Posting Date: Monday Sept. 23rd.

You need not hand in anything. Instead, be prepared to answer any of these problems on an upcoming take-home prelim exam. You may discuss your solutions with classmates up until the time that the prelim becomes available.

1. Recall that Newton's method for the general system of n nonlinear equations in n unknowns

$$0 = \underline{f}(\underline{x})$$

takes the iterative form $\underline{x}_{gnew} = \underline{x}_g$ - inv($d\underline{f}/d\underline{x}|_{xg}$)*$\underline{f}(\underline{x}_g)$. Use Newton's method to solve the following system of 2 nonlinear equations in 2 unknowns:

$$0 = x_1 + x_2 + x_1{*}x_2 + 5$$

$$0 = (x_1)^2 + 2{*}x_2 - (x_2)^2 - 2$$

Start Newton's method from the following 3 first guesses: $\underline{x}_{ginitial}$ = [4;-4], $\underline{x}_{ginitial}$ = [6;0], and $\underline{x}_{ginitial}$ = [-5;5]. Record your first 6 Newton $\underline{x}_g$ iterates for each initial guess along with the value of the norm of $\underline{f}(\underline{x}_g)$ that goes with each iterate.

2. Prove that, under suitable assumptions, there exists a value of the Levenberg-Marquardt parameter, λ, that yields a cost decrease of the nonlinear least-squares cost function. In other words, prove that there is a $\lambda \geq 0$ such that $J(\underline{x}_g + \Delta\underline{x}) < J(\underline{x}_g)$ if $\Delta\underline{x}$ = inv($H'{*}H + \lambda{*}I$)*H'* [$\underline{z} - \underline{h}(\underline{x}_g)$].

Hints: In place of λ use $1/\varepsilon$. Define a function $\tilde{J}(\varepsilon) = J[\underline{x}_g + \Delta\underline{x}(\varepsilon)]$. Compute the derivative of $\tilde{J}$ with respect to ε at $\varepsilon = 0$, and make an argument about the sign of this derivative. Finish by using the sign of this derivative to prove that a cost decrease occurs for some positive value of ε.

3. Use the Gauss-Newton method to solve, in a weighted least-squares sense, the following over-determined system of nonlinear equations for $\underline{x} = [x_1;x_2;x_3]$:

$z_j = x_1{*}\cos(x_2{*}t_j + x_3) + w_j$ for $j = 1,...,11$

where $E\{w_j\} = 0$, $E\{w_j^2\} = 1$, $E\{w_iw_j\} = 0.5$ if $|i-j| = 1$, $E\{w_iw_j\} = 0$ if $|i-j| \geq 2$. In other words, R is a matrix with 1's on its main diagonal, 0.5's on the diagonals just above and just below the main diagonal, and zeros everywhere else.

The sample times t_j are contained in the following "thist" vector:

thist = [0; 0.1000; 0.2000; 0.3000; 0.4000; 0.5000; 0.6000; 0.7000; 0.8000; 0.9000; 1.0000]

and the measurements z_j are contained in the following vector

zhist = [7.7969; 1.4177; -3.0970; -7.6810; -9.8749; -6.1828; -0.8212; 4.5074; 8.2259; 9.5369; 6.2827]

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Also, determine the estimation error covariance, P_{xx}, after you have minimized the appropriate weighted least-squares cost function (or the equivalent unweighted least-squares cost function). Make a record of your $\underline{\hat{x}}$ estimate and your P_{xx} estimation error covariance matrix.

4. Use the Gauss-Newton method to solve a modified missile tracking problem. The modified problem is like the problem presented in class, except that two radar are used instead of one, and each radar only measures range. Both radar are at zero altitude. Radar "a" is located at $y_1 = 4.1\times10^5$ m $= l_a$, and radar "b" is located at $y_1 = 4.4\times10^5$ m $= l_b$. Radar "a" has a range measurement error standard deviation of $\sigma_{\rho a} = 10$ m, but radar "b" has a measurement error standard deviation of $\sigma_{\rho b} = 30$ m because it is an older radar. Twelve samples of data are available from the two radar stations. The sample times are stored in the array "thist", and the measurements are stored in the arrays "rhoahist" and "rhobhist", all of which are contained in the MATLAB data file "radarmeasdata_missile.mat", which is available on the course web site.

   Estimate the missile initial position and velocity using this data and determine the Gauss-Newton approximation of the estimation error covariance.

   Hints: You may solve this problem by modifying the code for the range-elevation missile tracking problem that is given on the course web site. You need not deal with 2nd derivatives because you are implementing the Gauss-Newton method, not Newton's method. Get your initial guess for the solution in a manner similar to what is done in the software for the other missile problem: Assume perfect measurements and use the first and last measurements to derive positions for the missile at the corresponding times. Use those 2 positions to derive the corresponding initial conditions. You will have to solve two quadratic equations in two unknowns in order to go from two ranges to missile position coordinates. There is a clever trick that reduces these to one linear equation in one unknown. Assume that the missile's altitude is positive in order to resolve the ambiguity in the other unknown. Such ambiguity is inherent in any quadratic equation.

5. Use the Gauss-Newton method to solve a tracking problem for a tricycle cart. Consider the following diagram of the problem:

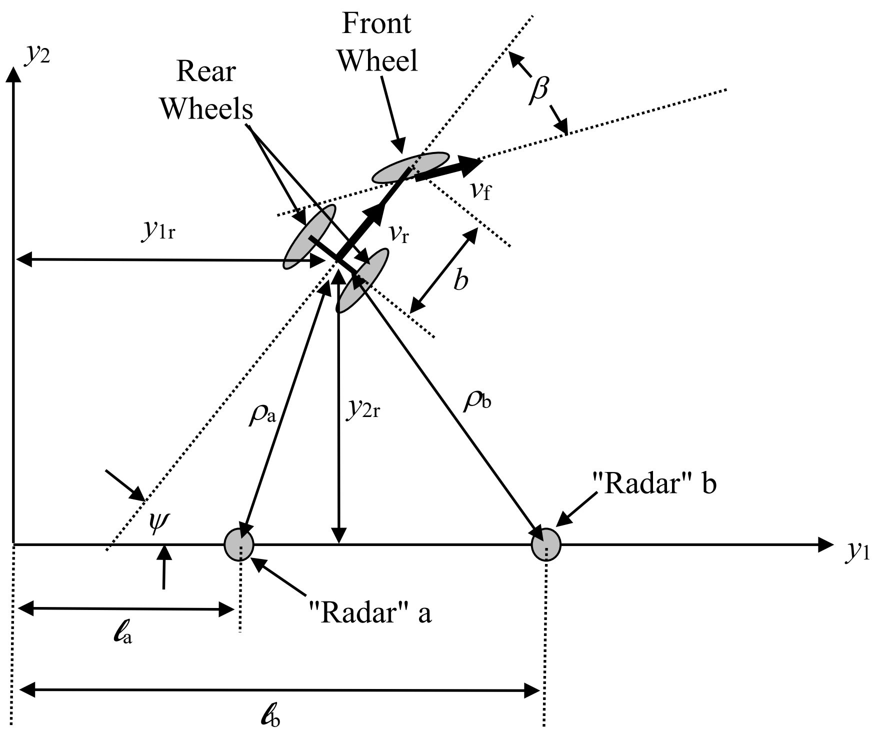


Fig. 1. Tricycle tracking problem, geometry of dynamics and sensing.

Figure 1 shows a tricycle that moves in the y_1-y_2 horizontal plane, where y_1 is the east coordinate and y_2 is the north coordinate. The state of the tricycle is given by its heading angle, ψ, and by the location of the midpoint between its two rear wheels, (y_{1r},y_{2r}). The mid-point between the rear wheels moves in the direction of the heading with a speed v_r. The steer angle of the front wheel relative to the center line of the tricycle is β. The ground contact point of the front wheel moves with a speed v_f with the heading $(\psi+\beta)$. The wheel base of the tricycle is b. Kinematics can be used to derive the following equations of motion for the tricycle:

$$\dot{\psi} = -\frac{v_r \tan\beta}{b}$$

$$\dot{y}_{1r} = v_r \cos\psi$$

$$\dot{y}_{2r} = v_r \sin\psi$$

Kinematics also dictates the following relationship:

$$v_f = v_r\sqrt{1+\tan^2\beta}$$

If the steer angle β and the speed v_r are constant, then the kinematic equations of motion can be integrated to yield the following solutions:

$$\psi(t) = \psi_0 + \dot{\psi}\, t$$

$$y_{1r}(t) = y_{1r0} + v_r t\, sa(\tfrac{1}{2}\dot{\psi}\, t)\, cos(\psi_0 + \tfrac{1}{2}\dot{\psi}\, t)$$

$$y_{2r}(t) = y_{2r0} + v_r t\, sa(\tfrac{1}{2}\dot{\psi}\, t)\, sin(\psi_0 + \tfrac{1}{2}\dot{\psi}\, t)$$

where $sa(q) = \sin(q)/q$ is a continuous differentiable function.

Suppose that you are given a time history of range data from radar stations "a" and "b" for the ranges to the mid-point between the tricycle's two rear wheels, i.e., suppose that you are given $\rho_a(t_j)$ and $\rho_b(t_j)$ for $j = 1,\ldots, k$. Then develop a Gauss-Newton nonlinear least-squares estimator for the unknown vector $\underline{x} = [\psi_0; y_{1r0}; y_{2r0}; \dot{\psi}; v_r]$.

Estimate $\underline{x}$ and calculate its estimation error covariance if $\ell_a = -1$ m, $\ell_b = 1$ m, and $\sigma_{\rho a} = \sigma_{\rho b} = 0.005$ m. Thirty-one samples of data are available from the two radar stations. The sample times are stored in the array "thist", and the measurements are stored in the arrays "rhoahist" and "rhobhist", all of which are contained in the MATLAB data file "radarmeasdata_cart.mat", which is available on the course web site.

<u>Hints and Help</u>: Code for a MATLAB function that computes $sa(q)$ and its first two derivatives with respect to q is given in the file "safunct.m", which is on the course web site. Feel free to re-use your code from Problem 4 or code from the course web site, making changes only as needed in order to solve the current problem.

You might want to use the following procedure in order to develop a first guess of $\underline{x}$:

a) Determine the cart locations at the first two samples, (y_{1r1}, y_{2r1}) and (y_{1r2}, y_{2r2}), which are given at the first two times in the array "thist", t_1 and t_2. You can determine these points from the first two pairs of radar range measurements just as you did for the missile in Problem 4. This assumes that the first two radar measurements have no error. You can assume that the cart has a positive north position in order to resolve any ambiguity in your first two time points.

b) Compute the approximate vector velocity of the mid-point between the two rear wheels by doing a finite-difference derivative using your computed positions at the first two radar measurement times: $\underline{v}_r = [(y_{1r2}-y_{1r1})/(t_2-t_1); (y_{2r2}-y_{2r1})/(t_2-t_1)]$. This can be a dangerous calculation if the sample times are too close to each other because the velocity component error standard deviations are $\sqrt{2}\,\sigma_\rho/(t_2-t_1)$. This approximation is reasonable in the present case because σ_ρ is relatively small, which keeps the velocity uncertainty from becoming too large.

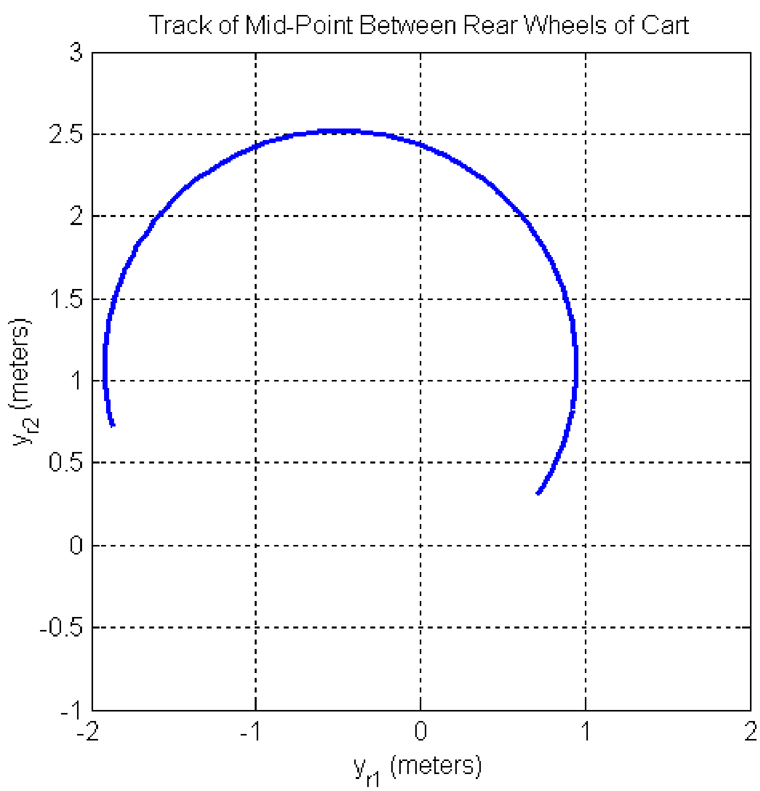
c) Approximate the initial position by propagating backwards in time under the assumption of a constant vector velocity: $\underline{y}_{r0} = [y_{1r1}; y_{2r1}] - \underline{v}_r t_1$. This assumes that $\dot{\psi}\, t_1$ is not very large. This $\underline{y}_{r0}$ provides the 2nd and 3rd elements of your initial guess of $\underline{x}$.

d) Compute the 1st and 5th elements of your initial guess of $\underline{x}$ as, respectively, $\psi_0 = \text{atan2}(v_{r2}, v_{r1})$ and $v_r = ||\underline{v}_r||$.

e) Guess $\dot{\psi} = 0$ for the 4th element of $\underline{x}$; i.e., guess that the turn rate is zero.

The efficacy of this first-guess procedure depends on the nature of the data. It is to be hoped that, even if this is not such a good first guess, your estimator will still yield a good estimate due to its robust convergence properties. If this is the case, then the goodness of the first guess only affects the number of Gauss-Newton iterations that are needed to converge, but not the accuracy of the final solution.

The following figure shows the true ground track of the mid-point between the cart's rear wheels. It extends from a time before t_1 to a time after t_{31}, which makes it very hard for you to derive a first guess directly from the figure. The function "psiy1y2cart.m" has been included on the course web site in order to help you make a similar plot using your estimate of $\underline{x}$. Make this plot and check whether your resulting path closely matches a sub-arc of this figure.



6. Suppose that one defines the following least-squares estimation cost function for k samples of data:

$$J(\underline{x},k) = \sum_{j=1}^{k} [\underline{z}(j) - H(j)\underline{x}]^{\mathrm{T}} R^{-1}(j)[\underline{z}(j) - H(j)\underline{x}]$$

where $R(j) = E[\underline{w}(j)\underline{w}^{\mathrm{T}}(j)]$ is the usual measurement error covariance matrix. Suppose, also, that $\hat{\underline{x}}(k,\underline{z}^k)$ and $\hat{P}(k,\underline{z}^k)$ are, respectively, the least-squares estimate that minimizes $J(\underline{x},k)$ and the estimation error covariance for the estimate. Then prove that

$$J(\underline{x},k) = [\underline{x} - \hat{\underline{x}}(k,\underline{z}^k)]^{\mathrm{T}} \hat{P}^{-1}(k,\underline{z}^k)[\underline{x} - \hat{\underline{x}}(k,\underline{z}^k)] + J[\hat{\underline{x}}(k,\underline{z}^k),k]$$

<u>Hints</u>: Both forms of $J(\underline{x},k)$ are quadratic in the argument $\underline{x}$. Equality of the two forms of $J(\underline{x},k)$ can be proved by proving that the respective coefficients of the constant, linear, and quadratic $\underline{x}$ terms are equal. You may need to try several of the alternate equivalent formulas for $\hat{P}(k,\underline{z}^k)$ in order to find the one that is most useful for purposes of this proof.

7. Re-consider the second part of problem 1-9 in Bar-Shalom, the hypothesis test with $n = 2$, and refer to the lecture notes on locally most powerful tests.

   a) Write MATLAB code that determines the power function

$$Power(\theta_1) = \int_{-\infty}^{-\beta_0} p(\beta \mid \theta = \theta_1)d\beta + \int_{\beta_0}^{\infty} p(\beta \mid \theta = \theta_1)d\beta$$

   on a grid of θ_1 values stretching from $\theta_1 = -10$ to $\theta_1 = 10$ with a grid spacing of $\Delta\theta_1 = 0.01$. Plot this function.

   Hint: You will want to use the function normcdf.m in order to do this.

   b) Write MATLAB code to plot the two probability density functions for the locally most powerful test statistic β, $p(\beta|\theta=0)$ and $p(\beta|\theta=\theta_1=4)$, on an appropriate grid of β values, and plot these functions. Also, plot the threshold value β_0 on this plot. Be sure to use different line types for the two probability density functions, and use the MATLAB "legend" command in order to label the two curves on the resulting plot.

   Hint: You will want to use the function normpdf.m in order to do this.

   c) Consider the alternative, sub-optimal acquisition statistic $\eta(\underline{z}) = z_1 - 0.3\, z_2 = [1, -0.3]\underline{z}$. Determine formulas for the probability density functions $p(\eta|\theta=0)$ and $p(\eta|\theta=\theta_1)$ and determine the threshold value η_0 that yields a false-alarm probability of $\alpha = 0.01$ (1%) for the test.

   Accept H_1 if $|\eta(\underline{z})| \geq \eta_0$; otherwise, accept H_0.

Note: this statistic is not an optimal Neyman-Pearson test statistic. It is an ad hoc test that your boss dreamed up based on some heuristic about the problem. It is your job to check whether it gives a good means of discriminating between hypotheses H_0 and H_1.

d) Repeat part a), but use the sub-optimal acquisition statistic of part c) and plot its power vs. θ_1 on the same graph as your result from part a), but using a different line type. Use the MATLAB "legend" command in order to label your two curves on the resulting plot.

e) Repeat part b), but use the sub-optimal acquisition statistic of part c) and plot these two new probability density functions together on one new graph along with η_0.

f) Compare and contrast the locally most powerful hypothesis test and the heuristic hypothesis test. Use your plots from the previous sections in order to do this comparison. Discuss the usefulness (or lack thereof) of the Neyman-Pearson test in comparison to heuristic tests.

8. Re-consider problem 1-9 of Bar-Shalom yet again. An alternative to the locally most powerful test is to let $p(\underline{z}|H_1) = p[\underline{z}|\hat{\theta}_1(\underline{z})]$ where $\hat{\theta}_1(\underline{z})$ is an optimal estimate of θ_1. This approach solves the problem of there being infinitely many possible θ_1 values within the hypothesis H_1 by using an estimate of θ_1 based on the data in $\underline{z}$. For problem 1-9 of Bar-Shalom, develop the Neyman-Pearson test statistic using this definition of $p(\underline{z}|H_1)$ with $\hat{\theta}_1(\underline{z})$ defined using maximum likelihood estimate, i.e., using $\hat{\theta}_1(\underline{z})$ that maximizes $p(\underline{z}|\theta_1)$. Show that this test statistic is equivalent to the original test statistic that you derived using the locally most powerful test. (Note that this equivalence between the estimation-based test statistic and the locally most powerful test statistic is not necessarily true in all situations even though it is true in this situation.)

Hint: the test criterion $|\beta(\underline{z})| \geq \beta_0$ and the criterion $[\beta(\underline{z})]^2 \geq [\beta_0]^2$ are equivalent.

9. The following is a simplified nonlinear GPS pseudorange measurement model:

$$z(j) = P^j = [(X-X^j)^2 + (Y-Y^j)^2 + (Z-Z^j)^2]^{0.5} + c\delta_R + w(j) \quad \text{for } j = 1, \ldots, N$$

where P^j is the measured pseudorange from the j^{th} GPS satellite to the receiver in meters units, $[X;Y;Z]$ is the unknown receiver location in ECEF Cartesian coordinates in meters units, $[X^j;Y^j;Z^j]$ is the known location of the j^{th} GPS satellite in ECEF Cartesian coordinates at the time of signal transmission in meters units, c is the speed of light in vacuum in meters/sec units, δ_R is the unknown offset of the receiver clock from true GPS time in seconds (it would be subtracted from the erroneous receiver clock time in order to determine the true GPS time of signal reception if it were known), and $w(j)$ is the pseudorange measurement error in the P^j measurement. N is the number of GPS satellites.

Develop and implement a Gauss-Newton solver for the unknown 4x1vector $\underline{x} = [X;Y;Z;(c\delta_R)]$, all of whose elements are given in meters units. Implement your solver in MATLAB. Also compute the estimation error covariance matrix $P_{xx} = (H^T R^{-1} H)^{-1}$. As part of your implementation, develop a MATLAB function called `hmeasgps.m`. Its function call form must be

```
>> [h,H] = hmeasgps(x,rsatsmat)
```

where `x` is the 4x1 vector of unknowns that are to be estimated and that has been defined above, `rsatsmat` is a 3xN matrix that contains the known Cartesian position vectors of the N satellites so that X^j = `rsatsmat(1,j)`, Y^j = `rsatsmat(2,j)`, and Z^j = `rsatsmat(3,j)`, `h` is the Nx1 vector that contains the modeled pseudoranges, and `H` is the Nx4 Jacobian first partial derivative of `h` with respect to `x`.

Assume that each $w(j)$ is a sample from a Gaussian distribution with a mean of 0 and a standard deviation of σ_{PR} meters so that $E\{w(j)\} = 0$ and $E\{[w(j)]^2\} = (\sigma_{PR})^2$. Also assume that the measurement errors of different satellites are uncorrelated so that $E\{w(j)w(k)\} = 0$ if $j \neq k$.

Solve the specific problem whose data are contained in the file `assign03prob09_data.mat`. In addition to the satellites' locations in the array `rsatsmat`, this file contains the measured pseudoranges in `PRvec` such that P^j = `PRvec(j,1)`, the pseudorange measurement error standard deviation σ_{PR} in `sigmaPR`, the number of satellites `N`, and the truth value of $\underline{x}$ in `xtrue`.

After you get to your nonlinear least-squares estimate, `xhat`, compute `(xhat - xtrue)'*inv(Pxx)*(xhat - xtrue)`. If a linearized form of the measurement model is accurate, then this non-negative scalar should a sample from a chi-squared distribution of degree 4. Check whether your value for this statistic is reasonable as a possible sample from such a distribution.

Also do the following problems from Bar-Shalom:

3-7, 3-9, 3-11 (note, the final formula for the requested ratio in Bar-Shalom is wrong. It should be $4[n\text{-}0.5]/[n+1]$. Bar-Shalom's answer would be correct if the question concerned the position error variance one full sample period T after the final data point.), 3-12, 3-13, 3-16