

# On Multiphase-Linear Ranking Functions

Xie Li

May 11, 2020

# Single Path Linear Constraint Loop

## Example

`while`  $(x \geq -z)$  `do`  $x' = x + y$ ,  $y' = y + z$ ,  $z' = z - 1$

`while`  $(x_2 - x_1 \leq 0, x_1 + x_2 \geq 1)$  `do`  $x_2' = x_2 - 2x_1 + 1$ ,  $x_1' = x_1$

## Definition (SLC)

*while*  $(B\mathbf{x} \leq \mathbf{b})$  *do*  $A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$

$$A'' = \begin{pmatrix} B & 0 \\ A \end{pmatrix}$$

$$\mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

# Ranking Functions

## Definition (Linear Ranking Function(LRF))

$f(x_1, \dots, x_n) = a_1x_1 + \dots a_nx_n + a_0$ , such that

- ▶  $f(\mathbf{x}) \geq 0$  for any  $\mathbf{x}$  satisfies the loop constraints.
- ▶  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  for any transition from  $\mathbf{x}$  to  $\mathbf{x}'$ .

## Example

`while (x - 1 > 0)do x' = x - 5`

Its LRF:  $f(x) = x - 1$

We can define a binary relation  $\mathbf{x} \succeq \mathbf{x}'$  iff  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  and  $f(\mathbf{x}) \geq 0$

# Nested r.f.

## Definition (Nested Ranking Function)

A tuple  $\langle f_1, \dots, f_d \rangle$  is a nested ranking function for  $T$  if the following requirements are satisfied for all  $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

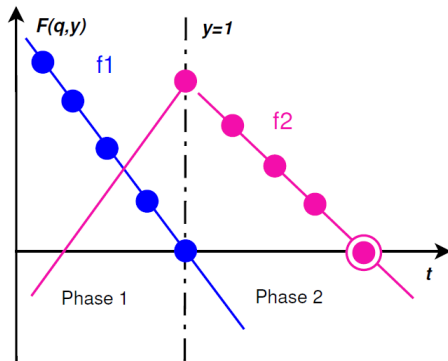
Let  $f_0 = 0$ .

## Example: Nested r.f.

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

while  $(q > 0)$  do  $q' = q - y, y' = y + 1$

- Above loop has Nested r.f.  $\langle 1 - y, q + 1 \rangle$



## Example: Multiphase Ranking Function

**while**  $(x > -z)$  **do**  $x' = x + y, y' = y + z, z = z - 1$

Attempt to use a ranking function that has several phases:

$\langle z + 1, y + 1, x \rangle$

| $x$ | $y$ | $z$ | $z + 1$  | $y + 1$  | $x$      |
|-----|-----|-----|----------|----------|----------|
| 1   | 1   | 1   | <b>2</b> | 2        | 1        |
| 2   | 2   | 0   | <b>1</b> | 3        | 2        |
| 4   | 2   | -1  | <b>0</b> | 3        | 4        |
| 6   | 1   | -2  | -1       | <b>2</b> | 6        |
| 7   | -1  | -3  | -2       | <b>0</b> | 7        |
| 6   | -4  | -4  | -3       | -3       | <b>6</b> |
| 2   | -8  | -5  | -4       | -7       | <b>2</b> |
| -6  | -13 | -6  | -5       | -12      | -6       |

# Multiphase Ranking Function

## Definition

Given a set of transitions  $T \subseteq \mathbb{Q}^{2n}$ , we say  $\langle f_1, \dots, f_d \rangle$  is a multiphase ranking function for  $T$  if for every  $\mathbf{x}'' \in T$ , there is an index  $i \in [1, d]$ , s.t.

$$\begin{aligned} \forall j \leq i . \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i . f_j(\mathbf{x}) &\leq 0. \end{aligned}$$

We say that  $\mathbf{x}''$  is ranked by  $f_i$  (for the minimal).

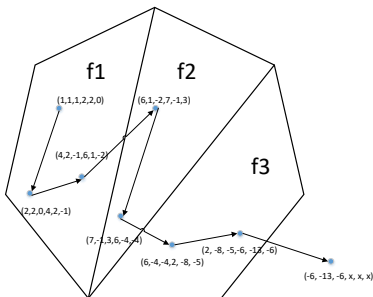
# Example Revisit

**while**  $(x > -z)$  **do**  $x' = x + y, y' = y + z, z = z - 1$

$\langle z + 1, y + 1, x \rangle$

$\mathbf{x}''$  is ranked by  $f_k$  when  $i = k$ .

$$\begin{aligned} \forall j \leq i. \Delta f_j(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \\ \forall j < i. f_j(\mathbf{x}) &\leq 0. \end{aligned}$$





# Motzkin's Transposition Theorem

## Theorem (Motzkin's Transposition Theorem)

*For  $A \in \mathbb{K}^{m \times n}$ ,  $C \in \mathbb{K}^{l \times n}$ ,  $b \in \mathbb{K}^m$ , and  $d \in \mathbb{K}^l$ . The formulae below are equivalent.*

- ▶  $\forall x \in \mathbb{K}^n. \neg(Ax \leq b \wedge Cx < d)$
- ▶  $\exists \lambda \in \mathbb{K}^m. \exists \mu \in \mathbb{K}^l.$   
 $\lambda \geq 0 \wedge \mu \geq 0$   
 $\wedge \lambda^T A + \mu^T C = 0 \wedge \lambda^T b + \mu^T d \leq 0$   
 $\wedge (\lambda^T b < 0 \vee \mu \neq 0)$

Intuition of Motzkin's transposition theorem:...

## Lemma (0)

*Given an non-empty polyhedron  $\mathcal{P}$  and linear functions  $f_1, \dots, f_k$  such that*

1.  $\mathbf{x} \in \mathcal{P} \rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee f_k(\mathbf{x}) \geq 0$
2.  $\mathbf{x} \in \mathcal{P} \not\rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0$

*There exists a non-negative constants  $\mu_1, \dots, \mu_{k-1}$  such that*

$$\mathbf{x} \in \mathcal{P} \rightarrow \mu_1 f_1(\mathbf{x}) + \mu_{k-1} f_{k-1}(\mathbf{x}) + f_k(\mathbf{x}) \geq 0$$

## Proof of Lemma (0)

Let  $\mathcal{P}$  be  $B\mathbf{x} \leq c$ ,  $f_i = \vec{a}_i\mathbf{x} - b_i$ . Then (i) is equivalent to the infeasibility of  $B\mathbf{x} \leq c \wedge A\mathbf{x} < b$ , by the theorem we have  $\vec{\lambda}, \vec{\mu}$  s.t.

$$\vec{\lambda}B + \vec{\mu}A = 0 \wedge \vec{\lambda}c + \vec{\mu}b \leq 0 \quad \wedge (\vec{\mu} \neq 0 \vee \vec{\lambda}c + \vec{\mu}b < 0)$$

Then for all  $\mathbf{x} \in \mathcal{P}$

$$\sum_i \mu_i f_i(\mathbf{x}) = \vec{\mu}A\mathbf{x} - \vec{\mu}b = -\vec{\lambda}B\mathbf{x} - \vec{\mu}b \geq -\vec{\lambda}c - \vec{\mu}b \geq 0$$

# MΦRF to Nested r.f.

## Theorem (1)

*If  $\mathcal{Q}$  has a MΦRF of depth  $d$ , then it has a nested ranking function of depth at most  $d$ .*

## Proof.

By induction on the depth  $d$ .

- ▶  $d = 1$ : MΦRF and nested r.f. are both LRF.
- ▶  $d > 1$ :  $d = 2$  e.g.  $\langle f_1, f_2 \rangle$ . When index  $i = 1$ , we do not impose bound on  $f_2(\mathbf{x})$ . However, a bound is needed for  $f'_2(\mathbf{x}s)$  in nested r.f.  $\langle f'_1, f'_2 \rangle$ .

To solve the problem that  $f_2(\mathbf{x})$  might goes under 0, when  $\mathbf{x}''$  is ranked by  $f_1$ . Consider  $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$

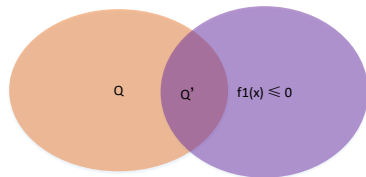


# MΦRF to Nested r.f.

## Lemma (1)

Let  $\tau = \langle f_1, \dots, f_d \rangle$  be an irredundant MΦRF for  $\mathcal{Q}$ , such that  $\langle f_2, \dots, f_d \rangle$  is a nested ranking function for  $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$ . Then there is a nested ranking function of depth  $d$  for  $\mathcal{Q}$ .

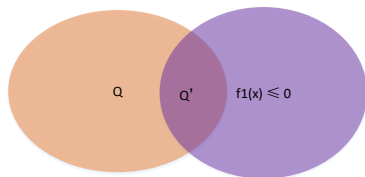
Prove by construction: construct a nested r.f.  $\langle f'_1, \dots, f'_d \rangle$



$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

If  $f_d$  is non-negative on  $\mathcal{Q}$ , then  $f'_d = f_d$ . Otherwise,  
 $\mathbf{x}'' \in \mathcal{Q} \rightarrow f_d(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) > 0$

## MΦRF to Nested r.f.



$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \quad \text{for all } i = 1, \dots, d. \end{aligned}$$

Assume  $f'_n(\mathbf{x}) = f_n(\mathbf{x}) + \mu_n f_1(\mathbf{x})$  and  $f'_d, \dots, f'_i$  has already been computed.

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

If above inequation also holds for  $\mathcal{Q}$ , then  $f'_{i-1} = f_{i-1}$ , Otherwise

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) \geq 0$$

$BM\Phi RF(\mathbb{Q}) \in PTIME$

Theorem (2)

$BM\Phi RF(\mathbb{Q}) \in PTIME.$

Proof.



# LLRF

Intuition: remind binary relation  $\mathbf{x} \succeq \mathbf{x}'$  iff  $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$  and  $f(\mathbf{x}) \geq 0$ .

Generalize it into several phases using lexicographical order of ranking functions.

$$\langle f_1, f_2, \dots, f_d \rangle$$

$$(2, 3, 1, 3) \geq (2, 1, 5, 4)$$

## Definition (LLRF)

Given a set of transitions  $T$  we say that  $\langle f_1, f_2, \dots, f_d \rangle$  is a LLRF (of depth  $d$ ) for  $T$  if for every  $\mathbf{x}'' \in T$  there is an index  $i$  such that

$$\begin{aligned} \forall j < i . \Delta f_j(\mathbf{x}'') &\geq 0, \\ \Delta f_i(\mathbf{x}'') &\geq 1, \\ f_i(\mathbf{x}) &\geq 0, \end{aligned}$$

A LLRF is weak if..



# Weak LLRF to $M\Phi$ RF

## Theorem (3)

*If  $\mathcal{Q}$  has a weak LLRF of depth  $d$ , it has a  $M\Phi$ RF of depth  $d$ .*

## Proof.

Prove by induction.

- ▶  $d = 1$ : For LLRF:  $\Delta f_1(\mathbf{x}'') > 0$ ,  $f_1(\mathbf{x}) \geq 0$  is a LRF due to the loop is linear.  
For  $M\Phi$ RF: is a LRF.
- ▶  $d > 1$ : Observe that for a given LLRF  $\langle f_1, f_2, \dots, f_d \rangle$ , after removing  $f_k$ ,  $\langle f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_d \rangle$  is also a LLRF.  
If we apply IH here, we get a  $M\Phi$ RF of depth  $d - 1$ .



## Weak LLRF to MΦRF

Now we want some techniques to use a MΦRF of depth  $d - 1$  and  $f_k$  we removed to prove there is a MΦRF of depth  $d$  on  $\mathcal{Q}$ .

### Lemma (2)

*Let  $f$  be a non-negative linear function over  $\mathcal{Q}$ . If  $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \mid \Delta f(\mathbf{x}'') \leq 0\}$  has a MΦRF of depth  $d$ , then  $\mathcal{Q}$  has a MΦRF of depth at most  $d + 1$ .*

### Proof.

Prove by construction: if the known MΦRF is  $\langle g_1, \dots, g_d \rangle$  and the function non-negative function is  $f$ , we wish to construct a MΦRF  $\langle g'_1, \dots, g'_n, f \rangle$  of depth  $d + 1$

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow \Delta f(\mathbf{x}'') > 0 \vee \Delta g_1(\mathbf{x}'') \geq 1$$

Then update  $g_2$  to  $g'_2$  when  $g'_1(\mathbf{x}) \leq -1$ , and so on...



## Weak LLRF to MΦRF

Remind the  $f_k$  we removed in the theorem, together with Lemma(2), we wish to construct a non-negative linear function  $g$  over  $\mathcal{Q}$  and  $g$  decrease on (at least) the same transitions of  $f_k$ .

### Lemma (3)

*Let  $\langle f_1, \dots, f_d \rangle$  be a weak LLRF for  $\mathcal{Q}$ . There is a linear function  $g$  that is positive over  $\mathcal{Q}$ , and decreasing on (at least) the same transitions of  $f_i$ , for some  $i \in [1, d]$ .*

### Proof.

Use Lemma(0) to find the  $i$ . □

# MΦRF and LLRFs over the Integers

- ▶ Actual programs with `int`.
- ▶ More important, conclusions for rational does not always applicable in on integer version.

## Example

`while ( $x_2 - x_1 \leq 0, x_1 + x_2 \geq 1$ ) do  $x'_2 = x_2 - 2x_1 + 1, x'_1 = x_1$`

For rationals:  $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}$

For integers: there exists a linear ranking function

$$f(x_1, x_2) = x_1 + x_2$$

# MΦRF and LLRFs over the Integers

- ▶ Integer case for LRF: completeness for the integer version was achieved by reducing the problem to the rational case. Intuitively, since  $Q_I$  is the convex combination of points in  $I(Q)$ .
- ▶ Theorems above does not apply to the integer versions, but in the following we will prove that the reduction from integer case to rational also works for LLRF and MΦRF.

# Weak LLRF: Integer to Rational

## Theorem (4)

*Let  $\langle f_1, \dots, f_d \rangle$  be a weak LLRF for  $I(\mathcal{Q})$ . Then there are constants  $c_1, \dots, c_d$  such that  $\langle f_1 + c_1, \dots, f_d + c_d \rangle$  is a weak LLRF for  $\mathcal{Q}_I$  (over the rationals).*

## Proof.

prove by induction:

- ▶  $d = 1$ , LRF.
- ▶  $d > 1$ , define

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$

$$\mathcal{Q}'' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \Delta f_1(\mathbf{x}'') = 0\}$$

by IH, the theorem holds on  $\mathcal{Q}'_I$  and  $\mathcal{Q}''_I$  for weak LLRF of depth  $d - 1$ . say,

$$\langle f_2 + c'_2, \dots, f_d + c'_d \rangle, \langle f_2 + c''_2, \dots, f_d + c''_d \rangle$$

## Proof Continue

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$

$$\mathcal{Q}'' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \Delta f_1(\mathbf{x}'') = 0\}$$

Then we wish to have a lower bound on  $f_1(\mathbf{x})$ .

$$\langle f_1 + c_1, f_2 + \max(c'_2, c''_2), \dots, f_d + \max(c'_d, c''_d) \rangle$$

This implies the above formula is a weak LLRF on  $\mathcal{Q}_I$ . i.e. given a  $\mathbf{x}'' \in \mathcal{Q}_I$ , either..., or...

Problem: how to prove the existence of the lower bound?

# Prove the Lower Bound

$\mathcal{Q}'_I, \mathcal{Q}''_I$ .

- ▶ If  $\mathcal{Q}'_I$  is empty, then by the definition of  $\mathcal{Q}'$   $f_1$  is lower bounded.
- ▶ Otherwise, prove the lower bound on  $\mathcal{Q}_I \setminus \mathcal{Q}'_I$

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

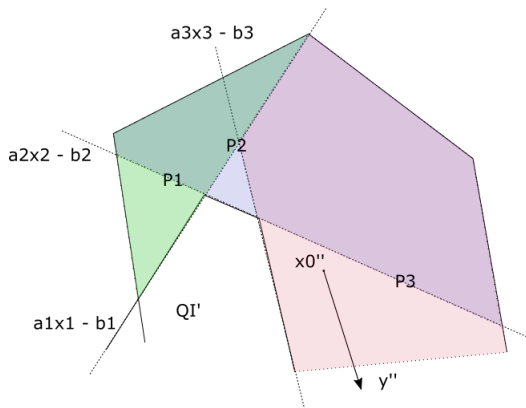
$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

For  $i \in [1, m]$ . Then clearly  $\mathcal{Q}_I \setminus \mathcal{Q}'_I \subseteq \bigcup_{i=1}^m \mathcal{P}_i$ , by construction all the integer points in  $\mathcal{P}_i$  are also in  $\mathcal{Q}_I \setminus \mathcal{Q}'_I$ .  
Proof target: for every  $i$ ,  $f_1$  is lower bounded in  $\mathcal{P}_i$  for every  $i$ .  
Fix  $i$  for the following arguments, s.t.  $\mathcal{P}_i$  is not empty.



# Intuition: Proof of the Lower Bound

$$Q' = Q_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$



# Prove the Lower Bound

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

Assume (prove by contradiction)  $\mathcal{P}_i$  does not lower bound  $f_1$ . Let  $\mathbf{x}''_0 \in \mathcal{P}_i$ .

$$f_1(\mathbf{x}) = \vec{\lambda} \cdot \mathbf{x} + \lambda_0$$

$$\mathcal{P}_i = \mathcal{O} + \mathcal{C}$$

There must be a vector  $\mathbf{y}'' \in \mathcal{C}$  s.t.  $\vec{\lambda} \cdot \mathbf{y} < 0$

# Prove the Lower Bound

$$\mathcal{Q}' = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid f_1(\mathbf{x}) \leq -1\}$$

$$\mathcal{Q}'_I = \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \leq b_i, i \in [1, m]\}$$

$$\mathcal{P}_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' > b_i\}$$

$$\mathcal{P}'_i = \mathcal{Q}_I \cap \{\mathbf{x}'' \mid \vec{a}_i \cdot \mathbf{x}'' \geq b_i\}$$

$\mathbf{x}''_0 + k\mathbf{y}''$  is in  $\mathcal{P}'_i$ , the set  $S = \{\mathbf{x}''_0 + k\mathbf{y}'' \mid k \in \mathbb{Z}_+\}$  is contained in  $\mathcal{P}_i$ .

Integer points of  $\mathcal{P}_i$  are all in  $\mathcal{Q}_I \setminus \mathcal{Q}'_I$ .

Contradiction.

Hence,  $f_1$  is bounded.

# The Depth of a MΦRF

Idea: pre-compute the depth  $d$  for MΦRF synthesis.

## Theorem (5)

*For integer  $B > 0$ , the following loop  $\mathcal{Q}_B$*

*while  $(x \geq 1, y \geq 1, x \geq y, 2^B y \geq x)$  do  $x' = 2x, y' = 3y$*

*needs at least  $B + 1$  components in any MΦRF.*

## Proof.

Define  $R_I = \{(2^i c, c, 2^{i+1} c, 3c) \mid c \geq 1\}$  and note that for  $i \in [0, B]$ , we have  $R_i \in \mathcal{Q}_B$ .

Assume the loop has a MΦRF with depth  $B$ , then it is obvious that there are  $R_i$  and  $R_j, i \neq j$  that are ranked by the same phase  $f_k$ , w.l.o.g., assume  $j > i$  and  $f_k(x, y) = a_1 x + a_2 y + a_0$ , we have



## Proof of Theorem (5)

$$j > i \text{ and } f_k(x, y) = a_1x + a_2y + a_0$$

$$f_k(2^i, 1) - f_k(2^{i+1}, 3) = -a_12^i - a_22 > 0$$

$$f_k(2^j, 1) - f_k(2^{j+1}, 3) = -a_12^j - a_22 > 0$$

$$f_k(2^i, 1) - f_k(0, 0) = a_12^i + a_2 \geq 0$$

$$f_k(2^j, 1) - f_k(0, 0) = a_12^j + a_2 \geq 0$$

# Iteration Bounds from MΦRFs

## Example

**while**  $(x \geq 0)$ **do**  $x' = x + y, y' = y - 1$

MΦRF:  $\langle y + 1, x \rangle$

When start from  $x = x_0$  and  $y = y_0 \dots$

$$x_0 + \frac{y_0(y_0 + 1)}{2} - 1$$

# Iteration Bounds from MΦRFs

Overview: Given a SLC loop and a corresponding MΦRF

$\tau = \langle f_1, \dots, f_d \rangle$ .

- ▶  $F_k(t)$ : the value of  $f_k$  after iteration  $t$ .
- ▶  $UB_k(t)$ : bound for  $f_k$ . For  $t > T_k$ ,  $UB_k(T_k)$  becomes negative.
- ▶  $T_k$ : an upper bound on the time in which the  $k$ -th phase ends.
- ▶ The whole loop must terminate before  $\max_k T_k$  iterations.

$\mathbf{x}_t$  be the state after iteration  $t$ . Define  $F_k(t) = f_k(\mathbf{x}_t)$ . Let  $M = \max(f_1(\mathbf{x}_0), \dots, f_d(\mathbf{x}_0))$

# Iteration Bounds from MΦRF

## Lemma (4)

*For all  $k \in [1, d]$ , there are  $\mu_1, \dots, \mu_{k-2} \geq 0$  and  $\mu_{k-1} > 0$  such that  $\mathbf{x}'' \in \mathcal{Q} \rightarrow \mu_1 f_1(\mathbf{x}) + \dots + \mu_{k-1} f_{k-1}(\mathbf{x}) + (\Delta f_k(\mathbf{x}'') - 1) \geq 0$ .*

**Proof.**

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow f_1(\mathbf{x}) \geq 0 \vee \dots \vee f_{k-1}(\mathbf{x}) \geq 0 \vee \Delta f_k(\mathbf{x}'') \geq 1.$$



## Lemma (5)

*For all  $k \in [1, d]$ , there are constants  $c_k, d_k > 0$  such that  $F_k(t) \leq c_k M t^{k-1} - d_k t^k$ , for all  $t \geq 1$ .*

[6] Proof Idea: Use the bound for  $-\Delta f_k(\mathbf{x}_i'')$  to bound  $F_k(t)$ .



# Proof of Lemma (6)

$$\begin{aligned} F_k(t) &= f_k(\mathbf{x}_0) + \sum_{i=0}^{t-1} (f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)) \\ &< M + \sum_{i=0}^{t-1} (\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} (\mu_1 F_1(i) + \cdots + \mu_{k-1} F_{k-1}(i)) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} \sum_{j=1}^{k-1} (\mu_j c_j M i^{j-1} - \mu_j d_j i^j) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} ((\sum_{j=1}^{k-1} \mu_j c_j M i^{j-1}) - \mu_{k-1} d_{k-1} i^{k-1}) \\ &\leq M(1 + \mu) + \sum_{i=1}^{t-1} (M(\sum_{j=1}^{k-1} \mu_j c_j) i^{k-2} - \mu_{k-1} d_{k-1} i^{k-1}) \\ &= M(1 + \mu) + M(\sum_{j=1}^{k-1} \mu_j c_j) (\sum_{i=1}^{t-1} i^{k-2}) - \mu_{k-1} d_{k-1} \sum_{i=1}^{t-1} i^{k-1} \\ &\leq M(1 + \mu) + M(\sum_{j=1}^{k-1} \mu_j c_j) \left( \frac{t^{k-1}}{k-1} \right) - \mu_{k-1} d_{k-1} \left( \frac{t^k}{k} - t^{k-1} \right) \\ &= c_k M t^{k-1} - d_k t^k \end{aligned}$$

where  $\mu_1 f_1(\mathbf{x}) + \cdots + \mu_{k-1} f_{k-1}(\mathbf{x}) \geq \Delta f_k(\mathbf{x}'') = f_k(\mathbf{x}_{i+1}) - f_k(\mathbf{x}_i)$

## Theorem (6)

*An SLC loop that has a  $M\Phi RF$  terminates in a number of iterations bounded by  $O(\|\mathbf{x}_0\|_\infty)$*

### Proof.

$F_k(t) \leq c_k M t^{k-1} - d_k t^k$ . For  $t > \max\{1, (c_k/d_k)M\}$ , we have  $F_k(t) < 0$ .

Thus, the loop terminates by the time  $\max\{1, (c_i/d_i)M, \dots, (c_k/d_k)M\}$  where  $M = \max(f_1(\mathbf{x}_0), \dots, f_k(\mathbf{x}_0))$ .

