On Multiphase-Linear Ranking Functions

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Single Path Linear Constraint Loop

Example

$$\texttt{while}\ (x \geq -z)\ \texttt{do}\ x' = x + y,\ y' = y + z,\ z' = z - 1$$

Let $B = (-1, 0, 1)$, $\mathbf{x} = (x, y, z)^T$, $\mathbf{b} = 0$.
Let $\mathbf{x}'' = (x, y, z, x', y', z')$,

$$A = \begin{bmatrix} 1 & 1 & 0-1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

and $\mathbf{c} = (0, 0, 1)^T$

Definition (SLC)

$$\mathit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \mathit{do}\ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Single Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while}\ (x - 1 > 0)\texttt{do}\ x' = x - 5$$

LRF: $f(x) = ax + b$.

- $ax + b \geq 0 \Rightarrow x \geq -\frac{b}{a} = 1.$
- $ax + b - (ax' + b) = a(x - x') = 5a \Rightarrow 5a \geq 1$

A possible SLRF: $f(x) = x - 1$

Limitation of SLRF

$$\texttt{while } (q > 0)\texttt{do } q' = q - y, y' = y + 1$$

Assume there is a LRF for this loop, say $f(q, y) = a_1 q + a_2 y + b$

$$f(q, y) - f(q', y') = a_1 y + a_2$$

Since y is not bounded, we cannot guarantee $\Delta f(q, y, q', y') > 0$
The loop does not has a SLRF, however, it does terminate.
We still wish to use q for ranking function, but to distinguish different "phase" of q base on either $y \geq 0$ or $y < 0$

Nested RF

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d. \end{aligned}$$

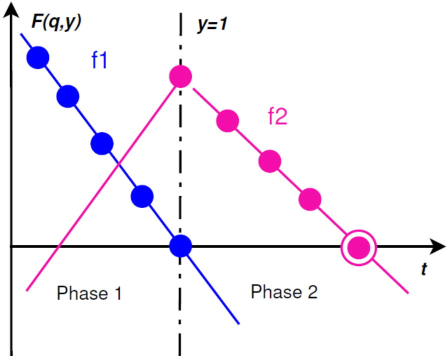
Let $f_0 = 0$.

Example: Nested RF

$$
\begin{aligned}
f_d(\mathbf{x}) &\geq 0 \\
(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d.
\end{aligned}
$$

$$\texttt{while } (q > 0)\texttt{do } q' = q - y, y' = y + 1$$

- Above loop has Nested RF $\langle 1 - y, q + 1 \rangle$



Linear Loop Program

Definition

A linear loop program $\texttt{LOOP}(x, x')$ is a binary relation defined by a formula with the free variables x and x' of the form

$$\bigvee_{i \in I} \left(A_i \binom{x}{x'} \leq b_i \ \wedge \ C_i \binom{x}{x'} < d_i \right)$$

for some finite index set I.

Example

$\texttt{while}\ (q > 0)\{\texttt{if}\ (y > 0) : q' = q - y - 1; \texttt{else}\ : q' = q + y - 1\}$

can be represented by

$$\begin{aligned} &(q > 0 \ \wedge \ y > 0 \ \wedge \ y' = y \ \wedge \ q' = q - y - 1) \\ \vee &(q > 0 \ \wedge \ y \leq 0 \ \wedge \ y' = y \ \wedge \ q' = q + y - 1) \end{aligned}$$

Limitation of Nested RF

Example

$$\texttt{while}\ (q > 0 \vee y > 0)$$
$$\{\texttt{if}\ (y > 0) : y' = y - 1; q' = q; \texttt{else}\ : q' = q - 1\}$$

This program does not have a nested ranking function for we require $f_d \geq 0$ but the guard is $q > 0 \vee y > 0$.
Howerver, this loop does terminate. Then we use a "multi-phase" ranking function $\langle y, q \rangle$ to prove the termination.

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \dots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

$$\begin{aligned} \forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1 \,, \\ f_i(\mathbf{x}) &\geq 0 \,, \\ \forall j < i \ . \quad f_j(\mathbf{x}) &\leq 0 \,. \end{aligned}$$

We say that $\mathbf{x}''$ is ranked by f_i(for the minimal).

Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases:
$\langle z+1, y+1, x \rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	**2**	2	1
2	2	0	**1**	3	2
4	2	-1	**0**	3	4
6	1	-2	-1	**2**	6
7	-1	-3	-2	**0**	7
6	-4	-4	-3	-3	**6**
2	-8	-5	-4	-7	**2**
-6	-13	-6	-5	-12	-6

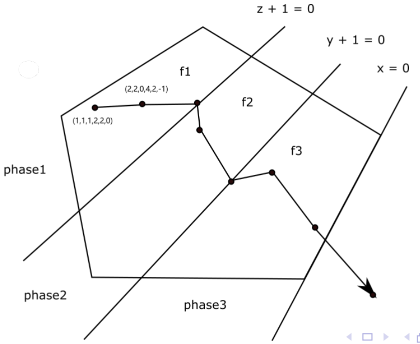
Example: Multiphase Ranking Function

$$\texttt{while}\ (x > -z)\texttt{do}\ \ x' = x + y, y' = y + z, z' = z - 1$$

$$\langle z + 1, y + 1, x \rangle$$

$\mathbf{x}''$ is ranked by f_k when $i = k$. In this example, $f_1(x, y, z) = z + 1$, $f_2(x, y, z) = y + 1$ and $f_3(x, y, z) = x$

$$\begin{aligned} \forall j \le i\ .\ \Delta f_j(\mathbf{x}'') &\ge 1\,, \\ f_i(\mathbf{x}) &\ge 0\,, \\ \forall j < i\ .\quad f_j(\mathbf{x}) &\le 0\,. \end{aligned}$$



MΦRF to Nested RF

Theorem (1)

If $\mathcal{Q}$ has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

Proof.

By induction on the depth d.

- $d = 1$: MΦRF and nested RF are both LRF.
- $d > 1$: $d = 2$ e.g. $\langle f_1, f_2 \rangle$. When index $i = 1$, we do not impose bound on $f_2(\mathbf{x})$. However, a bound is needed for $f_2'(\mathbf{x})$ in nested RF $\langle f_1', f_2' \rangle$.
  To solve the problem that $f_2(\mathbf{x})$ might goes under 0, when $\mathbf{x}''$ is ranked by f_1. Consider $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$

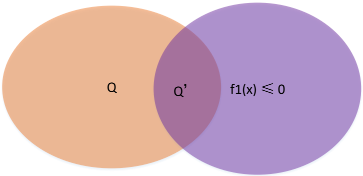
$\square$

MΦRF to Nested RF

Lemma (1)

Let $\tau = \langle f_1, \ldots, f_d \rangle$ be an irredundant MΦRF for $\mathcal{Q}$, such that $\langle f_2, \ldots, f_d \rangle$ is a nested ranking function for $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$. Then there is a nested ranking function of depth d for $\mathcal{Q}$.

Prove by construction: construct a nested RF $\langle f'_1, \ldots, f'_d \rangle$



$$
\begin{aligned}
f_d(\mathbf{x}) &\geq 0 \\
(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d.
\end{aligned}
$$

If f_d is non-negative on $\mathcal{Q}$, then $f'_d = f_d$. Otherwise,
$\mathbf{x}'' \in \mathcal{Q} \rightarrow f_d(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) > 0$

Lemma (0)

Given an non-empty polyhedron $\mathcal{P}$ *and linear functions* $f_1, \ldots, f_k$ *such that*

1. $\mathbf{x} \in \mathcal{P} \to f_1(\mathbf{x}) \geq 0 \vee \ldots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee f_k(\mathbf{x}) \geq 0$
2. $\mathbf{x} \in \mathcal{P} \not\to f_1(\mathbf{x}) \geq 0 \vee \ldots \vee f_{k-1}(\mathbf{x}) \geq 0$

There exists a non-negative constants $\mu_1, \ldots, \mu_{k-1}$ *such that*

$$\mathbf{x} \in \mathcal{P} \to \mu_1 f_1(\mathbf{x}) + \ldots + \mu_{k-1} f_{k-1}(\mathbf{x}) + f_k(\mathbf{x}) \geq 0$$

Motzkin's Transposition Theorem

Theorem (Motzkin's Transposition Theorem)

For $A \in \mathbb{K}^{m \times n}, C \in \mathbb{K}^{l \times n}, b \in \mathbb{K}^m$, *and* $d \in \mathbb{K}^l$. *The formulae below are equivalent.*

- $\forall x \in \mathbb{K}^n . \neg(Ax \leq b \wedge Cx < d)$
- $\exists \lambda \in \mathbb{K}^m . \exists \mu \in \mathbb{K}^l .$
  $\lambda \geq 0 \wedge \mu \geq 0$
  $\wedge \lambda^T A + \mu^T C = 0 \wedge \lambda^T b + \mu^T d \leq 0$
  $\wedge (\lambda^T b < 0 \vee \mu \neq 0)$

Proof of Lemma (0)

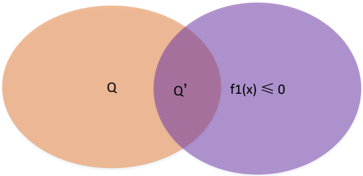
Let $\mathcal{P}$ be $B\mathbf{x} \leq c$, $f_i = \vec{a}_i\mathbf{x} - b_i$. Then (i) is equivalent to the infeasibility of $B\mathbf{x} \leq c \wedge A\mathbf{x} < b$, by the theorem we have $\vec{\lambda}, \vec{\mu}$ s.t.

$$\vec{\lambda}B + \vec{\mu}A = 0 \wedge \vec{\lambda}\mathbf{c} + \vec{\mu}\mathbf{b} \leq 0 \quad \wedge (\vec{\mu} \neq 0 \vee \vec{\lambda}\mathbf{c} + \vec{\mu}\mathbf{b} < 0)$$

Then for all $\mathbf{x} \in \mathcal{P}$

$$\sum_i \mu_i f_i(\mathbf{x}) = \vec{\mu}A\mathbf{x} - \vec{\mu}\mathbf{b} = -\vec{\lambda}B\mathbf{x} - \vec{\mu}\mathbf{b} \geq -\vec{\lambda}\mathbf{c} - \vec{\mu}\mathbf{b} \geq 0$$

MΦRF to Nested RF



$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \qquad \text{for all } i = 1, \ldots, d.$$

Assume $f'_n(\mathbf{x}) = f_n(\mathbf{x}) + \mu_n f_1(\mathbf{x})$ and $f'_d, \ldots, f'_i$ has already been computed.

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

If above inequation also holds for $\mathcal{Q}$, then $f'_{i-1} = f_{i-1}$, Otherwise

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) \geq 0$$

BMΦRF($\mathbb{Q}$) $\in$PTIME

Theorem (2)

BMΦRF($\mathbb{Q}$)$\in$$\mathtt{PTIME}$.

Proof.

Leike et al..Ranking Templates for Linear Loops.

$\square$

The Depth of a MΦRF

Idea: pre-compute the depth d for MΦRF synthesis.

Theorem (5)

For integer $B > 0$, the following loop $\mathcal{Q}_B$

$$\texttt{while } (x \geq 1,\ y \geq 1,\ x \geq y,\ 2^B y \geq x) \texttt{ do } x' = 2x,\ y' = 3y$$

needs at least $B+1$ components in any MΦRF.

Proof.

Define $R_I = \{(2^i c, c, 2^{i+1}c, 3c) \mid c \geq 1\}$ and note that for $i \in [0, B]$, we have $R_i \in \mathcal{Q}_B$.
Assume the loop has a MΦRF with depth B, then it is obvious that there are R_i and $R_j, i \neq j$ that are ranked by the same phase f_k, w.l.o.g., assume $j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$, we have

□

Proof of Theorem (5)

$j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$

$$\begin{aligned}
f_k(2^i, 1) - f_k(2^{i+1}, 3) &= -a_1 2^i - a_2 2 &> 0 \\
f_k(2^j, 1) - f_k(2^{j+1}, 3) &= -a_1 2^j - a_2 2 &> 0 \\
f_k(2^i, 1) - f_k(0, 0) &= a_1 2^i + a_2 &\geq 0 \\
f_k(2^j, 1) - f_k(0, 0) &= a_1 2^j + a_2 &\geq 0
\end{aligned}$$

Iteration Bounds from MΦRFs

Example

$$\texttt{while}\ (x \geq 0)\texttt{do}\ \ x' = x + y, y' = y - 1$$

MΦRF: $\langle y + 1, x \rangle$
When start from $x = x_0$ and $y = y_0$...
$x_0 + \dfrac{y_0(y_0 + 1)}{2} - 1$

Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF $\tau = \langle f_1, \dots, f_d \rangle$.

- $F_k(t)$: the value of f_k after iteration t.
- $UB_k(t)$: bound for f_k. For $t > T_k$, $UB_k(T_k)$ becomes negative.
- T_k: an upper bound on the time in which the k-th phase ends.
- The whole loop must terminate before $\max_k T_k$ iterations.

$\mathbf{x}_t$ be te state after iteration t. Define $F_k(t) = f_k(\mathbf{x}_t)$. Let $M = \max(f_1(\mathbf{x}_0), \dots, f_k(\mathbf{x}_0))$

Iteration Bounds from MΦRF

Lemma (4)

For all $k \in [1, d]$, there are $\mu_1, \ldots, \mu_{k-2} \geq 0$ and $\mu_{k-1} > 0$ such that $\mathbf{x}'' \in \mathcal{Q} \rightarrow \mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0.$

Proof.

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow f_1(\mathbf{x}) \geq 0 \vee \cdots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee \Delta f_k(\mathbf{x}'') \geq 1.$$

□

Lemma (5)

For all $k \in [1, d]$, there are constants $c_k, d_k > 0$ such that $F_k(t) \leq c_k M t^{k-1} - d_k t^k$, for all $t \geq 1$.

[6] Proof Idea: Use the bound for $-\Delta f_k(\mathbf{x}''_i)$ to bound $F_k(t)$.

Proof of Lemma (6)

$$
\begin{aligned}
F_k(t) &= f_k(\mathbf{x}_0) + \Sigma_{i=0}^{t-1}(f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)) \\
&< M + \Sigma_{i=0}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1}F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1}F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\Sigma_{j=1}^{k-1}\left(\mu_j c_j M i^{j-1} - \mu_j d_j i^j\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left((\Sigma_{j=1}^{k-1}\mu_j c_j M i^{j-1}) - \mu_{k-1}d_{k-1}i^{k-1}\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(M(\Sigma_{j=1}^{k-1}\mu_j c_j) i^{k-2} - \mu_{k-1}d_{k-1}i^{k-1}\right) \\
&= M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\Sigma_{i=1}^{t-1} i^{k-2}) - \mu_{k-1}d_{k-1}\Sigma_{i=1}^{t-1} i^{k-1} \\
&\leq M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\frac{t^{k-1}}{k-1}) - \mu_{k-1}d_{k-1}(\frac{t^k}{k} - t^{k-1}) \\
&= c_k M t^{k-1} - d_k t^k
\end{aligned}
$$

where $\mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) \geq \Delta f_k(\mathbf{x}'') = f_k(\mathbf{x}_{i+1} - f_k(\mathbf{x}_i)$

Theorem (6)

An SLC loop that has a $M\Phi RF$ terminates in a number of iterations bounded by $O(||\mathbf{x}_0||_\infty)$

Proof.

$F_k(t) \leq c_k M t^{k-1} - d_k t^k$. For $t > \max\{1, (c_k/d_k)M\}$, we have $F_k(t) < 0$.

Thus, the loop terminates by the time $\max\{1, (c_i/d_i)M, \ldots, (c_k/d_k)M\}$ where $M = \max(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0))$.

$\square$