2) Prove: $\frac{1}{2}\underline{\nu}^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1) = J_b[\hat{\underline{v}}(k), \hat{\underline{x}}(k+1), k]$

$$J_b[\underline{v}(k), \underline{x}(k+1), k] = J_a[F^{-1}(k)(\underline{x}(k+1) - G(k)\underline{u}(k) - \Gamma(k)\underline{v}(k)), \underline{v}(k), \underline{x}(k+1), k]$$

$$J_a[\underline{x}(k), \underline{v}(k), \underline{x}(k+1), k] = \frac{1}{2}[\underline{x}(k) - \hat{\underline{x}}(k)]^T P^{-1}(k)[\underline{x}(k) - \hat{\underline{x}}(k)] + \ldots$$

$$\frac{1}{2}\underline{v}^T(k)Q^{-1}(k)\underline{v}(k) + \frac{1}{2}[\underline{z}(k+1) - H(k+1)\underline{x}(k+1)]^T R^{-1}(k+1)[\ldots$$

$$\underline{z}(k+1) - H(k+1)\underline{x}(k+1)]$$

$$J_b[\hat{\underline{v}}(k), \hat{\underline{x}}(k+1), k] = \frac{1}{2}\underline{A}^T P^{-1}(k)\underline{A} + \frac{1}{2}\underline{C}^T Q^{-1}(k)\underline{C} + \frac{1}{2}\underline{B}^T R^{-1}(k+1)\underline{B}$$

$$\hat{\underline{x}}(k|k+1) = F^{-1}(k)[\hat{\underline{x}}(k+1) - G(k)\underline{u}(k) - \Gamma(k)\hat{\underline{v}}(k)]$$

$$\underline{A} = [\hat{\underline{x}}(k|k+1) - \hat{\underline{x}}(k)]$$
$$\underline{C} = \hat{\underline{v}}(k)$$
$$\underline{B} = [\underline{z}(k+1) - H(k+1)\hat{\underline{x}}(k+1)]$$

rewrite in terms of innovation, $\underline{\nu}(k+1)$

C) $\underline{C} = \hat{\underline{v}}(k) = Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[\hat{\underline{x}}(k+1) - F(k)\hat{\underline{x}}(k) - G(k)\underline{u}(k)]$

$\bar{\underline{x}}(k+1) = F(k)\hat{\underline{x}}(k) + G(k)\underline{u}(k)$ → state mean propagation

$\hat{\underline{x}}(k+1) = \bar{\underline{x}}(k+1) + \bar{P}(k+1)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1)$ → K.F. update

$$\underline{C} = Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[\bar{\underline{x}}(k+1) + \bar{P}(k+1)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1) - \ldots$$

$$F(k)F^{-1}(k)(\bar{\underline{x}}(k+1) - G(k)\underline{u}(k)) - G(k)\underline{u}(k)]$$

$$\underline{C} = Q(k)\Gamma^T(k)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1)$$

(A) $$\underline{A} = F^{-1}(k)[\hat{\underline{x}}(k+1) - G(k)\underline{u}(k) - \Gamma(k)\hat{\underline{v}}(k)] - \hat{\underline{x}}(k)$$

$$\underline{A} = F^{-1}(k)[\bar{\underline{x}}(k+1) + \bar{P}(k+1)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1) - \dots$$
$$G(k)\underline{u}(k) - \Gamma(k)\underline{\hat{v}}(k) - (\bar{\underline{x}}(k+1) - G(k)\underline{u}(k))]$$

$$\underline{A} = F^{-1}(k)[\bar{P}(k+1) - \Gamma(k)Q(k)\Gamma^T(k)]H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1)$$

$$\bar{P}(k+1) = F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k) \rightarrow \text{covariance propagation}$$

$$\underline{A} = P(k)F^T(k)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1)$$

(B) $$\underline{B} = \underline{z}(k) - H(k+1)\hat{\underline{x}}(k+1)$$

$$\underline{\nu}(k+1) = \underline{z}(k+1) - H(k+1)\bar{\underline{x}}(k+1)$$

$$\underline{B} = \underline{\nu}(k+1) + H(k+1)\bar{\underline{x}}(k+1) - H(k+1)[\bar{\underline{x}}(k+1) + \dots$$
$$\bar{P}(k+1)H^T(k+1)S^{-1}(k+1)\underline{\nu}(k+1)]$$

$$\underline{B} = [\underbrace{S(k+1) - H(k+1)\bar{P}(k+1)H^T(k+1)}_{R(k+1)}]S^{-1}(k+1)\underline{\nu}(k+1)$$

Factor Quadratic Terms

$$\underline{A} = \Theta_A S^{-1}(k+1)\underline{\nu}(k+1)$$
$$\underline{B} = \Theta_B S^{-1}(k+1)\underline{\nu}(k+1)$$
$$\underline{C} = \Theta_C S^{-1}(k+1)\underline{\nu}(k+1)$$

$$J_b[\,] = \frac{1}{2}\underline{\nu}(k+1)^T S^{-1}(k+1)^T[\underbrace{\Theta_A^T P^{-1}(k)\Theta_A}_{\Phi_A} + \underbrace{\Theta_C^T Q^{-1}(k)\Theta_C}_{\Phi_C} + \underbrace{\Theta_B^T R^{-1}(k+1)\Theta_B}_{\Phi_B}] \dots$$
$$S^{-1}(k+1)\underline{\nu}(k+1)$$

$$\Phi_A = \left(P(k)F^T(k)H^T(k+1)\right)^T P^{-1}(k)P(k)F^T(k)H^T(k+1)$$

$$\Phi_A = H(k+1)F(k)P^T(k)F^T(k)H^T(k+1)$$

$$\Phi_c = H(k+1)\Gamma(k)Q^T(k)Q^{-1}(k)Q(k)\Gamma^T(k)H^T(k+1)$$

$$\Phi_c = H(k+1)\left[\bar{P}(k+1) - F(k)P(k)F^T(k)\right]^T H^T(k+1)$$

$$\Phi_c = H(k+1)\bar{P}^T(k+1)H^T(k+1) - H(k+1)F(k)P^T(k)F^T(k)H^T(k+1)$$

$$\Phi_B = R^T(k+1)R^{-1}(k+1)R(k+1)$$

$$\Phi_B = R^T(k+1) = S^T(k+1) - H(k+1)\bar{P}^T(k+1)H^T(k+1)$$

Putting it all together:

$$J_b[\,] = \frac{1}{2}\nu^T(k+1)S^{-1}(k+1)^T\Big[H(k+1)F(k)P^T(k)F^T(k)H^T(k+1) + \ldots$$

$$H(k+1)\bar{P}^T(k+1)H^T(k+1) + \ldots$$

$$-H(k+1)F(k)P^T(k)F^T(k)H^T(k+1) + \ldots$$

$$S^T(k+1) - H(k+1)\bar{P}^T(k+1)H^T(k+1)\ldots$$

$$\Big]S^{-1}(k+1)\nu(k+1)$$

$$J_b[\hat{\underline{v}}(k), \hat{\underline{x}}(k+1), k] = \frac{1}{2}\nu^T(k+1)S^{-1}(k+1)^T\left[S^T(k+1)\right]S^{-1}(k+1)\underline{\nu}(k+1)$$

$$\boxed{J_b[\hat{\underline{v}}(k), \hat{\underline{x}}(k+1), k] = \frac{1}{2}\nu^T(k+1)S^{-1}(k+1)\nu(k+1)} \quad ✓$$

$$\left(S(k+1) = S^T(k+1)\right)$$