

On Multiphase-Linear Ranking Functions

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Single Path Linear Constraint Loop

Example

while $(x \geq -z)$ **do** $x' = x + y$, $y' = y + z$, $z' = z - 1$

Let $B = (-1, 0, 1)$, $\mathbf{x} = (x, y, z)^T$, $\mathbf{b} = 0$.

Let $\mathbf{x}'' = (x, y, z, x', y', z')$,

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \quad (1)$$

and $\mathbf{c} = (0, 0, 1)^T$

Definition (SLC)

while $(B\mathbf{x} \leq \mathbf{b})$ *do* $A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$

$$A'' = \begin{pmatrix} B & 0 \\ A \end{pmatrix}$$

$$\mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Single Linear Ranking Function(LRF))

$f(x_1, \dots, x_n) = a_1x_1 + \dots a_nx_n + a_0$, such that

- ▶ $f(\mathbf{x}) \geq 0$ for any \mathbf{x} satisfies the loop constraints.
- ▶ $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from \mathbf{x} to \mathbf{x}' .

Example

`while (x - 1 > 0)do x' = x - 5`

LRF: $f(x) = ax + b$.

- ▶ $ax + b \geq 0 \Rightarrow x \geq -\frac{b}{a} = 1$.
- ▶ $ax + b - (ax' + b) = a(x - x') = 5a \Rightarrow 5a \geq 1$

A possible SLRF: $f(x) = x - 1$

Limitation of SLRF

`while ($q > 0$)do $q' = q - y, y' = y + 1$`

Assume there is a LRF for this loop, say $f(q, y) = a_1q + a_2y + b$

$$f(q, y) - f(q', y') = a_1y + a_2$$

Since y is not bounded, we cannot guarantee $\Delta f(q, y, q', y') > 0$

The loop does not has a SLRF, however, it does terminate.

We still wish to use q for ranking function, but to distinguish different “phase” of q base on either $y \geq 0$ or $y < 0$

Nested RF

Definition (Nested Ranking Function)

A tuple $\langle f_1, \dots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

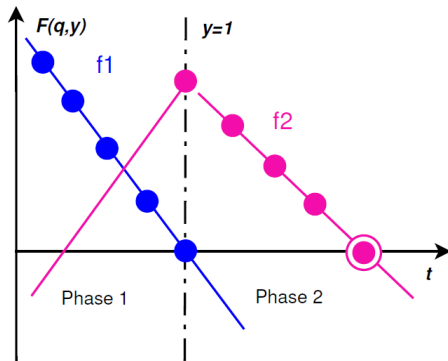
Let $f_0 = 0$.

Example: Nested RF

$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \quad \text{for all } i = 1, \dots, d.$$

while ($q > 0$)do $q' = q - y, y' = y + 1$

- Above loop has Nested RF $\langle 1 - y, q + 1 \rangle$



Linear Loop Program

Definition

A linear loop program $\text{LOOP}(x, x')$ is a binary relation defined by a formula with the free variables x and x' of the form

$$\bigvee_{i \in I} (A_i(x) \leq b_i \wedge C_i(x') < d_i)$$

for some finite index set I .

Example

`while (q > 0){if (y > 0) : q' = q - y - 1; else : q' = q + y - 1}`

can be represented by

$$(q > 0 \wedge y > 0 \wedge y' = y \wedge q' = q - y - 1) \\ \vee (q > 0 \wedge y \leq 0 \wedge y' = y \wedge q' = q + y - 1)$$

Limitation of Nested RF

Example

```
while ( $q > 0 \vee y > 0$ )  
  {if ( $y > 0$ ) :  $y' = y - 1$ ;  $q' = q$ ; else :  $q' = q - 1$ }
```

This program does not have a nested ranking function for we require $f_d \geq 0$ but the guard is $q > 0 \vee y > 0$.

However, this loop does terminate. Then we use a “multi-phase” ranking function $\langle y, q \rangle$ to prove the termination.

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \dots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

$$\begin{aligned} \forall j \leq i . \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i . f_j(\mathbf{x}) &\leq 0. \end{aligned}$$

We say that \mathbf{x}'' is ranked by f_i (for the minimal).

Example: Multiphase Ranking Function

while $(x > -z)$ **do** $x' = x + y, y' = y + z, z = z - 1$

Attempt to use a ranking function that has several phases:

$\langle z + 1, y + 1, x \rangle$

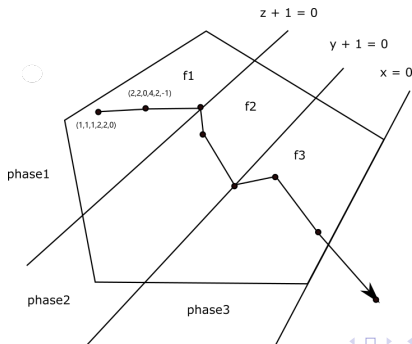
x	y	z	$z + 1$	$y + 1$	x
1	1	1	2	2	1
2	2	0	1	3	2
4	2	-1	0	3	4
6	1	-2	-1	2	6
7	-1	-3	-2	0	7
6	-4	-4	-3	-3	6
2	-8	-5	-4	-7	2
-6	-13	-6	-5	-12	-6

Example: Multiphase Ranking Function

while $(x > -z)$ **do** $x' = x + y, y' = y + z, z' = z - 1$
 $\langle z + 1, y + 1, x \rangle$

\mathbf{x}'' is ranked by f_k when $i = k$. In this example,
 $f_1(x, y, z) = z + 1$, $f_2(x, y, z) = y + 1$ and $f_3(x, y, z) = x$

$$\begin{aligned} \forall j \leq i. \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i. f_j(\mathbf{x}) &\leq 0. \end{aligned}$$



MΦRF to Nested RF

Theorem (1)

If \mathcal{Q} has a MΦRF of depth d , then it has a nested ranking function of depth at most d .

Proof.

By induction on the depth d .

- ▶ $d = 1$: MΦRF and nested RF are both LRF.
- ▶ $d > 1$: $d = 2$ e.g. $\langle f_1, f_2 \rangle$. When index $i = 1$, we do not impose bound on $f_2(\mathbf{x})$. However, a bound is needed for $f'_2(\mathbf{x})$ in nested RF $\langle f'_1, f'_2 \rangle$.

To solve the problem that $f_2(\mathbf{x})$ might goes under 0, when \mathbf{x}'' is ranked by f_1 . Consider $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$

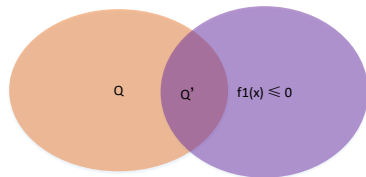


MΦRF to Nested RF

Lemma (1)

Let $\tau = \langle f_1, \dots, f_d \rangle$ be an irredundant MΦRF for \mathcal{Q} , such that $\langle f_2, \dots, f_d \rangle$ is a nested ranking function for $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$. Then there is a nested ranking function of depth d for \mathcal{Q} .

Prove by construction: construct a nested RF $\langle f'_1, \dots, f'_d \rangle$



$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

If f_d is non-negative on \mathcal{Q} , then $f'_d = f_d$. Otherwise,
 $\mathbf{x}'' \in \mathcal{Q} \rightarrow f_d(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) > 0$

Lemma (0)

Given an non-empty polyhedron \mathcal{P} and linear functions f_1, \dots, f_k such that

1. $\mathbf{x} \in \mathcal{P} \rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee f_k(\mathbf{x}) \geq 0$
2. $\mathbf{x} \in \mathcal{P} \not\rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0$

There exists a non-negative constants μ_1, \dots, μ_{k-1} such that

$$\mathbf{x} \in \mathcal{P} \rightarrow \mu_1 f_1(\mathbf{x}) + \dots + \mu_{k-1} f_{k-1}(\mathbf{x}) + f_k(\mathbf{x}) \geq 0$$

Motzkin's Transposition Theorem

Theorem (Motzkin's Transposition Theorem)

For $A \in \mathbb{K}^{m \times n}$, $C \in \mathbb{K}^{l \times n}$, $b \in \mathbb{K}^m$, and $d \in \mathbb{K}^l$. The formulae below are equivalent.

- ▶ $\forall x \in \mathbb{K}^n. \neg(Ax \leq b \wedge Cx < d)$
- ▶ $\exists \lambda \in \mathbb{K}^m. \exists \mu \in \mathbb{K}^l.$
 $\lambda \geq 0 \wedge \mu \geq 0$
 $\wedge \lambda^T A + \mu^T C = 0 \wedge \lambda^T b + \mu^T d \leq 0$
 $\wedge (\lambda^T b < 0 \vee \mu \neq 0)$

Proof of Lemma (0)

Let \mathcal{P} be $B\mathbf{x} \leq c$, $f_i = \vec{a}_i\mathbf{x} - b_i$. Then (i) is equivalent to the infeasibility of $B\mathbf{x} \leq c \wedge A\mathbf{x} < b$, by the theorem we have $\vec{\lambda}, \vec{\mu}$ s.t.

$$\vec{\lambda}B + \vec{\mu}A = 0 \wedge \vec{\lambda}c + \vec{\mu}b \leq 0 \quad \wedge (\vec{\mu} \neq 0 \vee \vec{\lambda}c + \vec{\mu}b < 0)$$

Then for all $\mathbf{x} \in \mathcal{P}$

$$\sum_i \mu_i f_i(\mathbf{x}) = \vec{\mu}A\mathbf{x} - \vec{\mu}b = -\vec{\lambda}B\mathbf{x} - \vec{\mu}b \geq -\vec{\lambda}c - \vec{\mu}b \geq 0$$

MΦRF to Nested RF

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

Assume $f'_n(\mathbf{x}) = f_n(\mathbf{x}) + \mu_n f_1(\mathbf{x})$ and f'_d, \dots, f'_i has already been computed.

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

If above inequation also holds for \mathcal{Q} , then $f'_{i-1} = f_{i-1}$, Otherwise

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) \geq 0$$

$BM\Phi RF(\mathbb{Q}) \in PTIME$

Theorem (2)

$BM\Phi RF(\mathbb{Q}) \in PTIME.$

Proof.

Leike et al..Ranking Templates for Linear Loops.



The Depth of a MΦRF

Idea: pre-compute the depth d for MΦRF synthesis.

Theorem (5)

For integer $B > 0$, the following loop \mathcal{Q}_B

while $(x \geq 1, y \geq 1, x \geq y, 2^B y \geq x)$ do $x' = 2x, y' = 3y$

needs at least $B + 1$ components in any MΦRF.

Proof.

Define $R_I = \{(2^i c, c, 2^{i+1} c, 3c) \mid c \geq 1\}$ and note that for $i \in [0, B]$, we have $R_i \in \mathcal{Q}_B$.

Assume the loop has a MΦRF with depth B , then it is obvious that there are R_i and $R_j, i \neq j$ that are ranked by the same phase f_k , w.l.o.g., assume $j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$, we have



Proof of Theorem (5)

$$j > i \text{ and } f_k(x, y) = a_1x + a_2y + a_0$$

$$f_k(2^i, 1) - f_k(2^{i+1}, 3) = -a_12^i - a_22 > 0$$

$$f_k(2^j, 1) - f_k(2^{j+1}, 3) = -a_12^j - a_22 > 0$$

$$f_k(2^i, 1) - f_k(0, 0) = a_12^i + a_2 \geq 0$$

$$f_k(2^j, 1) - f_k(0, 0) = a_12^j + a_2 \geq 0$$

$$j > i$$

$$a_12^{i-1} > 0 \Rightarrow a_1 > 0$$

$$a_1(2^i - 2^{j-1}) > 0 \Rightarrow i + 1 > j \Rightarrow i \geq j. \text{ Contradiction.}$$

Iteration Bounds from MΦRFs

Example

while $(x \geq 0)$ **do** $x' = x + y, y' = y - 1$

MΦRF: $\langle y + 1, x \rangle$

When start from $x = x_0$ and $y = y_0 \dots$

$$x_0 + \frac{y_0(y_0 + 1)}{2} - 1$$

Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF

$\tau = \langle f_1, \dots, f_d \rangle$.

- ▶ $F_k(t)$: the value of f_k after iteration t .
- ▶ $UB_k(t)$: bound for f_k . For $t > T_k$, $UB_k(T_k)$ becomes negative.
- ▶ T_k : an upper bound on the time in which the k -th phase ends.
- ▶ The whole loop must terminate before $\max_k T_k$ iterations.

\mathbf{x}_t be the state after iteration t . Define $F_k(t) = f_k(\mathbf{x}_t)$. Let $M = \max(f_1(\mathbf{x}_0), \dots, f_k(\mathbf{x}_0))$

Iteration Bounds from MΦRF

Lemma (4)

For all $k \in [1, d]$, there are $\mu_1, \dots, \mu_{k-2} \geq 0$ and $\mu_{k-1} > 0$ such that $\mathbf{x}'' \in \mathcal{Q} \rightarrow \mu_1 f_1(\mathbf{x}) + \dots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0$.

Proof.

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee \Delta f_k(\mathbf{x}'') \geq 1.$$



Lemma (5)

For all $k \in [1, d]$, there are constants $c_k, d_k > 0$ such that $F_k(t) \leq c_k M t^{k-1} - d_k t^k$, for all $t \geq 1$.

[6] Proof Idea: Use the bound for $-\Delta f_k(\mathbf{x}_i'')$ to bound $F_k(t)$.

Proof of Lemma (6)

$$\begin{aligned} F_k(t) &= f_k(\mathbf{x}_0) + \sum_{i=0}^{t-1} (f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)) \\ &< M + \sum_{i=0}^{t-1} (\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} (\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} \sum_{j=1}^{k-1} (\mu_j c_j M i^{j-1} - \mu_j d_j i^j) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} ((\sum_{j=1}^{k-1} \mu_j c_j M i^{j-1}) - \mu_{k-1} d_{k-1} i^{k-1}) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} (M(\sum_{j=1}^{k-1} \mu_j c_j) i^{k-2} - \mu_{k-1} d_{k-1} i^{k-1}) \\ &= M(1 + \mu) + M(\sum_{j=1}^{k-1} \mu_j c_j) (\sum_{i=1}^{t-1} i^{k-2}) - \mu_{k-1} d_{k-1} \sum_{i=1}^{t-1} i^{k-1} \\ &\leq M(1 + \mu) + M(\sum_{j=1}^{k-1} \mu_j c_j) \left(\frac{t^{k-1}}{k-1} \right) - \mu_{k-1} d_{k-1} \left(\frac{t^k}{k} - t^{k-1} \right) \\ &= c_k M t^{k-1} - d_k t^k \end{aligned}$$

where $\mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) \geq \Delta f_k(\mathbf{x}'') = f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)$

Theorem (6)

An SLC loop that has a $M\Phi RF$ terminates in a number of iterations bounded by $O(\|\mathbf{x}_0\|_\infty)$

Proof.

$F_k(t) \leq c_k M t^{k-1} - d_k t^k$. For $t > \max\{1, (c_k/d_k)M\}$, we have $F_k(t) < 0$.

Thus, the loop terminates by the time $\max\{1, (c_i/d_i)M, \dots, (c_k/d_k)M\}$ where $M = \max(f_1(\mathbf{x}_0), \dots, f_k(\mathbf{x}_0))$.

