Posting Date: Tuesday Aug. 27th.

You need not hand in anything. Instead, be prepared to answer any of these problems on an upcoming take-home prelim exam. You may discuss your solutions with classmates up until the time that the prelim becomes available.

1. Prove that a quadratic form with a non-symmetric matrix has an equivalent for any given $\underline{x}$ that uses a symmetric matrix, and derive the formula for the symmetric matrix in terms of the original non-symmetric matrix. In other words, prove that there exists a symmetric matrix P such that $\underline{x}^T S \underline{x} = \underline{x}^T P \underline{x}$ for all $\underline{x}$ even though $S \neq S^T$, and derive a formula for P in terms of S.
2. Prove that the vector 2-norm satisfies the 3 norm properties. Do not use Schwartz' inequality in order to prove the triangle inequality unless you also prove Schwartz' inequality. Hint: in order to prove the triangle inequality, it might be helpful to decompose one of the two vectors into two components, one that is parallel to the other vector and one that is perpendicular to the other vector.
3. Prove that if A is an mxn matrix, if rank(A) = $l < \min(m,n)$, and if the first l columns of A are linearly independent, then the upper triangular R matrix in the QR factorization of A has only an $l \times l$ nonsingular part and the last m-l rows of the matrix are identically zero. In other words, prove that

$$A = Q\begin{bmatrix} R_{l\times l} & X_{l\times(n-l)} \\ 0_{(m-l)\times l} & 0_{(m-l)\times(n-l)} \end{bmatrix}$$

   where Q is orthonormal, $R_{l\times l}$ is upper triangular and nonsingular, and $X_{l\times(n-l)}$ is some $lx(n$-$l)$ matrix. Hint: left-multiplication of a matrix by a nonsingular matrix does not change the linear dependence/independence of the columns of the matrix.
4. Prove that a Householder transformation is orthonormal.
5. Prove that Tr(AB) = Tr(BA) if A is an mxn matrix and B is an nxm matrix.
6. Determine two Householder transformations, H_1 and H_2, such that

$$H_1 H_2 \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 3 & -14 \\ -13 & 16 \end{bmatrix}$$

   Hint: You may want to review pp. 37-40 of Gill, Murray, and Wright.

7. Prove that the Gaussian distribution of a scalar random variable x:

$$p_x(\theta) = \frac{1}{\sqrt{2\pi}b} e^{-(\theta-a)^2/2b^2}$$

   has mean $\bar{x} = a$ and variance $\sigma^2 = b^2$; i.e., prove that E[x] = a and that E[$(x-a)^2$] = b^2

   Hints: Use variable transformation, integration by parts, and the fact that the integral of $p_x(\theta)$ from negative infinity to positive infinity is equal to 1.

8. Plot the histogram produced by 2 different 100,000-point sample realizations of a sinusoidal distribution. Also, plot the histogram that is produced by a 100,000-point sample realization of a variable that is the average of 100 independent sinusoidally distributed random variables. See if you can plot in probability density units instead of in histogram units. This will require you to get variables output from the histogram function, and you will need to use the "bar" function.

   Discuss how these results reflect on the central limit theorem. Do they help to convince you of its correctness?

   Help: The MATLAB .m-file centrall.m can be down-loaded from the course web site. It will help you to carry out this part of the assignment.

9. Prove that the Gaussian distribution of an n-dimensional random vector $\underline{x}$:

$$p(\underline{x}) = \frac{1}{(2\pi)^{n/2}[\det(P)]^{1/2}} \exp\{-\tfrac{1}{2}(\underline{x}-\overline{\underline{x}})^{\mathrm{T}} P^{-1}(\underline{x}-\overline{\underline{x}})\}$$

   has a mean of $\overline{\underline{x}}$ and a covariance matrix of P; i.e., prove that $E[\underline{x}] = \overline{\underline{x}}$ and that $E[(\underline{x} - \overline{\underline{x}})(\underline{x} - \overline{\underline{x}})^{\mathrm{T}}] = P$. Also, prove that this distribution is normalized; i.e., that its integral over the whole sample space is equal to 1.

   Hints: Use the transformation $\underline{z} = V^{\mathrm{T}}\underline{x}$, where V is the $n \times n$ orthonormal matrix whose columns are the eigenvectors of P so that $P = V\mathrm{diag}([\sigma_{z1}^2, \sigma_{z2}^2, ..., \sigma_{zn}^2])V^{\mathrm{T}}$. The elements of $\underline{z} = [z_1; z_2; ...; z_n]$ will be statistically independent, which will allow you to factorize the probability density function for the z_i components into n scalar Gaussian distributions. You can then use the known properties of scalar Gaussian distributions in order to prove the results that you need; that is, you can use what you know about various expectation-type integrals of scalar Gaussian distributions. It may prove helpful to recall that $[\det(V)]^2 = 1$ if $V^{\mathrm{T}}V = I$ and that the determinant of a product is the product of the determinants if all of the matrices in question are square. Also, use the fact that $\exp(a + b) = \exp(a)*\exp(b)$.

10. Given the joint probability distribution for $\underline{\eta}_1$ and $\underline{\eta}_2$:

$$p(\underline{\eta}_1,\underline{\eta}_2) = \frac{1}{(2\pi)^{m/2}[\det(P_\eta)]^{1/2}} \exp\{-\tfrac{1}{2}(\underline{\eta}_1^{\mathrm{T}} H_{11}\underline{\eta}_1 + 2\underline{\eta}_1^{\mathrm{T}} H_{12}\underline{\eta}_2 + \underline{\eta}_2^{\mathrm{T}} H_{22}\underline{\eta}_2)\}$$

   where $\underline{\eta}_1$ is an n-dimensional vector and $\underline{\eta}_2$ is an $(m-n)$-dimensional vector and where

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^{\mathrm{T}} & H_{22} \end{bmatrix} = P_\eta^{-1}$$

   Prove that the integral of $p(\underline{\eta}_1,\underline{\eta}_2)d\underline{\eta}_1$ over the whole $\underline{\eta}_1$ space is equal to

$$\frac{(2\pi)^{n/2}}{(2\pi)^{m/2}[\det(P_\eta)]^{1/2}[\det(H_{11})]^{1/2}} \exp\{-\tfrac{1}{2}\underline{\eta}_2^{\mathrm{T}}(H_{22} - H_{12}^{\mathrm{T}} H_{11}^{-1} H_{12})\underline{\eta}_2\}$$

Note that P_η, H_{11}, and H_{22} are all symmetric matrices, as one would expect for the covariance and inverse covariance matrices of a vector Gaussian distribution.

Hints: In the expression for $p(\underline{\eta}_1,\underline{\eta}_2)$ express the argument of the exponential function in the form $-\frac{1}{2}(\underline{\eta}_1 - \overline{\underline{\eta}}_1)^{\mathrm{T}} H_{11}(\underline{\eta}_1 - \overline{\underline{\eta}}_1) - \frac{1}{2}\underline{\eta}_2^{\mathrm{T}} G \underline{\eta}_2$ for an appropriately chosen vector $\overline{\underline{\eta}}_1$ and an appropriately chosen matrix G. Then use the fact that the integral over the whole $\underline{\eta}_1$ space of $\exp\{-\frac{1}{2}\underline{\eta}_1^{\mathrm{T}} H_{11}\underline{\eta}_1\}$ must equal $(2\pi)^{n/2}[\det(H_{11}^{-1})]^{1/2}$ in order for the corresponding multivariate Gaussian function to be normalized. Note that $\det(A^{-1}) = 1/\det(A)$. An appropriate choice of $\overline{\underline{\eta}}_1$ is $\overline{\underline{\eta}}_1 = A\underline{\eta}_2$. You will have to determine an appropriate A matrix for this choice.

11. Suppose that you roll a single die twice. What is the probability that the first roll is a 5 conditioned on the presumption that the sum of the two rolls is 9?

Also do the following problems from Bar Shalom:

1-1, 1-6, 1-7, 1-9, 1-15

Hints for 1-9: Create the vector $\underline{z} = \underline{e}\theta + \underline{w}$, where $\underline{e}$ is a vector of the same length as $\underline{z}$ and $\underline{w}$ and where its entries are all 1's. Be sure to check out the notes on locally most powerful tests. You should use matrix-vector notation in your derivation of the locally most powerful test statistic. You should be able to develop a locally most powerful test statistic that obeys a Gaussian distribution. This will enable you to solve for the correct threshold value using the MATLAB function norminv.m, but you will first have to use a symmetry argument about your H_0 probability distribution before you will be able to determine the threshold using norminv.m. You will need to install MATLAB's statistics toolbox in order to access this function.