AOE 5984 Handout on Locally Most Powerful Hypothesis Tests

1. One-Sided Locally Most Powerful Test

Often in hypothesis testing one has alternate hypotheses of the form

H_0: $\theta = 0$

H_1: $\theta > 0$

where H_1 is considered true for any $\theta > 0$, not just for a single value, and there is no *a priori* distribution given for the possible values of $\theta > 0$ under hypothesis H_1. Therefore, it is not possible to write down a specific probability density for the measurement z given H_1, even though it is possible to write down a probability density for z given θ equal to any particular positive value, say $\theta_1 > 0$. This type of situation often occurs when the parameter θ is something like the returned signal strength of a radar reflection. If there is any returned strength, no matter how small, then there is an aircraft or a missile "out there", even though the absolute strength of the signal, θ, can vary with the type of object, its orientation, and its distance.

One attacks this type of problem by using a type of hypothesis test that is called a locally most powerful test. It is developed by first forming the optimal Neyman-Pearson test statistic under the assumption that hypothesis H_1 corresponds to a specific value $\theta = \theta_1$. This statistic takes the form

$$q(z,\theta_1) \;=\; log[\frac{p(z\,|\,\theta=\theta_1)}{p(z\,|\,\theta=0)}]$$

(actually, it is not necessary to take the natural logarithm, but it is often helpful to do so). The next step is to expand this function in a Taylor series in θ_1:

$$q(z,\theta_1) \;=\; q(z,0)+\left[\left.\frac{\partial q}{\partial\theta_1}\right|_{(z,0)}\right]\theta_1+\left[\frac{1}{2}\left.\frac{\partial^2 q}{\partial\theta_1^2}\right|_{(z,0)}\right]\theta_1^2+O(\theta_1^3)$$

It is easy to show that $q(z,0) = 0$. Therefore

$$q(z,\theta_1) \;=\; \left[\left.\frac{\partial q}{\partial\theta_1}\right|_{(z,0)}\right]\theta_1+\left[\frac{1}{2}\left.\frac{\partial^2 q}{\partial\theta_1^2}\right|_{(z,0)}\right]\theta_1^2+O(\theta_1^3) \tag{1}$$

The locally most powerful test is the Neyman-Pearson test applied in the limit at $\theta_1 \to 0$. Thus, it minimizes the probability of a missed detection for a given probability of false alarm in the limit as $\theta_1 \to 0$. This is normally a very wise thing to do because the probability of a missed-detection is likely to be the highest when θ_1 is small because it is near the value that applies for hypothesis H_0. This test will not be optimal when θ_1 is large; that is, it will not minimize the probability of a missed detection for a given probability of false alarm. This normally is not a problem because the probability of a missed detection will be very low when θ_1 is large, even though it will not be the lowest that it could possibly be.

If the coefficient of the θ_1 term in the Taylor series expansion is non-zero, then the locally most powerful test takes the following form:

Accept H_1 if $\left[\left.\frac{\partial q}{\partial \theta_1}\right|_{(z,0)}\right]\theta_1 \geq q_0$; otherwise, accept H_0.

Unfortunately, this test still involves the unknown positive value of θ_1. Therefore, both sides of the inequality are divided by θ_1 in order to yield the following locally most power test:

Accept H_1 if $\beta(z) = \left[\left.\frac{\partial q}{\partial \theta_1}\right|_{(z,0)}\right] \geq \frac{q_0}{\theta_1} = \beta_0$; otherwise, accept H_0.

The condition on the derivative function $\beta(z)$ and its threshold value β_0 can be evaluated independent of the particular value of θ_1 that occurs. The threshold value is determined in the usual way. It must satisfy:

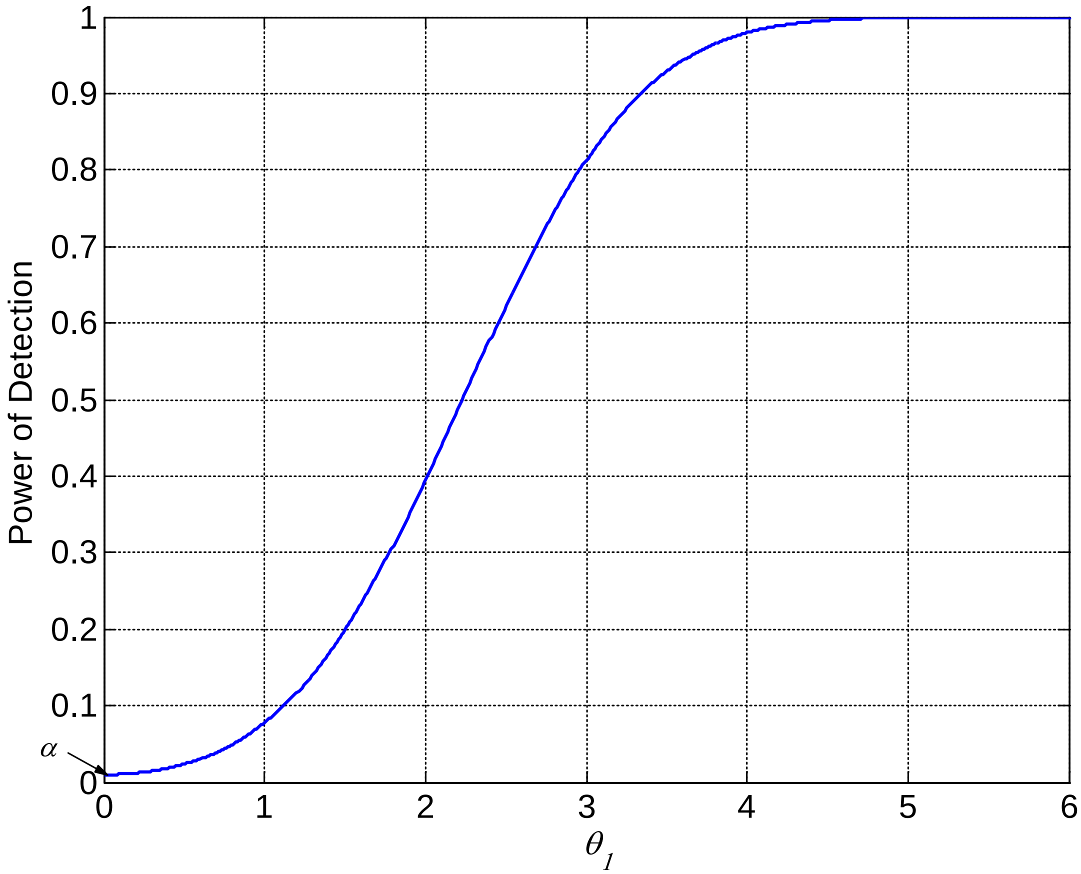
$$\alpha = P\{\beta \geq \beta_0 \mid H_0\} = \int_{\beta_0}^{\infty} p(\beta \mid H_0)d\beta = \int_{\beta_0}^{\infty} p(\beta \mid \theta = 0)d\beta \tag{2}$$

where α is the desired false alarm probability. The test is designed by picking a value for α and then solving this equation for β_0. Note that the test statistic β is random because it depends on the measurement z, which has randomness that is caused by its measurement error. The probability density function $p(z|\theta=0)$ that models the randomness in z under hypothesis H_0 is used along with the formula for $\beta(z)$ in order to derive the probability density function $p(\beta|\theta=0)$ that is used in the above formula.

It is often interesting in these situations to compute the power of the test as a function of θ_1. The power of the test is the probably of accepting hypothesis H_1 when $\theta = \theta_1 > 0$. This power function is computed as follows:

$$Power(\theta_1) = \int_{\beta_0}^{\infty} p(\beta \mid \theta = \theta_1)d\beta \tag{3}$$

A typical plot of this function might look like the following:



It is easy to show that $Power(0) = \alpha$.

If the coefficient of the θ_1 term in the Taylor series expansion of $q(z,\theta_1)$ is identically zero for all z, then the locally most powerful test takes the alternative form:

$$\text{Accept } H_1 \text{ if } \left[\frac{1}{2} \left. \frac{\partial^2 q}{\partial \theta_1^2} \right|_{(z,0)} \right] \theta_1^2 \geq q_0 \text{; otherwise, accept } H_0.$$

This test can be translated into the following test on the second derivative of $q(z,\theta_1)$:

$$\text{Accept } H_1 \text{ if } \gamma(z) = \left[\left. \frac{\partial^2 q}{\partial \theta_1^2} \right|_{(z,0)} \right] \geq \frac{2q_0}{\theta_1^2} = \gamma_0 \text{; otherwise, accept } H_0.$$

One can perform all of the usual threshold and power-of-test calculations for this test by using the function $\gamma(z)$ and the threshold γ_0 in place of the function $\beta(z)$ and the threshold β_0 in Eqs. (2) and (3).

2. Two-Sided Locally Most Powerful Test

Suppose, now, that the H_1 hypotheses is altered so that the two hypotheses are:

H_0: $\theta = 0$

H_1: $\theta \neq 0$

It is still possible to define a locally most powerful test using similar techniques in this case, only now the test must consider the two possibilities $\theta_1 > 0$ and $\theta_1 < 0$. Equation (1) still applies as does the limiting Neyman-Pearson test

$$\text{Accept } H_1 \text{ if } \left[\left. \frac{\partial q}{\partial \theta_1} \right|_{(z,0)} \right] \theta_1 \geq q_0 \text{; otherwise, accept } H_0.$$

The two sign possibilities for θ_1 force the following transformation of this test in order to eliminate its θ_1 dependence:

$$\text{Accept } H_1 \text{ if } |\beta(z)| = \left| \left[\left. \frac{\partial q}{\partial \theta_1} \right|_{(z,0)} \right] \right| \geq \frac{q_0}{|\theta_1|} = \beta_0 \text{; otherwise, accept } H_0.$$

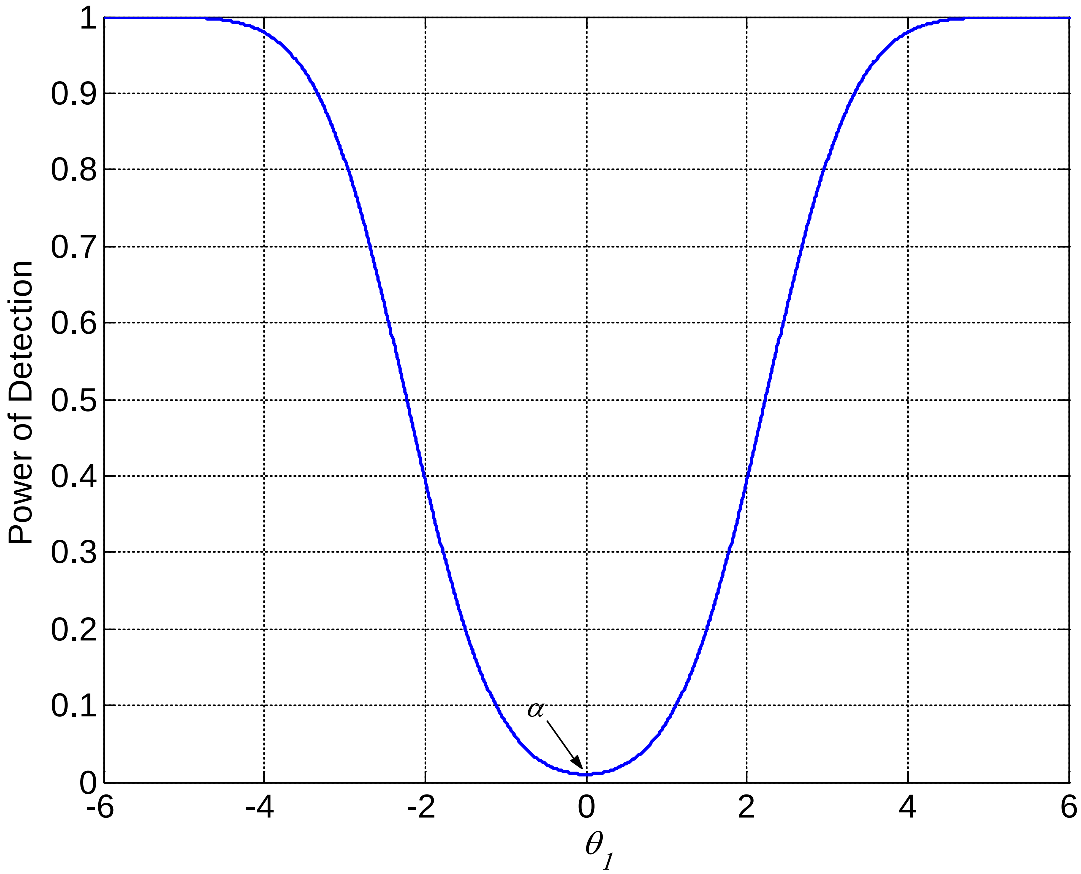
The threshold value β_0 is then evaluated by picking the false alarm probability α and solving the following equation for β_0:

$$\begin{aligned} \alpha = P\{|\beta(z)| \geq \beta_0 \mid H_0\} &= \int_{-\infty}^{-\beta_0} p(\beta \mid H_0) d\beta + \int_{\beta_0}^{\infty} p(\beta \mid H_0) d\beta \\ &= \int_{-\infty}^{-\beta_0} p(\beta \mid \theta = 0) d\beta + \int_{\beta_0}^{\infty} p(\beta \mid \theta = 0) d\beta \end{aligned}$$

The test's power as a function of θ_1 is determined by using the formula:

$$Power(\theta_1) = \int_{-\infty}^{-\beta_0} p(\beta \mid \theta = \theta_1) d\beta + \int_{\beta_0}^{\infty} p(\beta \mid \theta = \theta_1) d\beta$$

A typical plot of this two-sided power function looks like:



Again, it is easy to show that *Power*(0) = α.