AOE 5784 Prelim #1 Solution

Fall 2024

1) P.S. 1, #9, 15 Points:

$$\underline{x}_{mean} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x}\, p(\underline{x})\, dx_1 dx_2 \ldots dx_n$$

$$P = V \begin{bmatrix} \sigma_{z_1}^2 & & & 0 \\ & \sigma_{z_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{z_n}^2 \end{bmatrix} V^T$$

$$[\det V]^2 = 1 \qquad V^T V = I$$

$$\underline{z} = V^T \underline{x} \qquad \underline{x} = V \underline{z}$$

$$p(\underline{x})\, dx_1 \ldots dx_n = p(V\underline{z})\, dz_1 \ldots dz_n\, |\det V|$$

so

$$\underline{x}_{mean} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V\underline{z}\, p(V\underline{z})\, dz_1 \ldots dz_n$$

let $\bar{\underline{z}} = V^T \bar{\underline{x}}$ so $\bar{\underline{x}} = V\bar{\underline{z}}$

then

$$p(V\underline{z}) = \frac{1}{2\pi^{n/2} [\det P]^{1/2}} \exp\left\{-\tfrac{1}{2}(\underline{z} - \bar{\underline{z}})^T V^T P^{-1} V (\underline{z} - \bar{\underline{z}})\right\}$$

but $\det P = \det V \det \begin{bmatrix} \sigma_{z_1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{z_n}^2 \end{bmatrix} \det V^T$

$$= \sigma_{z_1}^2 \sigma_{z_2}^2 \cdots \sigma_{z_n}^2$$

and $V^T P^{-1} V = V^T V \begin{bmatrix} \frac{1}{\sigma_{z_1}^2} & & & 0 \\ & \frac{1}{\sigma_{z_2}^2} & & \\ & & \ddots & \\ 0 & & & 1/\sigma_{z_n}^2 \end{bmatrix} V^T V$

$$= \begin{bmatrix} \frac{1}{\sigma_{z_1}^2} & & & 0 \\ & \frac{1}{\sigma_{z_2}^2} & & \\ & & \ddots & \\ 0 & & & 1/\sigma_{z_n}^2 \end{bmatrix}$$

so $p(Vz) = \dfrac{1}{2\pi^{n/2} \sigma_{z_1} \sigma_{z_2} \cdots \sigma_{z_n}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} \left(\frac{z_i - \bar{z}_i}{\sigma_{z_i}} \right)^2 \right\}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{z_1}} \exp\left\{ -\frac{1}{2} \left(\frac{z_1 - \bar{z}_1}{\sigma_{z_1}} \right)^2 \right\} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_{z_2}} \exp\left\{ -\frac{1}{2} \left(\frac{z_2 - \bar{z}_2}{\sigma_{z_2}} \right)^2 \right\} \cdot$$

$$\cdots \frac{1}{\sqrt{2\pi}\,\sigma_{z_n}} \exp\left\{ -\frac{1}{2} \left(\frac{z_n - \bar{z}_n}{\sigma_{z_n}} \right)^2 \right\}$$

$$\underline{x}_{mean} = V \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} z\, p(Vz)\, dz_1 \ldots dz_n$$

$$= V \underline{z}_{mean}$$

The ith element of z_{mean} is

$$(z_{mean})_i = \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{z_1}} \exp\left\{-\frac{1}{2}\left(\frac{z_1 - \bar{z}_1}{\sigma_{z_1}}\right)^2\right\} dz_1\right] \cdot$$

$$\cdots \cdot \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{z_{i-1}}} \exp\left\{-\frac{1}{2}\left(\frac{z_{i-1} - \bar{z}_{i-1}}{\sigma_{z_{i-1}}}\right)^2\right\} dz_{i-1}\right] \cdot$$

$$\left[\int_{-\infty}^{\infty} \frac{z_i}{\sqrt{2\pi}\,\sigma_{z_i}} \exp\left\{-\frac{1}{2}\left(\frac{z_i - \bar{z}_i}{\sigma_{z_i}}\right)^2\right\} dz_i\right] \cdot$$

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{z_{i+1}}} \exp\left\{-\frac{1}{2}\left(\frac{z_{i+1} - \bar{z}_{i+1}}{\sigma_{z_{i+1}}}\right)^2\right\} dz_{i+1}\right] \cdot$$

$$\cdots \cdot \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{z_n}} \exp\left\{-\frac{1}{2}\left(\frac{z_n - \bar{z}_n}{\sigma_{z_n}}\right)^2\right\} dz_n\right]$$

All of the scalar integrals in this expression equal 1 by the pdf normalization, except for the ith integral, which equals $\bar{z}_i$ by the result of Problem set 1, #7.

Therefore $z_{mean} = \bar{z}$

and $x_{mean} = V z_{mean} = V\bar{z} = VV^T\bar{x} = \bar{x}$

so $x_{mean} = \bar{x}$

$$E\left[(\underline{x}-\bar{\underline{x}})(\underline{x}-\bar{\underline{x}})^T\right] = E\left[(V\underline{z}-V\bar{\underline{z}})(V\underline{z}-V\bar{\underline{z}})^T\right]$$

$$= V\,E\left[(\underline{z}-\bar{\underline{z}})(\underline{z}-\bar{\underline{z}})^T\right]V^T$$

$$= V\,P_{zz}\,V^T$$

where

$$(P_{zz})_{ij} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-\bar{z}_i)(z_j-\bar{z}_j)\,p(V\underline{z})\,dz_1\ldots dz_n$$

If $i \neq j$ then

$$(P_{zz})_{ij} = \left[\int_{-\infty}^{\infty}\frac{(z_i-\bar{z}_i)}{\sqrt{2\pi}\,\sigma_{z_i}}\exp\left\{-\frac{1}{2}\left(\frac{z_i-\bar{z}_i}{\sigma_{z_i}}\right)^2\right\}dz_i\right]\cdot$$

$$\left[\int_{-\infty}^{\infty}\frac{(z_j-\bar{z}_j)}{\sqrt{2\pi}\,\sigma_{z_j}}\exp\left\{-\frac{1}{2}\left(\frac{z_j-\bar{z}_j}{\sigma_{z_j}}\right)^2\right\}dz_j\right]\cdot$$

$$\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma_{z_k}}\exp\left\{-\frac{1}{2}\left(\frac{z_k-\bar{z}_k}{\sigma_{z_k}}\right)^2\right\}dz_k\right]$$

It is easy to show that the first two integrals in this expression both equal 0 while all of the others equal 1. Therefore

$$(P_{zz})_{ij} = 0 \quad \text{if} \quad i \neq j$$

$$(P_{zz})_{ii} = \left[\int_{-\infty}^{\infty} \frac{(z_i - \bar{z}_i)^2}{\sqrt{2\pi}\,\sigma_{z_i}} \exp\left\{-\frac{1}{2}\left(\frac{z_i - \bar{z}_i}{\sigma_{z_i}}\right)^2\right\} dz_i\right]$$

$$\prod_{\substack{k=1 \\ k \neq i}}^{n} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{z_k}} \exp\left\{-\frac{1}{2}\left(\frac{z_k - \bar{z}_k}{\sigma_{z_k}}\right)^2\right\} dz_k\right]$$

The first integral in this expression is $\sigma_{z_i}^2$ by the result of Problem set 1, #7, and all of the remaining integrals are 1 because of pdf unit normalization! Therefore

$$(P_{zz})_{ii} = \sigma_{z_i}^2$$

so

$$P_{zz} = \begin{bmatrix} \sigma_{z_1}^2 & & & 0 \\ & \sigma_{z_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{z_n}^2 \end{bmatrix}$$

and

$$E\left[(\underline{x}-\bar{\underline{x}})(\underline{x}-\bar{\underline{x}})^T\right] = V \begin{bmatrix} \sigma_{z_1}^2 & & & 0 \\ & \sigma_{z_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{z_n}^2 \end{bmatrix} V^T$$

$$= P \quad \checkmark$$

Normalization:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\underline{x})\, dx_1 \cdots dx_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(V\underline{z})\, dz_1 \cdots dz_n \, |\det(V)|$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma_{z_i}} \exp\left\{-\frac{1}{2}\left(\frac{z_i-\bar{z}_i}{\sigma_{z_i}}\right)^2\right\} dz_i \right]$$

$$= \prod_{i=1}^{n} \left[1 \right]$$

by normalization of scalar Gaussian distribution

$$= 1 \quad \checkmark$$

2) P.S 3 # 7a & b, 20 Points

Preparation

Test statistic

$$\log\left[\frac{p(z \mid \theta=\theta_1)}{p(z \mid \theta=0)}\right]$$

$$= -\frac{1}{2}\left[\underline{z} - \underline{e}\,\theta_1\right] P^{-1}\left[\underline{z} - \underline{e}\,\theta_1\right] + \frac{1}{2}\,\underline{z}\,P^{-1}\underline{z}$$

$$= \theta_1\,\underline{e}^T P^{-1}\underline{z} - \frac{\theta_1^2}{2}\,\underline{e}^T P^{-1}\underline{e} \qquad \underline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If $\beta(\underline{z}) = \underline{e}^T P^{-1}\underline{z}$, then the locally most powerful test is

Accept H_1 if $|\beta(z)| \geq \beta_0$, but accept H_0 if $|\beta(z)| < \beta_0$

$$\underline{e}^T P^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -0.5 \\ -0.5 & 1 \end{bmatrix}}{1.75}$$

$$= \begin{bmatrix} \frac{1.5}{1.75} & \frac{0.5}{1.75} \end{bmatrix} = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix}$$

Under hypothesis H_0 $\beta(\underline{z})$ is a Gaussian distribution because it is the sum of two variables that are Gaussian. Its mean and variance are

$$\bar{\beta}_{H_0} = E[\beta / H_0] = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} E[z / H_0] = 0$$

$$\sigma^2_{\beta_{H_0}} = E[\beta^2 / H_0] = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} E[z z^T / H_0] \begin{bmatrix} 6/7 \\ 2/7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} 6/7 \\ 2/7 \end{bmatrix}$$

$$= 56/49$$

Under hypothesis H_1 β is also Gaussian because it is the sum of Gaussian random variables. In this latter case its mean and variance are

$$\bar{\beta}_{H_1} = E[\beta / H_1] = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} E[z / H_1]$$

$$= \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \theta_1 = \frac{8}{7} \theta_1$$

$$\sigma^2_{\beta_{H_1}} = E[(\beta - \bar{\beta}_1)^2 / H_1] = E[\beta^2 / H_1] - \bar{\beta}^2_{H_1}$$

$$= \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} E[z z^T / H_1] \begin{bmatrix} 6/7 \\ 2/7 \end{bmatrix} - \bar{\beta}^2_{H_1}$$

$$= \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \theta_1^2 \right\} \begin{bmatrix} 6/7 \\ 2/7 \end{bmatrix} - \bar{\beta}^2_{H_1}$$

or $\bar{\beta}_{H_1}^2 = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} 6/7 \\ 2/7 \end{bmatrix} = \bar{\beta}_{H_0}^2 = \bar{\beta}^2$

$= 56/49$

because the other terms cancel each other

$$\int_{-\infty}^{-\beta_0} \frac{1}{\sqrt{2\pi}\,\sqrt{\frac{56}{49}}} \exp\left\{-\frac{1}{2}\left(\frac{\beta^2}{56/49}\right)\right\} d\beta = \frac{\alpha}{2} = 0.005$$

so

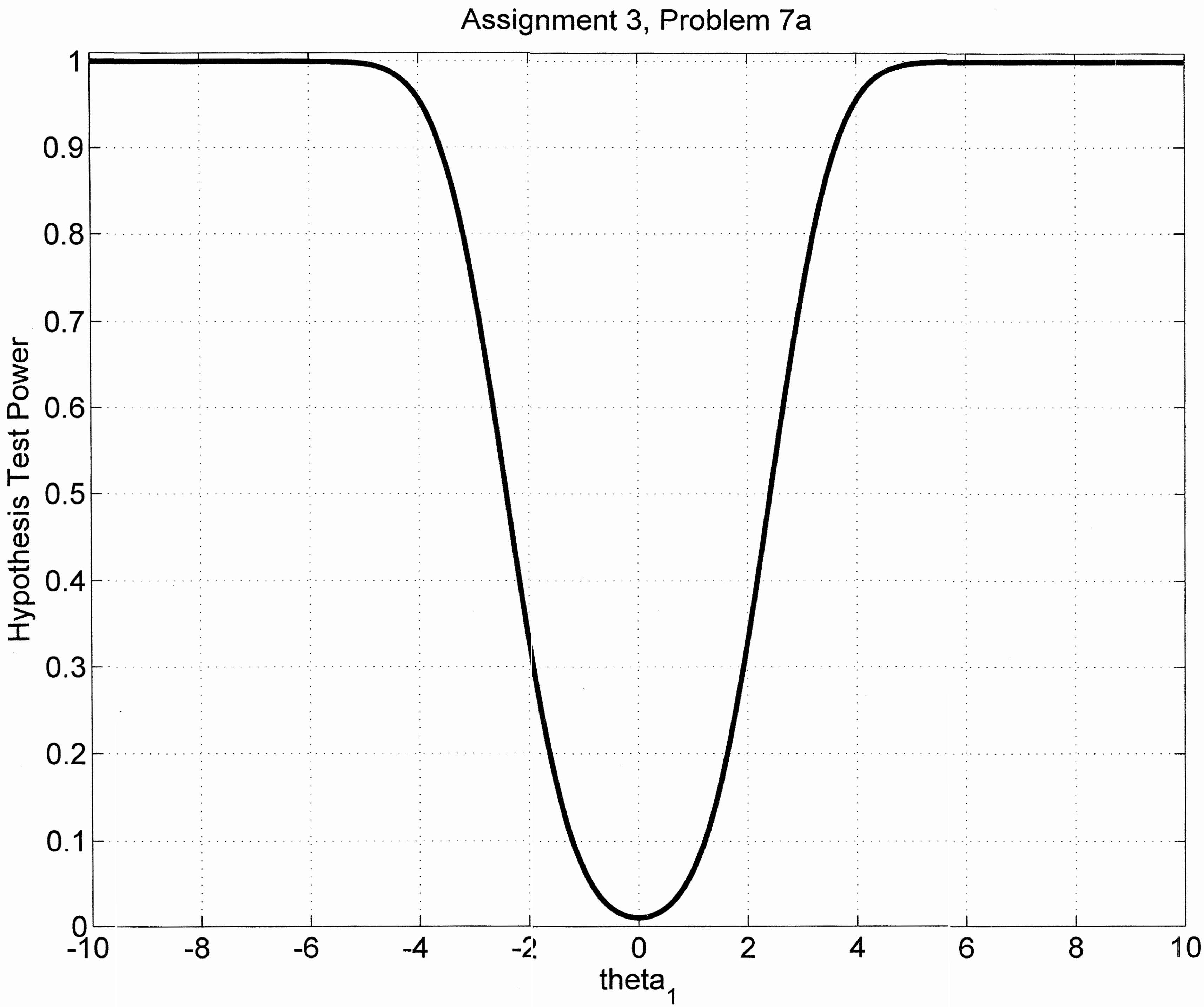
$$\beta_0 = -\text{norminv}\left(0.005, 0, \sqrt{\tfrac{56}{49}}\right)$$

$$= 2.75367735$$

a)

```
beta0 = -norminv(0.005,0,sqrt(56/49));
theta1grid = [-10:.01:10]';
Ngrid = size(theta1grid,1);
Powergrid = zeros(Ngrid,1);
for jj = 1:Ngrid
   betabarjj_H1 = (8/7)*theta1grid(jj,1);
   Powergrid(jj,1) = normcdf(-beta0,betabarjj_H1,sqrt(56/49)) + ...
                     (1 - normcdf(beta0,betabarjj_H1,sqrt(56/49)));
end
plot(theta1grid,Powergrid)
```

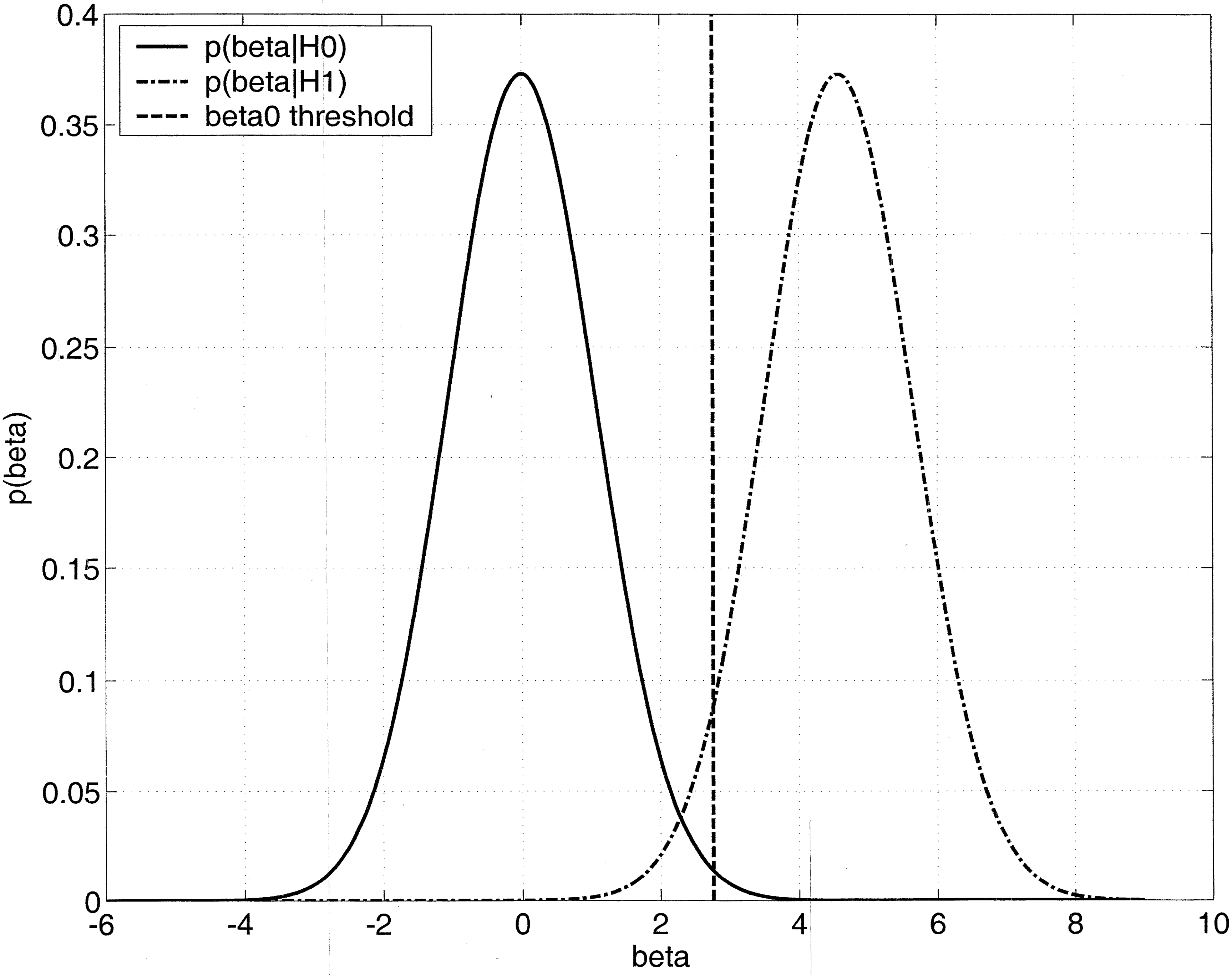
See part a plot on next sheet


Assignment 3, Problem 7a
Hypothesis Test Power
theta1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10

b)

```
betabar_H0 = 0;
sigmabeta_H0 = sqrt(56/49);
betabar_H1 = (8/7)*4;
sigmabeta_H1 = sigmabeta_H0;
betagrid_lolim = floor(betabar_H0 - 5*sigmabeta_H0);
betagrid_uplim = floor(betabar_H1 + 5*sigmabeta_H0);
betagrid = [betagrid_lolim:.01:betagrid_uplim]';
PH0grid = normpdf(betagrid,betabar_H0,sigmabeta_H0);
PH1grid = normpdf(betagrid,betabar_H1,sigmabeta_H1);
plot(betagrid,PH0grid,'b-',betagrid,PH1grid,'r-.');
xlabel('beta')
ylabel('p(beta)')
grid
beta0 = -norminv(0.005,0,sqrt(56/49));
hold on
dum = axis;
plot(beta0*[1;1],dum(1,3:4)','g--')
legend('p(beta|H0)','p(beta|H1)','beta0 threshold',0)
```

see next page for plot



p(beta|H0)
p(beta|H1)
beta0 threshold
p(beta)
beta
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-6
-4
-2
0
2
4
6
8
10

3) P.S. 2, #4, 15 Points

Sheet 13 of 21

```
function [xhat,P] = lsweight_14(z,H,R)
%
% Copyright (c) 2014 Mark L. Psiaki. All rights reserved.
%
% This function solves the weighted least-square problem
%
%        Min J(x) = 0.5*(z - H*x)'*inv(R)*(z - H*x)
%
% to produce the solution xhat. It also produces the covariance:
%
%  P = inv(H'*inv(R)*H).
%
% This function uses Cholesky factorization and QR factorization.
% Except for the calculation of P, it was an assignment for MAE 6760.
%
   n = size(H,2);
   Ra = chol(R);
   Ratrinv = (inv(Ra))';
   za = Ratrinv*z;
   Ha = Ratrinv*H;
   [Qb,Rb] = qr(Ha);
   Rb = Rb(1:n,:);
   zb = (Qb')*za;
   zb1 = zb(1:n,1);
   xhat = Rb\zb1;
   Rbinv = inv(Rb);
   P = Rbinv*(Rbinv');
end
%
% Matlab commands that solved Problem 4 of Assignment 2 with modifications
% as spelled out for Prelim 1 in 2014.
%
» z = [ -45.1800;...
          1.7900;...
        -31.3800;...
         26.7700;...
         27.6400] + 0.25*ones(5,1);
» H = [  -4.9300   -1.3100   -1.5900;...
         13.2600    9.7100   30.7000;...
        -17.0800  -11.9100  -12.1300;...
        -24.0300   -2.9900  -26.9500;...
         -2.4000   -8.7000    9.3900];
» R = [   5.9700   -0.9200   -1.1800   -7.0600   -1.7900;...
         -0.9200    3.4500    1.7100   -0.6000   -4.0500;...
         -1.1800    1.7100    1.1900    0.5600   -1.6700;...
         -7.0600   -0.6000    0.5600    9.9200    4.8500;...
         -1.7900   -4.0500   -1.6700    4.8500    6.8700] + 1.2*eye(5);
» xhat = lsweight_14(z,H,R)
xhat =
   1.384738044762571
   0.549473323694544
  -0.297443754139889
```

4) Problem 2-7 in Bar-Shalom, 15 pts:

$$J(\hat{x}) = E\left[|x-\hat{x}| \mid z\right]$$

$$= \int_{-\infty}^{\infty} |x-\hat{x}|\, p(x|z)\, dx$$

$$= \int_{-\infty}^{\hat{x}} (\hat{x}-x)\, p(x|z)\, dx + \int_{\hat{x}}^{\infty} (x-\hat{x})\, p(x|z)\, dx$$

Minimizing with respect to $\hat{x}$ yields the eq:

$$0 = \frac{dJ}{d\hat{x}} = (\hat{x}-\hat{x})\, p(\hat{x}|z) + \int_{-\infty}^{\hat{x}} p(x|z)\, dx$$

$$- (\hat{x}-\hat{x})\, p(\hat{x}|z) - \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

$$0 = \int_{-\infty}^{\hat{x}} p(x|z)\, dx - \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

but, from the normalization constraint we know that

$$1 = \int_{-\infty}^{\hat{x}} p(x|z)\, dx + \int_{\hat{x}}^{\infty} p(x|z)\, dx$$

Adding these last two equations we get

$$1 = 2\int_{-\infty}^{\hat{x}} p(x|z)\, dx \quad \text{or} \quad \int_{-\infty}^{\hat{x}} p(x|z)\, dx = \frac{1}{2}$$

5) <u>P.S. 3, #6, 20 Points</u>: The constant scalar in the new $J(\underline{x}, k)$ is correct because the new formula evaluated at $\underline{x} = \hat{\underline{x}}(k, Z^k)$ yields:

$$\left[\hat{\underline{x}}(k, Z^k) - \hat{\underline{x}}(k, Z^k)\right]^T \hat{P}^{-1}(k, Z^k)\left[\hat{\underline{x}}(k, Z^k) - \hat{\underline{x}}(k, Z^k)\right]$$

$$+ J\left[\hat{\underline{x}}(k, Z^k), k\right] = J\left[\hat{\underline{x}}(k, Z^k), k\right]$$

because the first term equals 0.

<u>Using the old formula</u>:

$$\frac{\partial J}{\partial \underline{x}} = -2\sum_{j=1}^{k}\left[\underline{z}(j) - H(j)\underline{x}\right]^T R^{-1}(j) H(j)$$

and
$$\left.\frac{\partial J}{\partial \underline{x}}\right|_{\underline{x}=0} = -2\sum_{j=1}^{k} \underline{z}^T(j) R^{-1}(j) H(j)$$

$$\frac{\partial^2 J}{\partial \underline{x}^2} = 2\sum_{j=1}^{k} H^T(j) R^{-1}(j) H(j)$$

Using the new formula

$$\frac{\partial J}{\partial \underline{x}} = 2\left[\underline{x} - \hat{\underline{x}}(k, Z^k)\right]^T \hat{P}^{-1}(k, Z^k)$$

$$\left.\frac{\partial J}{\partial \underline{x}}\right|_{\underline{x}=0} = -2\,\hat{\underline{x}}^T(k, Z^k)\,\hat{P}^{-1}(k, Z^k)$$

$$\frac{\partial^2 J}{\partial \underline{x}^2} = 2\,\hat{P}^{-1}(k, Z^k)$$

Because the two cost function formulas are linear/quadratic in $\underline{x}$, equivalence of the two functions can be proved by equating the functions and their first derivatives at any choices of $\underline{x}$ and by equating their constant 2nd derivatives. The equation of function values at $\underline{x} = \hat{\underline{x}}(k, \underline{z}^k)$ has already been done. I will leave to the last step the equation of the first derivatives evaluated at $\underline{x} = 0$. The equation of the 2nd derivatives proceeds as follows

We know from lecture that

$$\hat{P}(k, \underline{z}^k) = \left[(H^k)^T (R^k)^{-1} (H^k) \right]^{-1}$$

where $H^k = \begin{bmatrix} H(1) \\ H(2) \\ \vdots \\ H(k) \end{bmatrix}$ $R^k = \begin{bmatrix} R(1) & & 0 \\ & R(2) & \\ 0 & & \ddots \\ & & & R(k) \end{bmatrix}$

So, $(R^k)^{-1} = \begin{bmatrix} R^{-1}(1) & & 0 \\ & R^{-1}(2) & \\ 0 & & \ddots \\ & & & R^{-1}(k) \end{bmatrix}$

and

$$(H^K)^T (R^K)^{-1} (H^K) = \sum_{j=1}^{K} H^T(j) R^{-1}(j) H(j)$$

So, $\hat{P}(K, Z^K) = \left[\sum_{j=1}^{K} H^T(j) R^{-1}(j) H(j) \right]^{-1}$

and $\hat{P}^{-1}(K, Z^K) = \sum_{j=1}^{K} H^T(j) R^{-1}(j) H(j)$

which is the same as saying that the two functions 2nd derivatives are the same.

<u>1st derivatives.</u>

We also know from lecture that

$$\hat{\underline{x}}(K, \underline{Z}^K) = \left[(H^K)^T (R^K)^{-1} (H^K) \right]^{-1} (H^K)^T (R^K)^{-1} \underline{Z}^K$$

where $Z^K = \begin{bmatrix} Z(1) \\ Z(2) \\ \vdots \\ Z(K) \end{bmatrix}$

this is equivalent to

$$\hat{x}(K, Z^K) = \left[\sum_{j=1}^{K} H^T(j) R^{-1}(j) H(j) \right]^{-1} \left[\sum_{j=1}^{K} H^T(j) R^{-1}(j) Z(j) \right]$$

Therefore

$$-2\hat{\underline{x}}^T(k,Z^k)\hat{P}^{-1}(k,Z^k) =$$

$$-2\left[\sum_{j=1}^{k} z^T(j)R^{-1}(j)H(j)\right]\left[\sum_{j=1}^{k} H^T(j)R^{-1}(j)H(j)\right]^{-1}$$

$$\cdot\left[\sum_{j=1}^{k} H^T(j)R^{-1}(j)H(j)\right]$$

$$= -2\sum_{j=1}^{k} z^T(j)R^{-1}(j)H(j)$$

but this amounts to an equation of the $\partial J/\partial x|_{\underline{x}=0}$ values for the original form of the cost function.

6) Bar-Shalom 3-13, 15 Pts:

Following the developments of the LMMSE on pp 127-129 of Bar-Shalom, we need to compute $\bar{y}$, P_{xy}, and P_{yy} for use in eqs 3.3.2-10 and 3.3.2-12

$$y = z^2 = x^2 + 2xw + w^2$$

$$\bar{y} = E[y] = E[x^2] + 2E[xw] + E[w^2]$$

$$\bar{y} = E[(x-\bar{x})^2 + 2x\bar{x} - \bar{x}^2] + 2E[x]E[w] + E[w^2]$$

$$\bar{y} = E\left[(x-\bar{x})^2\right] + 2E[x]\bar{x} - \bar{x}^2 + 2E[x]E[w] + E[w^2]$$

$$\bar{y} = P_{xx} + 2\bar{x}^2 - \bar{x}^2 + 2\bar{x}\cdot 0 + P_{ww}$$

$$\bar{y} = P_{xx} + \bar{x}^2 + P_{ww}$$

$$P_{xy} = E\left[(x-\bar{x})(y-\bar{y})\right]$$

$$= E\left[(x-\bar{x})(x^2 + 2xw + w^2 - P_{xx} - \bar{x}^2 - P_{ww})\right]$$

$$= E\left[(x-\bar{x})(x^2-\bar{x}^2)\right] + 2E\left[(x-\bar{x})x\right]E[w] + E\left[(x-\bar{x})\right]E[w^2] - E\left[(x-\bar{x})\right](P_{xx}+P_{ww})$$

$$= E\left[(x-\bar{x})\{(x-\bar{x})^2 + 2(x-\bar{x})\bar{x}\}\right] + 2E\left[(x-\bar{x})x\right]\cdot 0$$

$$+ 0\cdot P_{ww} - 0\cdot(P_{xx}+P_{ww})$$

$$= E\left[(x-\bar{x})^3\right] + 2E\left[(x-\bar{x})^2\right]\bar{x}$$

$$= 0 + 2P_{xx}\bar{x}$$

$$= 2P_{xx}\bar{x}$$

$$P_{yy} = E\left[(y-\bar{y})(y-\bar{y})\right]$$

$$= E\left[(x^2+2xw+w^2-P_{xx}-\bar{x}^2-P_{ww})(x^2+2xw+w^2-P_{xx}-\bar{x}^2-P_{ww})\right]$$

$$P_{yy} = E\Big[\{(x-\bar{x})^2 + 2(x-\bar{x})\bar{x} + 2(x-\bar{x})w + 2\bar{x}w + w^2 - P_{xx} - P_{ww}\}\cdot$$
$$\{(x-\bar{x})^2 + 2(x-\bar{x})\bar{x} + 2(x-\bar{x})w + 2\bar{x}w + w^2 - P_{xx} - P_{ww}\}\Big]$$

$$\begin{aligned}
P_{yy} = {} & E[(x-\bar{x})^4] + 4E[(x-\bar{x})^3]\bar{x} \\
& + 4E[(x-\bar{x})^3]E[w] + 4E[(x-\bar{x})^2]\bar{x}E[w] \\
& + 2E[(x-\bar{x})^2]E[w^2] - 2E[(x-\bar{x})^2](P_{xx}+P_{ww}) \\
& + 4E[(x-\bar{x})^2]\bar{x}^2 + 8E[(x-\bar{x})^2]\bar{x}E[w] \\
& + 8E[(x-\bar{x})]\bar{x}^2E[w] + 4E[(x-\bar{x})]\bar{x}E[w^2] \\
& - 4E[(x-\bar{x})]\bar{x}(P_{xx}+P_{ww}) + 4E[(x-\bar{x})^2]E[w^2] \\
& + 8E[(x-\bar{x})]\bar{x}E[w^2] + 4E[(x-\bar{x})]E[w^3] \\
& - 4E[(x-\bar{x})]E[w](P_{xx}+P_{ww}) + 4\bar{x}^2E[w^2] \\
& + 4\bar{x}E[w^3] - 4\bar{x}E[w](P_{xx}+P_{ww}) \\
& + E[w^4] - 2E[w^2](P_{xx}+P_{ww}) \\
& + (P_{xx}^2 + 2P_{xx}P_{ww} + P_{ww}^2)
\end{aligned}$$

Useful facts: $E[(x-\bar{x})] = 0$, $E[(x-\bar{x})^2] = P_{xx}$
$E[(x-\bar{x})^3] = 0$, $E[(x-\bar{x})^4] = 3P_{xx}^2$
$E[w] = 0$, $E[w^2] = P_{ww}$
$E[w^3] = 0$, $E[w^4] = 3P_{ww}^2$

Substituting these facts into the formula for P_{yy} yields:

$$\begin{aligned}
P_{yy} = {} & 3P_{xx}^2 + 2P_{xx}P_{ww} - 2P_{xx}(P_{xx}+P_{ww}) + 4P_{xx}\bar{x}^2 \\
& + 4P_{xx}P_{ww} + 4\bar{x}^2P_{ww} + 3P_{ww}^2 - 2P_{ww}(P_{xx}+P_{ww}) \\
& + (P_{xx}^2 + 2P_{xx}P_{ww} + P_{ww}^2)
\end{aligned}$$

or

$$P_{yy} = 2P_{xx}^2 + 4P_{xx}P_{ww} + 2P_{ww}^2 + 4(P_{xx}+P_{ww})\bar{x}^2$$
$$= 2[(P_{xx}+P_{ww})^2 + 2(P_{xx}+P_{ww})\bar{x}^2] = 2(P_{xx}+P_{ww})(P_{xx}+P_{ww}+2\bar{x}^2)$$

From eq 3.3.2-10 on p. 127 of Bar-Shalom we have

$$\hat{x} = \bar{x} + P_{xy}P_{yy}^{-1}(y-\bar{y})$$

$$\hat{x} = \bar{x} + \frac{(2P_{xx}\bar{x})}{2(P_{xx}+P_{ww})(P_{xx}+P_{ww}+2\bar{x}^2)}\left[y - P_{xx} - P_{ww} - \bar{x}^2\right]$$

$$\hat{x} = \bar{x} + \left[\frac{P_{xx}\bar{x}}{(P_{xx}+P_{ww})(P_{xx}+P_{ww}+2\bar{x}^2)}\right]\left[y - P_{xx} - P_{ww} - \bar{x}^2\right]$$

From eq 3.3.2-12 on p 128 of Bar-Shalom we have

$$MSE = P_{xx} - P_{xy}P_{yy}^{-1}P_{xy}^T$$

$$= P_{xx} - \frac{(2P_{xx}\bar{x})^2}{2(P_{xx}+P_{ww})(P_{xx}+P_{ww}+2\bar{x}^2)}$$

or

$$MSE = P_{xx} - \frac{2P_{xx}^2\bar{x}^2}{(P_{xx}+P_{ww})(P_{xx}+P_{ww}+2\bar{x}^2)}$$