{P}^T H^T R^{-1} \underline{z}$$

$$= \underline{z}^T R^{-1T} \overbrace{H H^{-1}}^{I,\ I} R^T \overbrace{H^{-1T} H^T}^{I} R^{-1} \underline{z}$$

$$\hat{\underline{x}}^T \hat{P}^{-1} \hat{\underline{x}} = \underline{z}^T R^{-1} \underline{z}$$

$$\hat{\underline{x}}^T \hat{P}^{-1} = (\hat{P} H^T R^{-1} \underline{z})^T \hat{P}^{-1}$$

$$= (H^T R^{-1} \underline{z})^T \overbrace{\hat{P}^T \hat{P}^{-1}}^{I \text{ (symmetric)}}$$

$$\hat{\underline{x}}^T \hat{P}^{-1} = \underline{z}^T R^{-1} H$$

$$\hat{P}^{-1} = H^T R^{-1} H$$

$$J_p(\underline{x}, k) = \underline{x}^T (H^T R^{-1} H) \underline{x} - 2(\underline{z}^T R^{-1} H) \underline{x} + (\underline{z}^T R^{-1} \underline{z}) + J(\hat{\underline{x}}, k)$$

refer to ① to prove coefficients equality.

BS 3-13) $\quad x \sim N(\bar{x}, P_{xx})$

$$z = x + w$$

$$w \sim N(0, P_{ww}), \quad E[xw] = 0$$

$$y = z^2 = (x+w)^2$$

Find the LMMSE estimate of x in terms of y and the associated MSE

$$\hat{x} = \bar{x} + P_{xy} P_{yy}^{-1} (y - \bar{y})$$

$$\bar{y} = E[y] = E[z^2] = E[(x+w)^2] = E[x^2 + 2\overset{0}{xw} + w^2] = P_{xx} + \bar{x}^2 + P_{ww}$$

$$P_{xy} = E[(x-\bar{x})(y-\bar{y})] = E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}] = E[xy] - \bar{y}E[x] - \bar{x}E[y] + \bar{x}\bar{y}$$

$$P_{xy} = E[xy] - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} = E[xy] - \bar{y}\bar{x}$$

$$E[xy] = E[x(x+w)^2] = E[x^3 + \overset{0}{2x^2 w} + xw^2] = E[x^3] + \bar{x} P_{ww}$$

use characteristic function of scalar random variable:

$$M_{x'}(s) = e^{\frac{1}{2}s^2\sigma^2}, \quad x' = x - \bar{x}, \quad \sigma^2 = P_{xx}$$

$$\left.\frac{d^3 M_{x'}}{ds^3}\right|_{s=0} = (3(0)\sigma^4 + (0)^3\sigma^6)\, e^{\frac{1}{2}s^2\sigma^2} = 0 = E[x'^3]$$

$$E[x^3] = E[(x' + \bar{x})^3] = E[x'^3 + 3x'^2\bar{x} + 3\bar{x}^2 x' + \bar{x}^3]$$

$$= \overset{0}{E[x'^3]} + 3\bar{x}E[x'^2] + 3\bar{x}^2 \overset{0}{E[x']} + \bar{x}^3 = 3\bar{x}P_{xx} + \bar{x}^3$$

$$P_{xy} = 3\bar{x} P_{xx} + \bar{x}^3 - \bar{x}(P_{xx} + P_{ww} + \bar{x}^2) + \bar{x} P_{ww}$$

$$P_{YY} = E[(Y-\bar{Y})(Y-\bar{Y})] = E[Y^2 - 2Y\bar{Y} + \bar{Y}^2] = E[Y^2] - 2\bar{Y}E[Y] + \bar{Y}^2$$

$$P_{YY} = E[\overbrace{(x+w)^2(x+w)^2}^{Y^2}] \overbrace{- 2(P_{xx} + P_{ww} + \bar{x}^2)^2 + (P_{xx} + P_{ww} + \bar{x}^2)^2}^{=(1)\bar{Y}^2}$$

$$E[Y^2] = E[x^4 + \overset{\nearrow 0}{4x^3 w} + 6x^2w^2 + \overset{\nearrow 0}{4xw^3} + w^4] = E[x^4] + 6E[x^2w^2] + E[w^4] +$$

$$E[x^4] = E[(x'+\bar{x})^4] = E[x'^4 + 4x'^3\bar{x} + 6x'^2\bar{x}^2 + 4x'\bar{x}^3 + \bar{x}^4]$$

$$E[x^4] = E[x'^4] + 4\bar{x}\overset{\nearrow 0}{E[x'^3]} + 6\bar{x}^2 \overset{\nearrow P_{xx}}{E[x'^2]} + 4\bar{x}^3\overset{\nearrow 0}{E[x']} + \bar{x}^4$$

From characteristic function:

$$E[Y^2] = 3P_{xx}^2 + 6\bar{x}^2 P_{xx} + \bar{x}^4 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2$$

$$P_{YY} = 3P_{xx}^2 + 6\bar{x}^2 P_{xx} + \bar{x}^4 - (P_{xx} + P_{ww} + \bar{x}^2)^2 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}$$

LMMSE: $\hat{x} = \bar{x} + (3\bar{x}P_{xx} + \bar{x}^3 - \bar{x}(P_{xx} + P_{ww} + \bar{x}^2) + \bar{x}P_{ww}) \circ \ldots$

$(3P_{xx}^2 + 6\bar{x}^2 P_{xx} + \bar{x}^4 - (P_{xx} + P_{ww} + \bar{x}^2)^2 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2)^{-1} \circ \ldots$

$(Y - (P_{xx} + P_{ww} + \bar{x}^2))$

MSE: $E[\tilde{x}\tilde{x}] = P_{xx} - P_{xY}P_{YY}^{-1}P_{Yx}$

Restating for clarity:

$$\hat{x}_{LMMSE} = \bar{x} + P_{xY}P_{YY}^{-1}(Y - \bar{Y}), \quad \bar{Y} = P_{xx} + P_{ww} + \bar{x}^2$$

$$MSE = P_{xx} - P_{xY}P_{YY}^{-1}P_{Yx}$$

$$P_{xY} = 3\bar{x}P_{xx} + \bar{x}^3 + \bar{x}P_{ww} - \bar{x}\bar{Y}$$

$$P_{YY} = 3P_{xx}^2 + 6\bar{x}^2 P_{xx} + \bar{x}^4 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2 - \bar{Y}^2$$