HW2 | Spencer Freeman | AOE 5784 | 9/28/24

5.) $$p(x|H_0) = \frac{1}{2\sigma_A} e^{-\frac{|x|}{\sigma_A}}$$ $\sigma_B > \sigma_A$

$$p(x|H_1) = \frac{1}{2\sigma_B} e^{-\frac{|x|}{\sigma_B}}$$ $\{x_1, x_2, \dots x_n\}$

$\alpha = 0.001$, $n = 10$

$\sigma_B = 4\sigma_A$

Accept H_1 if $\dfrac{p(x_1, x_2 \dots x_n | H_1)}{p(x_1, x_2, \dots x_n | H_0)} \geq \Lambda$, otherwise accept H_0

$$p(x_1 \dots | H_1) = \frac{1}{2^n \sigma_B^n} \exp\left\{\frac{-1}{\sigma_B} \sum_{i=1}^{n} |x_i| \right\}$$

$$p(x_1 \dots | H_0) = \frac{1}{2^n \sigma_A^n} \exp\left\{\frac{-1}{\sigma_A} \sum_{i=1}^{n} |x_i| \right\}$$

$$\left(\frac{\sigma_A}{\sigma_B}\right)^n \exp\left\{\left(\frac{1}{\sigma_A} - \frac{1}{\sigma_B}\right) \sum_{i=1}^{n} |x_i| \right\} \geq \Lambda$$

$$q = \sum_{i=1}^{n} |x_i| \geq \ln\left\{\left(\frac{\sigma_B}{\sigma_A}\right)^n \Lambda\right\} \frac{1}{(1/\sigma_A - 1/\sigma_B)} = q_0$$

$$\frac{2q}{\sigma_B} \sim \chi^2_{2n}, \quad \frac{2q}{\sigma_A} \sim \chi^2_{2n}$$

$$\alpha = p(q \geq q_0 | H_0) = \int_{q_0}^{\infty} \frac{1}{2^{n_2/2} \Gamma(n_2/2) \frac{\sigma_A}{2}} \left(\frac{2q}{\sigma_A}\right)^{\left(\frac{n_2-2}{2}\right)} \exp\left\{-\frac{1}{2}\left(\frac{2q}{\sigma_A}\right)\right\}$$

$2n = n_2$

$\frac{2q}{\sigma_A} = \beta$ $$\alpha = \int_{\frac{2q_0}{\sigma_A}}^{\infty} \frac{1}{2^{n_2/2} \Gamma\left(\frac{n_2}{2}\right)} \beta^{\left(\frac{n_2-2}{2}\right)} \exp\left\{-\frac{\beta}{2}\right\} d\beta$$

$$\alpha = 1 - \int_0^{\frac{2q_0}{\sigma_A}} \frac{1}{2^{n_2/2}\Gamma\left(\frac{n_2}{2}\right)} \beta^{\left(\frac{n_2-2}{2}\right)} \exp\left\{\frac{-\beta}{2}\right\} d\beta$$

$$q_0 = \frac{\sigma_A}{2} \cdot \text{chi2inv}(1-\alpha, n_2) \; : \; [\text{MATLAB}]$$

$$q_0 = \frac{\sigma_A}{2} \text{chi2inv}(1-0.001, 2(10)) = \boxed{22.7\,(\sigma_A)}$$

$$P_{MD} = P(q < q_0 | H_1) = \int_0^{q_0} \frac{1}{2^{n_2/2}\Gamma\left(\frac{n_2}{2}\right)\frac{\sigma_B}{2}} \left(\frac{2q}{\sigma_B}\right)^{\left(\frac{n_2-2}{2}\right)} \exp\left\{-\frac{1}{2}\left(\frac{2q}{\sigma_B}\right)\right\} dq$$

$$P_{MD} = \int_0^{\frac{2q_0}{\sigma_B}} \frac{1}{2^{n_2/2}\Gamma\left(\frac{n_2}{2}\right)} \beta^{\left(\frac{n_2-2}{2}\right)} \exp\left\{-\frac{\beta}{2}\right\} d\beta$$

$$P_{MD} = \text{chi2cdf}\left(\frac{2q_0}{\sigma_B}, n_2\right) \; : \; [\text{MATLAB}]$$

$$P_{MD} = \text{chi2cdf}\left(\frac{2(22.7)\cancel{\sigma_A}}{4\cancel{\sigma_A}}, 2(10)\right) = \boxed{0.06}$$

$$\boxed{\text{Accept } H_1 \text{ if } \sum_{i=1}^{n} |x_i| \geq q_0 \text{, otherwise accept } H_0}$$