

# Incorrectness Logic

Peter W. O'Hearn

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# Overview

- ▶ Intuitive Introduction to Incorrectness Logic
- ▶ Incorrectness Logic: Syntax and Proof Rules
- ▶ Incorrectness Logic: Semantic
- ▶ Reasoning with the Logic

# Incorrectness Logic: Intuitive Introduction

This paper describes a simple logic for program incorrectness which is the other side of the coin to Hoare's logic of correctness.

Hoare's Logic:

$$\{precond\}code\{postcond\}$$

where *postcond* is an **over-approximation(superset)** of the final states reachable.

Incorrectness Logic:

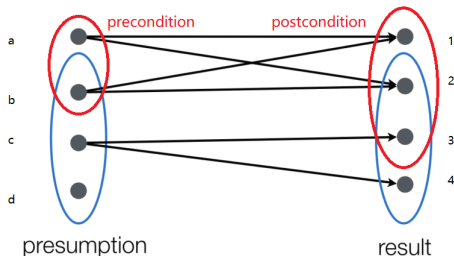
$$[presumption]code[result]$$

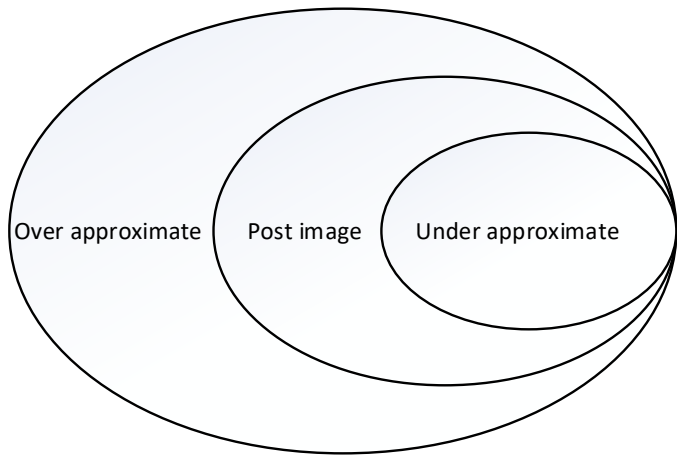
where *result* is a **under-approximation(subset)** of the final states reachable.

# Why Incorrectness Logic?

As is known, Hoare's logic is used to prove that programs do not contain bugs.

Incorrectness logic, however, is used to prove the presence of bugs but not their absence.





# Toolset

Website: [fbinfer.com](https://fbinfer.com)



The screenshot shows the homepage of the fbinfer.com website. The header is white with a hamburger menu icon and the 'Infer' logo on the left, and a search bar on the right. The main content area has a purple background with white text. The headline reads 'A tool to detect bugs in Java and C/C++/Objective-C code before it ships'. Below this, a paragraph explains that Infer is a static analysis tool that finds potential bugs in Java or C/C++/Objective-C code before they are shipped to users. At the bottom, there are two white buttons with black text: 'Get Started' and 'Learn More'.

☰ **Infer** 🔍 Search

## A tool to detect bugs in Java and C/C++/Objective-C code before it ships

Infer is a static analysis tool - if you give Infer some Java or C/C++/Objective-C code it produces a list of potential bugs. Anyone can use Infer to intercept critical bugs before they have shipped to users, and help prevent crashes or poor performance.

[Get Started](#) [Learn More](#)

# Beginning Examples

## Under-approximating Triples

For incorrectness logic, we use "presume" as to identify a pre-assertion and use "achieves" to identify the post-assertion. Consider an example below.

```
1  /* presumes: [z==11] */
2      if (x is even) {
3          if (y is odd) {
4              z=42;
5          }
6  /* achieves: [z==42] */
```

Question: is the assertion  $z == 42$  an under-approximation of the states reachable from executing code from states satisfying the presumption?

```
/* achieves: [z==42 && (x is even) && (y is odd) ] */
```

# Intuition: Inference

Inference rule of Hoare's logic:

$$\frac{\{p\}C\{q \wedge r\}}{\{p\}C\{q\}}$$

where the post condition  $q \wedge r$  can be relieved to  $q$ .

Inference rule of Incorrectness logic:

$$\frac{[p]C[q_1 \vee q_2]}{[p]C[q_1]}$$

where the result can only be strengthened.



# Beginning Examples

## Specifying Incorrectness

We can reason about errors by distinguishing the result-assertion by some tags. For example, we use "er: predicate" for erroneous or abnormal termination.

```
1  void foo(char* str)
2  /*  presumes: [*str[]]==s]
3      achieves: [er: *str[]==s && length(s) > 16 ] */
4      { char buf[16];
5        strcpy(buf, str);
6      }
7
8  int main(int argc, char *argv[])
9      { foo(argv[1]); }
```

In this paper, the author only considers errors caused by statement `error();`.

# Beginning Examples

## Under-approximate Success

Similar to the incorrectness specification, we also need tags for normal, not exceptional, termination of a program. Here we use "ok" for this purpose.

```
1  void mkeven()  
2  /*  presumes: [true],   wrong achieves: [ok: x==2 || x==4]      */  
3    { x=2; }  
4  
5  void usemkeven()  
6    { mkeven();  if (x==4) {error();} }
```

# Incorrectness Logic: Syntax and Proof Rules

Let  $\epsilon$  range over a collection of exit conditions, to include at least "ok" and "er". An *under-approximate triple* is of the form:

$$[p]C[\epsilon : q]$$

meaning  $q$  under-approximates the states when  $C$  exits via  $\epsilon$  starting from states in  $p$ .

Sometimes, we write  $[p]C[\text{ok} : q][\text{er} : r]$  as shorthand for  $[p]C[\text{ok} : q]$  and  $[p]C[\text{er} : r]$

An important point: the triple  $[p]C[\epsilon : q]$  express the reachability property that involves termination. **Every state in the result is reachable from some states in the presumption**

# Proof Rules

- ▶ *Empty under-approximate*:  $[p]C[\epsilon : \text{false}]$
- ▶ *Unit*:

$$[p]\text{skip}[\text{ok} : p][\text{er} : \text{false}]$$

- ▶ *Consequence*: 
$$\frac{p' \Leftarrow p, [p]C[\epsilon : q], q \Leftarrow q'}{[p']C[\epsilon : q']}$$

- ▶ *Disjunction*

$$\frac{[p_1]C[\epsilon : q_1], [p_2]C[\epsilon : q_2]}{[p_1 \vee p_2]C[\epsilon : q_1 \vee q_2]}$$

- ▶ *Sequencing(short-circuit)*

$$\frac{[p]C_1[\text{er} : r]}{[p]C_1; C_2[\text{er} : r]}$$

- ▶ *Sequencing(normal)*

$$\frac{[p]C_1[\text{ok} : q], [q]C_2[\epsilon : r]}{[p]C_1; C_2[\epsilon : r]}$$

# Proof Rules

- ▶ *Iterate zero*:  $[p]C^*[ok : p]$
- ▶ *Iterate non-zero*: 
$$\frac{[p]C^*; C[\epsilon : q]}{[p]C^*; [\epsilon : q]}$$
- ▶ *Backwards Variant*:

$$\frac{[p(n) \wedge nat(n)]C[ok : p(n+1) \wedge nat(n)]}{[p(0)]C^*[ok : \exists n. p(n) \wedge nat(n)]}$$

- ▶ *Choice* ( $i = 1$  or  $2$ ): 
$$\frac{[p]C_i[\epsilon : q]}{[p]C_1 + C_2[\epsilon : q]}$$
- ▶ *Error*:  $[p]error() [ok : false] [er : p]$
- ▶ *Assume*:  $[p]assume\ B [ok : p \wedge B] [er : false]$

# Program Conversion

$\text{while } B \text{ do } C \quad =_{def} \quad (\text{assume}(B); C)^{\star}; \text{assume}(\neg B)$

$\text{if } B \text{ then } C \text{ else } C' \quad =_{def} \quad (\text{assume}(B); C) + (\text{assume}(\neg B); C')$

$\text{assert}(B) \quad =_{def} \quad \text{assume}(B) + (\text{assume}(\neg B); \text{error}())$

# Rules for Variables and Mutations

*Assignment*

$$[p]x = e[\text{ok: } \exists x'. p[x'/x] \wedge x = e[x'/x]][\text{er: false}]$$

*Constancy*

$$\frac{[p]C[\epsilon: q]}{[p \wedge f]C[\epsilon: q \wedge f]} \quad \text{Mod}(C) \cap \text{Free}(f) = \emptyset$$

*Substitution I*

$$\frac{[p]C[\epsilon: q]}{([p]C[\epsilon: q])(e/x)} \quad (\text{Free}(e) \cup \{x\}) \cap \text{Free}(C) = \emptyset$$

*Nondet Assignment*

$$[p]x = \text{nondet}()[\text{ok: } \exists x' p][\text{er: false}]$$

*Local Variable*

$$\frac{[p]C(y/x)[\epsilon: q]}{[p]\text{local } x.C[\epsilon: \exists y.q]} \quad y \notin \text{Free}(p, C)$$

*Substitution II*

$$\frac{[p]C[\epsilon: q]}{([p]C[\epsilon: q])(y/x)} \quad y \notin \text{Free}(p, C, q)$$

where  $\text{Mod}(C)$  is the set of variables that program  $C$  modified,  $\text{Free}(r)$  means the set of free variables in assertion  $r$  and  $\text{nondet}()$  is a nondeterministic value.

# Incorrectness Logic: Semantic

A set of *states*  $\Sigma$  is a set of function from the set *Variables* to the set *Value*. (When reason about termination *Value* is the set of natural numbers).

## Definition (Post and Semantic Triples)

For any relation  $r \subseteq \Sigma \times \Sigma$  and predicate  $p, q \subseteq \Sigma$  define

- ▶ the *post-image* of  $r$ ,  $post(r) \in P(\Sigma) \rightarrow P(\Sigma)$ :

$$post(r)p = \{\sigma' \mid \exists \sigma \in p. (\sigma, \sigma') \in r\}$$

- ▶ the *under-approximate triple*:  
 $[p]r[q]$  is true iff  $q \subseteq post(r)p$
- ▶ the *over-approximate triple* (Hoare's logic):  
 $\{p\}r\{q\}$  is true iff  $post(r)p \subseteq q$



## Lemma (Characterization)

*The following statements are equivalent:*

- ▶  $[p]r[q]$  is true.
- ▶  $\forall \sigma_q \in q. \exists \sigma_p \in p. (\sigma_p, \sigma_q) \in r.$

The semantic of the triple  $[p]C[\epsilon : q]$ ? Relations can be regarded as the semantic of a program. Here  $\epsilon \in \{ok, er\}$ , hence  $\llbracket C \rrbracket_{ok}$  and  $\llbracket C \rrbracket_{er}$  are two different relations related to program  $C$ .

# Semantic

*Generic Semantics for arbitrary state sets  $\Sigma$*

$$\begin{aligned}\llbracket C \rrbracket \epsilon &\subseteq \Sigma \times \Sigma \\ \llbracket B \rrbracket &: \Sigma \rightarrow \text{Bool}\end{aligned}$$

$$\begin{aligned}\llbracket \text{skip} \rrbracket \text{ok} &= \{(\sigma, \sigma) \mid \sigma \in \Sigma\} & \llbracket \text{skip} \rrbracket \text{er} &= \emptyset \\ \llbracket \text{error}() \rrbracket \text{ok} &= \emptyset & \llbracket \text{error}() \rrbracket \text{er} &= \{(\sigma, \sigma) \mid \sigma \in \Sigma\} \\ \llbracket \text{assume } B \rrbracket \text{ok} &= \{(\sigma, \sigma) \mid \llbracket B \rrbracket \sigma = \text{true}\} & \llbracket \text{assume } B \rrbracket \text{er} &= \emptyset \\ \llbracket C^\star \rrbracket \epsilon &= \bigcup_{i \in \text{Nat}} \llbracket C^i \rrbracket \epsilon & \llbracket C_1 + C_2 \rrbracket \epsilon &= \llbracket C_1 \rrbracket \epsilon \cup \llbracket C_2 \rrbracket \epsilon \\ \llbracket C_1; C_2 \rrbracket \epsilon &= \{(\sigma_1, \sigma_3) \mid \exists \sigma_2. (\sigma_1, \sigma_2) \in \llbracket C_1 \rrbracket \text{ok} \text{ and } (\sigma_2, \sigma_3) \in \llbracket C_2 \rrbracket \epsilon\} \\ &\cup \left( \text{if } (\epsilon = \text{ok}) \text{ then } \emptyset \text{ else } \{(\sigma_1, \sigma_2) \mid (\sigma_1, \sigma_2) \in \llbracket C_1 \rrbracket \text{er}\} \right)\end{aligned}$$

*Semantics of mutation and local variables*

$$\begin{aligned}\Sigma &= \text{Variables} \rightarrow \text{Values} \\ \llbracket e \rrbracket &: \Sigma \rightarrow \text{Values}\end{aligned}$$

$$\begin{aligned}\llbracket x = e \rrbracket \text{ok} &= \{(\sigma, (\sigma \mid x \mapsto \llbracket e \rrbracket \sigma)) \mid \sigma \in \Sigma\} & \llbracket x = e \rrbracket \text{er} &= \emptyset \\ \llbracket x = \text{nondet}() \rrbracket \text{ok} &= \{(\sigma, (\sigma \mid x \mapsto v)) \mid \sigma \in \Sigma, v \in \text{Values}\} & \llbracket x = \text{nondet}() \rrbracket \text{er} &= \emptyset \\ \llbracket \text{local } x. C \rrbracket \epsilon &= \{((\sigma \mid x \mapsto v), (\sigma' \mid x \mapsto v)) \mid (\sigma, \sigma') \in \llbracket C \rrbracket \epsilon, v \in \text{Values}\}\end{aligned}$$

## Theorem

*All the listed properties in last section are true of the semantic.*

# Soundness and Completeness

## Definition (Interpretation of Specifications)

$[p]C[\epsilon : q]$  is true iff the semantic triple  $[p](\llbracket C \rrbracket \epsilon)[q]$  holds.

## Theorem (Soundness)

*The relational semantics last page validates all the rules introduced before: each axiom is true and each inference rule preserves truth.*

## Theorem (Completeness)

*Every true triple involving finitely-supported predicates is provable.*

where *finitely-supported* predicate means the predicate only related to finite number of variables in set *Variables*.

# Predicate Transformers

Predicate transformers are fundamental semantic tool in program logic and analysis, which are defined as functions that map predicates to predicates.

- Forward Transformers. As is known  $post(r)p$  can be given as the strongest predicate satisfying  $\{p\}r\{q\}$ . Symmetricly, we also have the weakest under-approximation to characterize that.

## Definition

For  $r \subseteq \Sigma \times \Sigma$ :

$$StrongestOverPost(r)p = \bigwedge \{q \mid \{p\}r\{q\} \text{ holds} \}$$

$$WeakestUnderPost(r)p = \bigvee \{q \mid [p]r[q] \text{ holds} \}$$

## Proposition

$$StrongestOverPost(r)p = WeakestUnderPost(r)p = post(r)$$

# Forward Transformer

Iteration:

$$\text{StrongestOverPost}(\llbracket C^* \rrbracket ok)p = \bigwedge \{I \mid p \Rightarrow I \wedge \{I\} C \{I\} \text{is true}\}$$

$$\text{WeakestUnderPost}(\llbracket C^* \rrbracket ok)p = \bigvee_{i \in \text{Nat}} \{q \mid [p] C^i [\epsilon : q] \text{is true}\}$$

$$\underline{\text{UnderPost}}(\llbracket C^* \rrbracket \epsilon)p = \bigvee_{i \leq \text{bound}} \{q \mid [p] C^i [\epsilon : q] \text{is true}\}$$

Nondeterministic choice:

$$\text{post}(\llbracket C_1 + C_2 \rrbracket \epsilon)p = \text{post}(\llbracket C_1 \rrbracket \epsilon)p \vee \text{post}(\llbracket C_2 \rrbracket \epsilon)p$$

$$\underline{\text{post}}(\llbracket C_1 + C_2 \rrbracket \epsilon)p = \underline{\text{post}}(\llbracket C_1 \rrbracket \epsilon)p \underline{\vee} \underline{\text{post}}(\llbracket C_2 \rrbracket \epsilon)p$$

where  $p \underline{\vee} q \Rightarrow p \vee q$

# Backward Transformers

## Fact (Valid presumptions need not exist)

*Given a relation  $r$  and assertion  $q$ , there need not exist any  $p$  such that  $[p]r[q]$  holds.*

Strongest under-approximate presumptions do not exist in general.

## Example

$C = (\text{assume } x == 1; x = 88) + (\text{assume } x == 2; x = 88)$

# Reasoning with the Logic

$$[p]foo()[ok : q][er : s] \vdash [p']C[ok : q'][er : s']$$

# Reasoning Details

## Reasoning about Loops

```
int x;

void loop0()
/* (default presumes is "true" when not specified)
   achieves: [ok: x>=0 ]    */
{ int n = nondet(); x = 0;
  while (n > 0) {
    x = x+n;
    n = nondet();
  } }

void client0()
/* achieves: [err: x==2,000,000 ]    */
{ loop0();
  if (x==2,000,000) {error();}
}
```

Use consequence rule and implication backward:

$$\frac{[true]loop0()[ok : x \geq 0], x \geq 0 \Leftarrow x==2000000}{[true]loop0()[ok : x==2000000]}$$



# Unroll Loop

```
[x==0]
  if (n>0) {
    [x==0 && n>0]      x=x+n;  n=nondet();  [x>0]
  } else
  { [x==0 && n<=0]      skip;
  }
[x>0 || (x==0 && n<=0)]
  assume (n<=0);
[(x>0 && n<=0) || x==0 && n<=0)]
[ok: x>=0 && n<=0]
```

```
void loop1()
/* achieves1: [ok: x==0 || x==1 || x==2 || x==3 ]
   achieves2: [ok: x>=0]    */
{ x = 0;
  Kleene-star{
    x = x+1;
  } }

void client1()
/* achieves: [er: x==2,000,000]  */
{ loop1();
  if (x==2,000,000) {error();}
}
```

# Conditionals, Expressiveness and Pruning

```
int x,y;

int difficult(int y)
{ return (y*y); /* or, return hash(y) ... or, unknown code */
}

void client()
/*achieves1: [ok: y==49 && x==1]
  achieves2: [ok: exists z. (y==difficult(z)&& x==1)|| (y!=difficult(z)&& x==2)]
  achieves3: [ok: x==1 || x==2]    */
{ int z = nondet();
  if (y == difficult(z))
    {x=1;}
  else
    {x=2;}
}

void test1()
/*achieves2: [ok: exists z. (y==difficult(z)&& x==1)|| (y!=difficult(z)&& x==2)]*/
{ client();
  if (x==1 || x==2) { error(); }
}

void test2()
{ client();
  if (x==2) {error();}
}
```

# Conclusion

This paper,

- ▶ described how under-approximate triples are relevant to proving the presence of bugs,
- ▶ designed a specific logic along with a semantics and proof theory,
- ▶ explored reasoning techniques that are concerned with automatic program analysis.