Analytical Solutions

Liam McCandlish

February 2020

1 First Analytical Solution

Our problem is to find an analytical solution to the potential boundary problem seen in 1. This problem displays a grounded circle of radius a surrounded by another circle of positive potential, with radius b. In order to solve this, we must use the Laplace equation (1).

$$\nabla^2 \vec{f} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1}$$

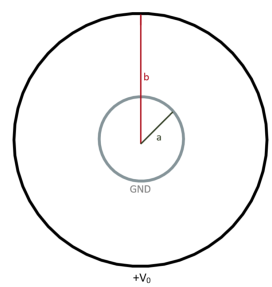

Figure 1: Diagram of first boundary problem

Due to the nature of our problem, being essentially two infinitely long concentric cylinders, it is useful to adapt (1) to cylindrical coordinates. This produces the following result (2).

$$\nabla^2 \vec{f} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{2}$$

Clearly, our problem is only dependent on r. This is because z is in third dimension and changing phi does not vary potential, since we have circle centred on z. This means our function of potential we have (3).

$$\nabla^2 \vec{V}(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r)}{\partial r}\right) = 0 \tag{3}$$

Now, upon looking at equation (3), we can see $1/r \neq 0$, and thus we can integrate both sides to get that $\frac{\partial V(r)}{\partial r} = \frac{A}{r}$. With A being some constant of integration and the $\frac{1}{r}$ having been carried over after integration. Once again we can integrate both sides to get (4), again with constant of integration B.

$$V(r) = A \log r + B \tag{4}$$

We can use our boundary conditions, which are $V(a) = 0$ and $V(b) = V_0$, to find the constants A and B. Firstly, use $V(a) = 0 = A \log a + B$ to get

$$B = -A \log a$$

Now for $V(b) = V$, we can substitute the result for B, giving the following:

$$V_0 = A(\log b - \log a) = A \left[\log \frac{b}{a} \right]$$

$$A = \frac{V_0}{\left(\log \frac{b}{a}\right)}$$

And thus, our final solution is (5)

$$V(r) = V_0 \frac{\log \frac{r}{a}}{\log \frac{b}{a}} \tag{5}$$

Clearly, $V(a) = 0$ and $V(b) = V_0$.

2 Second Analytical Solution

To solve the second boundary problem, we must implement a limitation, in that the circle enclosed by the field must have a radius much smaller than the distance l. This allows for two boundary conditions, one where V=0 at r=R, and another where as $\vec{r} \to |l|$, $V(r, \phi) \to E_0 rcos(\phi)$. This is from simple trigonometry, showing that if we define an x-axis as in figure 2, then the $\vec{E}$ field is acting in the way we defined.

From (2), there is a long and convoluted method to find a general solution to this for a circle (i.e when z=0). To give context, this is done by setting (**??**)= $V(r, \phi) = f_1(r)f_2(\phi)$, then solving for each function to get the general solution (6)

$$V(r, \phi) = (C_0 \ln r + D_0) + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi) \left(C_n r^n + D_n r^{-n} \right) \tag{6}$$

From this, we can set up boundary conditions. Our first boundary condition is that $V(r, \phi) = E_0 rcos(\phi)$ for $r >> R$, i.e $l >> R$. Therefore:

$$E_0 rcos(\phi) = (C_0 \ln r + D_0) + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi) \left(C_n r^n + D_n r^{-n} \right)$$

Clearly, this allows us to remove all terms but A_1 term. Giving:

$$V(r, \phi) = E_0 rcos(\phi) = \cos \phi \left(C' r^1 + D' r^{-1} \right) \tag{7}$$

where $C' = A_1 * C_1$ and $D' = A_1 * D_1$. For the first boundary condition, we have that $V(R, \phi) = 0 = \cos \phi \left(C' r + D' r^{-1} \right)$. We must set $\left(C' r + D' r^{-1} \right)$, which gives equation (8).

$$C' = -\frac{D'}{-R^2} \tag{8}$$

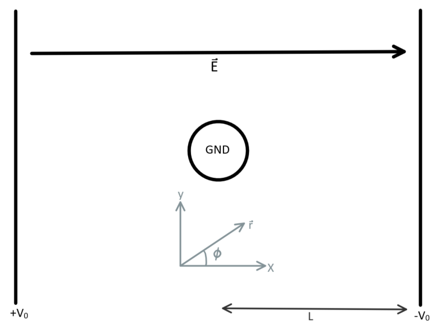

Figure 2: Diagram of second boundary problem

Combining, we have

$$V(r,\phi) = E_0 r cos(\phi) = C' \cos\phi \left(r - \frac{R^2}{r} \right)$$

Applying limit $r \to l >> R$, we can get rid of $\frac{R^2}{r}$ term and conclude that $C' = E_0$. While when $r = R$, we get $V(R,\phi) = 0$.

2.1 Approximations

This solution is fundamentally an approximation, it assumes that the circle in centre of plates is small and far from plates, this is to approximate the plates not as plates, but simply as a uniform electric field.