# $\boxed{\textbf{Multivariable Stochastic Processes}}$

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## $\boxed{\text{Multivariable Stochastic Processes}}$

For this lecture the following material was used:

- Chapter 5 of Lecture notes *Aircraft Responses to Atmospheric Turbulence*
- Chapter 4 of *Linear System Theory and Design*, by C.T. Chen (1984), Holt-Saunders International Eds.

Introduction to multivariable stochastic processes

Probability distribution function, p.d.f., covariance, correlation

Linear transformations

The covariance function matrix; the spectral density matrix

Dynamic multivariable system analysis in the frequency domain

Dynamic multivariable system analysis in the time domain
- Continuous time
- Discrete time

...The impulse response method... study at home

Miscellaneous

# Introduction to multivariable stochastic processes

Now the findings of previous chapters, which hold for <u>scalar</u> stochastic processes, will be extended to <u>multivariable</u> stochastic processes.

Consider a **state space** description of an LTI system in continuous time and in discrete time:

| Continuous time | Discrete time |
|---|---|
| $\begin{aligned}\dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t)\\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)\end{aligned}$ | $\begin{aligned}\underline{x}[k+1] &= \Phi\underline{x}[k] + \Gamma\underline{u}[k]\\ \underline{y}[k] &= C\underline{x}[k] + D\underline{u}[k]\end{aligned}$ |

**Q:** What will happen to the state vector $\underline{x}$ and the output vector $\underline{y}$ if a *stochastic input vector* $\underline{\bar{u}}$ acts on the system?

## $\boxed{\text{The probability distribution and probability density functions}}$

Consider an $n$-dimensional SV $\underline{\bar{x}}$ and an $m$-dimensional SV $\underline{\bar{y}}$.

$$\underline{\bar{x}} = [\bar{x}_1 \ \bar{x}_2 \ ... \ \bar{x}_n]^T \quad \text{and} \quad \underline{\bar{y}} = [\bar{y}_1 \ \bar{y}_2 \ ... \ \bar{y}_m]^T$$

Similar to scalar stochastic processes we can define the probability distribution function and the p.d.f.:

$$F_{\underline{\bar{x}}}(\underline{x}) = \mathrm{Pr}\left\{\bar{x}_1 \leq x_1 \wedge \bar{x}_2 \leq x_2 \wedge ... \wedge \bar{x}_n \leq x_n\right\} \tag{5.6}$$

$$f_{\underline{\bar{x}}}(\underline{x}) = \frac{\partial^n F_{\underline{\bar{x}}}(\underline{x})}{\partial x_1 \partial x_2 ... \partial x_n} \tag{5.7}$$

In general, similar to the scalar case, these probability functions are unknown, and we confine ourselves to the (joint) (central) *moments* of SVs $\underline{\bar{x}}$ and $\underline{\bar{y}}$.

we assume ergodicity, normal PDF

The **mean** of a stochastic $n \times 1$ vector $\underline{\bar{x}}$ is defined as:

$$\boxed{\mathsf{E}\{\underline{\bar{x}}\} = [\mathsf{E}\{\bar{x}_1\} \;\; \mathsf{E}\{\bar{x}_2\} \;\; \ldots \; \mathsf{E}\{\bar{x}_n\}]^T = \mu_{\underline{\bar{x}}}} \tag{5.9}$$

The **auto-covariance** becomes an $n \times n$ matrix:

$$\boxed{\begin{aligned} C_{\underline{\bar{x}}\underline{\bar{x}}} &= \mathsf{E}\left\{(\underline{\bar{x}} - \mu_{\underline{\bar{x}}}) \cdot (\underline{\bar{x}} - \mu_{\underline{\bar{x}}})^T\right\} \\ &= \begin{bmatrix} \sigma^2_{\bar{x}_1} & C_{\bar{x}_1\bar{x}_2} & \cdots & C_{\bar{x}_1\bar{x}_n} \\ C_{\bar{x}_2\bar{x}_1} & \sigma^2_{\bar{x}_2} & \cdots & C_{\bar{x}_2\bar{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{\bar{x}_n\bar{x}_1} & C_{\bar{x}_n\bar{x}_2} & \cdots & \sigma^2_{\bar{x}_n} \end{bmatrix} \end{aligned}} \tag{5.12}$$

We see that the $ij^{th}$ element of the auto-covariance matrix equals the scalar (!) cross-covariance of SVs $\bar{x}_i$ and $\bar{x}_j$.

The elements on the diagonal are the variances of the SVs $\bar{x}_i$ ($i$ =1...$n$). It is clear that the auto-covariance matrix is *symmetric*: $C_{\underline{\bar{x}\bar{x}}} = C^T_{\underline{\bar{x}\bar{x}}}$. When the elements of the stochastic vector are all uncorrelated, the auto-covariance matrix becomes a diagonal matrix with the variances on the diagonal: $C_{\underline{\bar{x}\bar{x}}} = \text{diag}[\sigma^2_{\bar{x}_1}, \sigma^2_{\bar{x}_2}, ...\sigma^2_{\bar{x}_n}]$.

If we are interested in the correlations between the different elements of the stochastic vector $\underline{\bar{x}}$ we scale the auto-covariance matrix and introduce the **auto-correlation** matrix $K_{\underline{\bar{x}\bar{x}}}$:

$$\boxed{K_{\underline{\bar{x}\bar{x}}} = \begin{bmatrix} 1 & \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \ldots & \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} \\ \frac{C_{\bar{x}_2\bar{x}_1}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_1}} & 1 & \ldots & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{\bar{x}_n\bar{x}_1}}{\sigma_{\bar{x}_n}\sigma_{\bar{x}_1}} & \frac{C_{\bar{x}_n\bar{x}_2}}{\sigma_{\bar{x}_n}\sigma_{\bar{x}_2}} & \ldots & 1 \end{bmatrix}} \tag{5.13}$$

When the elements of the stochastic vector are all uncorrelated, the auto-correlation matrix equals the identity matrix $I_n$.

Similar relations can be defined for the $n \times m$ **cross-covariance** matrix between two stochastic vectors $\underline{\bar{x}}$ and $\underline{\bar{y}}$:

$$
\begin{aligned}
C_{\underline{\bar{x}}\underline{\bar{y}}} &= \mathsf{E}\left\{(\underline{\bar{x}} - \mu_{\underline{\bar{x}}}) \cdot (\underline{\bar{y}} - \mu_{\underline{\bar{y}}})^T\right\} \\
&= \begin{bmatrix} C_{\bar{x}_1\bar{y}_1} & C_{\bar{x}_1\bar{y}_2} & \cdots & C_{\bar{x}_1\bar{y}_m} \\ C_{\bar{x}_2\bar{y}_1} & C_{\bar{x}_2\bar{y}_2} & \cdots & C_{\bar{x}_2\bar{y}_m} \\ \vdots & \vdots & \ddots & \vdots \\ C_{\bar{x}_n\bar{y}_1} & C_{\bar{x}_n\bar{y}_2} & \cdots & C_{\bar{x}_n\bar{y}_m} \end{bmatrix}
\end{aligned}
\tag{5.10}
$$

Note that this matrix is <u>not</u> symmetric (also for $n = m$)!.

The $n \times m$ **cross-correlation** matrix $K_{\underline{\bar{x}}\underline{\bar{y}}}$ can be obtained by scaling the cross-covariance matrix (5.10) with the standard deviations of the elements in the stochastic vectors $\underline{\bar{x}}$ and $\underline{\bar{y}}$.

When $\underline{\bar{x}}$ and $\underline{\bar{y}}$ are uncorrelated, the cross-covariance matrix as well as the cross-correlation matrix become zero.

# Linear transformations

It can be shown that when $\underline{\bar{y}} = A\underline{\bar{x}}$, where $\underline{\bar{y}}$ is $m \times 1$, $\underline{\bar{x}}$ is $n \times 1$ and $A$ is $m \times n$:

$$\boxed{\mathsf{E}\left\{\underline{\bar{y}}\right\} \;=\; A \cdot \mathsf{E}\left\{\underline{\bar{x}}\right\}} \tag{5.15}$$

and

$$\boxed{C_{\underline{\bar{y}}\underline{\bar{y}}} \;=\; A \cdot C_{\underline{\bar{x}}\underline{\bar{x}}} \cdot A^T} \tag{5.16}$$

**Proof:**
$$\left\{\begin{array}{rcl} C_{\underline{\bar{y}}\underline{\bar{y}}} & = & \mathsf{E}\left\{(\underline{\bar{y}} - \mathsf{E}\left\{\underline{\bar{y}}\right\}) \cdot (\underline{\bar{y}} - \mathsf{E}\left\{\underline{\bar{y}}\right\})^T\right\} \\ & = & \mathsf{E}\left\{(A\underline{\bar{x}} - A\mathsf{E}\left\{\underline{\bar{x}}\right\}) \cdot (A\underline{\bar{x}} - A\mathsf{E}\left\{\underline{\bar{x}}\right\})^T\right\} \\ & = & \mathsf{E}\left\{A(\underline{\bar{x}} - \mathsf{E}\left\{\underline{\bar{x}}\right\}) \cdot (\underline{\bar{x}} - \mathsf{E}\left\{\underline{\bar{x}}\right\})^T A^T\right\} \\ & = & A \cdot \mathsf{E}\left\{(\underline{\bar{x}} - \mathsf{E}\left\{\underline{\bar{x}}\right\}) \cdot (\underline{\bar{x}} - \mathsf{E}\left\{\underline{\bar{x}}\right\})^T\right\} \cdot A^T \\ & = & A \cdot C_{\underline{\bar{x}}\underline{\bar{x}}} \cdot A^T \end{array}\right.$$

# The covariance function matrix

Again, similar to the scalar case, we are particularly interested in stochastic processes as they evolve in time. Therefore, consider the coherence between stochastic vector $\underline{\bar{x}}$ at time $t$ and stochastic vector $\underline{\bar{y}}$ at time $t + \tau$ (remember that we are only dealing with stationary stochastic processes). We then get for the $n \times m$ **cross-covariance function matrix**:

$$C_{\underline{\bar{x}}\underline{\bar{y}}}(\tau) = \mathsf{E}\left\{(\underline{\bar{x}}(t) - \mu_{\underline{\bar{x}}}) \cdot (\underline{\bar{y}}(t+\tau) - \mu_{\underline{\bar{y}}})^T\right\} \tag{5.17}$$

Assuming zero-mean processes yields:

$$C_{\underline{\bar{x}}\underline{\bar{y}}}(\tau) = R_{\underline{\bar{x}}\underline{\bar{y}}}(\tau) = \mathsf{E}\left\{\underline{\bar{x}}(t) \cdot \underline{\bar{y}}^T(t+\tau)\right\} \tag{5.18}$$

which can be written as:

$$C_{\underline{\bar{x}}\underline{\bar{y}}}(\tau) = \begin{bmatrix} C_{\bar{x}_1\bar{y}_1}(\tau) & C_{\bar{x}_1\bar{y}_2}(\tau) & \dots & C_{\bar{x}_1\bar{y}_m}(\tau) \\ C_{\bar{x}_2\bar{y}_1}(\tau) & C_{\bar{x}_2\bar{y}_2}(\tau) & \dots & C_{\bar{x}_2\bar{y}_m}(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ C_{\bar{x}_n\bar{y}_1}(\tau) & C_{\bar{x}_n\bar{y}_2}(\tau) & \dots & C_{\bar{x}_n\bar{y}_m}(\tau) \end{bmatrix}$$

Note that the elements in this matrix are all <u>scalar functions of $\tau$</u>!

The $n \times n$ auto-covariance function matrix $C_{\underline{\bar{x}}\underline{\bar{x}}}(\tau)$ is defined similarly:

$$C_{\underline{\bar{x}}\underline{\bar{x}}}(\tau) = \begin{bmatrix} C_{\bar{x}_1\bar{x}_1}(\tau) & C_{\bar{x}_1\bar{x}_2}(\tau) & \dots & C_{\bar{x}_1\bar{x}_n}(\tau) \\ C_{\bar{x}_2\bar{x}_1}(\tau) & C_{\bar{x}_2\bar{x}_2}(\tau) & \dots & C_{\bar{x}_2\bar{x}_n}(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ C_{\bar{x}_n\bar{x}_1}(\tau) & C_{\bar{x}_n\bar{x}_2}(\tau) & \dots & C_{\bar{x}_n\bar{x}_n}(\tau) \end{bmatrix}$$

Q: is this matrix symmetric?

# The spectral density matrix function

The spectral density function is defined as the Fourier transform of the covariance function. If we define the Fourier transform of a matrix to be a matrix with the Fourier transform of the elements, we get for the multi-dimensional $n \times m$ **cross-PSD matrix**:

$$S_{\underline{\overline{x}}\,\underline{\overline{y}}}(\omega) = \begin{bmatrix} \mathcal{F}\{C_{\overline{x}_1\overline{y}_1}(\tau)\} & \mathcal{F}\{C_{\overline{x}_1\overline{y}_2}(\tau)\} & \dots & \mathcal{F}\{C_{\overline{x}_1\overline{y}_m}(\tau)\} \\ \mathcal{F}\{C_{\overline{x}_2\overline{y}_1}(\tau)\} & \mathcal{F}\{C_{\overline{x}_2\overline{y}_2}(\tau)\} & \dots & \mathcal{F}\{C_{\overline{x}_2\overline{y}_m}(\tau)\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{F}\{C_{\overline{x}_n\overline{y}_1}(\tau)\} & \mathcal{F}\{C_{\overline{x}_n\overline{y}_2}(\tau)\} & \dots & \mathcal{F}\{C_{\overline{x}_n\overline{y}_m}(\tau)\} \end{bmatrix}$$

which then equals:

$$S_{\underline{\overline{x}}\,\underline{\overline{y}}}(\omega) = \begin{bmatrix} S_{\overline{x}_1\overline{y}_1}(\omega) & S_{\overline{x}_1\overline{y}_2}(\omega) & \dots & S_{\overline{x}_1\overline{y}_m}(\omega) \\ S_{\overline{x}_2\overline{y}_1}(\omega) & S_{\overline{x}_2\overline{y}_2}(\omega) & \dots & S_{\overline{x}_2\overline{y}_m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ S_{\overline{x}_n\overline{y}_1}(\omega) & S_{\overline{x}_n\overline{y}_2}(\omega) & \dots & S_{\overline{x}_n\overline{y}_m}(\omega) \end{bmatrix} \tag{5.19}$$

The $n \times n$ **auto-PSD matrix** can be written as:

$$S_{\underline{\bar{x}\bar{x}}}(\omega) = \begin{bmatrix} S_{\bar{x}_1\bar{x}_1}(\omega) & S_{\bar{x}_1\bar{x}_2}(\omega) & \dots & S_{\bar{x}_1\bar{x}_n}(\omega) \\ S_{\bar{x}_2\bar{x}_1}(\omega) & S_{\bar{x}_2\bar{x}_2}(\omega) & \dots & S_{\bar{x}_2\bar{x}_n}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ S_{\bar{x}_n\bar{x}_1}(\omega) & S_{\bar{x}_n\bar{x}_2}(\omega) & \dots & S_{\bar{x}_n\bar{x}_n}(\omega) \end{bmatrix}$$

This is a square matrix (but not symmetric!), with the auto-PSDs of the vector elements on the diagonal.

Note that the elements in this matrix are all <u>scalar functions</u> of $\omega$!

## Dynamic multivariable system analysis in the frequency domain

Suppose that $\underline{\bar{y}}$ and $\underline{\bar{u}}$ are related through the $m \times \ell$ impulse response function matrix $h_{\underline{\bar{y}}\underline{\bar{u}}}(t)$:

$$\underline{\bar{y}}(t) = h_{\underline{\bar{y}}\underline{\bar{u}}}(t) \star \underline{\bar{u}}(t), \qquad \text{or:}$$

$$\begin{bmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \\ \vdots \\ \bar{y}_m(t) \end{bmatrix} = \begin{bmatrix} h_{11}(t) & \ldots & h_{1\ell}(t) \\ \vdots & \ddots & \vdots \\ h_{m1}(t) & \ldots & h_{m\ell}(t) \end{bmatrix} \star \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \\ \vdots \\ \bar{u}_\ell(t) \end{bmatrix} \tag{5.21}$$

In the frequency domain this becomes:

$$\underline{\bar{Y}}(\omega) = H_{\underline{\bar{y}}\underline{\bar{u}}}(\omega) \cdot \underline{\bar{U}}(\omega)$$

or:

$$\begin{bmatrix} \bar{Y}_1(\omega) \\ \bar{Y}_2(\omega) \\ \vdots \\ \bar{Y}_m(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & \dots & H_{1\ell}(\omega) \\ \vdots & \ddots & \vdots \\ H_{m1}(\omega) & \dots & H_{m\ell}(\omega) \end{bmatrix} \cdot \begin{bmatrix} \bar{U}_1(\omega) \\ \bar{U}_2(\omega) \\ \vdots \\ \bar{U}_\ell(\omega) \end{bmatrix} \tag{5.22}$$

In order to find the relations between the input PSD matrix $S_{\underline{\bar{u}}\underline{\bar{u}}}(\omega)$ and the output PSD matrix $S_{\underline{\bar{y}}\underline{\bar{y}}}(\omega)$ first the relations between the covariance functions are derived.

It can be shown that (see lecture notes):

$$\boxed{\begin{aligned} C_{\underline{\bar{u}}\underline{\bar{y}}}(\tau) &= C_{\underline{\bar{u}}\underline{\bar{u}}}(\tau) \star h^T_{\underline{\bar{y}}\underline{\bar{u}}}(\tau) && (5.23) \\ C_{\underline{\bar{y}}\underline{\bar{u}}}(\tau) &= h_{\underline{\bar{y}}\underline{\bar{u}}}(-\tau) \star C_{\underline{\bar{u}}\underline{\bar{u}}}(\tau) && (5.24) \\ C_{\underline{\bar{y}}\underline{\bar{y}}}(\tau) &= h_{\underline{\bar{y}}\underline{\bar{u}}}(-\tau) \star C_{\underline{\bar{u}}\underline{\bar{u}}}(\tau) \star h^T_{\underline{\bar{y}}\underline{\bar{u}}}(\tau) && (5.25) \end{aligned}}$$

In the frequency domain this becomes:

$$\boxed{\begin{aligned} S_{\underline{u}\underline{y}}(\omega) &= S_{\underline{u}\underline{u}}(\omega) \cdot H^T_{\underline{y}\underline{u}}(\omega) && (5.26)\\ S_{\underline{y}\underline{u}}(\omega) &= H_{\underline{y}\underline{u}}(-\omega) \cdot S_{\underline{u}\underline{u}}(\omega) && (5.27)\\ S_{\underline{y}\underline{y}}(\omega) &= H_{\underline{y}\underline{u}}(-\omega) \cdot S_{\underline{u}\underline{u}}(\omega) \cdot H^T_{\underline{y}\underline{u}}(\omega) && (5.28) \end{aligned}}$$

In the scalar case, these functions are all equal to the ones derived in Chapter 3 of the lecture notes.

Note that then the transpose disappears, and that the order of the multiplications do not matter anymore.

## Dynamic multivariable system analysis in the **time domain**: the continuous time case

The dynamic behaviour of a stochastic process can be studied in the frequency domain, using the PSD functions, but also in the time domain. Consider the following well-known description of a multivariable LTI system in continuous time:

$$\boxed{\begin{array}{rcl} \dot{\underline{x}}(t) & = & A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) & = & C\underline{x}(t) + D\underline{u}(t) \end{array}}$$

Now we are interested in studying the statistical properties of the state vector $\underline{x}$, and the output vector $\underline{y}$ as a function of time, when this system is driven by a stochastic input vector $\underline{\bar{u}}$.

Then, and this is important to realize, when a linear system is driven by a stochastic signal, **all other signals are stochastic as well**, i.e., the state and output vectors become stochastic variables $\underline{\bar{x}}$ and $\underline{\bar{y}}$. In many, if not all, practical applications this means that we are **always dealing with stochastic processes**. I.e., all outcomes of a particular 'experiment' or 'trial' or 'test' are in fact **realizations**.

First the continuous-time case will be discussed, followed by a description of the discrete-time case. It will be shown that there are differences in both situations and we have to be very cautious about our simulations, which are generally, because we use computers and software like Matlab, in discrete-time (even when considering physical systems like aircraft which are of course continuous-time).

CT system: `lsim`, DT system: `dsim`

Consider the system as being fully described by the following linear vector differential equation:

$$\dot{\underline{x}}(t) \;=\; A\underline{x}(t) \;+\; B\underline{u}(t) \tag{5.1}$$

Solving this equation yields:

$$\boxed{\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B\underline{u}(\tau)\mathrm{d}\tau}, \tag{5.29}$$

with $\Phi(t, t_0)$ the **transition matrix**:

$$\Phi(t, t_0) = e^{A(t-t_0)} = I + A(t - t_0) + \frac{1}{2!}A^2(t - t_0)^2 + ... \tag{5.30}$$

**Proof**

Use the following property:

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)\mathrm{d}\tau = f(t,\tau)|_{\tau=t} + \int_{t_0}^{t}\frac{\partial}{\partial t}f(t,\tau)\mathrm{d}\tau$$

Then, assume that the system is at rest at $t_0$, differentiate (5.29):

$$\begin{aligned}
\underline{\dot{x}}(t) &= \dot{\Phi}(t,t_0)\underline{x}(t_0) + \frac{\partial}{\partial t}\int_{t_0}^{t}\Phi(t,\tau)B\underline{u}(\tau)\mathrm{d}\tau \\
&= A\cdot\Phi(t,t_0)\underline{x}(t_0) + \Phi(t,t)B\underline{u}(t) + \int_{t_0}^{t}\frac{\partial}{\partial t}\Phi(t,\tau)B\underline{u}(\tau)\mathrm{d}\tau \\
&= A\cdot\Phi(t,t_0)\underline{x}(t_0) + B\underline{u}(t) + \int_{t_0}^{t}A\Phi(t,\tau)B\underline{u}(\tau)\mathrm{d}\tau \\
&= A\cdot\Phi(t,t_0)\underline{x}(t_0) + B\underline{u}(t) + A\cdot\left(\int_{t_0}^{t}\Phi(t,\tau)B\underline{u}(\tau)\mathrm{d}\tau\right) \\
&= A\cdot\underbrace{\left(\Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t}\Phi(t,\tau)B\underline{u}(\tau)\mathrm{d}\tau\right)}_{x(t)} + B u(t) \\
&= A\underline{x}(t) + B\underline{u}(t)
\end{aligned}$$

q.e.d

Now consider the stochastic problem where the input vector $\underline{u}(t)$ is chosen to be white noise $\underline{\bar{w}}(t)$ with zero mean $(\mathsf{E}\{\underline{\bar{w}}(t)\} = 0)$ and auto-covariance matrix:

$$C_{\underline{\bar{w}\bar{w}}}(\tau) = W\delta(\tau), \tag{5.32}$$

where $W$ denotes the **noise intensity matrix**. $W = \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_\ell \end{pmatrix}$

Then, the **mean** of the state vector $\underline{\bar{x}}$ is given by:

$$\begin{aligned} \mathsf{E}\{\underline{\bar{x}}(t)\} &= \mathsf{E}\left\{\Phi(t,t_0)\underline{\bar{x}}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B\underline{\bar{w}}(\tau)\mathrm{d}\tau\right\} \\ &= \Phi(t,t_0)\mathsf{E}\{\underline{\bar{x}}(t_0)\} + \int_{t_0}^{t} \Phi(t,\tau)B\mathsf{E}\{\underline{\bar{w}}(\tau)\}\mathrm{d}\tau \\ &= \Phi(t,t_0)\mathsf{E}\{\underline{\bar{x}}(t_0)\} \end{aligned} \tag{5.33}$$

The **auto-covariance matrix** of the state vector is then:

$$
\begin{aligned}
\mathsf{E}\left\{\underline{\bar{x}}(t_1)\underline{\bar{x}}^T(t_2)\right\} &= \mathsf{E}\Bigg\{\left(\Phi(t_1,t_0)\underline{\bar{x}}(t_0)+\int_{t_0}^{t_1}\Phi(t_1,\tau_1)B\underline{\bar{w}}(\tau_1)\mathrm{d}\tau_1\right) \\
&\quad \cdot\left(\Phi(t_2,t_0)\underline{\bar{x}}(t_0)+\int_{t_0}^{t_2}\Phi(t_2,\tau_2)B\underline{\bar{w}}(\tau_2)\mathrm{d}\tau_2\right)^T\Bigg\} \\
&= \mathsf{E}\left\{\Phi(t_1,t_0)\underline{\bar{x}}(t_0)\underline{\bar{x}}^T(t_0)\Phi^T(t_2,t_0)\right\} \\
&\quad +\mathsf{E}\left\{\Phi(t_1,t_0)\underline{\bar{x}}(t_0)\cdot\int_{t_0}^{t_2}\underline{\bar{w}}^T(\tau_2)B^T\Phi^T(t_2,\tau_2)\mathrm{d}\tau_2\right\} \\
&\quad +\mathsf{E}\left\{\int_{t_0}^{t_1}\Phi(t_1,\tau_1)B\underline{\bar{w}}(\tau_1)\mathrm{d}\tau_1\cdot\underline{\bar{x}}^T(t_0)\Phi^T(t_2,t_0)\right\} \\
&\quad +\mathsf{E}\left\{\int_{t_0}^{t_1}\Phi(t_1,\tau_1)B\underline{\bar{w}}(\tau_1)\mathrm{d}\tau_1\cdot\int_{t_0}^{t_2}\underline{\bar{w}}^T(\tau_2)B^T\Phi^T(t_2,\tau_2)\mathrm{d}\tau_2\right\}
\end{aligned}
$$

Set $t_1 = t_2 = t$ and we get:

$$
\begin{aligned}
C_{\underline{\bar{x}}\underline{\bar{x}}}(t) \quad = \quad & \Phi(t,t_0)\mathsf{E}\left\{\underline{\bar{x}}(t_0)\underline{\bar{x}}^T(t_0)\right\}\Phi^T(t,t_0) \\
& +\Phi(t,t_0)\cdot\int_{t_0}^{t}\mathsf{E}\left\{\underline{\bar{x}}(t_0)\underline{\bar{w}}^T(\tau)\right\}B^T\Phi^T(t,\tau)\mathrm{d}\tau \\
& +\int_{t_0}^{t}\Phi(t,\tau)B\mathsf{E}\left\{\underline{\bar{w}}(\tau)\underline{\bar{x}}^T(t_0)\right\}\mathrm{d}\tau\cdot\Phi^T(t,t_0) \\
& +\int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau_1)B\mathsf{E}\left\{\underline{\bar{w}}(\tau_1)\underline{\bar{w}}^T(\tau_2)\right\}B^T\Phi^T(t,\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1
\end{aligned}
$$

And because we are dealing with white noise, which is uncorrelated with everything except itself (for zero $\tau$) we get our main result:

$$
\boxed{C_{\underline{\bar{x}}\underline{\bar{x}}}(t) = \Phi(t,t_0)C_{\underline{\bar{x}}\underline{\bar{x}}}(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)BWB^T\Phi^T(t,\tau)\mathrm{d}\tau} \quad (5.35)
$$

In most cases, we are in particular interested in the covariance matrix of the **steady-state**, i.e., after the transients have died out. Hence, when $t \to \infty$:

$$
\begin{aligned}
C_{\underline{\bar{x}\bar{x}}_{ss}} &= \lim_{t\to\infty} C_{\underline{\bar{x}\bar{x}}}(t) \\
&= \lim_{t\to\infty} \left\{ e^{A(t-t_0)} C_{\underline{\bar{x}\bar{x}}}(t_0) e^{A^T(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} BWB^T e^{A^T(t-\tau)} \mathrm{d}\tau \right\}
\end{aligned}
$$

If and only if $A$ is asymptotically stable (all eigenvalues have strictly negative parts), the first hand term tends to zero for arbitrary $C_{\underline{\bar{x}\bar{x}}}(t_0)$ and we get:

$$\boxed{C_{\underline{\bar{x}\bar{x}}_{ss}} = \int_0^{\infty} e^{A\sigma} BWB^T e^{A^T\sigma} \mathrm{d}\sigma} \tag{5.36}$$

WHY do we obtain the integral from 0 to $\infty$??

$C_{\underline{\bar{x}\bar{x}}_{ss}}$ can be computed through the so-called **Lyapunov equation**:

$$\boxed{0 = AC_{\underline{\bar{x}\bar{x}}_{ss}} + C_{\underline{\bar{x}\bar{x}}_{ss}} A^T + BWB^T} \tag{5.37}$$

Matlab: `lyap`, `dlyap`

**Proof**

It can be shown that $C_{\underline{\overline{x}\overline{x}}_{ss}}$ (5.36) is the unique solution of the Lyapunov equation (5.37). First differentiate (5.35) with respect to time $t$:

$$\begin{aligned}\dot{C}_{\underline{\overline{x}\overline{x}}}(t) \quad &= \quad \dot{\Phi}(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)\dot{C}_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\dot{\Phi}^T(t,t_0) \\ &\quad + \frac{\partial}{\partial t}\int_{t_0}^{t}\Phi(t,\tau)BWB^T\Phi^T(t,\tau)\mathrm{d}\tau \end{aligned},$$

where $\dot{C}_{\underline{\overline{x}\overline{x}}}(t_0) = 0$. In steady-state $\dot{C}_{\underline{\overline{x}\overline{x}}}(t)$ becomes zero and we get:

$$\begin{aligned} 0 \quad &= \quad A\cdot\Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0)\cdot A^T \\ &\quad + \Phi(t,t)BWB^T\Phi^T(t,t) + \int_{t_0}^{t}\frac{\partial}{\partial t}\Phi(t,\tau)BWB^T\Phi^T(t,\tau)\mathrm{d}\tau \\ &= \quad A\cdot\Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0)\cdot A^T \\ &\quad + BWB^T + \int_{t_0}^{t}\left(A\cdot\Phi(t,\tau)BWB^T\Phi^T(t,\tau) + \Phi(t,\tau)BWB^T\Phi^T(t,\tau)A^T\right)\mathrm{d}\tau \\ &= \quad A\cdot\left(\Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)BWB^T\Phi^T(t,\tau)\mathrm{d}\tau\right) \\ &\quad + \left(\Phi(t,t_0)C_{\underline{\overline{x}\overline{x}}}(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)BWB^T\Phi^T(t,\tau)\mathrm{d}\tau\right)\cdot A^T + BWB^T \\ &= \quad A\cdot C_{\underline{\overline{x}\overline{x}}_{ss}} + C_{\underline{\overline{x}\overline{x}}_{ss}}\cdot A^T + BWB^T \end{aligned}$$

q.e.d.

Learn to use the Lyapunov equation (5.37) to obtain the results summarized in Tables 3.5 and 3.6 of Chapter 3.

Finally, note that we are dealing with the covariance function matrix as a function of time for zero $\tau$'s only. That is, the auto-covariance function matrix $C_{\underline{\bar{x}\bar{x}}}(\tau)$ of (5.17, the cross-covariance function, the same holds for the auto-covariance function), is being evaluated as a function of time $t$, for zero values of $\tau$ only. This is reflected in the derivation of Equation (5.35), where $t_1$ and $t_2$ were both set equal to $t$.

Hence, to study the general case, we should be looking at how the cross-covariance function $C_{\underline{\bar{x}\bar{x}}}$ evolves in time, *for different values of* $\tau$, i.e., $C_{\underline{\bar{x}\bar{x}}}(\tau; t)$. This is beyond the interest for this course.

## Dynamic multivariable system analysis in the time domain: the discrete time case

Consider the linear difference equation describing the dynamics of an LTI system in discrete-time:

$$\boxed{\underline{x}[k+1] = \Phi\underline{x}[k] + \Gamma\underline{u}[k]} \tag{5.3}$$

The relation with the continuous-time counterpart is most conveniently expressed in a Taylor series expansion form:

$$\Phi = I + \triangle t A + \frac{\triangle t^2}{2!}A^2 + \frac{\triangle t^3}{3!}A^3 + ... \tag{5.38}$$

$$\Gamma = \triangle t B + \frac{\triangle t^2}{2!}AB + \frac{\triangle t^3}{3!}A^2B + ... \tag{5.39}$$

with $\triangle t$ the sample time. For the formal derivations of these matrices, see the lecture notes.

Now, just as in the continuous-time case, we will develop expressions for the statistical properties of the system state $\underline{\bar{x}}$ at time $t_k$ when the system input signal $\underline{\bar{u}}$ is a discrete-time zero-mean white noise process $\underline{\bar{w}}[k]$ with auto-covariance $C_{\underline{\bar{w}\bar{w}}}$.

The **mean** is calculated as follows:

$$\mathsf{E}\{\underline{\bar{x}}[k+1]\} = \mathsf{E}\{\Phi\underline{\bar{x}}[k] + \Gamma\underline{\bar{w}}[k]\} = \Phi\mathsf{E}\{\underline{\bar{x}}[k]\} + \Gamma\mathsf{E}\{\underline{\bar{w}}[k]\},$$

resulting in the following recursive expression:

$$\boxed{\mathsf{E}\{\underline{\bar{x}}[k+1]\} = \Phi\mathsf{E}\{\underline{\bar{x}}[k]\}} \qquad (5.40)$$

Note that the average value of the state will remain zero for all $k$ when the initial expected state vector is zero.

The **auto-covariance** matrix can be calculated with:

$$\begin{aligned}
C_{\underline{\bar{x}}\underline{\bar{x}}}[k+1] &= \mathsf{E}\left\{\underline{\bar{x}}[k+1]\underline{\bar{x}}^T[k+1]\right\} \\
&= \mathsf{E}\left\{(\Phi\underline{\bar{x}}[k] + \Gamma\underline{\bar{w}}[k])(\Phi\underline{\bar{x}}[k] + \Gamma\underline{\bar{w}}[k])^T\right\} \\
&= \mathsf{E}\{\Phi\underline{\bar{x}}[k]\underline{\bar{x}}^T[k]\Phi^T + \Gamma\underline{\bar{w}}[k]\underline{\bar{w}}^T[k]\Gamma^T \\
&\quad +\Gamma\underline{\bar{w}}[k]\underline{\bar{x}}^T[k]\Phi^T + \Phi\underline{\bar{x}}[k]\underline{\bar{w}}^T[k]\Gamma^T\}
\end{aligned}$$

We are dealing with zero-mean white noise that is uncorrelated with everything except itself (for $\tau = 0$), yielding:

$$\boxed{C_{\underline{\bar{x}}\underline{\bar{x}}}[k+1] = \Phi C_{\underline{\bar{x}}\underline{\bar{x}}}[k]\Phi^T + \Gamma C_{\underline{\bar{w}}\underline{\bar{w}}}[k]\Gamma^T} \tag{5.41}$$

We may ask ourselves: *what is the auto-covariance matrix $C_{\underline{\bar{w}\bar{w}}}$ of the discrete-time white noise signal?* When the system under consideration is *truly* a discrete-time system, this is a trivial question. But when the system under consideration is a discrete-time **model** of a continuous-time system, the choice for the white noise intensity is no longer evident.

The question is how to choose the auto-covariance of the discrete white noise signal in order to truly simulate the continuous time situation? It is clear that what is required is that the covariance matrix of the system state should increase with an equal amount in discrete time as it would in continuous time.

In discrete time, the auto-covariance matrix $C_{\underline{\bar{x}\bar{x}}}[k]$ from time $t_k = k\triangle t$ to time $t_{k+1} = (k+1)\triangle t$ equals:

$$\triangle C_{\underline{\bar{x}\bar{x}}_{dis}} = C_{\underline{\bar{x}\bar{x}}}[k+1] - C_{\underline{\bar{x}\bar{x}}}[k] = \Phi C_{\underline{\bar{x}\bar{x}}}[k]\Phi^T + \Gamma C_{\underline{\bar{w}\bar{w}}}[k]\Gamma^T - C_{\underline{\bar{x}\bar{x}}}[k]$$

In continuous time, this increase of the covariance matrix is:

$$\begin{aligned}
\triangle C_{\underline{\bar{x}\bar{x}}_{con}} &= C_{\underline{\bar{x}\bar{x}}}(t_k + \triangle t) - C_{\underline{\bar{x}\bar{x}}}(t_k) \\
&= \Phi(t_k + \triangle t, t_k) C_{\underline{\bar{x}\bar{x}}}(t_k) \Phi^T(t_k + \triangle t, t_k) \\
&\quad + \int_{t_k}^{t_k+\triangle t} \Phi(t_k + \triangle t, \tau) B W B^T \Phi^T(t_k + \triangle t, \tau) \mathrm{d}\tau - C_{\underline{\bar{x}\bar{x}}}(t_k)
\end{aligned}$$

Now, the first and last terms are equal (use the Taylor expansion), so when we want an equal increase in the covariance matrix, the middle terms must be equal as well, i.e.:

$$\Gamma C_{\underline{\bar{w}\bar{w}}}[k]\Gamma^T = \int_{t_k}^{t_k+\triangle t} \Phi(t_k + \triangle t, \tau) B W B^T \Phi^T(t_k + \triangle t, \tau) \mathrm{d}\tau$$

with $\Gamma \approx B\triangle t$ and $\Phi(t_k + \triangle t, \tau) \approx I$ on the interval $[t_k, t_k + \triangle t]$ we get:

$$\triangle t^2 B C_{\underline{\bar{w}\bar{w}}} B^T = BWB^T \triangle t \qquad \Rightarrow \frac{\triangle t^2}{\triangle t} B C_{\underline{\bar{w}\bar{w}}} B^T = BWB^T$$

or, our main result:

$$\boxed{C_{\underline{\bar{w}\bar{w}}} = \frac{W}{\triangle t}} \tag{5.45}$$

Concluding, when a continuous time physical stochastic process driven by continuous time white noise with a noise intensity matrix $W$ is to be *modelled by a discrete-time model*, **then the discrete time system should be driven by a discrete time random time series with covariance matrix** $\frac{W}{\triangle t}$.

NOTE: when using Matlab's lsim-function to simulate the response of a CT system to white noise one <u>also</u> has to scale the white noise intensity to get the right results. Carefully study Example 5.1 of the lecture notes.

**Note** that this was the second mistake in the "old" lecture notes, <u>masked</u> by the first mistake made in Chapter 4 (still remember that one??

The discrete time method for calculating the auto-covariance matrix is a very powerful mathematical tool. It allows the treatment of virtually all possible situations since the input auto-covariance $C_{\underline{\bar{w}\bar{w}}}$ as well as the system matrices $\Phi$, $\Gamma$, $C$ and $D$ are *not* restricted to be time-invariant.

The auto-covariance matrix $C_{\underline{\bar{x}\bar{x}}}$ tends towards a constant matrix $C_{\underline{\bar{x}\bar{x}}_{ss}}$ when $t \to \infty$. In the steady-state condition we get:

$$C_{\underline{\bar{x}\bar{x}}_{ss}} = \Phi C_{\underline{\bar{x}\bar{x}}_{ss}} \Phi^T + \Gamma \frac{W}{\triangle t} \Gamma^T, \tag{5.46}$$

which is a discrete-time Lyapunov equation. A method to solve this equation is simply to transform the discrete pair $(\Phi, \Gamma)$ to the continuous form $(A, B)$ and solve the continuous-time Lyapunov equation.

or use `dlyap`

A more practical approach would be to run (5.43) until

$$C_{\underline{\bar{x}\bar{x}}}[k+1] - C_{\underline{\bar{x}\bar{x}}}[k] < \epsilon, \tag{5.47}$$

for some matrix $\epsilon$.

Note that with the increased use of digital computers and software packages like Matlab, continuous time systems are usually converted into their discrete-time counterpart. The statistical properties are generally obtained (estimated) using these discrete-time simulations.

Yet, still another method exists, the so-called **impulse response method**, which is suitable in particular for analog computers. This method is discussed in §5.7 of the lecture notes and will not be discussed any further here.

Concluding, when we are dealing with stochastic processes, it is often not possible to derive the probability distribution function or the p.d.f. of all variables. The first and second order moments, i.e. the mean and the auto-covariance matrix, however, can generally be estimated quite well. Then, it is often assumed that all signals involved have a Normal (Gaussian) distribution, as this distribution is completely determined by its mean and covariance. When dealing with *deviations* from a certain datum, as often is the case, the average of $\underline{\bar{x}}$ is zero and only the auto-covariance matrix $C_{\underline{\bar{x}}\underline{\bar{x}}}$ needs to be determined.

As we have seen in Chapters 2 to 5, there are now a couple of methods available for obtaining *an estimate* of the covariance matrix.

Recapitulating:

1. In the time domain, using the raw data, and under the assumption that the stochastic process under study is *ergodic*, $C_{\underline{\bar{x}\bar{x}}}$ can be obtained by a sampling of a realization:

$$C_{\underline{\bar{x}\bar{x}}} = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{x}[i] - \mu_{\underline{\bar{x}}}) \cdot (\bar{x}[i] - \mu_{\underline{\bar{x}}})^T$$

   This method is often referred to as the **Monte Carlo method**, yields acceptable results only for a very large number of samples $(N \to \infty)$ and can also only be used to obtain an estimate for the *steady-state* auto-covariance matrix.

2. In the frequency domain, through integrating the estimated PSD matrix function:

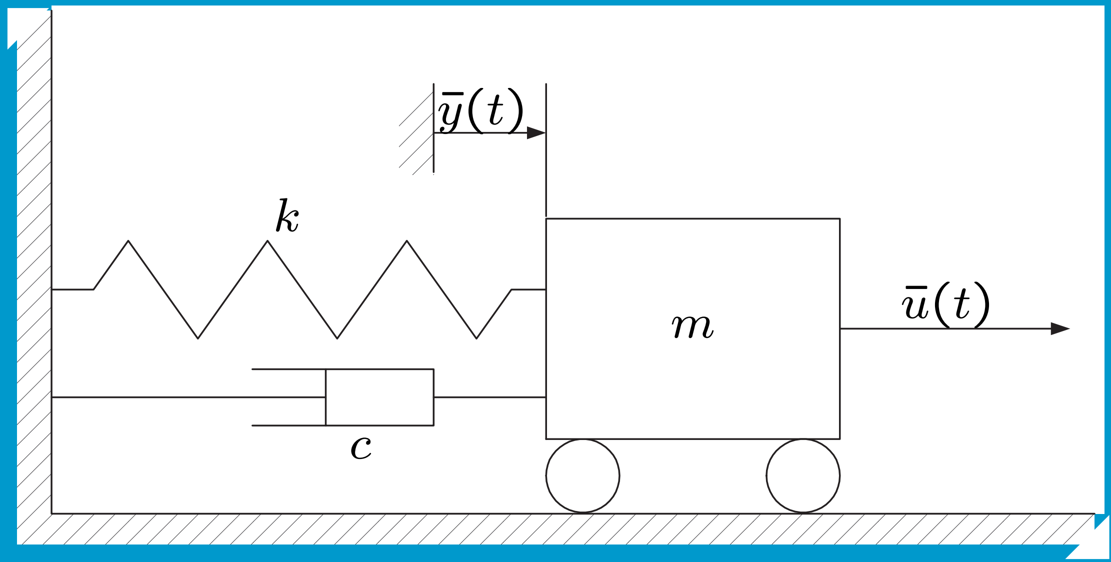
$$C_{\underline{\bar{x}\bar{x}}}(\tau = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}_{\underline{\bar{x}\bar{x}}}(\omega) \mathrm{d}\omega$$

   It is clear that also this method is only useful to obtain an estimate for the *steady-state* auto-covariance matrix.

3 In continuous time, when the stochastic process can be described using an LTI model driven by white noise we can simply integrate equation (5.35) to obtain the *transient* as well as the *steady-state* (for large integration times) of the auto-covariance function $C_{\underline{\bar{x}\bar{x}}}(t)$. The steady-state matrix can also be obtained through solving the Lyapunov equation (5.36).

4 In discrete time, when the stochastic process can be described using an LTI model driven by white noise we can simply recursively compute equation (5.41) to obtain the *transient* as well as the *steady-state* (for large calculation periods) of the auto-covariance function $C_{\underline{\bar{x}\bar{x}}}[k]$. The steady-state matrix can also be obtained through transforming the system dynamics to continuous time and then solving the Lyapunov equation (5.36).

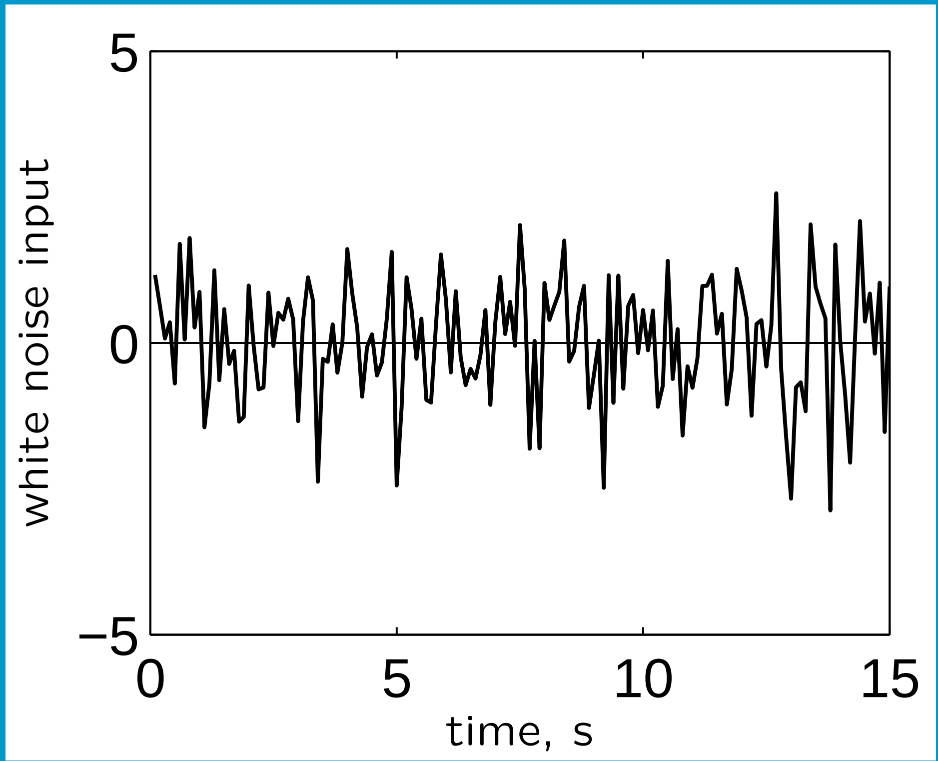
## Example 5.2



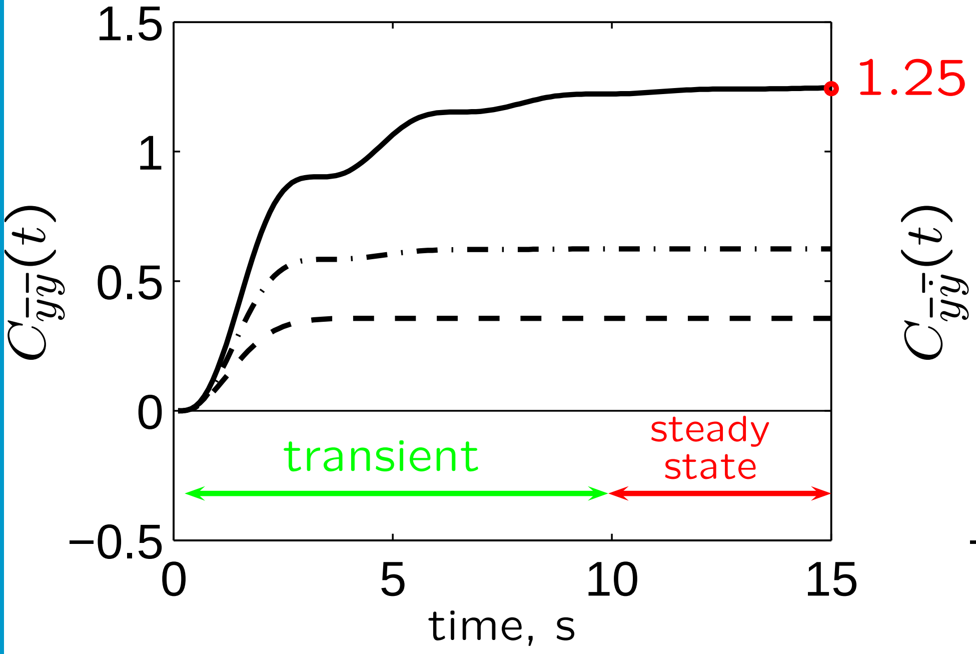

Calculate the auto-covariance matrix of this mass-spring-damper system (a $2^{nd}$ order system) in various situations, using various methods.

Note: $m\ddot{y} + c\dot{y} + ky = u$

[1] Response to white noise input, autocovariance matrix zero:




$$C_{\bar{y}\bar{y}}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



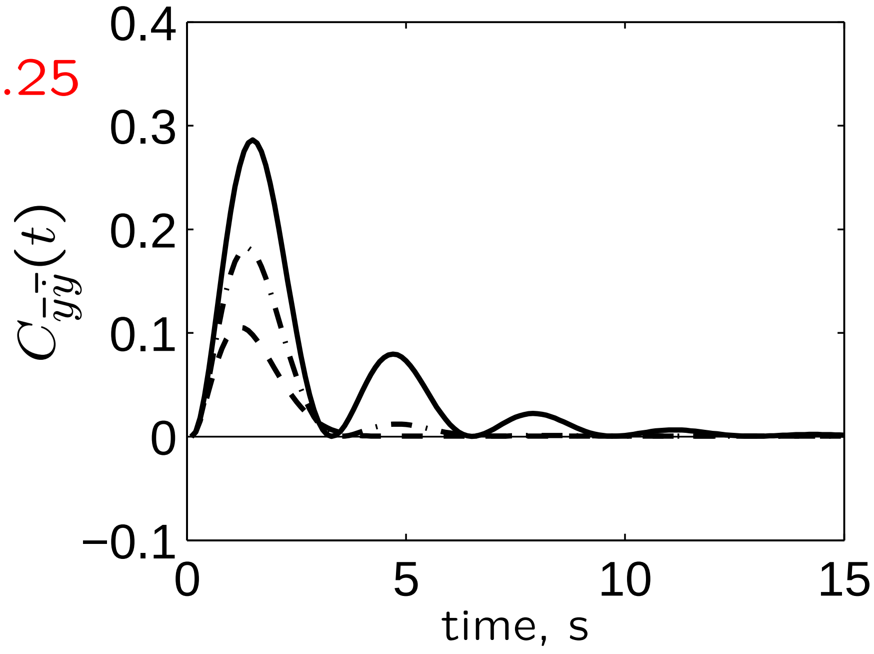
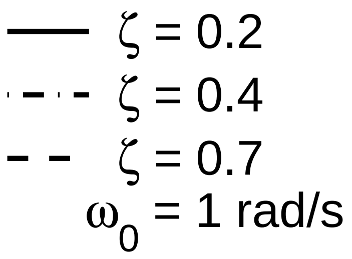
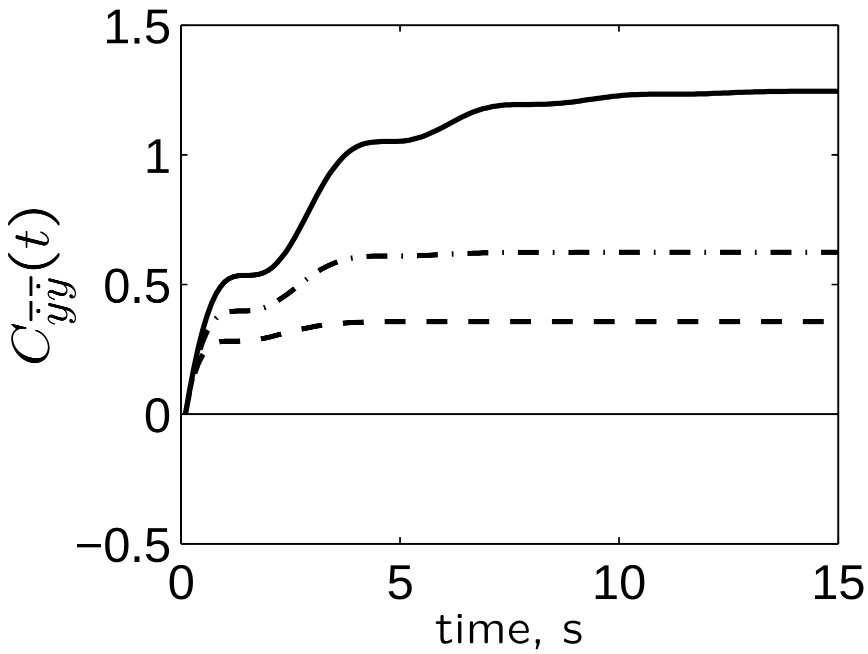
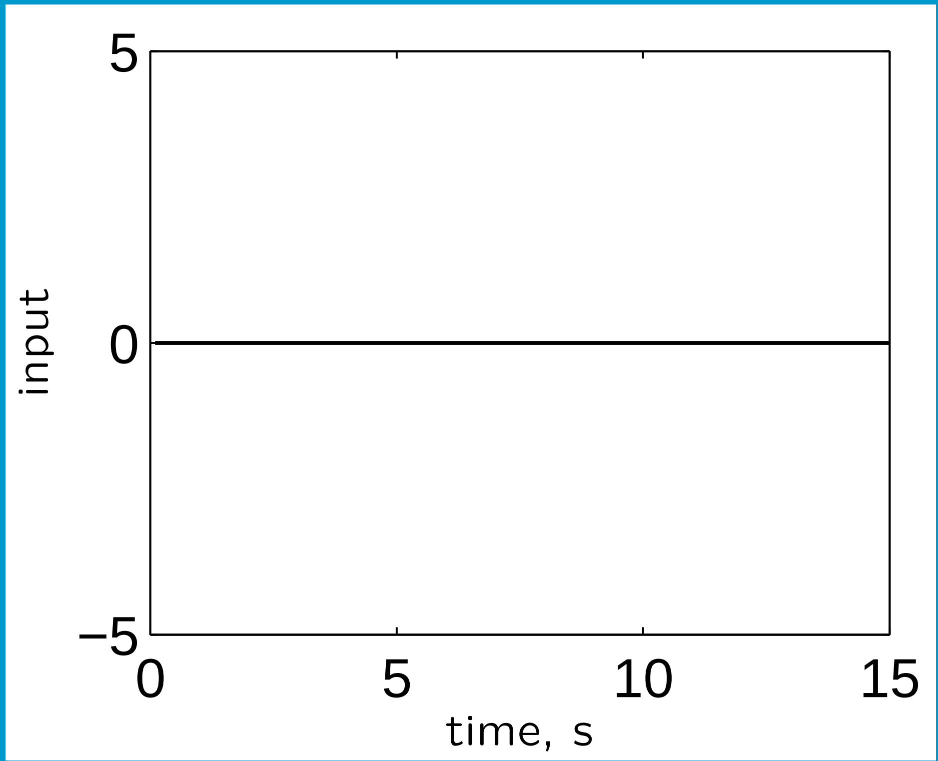




Table 3.6: $\frac{\omega_0}{4\zeta}$

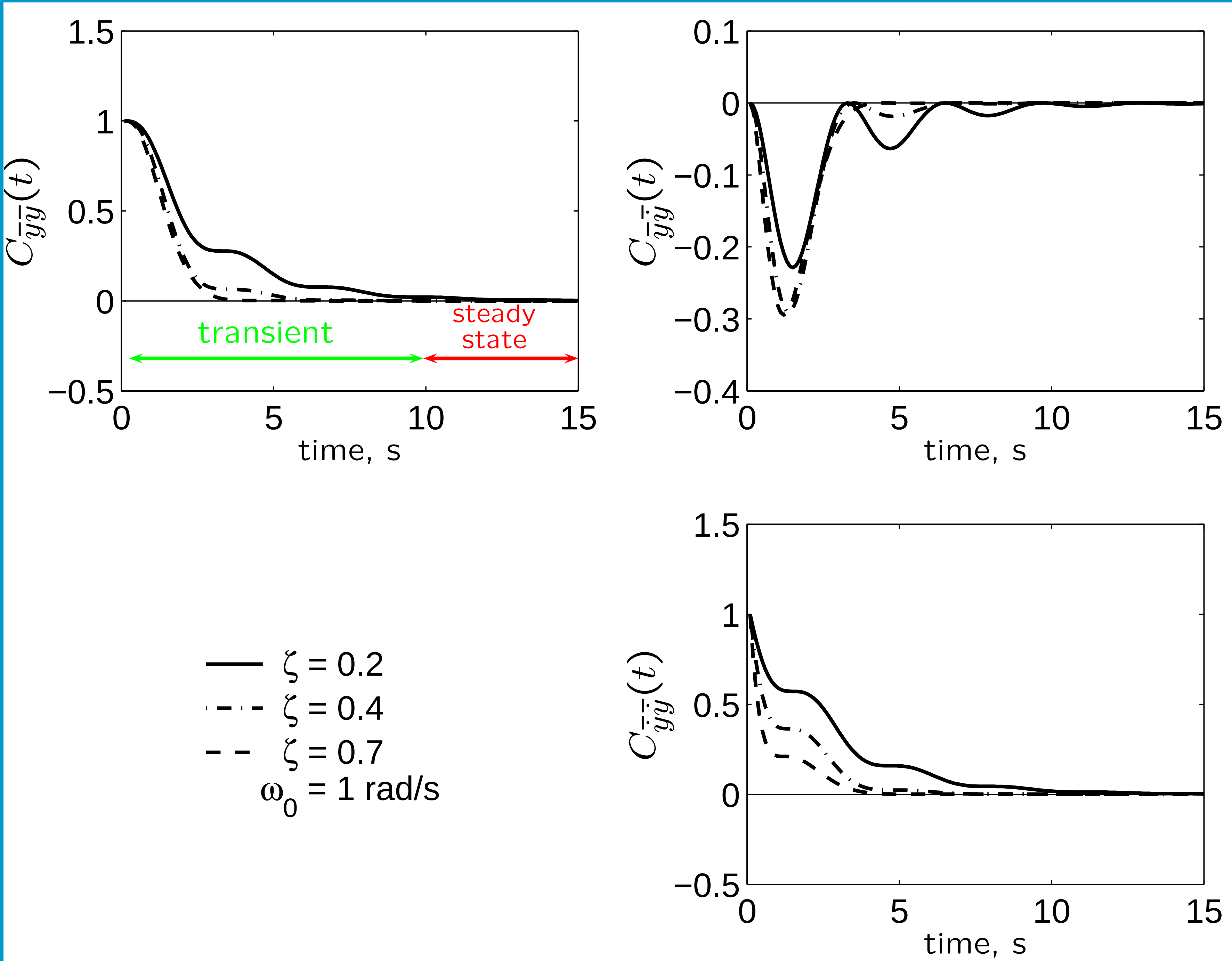


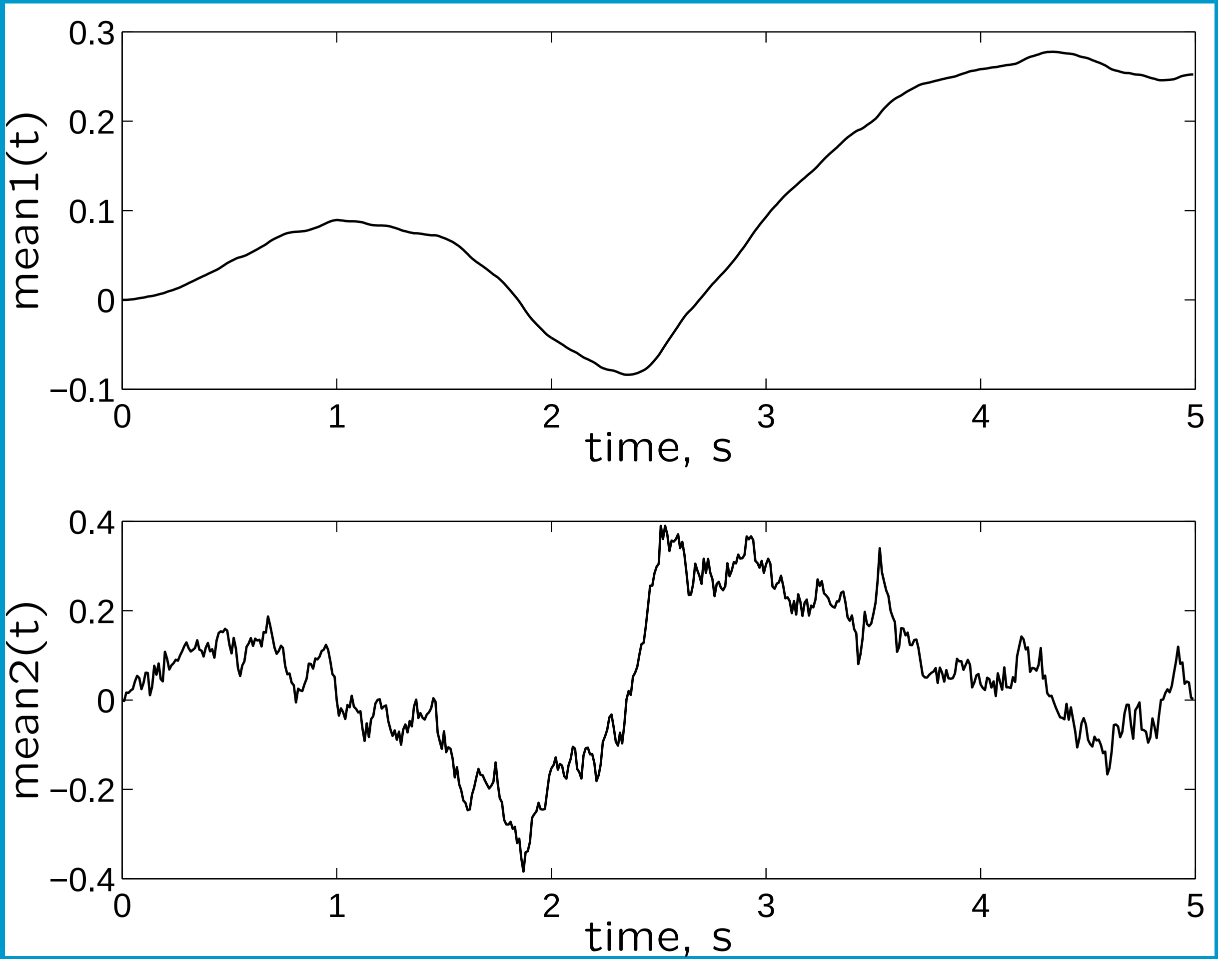
[2] Response to zero input signal, autocovariance matrix non-zero:




$$C_{\overline{y}\overline{y}}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$


1.5
1
0.5
0
−0.5
0
5
10
15
time, s
$C_{\bar{y}\bar{y}}(t)$
transient
steady state
0.1
0
−0.1
−0.2
−0.3
−0.4
$C_{\bar{y}\dot{\bar{y}}}(t)$
$C_{\dot{\bar{y}}\dot{\bar{y}}}(t)$
ζ = 0.2
ζ = 0.4
ζ = 0.7
ω0 = 1 rad/s

## **Example 5.4:** Monte Carlo simulation, $N$=15

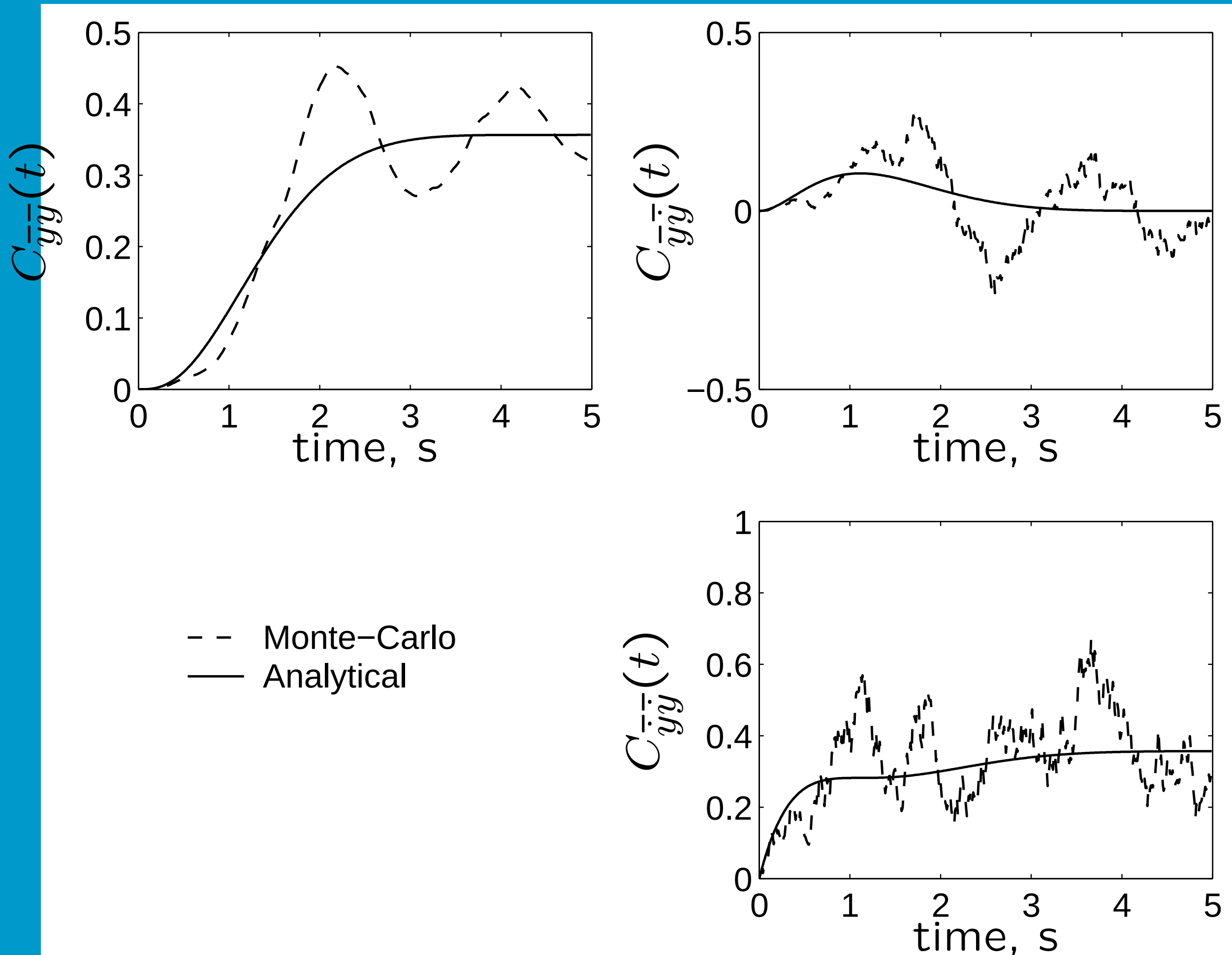



In this simulation: $\omega_n =$ 1 rad/s, $\zeta = 0.7$, $W = 1$ and $\triangle t = 0.01$ s.

We have averaged over 15 realizations. Note that, theoretically, the means should be zero (as $\mu_{\bar{u}} = 0$).

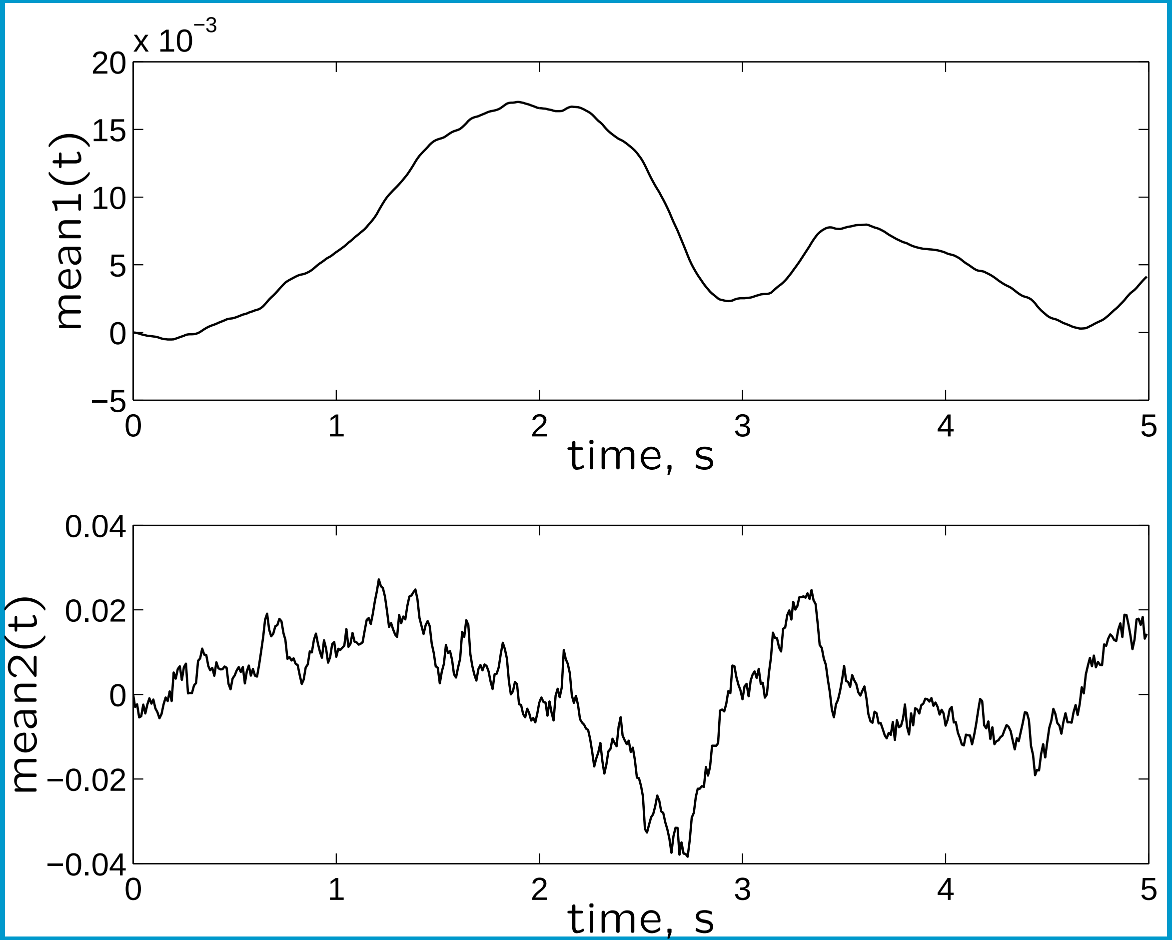
The top figure shows the mean of $\bar{y}$, the bottom figure shows the mean of $\bar{\dot{y}}$.

## **Example 5.4:** Monte Carlo simulation, $N$=15



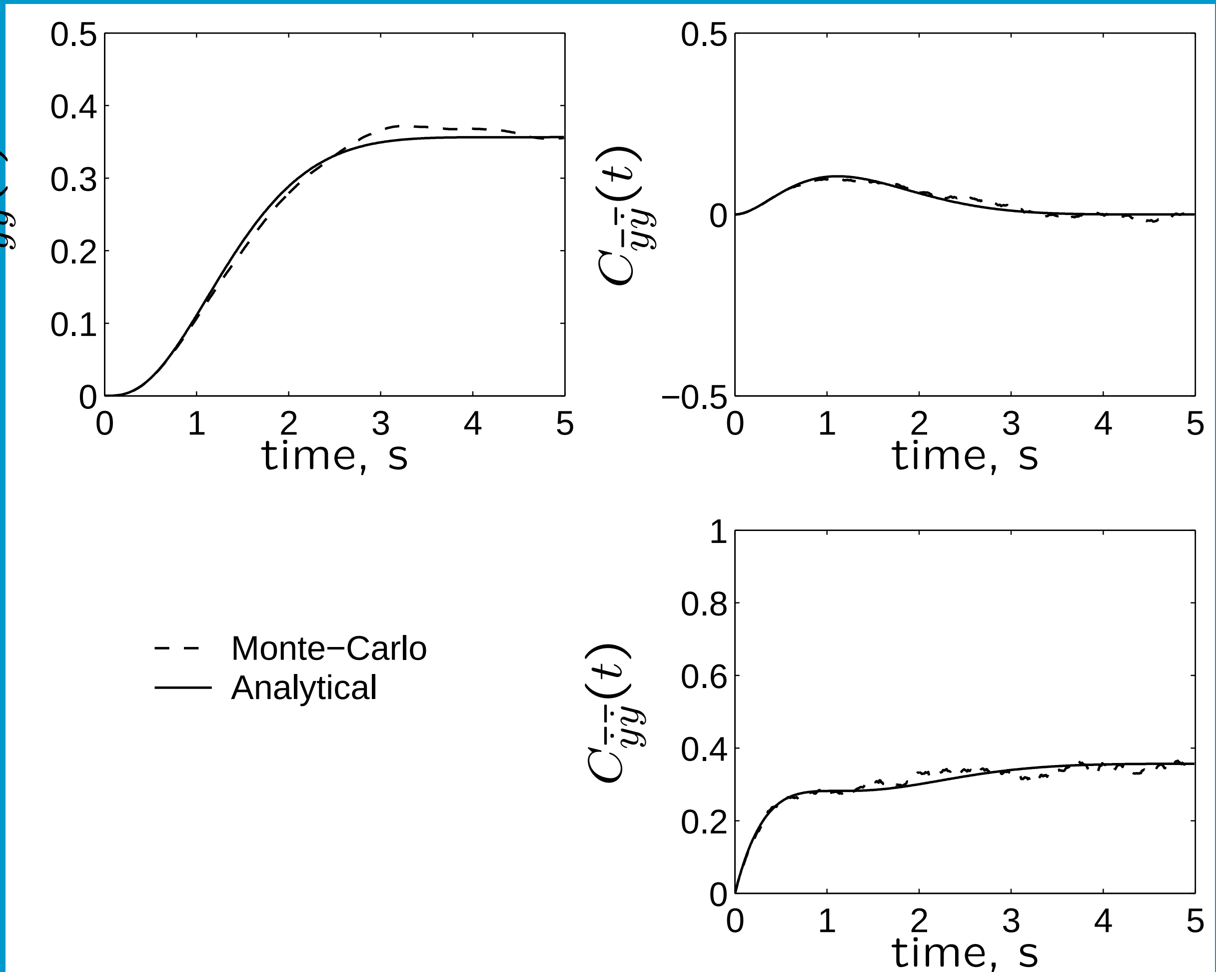

The continuous lines show the analytical auto- and crosscovariance functions as a function of time, the dashed lines show the results of the Monte Carlo analysis (averaging over 15 realizations).

## **Example 5.4:** Monte Carlo simulation, $N$=1500




We have averaged over 1500 realizations. Note that, theoretically, the means should be zero (as $\mu_{\bar{u}} = 0$). And indeed, with $N = 1500$ the means are much closer to zero.

## **Example 5.4:** Monte Carlo simulation, $N$=1500




We have averaged over 1500 realizations. The analytical response is approximated quite well.

## Remainder of this course

In the remainder of the lecture ae4-304 "*Aircraft Responses to Atmospheric Turbulence*" we will apply the techniques learned in the first five chapters to compute the response of aircraft to stochastic input signals.

The linearized aircraft flight dynamics are known from the third-year course "*Flight Dynamics*":

$$\begin{cases} \dot{\underline{x}}_a(t) &= A_a\, \underline{x}_a(t) + B_a\, \underline{u}_a(t), \\ \\ \underline{y}_a(t) &= C_a\, \underline{x}_a(t) + D_a\, \underline{u}_a(t). \end{cases}$$

In Chapter 6 of this lecture we will see that atmospheric turbulence can be modelled using so-called **Dryden** spectra. These may be considered as <u>shaping filters</u> that transform white noise into colored noise that has similar stochastic characteristics as atmospheric turbulence.

Hence, they can also be described in state-space form:

$$\begin{cases} \dot{\underline{x}}_t(t) & = & A_t\,\underline{x}_t(t) + B_t\,\underline{w}_t(t), \\ \underline{y}_t(t) & = & C_t\,\underline{x}_t(t) + D_t\,\underline{w}_t(t), \end{cases}$$

with $\underline{y}_t(t)$ the atmospheric turbulence and $\underline{w}_t(t)$ zero-mean Gaussian white noise.

We can combine both LTI dynamic systems and get the complete system, describing the aircraft response to atmospheric turbulence.

With $\underline{x}(t) = \begin{pmatrix} \underline{x}_a(t) \\ \underline{x}_t(t) \end{pmatrix}$, $\underline{u}(t) = \underline{u}_a(t)$, $\underline{y}(t) = \underline{y}_a(t)$, and $\underline{w}(t) = \underline{w}_t(t)$, one obtains:

$$\left\{ \begin{aligned} \underline{\dot{x}}(t) &= \begin{pmatrix} A_a & \Xi \\ 0 & A_t \end{pmatrix} \begin{pmatrix} \underline{x}_a \\ \underline{x}_t \end{pmatrix}(t) + \begin{pmatrix} B_a \\ 0 \end{pmatrix} \underline{u}(t) + \begin{pmatrix} 0 \\ B_t \end{pmatrix} \underline{w}(t) \\ \underline{y}(t) &= (C_a \ 0) \begin{pmatrix} \underline{x}_a \\ \underline{x}_t \end{pmatrix}(t) + D_a \, \underline{u}(t) \end{aligned} \right. \qquad (*)$$

The remaining question then is, how the aircraft state $\underline{x}_a$ is affected by the turbulence shaping filter state $\underline{x}_t$ ??

That question boils down to **what is matrix** $\Xi$ in Equation (*) ???

. . . . . . . . . . . . . . . . . . . . . . . . this will be the subject of Chapters 7 and 8.