On Multiphase-Linear Ranking Functions

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Contributions

- Equivalence of different classes of ranking function.
- Algorithms for converting between ranking functions.
- Complete solution for ranking functions on integer.
- Depth bound and iteration bound for MΦRF.

Single Path Linear Constraint Loop

Example

$$\texttt{while}\ (x \geq -z)\ \texttt{do}\ x' = x + y,\ y' = y + z,\ z' = z - 1$$

$$\texttt{while}\ (x_2 - x_1 \leq 0,\ x_1 + x_2 \geq 1)\ \texttt{do}\ x_2' = x_2 - 2x_1 + 1,\ x_1' = x_1$$

Definition (SLC)

$$\mathit{while}\ (B\mathbf{x} \leq \mathbf{b})\ \mathit{do}\ A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix} \qquad \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

Ranking Functions

Definition (Linear Ranking Function(LRF))

$f(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n + a_0$, such that

- $f(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ satisfies the loop constraints.
- $f(\mathbf{x}) - f(\mathbf{x}') \geq 1$ for any transition from $\mathbf{x}$ to $\mathbf{x}'$.

Example

$$\texttt{while } (x - 1 > 0)\texttt{do } x' = x - 1$$

Its LRF: $f(x) = x - 1$

Example: Multiphase Ranking Function

Problem: LRF is not strong enough for all loops.

Example

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$f(x, y, z) = a_1x + a_2y + a_3z + b$
y cannot be used for non-existence of its lower bound.
$f(x, y, z) = x + z$
Problem?

Example: Multiphase Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Attempt to use a ranking function that has several phases:
$\langle z + 1, y + 1, x \rangle$

x	y	z	$z+1$	$y+1$	x
1	1	1	2	2	1
2	2	0	1	3	2
4	2	-1	0	3	4
6	1	-2	-1	2	6
7	-1	-3	-2	0	7
6	-4	-4	-3	-3	6
2	-8	-5	-4	-7	2
-6	-13	-6	-5	-12	-6

Multiphase Ranking Function

Definition

Given a set of transitions $T \subseteq \mathbb{Q}^{2n}$, we say $\langle f_1, \ldots, f_d \rangle$ is a multiphase ranking function for T if for every $\mathbf{x}'' \in T$, there is an index $i \in [1, d]$, s.t.

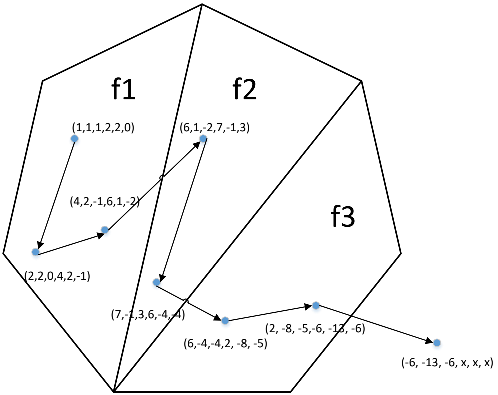
$$
\begin{aligned}
\forall j \leq i \ . \ \Delta f_j(\mathbf{x}'') &\geq 1 \,, \\
f_i(\mathbf{x}) &\geq 0 \,, \\
\forall j < i \ . \quad f_j(\mathbf{x}) &\leq 0 \,.
\end{aligned}
$$

We say that $\mathbf{x}''$ is ranked by f_i(for the minimal).

Example Revisit

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

$$\begin{aligned} \forall j \le i \,.\, \Delta f_j(\mathbf{x}'') &\ge 1\,, \\ f_i(\mathbf{x}) &\ge 0\,, \\ \forall j < i \,.\quad f_j(\mathbf{x}) &\le 0\,. \end{aligned}$$



Nested Ranking Function

$$\texttt{while } (x > -z)\texttt{do } x' = x + y, y' = y + z, z = z - 1$$

Loop condition: $x + z > 0$. We only want to use this constraint for the ranking function.
$\langle z + 1, y + 1, x + z \rangle$

Definition (Nested Ranking Function)

A tuple $\langle f_1, \ldots, f_d \rangle$ is a nested ranking function for T if the following requirements are satisfied for all $\mathbf{x}'' \in T$

$$\begin{aligned} f_d(\mathbf{x}) &\geq 0 \\ (\Delta f_i(\mathbf{x}'') - 1) + f_{i-1}(\mathbf{x}) &\geq 0 \qquad \text{for all } i = 1, \ldots, d. \end{aligned}$$

Let $f_0 = 0$.