On Multiphase-Linear Ranking Functions

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Contributions

- Equivalence of different classes of ranking function.
- Algorithms for converting between ranking functions.
- Converting ranking functions on integers to rational.
- Depth bound and iteration bound for MΦRF.

Single Path Linear Constraint Loop

Example

$$\texttt{while}\ (x \geq -z)\ \texttt{do}\ x' = x + y,\ y' = y + z,\ z' = z - 1$$

$$\texttt{while}\ (x_2 - x_1 \leq 0,\ x_1 + x_2 \geq 1)\ \texttt{do}\ x_2' = x_2 - 2x_1 + 1,\ x_1' = x_1$$

Definition (SLC)

$$\textit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \textit{do}\ A\begin{pmatrix}\mathbf{x}\\ \mathbf{x}'\end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1x_1 + \ldots a_nx_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while } (x - 1 > 0)\texttt{do } x' = x - 5$$

Its LRF: $f(x) = x - 1$

We can define a binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$

Example: Nested Ranking Function

Example: Multiphase Ranking Function

Problem: LRF is not strong enough for all loops.

Example

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$f(x, y, z) = a_1 x + a_2 y + a_3 z + b$
assume a_1, a_2, a_3 and b are given,

$$f(x, y, z) - f(x', y', z') = a_3 + ya_1 + za_2$$

variable y, z do not have bound and cannot be used in the ranking function.

Example: Multiphase Ranking Function

$$\texttt{while}\ (x > -z)\texttt{do}\ x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases: $\langle z + 1, y + 1, x\rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	**2**	2	1
2	2	0	**1**	3	2
4	2	-1	**0**	3	4
6	1	-2	-1	**2**	6
7	-1	-3	-2	**0**	7
6	-4	-4	-3	-3	**6**
2	-8	-5	-4	-7	**2**
-6	-13	-6	-5	-12	-6

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

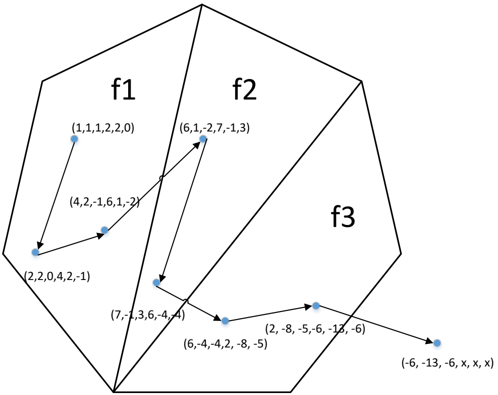
$$\begin{aligned} \forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \\ \forall j < i \ . \quad f_j(\mathbf{x}) &\leq 0\,. \end{aligned}$$

We say that $\mathbf{x}''$ is ranked by f_i(for the minimal).

Example Revisit

$$\texttt{while}\ (x > -z)\texttt{do}\ \ x' = x + y, y' = y + z, z = z - 1$$

$$\begin{aligned} \forall j \le i\ .\ \Delta f_j(\mathbf{x}'') \ge 1\,, \\ f_i(\mathbf{x}) \ge 0\,, \\ \forall j < i\ .\quad f_j(\mathbf{x}) \le 0\,. \end{aligned}$$



Nested Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Loop condition: $x + z > 0$. We only want to use this constraint for the ranking function.
$\langle z + 1, y + 1, x + z \rangle$

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d. \end{aligned}$$

Let $f_0 = 0$.

Nested is Multiphase, but not the opposite.
$(x = -1, y = 0, z = 1)$

MΦRF to Nested Ranking Function?

Theorem

If T has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

Sythesising nested ranking function is in `PTIME`, with theorem above we have..

Lexicography Linear Ranking Function

Intuition: remind binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$.
Generalize it into several phases using lexicographical order of ranking functions.
$\langle f_1, f_2, \ldots, f_d \rangle$
$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

Definition (LLRF)

Given a set of transitions T we say that $\langle f_1, f_2, \ldots, f_d \rangle$ is a LLRF (of depth d) for T if for every $\mathbf{x}'' \in T$ there is an index i such that

$$\begin{aligned} \forall j < i \ . \ \Delta f_j(\mathbf{x}'') &\geq 0 \,, \\ \Delta f_i(\mathbf{x}'') &\geq 1 \,, \\ f_i(\mathbf{x}) &\geq 0 \,, \end{aligned}$$

A LLRF is weak if..

Example: MΦRF is a LLRF

$$\begin{aligned} \forall j < i \;.\; \Delta f_j(\mathbf{x}'') &\geq 0\,, \\ \Delta f_i(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \end{aligned}$$

$$\texttt{while}\ (x > -z)\texttt{do}\ \ x' = x + y, y' = y + z, z = z - 1$$

x	y	z	$z+1$	$y+1$	x
1	1	1	2	2	1
2	2	0	1	3	2
4	2	-1	0	3	4
6	1	-2	-1	2	6
7	-1	-3	-2	0	7
6	-4	-4	-3	-3	6
2	-8	-5	-4	-7	2
-6	-13	-6	-5	-12	-6

LLRF to MΦRF?

Theorem (weak LLRF to MΦRF)

If T has a weak $LLRF$ of depth d, it has a MΦRF of depth d.

Ranking Function Over Integers

$$A''\mathbf{x}'' \leq \mathbf{c}$$

- Rational convex polyhedra: polyhedra defined by $\mathcal{P} = \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid A''\mathbf{x}'' \leq \mathbf{c}\}$.
- Integers: $I(\mathcal{P}) = \mathcal{P} \cap \mathbb{Z}^{2n}$
- Integer hull: $\mathcal{Q}_I$ is the space of convex combinition of points in $I(\mathcal{P})$.

Why consider integer?

- Actual programs with `int`.
- More important, conclusions for rational does not always applicable in on integer version.

Example

$$\texttt{while}\ (x_2 - x_1 \leq 0,\ x_1 + x_2 \geq 1)\ \texttt{do}\ x_2' = x_2 - 2x_1 + 1,\ x_1' = x_1$$

For rationals: $x_1 = \dfrac{1}{2}, x_2 = \dfrac{1}{2}$
For integers: there exists a linear ranking function $f(x_1, x_2) = x_1 + x_2$

Integer to Rational

Theorem

Let $\langle f_1, f_2, \ldots, f_d \rangle$ *be a weak LLRF for* $I(\mathcal{P})$*. Then there are constants* $c_1, \ldots, c_d$ *such that* $\langle f_1 + c_1, f_2 + c_2, \ldots, f_d + c_d \rangle$ *is a weak LLRF for* $\mathcal{Q}_I$

The Depth of a MΦRF

Idea: precompute an upper bound of depth $\rightarrow$ a decision procedure for MΦRF in general

Theorem

For integer $B > 0$, following loop needs at lease $B + 1$ components in any MΦRF.

$$\texttt{while}\ (x \geq 1,\ y \geq 1,\ x \geq y,\ 2^B y \geq x)\ \texttt{do}\ x' = 2x,\ y' = 3y$$

Example

$$\texttt{while}\ (x \geq 1,\ y \geq 1,\ x \geq y,\ 4y \geq x)\ \texttt{do}\ x' = 2x,\ y' = 3y$$

Iteration Bound

Theorem

An SLC loop that has a MΦRF terminates in a number of iterations bounded by $O(||x_0||_\infty)$

Future Work & Questions