On Multiphase-Linear Ranking Functions

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Single Path Linear Constraint Loop

Example

$$\texttt{while } (x \geq -z) \texttt{ do } x' = x + y,\ y' = y + z,\ z' = z - 1$$

Let $B = (-1, 0, 1)$, $\mathbf{x} = (x, y, z)^T$, $\mathbf{b} = 0$.
Let $\mathbf{x}'' = (x, y, z, x', y', z')$,

$$A = \begin{bmatrix} 1 & 1 & 0-1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

and $\mathbf{c} = (0, 0, 1)^T$

Single Path Linear Constraint Loop

Definition (SLC)

$$\mathit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \mathit{do}\ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while} \;\; (x - 1 > 0)\texttt{do} \;\; x' = x - 5$$

Its LRF: $f(x) = x - 1$

We can define a binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$

Nested r.f.

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$
\begin{aligned}
f_d(\mathbf{x}) &\geq 0 \\
(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d.
\end{aligned}
$$

Let $f_0 = 0$.

Limitation of LRF

$$\texttt{while}\ (q > 0)\texttt{do}\ \ q' = q - y, y' = y + 1$$

Assume there is a LRF for this loop, say $f(q, y) = a_1 q + a_2 y + b$

$$f_(q, y) - f(q', y') = a_1 y + a_2$$

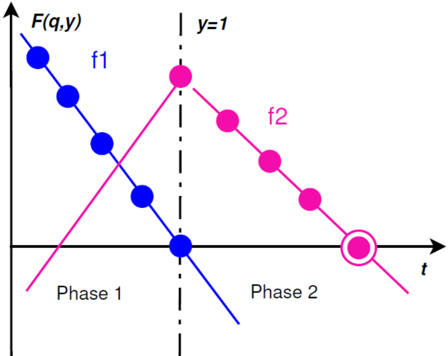
Since y is not bounded, we cannot guarantee $\Delta f(q, y, q', y') > 0$

Example: Nested r.f.

$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \qquad \text{for all } i = 1, \ldots, d.$$

$$\texttt{while } (q > 0)\texttt{do } q' = q - y, y' = y + 1$$

- Above loop has Nested r.f. $\langle 1 - y, q + 1 \rangle$



Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases: $\langle z+1, y+1, x\rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	**2**	2	1
2	2	0	**1**	3	2
4	2	−1	**0**	3	4
6	1	−2	−1	**2**	6
7	−1	−3	−2	**0**	7
6	−4	−4	−3	−3	**6**
2	−8	−5	−4	−7	**2**
−6	−13	−6	−5	−12	−6

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

$$\begin{aligned} \forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \\ \forall j < i \ . \qquad f_j(\mathbf{x}) &\leq 0\,. \end{aligned}$$

We say that $\mathbf{x}''$ is ranked by f_i(for the minimal).

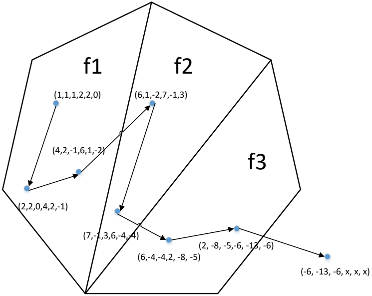
Example Revisit

$$\texttt{while } (x > -z) \texttt{do } x' = x + y, y' = y + z, z' = z - 1$$

$$\langle z + 1, y + 1, x \rangle$$

$\mathbf{x}''$ is ranked by f_k when $i = k$. In this example, $f_1(x, y, z) = z + 1$, $f_2(x, y, z) = y + 1$ and $f_3(x, y, z) = x$

$$\begin{aligned} \forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1 \,, \\ f_i(\mathbf{x}) &\geq 0 \,, \\ \forall j < i \ . \quad f_j(\mathbf{x}) &\leq 0 \,. \end{aligned}$$



Motzkin's Transposition Theorem

Theorem (Motzkin's Transposition Theorem)

For $A \in \mathbb{K}^{m \times n}, C \in \mathbb{K}^{l \times n}, b \in \mathbb{K}^m$, *and* $d \in \mathbb{K}^l$. *The formulae below are equivalent.*

- $\forall x \in \mathbb{K}^n . \neg(Ax \leq b \wedge Cx < d)$
- $\exists \lambda \in \mathbb{K}^m . \exists \mu \in \mathbb{K}^l .$
  $\lambda \geq 0 \wedge \mu \geq 0$
  $\wedge \lambda^T A + \mu^T C = 0 \wedge \lambda^T b + \mu^T d \leq 0$
  $\wedge (\lambda^T b < 0 \vee \mu \neq 0)$

Intuition of Motzkin's transposition theorem:...

Lemma (0)

Given an non-empty polyhedron $\mathcal{P}$ *and linear functions* $f_1, \ldots, f_k$ *such that*

1. $\mathbf{x} \in \mathcal{P} \to f_1(\mathbf{x}) \geq 0 \vee \ldots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee f_k(\mathbf{x}) \geq 0$
2. $\mathbf{x} \in \mathcal{P} \not\to f_1(\mathbf{x}) \geq 0 \vee \ldots f_{k-1}(\mathbf{x}) \geq 0$

There exists a non-negative constants $\mu_1, \ldots, \mu_{k-1}$ *such that*

$$\mathbf{x} \in \mathcal{P} \to \mu_1 f_1(\mathbf{x}) + \mu_{k-1} f_{k-1}(\mathbf{x}) + f_k(\mathbf{x}) \geq 0$$

Proof of Lemma (0)

Let $\mathcal{P}$ be $B\mathbf{x} \leq c$, $f_i = \vec{a}_i\mathbf{x} - b_i$. Then (i) is equivalent to the infeasibility of $B\mathbf{x} \leq c \wedge A\mathbf{x} < b$, by the theorem we have $\vec{\lambda}, \vec{\mu}$ s.t.

$$\vec{\lambda}B + \vec{\mu}A = 0 \wedge \vec{\lambda}\mathbf{c} + \vec{\mu}\mathbf{b} \leq 0 \quad \wedge (\vec{\mu} \neq 0 \vee \vec{\lambda}\mathbf{c} + \vec{\mu}\mathbf{b} < 0)$$

Then for all $\mathbf{x} \in \mathcal{P}$

$$\sum_i \mu_i f_i(\mathbf{x}) = \vec{\mu}A\mathbf{x} - \vec{\mu}\mathbf{b} = -\vec{\lambda}B\mathbf{x} - \vec{\mu}\mathbf{b} \geq -\vec{\lambda}\mathbf{c} - \vec{\mu}\mathbf{b} \geq 0$$

MΦRF to Nested r.f.

Theorem (1)

If $\mathcal{Q}$ has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

Proof.

By induction on the depth d.

- $d = 1$: MΦRF and nested r.f. are both LRF.
- $d > 1$: $d = 2$ e.g. $\langle f_1, f_2 \rangle$. When index $i = 1$, we do not impose bound on $f_2(\mathbf{x})$. However, a bound is needed for $f_2'(\mathbf{x}s)$ in nested r.f. $\langle f_1', f_2' \rangle$.
  To solve the problem that $f_2(\mathbf{x})$ might goes under 0, when $\mathbf{x}''$ is ranked by f_1. Consider $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$

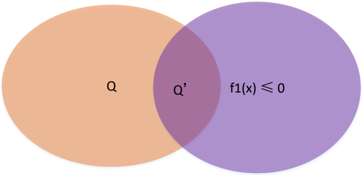
$\square$

MΦRF to Nested r.f.

Lemma (1)

Let $\tau = \langle f_1, \ldots, f_d \rangle$ be an irredundant MΦRF for $\mathcal{Q}$, such that $\langle f_2, \ldots, f_d \rangle$ is a nested ranking function for $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \le 0\}$. Then there is a nested ranking function of depth d for $\mathcal{Q}$.

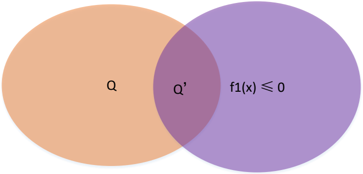
Prove by construction: construct a nested r.f. $\langle f'_1, \ldots, f'_d \rangle$



$$f_d(\mathbf{x}) \ge 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \ge 0 \qquad \text{for all } i = 1, \ldots, d.$$

If f_d is non-negative on $\mathcal{Q}$, then $f'_d = f_d$. Otherwise, $\mathbf{x}'' \in \mathcal{Q} \to f_d(\mathbf{x}) \ge 0 \vee f_1(\mathbf{x}) > 0$

MΦRF to Nested r.f.



$$
\begin{aligned}
f_d(\mathbf{x}) &\geq 0 \\
(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d.
\end{aligned}
$$

Assume $f'_n(\mathbf{x}) = f_n(\mathbf{x}) + \mu_n f_1(\mathbf{x})$ and $f'_d, \ldots, f'_i$ has already been computed.

$$\mathbf{x}'' \in \mathcal{Q}' \to (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

$$\mathbf{x}'' \in \mathcal{Q}' \to (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

If above inequation also holds for $\mathcal{Q}$, then $f'_{i-1} = f_{i-1}$, Otherwise

$$\mathbf{x}'' \in \mathcal{Q} \to (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) \geq 0$$

BMΦRF($\mathbb{Q}$) $\in$PTIME

Theorem (2)

BMΦRF($\mathbb{Q}$)$\in$$\mathtt{PTIME}$.

Proof.

Leike et al. .Ranking Templates for Linear Loops.

$\square$

LLRF

Intuition: remind binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$.
Generalize it into several phases using lexicographical order of ranking functions.
$\langle f_1, f_2, \ldots, f_d \rangle$
$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

Definition (LLRF)

Given a set of transitions T we say that $\langle f_1, f_2, \ldots, f_d \rangle$ is a LLRF (of depth d) for T if for every $\mathbf{x}'' \in T$ there is an index i such that

$$\begin{aligned} \forall j < i \ . \ \Delta f_j(\mathbf{x}'') &\geq 0\,, \\ \Delta f_i(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \end{aligned}$$

A LLRF is weak if..

Weak LLRF to MΦRF

Theorem (3)

If $\mathcal{Q}$ has a weak LLRF of depth d, it has a MΦRF of depth d.

Proof.

Prove by induction.

- $d = 1$: For LLRF: $\Delta f_1(\mathbf{x}'') > 0$, $f_1(\mathbf{x}) \geq 0$ is a LRF due to the loop is linear.
  For MΦRF: is a LRF.
- $d > 1$: Observe that for a given LLRF $\langle f_1, f_2, \ldots, f_d \rangle$, after removing f_k, $\langle f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_d \rangle$ is also a LLRF.
  If we apply IH here, we get a MΦRF of depth $d - 1$.

□

Weak LLRF to MΦRF

Now we want some techniques to use a MΦRF of depth $d-1$ and f_k we removed to prove there is a MΦRF of depth d on $\mathcal{Q}$.

Lemma (2)

Let f be a non-negative linear function over $\mathcal{Q}$. If $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \mid \Delta f(\mathbf{x}'') \leq 0\}$ has a MΦRF of depth d, then $\mathcal{Q}$ has a MΦRF of depth at most $d+1$.

Proof.

Prove by construction: if the known MΦRF is $\langle g_1, \ldots, g_d \rangle$ and the funtion non-negative function is f, we wish to construct a MΦRF $\langle g_1', \ldots, g_n', f \rangle$ of depth $d+1$

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow \Delta f(\mathbf{x}'') > 0 \vee \Delta g_1(\mathbf{x}'') \geq 1$$

Then update g_2 to g_2' when $g_1'(\mathbf{x}) \leq -1$, an so on...

□

Weak LLRF to MΦRF

Remind the f_k we removed in the theorem, together with Lemma(2), we wish to construct a non-negative linear function g over $\mathcal{Q}$ and g decrease on (at least) the same transitions of f_k.

Lemma (3)

Let $\langle f_1, \ldots, f_d \rangle$ be a weak LLRF for $\mathcal{Q}$. There is a linear function g that is positive over $\mathcal{Q}$, and decreasing on (at least) the same transitions of f_i, for some $i \in [1, d]$.

Proof.

Use Lemma(0) to find the i. □

MΦRF and LLRFs over the Integers

- Actual programs with `int`.
- More important, conclusions for rational does not always applicable in on integer version.

Example

$$\texttt{while } (x_2 - x_1 \leq 0,\, x_1 + x_2 \geq 1) \texttt{ do } x_2' = x_2 - 2x_1 + 1,\, x_1' = x_1$$

For rationals: $x_1 = \dfrac{1}{2}, x_2 = \dfrac{1}{2}$
For integers: there exists a linear ranking function $f(x_1, x_2) = x_1 + x_2$

MΦRF and LLRFs over the Integers

- Integer case for LRF: completeness for the integer version was achieved by reducing the problem to the rational case. Intuitionly, since $\mathcal{Q}_I$ is the convex combinition of points in $I(\mathcal{Q})$.
- Theorems above does not apply to the integer versions, but in the following we will prove that the reduction from integer case to rational also works for LLRF and MΦRF.

Weak LLRF: Integer to Rational

Theorem (4)

Let $\langle f_1, \ldots, f_d \rangle$ be a weak LLRF for $I(\mathcal{Q})$. Then there are constants $c_1, \ldots, c_d$ such that $\langle f_1 + c_1, \ldots, f_d + c_d \rangle$ is a weak LLRF for $\mathcal{Q}_I$(over the rationals).

Proof.

prove by induction:

- $d = 1$, LRF.
- $d > 1$, define

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$

$$\mathcal{Q}'' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \Delta f_1(\mathbf{x}'') = 0\}$$

by IH, the theorem holds on $\mathcal{Q}'_I$ and $\mathcal{Q}''_I$ for weak LLRF of depth $d-1$. say,

$$\langle f_2 + c'_2, \ldots, f_d + c'_d \rangle, \langle f_2 + c''_2, \ldots, f_d + c''_d \rangle$$

Proof Continue

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$
$$\mathcal{Q}'' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \Delta f_1(\mathbf{x}'') = 0\}$$

Then we wish to have a lower bound on $f_1(\mathbf{x})$.

$$\langle f_1 + c_1, f_2 + \max(c_2', c_2''), \ldots, f_d + \max(c_d', c_d'')\rangle$$

This implies the above formula is a weak LLRF on $\mathcal{Q}_I$. i.e. given a $\mathbf{x}'' \in \mathcal{Q}_I$, either..., or...
Problem: how to prove the existence of the lower bound?

Prove the Lower Bound

$\mathcal{Q}'_I, \mathcal{Q}''_I$.

- If $\mathcal{Q}'_I$ is empty, then by the definition of $\mathcal{Q}'$ f_1 is lower bounded.
- Otherwise, prove the lower bound on $\mathcal{Q}_I \setminus \mathcal{Q}'_I$

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

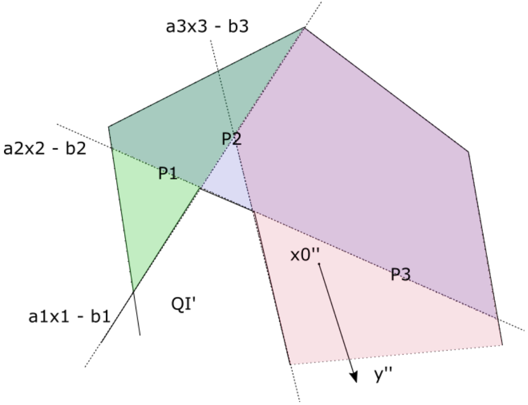
$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

For $i \in [1, m]$. Then clearly $\mathcal{Q}_I \setminus \mathcal{Q}'_I \subseteq \bigcup_{i=1}^m \mathcal{P}_i$, by construction all the integer points in $\mathcal{P}_i$ are also in $\mathcal{Q}_I \setminus \mathcal{Q}'_I$.
Proof target: for every i, f_1 is lower bounded in $\mathcal{P}_i$ for every i.
Fix i for the following arguments, s.t. $\mathcal{P}_i$ is not empty.

Intuition: Proof of the Lower Bound

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$



Prove the Lower Bound

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

Assume (prove by contradiction) $\mathcal{P}_i$ does not lower bound f_1. Let $\mathbf{x}''_0 \in \mathcal{P}_i$.

$$f_1(\mathbf{x}) = \vec{\lambda} \cdot \mathbf{x} + \lambda_0$$

$$\mathcal{P}_i = \mathcal{O} + \mathcal{C}$$

There must be a vector $\mathbf{y}'' \in \mathcal{C}$ s.t. $\vec{\lambda} \cdot \mathbf{y} < 0$

Prove the Lower Bound

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

$\mathbf{x}''_0 + k\mathbf{y}''$ is in $\mathcal{P}'_i$, the set $S = \{\mathbf{x}''_0 + k\mathbf{y}'' \mid k \in \mathbb{Z}_+\}$ is contained in $\mathcal{P}_i$.

Integer points of $\mathcal{P}_i$ are all in $\mathcal{Q}_I \setminus \mathcal{Q}'_I$.

Contradiction.

Hence, f_1 is bounded.

The Depth of a MΦRF

Idea: pre-compute the depth d for MΦRF synthesis.

Theorem (5)

For integer $B > 0$, the following loop $\mathcal{Q}_B$

$$\texttt{while } (x \geq 1,\ y \geq 1,\ x \geq y,\ 2^B y \geq x) \texttt{ do } x' = 2x,\ y' = 3y$$

needs at least $B + 1$ components in any MΦRF.

Proof.

Define $R_I = \{(2^i c, c, 2^{i+1} c, 3c) \mid c \geq 1\}$ and note that for $i \in [0, B]$, we have $R_i \in \mathcal{Q}_B$.

Assume the loop has a MΦRF with depth B, then it is obvious that there are R_i and $R_j, i \neq j$ that are ranked by the same phase f_k, w.l.o.g., assume $j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$, we have

□

Proof of Theorem (5)

$j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$

$$\begin{aligned} f_k(2^i, 1) - f_k(2^{i+1}, 3) &= -a_1 2^i - a_2 2 & > 0 \\ f_k(2^j, 1) - f_k(2^{j+1}, 3) &= -a_1 2^j - a_2 2 & > 0 \\ f_k(2^i, 1) - f_k(0, 0) &= a_1 2^i + a_2 & \geq 0 \\ f_k(2^j, 1) - f_k(0, 0) &= a_1 2^j + a_2 & \geq 0 \end{aligned}$$

Iteration Bounds from MΦRFs

Example

$$\texttt{while } (x \geq 0)\texttt{do } x' = x + y, y' = y - 1$$

MΦRF: $\langle y + 1, x \rangle$
When start from $x = x_0$ and $y = y_0$...
$x_0 + \dfrac{y_0(y_0 + 1)}{2} - 1$

Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF $\tau = \langle f_1, \ldots, f_d \rangle$.

- $F_k(t)$: the value of f_k after iteration t.
- $UB_k(t)$: bound for f_k. For $t > T_k$, $UB_k(T_k)$ becomes negative.
- T_k: an upper bound on the time in which the k-th phase ends.
- The whole loop must terminate before $\max_k T_k$ iterations.

$\mathbf{x}_t$ be te state after iteration t. Define $F_k(t) = f_k(\mathbf{x}_t)$. Let $M = \max(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0))$

Iteration Bounds from MΦRF

Lemma (4)

For all $k \in [1, d]$, *there are* $\mu_1, \ldots, \mu_{k-2} \geq 0$ *and* $\mu_{k-1} > 0$ *such that* $\mathbf{x}'' \in \mathcal{Q} \to \mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0$.

Proof.

$$\mathbf{x}'' \in \mathcal{Q} \to f_1(\mathbf{x}) \geq 0 \vee \cdots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee \Delta f_k(\mathbf{x}'') \geq 1.$$

□

Lemma (5)

For all $k \in [1, d]$, *there are constants* $c_k, d_k > 0$ *such that* $F_k(t) \leq c_k M t^{k-1} - d_k t^k$, *for all* $t \geq 1$.

[6] Proof Idea: Use the bound for $-\Delta f_k(\mathbf{x}_i'')$ to bound $F_k(t)$.

Proof of Lemma (6)

$$
\begin{aligned}
F_k(t) &= f_k(\mathbf{x}_0) + \Sigma_{i=0}^{t-1}(f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)) \\
&< M + \Sigma_{i=0}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1}F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1}F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\Sigma_{j=1}^{k-1}\left(\mu_j c_j M i^{j-1} - \mu_j d_j i^j\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left((\Sigma_{j=1}^{k-1}\mu_j c_j M i^{j-1}) - \mu_{k-1}d_{k-1}i^{k-1}\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(M(\Sigma_{j=1}^{k-1}\mu_j c_j) i^{k-2} - \mu_{k-1}d_{k-1}i^{k-1}\right) \\
&= M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\Sigma_{i=1}^{t-1} i^{k-2}) - \mu_{k-1}d_{k-1}\Sigma_{i=1}^{t-1} i^{k-1} \\
&\leq M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\frac{t^{k-1}}{k-1}) - \mu_{k-1}d_{k-1}(\frac{t^k}{k} - t^{k-1}) \\
&= c_k M t^{k-1} - d_k t^k
\end{aligned}
$$

where $\mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1}f_{k-1}(\mathbf{x}) \geq \Delta f_k(\mathbf{x}'') = f_k(\mathbf{x}_{i+1} - f_k(\mathbf{x}_i)$

Theorem (6)

An SLC loop that has a MΦRF terminates in a number of iterations bounded by $O(||\mathbf{x}_0||_\infty)$

Proof.

$F_k(t) \leq c_k M t^{k-1} - d_k t^k$. For $t > \max\{1, (c_k/d_k)M\}$, we have $F_k(t) < 0$.

Thus, the loop terminates by the time $\max\{1, (c_i/d_i)M, \ldots, (c_k/d_k)M\}$ where $M = \max(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0))$.

$\square$