# On Multiphase-Linear Ranking Functions

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# Contributions

- Equivalence of different classes of ranking function.
- Algorithms for converting between ranking functions.
- Converting ranking functions on integers to rational.
- Depth bound and iteration bound for MΦRF.

# Single Path Linear Constraint Loop

## Example

$$\texttt{while}\ (x \geq -z)\ \texttt{do}\ x' = x + y,\ y' = y + z,\ z' = z - 1$$

$$\texttt{while}\ (x_2 - x_1 \leq 0,\ x_1 + x_2 \geq 1)\ \texttt{do}\ x_2' = x_2 - 2x_1 + 1,\ x_1' = x_1$$

## Definition (SLC)

$$\textit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \textit{do}\ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ A & \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

# Ranking Functions

## Definition (Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

## Example

$$\texttt{while } (x - 1 > 0)\texttt{do } x' = x - 5$$

Its LRF: $f(x) = x - 1$

We can define a binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$

# Example: Multiphase Ranking Function

Problem: LRF is not strong enough for all loops.

Example

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$f(x, y, z) = a_1 x + a_2 y + a_3 z + b$
assume $a_1, a_2, a_3$ and $b$ are given,

$$f(x, y, z) - f(x', y', z') = a_3 + y a_1 + z a_2$$

variable $y, z$ do not have bound and cannot be used in the ranking function.

# Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases: $\langle z + 1, y + 1, x \rangle$

| $x$ | $y$ | $z$ | $z+1$ | $y+1$ | $x$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | **2** | 2 | 1 |
| 2 | 2 | 0 | **1** | 3 | 2 |
| 4 | 2 | $-1$ | **0** | 3 | 4 |
| 6 | 1 | $-2$ | $-1$ | **2** | 6 |
| 7 | $-1$ | $-3$ | $-2$ | **0** | 7 |
| 6 | $-4$ | $-4$ | $-3$ | $-3$ | **6** |
| 2 | $-8$ | $-5$ | $-4$ | $-7$ | **2** |
| $-6$ | $-13$ | $-6$ | $-5$ | $-12$ | $-6$ |

# Multiphase Ranking Function

## Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for $T$ if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

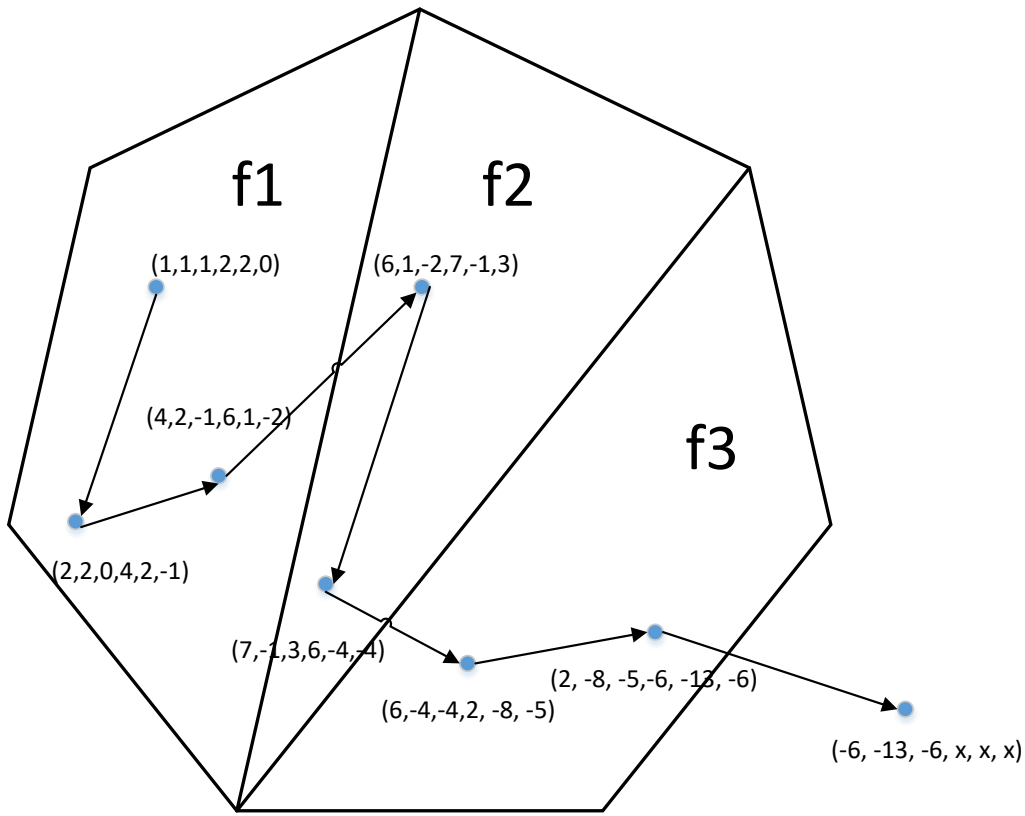
$$\begin{aligned} \forall j \leq i \;.\; \Delta f_j(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \\ \forall j < i \;.\quad f_j(\mathbf{x}) &\leq 0\,. \end{aligned}$$

We say that $\mathbf{x}''$ is ranked by $f_i$(for the minimal).

# Example Revisit

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$$\begin{aligned} \forall j \leq i \;.\; \Delta f_j(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \\ \forall j < i \;.\quad f_j(\mathbf{x}) &\leq 0\,. \end{aligned}$$



# Nested Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Loop condition: $x + z > 0$. We only want to use this constraint for the ranking function.
$\langle z + 1, y + 1, x + z \rangle$

## Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for $T$ if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d. \end{aligned}$$

Let $f_0 = 0$.

Nested is Multiphase, but not the opposite.
$(x = -1, y = 0, z = 1)$

# MΦRF to Nested Ranking Function?

**Theorem**

*If $T$ has a MΦRF of depth $d$, then it has a nested ranking function of depth at most $d$.*

Sythesising nested ranking function is in `PTIME`, with theorem above we have..

# Lexicography Linear Ranking Function

Intuition: remind binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$.
Generalize it into several phases using lexicographical order of ranking functions.
$\langle f_1, f_2, \ldots, f_d \rangle$
$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

## Definition (LLRF)

Given a set of transitions $T$ we say that $\langle f_1, f_2, \ldots, f_d \rangle$ is a LLRF (of depth $d$) for $T$ if for every $\mathbf{x}'' \in T$ there is an index $i$ such that

$$\begin{aligned} \forall j < i \;.\; \Delta f_j(\mathbf{x}'') &\geq 0\,, \\ \Delta f_i(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \end{aligned}$$

A LLRF is weak if..

# Example: MΦRF is a LLRF

$$\begin{aligned} \forall j < i \;.\; \Delta f_j(\mathbf{x}'') &\geq 0\,, \\ \Delta f_i(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \end{aligned}$$

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

| $x$ | $y$ | $z$ | $z+1$ | $y+1$ | $x$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 1 |
| 2 | 2 | 0 | 1 | 3 | 2 |
| 4 | 2 | $-1$ | 0 | 3 | 4 |
| 6 | 1 | $-2$ | $-1$ | 2 | 6 |
| 7 | $-1$ | $-3$ | $-2$ | 0 | 7 |
| 6 | $-4$ | $-4$ | $-3$ | $-3$ | 6 |
| 2 | $-8$ | $-5$ | $-4$ | $-7$ | 2 |
| $-6$ | $-13$ | $-6$ | $-5$ | $-12$ | $-6$ |

# LLRF to MΦRF?

**Theorem (weak LLRF to MΦRF)**

*If $T$ has a weak $LLRF$ of depth $d$, it has a MΦRF of depth $d$.*

# Ranking Function Over Integers

$$A''\mathbf{x}'' \leq \mathbf{c}$$

- Rational convex polyhedra: polyhedra defined by $\mathcal{P} = \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid A''\mathbf{x}'' \leq \mathbf{c}\}$.
- Integers: $I(\mathcal{P}) = \mathcal{P} \cap \mathbb{Z}^{2n}$
- Integer hull: $\mathcal{Q}_I$ is the space of convex combinition of points in $I(\mathcal{P})$.

# Why consider integer?

- Actual programs with `int`.
- More important, conclusions for rational does not always applicable in on integer version.

## Example

$$\texttt{while } (x_2 - x_1 \leq 0,\ x_1 + x_2 \geq 1) \texttt{ do } x_2' = x_2 - 2x_1 + 1,\ x_1' = x_1$$

For rationals: $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}$

For integers: there exists a linear ranking function $f(x_1, x_2) = x_1 + x_2$

# Integer to Rational

**Theorem**

*Let $\langle f_1, f_2, \ldots, f_d \rangle$ be a weak LLRF for $I(\mathcal{P})$. Then there are constants $c_1, \ldots, c_d$ such that $\langle f_1 + c_1, f_2 + c_2, \ldots, f_d + c_d \rangle$ is a weak LLRF for $\mathcal{Q}_I$*

# The Depth of a MΦRF

Idea: precompute an upper bound of depth $\rightarrow$ a decision procedure for MΦRF in general

## Theorem

*For integer $B > 0$, following loop needs at lease $B + 1$ components in any MΦRF.*

$$\texttt{while } (x \geq 1,\ y \geq 1,\ x \geq y,\ 2^B y \geq x) \texttt{ do } x' = 2x,\ y' = 3y$$

## Example

$$\texttt{while } (x \geq 1,\ y \geq 1,\ x \geq y,\ 4y \geq x) \texttt{ do } x' = 2x,\ y' = 3y$$

# Iteration Bound

**Theorem**

*An SLC loop that has a M$\Phi$RF terminates in a number of iterations bounded by* $O(||x_0||_\infty)$

# Future Work & Questions