

# On Multiphase-Linear Ranking Functions

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May 15, 2020

# Single Path Linear Constraint Loop

## Example

**while**  $(x \geq -z)$  **do**  $x' = x + y$ ,  $y' = y + z$ ,  $z' = z - 1$

Let  $B = (-1, 0, 1)$ ,  $\mathbf{x} = (x, y, z)^T$ ,  $\mathbf{b} = 0$ .

Let  $\mathbf{x}'' = (x, y, z, x', y', z')$ ,

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \quad (1)$$

and  $\mathbf{c} = (0, 0, 1)^T$

## Definition (SLC)

*while*  $(B\mathbf{x} \leq \mathbf{b})$  *do*  $A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$

$$A'' = \begin{pmatrix} B & 0 \\ A \end{pmatrix}$$

$$\mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

# Ranking Functions

## Definition (Single Linear Ranking Function(LRF))

$f(x_1, \dots, x_n) = a_1x_1 + \dots a_nx_n + a_0$ , such that

- ▶  $f(\mathbf{x}) \geq 0$  for any  $\mathbf{x}$  satisfies the loop constraints.
- ▶  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  for any transition from  $\mathbf{x}$  to  $\mathbf{x}'$ .

## Example

`while (x - 1 > 0)do x' = x - 5`

LRF:  $f(x) = ax + b$ .

- ▶  $ax + b \geq 0 \Rightarrow x \geq -\frac{b}{a} = 1$ .
- ▶  $ax + b - (ax' + b) = a(x - x') = 5a \Rightarrow 5a \geq 1$

A possible SLRF:  $f(x) = x - 1$

We can define a binary relation  $\mathbf{x} \succeq \mathbf{x}'$  iff  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  and  $f(\mathbf{x}) \geq 0$

# Limitation of SLRF

`while ( $q > 0$ )do  $q' = q - y, y' = y + 1$`

Assume there is a LRF for this loop, say  $f(q, y) = a_1q + a_2y + b$

$$f(q, y) - f(q', y') = a_1y + a_2$$

Since  $y$  is not bounded, we cannot guarantee  $\Delta f(q, y, q', y') > 0$

The loop does not has a SLRF, however, it does terminate.

We still wish to use  $q$  for ranking function, but to distinguish different “phase” of  $q$  base on either  $y \geq 0$  or  $y < 0$

# Nested RF

## Definition (Nested Ranking Function)

A tuple  $\langle f_1, \dots, f_d \rangle$  is a nested ranking function for  $T$  if the following requirements are satisfied for all  $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

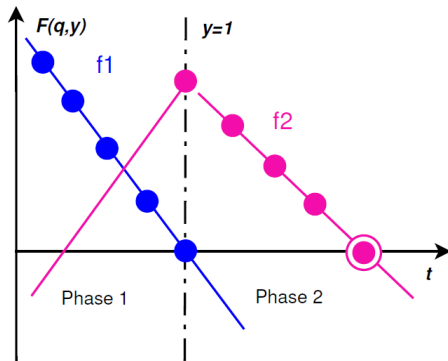
Let  $f_0 = 0$ .

## Example: Nested RF

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

while ( $q > 0$ )do  $q' = q - y, y' = y + 1$

- Above loop has Nested RF  $\langle 1 - y, q + 1 \rangle$



# Linear Loop Program

## Definition

A linear loop program  $\text{LOOP}(x, x')$  is a binary relation defined by a formula with the free variables  $x$  and  $x'$  of the form

$$\bigvee_{i \in I} (A_i(x) \leq b_i \wedge C_i(x') < d_i)$$

for some finite index set  $I$ .

## Example

`while (q > 0){if (y > 0) : q' = q - y - 1; else : q' = q + y - 1}`

can be represented by

$$(q > 0 \wedge y > 0 \wedge y' = y \wedge q' = q - y - 1) \\ \vee (q > 0 \wedge y \leq 0 \wedge y' = y \wedge q' = q + y - 1)$$

# Limitation of Nested RF

## Example

```
while ( $q > 0 \vee y > 0$ )  
{if ( $y > 0$ ) :  $y' = y - 1$ ;  $q' = q$ ; else :  $q' = q - 1$ }
```

This program does not have a nested ranking function for we require  $f_d \geq 0$  but the guard is  $q > 0 \vee y > 0$ .

However, this loop does terminate. Then we use a “multi-phase” ranking function  $\langle y, q \rangle$  to prove the termination.

$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \quad \text{for all } i = 1, \dots, d.$$



# Multiphase Ranking Function

## Definition

Given a set of transitions  $T \subseteq \mathbb{Q}^{2n}$ , we say  $\langle f_1, \dots, f_d \rangle$  is a multiphase ranking function for  $T$  if for every  $\mathbf{x}'' \in T$ , there is an index  $i \in [1, d]$ , s.t.

$$\begin{aligned} \forall j \leq i . \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i . f_j(\mathbf{x}) &\leq 0. \end{aligned}$$

We say that  $\mathbf{x}''$  is ranked by  $f_i$  (for the minimal).

## Example: Multiphase Ranking Function

**while**  $(x > -z)$  **do**  $x' = x + y, y' = y + z, z = z - 1$

Attempt to use a ranking function that has several phases:

$\langle z + 1, y + 1, x \rangle$

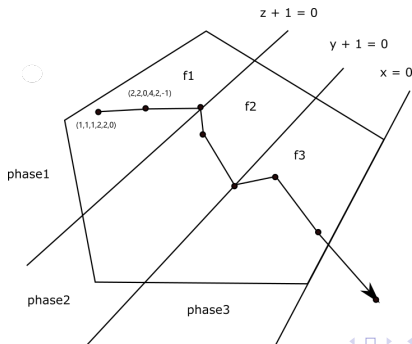
| $x$ | $y$ | $z$ | $z + 1$  | $y + 1$  | $x$      |
|-----|-----|-----|----------|----------|----------|
| 1   | 1   | 1   | <b>2</b> | 2        | 1        |
| 2   | 2   | 0   | <b>1</b> | 3        | 2        |
| 4   | 2   | -1  | <b>0</b> | 3        | 4        |
| 6   | 1   | -2  | -1       | <b>2</b> | 6        |
| 7   | -1  | -3  | -2       | <b>0</b> | 7        |
| 6   | -4  | -4  | -3       | -3       | <b>6</b> |
| 2   | -8  | -5  | -4       | -7       | <b>2</b> |
| -6  | -13 | -6  | -5       | -12      | -6       |

## Example: Multiphase Ranking Function

**while**  $(x > -z)$  **do**  $x' = x + y, y' = y + z, z' = z - 1$   
 $\langle z + 1, y + 1, x \rangle$

$\mathbf{x}''$  is ranked by  $f_k$  when  $i = k$ . In this example,  
 $f_1(x, y, z) = z + 1$ ,  $f_2(x, y, z) = y + 1$  and  $f_3(x, y, z) = x$

$$\begin{aligned} \forall j \leq i. \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i. f_j(\mathbf{x}) &\leq 0. \end{aligned}$$



# MΦRF to Nested RF

## Theorem (1)

*If  $\mathcal{Q}$  has a MΦRF of depth  $d$ , then it has a nested ranking function of depth at most  $d$ .*

$BM\Phi RF(\mathbb{Q}) \in PTIME$

Theorem (2)

$BM\Phi RF(\mathbb{Q}) \in PTIME.$

Proof.

Leike et al..Ranking Templates for Linear Loops.



# LLRF

Intuition: remind binary relation  $\mathbf{x} \succeq \mathbf{x}'$  iff  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  and  $f(\mathbf{x}) \geq 0$ .

Generalize it into several phases using lexicographical order of ranking functions.

$\langle f_1, f_2, \dots, f_d \rangle$

$(2, 3, 1, 3) \geq (2, 1, 5, 4)$

## Definition (LLRF)

Given a set of transitions  $T$  we say that  $\langle f_1, f_2, \dots, f_d \rangle$  is a LLRF (of depth  $d$ ) for  $T$  if for every  $\mathbf{x}'' \in T$  there is an index  $i$  such that

$$\begin{aligned} \forall j < i . \Delta f_j(\mathbf{x}'') &\geq 0, \\ \Delta f_i(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \end{aligned}$$

A LLRF is weak if..

# Weak LLRF to $M\Phi RF$

## Theorem (3)

*If  $\mathcal{Q}$  has a weak LLRF of depth  $d$ , it has a  $M\Phi RF$  of depth  $d$ .*

# Weak LLRF: Integer to Rational

## Theorem (4)

*Let  $\langle f_1, \dots, f_d \rangle$  be a weak LLRF for  $I(\mathbb{Q})$ . Then there are constants  $c_1, \dots, c_d$  such that  $\langle f_1 + c_1, \dots, f_d + c_d \rangle$  is a weak LLRF for  $\mathcal{Q}_I$  (over the rationals).*



# The Depth of a MΦRF

Idea: pre-compute the depth  $d$  for MΦRF synthesis.

## Theorem (5)

*For integer  $B > 0$ , the following loop  $\mathcal{Q}_B$*

*while  $(x \geq 1, y \geq 1, x \geq y, 2^B y \geq x)$  do  $x' = 2x, y' = 3y$*

*needs at least  $B + 1$  components in any MΦRF.*

## Proof.

Define  $R_I = \{(2^i c, c, 2^{i+1} c, 3c) \mid c \geq 1\}$  and note that for  $i \in [0, B]$ , we have  $R_i \in \mathcal{Q}_B$ .

Assume the loop has a MΦRF with depth  $B$ , then it is obvious that there are  $R_i$  and  $R_j, i \neq j$  that are ranked by the same phase  $f_k$ , w.l.o.g., assume  $j > i$  and  $f_k(x, y) = a_1 x + a_2 y + a_0$ , we have



## Proof of Theorem (5)

$$j > i \text{ and } f_k(x, y) = a_1x + a_2y + a_0$$

$$f_k(2^i, 1) - f_k(2^{i+1}, 3) = -a_12^i - a_22 > 0$$

$$f_k(2^j, 1) - f_k(2^{j+1}, 3) = -a_12^j - a_22 > 0$$

$$f_k(2^i, 1) - f_k(0, 0) = a_12^i + a_2 \geq 0$$

$$f_k(2^j, 1) - f_k(0, 0) = a_12^j + a_2 \geq 0$$

$$j > i$$

$$a_12^{i-1} > 0 \Rightarrow a_1 > 0$$

$$a_1(2^i - 2^{j-1}) > 0 \Rightarrow i + 1 > j \Rightarrow i \geq j. \text{ Contradiction.}$$

# Iteration Bounds from MΦRFs

## Example

**while**  $(x \geq 0)$ **do**  $x' = x + y, y' = y - 1$

MΦRF:  $\langle y + 1, x \rangle$

When start from  $x = x_0$  and  $y = y_0 \dots$

$$x_0 + \frac{y_0(y_0 + 1)}{2} - 1$$

# Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF

$\tau = \langle f_1, \dots, f_d \rangle$ .

- ▶  $F_k(t)$ : the value of  $f_k$  after iteration  $t$ .
- ▶  $UB_k(t)$ : bound for  $f_k$ . For  $t > T_k$ ,  $UB_k(T_k)$  becomes negative.
- ▶  $T_k$ : an upper bound on the time in which the  $k$ -th phase ends.
- ▶ The whole loop must terminate before  $\max_k T_k$  iterations.

$\mathbf{x}_t$  be the state after iteration  $t$ . Define  $F_k(t) = f_k(\mathbf{x}_t)$ . Let  $M = \max(f_1(\mathbf{x}_0), \dots, f_d(\mathbf{x}_0))$

# Iteration Bounds from MΦRFs

## Lemma (4)

*For all  $k \in [1, d]$ , there are  $\mu_1, \dots, \mu_{k-2} \geq 0$  and  $\mu_{k-1} > 0$  such that  $\mathbf{x}'' \in \mathcal{Q} \rightarrow \mu_1 f_1(\mathbf{x}) + \dots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0$ .*

## Lemma (5)

*For all  $k \in [1, d]$ , there are constants  $c_k, d_k > 0$  such that  $F_k(t) \leq c_k M t^{k-1} - d_k t^k$ , for all  $t \geq 1$ .*

## Theorem (6)

*An SLC loop that has a  $M\Phi RF$  terminates in a number of iterations bounded by  $O(\|\mathbf{x}_0\|_\infty)$*