AOE 5784 Prelim #1 Solution
Fall 2022

1) [10 Pts] P.S. #1, Problem #11

There are 36 equi-probable outcomes of rolling a single die twice. Of those, 4 of the outcomes produce a sum equal to 9. They are

First Roll	Second Roll
3	6
4	5
5	4
6	3

Thus, the condition that the sum equal 9 restricts us to consider only these 4 equi-probable outcomes. Only one of these outcomes has a first roll of 5. Therefore, the answer is $\boxed{1/4}$

Alternate analysis $A = \{Sum = 9\}$ $B = \ldots$ $\{$1st roll is five$\}$. $P\{B|A\} = P\{A|B\}P\{B\}/P\{A\}$ $P\{B\} = 1/6$ - obviously. $P\{A\} = 4/36$ from above analysis. $P\{A|B\} = (\text{Probability that 2nd roll} = 4) = 1/6$. Therefore, answer is

$$P\{B/A\} = \left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)/\left(4/36\right) = \boxed{1/4}$$

2) [15 Pts] Problem 1-1 in Bar-Shalom

1-1.1 : The key word is "arbitrary". This implies that the assertion is true only if it is true in every imaginable case. The assertion is

$$\boxed{\text{NOT TRUE}}$$

because one or more counter-examples can be constructed. Taking a hint from part 2 and modifying it slightly, let $x \sim N(0,1)$ and $y \sim N(0,1)$ be independent so that

$$p(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right)$$

but let $z = x + y + w$ where w is $w \sim N(0,1)$ with w independent of x and y, but not independent of z. Then

$$p(x,y,z) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}\left[x^2+y^2+\{z-x-y\}^2\right]}$$

$$p(x,y|z) = p(x,y,z)/p(z)$$

$$p(z) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}\left[x^2+y^2+\{z-x-y\}^2\right]}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left[2x^2+z^2-2xz-\frac{1}{2}(z-x)^2\right]} \quad *$$

$$\int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}\left[2\left(y-\frac{1}{2}(z-x)\right)^2\right]}$$

letting $\eta = \sqrt{2}\left[y - \frac{1}{2}(z-x)\right]$

and $d\eta = \sqrt{2}\, dy$ in the last integral

then $$p(z) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left[2x^2+z^2-2xz-\frac{1}{2}(z-x)^2\right]} \quad *$$

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} d\eta\, e^{-\frac{1}{2}\eta^2}$$

$$\therefore\ p(z) = \frac{1}{\sqrt{2}\,(2\pi)} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left[2x^2+z^2-2xz-\frac{1}{2}(z-x)^2\right]}$$

$$= \frac{1}{\sqrt{2}\,(2\pi)} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left[x^2+\frac{1}{2}(z-x)^2\right]}$$

$$= \frac{1}{\sqrt{2}\,(2\pi)} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\left[1.5\left(x-\frac{1}{3}z\right)^2+\frac{1}{3}z^2\right]}$$

Let $\alpha = \sqrt{\frac{3}{2}}\left(x - \frac{1}{3} z\right)$

$$d\alpha = \sqrt{\frac{3}{2}}\, dx$$

Then $p(z) = \dfrac{1}{\sqrt{2}\,(2\pi)} \sqrt{\dfrac{2}{3}} \left[\displaystyle\int_{-\infty}^{\infty} d\alpha\, e^{-\frac{1}{2}\alpha^2}\right] e^{-\frac{1}{2}\left[\frac{1}{3} z^2\right]}$

$$= \frac{1}{\sqrt{2\pi}\,\sqrt{3}}\, e^{-\frac{1}{2}\left[\frac{z}{\sqrt{3}}\right]^2}$$

which is a Gaussian distribution with a mean of zero and a standard deviation of $\sigma_z = \sqrt{3}$, as expected.

So $p(x,y\,|\,z) = \dfrac{\left(\dfrac{1}{(2\pi)^{3/2}}\right) e^{-\frac{1}{2}\left\{x^2 + y^2 + \{z - x - y\}^2\right\}}}{\left(\dfrac{1}{\sqrt{2\pi}\,\sqrt{3}}\right) e^{-\frac{1}{2}\left(\frac{z}{\sqrt{3}}\right)^2}}$

$$= \frac{\sqrt{3}}{2\pi}\, e^{-\frac{1}{2}\left\{2x^2 + 2y^2 + \frac{2}{3} z^2 - 2xz - 2yz + 2xy\right\}}$$

$$= \frac{\sqrt{3}}{2\pi}\, e^{-\frac{1}{2}\left(\left\{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} z\right\}^T \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}^{-1} \left\{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} z\right\}\right)}$$

therefore $\begin{bmatrix} x \\ y \end{bmatrix}$ conditioned on z obeys a joint Gaussian distribution with mean equal to $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} z$ and

with covariance equal to

$$E\left[\left\{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} z\right\}\left\{\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} z\right\}^T \Big/ z\right]$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

The non-zero off-diagonal terms in this matrix prove that x and y are not independent when conditioned on z and, therefore, that it is impossible to express $p(x, y | z)$ in the form

$$p(x, y, z) = p(x|z)\,p(y|z)$$

1.1-2 $$p(x,y,z) = p(x,y)\,\delta(z-x-y)$$

$$p(x,y|z) = \frac{p(x,y,z)}{p(z)}$$

$$p(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, p(x,y,z)$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, p(x,y)\,\delta(z-x-y)$$

$$= \int_{-\infty}^{\infty} dx\, p(x, z-x)$$

$$= \int_{-\infty}^{\infty} dy\, p(z-y, y)$$ [alternate equivalent formula]

therefore

$$p(x,y|z) = \frac{p(x,y)\,\delta(z-x-y)}{\int_{-\infty}^{\infty} dy\, p(y, z-y)}$$

or

$$p(x,y|z) = \frac{p(x,y)\,\delta(z-x-y)}{\int_{-\infty}^{\infty} d\alpha\, p(z-\alpha, \alpha)}$$

$$p(z) = \int_{-\infty}^{\infty} dx \, \frac{1}{2\pi} e^{-\frac{1}{2}\left[x^2 + (z-x)^2\right]}$$

$$= \int_{-\infty}^{\infty} dx \, \frac{1}{2\pi} e^{-\frac{1}{2}\left[2x^2 - 2xz + z^2\right]}$$

$$= \int_{-\infty}^{\infty} dx \, \frac{1}{2\pi} e^{-\frac{1}{2}\left[2\left(x - \frac{1}{2}z\right)^2 + \frac{1}{2}z^2\right]}$$

letting $\quad \eta = \sqrt{2}\left(x - \frac{1}{2}z\right)$

$\quad d\eta = \sqrt{2}\, dx$

$$p(z) = \frac{1}{2\pi\sqrt{2}} \left[\int_{-\infty}^{\infty} d\eta \, e^{-\frac{1}{2}\eta^2}\right] e^{-\frac{1}{4}z^2}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{2}}\right)^2}$$

therefore

$$p(x,y|z) = \frac{\left(\frac{1}{2\pi}\right) e^{-\frac{1}{2}(x^2+y^2)}}{\left(\frac{1}{\sqrt{2\pi}\sqrt{2}}\right) e^{-\frac{1}{2}\left(\frac{z}{\sqrt{2}}\right)^2}} \delta(z-x-y)$$

$$\therefore \boxed{p(x,y|z) = \frac{\sqrt{2}}{\sqrt{2\pi}} e^{\frac{1}{4}z^2} e^{-\frac{1}{2}(x^2+y^2)} \delta(z-x-y)}$$

or recognizing that it is nonzero only when $z-x-y=0$

$$p(x,y|z)=\frac{\sqrt{2}}{\sqrt{2\pi}}\,e^{\frac{1}{4}z^2-\frac{1}{2}\left[(z-y)^2+y^2\right]}\,\delta(z-x-y)$$

$$=\frac{\sqrt{2}}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left\{2\left(y-\frac{1}{2}z\right)^2\right\}}\,\delta(z-x-y)$$

$$=\frac{\sqrt{2}}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left\{2\left[x-\frac{1}{2}z\right]^2\right\}}\,\delta(z-x-y)$$

3) [10 pts] P.S. #2, Problem #2

Let $\underline{\eta}=\underline{x}-\underline{\bar{x}}$. Then

$$C(\underline{x}|\underline{z})=\frac{1}{2}\underline{\eta}^T V_{xx}\underline{\eta}+\left[V_{xz}(\underline{z}-\underline{\bar{z}})\right]^T\underline{\eta}$$

$$+\frac{1}{2}(\underline{z}-\underline{\bar{z}})^T V_{zz}(\underline{z}-\underline{\bar{z}}) \;=\; J_{new}(\underline{\eta})$$

This is like the function in P.S. #2, Problem #1, except $\underline{\eta}$ replaces $\underline{x}$, V_{xx} replaces P, $[V_{xz}(\underline{z}-\underline{\bar{z}})]$ replaces g and the constant $\frac{1}{2}(\underline{z}-\underline{\bar{z}})^T V_{zz}(\underline{z}-\underline{\bar{z}})$ has been added to the function. This constant will not affect the values of any derivatives with respect to η. Therefore

$$\left(\frac{\partial C}{\partial \underline{x}}\right)^T = \left(\frac{\partial J_{new}}{\partial \eta} \frac{\partial \eta}{\partial \underline{x}}\right)^T \qquad \text{(Chain rule)}$$

and $$\frac{\partial^2 C}{\partial \underline{x}^2} = \left(\frac{\partial \eta}{\partial \underline{x}}\right)^T \frac{\partial^2 J_{new}}{\partial \eta^2} \left(\frac{\partial \eta}{\partial \underline{x}}\right)$$

but $$\frac{\partial \eta}{\partial \underline{x}} = I$$

$$\frac{\partial J_{new}}{\partial \eta} = \eta^T V_{xx} + (\underline{z} - \bar{\underline{z}})^T V_{xz}^T$$

(from Problem # 1 of P.S. 2)

$$= (\underline{x} - \bar{\underline{x}})^T V_{xx} + (\underline{z} - \bar{\underline{z}})^T V_{xz}^T$$

$$\frac{\partial^2 J_{new}}{\partial \eta^2} = V_{xx}$$

Therefore $$\left(\frac{\partial C}{\partial \underline{x}}\right)^T = V_{xx}(\underline{x} - \bar{\underline{x}}) + V_{xz}(\underline{z} - \bar{\underline{z}})$$

$$\frac{\partial^2 C}{\partial \underline{x}^2} = V_{xx}$$

4) [20 Pts] P.S. #2, Problem # 5

$$p(x_1,\ldots,x_n|H_0)=\frac{1}{2^n\sigma_A^n}\exp\left\{-\frac{1}{\sigma_A}\sum_{i=1}^{n}|x_i|\right\}$$

$$p(x_1,\ldots,x_n|H_1)=\frac{1}{2^n\sigma_B^n}\exp\left\{-\frac{1}{\sigma_B}\sum_{i=1}^{n}|x_i|\right\}$$

Optimal Test: Accept H_1 if

$$\frac{p(x_1,\ldots,x_n|H_1)}{p(x_1,\ldots,x_n|H_0)}\geq\lambda$$

or

$$\frac{\sigma_A^n}{\sigma_B^n}\exp\left\{\left(\frac{1}{\sigma_A}-\frac{1}{\sigma_B}\right)\sum_{i=1}^{n}|x_i|\right\}\geq\lambda$$

or $$\exp\left\{\left(\frac{\sigma_B-\sigma_A}{\sigma_A\sigma_B}\right)\sum_{i=1}^{n}|x_i|\right\}\geq\left(\frac{\sigma_B}{\sigma_A}\right)^n\lambda$$

or $$\left(\frac{\sigma_B-\sigma_A}{\sigma_A\sigma_B}\right)\sum_{i=1}^{n}|x_i|\geq n\log\left(\frac{\sigma_B}{\sigma_A}\right)+\log\lambda$$

or $$q=\sum_{i=1}^{n}|x_i|\geq\left[\frac{\sigma_A\sigma_B}{\sigma_B-\sigma_A}\right]\left\{n\log\left(\frac{\sigma_B}{\sigma_A}\right)+\log\lambda\right\}$$

or Accept H_1 if

$$q=\sum_{i=1}^{n}|x_i|\geq q_{thresh}$$

Otherwise accept H_0. Given that $2q/\sigma_A = \eta$ is a sample from $p_{\chi^2_{2n}}(\eta)$ - a degree-$2n$ chi-squared distribution if H_0 is valid, we choose q_{thresh} so that

$$\int_{2q_{thresh}/\sigma_A}^{\infty} p_{\chi^2_{2n}}(\eta)\,d\eta = \alpha = \text{False alarm probability}$$

Using Matlab: $\eta_{THRESH} = \text{chi2inv}(1-\alpha, 2*n)$
and $q_{THRESH} = (\sigma_A/2)\eta_{THRESH}$

Given that $2q/\sigma_B = \beta$ is a sample from $p_{\chi^2_{2n}}(\beta)$ if H_1 is valid, the probability of a missed detection is then

$$P_{MD} = \int_0^{2q_{THRESH}/\sigma_B} p_{\chi^2_{2n}}(\beta)\,d\beta$$

In MATLAB $P_{MD} = \text{chi2cdf}(2*q_{THRESH}/\sigma_B, 2*n)$

For $n = 10$ & $\alpha = 0.001$ $\eta_{THRESH} = 45.3147$

$\boxed{q_{THRESH} = 22.6574\ \sigma_A.}$

$$P_{MD} = \text{chi2cdf}\left(2*22.6574*\left(\frac{\sigma_A}{\sigma_B}\right), 20\right)$$

$$= \text{chi2cdf}\left(2*22.6574*\left(\frac{1}{4}\right), 20\right) = \boxed{0.0627508}$$

5) [20 pts] P.S. #3, Problem #1 with modifications

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F & df/dx function evaluations:

```
function [f,dfdx] = fnewt02(x)
%
%  Copyright (c) 2009 Mark L. Psiaki.  All rights reserved.
%
%
%  This function computes f(x) and its Jacobian for an example problem
%  in applying Newton's method.
%

%
   f = [(x(1) + x(2) + x(1)*x(2) + 5.1);(x(1)^2 + 2*x(2) - x(2)^2 - 1.9)];
   dfdx = [(1 + x(2)),(1 + x(1));(2*x(1)),(2 - 2*x(2))];
```

Script to execute Newton iterations from 3 different Initial guesses

```
% script_fnewt02.m
%
%  Copyright (c) 2009 Mark L. Psiaki.  All rights reserved.
%
%
%  This script solves problem set 3, problem 1 with altered
%  constant values in the equations.
%

%
   clear
%
   xghist_a = [[4;-4],zeros(2,6)];
   for jj = 2:7
      [fjj,dfdxjj] = fnewt02(xghist_a(:,jj-1));
      xghist_a(:,jj) = xghist_a(:,jj-1) - dfdxjj\fjj;
   end
   fnormvec_a = zeros(1,7);
   for jj = 1:7
      fjj = fnewt02(xghist_a(:,jj));
      fnormvec_a(1,jj) = norm(fjj);
   end
%
   xghist_b = [[6;0],zeros(2,6)];
   for jj = 2:7
      [fjj,dfdxjj] = fnewt02(xghist_b(:,jj-1));
      xghist_b(:,jj) = xghist_b(:,jj-1) - dfdxjj\fjj;
   end
   fnormvec_b = zeros(1,7);
   for jj = 1:7
      fjj = fnewt02(xghist_b(:,jj));
      fnormvec_b(1,jj) = norm(fjj);
   end
%
   xghist_c = [[-5;5],zeros(2,6)];
   for jj = 2:7
      [fjj,dfdxjj] = fnewt02(xghist_c(:,jj-1));
      xghist_c(:,jj) = xghist_c(:,jj-1) - dfdxjj\fjj;
   end
   fnormvec_c = zeros(1,7);
   for jj = 1:7
      fjj = fnewt02(xghist_c(:,jj));
      fnormvec_c(1,jj) = norm(fjj);
   end
%
   clear jj fjj dfdxjj
   save fnewt02
```

Newton Iteration Results

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Case 1

Iteration:	0	1	2	3	4	5	6
x1:	4.000000000000000	3.150000000000000	3.124907435976550	3.137738510489899	3.137732910785439	3.137732910838087	3.137732910838087
x2:	-4.000000000000000	-2.330000000000000	-1.995993520518358	-1.990863527572901	-1.990880776576063	-1.990880776586799	-1.990880776586799
||f||:	14.724808997063425	2.506988075360555	0.111246846319897	0.000153183008289	0.000000000283153	0	0

Case 2

Iteration:	0	1	2	3	4	5	6
x1:	6.000000000000000	3.359756097560976	3.031143295897127	3.139820847552099	3.137733618522335	3.137732910838158	3.137732910838087
x2:	0	-1.208536585365854	-1.956137843100565	-1.991304200035350	-1.990880820683151	-1.990880776586790	-1.990880776586799
||f||:	35.861121008691299	6.367504252497271	0.513502091862765	0.011243597802120	0.000004269722873	0.000000000000501	0.000000000000001

Case 3

Iteration:	0	1	2	3	4	5	6
x1:	-5.000000000000000	-2.822727272727273	-2.142728418502635	-2.068793385194184	-2.068294878539138	-2.068294995763746	-2.068294995763745
x2:	5.000000000000000	3.290909090909091	2.850171866304383	2.837011939305650	2.837891245221337	2.837891234404620	2.837891234404623
||f||:	21.485343841791313	4.142183008622919	0.402149892160641	0.005381887545047	0.000000683681539	0.000000000000015	0.000000000000001

Notice how the first 2 cases converged to the same solution for x1 and x2, but the 3rd case converged to a different solution.
The different solution for Case 3 is equally valid as evidenced by the norm of f.

6) [25 Pts] P.S. #3, Problem #5 w/different data + w/ defined first guess

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```
% First Gauss-Newton Increment:
delxgn_first = [  0.364793046765476;...
                 -0.204346732132889;...
                  0.622977370351628;...
                 -1.175524557233098;...
                 -0.994482780298112]

% Final Estimate of Gauss-Newton Procedure:
xesttgn_final = [ 3.223825160154675;...
                  0.899792908774152;...
                  1.496338379532975;...
                 -0.803805936404100;...
                  2.198857133678682]

% Initial and Final Weighted Nonlinear Least-Squares Costs
Jorig = 1.878158484193378e+005
Jopt = 23.993211198713063

%  Note: given that there are 31*2 = 62 measurements and
%  5 estimated quantities, Jopt should be half the value
%  of a sample from a chi-squared distribution of degree
%  2*31 - 5 = 57.  Therefore, the expectation value of
%  Jopt is 0.5*57 = 28.5.  It is lower than this.  95% of the
%  time Jopt should lie in the range from r1 = chi2inv(0.025,57)/2 = ...
%  19.013370478108758 tp r2 = chi2inv(1-0.025,57)/2 = ...
%  39.876096140145201.  Clearly, it lies in this range.
```

```
function [hj,Hj] = hjcart(x,tj,lradara,lradarb,i1stdrv)
%
%  Copyright (c) 2002 Mark L. Psiaki.  All rights reserved.
%
%  This function gives the measurement function h(x(j),tj) and its
%  first derivatives with respect to x(j), Hj = dhj/dx.
%  It is for use in the nonlinear least-squares problem
%  that does tricycle cart tracking.  The measurements are range
%  to two different radars at two different known locations.  The
%  tricycle is moving at constant velocity along a circle.
%
%
%  Inputs:
%
%    x         = [psi0;y10;y20;psidot;vrear], the vector of initial
%              conditions and rates of the tricycle cart at time 0.  psi0
%              is the initial heading angle in rad, y10 is the initial
%              east position in meters, y20 is the initial north position
%              in meters, psidot is the turn rate in rad/sec, and vrear
%              is the speed of the midpoint between the two rear
%              wheels in m/sec.
%
%    tj        The time in seconds of the radar measurement.
%
%    lradara   The east position of radar station a, in meters.
%              This radar station is assumed to be located at zero
%              north position.
%
%    lradarb   The east position of radar station b, in meters.
%              This radar station is assumed to be located at zero
%              north position.
%
%    i1stdrv   A flag that tells whether (i1stdrv = 1) or not
%              (i1stdrv = 0) Hj needs to get computed.
%
%  Outputs:
%
%    hj        = [rhoa_j;rhob_j], the 2x1 radar output vector.  rhoa_j
%              is the measured distance from radar a to the cart
%              at time tj given in meters.  rhob_j is the measured
%              distance from radar b to the cart at time tj in meters.
%              These distances are to the mid point between the cart's
%              two rear wheels.
%
%    Hj        = dhj/dx.  Hj is a 2x5 matrix Hj(1,i) is the derivative
%              of rhoa_j with respect to x(i).  Hj(2,i) is the derivative
%              of rhob_j with respect to x(i).  This output will
%              be needed to do Newton's method or to do the Gauss
%              Newton method.
%

%
```

```
%   Set up output arrays as needed.
%
    hj = zeros(2,1);
    if i1stdrv == 1
       Hj = zeros(2,5);
    else
       Hj = [];
    end
%
%   Compute the position of the cart relative to the two radars.
%
    q = x(4)*tj*0.5;
    [sa,dsadq,d2sadq2] = safunct(q);
    pfac = x(5)*tj*sa;
    gamma = x(1) + q;
    cosgamma = cos(gamma);
    singamma = sin(gamma);
    y1 = x(2) + pfac*cosgamma;
    y2 = x(3) + pfac*singamma;
    dely1a = lradara - y1;
    dely1b = lradarb - y1;
    dely2 = y2;
%
%   Compute the hj outputs.
%
    dely1asq = dely1a^2;
    dely1bsq = dely1b^2;
    dely2sq = dely2^2;
    hj(1,1) = sqrt(dely1asq + dely2sq);
    hj(2,1) = sqrt(dely1bsq + dely2sq);
%
%   Return if neither first derivatives nor second derivatives
%   need to be calculated.
%
    if i1stdrv == 0
       return
    end
%
%   Calculate the first derivatives.  Use analytic formulas.
%
    dsadx = dsadq*[0,0,0,(tj*0.5),0];
    dpfacdx = x(5)*tj*dsadx + [0,0,0,0,(tj*sa)];
    dcosgammadx = [(-singamma),0,0,(-singamma*tj*0.5),0];
    dsingammadx = [(cosgamma),0,0,(cosgamma*tj*0.5),0];
    dy1dx = [0,1,0,0,0] + dpfacdx*cosgamma + pfac*dcosgammadx;
    dy2dx = [0,0,1,0,0] + dpfacdx*singamma + pfac*dsingammadx;
    ddely1adx = -dy1dx;
    ddely1bdx = -dy1dx;
    ddely2dx = dy2dx;
    one_over_hja = 1/hj(1,1);
    Hj(1,:) = one_over_hja*(dely1a*ddely1adx + dely2*ddely2dx);
%
```

```
one_over_hjb = 1/hj(2,1);
Hj(2,:) = one_over_hjb*(dely1b*ddely1bdx + dely2*ddely2dx);
```

Problem Set 3, Problem 5, Solution w/alternate data