Project 1: FVM for 2D advection-diffusion equation

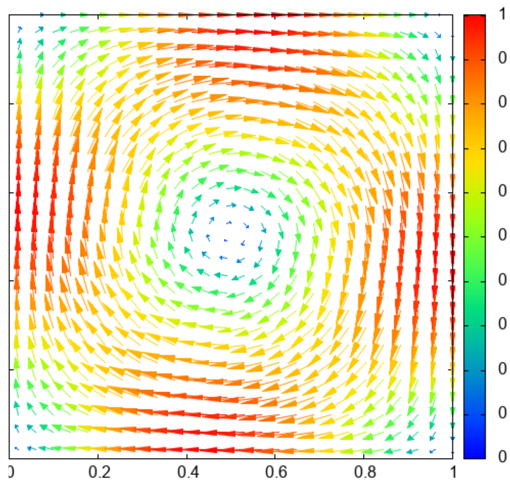
- Let's extend to the AD equation to 2D, and we use T to represent the unknown variable, and $\mathbf{u}$ to represents the velocity vector.

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nu \nabla^2 T$$

- To keep the equation linear, the velocity is independent of T and here we employ the non-decaying Taylor-Green vortex

$$u = -\sin \pi x \cos \pi y, \quad v = \cos \pi x \sin \pi y$$

For a square domain $0 < x < 1$ and $0 < y < 1$, the flow is a clockwise vortex occupying the whole domain.



- The Taylor-Green vortex is an exact solution of the NS equation and it satisfy the divergence free condition $\nabla \cdot \mathbf{u} = 0$, therefore, we can rewrite the equation in conservative form as

$$\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{u}T - \nu \nabla T) = 0$$

or

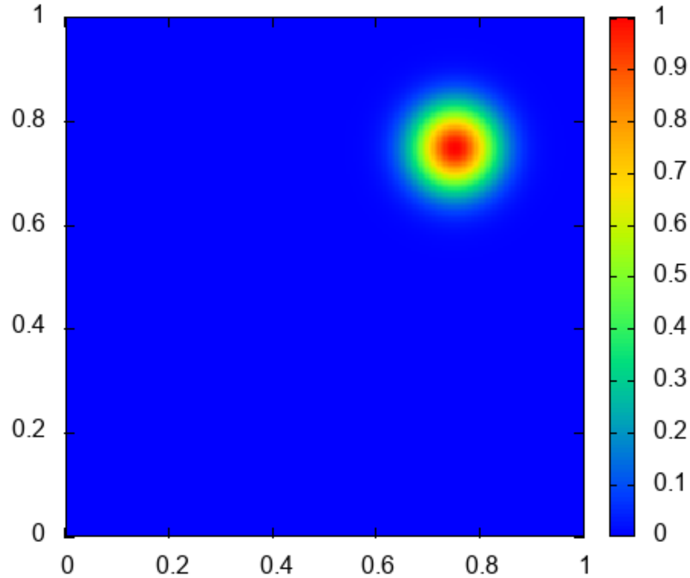
$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

where $U = T$ and $\mathbf{F} = \mathbf{u}T - \nu \nabla T$.

- The unsteady AD equation requires initial condition for T, here as an demonstration example, we use

$$T(x, y, t = 0) = \exp\left[-\left(\frac{x - 0.75}{0.025\pi}\right)^2 - \left(\frac{y - 0.75}{0.025\pi}\right)^2\right]$$

which give non-zero T in a approximate circular region at $(0.75, 0.75)$.



- One can view the variable T is a passive tracer in the flow and here we simply give the initial distribution of the tracer, like we initially "color" the fluid by some means and then see how does the color evolves in time.
- The advection mechanism should simply convect the fluid with non-zero T around and one can see the deformation of the non-zero T region.
- Simultaneously, the diffusion mechanism will further smoothen the difference between the non-zero and zero regions and "mix" the fluids with different T together.
- With finite volume method, we solve the 2D AD Eqn in integral form

$$\int_V \frac{\partial U}{\partial t} dV + \int_V \nabla \cdot \mathbf{F} dV = 0$$

$$\Delta V \frac{d\overline{U}}{dt} + \int_S \mathbf{F} \cdot \mathbf{n} dS = 0$$

- For a given cell volume i, j, where i and j denotes the x and y directions, respectively.

$$\Delta x \Delta y \left(\frac{d\overline{U}}{dt} \right)_{i,j} + (F^n_{i+1}\Delta y - F^n_i \Delta y) + (G^n_{j+1}\Delta x - G^n_j \Delta x) = 0$$

where $F = Tu - \nu\frac{\partial T}{\partial x}$ and $G = Tv - \nu\frac{\partial T}{\partial y}$ are the x- and y- components of the flux.

- If we apply the forward Euler for time stepping,

$$\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} + \frac{F^n_{i+1} - F^n_i}{\Delta x} + \frac{G^n_{j+1} - G^n_j}{\Delta y} = 0,$$

Project Tasks

1. For the project, we are going to try to solve the equation with $\nu = 0$ and 1.
2. Please discretize the domain into a Cartesian mesh, such as 128×128.
3. We will solve the equation from $t = 0$ to 10. Please determine the time step and justify your choice.
4. Please use central and upwind schemes to calculate the flux.
5. Plot the time snapshots of T and comment on the effects of ν and flux calculation schemes.