BS 3-13)

$$x \sim N(\bar{x}, P_{xx})$$

$$z = x + w$$

$$w \sim N(0, P_{ww}), \quad E[xw] = 0$$

$$y = z^2 = (x+w)^2$$

Find the LMMSE estimate of x in terms of y and the associated MSE

$$\hat{x} = \bar{x} + P_{xy} P_{yy}^{-1} (y - \bar{y})$$

$$\bar{y} = E[y] = E[z^2] = E[(x+w)^2] = E[x^2 + 2\overset{\nearrow 0}{xw} + w^2] = P_{xx} + \bar{x}^2 + P_{ww}$$

$$P_{xy} = E[(x-\bar{x})(y-\bar{y})] = E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}] = E[xy] - \bar{y}E[x] - \bar{x}E[y] + \bar{x}\bar{y}$$

$$P_{xy} = E[xy] - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} = E[xy] - \bar{y}\bar{x}$$

$$E[xy] = E[x(x+w)^2] = E[x^3 + 2\overset{\nearrow 0}{x^2 w} + xw^2] = E[x^3] + \bar{x} P_{ww}$$

use characteristic function of scalar random variable:

$$M_{x'}(s) = e^{\frac{1}{2}s^2\sigma^2}, \quad x' = x - \bar{x}, \quad \sigma^2 = P_{xx}$$

$$\left.\frac{d^3 M_{x'}}{ds^3}\right|_{s=0} = (3(0)\sigma^4 + (0)^3\sigma^6)\, e^{\frac{1}{2}s^2\sigma^2} = 0 = E[x'^3]$$

$$E[x^3] = E[(x' + \bar{x})^3] = E[x'^3 + 3x'^2\bar{x} + 3\bar{x}^2 x' + \bar{x}^3]$$

$$= \overset{\nearrow 0}{E[x'^3]} + 3\bar{x}E[x'^2] + 3\bar{x}^2 \overset{\nearrow 0}{E[x']} + \bar{x}^3 = 3\bar{x}P_{xx} + \bar{x}^3$$

$$P_{xy} = 3\bar{x}P_{xx} + \bar{x}^3 - \bar{x}(P_{xx} + P_{ww} + \bar{x}^2) + \bar{x}P_{ww}$$

$$P_{yy} = E[(y-\bar{y})(y-\bar{y})] = E[y^2 - 2y\bar{y} + \bar{y}^2] = E[y^2] - 2\bar{y}E[y] + \bar{y}^2$$

$$P_{yy} = E[\overbrace{(x+w^2)(x+w^2)}^{y^2}] - 2\overbrace{(P_{xx} + P_{ww} + \bar{x}^2)^2 + (P_{xx} + P_{ww} + \bar{x}^2)^2}^{-(1)\bar{y}^2}$$

$$E[y^2] = E[x^4 + 4x^3w \nearrow^0 + 6x^2w^2 + 4xw^3 \nearrow^0 + w^4] = E[x^4] + 6E[x^2w^2] + E[w^4] +$$

$$E[x^4] = E[(x'+\bar{x})^4] = E[x'^4 + 4x'^3\bar{x} + 6x'^2\bar{x}^2 + 4x'\bar{x}^3 + \bar{x}^4]$$

$$E[x^4] = E[x'^4] + 4\bar{x}E[x'^3] \nearrow^0 + 6\bar{x}^2 E[x'^2] \nearrow^{P_{xx}} + 4\bar{x}^3 E[x'] \nearrow^0 + \bar{x}^4$$

From characteristic function: $E[y^2] = 3P_{xx}^2 + 6\bar{x}^2P_{xx} + \bar{x}^4 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2$

$$P_{yy} = 3P_{xx}^2 + 6\bar{x}^2P_{xx} + \bar{x}^4 - (P_{xx} + P_{ww} + \bar{x}^2)^2 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2$$

LMMSE: $\hat{x} = \bar{x} + (3\bar{x}P_{xx} + \bar{x}^3 - \bar{x}(P_{xx} + P_{ww} + \bar{x}^2) + \bar{x}P_{ww}) \circ \ldots$

$(3P_{xx}^2 + 6\bar{x}^2P_{xx} + \bar{x}^4 - (P_{xx} + P_{ww} + \bar{x}^2)^2 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2)^{-1} \circ \ldots$

$(y - (P_{xx} + P_{ww} + \bar{x}^2))$

MSE: $E[\tilde{x}\tilde{x}] = P_{xx} - P_{xy}P_{yy}^{-1}P_{yx}$

Restating for clarity:

$$\boxed{\begin{aligned} \hat{x}_{LMMSE} &= \bar{x} + P_{xy}P_{yy}^{-1}(y-\bar{y}), \quad \bar{y} = P_{xx} + P_{ww} + \bar{x}^2 \\ MSE &= P_{xx} - P_{xy}P_{yy}^{-1}P_{yx} \\ P_{xy} &= 3\bar{x}P_{xx} + \bar{x}^3 + \bar{x}P_{ww} - \bar{x}\bar{y} \\ P_{yy} &= 3P_{xx}^2 + 6\bar{x}^2P_{xx} + \bar{x}^4 + 6(P_{xx} + \bar{x}^2)P_{ww} + 3P_{ww}^2 - \bar{y}^2 \end{aligned}}$$