On Multiphase-Linear Ranking Functions

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Motzkin's Transposition Theorem

Theorem (Motzkin's Transposition Theorem)

For $A \in \mathbb{K}^{m\times n}, C \in \mathbb{K}^{l\times n}, b \in \mathbb{K}^m$, *and* $d \in \mathbb{K}^l$. *The formulae below are equivalent.*

- $\forall x \in \mathbb{K}^n . \neg(Ax \leq b \land Cx < d)$
- $\exists \lambda \in \mathbb{K}^m . \exists \mu \in \mathbb{K}^l .$
  $\lambda \geq 0 \land \mu \geq 0$
  $\land \lambda^T A + \mu^T C = 0 \land \lambda^T b + \mu^T d \leq 0$
  $\land (\lambda^T b < 0 \lor \mu \neq 0)$

Intuition of Motzkin's transposition theorem:...

Lemma (0)

Given an non-empty polyhedron $\mathcal{P}$ *and linear functions* $f_1, \ldots, f_k$ *such that*

1. $\mathbf{x} \in \mathcal{P} \rightarrow f_1(\mathbf{x}) > 0 \vee \ldots \wedge f_{k-1}(\mathbf{x}) > 0 \vee f_k(\mathbf{x}) \geq 0$
2. $\mathbf{x} \in \mathcal{P} \not\rightarrow f_1(\mathbf{x}) > 0 \vee \ldots f_{k-1}(\mathbf{x}) > 0$

There exists a non-negative constants $\mu_1, \ldots, \mu_{k-1}$ *such that*

$$\mathbf{x} \in \mathcal{P} \rightarrow \mu_1 f_1(\mathbf{x}) + \mu_{k-1} f_{k-1}(\mathbf{x}) + f_k(\mathbf{x}) \geq 0$$

LRF, Nested r.f. and MΦRF

MΦRF to Nested r.f.

Theorem (1)

If $\mathcal{Q}$ has a MΦRF of depth d, then it has a nested ranking function of depth at most d.

Proof.

By induction on the depth d.

- $d = 1$: MΦRF and nested r.f. are both LRF.
- $d > 1$: $d = 2$ e.g. $\langle f_1, f_2 \rangle$. When index $i = 1$, we do not impose bound on $f_2(\mathbf{x})$. However, a bound is needed for $f_2'(\mathbf{x}s)$ in nested r.f. $\langle f_1', f_2' \rangle$.
  To solve the problem that $f_2(\mathbf{x})$ might goes under 0, when $\mathbf{x}''$ is ranked by f_1. Consider $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}'') \leq 0\}$

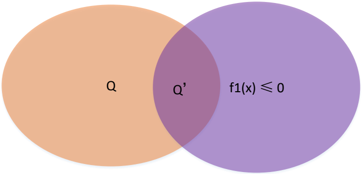
□

MΦRF to Nested r.f.

Lemma (1)

Let $\tau = \langle f_1, \ldots, f_d \rangle$ be an irredundant MΦRF for $\mathcal{Q}$, such that $\langle f_2, \ldots, f_d \rangle$ is a nested ranking function for $\mathcal{Q}' = \mathcal{Q} \cap \{\mathbf{x}'' \in \mathbb{Q}^{2n} \mid f_1(\mathbf{x}) \leq 0\}$. Then there is a nested ranking function of depth d for $\mathcal{Q}$.

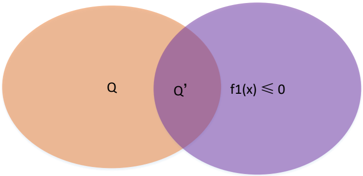
Prove by construction: construct a nested r.f. $\langle f'_1, \ldots, f'_d \rangle$



$$
\begin{aligned}
f_d(\mathbf{x}) &\geq 0 \\
(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d.
\end{aligned}
$$

If f_d is non-negative on $\mathcal{Q}$, then $f'_d = f_d$. Otherwise,
$\mathbf{x}'' \in \mathcal{Q} \to f_d(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) > 0$

MΦRF to Nested r.f.



$$f_d(\mathbf{x}) \geq 0$$
$$(\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \qquad \text{for all } i = 1, \dots, d.$$

Assume $f'_n(\mathbf{x}) = f_n(\mathbf{x}) + \mu_n f_1(\mathbf{x})$ and $f'_d, \dots, f'_i$ has already been computed.

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

$$\mathbf{x}'' \in \mathcal{Q}' \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0$$

If above inequation also holds for $\mathcal{Q}$, then $f'_{i-1} = f_{i-1}$, Otherwise

$$\mathbf{x}'' \in \mathcal{Q} \rightarrow (\Delta f'_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) \geq 0 \vee f_1(\mathbf{x}) \geq 0$$

BMΦRF($\mathbb{Q}$) $\in$PTIME

Theorem (2)

BMΦRF($\mathbb{Q}$)$\in$PTIME.

Proof.

$\square$

LLRF

Weak LLRF to MΦRF

Theorem (3)

If $\mathcal{Q}$ has a weak LLRF of depth d, it has a MΦRF of depth d.

Proof.

Prove by induction.

- $d = 1$: For LLRF: $\Delta f_1(\mathbf{x}'') > 0$, $f_1(\mathbf{x}) \geq 0$ is a LRF due to the loop is linear.
  For MΦRF: is a LRF.
- $d > 1$: Observe that for a given LLRF $\langle f_1, f_2, \ldots, f_d \rangle$, after removing f_k, $\langle f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_d \rangle$ is also a LLRF.
  If we apply IH here, we get a MΦRF of depth $d - 1$

□