1) Problem 4-6 in Bar Shalom [15 pts]:

$$\bar{x}(t) = e^{A(t-t_0)}\bar{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} D\bar{v}(\tau)\,d\tau$$

by linearity and the linearity of the expectation operator.

Therefore

$$[x(t)-\bar{x}(t)] = e^{A(t-t_0)}[x(t_0)-\bar{x}(t_0)] + \int_{t_0}^{t} e^{A(t-\tau)} D[v(\tau)-\bar{v}(\tau)]\,d\tau$$

using the known integral formula for $x(t)$, which is very similar to the integral formula for $\bar{x}(t)$ given above

Let $P_{xx}(t) = V_{xx}(t,t) = E\{[x(t)-\bar{x}(t)][x(t)-\bar{x}(t)]^T\}$

Using the integral formula for $[x(t)-\bar{x}(t)]$ above in an analysis similar to what has been done in lecture:

$$P_{xx}(t) = e^{A(t-t_0)} E\{[x(t_0)-\bar{x}(t_0)][x(t_0)-\bar{x}(t_0)]^T\} e^{A^T(t-t_0)}$$
$$+ \int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2\, e^{A(t-\tau_1)} D E\{[v(\tau_1)-\bar{v}(\tau_1)][v(\tau_2)-\bar{v}(\tau_2)]^T\} D^T e^{A^T(t-\tau_2)}$$

which reduces to (by substituting in the expectations and by exploiting the Dirac delta functions properties:

$$P_{xx}(t) = e^{A(t-t_0)} P_{xx}(t_0) e^{A^T(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} D Q(\tau) D^T e^{A^T(t-\tau)} \, d\tau$$

If $t \geq \tau$, then one can write

$$[x(t)-\bar{x}(t)] = e^{A(t-\tau)} [x(\tau)-\bar{x}(\tau)] + \int_{\tau}^{t} e^{A(t-\sigma)} D [w(\sigma)-\bar{w}(\sigma)] \, d\sigma$$

Therefore

$$E\{[x(t)-\bar{x}(t)][x(\tau)-\bar{x}(\tau)]\} =$$

$$e^{A(t-\tau)} E\{[x(\tau)-\bar{x}(\tau)][x(\tau)-\bar{x}(\tau)]^T\} + \int_{\tau}^{t} e^{A(t-\sigma)} D E\{[w(\sigma)-\bar{w}(\sigma)][x(\tau)-\bar{x}(\tau)]^T\} \, d\sigma$$

but $\sigma \geq \tau$ in the integral. Therefore

$$E\{[w(\sigma)-\bar{w}(\sigma)][x(\tau)-\bar{x}(\tau)]^T\} = 0$$

by the whiteness of $w(\sigma) - \bar{w}(\sigma)$.

Therefore

$$V_{xx}(t,\tau) = e^{A(t-\tau)} P_{xx}(\tau)$$

Substitution of the formula for $P_{xx}(\tau)$ yields

$$V_{xx}(t,\tau) = e^{A(t-\tau)} e^{A(\tau-t_0)} P_{xx}(t_0) e^{A^T(\tau-t_0)}$$
$$+ e^{A(t-\tau)} \int_{t_0}^{\tau} e^{A(\tau-\sigma)} D\,Q(\sigma) D^T e^{A^T(\tau-\sigma)} d\sigma$$

where t in the formula at the top of sheet 2 has been replaced by τ here, and the dummy integration variable τ_1 in that formula has been replaced by σ here.

Finally, recognizing that $e^{A(t-\tau)} e^{A(\tau-t_0)} = e^{A(t-t_0)}$

and that $e^{A(t-\tau)} e^{A(\tau-\sigma)} = e^{A(t-\sigma)}$

the final result becomes

$$V_{xx}(t,\tau) = e^{A(t-t_0)} P_{xx}(t_0) e^{A^T(\tau-t_0)}$$
$$+ \int_{t_0}^{\tau} e^{A(t-\sigma)} D\,Q(\sigma) D^T e^{A^T(\tau-\sigma)} d\sigma$$

If $t < \tau$ then one gets the correct result by interchanging the roles of t & τ in this formula and then transposing the result. The resulting formula differs from that given above only in the upper limit of the integral. Therefore, the general result is

$$\boxed{V_{xx}(t,\tau) = e^{A(t-t_0)} P_{xx}(t_0)\, e^{A^T(\tau-t_0)} + \int_{t_0}^{\min[t,\tau]} e^{A(t-\sigma)} D Q(\sigma) D^T e^{A^T(\tau-\sigma)}\, d\sigma}$$

Some analyses that got nearly the eq. at the bottom of Sheet 3 did not carefully explain why the correct upper limit of integration is τ and not t when $t > \tau$, and they lost credit because of this missing justification.

2) Problem Set 5, Problem 1 [20 PTS]:

$$J_a = \frac{1}{2}\left[\underline{x}^T(k), \underline{v}^T(k), \underline{x}^T(k+1)\right] \begin{bmatrix} P^{-1}(k) & 0 & 0 \\ 0 & Q^{-1}(k) & 0 \\ 0 & 0 & \{H^T(k+1)R^{-1}(k+1)H(k+1)\} \end{bmatrix} \begin{bmatrix} \underline{x}(k) \\ \underline{v}(k) \\ \underline{x}(k+1) \end{bmatrix}$$

$$+ \left[\{-\hat{\underline{x}}^T(k)P^{-1}(k)\}, 0, \{-\underline{z}^T(k+1)R^{-1}(k+1)H(k+1)\}\right] \begin{bmatrix} \underline{x}(k) \\ \underline{v}(k) \\ \underline{x}(k+1) \end{bmatrix}$$

$$+ \left\{\frac{1}{2}\hat{\underline{x}}^T(k)P^{-1}(k)\hat{\underline{x}}(k) + \frac{1}{2}\underline{z}^T(k+1)R^{-1}(k+1)\underline{z}(k+1)\right\}$$

and that

$$\begin{bmatrix} \underline{x}(k) \\ \underline{v}(k) \\ \underline{x}(k+1) \end{bmatrix} = \begin{bmatrix} \{F^{-1}(k)[I - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)]\} \\ \{Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\} \\ I \end{bmatrix} \underline{x}(k+1)$$

$$+ \begin{bmatrix} \{-F^{-1}(k)\{G(k)\underline{u}(k) - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[F(k)\bar{\underline{x}}(k)+G(k)\underline{u}(k)]\}\} \\ \{-Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[F(k)\bar{\underline{x}}(k) + G(k)\underline{u}(k)]\} \\ 0 \end{bmatrix}$$

Thus, the original J_a cost function has terms that are quadratic and linear in the vector

$$\begin{bmatrix} \underline{x}(k) \\ \underline{v}(k) \\ \underline{x}(k+1) \end{bmatrix}$$

plus a constant term that does not depend on this vector. The inverse dynamics equation and the formula for the optimal $\underline{v}(k)$ as a function of $\underline{x}(k+1)$ combine to yield a formula for the vector

$\begin{bmatrix} \underline{x}(k) \\ \underline{v}(k) \\ \underline{x}(k+1) \end{bmatrix}$ that has a term that is linear in $\underline{x}(k+1)$

and a constant term. – the top formula on this sheet. Therefore, substitution of this vector formula into the J_a formula yields a formula that has quadratic and linear terms in $\underline{x}(k+1)$ plus a constant term. It takes the form

$$J_a = \frac{1}{2}\underline{x}^T(k+1)\,A\,\underline{x}(k+1) + \underline{b}^T\underline{x}(k+1) + c$$

where, noting that $F^{-1}(k)\left[I - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\right] = F^{-1}(k)\left[F(k)P(k)F^T(k)\right]\bar{P}^{-1}(k+1)$
$= P(k)F^T(k)\bar{P}^{-1}(k+1)$

$$A = \left[\{\bar{P}^{-1}(k+1)F(k)P(k)\}, \{\bar{P}^{-1}(k+1)\Gamma(k)Q(k)\}, I\right] \cdot \left(\begin{bmatrix} P^{-1}(k) & 0 & 0 \\ 0 & Q^{-1}(k) & 0 \\ 0 & 0 & \{H^T(k+1)R^{-1}(k+1)H(k+1)\} \end{bmatrix} \cdot \begin{bmatrix} \{P(k)F^T(k)\bar{P}^{-1}(k+1)\} \\ \{Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\} \\ I \end{bmatrix} \right)$$

$$\underline{b} = \left[\{\bar{P}^{-1}(k+1)F(k)P(k)\}, \{\bar{P}^{-1}(k+1)\Gamma(k)Q(k)\}, I\right] \cdot \left(-\begin{bmatrix} P^{-1}(k)\hat{\underline{x}}(k) \\ 0 \\ H^T(k+1)R^{-1}(k+1)\underline{z}(k+1) \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} P^{-1}(k) & 0 & 0 \\ 0 & Q^{-1}(k) & 0 \\ 0 & 0 & \{H^T(k+1)R^{-1}(k+1)H(k+1)\} \end{bmatrix} \begin{bmatrix} \{-F^{-1}(k)(G(k)\underline{u}(k) - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[F(k)\hat{\underline{x}}(k) + G(k)\underline{u}(k)])\} \\ \{-Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[F(k)\hat{\underline{x}}(k) + G(k)\underline{u}(k)]\} \\ 0 \end{bmatrix} \right)$$

This transformed J_a and the final form of J_c both contain only quadratic and linear terms in $\underline{x}(k+1)$ plus a constant term. In order to show that two such functions are identical, it suffices to show that their quadratic, linear, and constant terms are identical. This is equivalent to showing that their constant 2nd derivative Hessian matrices are the same, that their vector first derivatives are the same at any convenient choice of $\underline{x}(k+1)$, and that their function values are the same at any convenient value of $\underline{x}(k+1)$

<u>2nd derivative,</u>

$$\frac{\partial J_a}{\partial \underline{x}(k+1)} = \underline{x}^T(k+1) A + \underline{b}^T$$

$$\frac{\partial^2 J_a}{\partial [\underline{x}(k+1)]^2} = A$$

$$\frac{\partial J_c}{\partial \underline{x}(k+1)} = [\underline{x}(k+1) - \hat{\underline{x}}(k+1)]^T P^{-1}(k+1)$$

$$\frac{\partial^2 J_c}{\partial [\underline{x}(k+1)]^2} = P^{-1}(k+1)$$

Therefore, we need to show that $A = \bar{P}^{-1}(k+1)$

Expanding the formula for A

$$A = \bar{P}^{-1}(k+1)\, F(k)\, P(k)\, P^{-1}(k)\, P(k)\, F^T(k)\, \bar{P}^{-1}(k+1)$$

$$+ \{\bar{P}^{-1}(k+1)\, \Gamma(k)\, Q(k)\}\, Q^{-1}(k)\, \{Q(k)\, \Gamma^T(k)\, \bar{P}^{-1}(k+1)\}$$

$$+ H^T(k+1)\, R^{-1}(k+1)\, H(k+1)$$

or $$A = \bar{P}^{-1}(k+1)\left[F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k)\right]\bar{P}^{-1}(k+1)$$

$$+ H^T(k+1)R^{-1}(k+1)H(k+1)$$

or $$A = \bar{P}^{-1}(k+1)\, \bar{P}(k+1)\, \bar{P}^{-1}(k+1) + H^T(k+1)R^{-1}(k+1)H(k+1)$$

or $$A = \bar{P}^{-1}(k+1) + H^T(k+1)R^{-1}(k+1)H(k+1) = P^{-1}(k+1)$$

from one of our covariance update formulas.

Therefore, the second derivatives are equivalent for the two functions.

Let's choose to equate first derivatives at $\underline{x}(k+1) = \hat{\underline{x}}(k+1)$. Therefore

$$\left.\frac{\partial J_a}{\partial \underline{x}(k+1)}\right|_{\hat{\underline{x}}(k+1)} = 0$$

Therefore, we need to check whether

$$0 \stackrel{?}{=} \left[\frac{\partial J_a}{\partial \underline{x}(k+1)} \Big|_{\hat{\underline{x}}(k+1)} \right]^T = A \hat{\underline{x}}(k+1) + \underline{b}$$

$$0 \stackrel{?}{=} \bar{P}^{-1}(k+1) \hat{\underline{x}}(k+1) + \underline{b}$$

Expanding the formula for $\underline{b}$ from sheet 6:

$$\underline{b} = \{ \bar{P}^{-1}(k+1) F(k) P(k) \} P^{-1}(k) \Big[-\hat{\underline{x}}(k)$$

$$- F^{-1}(k) \{ G(k) \underline{u}(k) - \Gamma(k) Q(k) \Gamma^T(k) \bar{P}^{-1}(k+1) \bar{\underline{x}}(k+1) \} \Big]$$

$$+ \{ \bar{P}^{-1}(k+1) \Gamma(k) Q(k) \} Q^{-1}(k) \Big[-Q(k) \Gamma^T(k) \bar{P}^{-1}(k+1) \bar{\underline{x}}(k+1) \Big]$$

$$- \{ H^T(k+1) R^{-1}(k+1) \underline{z}(k+1) \}$$

where we have used the fact that $\bar{\underline{x}}(k+1) = F(k)\hat{\underline{x}}(k) + G(k)\underline{u}(k)$ in order to simplify things. The first two main terms can be re-arranged to yield

$$\underline{b} = \bar{P}^{-1}(k+1) \Big[-F(k)\hat{\underline{x}}(k) - G(k)\underline{u}(k) + \Gamma(k) Q(k) \Gamma^T(k) \bar{P}^{-1}(k+1) \bar{\underline{x}}(k+1) \Big]$$

$$- \bar{P}^{-1}(k+1) \{ \Gamma(k) Q(k) \Gamma^T(k) \bar{P}^{-1}(k+1) \bar{\underline{x}}(k+1) \}$$

$$- \{ H^T(k+1) R^{-1}(k+1) \underline{z}(k+1) \}$$

or

$$\underline{b} = \bar{P}^{-1}(k+1)\left[-I + \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\right]\bar{x}(k+1)$$

$$- \bar{P}^{-1}(k+1)\left(\Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\right)\bar{x}(k+1)$$

$$- H^T(k+1)R^{-1}(k+1)z(k+1)$$

or recognizing that two of the terms cancel:

$$\underline{b} = -\bar{P}^{-1}(k+1)\bar{x}(k+1) - H^T(k+1)R^{-1}(k+1)z(k+1)$$

Then $A\hat{x}(k+1) + \underline{b} = P^{-1}(k+1)\hat{x}(k+1) - \bar{P}^{-1}(k+1)\bar{x}(k+1)$
$- H^T(k+1)R^{-1}(k+1)z(k+1)$

but from lecture we know that

$$\hat{x}(k+1) = \bar{x}(k+1) + W(k+1)\left[z(k+1) - H(k+1)\bar{x}(k+1)\right]$$
$$= \bar{x}(k+1) + P(k+1)H^T(k+1)R^{-1}(k+1)\left[z(k+1) - H(k+1)\bar{x}(k+1)\right]$$

Therefore $A\hat{x}(k+1) + \underline{b} = P^{-1}(k+1)\bar{x}(k+1)$

$$+ H^T(k+1)R^{-1}(k+1)\left[z(k+1) - H(k+1)\bar{x}(k+1)\right]$$
$$- \bar{P}^{-1}\bar{x}(k+1) - H^T(k+1)R^{-1}(k+1)z(k+1)$$

or

$$A\hat{x}(k+1) + \underline{b} = \left\{P^{-1}(k+1) - \bar{P}^{-1}(k+1) - H^T(k+1)R^{-1}(k+1)H(k+1)\right\}\bar{x}(k+1)$$
$$+ H^T(k+1)R^{-1}(k+1)\left\{z(k+1) - z(k+1)\right\}$$
$$= \{0\}\bar{x}(k+1) + H^T(k+1)R^{-1}(k+1)\{0\} = 0$$

One can check the constant terms by evaluating J_c at $\underline{x}(k+1) = \hat{\underline{x}}(k+1)$. It is obvious that

$$J_c\left[\hat{\underline{x}}(k+1), k+1\right] = J_6\left[\hat{\underline{v}}(k), \hat{\underline{x}}(k+1), k\right]$$

Given the definition of J_c in terms of J_6 and given that $\hat{\underline{v}}(k) = Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\left[\hat{\underline{x}}(k+1) - F(k)\hat{\underline{x}}(k) - G(k)\underline{u}(k)\right]$, the equality of these two constant terms is obvious.

3) Problem Set 5, Problem 4 (15 PTS)

```
% script_prelim2_prob3_f2015.m
%
%  This Matlab script solves Problem 3 of MAE 6760 Prelim 2 for
%  the Fall of 2015.  This is the solution to Problem 4 of
%  Problem Set 5, except with Qk increased from 4 to 40.
%
   clear
   Fk     = [  0.81671934103521,  0.08791146849849;...
              -3.47061412053765,  0.70624978972000];     % for all k
   Gammak = [  0.00464254201630;...
               0.08791146849849];                        % for all k
   Hkp1   = [  2.00000000000000,  0.30000000000000];     % for all k
%
   Qk     =   40.00000000000000;                         % for all k
   Rkp1   =    0.01000000000000;                         % for all k
%
   P0      = [  0.25000000000000,  0.08000000000000;...
                0.08000000000000,  0.50000000000000];
%
%  Do 50 iterations of the covariance dynamic propagations and
%  measurement updates in order to determine the approximate
%  steady-state P as in Problem 3 of the same problem set.
%  This is needed for comparison purposes.  Note that the
%  state estimate propagation and measurement update are not
%  needed in order to propagate and update the covariance
%  when dealing with a linear Kalman filter.
%
   Pkp1 = P0;
   n = size(Fk,1);
   for k = 0:49
      Pk = Pkp1;
      Pbarkp1 = Fk*Pk*(Fk') + Gammak*Qk*(Gammak');
      Skp1 = Hkp1*Pbarkp1*(Hkp1') + Rkp1;
      Wkp1 = Pbarkp1*(Hkp1')*inv(Skp1);
      eyemWHkp1 = eye(n) - Wkp1*Hkp1;
      Pkp1 = eyemWHkp1*Pbarkp1*(eyemWHkp1') + Wkp1*Rkp1*(Wkp1');
   end
%
%  Do the dlqe calculations:
%
   [Wss,Pbarss,Pss,evals_dt] = dlqe(Fk,Gammak,Hkp1,Qk,Rkp1);
%
%  Display results:
%
   format long
   Pkp1
   Pss
   normPerror = norm(Pkp1 - Pss)/norm(Pss)
   Wss
   Pbarss
   evals_dt
```

```
   evals_dt_theory = eig((eye(n) - Wss*Hkp1)*Fk)
   absevals_dt_theory = abs(evals_dt_theory)
%
%  Confirming output as sent to display:
%
%    Pkp1 = [0.000608413729247   0.000278806265465;...
%            0.000278806265465   0.064687255535901]
%
%    Pss =  [0.000608413729247   0.000278806265465;...
%            0.000278806265465   0.064687255535901]
%
%    normPerror = 4.290652757469403e-016
%
%  The preceding two matrices agree in all significant digits shown.
%  The relative norm of the error between the two matrices is
%  on the order of the machine precision, which indicates very
%  good agreement.
%
%    Wss =  [0.130046933813419;...
%            1.996378919169955]
%
%    Pbarss = [0.001807925064779   0.018692767082072;...
%              0.018692767082072   0.347363994730991]
%
%  Be careful not to confuse the Pbarss output of dlqe.m with
%  the Pss output.  You were asked for the latter, which obviously
%  differs from Pbarss.
%
%    evals_dt =          [0.335978196770240 + 0.107060561728605i;...
%                         0.335978196770240 - 0.107060561728605i]
%
%    evals_dt_theory = [0.335978196770239 + 0.107060561728604i;...
%                        0.335978196770239 - 0.107060561728604i]
%
%    absevals_dt_theory = [0.352623471400623;...
%                           0.352623471400623]
%
%  The discrete-time eigenvalues from dlqe.m agree with those
%  of the theoretical values for the error dynamics state
%  transition matrix, and both of the eigenvalue absolute
%  values are significantly less than 1, which indicates
%  stability of the steady-state error dynamics.
%
```

4) Problem 5-1.1 in Bar-Shalom [15 pts]

$$P_\infty = [\bar{P}_\infty^{-1} + H^T R^{-1} H]^{-1}$$

$$\bar{P}_\infty = F P_\infty F^T + \Gamma Q \Gamma^T$$

In this scalar problem $P_\infty = p_\infty$, $\bar{P}_\infty = \bar{p}_\infty$

$F = f$, $H = h$, $\Gamma = 1$, $Q = q$, and $R = r$

so $\bar{p}_\infty = f^2 p_\infty + q$

$$p_\infty = \frac{1}{\frac{1}{\bar{p}_\infty} + \frac{h^2}{r}} = \frac{\bar{p}_\infty r}{r + h^2 \bar{p}_\infty}$$

Substituting into the $\bar{p}_\infty$ equation yields

$$\bar{p}_\infty = \frac{f^2 r \bar{p}_\infty}{r + h^2 \bar{p}_\infty} + q$$

multiplying both sides by $(r + h^2 \bar{p}_\infty)$ yields

$$\bar{p}_\infty [r + h^2 \bar{p}_\infty] = f^2 r \bar{p}_\infty + q[r + h^2 \bar{p}_\infty]$$

or $$[h^2] \bar{p}_\infty^2 + [r(1 - f^2) - q h^2] \bar{p}_\infty - [q r] = 0$$

This is just a quadratic equation in $\bar{p}_\infty$

Its solution is

$$\bar{p}_\infty = \frac{\left[qh^2 + r\left(F^2 - 1\right)\right] \pm \sqrt{\left[qh^2 + r\left(F^2-1\right)\right]^2 + 4h^2 qr}}{2h^2}$$

or

$$\bar{p}_\infty = \frac{\left[qh^2 + r\left(F^2 - 1\right)\right] \pm \sqrt{q^2h^4 + r^2F^4 + r^2 + 2qrh^2F^2 + 2qrh^2 - 2r^2F^2}}{2h^2}$$

It is apparent that the + solution must apply in order to maintain $\bar{p}_\infty > 0$

and

$$p_\infty = \frac{r\left\{\frac{\left[qh^2 + r\left(F^2-1\right)\right] + \sqrt{q^2h^4 + r^2F^4 + r^2 + 2qrh^2F^2 + 2qrh^2 - 2r^2F^2}}{2h^2}\right\}}{r + h^2\left\{\frac{\left[qh^2 + r\left(F^2-1\right)\right] + \sqrt{q^2h^4 + r^2F^4 + r^2 + 2qrh^2F^2 + 2qrh^2 - 2r^2F^2}}{2h^2}\right\}}$$

or

$$\boxed{p_\infty = \frac{rqh^2 + r^2\left(F^2 - 1\right) + r\sqrt{q^2h^4 + r^2F^4 + r^2 + 2qrh^2F^2 + 2qrh^2 - 2r^2F^2}}{qh^4 + rh^2F^2 + rh^2 + h^2\sqrt{q^2h^4 + r^2F^4 + r^2 + 2qrh^2F^2 + 2qrh^2 - 2r^2F^2}}}$$

An algebraically equivalent answer is

$$\boxed{P_\infty = \frac{r f^2 - r - q h^2 + \sqrt{q^2 h^4 + r^2 f^4 + r^2 + 2 q r h^2 f^2 + 2 q r h^2 - 2 r^2 f^2}}{2 f^2 h^2}}$$

5) Problem 5.11 in Bar-Shalom [15 pts]

$$\bar{\underline{x}}(k+1) = E\{\underline{x}(k+1) \,/\, Z^k\}$$

$$= E\{F(k)\underline{x}(k) + G(k)\underline{u}(k) + \Gamma(k)\underline{v}(k) \,/\, Z^k\}$$

$$\bar{\underline{x}}(k+1) = F(k)\hat{\underline{x}}(k) + G(k)\underline{u}(k) + \underbrace{\Gamma(k)\bar{\underline{v}}(k)}_{\text{Difference from reg KF}}$$

$$\underline{x}(k+1) - \bar{\underline{x}}(k+1) = F(k)[\underline{x}(k) - \hat{\underline{x}}(k)] + \Gamma(k)[\underline{v}(k) - \bar{\underline{v}}(k)]$$

$$\bar{P}(k+1) = E\{[\underline{x}(k+1) - \bar{\underline{x}}(k+1)][\underline{x}(k+1) - \bar{\underline{x}}(k+1)]^T \,/\, Z^k\}$$

$$= F(k)E\{[\underline{x}(k) - \hat{\underline{x}}(k)][\underline{x}(k) - \hat{\underline{x}}(k)]^T \,/\, Z^k\}F^T(k)$$
$$+ \Gamma(k)E\{[\underline{v}(k) - \bar{\underline{v}}(k)][\underline{v}(k) - \bar{\underline{v}}(k)]^T \,/\, Z^k\}\Gamma^T(k)$$

because $E\{[\underline{x}(k) - \hat{\underline{x}}(k)][\underline{v}(k) - \bar{\underline{v}}(k)]^T / Z^k\} = 0$

Therefore $\bar{P}(k+1) = F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k)$
as before

$$\bar{z}(k+1) = E\{z(k+1) / Z^k\}$$
$$= E\{H(k+1)x(k+1) + w(k+1) / Z^k\}$$
$$= H(k+1)E\{x(k+1)/Z^k\} + E\{w(k+1)/Z^k\}$$

or $\bar{z}(k+1) = H(k+1)\bar{x}(k+1) + \underbrace{\bar{w}(k+1)}_{\text{difference from reg K.F}}$

$$z(k+1) - \bar{z}(k+1) = H(k+1)[x(k+1) - \bar{x}(k+1)] + [w(k+1) - \bar{w}(k+1)]$$

$$\bar{P}_{zz}(k+1) = E\{[z(k+1) - \bar{z}(k+1)][z(k+1) - \bar{z}(k+1)]^T / Z^k\}$$
$$= H(k+1)E\{[x(k+1) - \bar{x}(k+1)][x(k+1) - \bar{x}(k+1)]^T / Z^k\}H^T(k+1)$$
$$+ E\{[w(k+1) - \bar{w}(k+1)][w(k+1) - \bar{w}(k+1)]^T / Z^k\}$$

because $E\{[x(k+1) - \bar{x}(k+1)][w(k+1) - \bar{w}(k+1)]^T / Z^k\} = 0$

Therefore $\bar{P}_{zz}(k+1) = H(k+1)\bar{P}(k+1)H^T(k+1) + R(k+1)$ as before

$$\bar{P}_{xz}(k+1) = E\{[x(k+1) - \bar{x}(k+1)][z(k+1) - \bar{z}(k+1)]^T / Z^k\}$$
$$= E\{[x(k+1) - \bar{x}(k+1)][x(k+1) - \bar{x}(k+1)]^T / Z^k\}H^T(k+1)$$
$$= \bar{P}(k+1)H^T(k+1)$$

as before

Therefore $\hat{x}(k+1) = \bar{x}(k+1) + \bar{P}_{xz}(k+1)\bar{P}_{zz}^{-1}(k+1)[z(k+1) - \bar{z}(k+1)]$

$$= \bar{x}(k+1) + \bar{P}(k+1)H^T(k+1)S^{-1}(k+1)[z(k+1) - H(k+1)\bar{x}(k+1) - \underbrace{\bar{w}(k+1)}_{\text{difference from reg. K.F.}}]$$

and

$$P(k+1) = \bar{P}(k+1) - \bar{P}_{xz}(k+1)\,\bar{P}_{zz}^{-1}(k+1)\,\bar{P}_{xz}^{T}(k+1)$$

or $$P(k+1) = \bar{P}(k+1) - \bar{P}(k+1)H^{T}(k+1)S^{-1}(k+1)H(k+1)\bar{P}(k+1)$$

where $$S(k+1) = H(k+1)\bar{P}(k+1)H^{T}(k+1) + R(k+1)$$

Summarizing:

A) Dynamic Propagation

$$\bar{x}(k+1) = F(k)\hat{x}(k) + G(k)u(k) + \underline{\Gamma(k)\underline{\bar{v}}(k)}$$

Different

$$\bar{P}(k+1) = F(k)P(k)F^{T}(k) + \Gamma(k)Q(k)\Gamma^{T}(k)$$

B) Measurement update

$$\underline{\nu}(k+1) = z(k+1) - H(k+1)\bar{x}(k+1) - \underline{\bar{w}(k+1)}$$

Different

$$S(k+1) = H(k+1)\bar{P}(k+1)H^{T}(k+1) + R(k+1)$$
$$W(k+1) = \bar{P}(k+1)H^{T}(k+1)S^{-1}(k+1)$$
$$\hat{x}(k+1) = \bar{x}(k+1) + W(k+1)\underline{\nu}(k+1)$$
$$P(k+1) = \bar{P}(k+1) - W(k+1)S(k+1)W^{T}(k+1)$$

6) Problem Set 6, Problem 4 [20 pts]

From lecture:

$$P_{vv}^*(k) = Q(k) - Q(k)\,\Gamma^T(k)\,\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]\bar{P}^{-1}(k+1)\,\Gamma(k)\,Q(k)$$

$$P_{vx}^*(k+1) = Q(k)\,\Gamma^T(k)\,\bar{P}^{-1}(k+1)\,P^*(k+1)$$

So

$$\begin{aligned} P^*(k) = F^{-1}(k)\Big\{ & P^*(k+1) - P^*(k+1)\,\bar{P}^{-1}(k+1)\,\Gamma(k)\,Q(k)\,\Gamma^T(k) \\ & - \Gamma(k)\,Q(k)\,\Gamma^T(k)\,\bar{P}^{-1}(k+1)\,P^*(k+1) \\ & + \Gamma(k)\,Q(k)\,\Gamma^T(k) \\ & - \Gamma(k)\,Q(k)\,\Gamma^T(k)\,\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]\cdot \\ & \quad \bar{P}^{-1}(k+1)\,\Gamma(k)\,Q(k)\,\Gamma^T(k)\Big\}\,F^{-T}(k) \end{aligned}$$

One can use the fact that

$$I - \bar{P}^{-1}(k+1)\,\Gamma(k)\,Q(k)\,\Gamma^T(k) = \bar{P}^{-1}(k+1)\,F(k)\,P(k)\,F^T(k)$$

to combine the first two terms in the braces on the RHS and one can -multiply the 4th term in the braces on the RHS by $\bar{P}^{-1}(k+1)\,\bar{P}(k+1)$ and combine it with the third term to get

$$P^*(k) = F^{-1}(k)\Big\{P^*(k+1)\bar{P}^{-1}(k+1)F(k)P(k)F^T(k)$$

$$+\Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]$$

$$-\Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]\bar{P}^{-1}(k+1)\Gamma(k)Q(k)\Gamma^T(k)\Big\}\cdot$$

$$F^{-T}(k)$$

one can next re-use the expression for $I - \bar{P}^{-1}(k+1)\Gamma(k)Q(k)\Gamma^T(k)$ in order to combine the 2nd & 3rd terms in the braces on the RHS to yield:

$$P^*(k) = F^{-1}(k)\Big\{P^*(k+1)\bar{P}^{-1}(k+1)F(k)P(k)F^T(k)$$

$$+\Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]\bar{P}^{-1}(k+1)F(k)P(k)F^T(k)\Big\}\cdot$$

$$F^{-T}(k)$$

now factoring $\bar{P}^{-1}(k+1)F(k)P(k)F^T(k)$ out from the right of the expression in braces on the RHS yields:

$$P^*(k) = F^{-1}(k)\Big\{P^*(k+1) + \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)\left[\bar{P}(k+1) - P^*(k+1)\right]\Big\}\cdot$$

$$\bar{P}^{-1}(k+1)F(k)P(k)$$

One can add $P(k) - F^{-1}(k)F(k)P(k)P^{-1}(k)F^{-1}(k)\bar{P}(k+1)\bar{P}^{-1}(k+1)F(k)P(k)$ $= 0$ to the RHS of the equation without affecting it. The term on the right is added onto the expression in braces on the RHS. The Result is:

$$P^*(k) = P(k) + F^{-1}(k)\{P^*(k+1) - F(k)P(k)P^{-1}(k)F^{-1}(k)\bar{P}(k+1) + \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[\bar{P}(k+1) - P^*(k+1)]\} \cdot \bar{P}^{-1}(k+1)F(k)P(k)$$

Now the three terms in the braces can be combined after simplifying the 2nd term to $\bar{P}(k+1)$. The result is,

$$P^*(k) = P(k) + F^{-1}(k)\{[-I + \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)][\bar{P}(k+1) - P^*(k+1)]\} \cdot \bar{P}^{-1}(k+1)F(k)P(k)$$

The term $[-I + \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)] = -F(k)P(k)F^T(k)\bar{P}^{-1}(k+1)$. Substitution of this result into the first term in the braces on the RHS yields the desired result:

$$P^*(k) = P(k) - P(k)F^T(k)\bar{P}^{-1}(k+1)[\bar{P}(k+1) - P^*(k+1)]\bar{P}^{-1}(k+1)F(k)P(k)$$