First Analytical Solution

February 2020

Our problem is to find an analytical solution to the potential boundary problem seen in 1. This problem displays a grounded circle of radius a surrounded by another circle of positive potential, with radius b. In order to solve this, we must use the Laplace equation (1).

$$\nabla^2 \vec{f} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1}$$

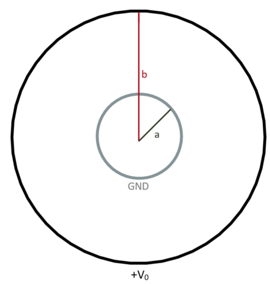

Figure 1: Diagram of our problem

Due to the nature of our problem, being essentially two infinitely long concentric cylinders, it is useful to adapt (1) to cylindrical coordinates. This produces the following result (2).

$$\nabla^2 \vec{f} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{2}$$

Clearly, our problem is only dependent on r. This is because z is in third dimension and changing phi does not vary potential, since we have circle centred on z. This means our function of potential we have (3).

$$\nabla^2 \vec{V}(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r)}{\partial r}\right) = 0 \tag{3}$$

Now, upon looking at equation (3), we can see $1/r \neq 0$, and thus we can integrate both sides to get that $\frac{\partial V(r)}{\partial r} = \frac{A}{r}$. With A being some constant of integration and the $\frac{1}{r}$ having been carried over after integration. Once again we can integrate both sides to get (4), again with constant of integration B.

$$V(r) = A \log r + B \tag{4}$$

We can use our boundary conditions, which are $V(a) = 0$ and $V(b) = V_0$, to find the constants A and B. Firstly, use $V(a) = 0 = A \log a + B$ to get

$$B = -A \log a$$

Now for $V(b) = V$, we can substitute the result for B, giving the following:

$$V_0 = A(\log b - \log a) = A\left[\log \frac{b}{a}\right]$$

$$A = \frac{V_0}{\left(\log \frac{b}{a}\right)}$$

And thus, our final solution is (5)

$$V(r) = V_0 \frac{\log \frac{r}{a}}{\log \frac{b}{a}} \tag{5}$$

Clearly, $V(a) = 0$ and $V(b) = V_0$.