On Multiphase-Linear Ranking Functions

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Motzkin's Transposition Theorem

Theorem (Motzkin's Transposition Theorem)

For $A \in \mathbb{K}^{m \times n}, C \in \mathbb{K}^{l \times n}, b \in \mathbb{K}^m$, and $d \in \mathbb{K}^l$. The formulae below are equivalent.

- $\forall x \in \mathbb{K}^n . \neg(Ax \leq b \land Cx < d)$
- $\exists \lambda \in \mathbb{K}^m . \exists \mu \in \mathbb{K}^l .$
  $\lambda \geq 0 \land \mu \geq 0$
  $\land \lambda^T A + \mu^T C = 0 \land \lambda^T b + \mu^T d \leq 0$
  $\land (\lambda^T b < 0 \lor \mu \neq 0)$

LRF, Nested r.f. and MΦRF

MΦRF to Nested r.f.

Theorem

If $\mathcal{Q}$ has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

Proof.

By induction on the depth d.

- $d = 1$: MΦRF and nested r.f. are both LRF.
- $d > 1$: $d = 2$ e.g. $\langle f_1, f_2 \rangle$. When index $i = 1$, we do not impose bound on $f_2(\mathbf{x})$. However, a bound is needed for $f_2'(\mathbf{x}s)$ in nested r.f. $\langle f_1', f_2' \rangle$.
  To solve the problem that $f_2(\mathbf{x})$ might goes under 0, when $\mathbf{x}''$ is ranked by f_1. Consider $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}'') \leq 0\}$

$\square$

MΦRF to Nested r.f.

Lemma (1)

Let $\tau = \langle f_1, \ldots, f_n \rangle$ be an irredundant MΦRF for $\mathcal{Q}$, such that $\langle f_2, \ldots, f_d \rangle$ is a nested ranking function for $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}'') \leq 0\}$. Then there is a nested ranking function of depth d for $\mathcal{Q}$.

Prove by construction: construct a nested r.f. $\langle f'_1, \ldots, f'_n \rangle$