A simple randomized algorithm for approximating the spectral norm of streaming data

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April 2023

Motivation: Why approximate the spectral norm?

Spectral norms can be used as an error estimator when trying to approximate matrices.

Remark 2.1 (Martinson et. al., 2020)

For example, let us consider a variant of the spiked covariance model that is common in statistics applications. Suppose we need to approximate a rank-one matrix contaminated with additive noise: $A = \vec{u}\vec{u}^* + \in \mathbb{R}^{n\times n}$, where $\|\vec{u}\| = 1$ and $G \in \mathbb{R}^{n\times n}$ has independent entries from $\mathcal{N}(0, n^{-1})$ entries. With respect to the Frobenius norm, the zero matrix is almost as good an approximation of A as the rank-one matrix uu^*:

$$\mathbb{E}[\|A - \vec{u}\vec{u}^*\|_F^2] = \varepsilon^2 n \text{ and } \mathbb{E}[\|A - 0\|_F^2] = \varepsilon^2 n$$

An Existing Approach: Power Method

Liberty et. al., (2007)

Suppose A is an $m \times n$ complex-valued matrix and $\vec{\omega}$ is a $n \times 1$ column vector with i.i.d. entries from a complex gaussian distribution. With $\vec{\nu} = \frac{\vec{\omega}}{\|\vec{\omega}\|_2}$, we define

$$p_j(A) = \sqrt{\frac{\|(A^* A)^j \vec{\nu}\|_2}{\|(A^* A)^{j-1} \vec{\nu}\|_2}}.$$

Then $p_j(A) \geq \|A\|/10$ with probability greater than $1 - 4\sqrt{n/(j-1)}100^{-j}$, and $p_j(A) \leq \|A\|$ for all j.

Streaming Data

Let A be an $m \times n$ matrix of data, and suppose we went to append it with an $m \times k$ dataset B.

$$C = \begin{bmatrix} A \mid B \end{bmatrix} = \begin{bmatrix} A \mid 0_{m\times k} \end{bmatrix} + \begin{bmatrix} 0_{m\times n} \mid B \end{bmatrix} = A' + B'$$

By writing $C = A' + B'$ as above, we run into a potential storage issue. We must store both the old data A and new data B in order to calculate the spectral norm approximation:

$$p_j(C) = \sqrt{\frac{\|\left((A'+B')^*(A'+B')\right)^j \vec{\nu}\|_2}{\|\left((A'+B')^*(A'+B')\right)^{j-1} \vec{\nu}\|_2}}$$

A Different Approach:

Lemma 4.1 (Halko et. al., 2011)

Let A be a real $m \times n$ matrix. Fix a positive integer r and a real number $\alpha > 1$. Draw an independent family $\{\vec{\omega}_i : i = 1, 2, \ldots, N\}$ of standard Gaussian vectors. Then

$$\|A\| \leq \alpha \max_{i=1,2,\ldots,N} \|A\vec{\omega}_i\|$$

except with probability α^{-N}

Efficient Storage for Streaming Data

Let $\Omega_A = \begin{bmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \dots & \vec{\omega}_N \end{bmatrix}$ be an $n \times N$ matrix whose columns are independent standard Gaussian vectors, and define

$$Y_A = A\Omega_A = \begin{bmatrix} A\vec{\omega}_1 & A\vec{\omega}_2 & \dots & A\vec{\omega}_N \end{bmatrix}.$$

To achieve the bound on the previous slide, calculate $\max_{i=1,2,\dots,N} \|A\vec{\omega}_i\|$

Suppose now that we append the $m \times k$ matrix B to A to get $C = \begin{bmatrix} A \mid B \end{bmatrix}$. We let Ω_B be a $k \times N$ matrix whose columns are independent standard Gaussian vectors, and define $\Omega_C = \begin{bmatrix} \Omega_A \\ \hline \Omega_B \end{bmatrix}$. Then

$$Y_C = C\Omega_C = \begin{bmatrix} A \mid B \end{bmatrix} \begin{bmatrix} \Omega_A \\ \hline \Omega_B \end{bmatrix} = A\Omega_A + B\Omega_B = Y_A + B\Omega_B,$$

implying that we only need to calculate $B\Omega_B$ after storing the $m \times N$ matrix Y_A.

Frobenius Norm is off by factor of $r^{1/2}$

We can write the Frobenius norm as the ℓ_2-norm of the singular values: $\|A\|_F = \sqrt{\sum_{i=1}^r \sigma_i^2}$. Using this and the fact that the spectral norm of A is the largest singular value of A, we have

$$\|A\| \leq \|A\|_F \leq r^{1/2}\|A\|$$

since

$$\sigma_{\max} \leq (\sum_{j=1}^{r} \sigma_r^2)^{1/2} \leq r^{1/2}\sigma_{\max}.$$

This tells us that the Frobenius norm can be off from the spectral norm by a factor of $r^{1/2}$.

Estimate is greater than Frobenius norm

We show $\mathbb{E}[\|A\vec{\omega}\|^2] = \|A\|_F^2$:

$$\mathbb{E}[\|A\vec{\omega}\|^2] = \mathbb{E}[\vec{\omega}^T A^T A\vec{\omega}] = \mathrm{Tr}(\mathbb{E}[\vec{\omega}^T A^T A\vec{\omega}]) = \mathbb{E}[\mathrm{Tr}(\vec{\omega}^T A^T A\vec{\omega})]$$

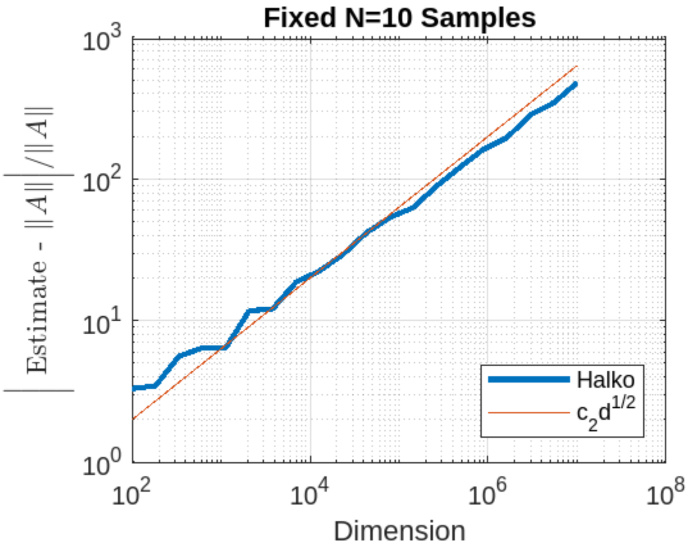
$$= \mathbb{E}[\mathrm{Tr}(A^T A\vec{\omega}\vec{\omega}^T)] = \mathrm{Tr}(\mathbb{E}[A^T A\vec{\omega}\vec{\omega}^T]) = \mathrm{Tr}(A^T A\mathbb{E}[\vec{\omega}\vec{\omega}^T])$$

$$= \mathrm{Tr}(A^T A) = \|A\|_F^2$$

Analyzing the bound given by Halko et. al. (2011), we see

$$\mathbb{E}[\max_{i=1,2,\dots,N} \|A\vec{\omega}_i\|^2] \geq \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} \|A\vec{\omega}_i\|^2] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[\|A\vec{\omega}_i\|^2] = \|A\|_F^2$$

Plot of Error



The ℓ_4 norm is better

By the same arguement as before,

$$\|A\| \leq \|\vec{\sigma}\|_4 \leq r^{1/4}\|A\|.$$

One idea is to approximate the ℓ_4-norm of the singular values since this is a tighter bound.

Let $\vec{\omega}, \vec{\nu} \in \mathcal{N}(0, I_r)$ be independent gaussian random vectors. Define the random variable $X = (A\vec{\omega})^T A\vec{\nu}$. We will show $\mathbb{E}[X^2] = \|\vec{\sigma}\|_4^4$

WLOG, use diagonal matrices

Let $A = U\Sigma V^T$ be the singular value composition of our $m \times n$ matrix A. By orthogonality,

$$\mathbb{E}[X^2] = \mathbb{E}[((A\vec{\omega})^T A\vec{\nu})^2] = \mathbb{E}[(\vec{\omega}^T A^T A\vec{\nu})^2] = \mathbb{E}[(\vec{\omega}^T V\Sigma U^T U\Sigma V^T\vec{\nu})^2]$$

$$= \mathbb{E}[(\vec{\omega}^T V\Sigma^2 V^T\vec{\nu})^2] = \mathbb{E}[((V^T\vec{\omega})^T\Sigma^2 V^T\vec{\nu})^2]$$

Since $\vec{\omega}, \vec{\nu} \in \mathcal{N}(0, I_n)$, we have that $V^T\vec{\omega}, V^T\vec{\nu} \in \mathcal{N}(0, V^T V) = \mathcal{N}(0, I_n)$. Thus,

$$\mathbb{E}[((A\vec{\omega})^T A\vec{\nu})^2] = \mathbb{E}[((\Sigma\omega)^T\Sigma\nu)^2]$$

Furthermore, since Σ only has r non-zero values along it's diagonal, without loss of generality, we can let Σ be an $r \times r$ diagonal matrix from here on and have $\vec{\omega}, \vec{\nu} \in \mathcal{N}(0, I_r)$, and later on we will asume A to be the same.

Calculating the ℓ_4 norm:

$$\mathbb{E}[X^2] = \mathbb{E}[((\Sigma\vec{\omega})^T\Sigma\vec{\nu})^2] = \mathbb{E}[(\sum_{j=1}^{r}\sigma_j^2\omega_j\nu_j)^2]$$

$$= \mathbb{E}[\sum_{j=1}^{r}\sum_{k=1}^{r}\sigma_j^2\sigma_k^2\omega_j\omega_k\nu_j\nu_k] = \sum_{j=1}^{r}\sum_{k=1}^{r}\sigma_j^2\sigma_k^2\mathbb{E}[\omega_j\omega_k\nu_j\nu_k]$$

$$= \sum_{j=1}^{r}\sum_{k=1}^{r}\sigma_j^2\sigma_k^2\mathbb{E}[\omega_j\omega_k]\mathbb{E}[\nu_j\nu_k] = \sum_{j=1}^{r}\sigma_j^4 = \|\vec{\sigma}\|_4^4.$$

Practically speaking, we draw random vectors from a Gaussian distribution to create a sample mean to approximate $\mathbb{E}[X^2]$. Thus, we would like to show that the difference $\left|\frac{1}{N}\sum_{j=1}^{N}X_j^2 - \mathbb{E}[X^2]\right|$ is small with high probability.

Sub-Weibull Random Variables

We define X to be sub-Weibull random variable with tail parameter θ if

$$\mathbb{P}(|X| \geq x) \leq a\exp(-bx^{1/\theta}) \text{ for all } x > 0, \text{ for some } \theta, a, b > 0$$

Equivalently, a random variable is a sub-Weibull with tail parameter θ if there exists some constant $K_2 > 0$ such that

$$\|X\|_p := (\mathbb{E}[|X|^p])^{1/p} \leq K_2 p^\theta$$

for all $p \geq 1$.

Examples

Sub-Gaussian random variables have $\theta = 1/2$

Sub-Exponential have $\theta = 1$

$X = (D\vec{\omega})^T D\vec{\nu}$ is sub-exponential

Let A be a diagonal $r \times r$ matrix with positive diagonal entries σ_i, and let $\omega_i, \nu_i \in \mathcal{N}(0,1)$. Since ω_i, ν_i are sub-Gaussian, there exists a constant k such that for all $p \geq 1$,

$$\|\omega_i\|_p \leq kp^{1/2}.$$

Since $\|\cdot\|_p$ is a norm, we can use the triangle inequality on X:

$$\|X\|_p = \|\sum_{i=1}^{r} \sigma_i^2 \omega_i \nu_i\|_p \leq \sum_{i=1}^{r} \sigma_i^2 \|\omega_i \nu_i\|_p = \sum_{i=1}^{r} \sigma_i^2 (\mathbb{E}[|\omega_i|^p |\nu_i|^p])^{1/p}.$$

By independence, the above equals

$$\sum_{i=1}^{r} \sigma_i^2 (\mathbb{E}[|\omega_i|^p])^{1/p} (\mathbb{E}[|\nu_i|^p])^{1/p} \leq \sum_{i=1}^{r} \sigma_i^2 (kp^{1/2})(kp^{1/2}) = k^2 p \|A\|_F^2$$

We care about X^2, but there's a problem

$X^2 = \left((D\vec{\omega})^T D\vec{\nu}\right)^2$ is sub-Weibull with parameter $\theta = 2$:

$$\|X^2\|_p = (\mathbb{E}[|X^2|^p])^{1/p} = \left((\mathbb{E}[|X|^{2p}])^{1/2p}\right)^2 = \left(\|X\|_{2p}\right)^2$$

$$\leq \left(\|A\|_F^2 k^2 (2p)\right)^2 = 4k^4 \|A\|_F^4 p^2.$$

We would like to use concentration properties of sub-Weibull random variables to show the difference $\left|\frac{1}{N}\sum_{j=1}^N X_j^2 - \mathbb{E}[X^2]\right|$ is small with high probability.

Sub-Weibull Theorems

Corollay 3.1 (Vladimirova et. al., 2020)

Let $X_1, ..., X_n$ be identically distributed sub-Weibull random variables with tail parameter θ. Then, for all $x \geq NK_\theta$, we have

$$\mathbb{P}(|\sum_{i=1}^{N} X_i| \geq x) \leq \exp(-(\frac{x}{NK_\theta}))$$

for some constant K_θ dependent on θ.

The problem is that for our situation, K_θ is proportional to $1/N$.

Sub-Weibull theorems

Theorem 3.1 (Kuchibhotla et. al., 2022)

If $X_1, \ldots, X_n$ are independent mean zero random variables with $\|X_i\|_{\psi_\alpha} < \infty$ for all $1 \leq i \leq n$ and some $\alpha > 0$, then for any vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$, then we have

$$\mathbb{P}(|\sum_{i=1}^{n} a_i X_i| \geq 2eC(\alpha)\|b\|_2\sqrt{t} + 2eL_n^*(\alpha)t^{1/\alpha}\|b\|_{\beta(\alpha)}) \leq 2e^{-t}$$

for all $t \geq 0$, where $b = (a_1\|X_1\|_{\psi_\alpha}, \ldots, a_n\|X_n\|_{\psi_\alpha}) \in \mathbb{R}^n$.

Another attempt

Theorem:

Let A be an $m \times n$ real-valued matrix with rank $r > 16$. Draw $\vec{\omega}_i$ and $\vec{\nu}_i$ independently from $\mathcal{N}(0, I_n)$ for all $i \in \{1, ..., N\}$. If we define $X_i = (A\vec{\omega}_i)^T A\vec{\nu}_i$, then there exists a constant $K > 0$ such that for any $t > 0$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} \left| X_i \right|^{1/2} - \|A\| \right| \leq (r^{1/4} - 1)\|A\| + t,$$

with probability greater than $1 - 2\exp(-\frac{Nt^2}{Kr\|A\|^2})$.

This theorem is far from ideal.

If $\|A\| \leq \frac{1}{N} \sum_{i=1}^{N} \left|X_i\right|^{1/2}$, we have that $\frac{1}{N} \sum_{i=1}^{N} \left|X_i\right|^{1/2} \leq r^{1/4}\|A\| + t$ and is actually a slightly better approximation than our estimator $\frac{1}{N} \sum_{i=1}^{N} X_i^2$.

However, it is not guaranteed that $\|A\| \leq \frac{1}{N} \sum_{i=1}^{N} \left|X_i\right|^{1/2}$.

(Proof) Concave Jensen

We use the concave version of Jensen's inequality:

$$\mathbb{E}[|X|^{1/2}] = \mathbb{E}[|X|^{2/4}] \le \left(\mathbb{E}[X^2]\right)^{1/4} = \|\vec{\sigma}\|_4$$

If $\|A\| \le \mathbb{E}[|X|^{1/2}]$,

$$\mathbb{E}[|X|^{1/2}] - \|A\| \le r^{1/4}\|A\| - \|A\| = (r^{1/4} - 1)\|A\|,$$

and if $\|A\| \ge \mathbb{E}[|X|^{1/2}]$,

$$\|A\| - \mathbb{E}[|X|^{1/2}] \le \|A\| \le (r^{1/4} - 1)\|A\|$$

Thus we have a bound on the absolute value of the error.

(Proof) $X^{1/2}$ is sub-Gaussian

The advantage of using $|X|^{1/2}$ is that it is sub-Gaussian with constant proportional to $\|A\|_F$. Using Jensen's inequality again, we see

$$\||X|^{1/2}\|_p = (\mathbb{E}[|X|^{p/2}])^{1/p} \leq \left((\mathbb{E}[|X|^p])^{1/p}\right)^{1/2} = \left(\|X\|_p\right)^{1/2} \leq k\|A\|_F p^{1/2}$$

Thus, we will apply general Hoeffding's inequality to show $\mathbb{E}[|X|^{1/2}]$ can be closely approximated by $\frac{1}{N}\sum_{j=1}^{N}\left|X_j\right|^{1/2}$ with high probability.

(Proof) General Hoeffding's Inequality

Given a random variable X, we define the sub-Gaussian norm of X to be

$$\|X\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}[\exp(X^2/t^2) \leq 2]$$

General Hoeffding's Inequality (Vershynin, 2018)

Let $X_1, X_2, ..., X_N$ be independent, mean zero, sub-gaussian random variables, and $a = (a_1, a_2, ..., a_N) \in \mathbb{R}^N$. Then for every $t \geq 0$

$$\mathbb{P}\left(\left|\sum_{j=1}^{N} a_j X_j\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{K^2\|a\|_2^2}\right)$$

where $K = \max_j \|X_j\|_{\psi_2}$

(Proof) Applying Hoeffding

Using the triangle inequality,

$$\||X|^{1/2} - \mathbb{E}[|X|^{1/2}]\|_p \leq \||X|^{1/2}\|_p + \|\mathbb{E}[|X|^{1/2}]\|_p \leq k\|A\|_F p^{1/2} + \mathbb{E}[|X|^{1/2}]$$

$$\leq k\|A\|_F p^{1/2} + r^{1/4}\|A\| p^{1/2} \leq r^{1/2}(k+1)\|A\| p^{1/2}.$$

We can assert that $\||X|^{1/2} - \mathbb{E}[|X|^{1/2}]\|_{\psi_2} = Cr^{1/2}(k+1)\|A\|$ for some constant $C > 0$.

Applying Hoeffding

This lets us apply Hoeffding to the subgaussian random variables $\tilde{X}_j = |X_j|^{1/2} - \mathbb{E}[|X|^{1/2}]$ with $a_j = 1/N$ for all j and $K = C^2(k+1)^2/c$:

$$\mathbb{P}\Big(|\frac{1}{N}\sum_{j=1}^{N}|X_j|^{1/2} - \mathbb{E}[|X|^{1/2}]| \geq t\Big) \leq 2\exp\left(-\frac{Nt^2}{Kr\|A\|^2}\right)$$

(Proof) Conclusion

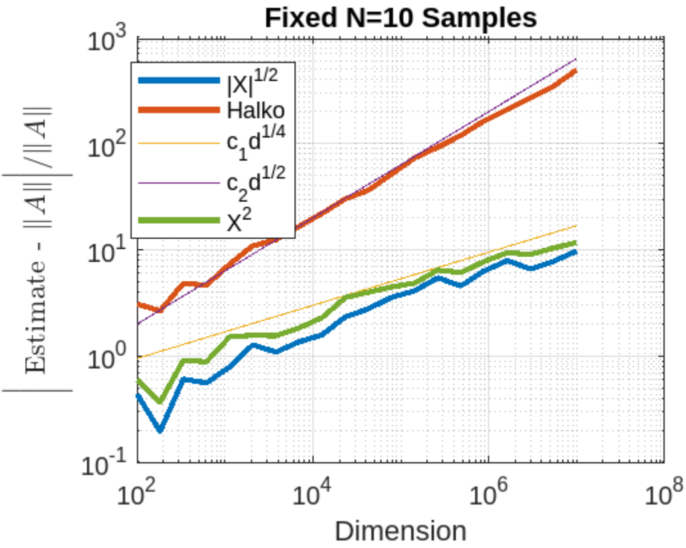
Finally, by the triangle inequality,

$$\left|\frac{1}{N}\sum_{i=1}^{N}\left|X_i\right|^{1/2} - \|A\|\right| \leq \left|\mathbb{E}[|X|^{1/2}] - \|A\|\right| + \left|\frac{1}{N}\sum_{i=1}^{N}\left|X_i\right|^{1/2} - \mathbb{E}[|X|^{1/2}]\right|$$

$$\leq (r^{1/4} - 1)\|A\| + t$$

with probability greater than $1 - 2\exp(-\frac{Nt^2}{Kr\|A\|^2})$.

Conclusion



Fixed N=10 Samples
10^3
10^2
10^1
10^0
10^{-1}
$|X|^{1/2}$
Halko
$c_1 d^{1/4}$
$c_2 d^{1/2}$
X^2
$\left| \text{Estimate} - \|A\| \right| / \|A\|$
10^2
10^4
10^6
10^8
Dimension

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Highlighting text

In this slide, some important text will be highlighted because it's important. Please, don't abuse it.

Remark

Sample text

Important theorem

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Examples

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