Altering the Trefoil Knot

Spencer Shortt
Georgia College

December 19, 2018

Abstract

A mathematical knot K is defined to be a topological imbedding of the circle into the 3-dimensional Euclidean space. Conceptually, a knot can be pictured as knotted shoe lace with both ends glued together. Two knots are said to be equivalent if they can be continuously deformed into each other. Different knots have been tabulated throughout history, and there are many techniques used to show if two knots are equivalent or not. The knot group is defined to be the fundamental group of the knot complement in the 3-dimensional Euclidean space. It is known that equivalent knots have isomorphic knot groups, although the converse is not necessarily true. This research investigates how piercing the space with a line changes the trefoil knot group based on different positions of the line with respect to the knot. This study draws comparisons between the fundamental groups of the altered knot complement space and the complement of the trefoil knot linked with the unknot.

Contents

1 Introduction to Concepts in Knot Theory

1.1 What is a Knot?

In the context of this work, a **knot** can be pictured as a rope or string that has been intertwined and tangled with its ends glued together. Some examples are shown below in Figure 1 (the first one being the unknot, the second being the trefoil knot):

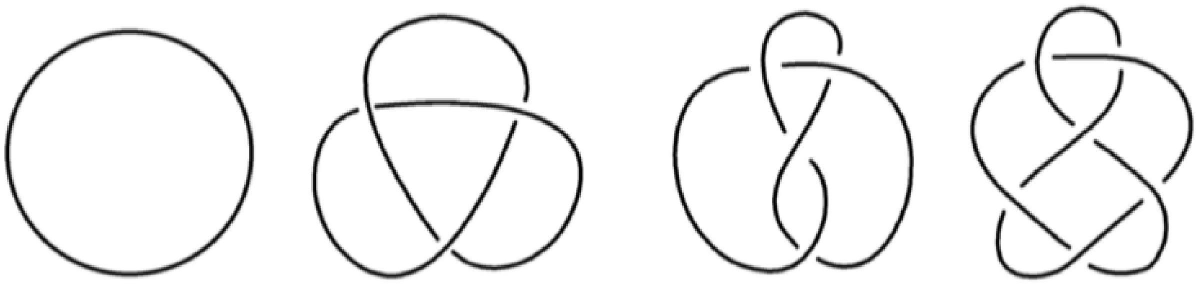

Figure 1:

The "string" that makes up the knot is thought to have infinitesimally small thickness. Another property of a knot is that any part of the "string" cannot intersect with any other part of the string. Analogously, a person in the real world cannot untie their shoe laces by pulling them through each other because they are physical objects. Two knots are said to be **equivalent** if one can be tangled or untangled into the other without ever having to intersect the components of the string. An example of two unknots can be seen in Figure 2.

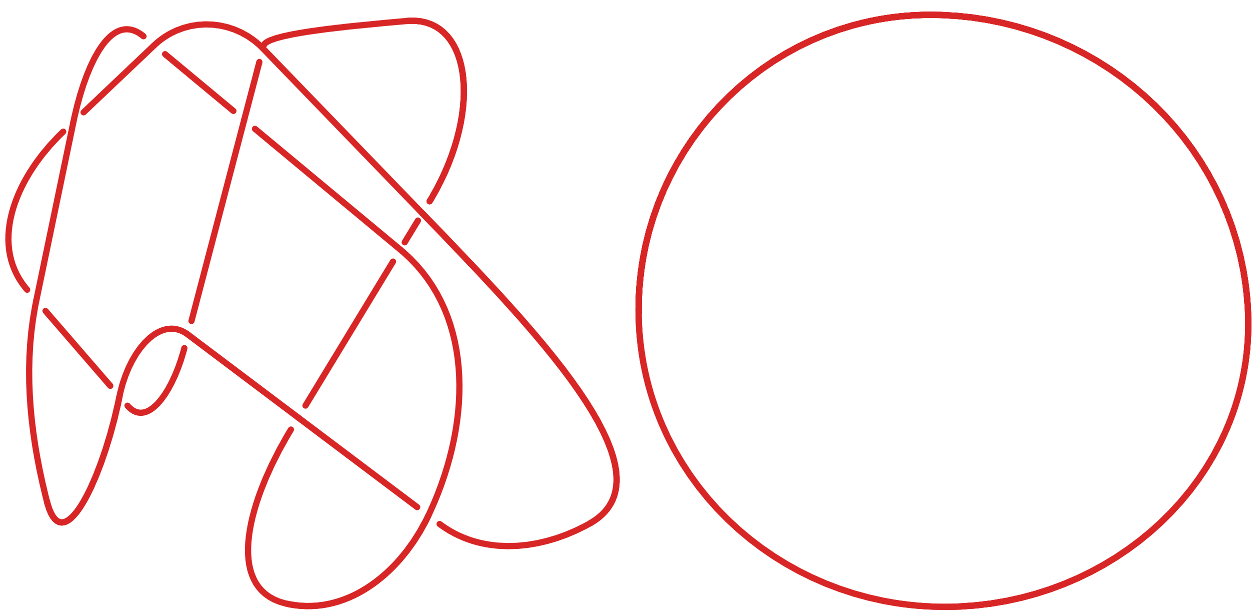

Figure 2:

1.2 Rolfsen Knot Tables

The Rolfsen Knot Tables (2009) are a tabulation of distinct knots (shown in Figure 3).

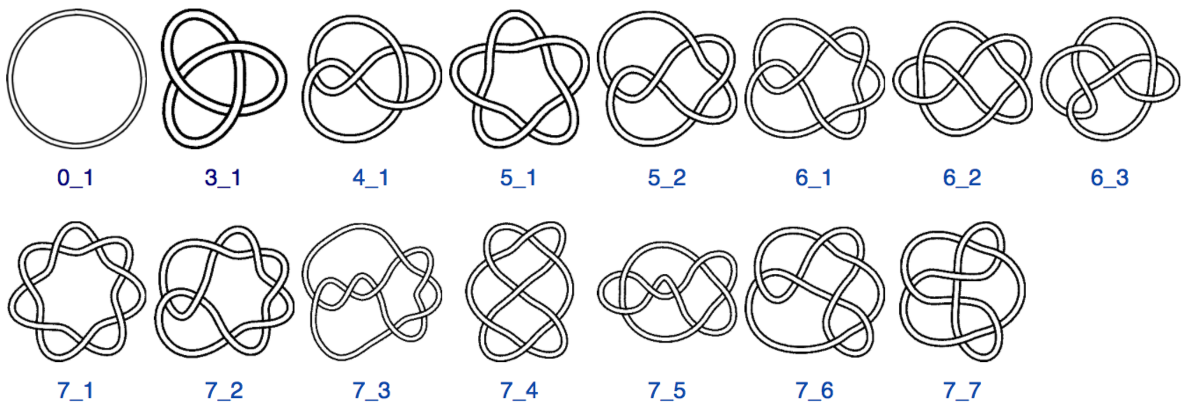

Figure 3:

The first number below each knot indicates the amount of crossings that knot contains, and the second number indicates which position it has in the Rolfsen Table. The tables for knots with 8 and 9 crossings are shown below in Figures 4 and 5:

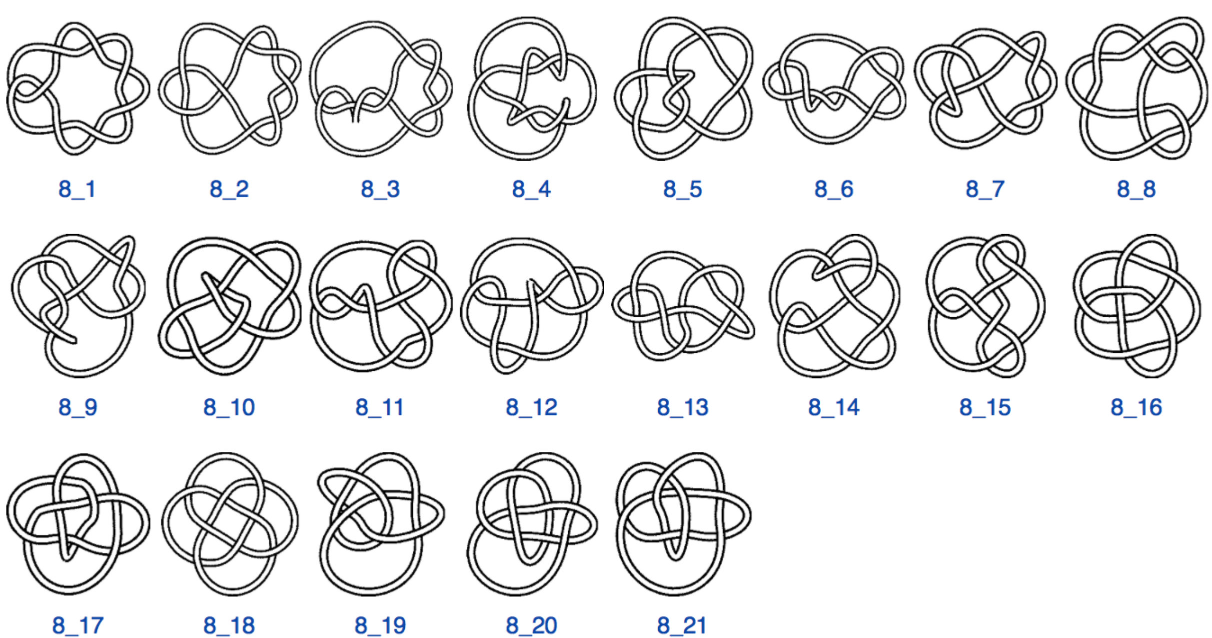

Figure 4:

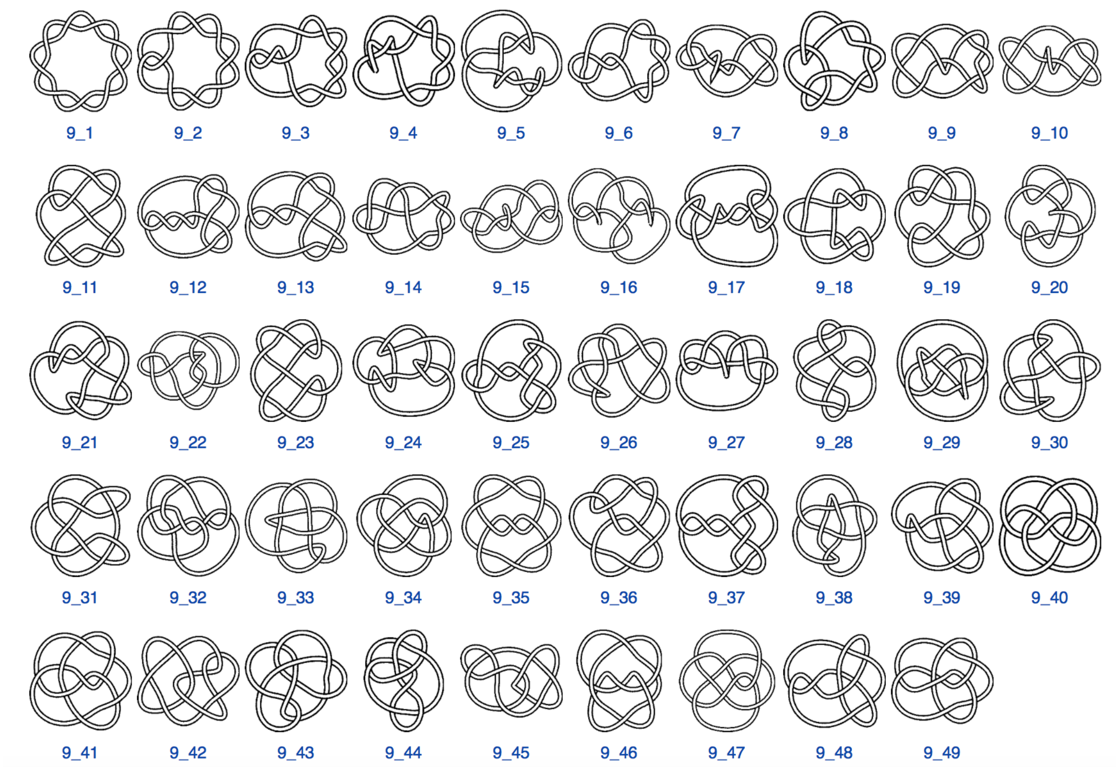

Figure 5:

1.3 Links

A **link** is a set of tangled knots that do not intersect. The Rolfsen Link table is shown in Figure 6:

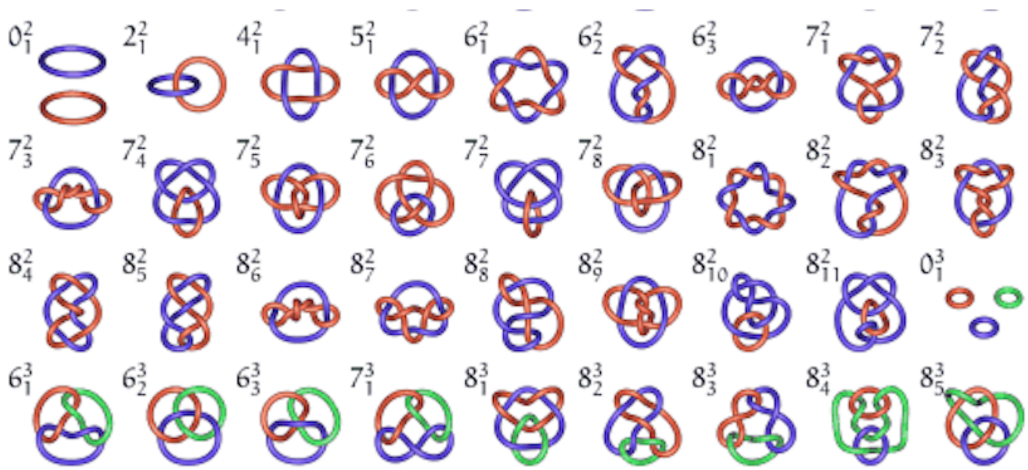

Figure 6:

1.4 Knot Composition

Given two knots K_1 and K_2, we can define a new knot by removing a small arc from each knot and then connecting the four endpoints by two new arcs as in the example in Figure 7. The resulting knot is called the **composition** of the two knots and is written as $K_1 \# K_2$ (Adams 2004). If a knot K_3 can be written as the composition of two knots K_1 and K_2, then

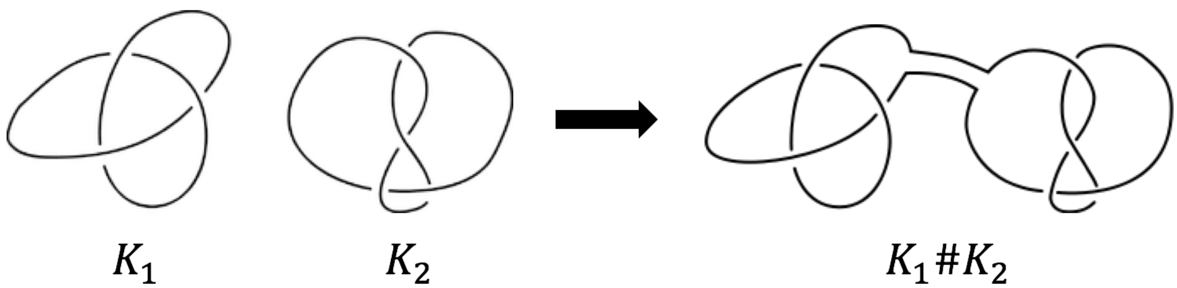

Figure 7:

K_1 and K_2 are called the **factor knots** of K_3. It is interesting to note that the composition between any knot K and the unknot is simply the knot K. Because of this the unknot is sometimes called the **trivial knot**. If a knot is not the composition of two nontrivial knots, it is called a **prime knot**.

1.5 Unknotting Number

A crossing like the one in Figure 8 is said to be **changed** if the line on the bottom of the crossing is switched to the top. It is said that a knot K has a **unknotting number** n if

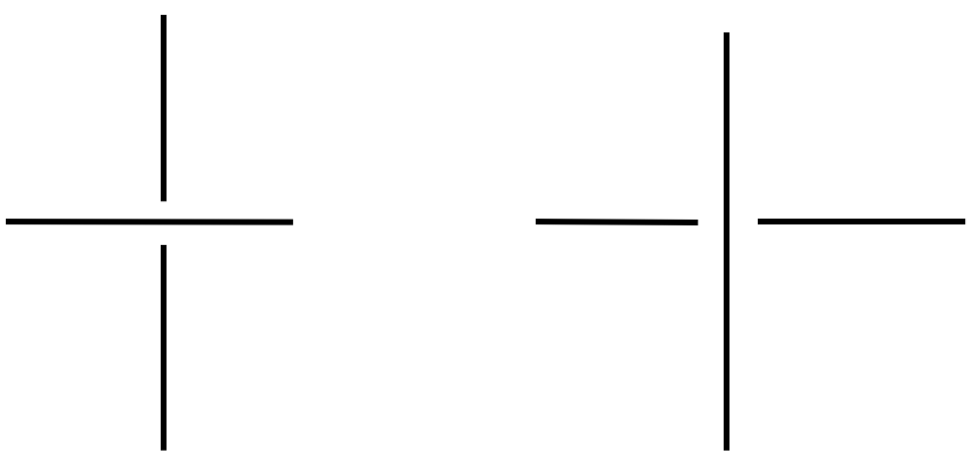

Figure 8:

K or any knot equivalent to K can be turned into the unknot by changing n crossings and there does not exist a knot equivalent to K such that fewer changes would turn it into the unknot.

2 Relevant Mathematics

All of the definitions and theorems in this section are taken from Munkres (2000).

2.1 Continuity, Homeomorphisms, and Topological Imbeddings

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. Now suppose f is a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**. Any property of X that is entirely expressed in terms of the topology of X (that is, in terms of the open sets of X) yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a **topological property** of X. A homeomorphism can be thought of as a bijective correspondence that preserves the topological structure involved.

Now suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set of $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological imbedding** of X in Y.

2.2 Paths and Path Homotopy

Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

If f and f' are continuous maps of the space X into the space Y, we say that f is ***homotopic*** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x)$$

for each x. The map F is called a ***homotopy*** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is ***nulhomotopic***.

Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be ***path homotopic*** if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that

$$F(s, 0) = f(s) \text{ and } F(s, 1) = f'(s),$$

$$F(0, t) = x_0 \text{ and } F(1, t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a ***path homotopy*** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$. An illustration is given by Figure 9.

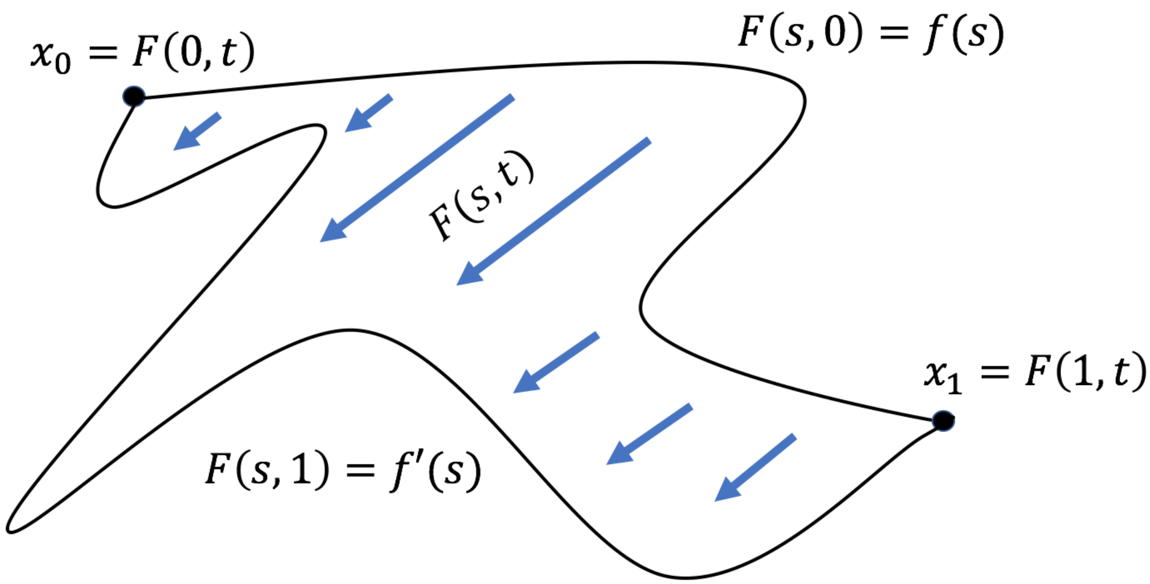

Figure 9:

2.3 Product Operation

If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the ***product*** $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}], \\ g(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

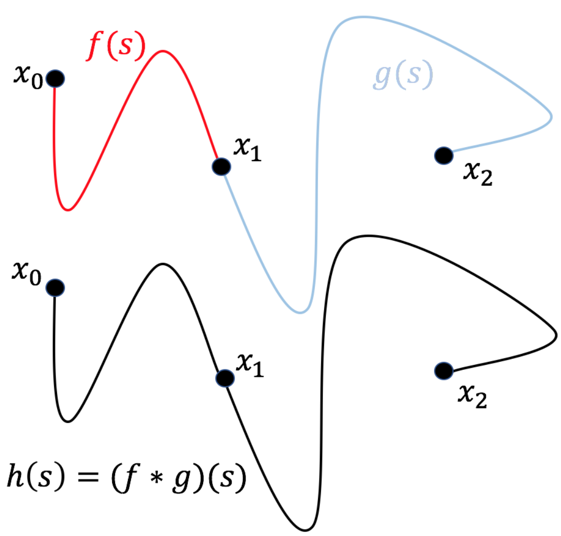

Figure 10:

An illustration is given in Figure 10. The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g].$$

2.4 Fundamental Groups

Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a ***loop*** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the ***fundamental group*** of X relative to the ***base point*** x_0. It is denoted by $\pi_1(X, x_0)$. If $[f] \in \pi_1(X, x_0)$, it's inverse is written as $[\bar{f}]$. The identity element is the constant loop. An example is given in Figure 11, where X is a plane pierced by two points.

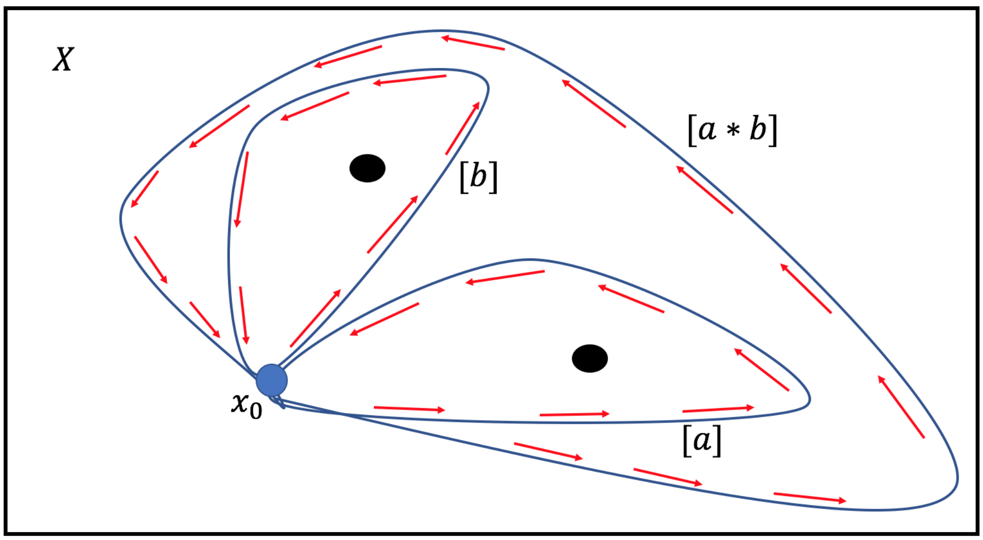

Figure 11:

2.5 Induced Homomorphisms

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map, where $h(x_0) = y_0$. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the ***homomorphism induced by h***, relative to the base point x_0. The fact that h_* is a homomorphism follows from the equation

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

When dealing with a function $h : X \to Y$ with respect to multiple points in X, it is sometimes helpful to use the notation $(h_{x_0})_*$ if x_0 is the base point under consideration.

2.6 Deformation Retracts

Let A be a subspace of X. We say that A is ***deformation retract*** of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. This means that there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$. The homotopy H is called a ***deformation retraction*** of X onto A. The map $r : X \to A$ defined by the equation $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$, where $j : A \to X$ is an inclusion.

Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion map

$$j : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

2.7 Generators

Let G be a group; let $\{a_\alpha\}$ be a family of elements of G, for $\alpha \in J$. We say the elements $\{a_\alpha\}$ **generate** G if every element of G can be written as a product of powers of the elements a_α. If the family is finite, we say G is **finitely generated**. For example, the generators for the fundamental group of the space in Figure 11 are $[a]$ and $[b]$.

2.8 The Seifert-van Kampen Theorem

Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let

$$\phi_1 : \pi_1(U, x_0) \to H \text{ and } \phi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by inclusion.

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

2.9 Mathematical Definition of a Knot

A knot K is defined as the topological imbedding of the circle S^1 in $\mathbb{R}^3$. This means that there exists an injective continuous map $f : S^1 \to \mathbb{R}^3$ such that the restriction $f|K$ is homeomorphic (bijective and continuous with a continuous inverse), where $K \subset \mathbb{R}^3$. This concept is shown Figure 12.

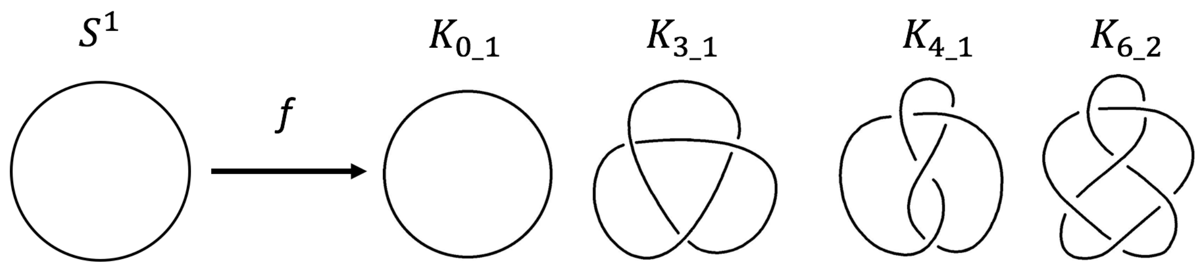

Figure 12:

2.10 Knot Groups and Wirtinger Presentations

Let K be a set describing a mathematical knot in $\mathbb{R}^3$. The **knot group** is defined as the fundamental group of the knot complement, $\pi_1(\mathbb{R}^3 - K)$.

Knot groups are often represented with Wirtinger presentations

$$< S|R >,$$

where S is a set of generators of $\pi_1(\mathbb{R}^3 - K)$ and R is a set of their relations. If two knots have the same Wirtinger presentations, their knot groups are isomorphic and the knots are considered equivalent. However, if two knots have different Wirtinger presentations, **they are not necessarily inequivalent**.

3 The Trefoil Knot Group

3.1 Constructing the Knot Group with Pictures

In this section, we find the knot group of the trefoil knot by drawing the loops belonging to the generators of $\pi_1(\mathbb{R}^3 - K_{3\_1})$. We will establish relations between their homotopy classes and reduce the presentation to its simplest form. We begin by dividing the trefoil knot into three sections and giving each section its own generator, as shown in Figure 13. It should

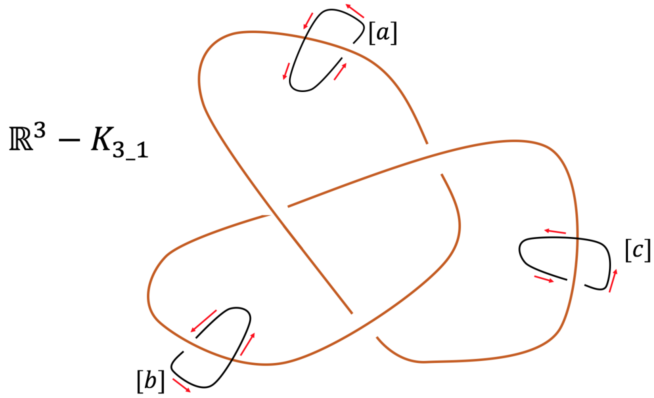

Figure 13:

be pointed out that all of these generators should technically originate from the same fixed base point, but since the space is path connected, it does not matter where the base point is. We then look at each of the three crossings of the trefoil knot to detrmine relations between the generators. In Figure 14, we see that the two loops at the crossing can be continuously deformed into each other. This implies that the two loops belong to the same homotopy class. We now want to decompose each loop into some combination of the three generators.

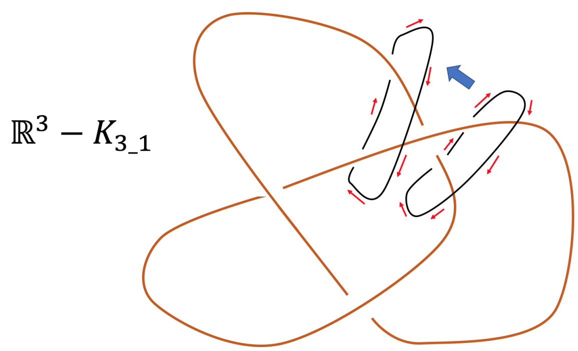

Figure 14:

To illustrate this process, take the loop given in Figure 15a. Notice that this loop can be continuously deformed into that of Figure 15b. This loop can be further deformed into that

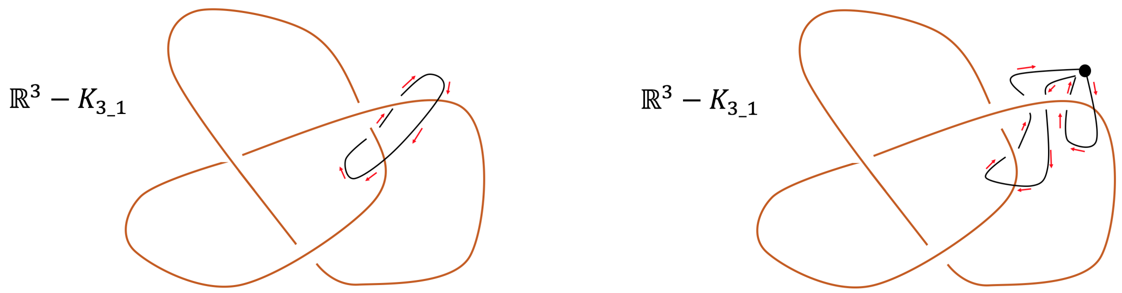

Figure 15: Left (a); Right (b)

of Figure 16a. Finally, we see that the loop belongs to the homotopy class $[c * a]$ in Figure 16b. By doing this same process for the other loop from Figure 14, we get the labeling in

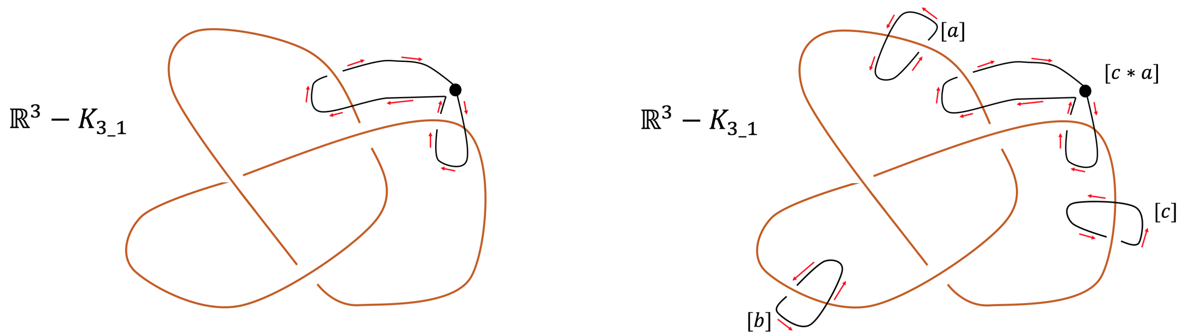

Figure 16: Left (a); Right (b)

Figure 17b. Since the two loops belong to the same homotopy equivalence class, we can say that $[a * b] = [c * a]$. We then repeat the process on the other crossings shown in Figure 18.

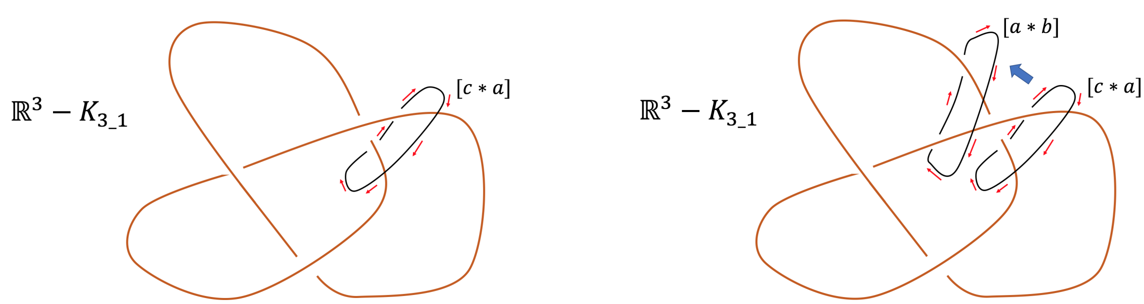

Figure 17: Left (a); Right (b)

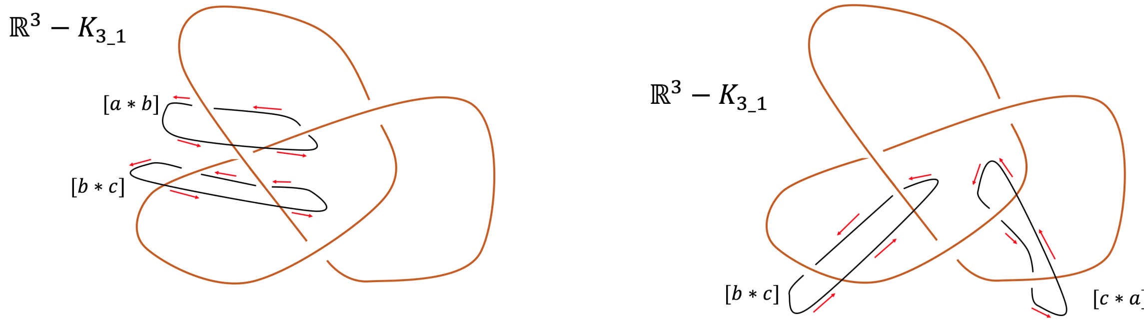

Figure 18: Left (a); Right (b)

Therefore, we say that the trefoil knot group is isomorphic to the following Wirtinger Presentation:

$$\pi_1(\mathbb{R}^3 - K_{3\_1}) \cong < a, b, c | a * b = c * a, b * c = a * b, c * a = b * c >$$

The relations for this group are as follows:

- $a * b = c * a$
- $b * c = a * b$
- $c * a = b * c$

By plugging in $c = a * b * a^{-1}$, the equations can be reduced to (Salomone 2018):

$$b * a * b * a^{-1} = a * b$$

$$\implies b * a * b = a * b * a.$$

Thus,

$$\pi_1(\mathbb{R}^3 - K_{3\_1}) \cong < a, b | b * a * b = a * b * a >$$

Sometimes you will see the presentation written as

$$< p, q | p^3 = q^2 > .$$

This comes from

$$b * a * b = a * b * a$$

$$\implies (a * b * a) * b * a * b = (a * b * a) * a * b * a$$

$$\implies (a * b) * (a * b) * (a * b) = (a * b * a) * (a * b * a).$$

By letting $p = a * b$ and $q = a * b * a$, we see that

$$p^3 = q^2.$$

4 Piercing the Unknot Complement

Up until this point, no original ideas have been presented. From here on, the idea of the knot group is explored by piercing the complement of a knot with a line L.

4.1 Piercing the Space Outside of the Unknot

Let S^1 be the unknot in $\mathbb{R}^3$. By piercing the complement of the unknot such that the line L does not link with S^1, we get an image shown in Figure 19. Let us construct the

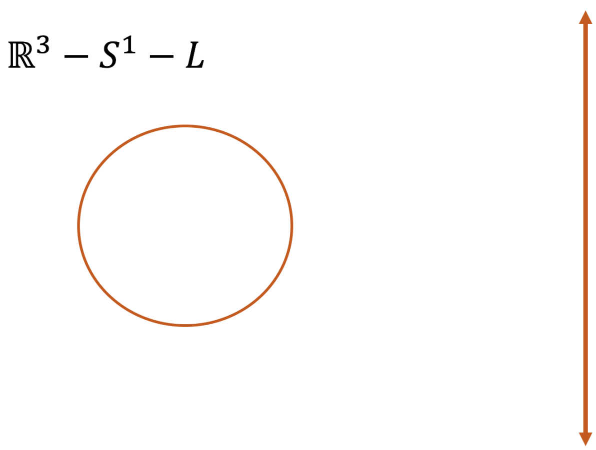

Figure 19:

fundamental group of $\mathbb{R}^3 - S^1 - L$. Similarly to the construction of the trefoil knot group, we want to begin by drawing loops. The obvious generator is $[l]$, the homotopy class of loops around the line L. Now consider the classes $[s_+]$ and $[s_-]$ as shown in Figure 20, where $[s_+]$ is the class of loops crossing over L and then around S^1, and $[s_-]$ is the class of loops crossing under L and then around S^1. Is there any way to continuously deform s_+ into s_- without

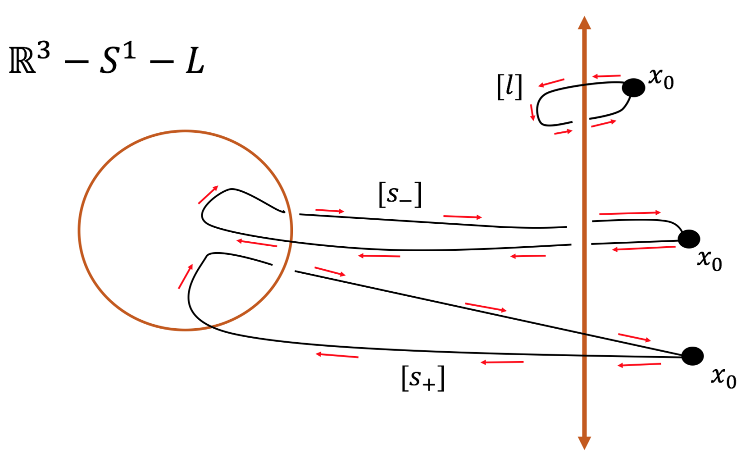

Figure 20:

intersecting the loop with L? Since the line L is infinite, the answer is no. However, this does not mean that $\pi_1(\mathbb{R}^3 - S^1 - L)$ has a minimum of three generators. Figure 21 shows that $[s_+]$ can be written in terms of $[l]$ and $[s_-]$. Therefore, since $[s_+] = [l * s_- * l^{-1}]$ we see

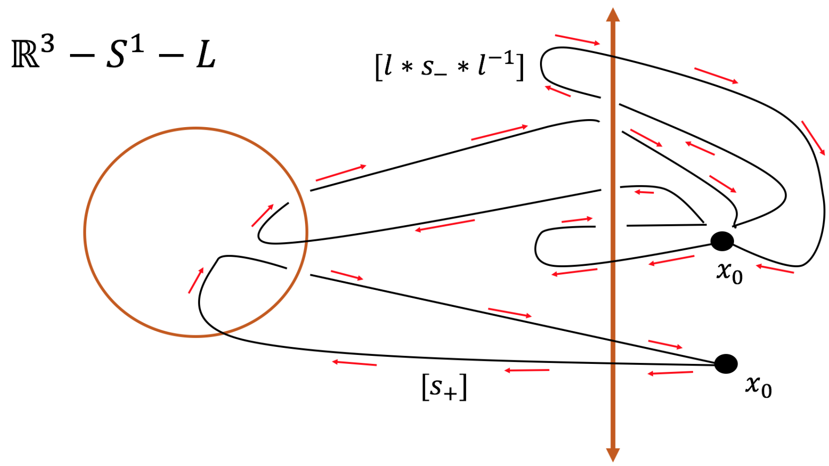

Figure 21:

that the fundamental group is isomorphic to the following Wirtinger presentation, where s is an arbitrary choice of s_+ or s_-:

$$\pi_1(\mathbb{R}^3 - S^1 - L) \cong < l, s | \emptyset >$$

4.2 Relating the Space to Two Unknots

How does the fundamental group of the space in the previous section compare to that of the complement of two unknots in $\mathbb{R}^3$? Figure 22 illustrates $\mathbb{R}^3 - S_1^1 - S_2^1$ and shows the generators around the two circles. We see that the fundamental group of this space

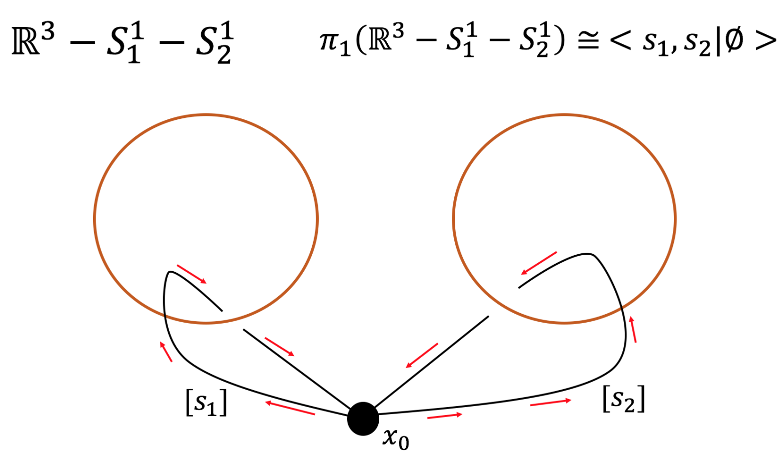

Figure 22:

is relatively simple to determine, since there are only two generators, and because these generators are unrelated. Thus,

$$\pi_1(\mathbb{R}^3 - S_1^1 - S_2^1) \cong < s_1, s_2 | \emptyset > .$$

Because the Wirtinger presentations of the fundamental groups of $\mathbb{R}^3 - S_1^1 - S_2^1$ and $\mathbb{R}^3 - S^1 - L$ are identical in that they both have two generators and no relations, we see that

$$< s_1, s_2 | \emptyset > \cong < s, l | \emptyset >,$$

which implies that

$$\pi_1(\mathbb{R}^3 - S_1^1 - S_2^1) \cong \pi_1(\mathbb{R}^3 - S^1 - L).$$

Therefore, the fundamental group of the unknot complement pierced by a line outside of the circle is isomorphic to that of the complement of two unlink unknots.

4.3 Piercing the Space Through the Unknot

We just looked at how piercing the unknot complement with a line affected its fundamental group. Now we investigate whether the position of the piercing matters. Suppose $\mathbb{R}^3 - S^1 - L$ describes the space illustrated in Figure 23, where L crosses through S^1. As

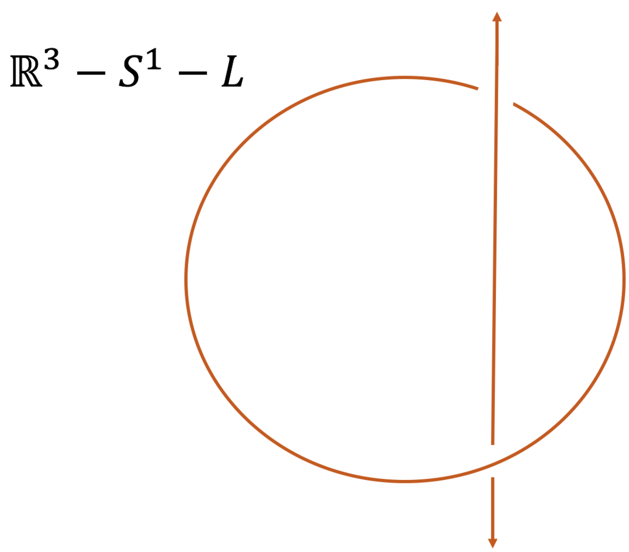

Figure 23:

always, we determine the Wirtinger of the presentation by first drawing the generators, followed by any relations that may exist. Figure 24 shows the space containing two generators $[s]$ and $[l]$. This time, we see that there is indeed a relation given by the crossing:

$$[s * l] = [l * s].$$

When the line pierces through the circle, the fundamental group of its complement takes the presentation

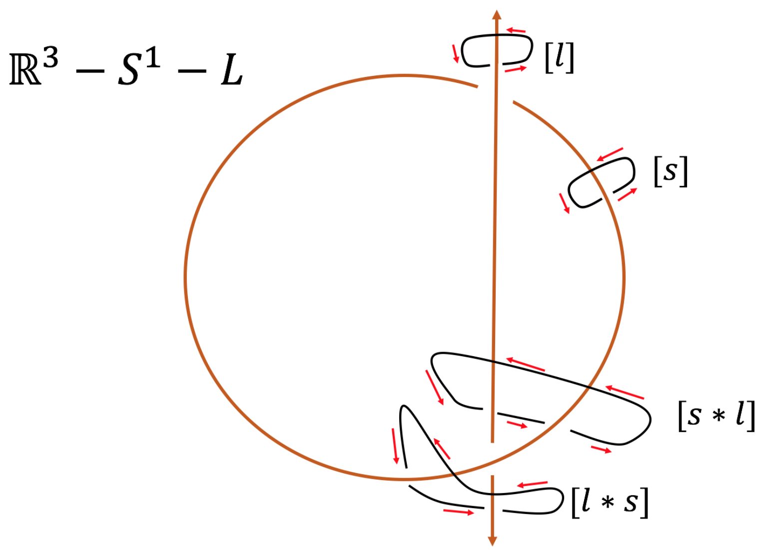

Figure 24:

$$\pi_1(\mathbb{R}^3 - S^1 - L) \cong < s, l | s * l = l * s >$$

Therefore, because of this extra relation, the fundamental group of this space is not isomorphic to that of the circle with a line **outside** of it. Rather than having two unlinked unknots, we now investigate the fundamental group of the complement of two linked unknots in Figure 25. The fundamental group of this linked unknot complement takes the Wirtinger

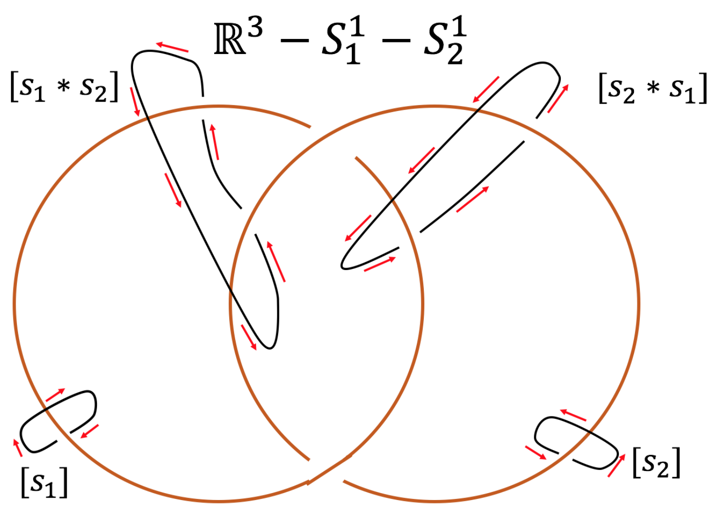

Figure 25:

presentation

$$< s_1, s_2 | s_1 * s_2 = s_2 * s_1 > .$$

Since the pierced unknot has the presentation $< s, l | s * l = l * s >$, this implies

$$\pi_1(\mathbb{R}^3 - S^1 - L) \cong \pi_1(\mathbb{R}^3 - S^1_1 - S^1_2).$$

Therefore, the fundamental group of the unknot complement pierced by a line through the circle is isomorphic to that of the complement of two linked unknots.

5 Piercing the Trefoil Knot Complement

The unknot is the simplest knot, the next simplest being the trefoil knot. Here we investigate how piercing the complement of the trefoil knot affects the fundamental group of the space.

5.1 Piercing Outside the Trefoil Knot

We begin by piercing the trefoil complement with a line L such that L is not going through the knot. Since it was established earlier that it does not matter whether the generators looping around the knot go over or under the line L, we can simply draw three generators around each section of the trefoil knot, plus one generator $[l]$ around the line. This is shown in Figure 26. Thus, the only difference between the fundamental group of the trefoil knot

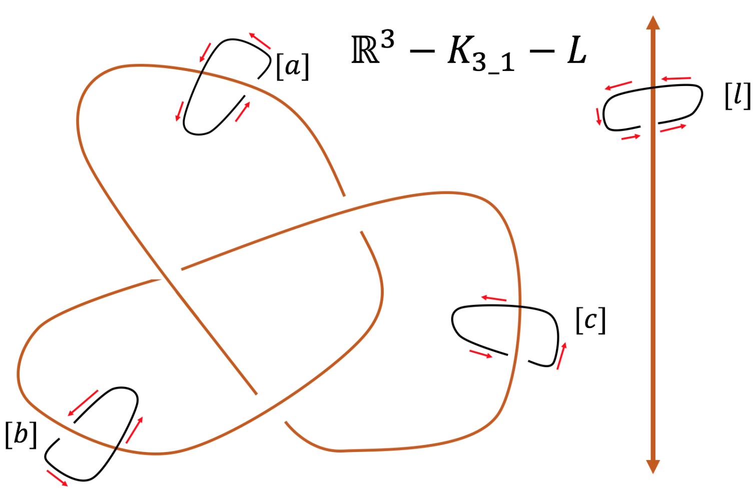

Figure 26:

complement with a line pierced outside of the knot and the trefoil knot group itself is the extra generator $[l]$. Instead of piercing the outside of the trefoil knot with a line, what is the fundamental group of the trefoil knot and the unlinked unknot? Figure 27 shows that the only difference is the extra generator $[s]$. Therefore,

$$\pi_1(\mathbb{R}^3 - K_{3\_1} - L) \cong < a, b, l | b * a * b = a * b * a >$$

and

$$\pi_1(\mathbb{R}^3 - K_{3\_1} - S^1) \cong < a, b, s | b * a * b = a * b * a >$$

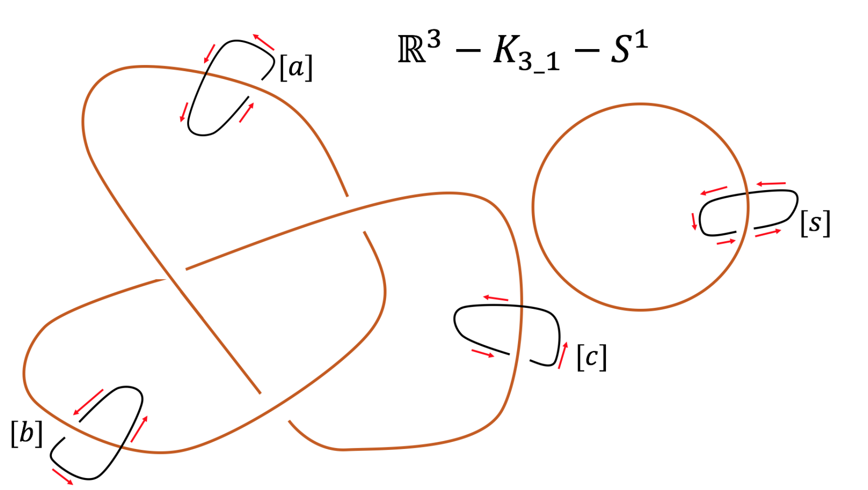

Figure 27:

imply that

$$\pi_1(\mathbb{R}^3 - K_{3\_1} - L) \cong \pi_1(\mathbb{R}^3 - K_{3\_1} - S^1).$$

5.2 Piercing Through a Loop of the Trefoil Knot

Finally, this work analyzes how the fundamental group is affected when a line L pierces through a loop of the trefoil knot. Figure 28 illustrates this space, along with the same generators as the last section. Similarly to the line piercing through the unknot, the line

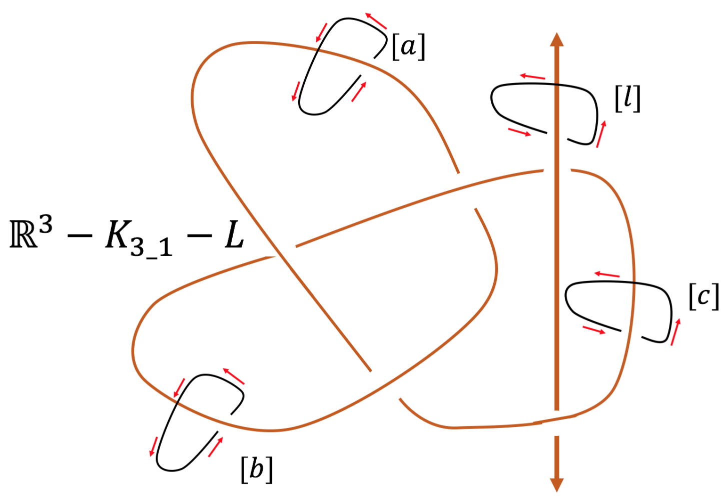

Figure 28:

piercing through one loop of the trefoil knot adds one generator l, which commutes with c, the generator around the pierced loop. In this case we wrote c in terms of a and b, although it is somewhat arbitrary. Therefore the presentation of $\pi_1(\mathbb{R}^3 - K_{3\_1} - L)$ is:

$$< a, b, l | b * a * b = a * b * a, a * b * a^{-1} * l = l * a * b * a^{-1} >$$

How does this fundamental group compare to that of the unknot linked with the same loop of the trefoil knot? Figure 29 shows that the generators and relations will be the same as with L. Therefore the unknot linked through one loop of the trefoil knot can be presented

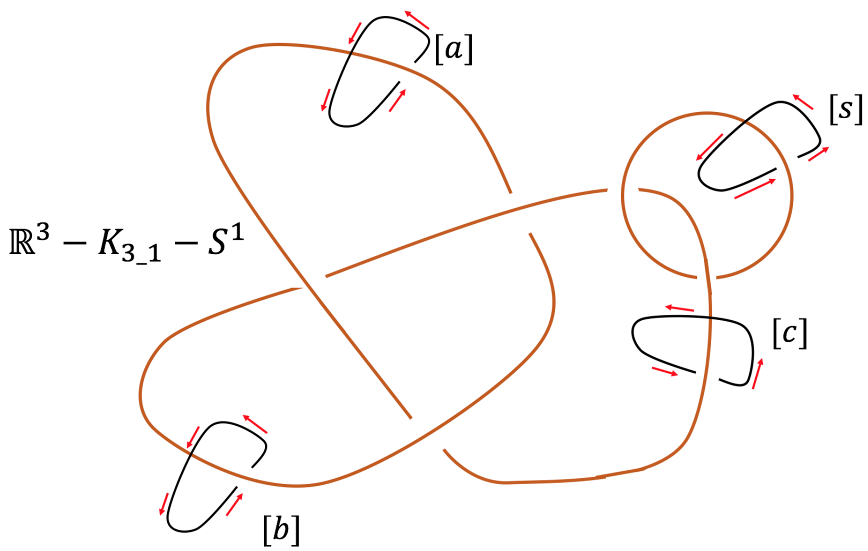

Figure 29:

as:

$$< a, b, s | b * a * b = a * b * a, a * b * a^{-1} * s = s * a * b * a^{-1} > .$$

Thus, we see that

$$\pi_1(\mathbb{R}^3 - K_{3\_1} - L) \cong \pi_1(\mathbb{R}^3 - K_{3\_1} - S^1)$$

6 Conclusion

In summary, this research has analyzed the definition of the knot group by asking a simple question: how does piercing the trefoil knot complement with a line affect the fundamental group of the space? The position dependent effects on this change were also investigated by piercing the trefoil knot complement through one its loops. Furthermore, this work compared the fundamental groups of the pierced spaces with that of the trefoil knot linked and unlinked with the unknot. The two spaces illustrated in Figure 30 were found to have isomorphic fundamental groups. Similarly, the two spaces illustrated in Figure 31 were found to have isomorphic fundamental groups. This work shows that the spaces in Figure 30 do not have isomorphic fundamental groups to the spaces in Figure 31, arguing that the effects due to alteration are dependent on the piercing position or the linking status.

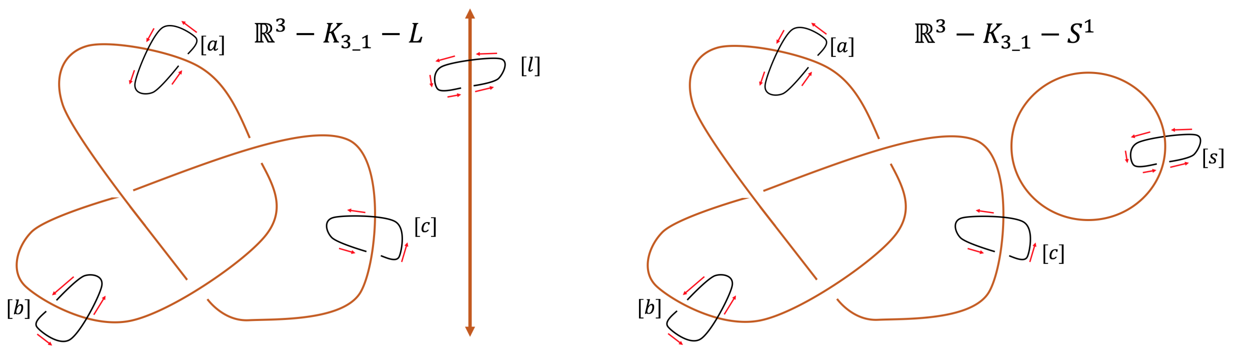

Figure 30:

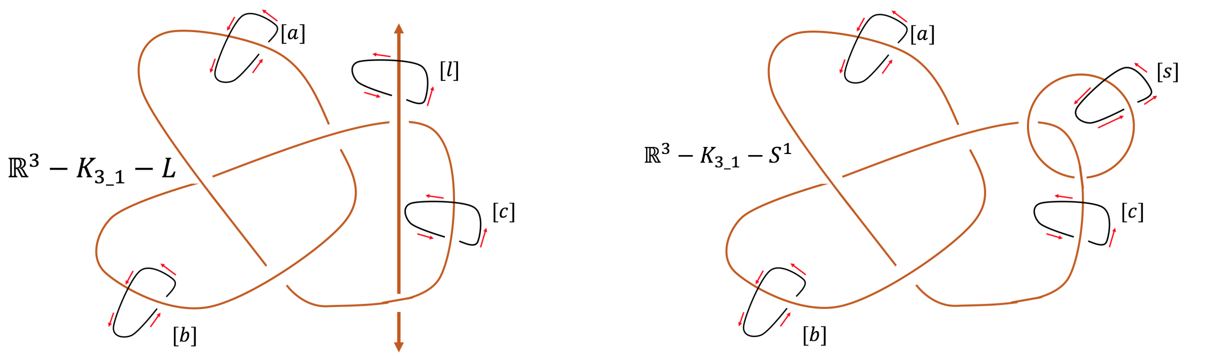

Figure 31:

7 Acknowledgments

I would like to thank the Georgia College Mathematics Department for letting me explore my interests. In particular, I would like to thank Dr. Rodica Cazacu for guiding me through different areas of geometry and topology, and for working with me throughout my undergraduate career.

8 References

- Adams, C. (2004) The Knot Book, American Mathematical Society
- Munkres, J. R. (2000) Topology, Prentice Hall, Incorporated
- Salomone, M. (2018) Simplifying the Trefoil Knot Group https://www.youtube.com/watch?v=tP5uIShGSsE
- Sullivan, M. (2016) The Knot Group http://galileo.math.siu.edu/mikesullivan/Courses/531/S16/knotgroup.pdf
- The Knot Atlas (2009) http://katlas.org/wiki/The Rolfsen Knot Table