On Multiphase-Linear Ranking Functions

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May 7, 2020

Contributions

- Equivalence of different classes of ranking function.
- Algorithms for converting between ranking functions.
- Complete solution for ranking functions on integer.
- Depth bound and iteration bound for MΦRF.

Single Path Linear Constraint Loop

Example

$$\texttt{while } (x \geq -z) \texttt{ do } x' = x + y, \ y' = y + z, \ z' = z - 1$$

$$\texttt{while } (x_2 - x_1 \leq 0, \ x_1 + x_2 \geq 1) \texttt{ do } x_2' = x_2 - 2x_1 + 1, \ x_1' = x_1$$

Definition (SLC)

$$while \ (B\mathbf{x} \leq \mathbf{b}) \ do \ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad\qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while } (x - 1 > 0)\texttt{do } x' = x - 1$$

Its LRF: $f(x) = x - 1$

We can define a binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$

Example: Multiphase Ranking Function

Problem: LRF is not strong enough for all loops.

Example

$$\texttt{while } (x > -z) \texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$f(x, y, z) = a_1 x + a_2 y + a_3 z + b$

y cannot be used for non-existence of its lower bound.

$f(x, y, z) = x + z$

Problem?

Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases:
$\langle z + 1, y + 1, x \rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	2	2	1
2	2	0	1	3	2
4	2	−1	0	3	4
6	1	−2	−1	2	6
7	−1	−3	−2	0	7
6	−4	−4	−3	−3	6
2	−8	−5	−4	−7	2
−6	−13	−6	−5	−12	−6

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.
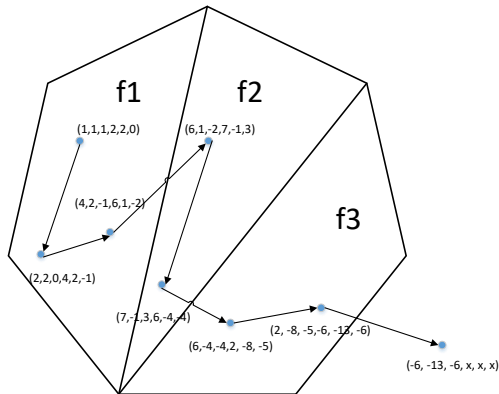
$$
\begin{aligned}
\forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1 \, , \\
f_i(\mathbf{x}) &\geq 0 \, , \\
\forall j < i \ . \quad f_j(\mathbf{x}) &\leq 0 \, .
\end{aligned}
$$

We say that $\mathbf{x}''$ is ranked by f_i (for the minimal).

Example Revisit

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$$\forall j \leq i \, . \, \Delta f_j(\mathbf{x}'') \geq 1 \, ,$$
$$f_i(\mathbf{x}) \geq 0 \, ,$$
$$\forall j < i \, . \quad f_j(\mathbf{x}) \leq 0 \, .$$

Nested Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Loop condition: $x + z > 0$. We only want to use this constraint for the ranking function.

$\langle z + 1, y + 1, x + z \rangle$

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \qquad \text{for all } i = 1, \ldots, d.$$

Let $f_0 = 0$.

Nested is Multiphase, but not the opposite.

Lexicography Linear Ranking Function

Intuition: remind binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$.

Generalize it into several phases using lexicographical order of ranking functions.

$\langle f_1, f_2, \ldots, f_d \rangle$

$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

Definition (LLRF)

Given a set of transitions T we say that $\langle f_1, f_2, \ldots, f_d \rangle$ is a LLRF (of depth d) for T if for every $\mathbf{x}'' \in T$ there is an index i such that

$$\forall j < i . \ \Delta f_j(\mathbf{x}'') \geq 0 ,$$
$$\Delta f_i(\mathbf{x}'') \geq 1 ,$$
$$f_i(\mathbf{x}) \geq 0 ,$$

A LLRF is weak if..

Example: LLRF

Example

$$\texttt{while } (x \geq 0, y \leq 10, z \geq 0, z \leq 1)\texttt{do}$$

$$x' = x + y + z - 10, y' = y + z, z' = z - 1$$

$\langle 4y, 4x - 4z + 1 \rangle$

Example: MΦRF is a LLRF

$$\forall j < i \;.\; \Delta f_j(\mathbf{x}'') \geq 0\,,$$
$$\Delta f_i(\mathbf{x}'') \geq 1\,,$$
$$f_i(\mathbf{x}) \geq 0\,,$$

`while` $(x > -z)$ `do` $x' = x + y, y' = y + z, z = z - 1$

Attempt to use a ranking function that has several phases:
$\langle z + 1, y + 1, x \rangle$

x	y	z	$z + 1$	$y + 1$	x
1	1	1	2	2	1
2	2	0	1	3	2
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6	1	-2	-1	2	6
7	-1	-3	-2	0	7
6	-4	-4	-3	-3	6
2	-8	-5	-4	-7	2
-6	-13	-6	-5	-12	-6

LLRF to MΦRF?

Theorem (weak LLRF to MΦRF)

If T has a weak LLRF of depth d, it has a MΦRF of depth d.