Reachability of One-Counter Automata

Reporter: Xie Li

February 27, 2019

Outline

- NP Lower Bound
- Weighted Graph
  Weighted graph, path, cycle, v-cycle, positive v-cycle template.
  An algorithm to decide positive cycle in a weighted graph.
- Path Flows
  Flow, s-t path flow, weight of a flow, support of a flow, s-t support.
  Lemma 4.1.6.
- The NP Upper Bound

Path Flows

Question: Why are interested in path flows?

Lemma (4.1.6)

A flow f is a s-t path flow iff f satisfies the following conditions:

- *If $s = t$ then*
  1. $\Sigma_{w \in out(v)} f(v,w) = \Sigma_{w \in in(v)} f(w,v)$ *for all* $v \in V$
  2. $F(f)$ *is a s-t support*
- *If $s \neq t$ then*
  1. $\Sigma_{w \in out(v)} f(v,w) = \Sigma_{w \in out(v)} f(w,v)$ *for all* $v \in V - \{s,t\}$
     $\Sigma_{w \in out(s)} f(s,w) = \Sigma_{w \in out(s)} f(w,s) + 1$
     $\Sigma_{w \in out(t)} = \Sigma_{w \in out(t)} f(w,t) - 1$
  2. *F(f) is connected.*

Lemma (4.1.7)

Let $s, t \in V, n, n' \in \mathbb{N}$ and F be an s-t support, there exists a QFPA formula $\phi(G, F < s, t)(c, c')$ such that $\phi[n/c, n'/c']$ iff thre exits an s-t path flow f with support F and weight$(f) = n' - n$. Moreover, $|\phi| = O(|G|^2)$.

Proof.

Proof idea: regard each $f(e)$ as a variable in the formula.
Let $\psi(f(e_1), \ldots, f(e_k))$ be the formula derived from lemma4.1.6 regarding $s = t$ or not.
$\psi'(f(e_1), \ldots, f(e_k)) = \Sigma_{e \in F} f(e)\mu(e) = c' - c \wedge \bigwedge_{e \in E-F} f(e) = 0$
The final formula:
$\phi(G, F, s, t) = \exists_{e \in E} f(e).\psi \wedge \psi'$

□

remark: path flows are additive regarding to the concatenation of paths.

Lemma (4.1.8 & 4.1.9 & 4.1.10)

4.1.8 *Let f, f' be path flows, if*

- *f is an s-v path flow and f' is a v-t path flow, or*
- *f is an s-t path flow, f' is a v-v path flow and $F \cup F'$ is an s-t support.*

$f + f'$ is an s-t path flow.

4.1.9 *A flow f a v-v path flow iff there are path flows $f_1, \dots f_j, j \in [|G|]$ such that each G/F_i is a loop and $f = \Sigma_{i \in [j]} f_i$*

4.1.10 *A flow f is an s-t path flow iff there are j path flows f_i with $j \in O(|G|^2)$ and a path flow f_0 such that $f = f_0 + \Sigma_{i \in [j]} f_i$ where G/F_0 is a simple s-t path, G/F_i is a simple cycle and $G/\cup_i F_i)$ is connected.*

Proof.

4.1.8 A directly application Lemma 4.1.6.

4.1.9 $\Rightarrow$: By induction on the number of edges in G/F

4.1.10 $\Rightarrow$: By induction, deleting cycle in path and lemma 4.1.9.

Definition (Support Edge Decomposition)

Given an s-t support F, a support-edge decomposition of F is a squence of tuples $(F_i, v_i, w_i, e_i)_{i\in[m]}$ with $F_i \subseteq F, v_1 = s, v_{m+1} = t$ such that

- F_i is a $v_i - w_i$ support, $e_i = (w_i, v_{i+1})$, $i \in [m]$
- $e_i \notin F_j$ for all $1 \leq i < j \leq m$
- $F = \cup_{i\in[m]}\{e_i\}$

An *edge decomposition* is a nsequence of tuples $(f_i, e_i)_{i\in[m]}$ where f_i is $v_i - w_i$ path flow, $v_1 = s, v_{m+1} = t, e_i = (w_i, v_{i+1})$ s.t. ...

The NP Upper Bound

First consider one-counter automata without zero-test. Under this circumstance, it can be viewed as weighted graph.
$\mathcal{A} = (Q, \Lambda, q_0, F, \delta, \lambda, \epsilon)$ corresponds to $G_{\mathcal{A}} = (Q, \Delta, \lambda)$.
Then we can relate runs in $T(\mathcal{A})$with paths in $G_{\mathcal{A}}$.
However a path in $G_{\mathcal{A}}$ does not gurantee a run between two configurations of $T(\mathcal{A})$.

Lemma (4.1.11)

Let π be a $q - q'$ path in $G_{\mathcal{A}}$ and $n, n' \in \mathbb{N}$. There is a run $(q, n) \rightarrow^{*}_{\mathcal{A}} (q', n')$ that π corresponds to iff $drop(\pi) \geq -n$ and $weight(\pi) = n' - n$.

Proof.

By induction on $|\pi|$.

$\square$

Reachability Certificate to Path

Definition (Rechability Certificate)

Let G be a graph, f is a s-t path flow and $n, n' \in \mathbb{N}$. The n (G, f, n, n') fulfills the

1. type-1 reachability criteria if
   - G/F does not contain positive cycles
   - $weight(f) = n' - n$
   - f has an edge decomposition $(f_i, e_i)_{i \in [m]}$ such that $\Sigma_{i \in [j]} weight(f + f_{e_i}) \geq -n$ for all $j \in [m]$
2. type-2 reachability criteria if $(G^{op}, f^{op}, n' n)$ fulfills the type-1 reachability criteria.
3. type-3 reachability criteria if
   - $weight(f = n' - n)$
   - there is a positive s-cycle template l in G with respect to n
   - there is a positive t-cycle template l' in G^{op} with respect to n'

then we call (G, f, n, n') type-i reachability certificate correspondingly.

Lemma (4.1.12)

Let $(q, n), (q', n')$ be configurations of $\mathcal{A}$ and $G_{\mathcal{A}}$ the graph corresponding to $\mathcal{A}$.
If $(G_{\mathcal{A}}, f, n, n')$ is a type-i reachability certificate then $f = f_\pi$ for some path π corresponding to a run $\rho : (q, n) \rightarrow^*_{\mathcal{A}} (q', n')$ in $T(\mathcal{A})$.

Proof.

Make use of the result of lemma 4.1.11 $\square$

Path to Reachability Certificate

Lemma (4.1.13)

Let $\rho : (q, n) \to^*_{\mathcal{A}}$ be a run in $T(\mathcal{A})$ with the corresponding path π in $G_{\mathcal{A}}$ and let F be the support of $f_p i$. If π does not contain any positive Cycle then either $G_{\mathcal{A}}/F$ does not contain any positive cycles or there is a path π' in $G_{\mathcal{A}}$ that factors as $\pi' = \pi_1 \cdot \pi_2 \cdot \pi_3$ and corresponds to a run $\rho' : (q, n) \to (q', n')$ in $T(\mathcal{A})$ such that $|\pi_1| < |\pi|$ and π_2 is a positive cycle.

Proof.

Assume $G_{\mathcal{A}}/F$ contains a positive cycle l.
Let p be the first vertex of l that occurs in π and let (p, m) be the configuration first reached by ρ.
We claim that there is a posotive cycle at p in $G_{\mathcal{A}}$ that corresponds to a run $(p, m) \to^*_{\mathcal{A}} (p, m')$ for some $m' > m$.
If not argue as follows...
Next we observe that the first occurence of p in π lies on a negative cycle in π. We can decompose $\rho = \rho_1 \cdot \rho_2 \cdot \rho_3$ □

Lemma (4.1.14)

There is a run $\rho : (q, n) \to (q', n')$ *in* $T(\mathcal{A})$ *iff there is a q-q' path* π *in* $G_{\mathcal{A}}$ *that can be written as* $\pi = \pi_1 \cdot \pi_2 \cdot \pi_3$ *such that there are* $n_1, n_2 \in \mathbb{N}$ *such that*

- *if* $|\pi_1| > 0$*, then* $(G_{\mathcal{A}}, f_{\pi_1}, n, n_1)$ *is a type-1 reachability certificate.*
- *if* $|\pi_2| > 0$*, then* $(G_{\mathcal{A}}, f_{\pi_2}, n_1, n_2)$ *is a type-3 reachability certificate.*
- *if* $|\pi_3| > 0$*, then* $(G_{\mathcal{A}}, f_{\pi_3}, n_2, n')$ *is a type-2 reachability certificate.*

Proof.

$\Rightarrow$:

- $G_{\mathcal{A}}/F$ does not contain any positive cycle.
- Otherwise, let π' be a path in $G_{\mathcal{A}}$ corresponding to some run $(q, n) \to^*_{\mathcal{A}} (q', n')$. By repeatly applying lemma 4.1.13 to π', we can obbtain $\pi_1 \cdot \pi'_2 \cdot \pi'_3$. Let $\pi'' = \pi_2 \cdot \pi_3$, if $G_{\mathcal{A}}/F''_{\pi}$ does not contain negative cycles...

By now we have shown that deciding reachability in $T(\mathcal{A})$ can be reduced to checking the exisitence of at most three reachability certificates in $G_{\mathcal{A}}$.
Now we need to phrased in terms of an open formula in QFPA.

Definition (QFPA definable)

Given a set $M \subseteq \mathbb{N}^k$, we say M is QFPA-definable if there exists a finite set R of QFPA formulae, each with free variables $x_1, \ldots, x_k$ such that

$$M = \bigcup_{\phi(x_1,\cdots,x_k)\in R}\{(n_1,\ldots,n_k) \in \mathbb{N}^k :\models \phi[n_1/x_1,\ldots,n_k/x_k]\}$$

Lemma (4.1.15)

Given a graph G and vertices s, t, the sets

$$\{(n, n') : \textit{thre exists an } s\text{-}t \textit{ path flow } f \textit{ such that } (G, f, n, n') \textit{ is a type-1 reachability certificate } \}$$

$$\{(n, n') : \textit{thre exists an } s\text{-}t \textit{ path flow } f \textit{ such that } (G, f, n, n') \textit{ is a type-2 reachability certificate } \}$$

are definable via sets $R_1(G, s, t)$ and $R_2(G, s, t)$ of QFPA formulae, where $|\phi| = O(|G|^4)$ for each $\phi \in R_1(G, s, t) \cup R_2(G, s, t)$

Proof.

$$\varphi \stackrel{\text{def}}{=} \exists_{i \in [m]} c_i, c_i'. \quad \underbrace{\varphi(G/F)}_{\text{no positive cycles}} \quad \wedge \quad \underbrace{\bigwedge_{i \in [m]} \varphi(G, F_i, v_i, v_i')(c_i, c_i')}_{\text{there are path flows } f_i \text{ with weight } c_i' - c_i} \quad \wedge$$

$$\wedge \quad \underbrace{\bigwedge_{i \in [m]} \sum_{j \in [i]} c_i' - c_i + \mu(e_i) \geq -c}_{\text{weights of the edge decomposition sum up correctly}} \quad \wedge \underbrace{\sum_{i \in [m]} c_i' - c_i + \mu(e_i) = c' - c}_{\text{total weight matches}}$$