BS, P2-7)

Show Bayesian estimation that minimizes the expected value of the cost function

$$C(x-\hat{x}) \triangleq |x-\hat{x}|$$

yields $\hat{x} = x_m$, the median of x:

$$\int_{-\infty}^{x_m} p(x|z)\,dx = \frac{1}{2}$$

$$\hat{x}_{mmse}(z) = \min_{\hat{x}} E\big[(|x-\hat{x}|\,|\,z)\big] = \min_{\hat{x}} \int_{-\infty}^{\infty} |x-\hat{x}|\,p(x|z)\,dx$$

$$0 = \frac{d}{d\hat{x}}\Big[\int_{-\infty}^{\infty} |x-\hat{x}|\,p(x|z)\,dx\Big], \quad |x-\hat{x}| = \begin{cases} -(x-\hat{x}), & x<\hat{x} \\ x-\hat{x}, & x \geq \hat{x} \end{cases}$$

$$0 = \frac{d}{dx}\Big[\int_{-\infty}^{\hat{x}} -(x-\hat{x})\,p(x|z)\,dx + \int_{\hat{x}}^{\infty} (x-\hat{x})\,p(x|z)\,dx\Big]$$

Leibniz rule:

$$0 = (\hat{x}-\infty)\,p(x|z)\overset{0}{\frac{d}{d\hat{x}}(\infty)} - (\overset{0}{\hat{x}-\hat{x}})\,p(x|z)\frac{d}{d\hat{x}}(\hat{x}) + \int_{-\infty}^{\hat{x}} \overset{1}{\frac{d}{d\hat{x}}(\hat{x}-x)}\,p(x|z)\,dx + \dots$$

$$(\overset{0}{\hat{x}-\hat{x}})\,p(x|z)\frac{d}{d\hat{x}}(\hat{x}) - (\infty-\hat{x})\,p(x|z)\overset{0}{\frac{d}{d\hat{x}}(\infty)} - \int_{\hat{x}}^{\infty} p(x|z)\,dx$$

$$0 = \int_{-\infty}^{\hat{x}} p(x|z)\,dx - \int_{\hat{x}}^{\infty} p(x|z)\,dx, \quad 1 = \int_{-\infty}^{\hat{x}} p(x|z)\,dx + \int_{\hat{x}}^{\infty} \overset{\text{PDF}}{p(x|z)}\,dx$$

$$1 = (\overset{2}{1+1})\int_{-\infty}^{\hat{x}} p(x|z)\,dx + (\overset{0}{-1+1})\int_{\hat{x}}^{\infty} p(x|z)\,dx$$

$$\boxed{\frac{1}{2} = \int_{-\infty}^{\hat{x}} p(x|z)\,dx}$$