On Multiphase-Linear Ranking Functions

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Single Path Linear Constraint Loop

Example

$$\texttt{while}\ (x \geq -z)\ \texttt{do}\ x' = x + y,\ y' = y + z,\ z' = z - 1$$

Let $B = (-1, 0, 1)$, $\mathbf{x} = (x, y, z)^T$, $\mathbf{b} = 0$.
Let $\mathbf{x}'' = (x, y, z, x', y', z')$,

$$A = \begin{bmatrix} 1 & 1 & 0 - 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

and $\mathbf{c} = (0, 0, 1)^T$

Definition (SLC)

$$\textit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \textit{do}\ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Single Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1x_1 + \ldots a_nx_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while } (x - 1 > 0)\texttt{do } x' = x - 5$$

LRF: $f(x) = ax + b$.

- $ax + b \geq 0 \Rightarrow x \geq -\frac{b}{a} = 1$.
- $ax + b - (ax' + b) = a(x - x') = 5a \Rightarrow 5a \geq 1$

A possible SLRF: $f(x) = x - 1$

Limitation of SLRF

$$\texttt{while } (q > 0)\texttt{do } q' = q - y, y' = y + 1$$

Assume there is a LRF for this loop, say $f(q, y) = a_1 q + a_2 y + b$

$$f(q, y) - f(q', y') = a_1 y + a_2$$

Since y is not bounded, we cannot guarantee $\Delta f(q, y, q', y') > 0$
The loop does not has a SLRF, however, it does terminate.
We still wish to use q for ranking function, but to distinguish different "phase" of q base on either $y \geq 0$ or $y < 0$

Nested RF

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d. \end{aligned}$$

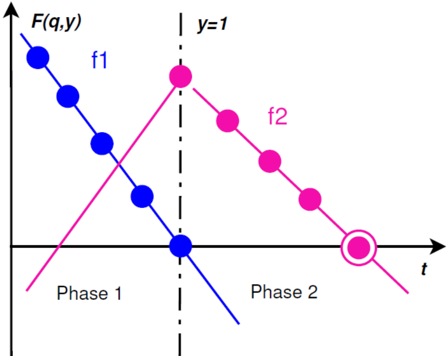
Let $f_0 = 0$.

Example: Nested RF

$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \qquad \text{for all } i = 1, \ldots, d.$$

$$\texttt{while } (q > 0)\texttt{do } q' = q - y, y' = y + 1$$

- Above loop has Nested RF $\langle 1 - y, q + 1 \rangle$



Linear Loop Program

Definition

A linear loop program $\mathtt{LOOP}(x, x')$ is a binary relation defined by a formula with the free variables x and x' of the form

$$\bigvee_{i \in I} \left(A_i \binom{x}{x'} \leq b_i \ \wedge \ C_i \binom{x}{x'} < d_i \right)$$

for some finite index set I.

Example

$$\mathtt{while}\ \ (q > 0)\{\mathtt{if}\ \ (y > 0) : q' = q - y - 1; \mathtt{else}\ \ : q' = q + y - 1\}$$

can be represented by

$$\begin{aligned} &(q > 0 \ \wedge \ y > 0 \ \wedge \ y' = y \ \wedge \ q' = q - y - 1) \\ \vee\, &(q > 0 \ \wedge \ y \leq 0 \ \wedge \ y' = y \ \wedge \ q' = q + y - 1) \end{aligned}$$

Limitation of Nested RF

Example

$$\texttt{while}\ (q > 0 \vee y > 0)$$

$$\{\texttt{if}\ (y > 0) : y' = y - 1; q' = q; \texttt{else}\ : q' = q - 1\}$$

This program does not have a nested ranking function for we require $f_d \geq 0$ but the guard is $q > 0 \vee y > 0$.
Howerver, this loop does terminate. Then we use a "multi-phase" ranking function $\langle y, q \rangle$ to prove the termination.

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

$$\begin{aligned} \forall j \le i \ . \ \Delta f_j(\mathbf{x}'') &\ge 1\,, \\ f_i(\mathbf{x}) &\ge 0\,, \\ \forall j < i \ . \quad f_j(\mathbf{x}) &\le 0\,. \end{aligned}$$

We say that $\mathbf{x}''$ is ranked by f_i(for the minimal).

Example: Multiphase Ranking Function

$$\texttt{while}\ (x > -z)\texttt{do}\ \ x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases:
$\langle z + 1, y + 1, x \rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	$\mathbf{2}$	2	1
2	2	0	$\mathbf{1}$	3	2
4	2	-1	$\mathbf{0}$	3	4
6	1	-2	-1	$\mathbf{2}$	6
7	-1	-3	-2	$\mathbf{0}$	7
6	-4	-4	-3	-3	$\mathbf{6}$
2	-8	-5	-4	-7	$\mathbf{2}$
-6	-13	-6	-5	-12	-6

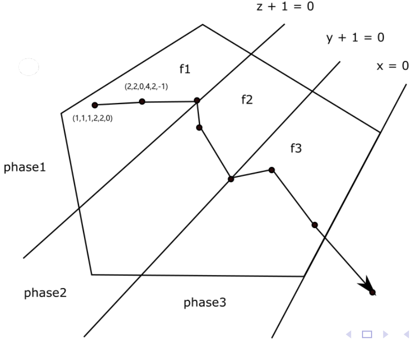
Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z' = z - 1$$

$$\langle z + 1, y + 1, x \rangle$$

$\mathbf{x}''$ is ranked by f_k when $i = k$. In this example, $f_1(x, y, z) = z + 1$, $f_2(x, y, z) = y + 1$ and $f_3(x, y, z) = x$

$$\begin{aligned} \forall j \leq i \,.\, \Delta f_j(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \\ \forall j < i \,.\quad f_j(\mathbf{x}) &\leq 0\,. \end{aligned}$$



MΦRF to Nested RF

Theorem (1)

If $\mathcal{Q}$ has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

$\mathrm{BM\Phi RF}(\mathbb{Q}) \in \mathtt{PTIME}$

Theorem (2)

$BM\Phi RF(\mathbb{Q}) \in \mathtt{PTIME}$.

Proof.

Leike et al..Ranking Templates for Linear Loops.

$\square$

LLRF

Intuition: remind binary relation $\mathbf{x} \succeq \mathbf{x}'$ iff $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ and $f(\mathbf{x}) \geq 0$.

Generalize it into several phases using lexicographical order of ranking functions.

$\langle f_1, f_2, \ldots, f_d \rangle$

$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

Definition (LLRF)

Given a set of transitions T we say that $\langle f_1, f_2, \ldots, f_d \rangle$ is a LLRF (of depth d) for T if for every $\mathbf{x}'' \in T$ there is an index i such that

$$\begin{aligned} \forall j < i \;.\; \Delta f_j(\mathbf{x}'') &\geq 0\,, \\ \Delta f_i(\mathbf{x}'') &\geq 1\,, \\ f_i(\mathbf{x}) &\geq 0\,, \end{aligned}$$

A LLRF is weak if..

Weak LLRF to MΦRF

Theorem (3)

If $\mathcal{Q}$ has a weak LLRF of depth d, it has a MΦRF of depth d.

Weak LLRF: Integer to Rational

Theorem (4)

Let $\langle f_1, \ldots, f_d \rangle$ be a weak LLRF for $I(\mathcal{Q})$. Then there are constants $c_1, \ldots, c_d$ such that $\langle f_1 + c_1, \ldots, f_d + c_d \rangle$ is a weak LLRF for $\mathcal{Q}_I$(over the rationals).

The Depth of a MΦRF

Idea: pre-compute the depth d for MΦRF synthesis.

Theorem (5)

For integer $B > 0$, the following loop $\mathcal{Q}_B$

$$\texttt{while}\ (x \geq 1,\ y \geq 1,\ x \geq y,\ 2^B y \geq x)\ \texttt{do}\ x' = 2x,\ y' = 3y$$

needs at least $B+1$ components in any MΦRF.

Proof.

Define $R_I = \{(2^i c, c, 2^{i+1} c, 3c) \mid c \geq 1\}$ and note that for $i \in [0, B]$, we have $R_i \in \mathcal{Q}_B$.
Assume the loop has a MΦRF with depth B, then it is obvious that there are R_i and $R_j, i \neq j$ that are ranked by the same phase f_k, w.l.o.g., assume $j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$, we have

□

Proof of Theorem (5)

$j > i$ and $f_k(x, y) = a_1 x + a_2 y + a_0$

$$
\begin{aligned}
f_k(2^i, 1) - f_k(2^{i+1}, 3) &= -a_1 2^i - a_2 2 &> 0 \\
f_k(2^j, 1) - f_k(2^{j+1}, 3) &= -a_1 2^j - a_2 2 &> 0 \\
f_k(2^i, 1) - f_k(0, 0) &= a_1 2^i + a_2 &\geq 0 \\
f_k(2^j, 1) - f_k(0, 0) &= a_1 2^j + a_2 &\geq 0
\end{aligned}
$$

$j > i$
$a_1 2^{i-1} > 0 \Rightarrow a_1 > 0$
$a_1(2^i - 2^{j-1}) > 0 \Rightarrow i + 1 > j \Rightarrow i \geq j$. Contradiction.

Iteration Bounds from MΦRFs

Example

$$\texttt{while } (x \geq 0)\texttt{do } x' = x + y, y' = y - 1$$

MΦRF: $\langle y + 1, x \rangle$
When start from $x = x_0$ and $y = y_0$...
$x_0 + \dfrac{y_0(y_0 + 1)}{2} - 1$

Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF $\tau = \langle f_1, \ldots, f_d \rangle$.

- $F_k(t)$: the value of f_k after iteration t.
- $UB_k(t)$: bound for f_k. For $t > T_k$, $UB_k(T_k)$ becomes negative.
- T_k: an upper bound on the time in which the k-th phase ends.
- The whole loop must terminate before $\max_k T_k$ iterations.

$\mathbf{x}_t$ be te state after iteration t. Define $F_k(t) = f_k(\mathbf{x}_t)$. Let $M = \max(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0))$

Iteration Bounds from MΦRF

Lemma (4)

For all $k \in [1,d]$, there are $\mu_1, \ldots, \mu_{k-2} \geq 0$ and $\mu_{k-1} > 0$ such that $\mathbf{x}'' \in \mathcal{Q} \to \mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0$.

Proof.

$$\mathbf{x}'' \in \mathcal{Q} \to f_1(\mathbf{x}) \geq 0 \vee \cdots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee \Delta f_k(\mathbf{x}'') \geq 1.$$

□

Lemma (5)

For all $k \in [1,d]$, there are constants $c_k, d_k > 0$ such that $F_k(t) \leq c_k M t^{k-1} - d_k t^k$, for all $t \geq 1$.

Proof Idea: Use the bound for $-\Delta f_k(\mathbf{x}_i'')$ to bound $F_k(t)$.

Proof of Lemma (6)

$$
\begin{aligned}
F_k(t) &= f_k(\mathbf{x}_0) + \Sigma_{i=0}^{t-1}(f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)) \\
&< M + \Sigma_{i=0}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\Sigma_{j=1}^{k-1}\left(\mu_j c_j M i^{j-1} - \mu_j d_j i^j\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left((\Sigma_{j=1}^{k-1}\mu_j c_j M i^{j-1}) - \mu_{k-1} d_{k-1} i^{k-1}\right) \\
&\leq M(1+\mu) + \Sigma_{i=1}^{t-1}\left(M(\Sigma_{j=1}^{k-1}\mu_j c_j) i^{k-2} - \mu_{k-1} d_{k-1} i^{k-1}\right) \\
&= M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\Sigma_{i=1}^{t-1} i^{k-2}) - \mu_{k-1} d_{k-1} \Sigma_{i=1}^{t-1} i^{k-1} \\
&\leq M(1+\mu) + M(\Sigma_{j=1}^{k-1}\mu_j c_j)(\frac{t^{k-1}}{k-1}) - \mu_{k-1} d_{k-1}(\frac{t^k}{k} - t^{k-1}) \\
&= c_k M t^{k-1} - d_k t^k
\end{aligned}
$$

where $\mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) \geq \Delta f_k(\mathbf{x}'') = f_k(\mathbf{x}_{i+1} - f_k(\mathbf{x}_i)$

Theorem (6)

An SLC loop that has a MΦRF terminates in a number of iterations bounded by $O(||\mathbf{x}_0||_\infty)$

Proof.

$F_k(t) \leq c_k M t^{k-1} - d_k t^k$. For $t > \max\{1, (c_k/d_k)M\}$, we have $F_k(t) < 0$.

Thus, the loop terminates by the time $\max\{1, (c_i/d_i)M, \ldots, (c_k/d_k)M\}$ where $M = \max(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0))$.

□