1-9) $H_0 : \theta = 0$ $\quad H_1 : \theta \neq 0$

$z_i = \theta + w_i \qquad i = 1, \dots n$

$w_i \sim N(0, \sigma) \qquad \underline{w} = [w_1 \dots w_n]^T$

$E[\underline{w}\underline{w}^T] = P$

① Optimal hypothesis test for false alarm prob. α:

$\boxed{\text{Two-sided locally most powerful test}}$

② $n = 2$, $P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $\alpha = 1\%$

$$q(\underline{z}, \theta_1) \approx \left[\frac{dq}{d\theta} \bigg|_{\underline{z},0} \right] \theta_1 + \frac{1}{2} \left[\frac{d^2 q}{d\theta^2} \bigg|_{\underline{z},0} \right] \theta_1^2 + O(\theta_1^3)$$

$$q(\underline{z}, \theta_1) = \ln\left(\frac{p(\underline{z} | \theta = \theta_1)}{p(\underline{z} | \theta = 0)} \right)$$

$$\underline{z} = [z_1 \dots z_n]^T = \underline{e}\,\theta + \underline{w}$$

test: accept H_1 if $|\beta(\underline{z})| = \left| \left[\frac{dq}{d\theta_1} \bigg|_{\underline{z},0} \right] \right| \geq \frac{q_0}{|\theta_1|} = \beta_0$;

otherwise, accept H_0

$$\alpha = \int_{-\infty}^{-\beta_0} p(\beta | \theta = 0)\, d\beta + \int_{\beta_0}^{\infty} p(\beta | \theta = 0)\, d\beta$$

$$p(\underline{z} \mid \theta = 0) = p(\underline{e}(0) + \underline{w}) = p(\underline{w}) = \frac{1}{(2\pi)^n \sqrt{|P|}} \cdot e^{-\frac{1}{2}(\underline{z} - \underline{\mu_w})^T P^{-1} (\underline{z} - \underline{\mu_w})}$$

$$p(\underline{z} \mid \theta = 0) = \frac{1}{(2\pi)^n \sqrt{|P|}} \cdot e^{-\frac{1}{2} \underline{z}^T P^{-1} \underline{z}}$$

$$p(\underline{z} \mid \theta = \theta_1) = p(\underline{e}\theta_1 + \underline{w}) \quad \underline{z} \sim N(\underline{e}\theta_1, P) \quad = \frac{1}{(2\pi)^n \sqrt{|P|}} \cdot e^{-\frac{1}{2}(\underline{z} - \theta_1 \underline{e})^T P^{-1} (\underline{z} - \theta_1 \underline{e})}$$

$$p(\underline{z} \mid \theta = \theta_1) = \frac{1}{(2\pi)^n \sqrt{|P|}} \cdot e^{-\frac{1}{2}(\underline{z} - \theta_1 \underline{e})^T P^{-1} (\underline{z} - \theta_1 \underline{e})}$$

$$q(\underline{z}, \theta_1) = \ln\left(e^{-\frac{1}{2}(\underline{z} - \theta_1 \underline{e})^T P^{-1} (\underline{z} - \theta_1 \underline{e}) + \frac{1}{2} \underline{z}^T P^{-1} \underline{z}} \right)$$

$$q(\underline{z}, \theta_1) = \frac{1}{2}\left(\underline{z}^T P^{-1} \underline{z} - (\underline{z} - \theta_1 \underline{e})^T P^{-1} (\underline{z} - \theta_1 \underline{e}) \right)$$

$$\beta(\underline{z}) = \left. \frac{dq}{d\theta_1} \right|_{\underline{z}, 0} = -\frac{1}{2} \frac{d}{d\theta_1}\left[(\underline{z} - \theta_1 \underline{e})^T P^{-1} (\underline{z} - \theta_1 \underline{e}) \right]$$

$$\beta(\underline{z}) = -\frac{1}{2} D(f \circ g)(\theta_1) = -\frac{1}{2} \left(D f(g(\theta_1)) \right) \left[D g(\theta_1) \right] \rightarrow \text{chain rule}$$

$$g(\theta_1) = \underline{z} - \theta_1 \underline{e}, \quad f(g(\theta_1)) = g(\theta_1)^T P^{-1} g(\theta_1) \rightarrow \text{quadratic form}$$

$$D f(g(\theta_1)) = g(\theta_1)^T (P^{-1} + P^{-1^T}) \rightarrow \text{derivative of quadratic form}$$

$$D g(\theta_1) = -\underline{e} \qquad P^{-1^T} = P^{-1}$$

$$\beta(\underline{z}) = -\frac{1}{2}(\underline{z} - \underbrace{\theta_1}_{\theta_1 = 0} \underline{e})^T (P^{-1} + P^{-1^T})(-\underline{e}) = \underline{\underline{\underline{z}^T P^{-1} \underline{e}}}$$

$p(\beta(\underline{z}) | \theta = 0)$, $\beta(\underline{z}) = \underline{z}^T P^{-1} \underline{e} = \underline{z}^T \underline{P_e}$

$\underset{(\theta=0)}{\beta(\underline{z})} = \underline{w}^T \underline{P_e}$

distribution of sum of jointly normal random variables

$\underset{(\theta=0)}{\beta(\underline{z})} \sim N(\overset{0}{\underline{\mu_w}^T} \underline{P_e} , \underline{P_e}^T P \underline{P_e})$

$$p(\beta(\underline{z}) | \theta = 0) = \frac{1}{\underline{P_e}^T P \underline{P_e} \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{\beta}{\underline{P_e}^T P \underline{P_e}}\right)^2}$$

$$\alpha = \int_{-\infty}^{-\beta_0} p(\beta(\underline{z}) | \theta = 0) d\beta + \int_{\beta_0}^{\infty} p(\beta(\underline{z}) | \theta = 0) d\beta$$

$$\boxed{\beta_0 = -\text{norminv}(\alpha/2, \mu_\beta, \sigma_\beta), \quad \mu_\beta = 0, \quad \sigma_\beta = \sqrt{\underline{P_e}^T P \underline{P_e}}}$$

$P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

$\alpha = 0.01$

$\mu_\beta = 0$

$\sigma_\beta = 1.1547$

$\beta_0 = 2.9743$

$$\boxed{\text{accept } H_1 \text{ if } |\underline{z}^T P^{-1} \underline{e}| \geq \beta_0}$$