# Spectral Analysis of Continuous-time SPs

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$\boxed{\text{Spectral Analysis of Continuous-time Stochastic Processes}}$

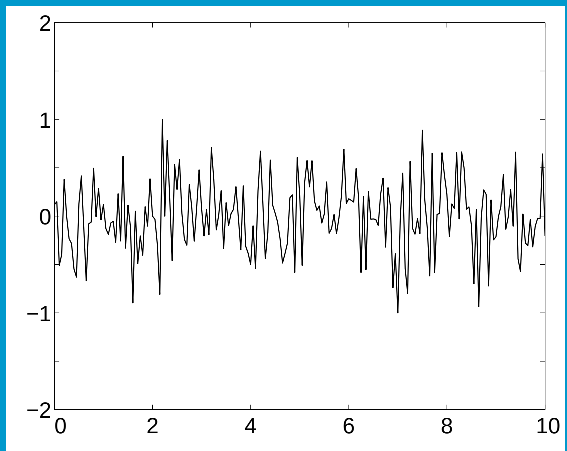
For this lecture the following material was used:

- Chapter 3 of Lecture notes *Atmospheric Flight Dynamics*
- MIT course "Signals & Systems", by prof. Weiss (put on Brightspace)
- Book *Systems & Signals* by Oppenheim, Wilsky & Young

# Introduction to spectral analysis

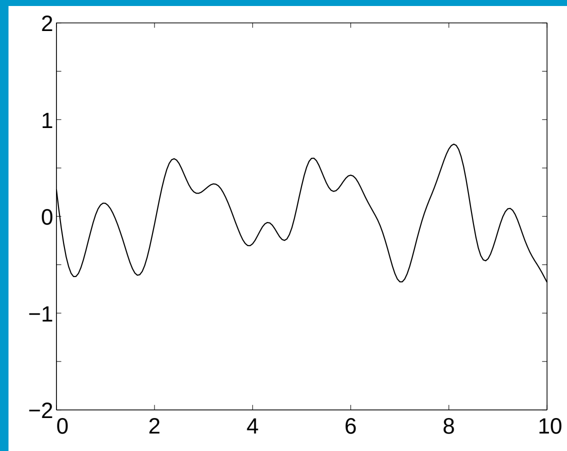
In Chapter 2 we have discussed the various properties (average, variance, covariance, correlation) of stochastic processes in the *time domain*.

process $\bar{x}_1(t)$




time, s

process $\bar{x}_2(t)$




time, s

An intuitively attractive approach in the search for useful tools to analyze stochastic processes would be to investigate the process in the *frequency domain*.

It will be shown that a realization of a stochastic process $\bar{x}(t)$, can be transformed into the frequency domain, yielding its Fourier transform $\bar{X}(\omega)$. However, since the time histories of the various realizations are different, so will be their counterparts in the frequency domain.

It turns out that the quantity $|\bar{X}(\omega)|^2$ as a function of frequency $\omega$ is "characteristic" (not the same!) for all realizations.

This quantity reflects the *distribution of energy over different frequencies* $\omega$ and is an *estimate* of the 'true' **spectral density function** (PSD) $S_{\bar{x}\bar{x}}(\omega)$ of the stochastic process. The power spectral density function fully characterizes the stochastic process.

In practice, averaging (in the frequency domain) the PSD estimates from $N$ realizations improves the estimate. In the limit case, i.e., $N \to \infty$ (or, an infinite observation length), the PSD estimate corresponds with the 'true' PSD:

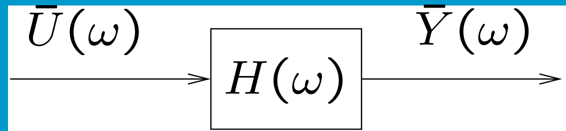
$$S_{\bar{x}\bar{x}}(\omega) = \lim_{N \to \infty} \widehat{S}_{\bar{x}\bar{x}}(\omega)$$

In this chapter the continuous time spectral analysis will be discussed. The discrete time spectral analysis is the subject of the next chapter.

# Fourier analysis

A powerful technique in signal analysis is the so-called **signal decomposition** method, where the signal concerned is viewed as a linear combination of certain **basis functions**.

The Fourier series expansion relies on **sinusoids** as its basic components and this has proven to be an extremely suitable choice.




First, the response of a linear time-invariant (LTI) system to a sinusoidal input signal is well-known: it yields a sinusoidal output signal with the same frequency. Only the amplitude and the phase may change relative to the input signal (see the sheets for lecture #1).

Second, the superposition property of linear systems states that the response of a linear system to a sum of input signals equals the sum of the responses of that system to each individual input signal.

Hence, when we can describe the input signal using a sum of 'basis functions', like sinusoids, then the response of the system to that input signal will be equal to the sum of the system response to each individual basis function

Fourier analysis allows us to describe linear dynamic systems in the frequency domain, using **frequency response functions** (FRFs).

# The Fourier Series (FS) expansion

Suppose a signal $\bar{x}(t)$ periodic over the time interval $[t_0, t_0 + T]$ and approximated by $\tilde{\bar{x}}(t)$, through a Fourier Series (FS) expansion, an array of $N$ sine and cosine functions:

$$\boxed{\tilde{\bar{x}}(t) = \sum_{k=0}^{N-1} [a_k \cos k\omega_0 t + b_k \sin k\omega_0 t]} \tag{3.1}$$

or: $$\tilde{\bar{x}}(t) = a_0 + \sum_{k=1}^{N-1} [a_k \cos k\omega_0 t + b_k \sin k\omega_0 t], \text{ and } b_0{=}0$$

The **fundamental frequency** $\omega_0$ is defined in such a way that all basic functions 'fit' an <u>integer</u> number of times in the interval $[t_0, t_0 + T]$:

$$\boxed{\omega_0 = \frac{2\pi}{T}} \tag{3.2}$$

In this case, all basic cosine and sine functions (with frequencies an integer number times the fundamental frequency $\omega_0$) are orthogonal at the interval $[t_0, t_0 + T]$:

$$\int_{t_0}^{t_0+T} \sin k\omega_0 t \cos \ell\omega_0 t \, \mathrm{d}t = 0$$

$$\int_{t_0}^{t_0+T} \sin k\omega_0 t \sin \ell\omega_0 t \, \mathrm{d}t = \begin{cases} 0 \text{ if } k \neq \ell \\ \frac{T}{2} \text{ if } k = \ell \end{cases}$$

$$\int_{t_0}^{t_0+T} \cos k\omega_0 t \cos \ell\omega_0 t \, \mathrm{d}t = \begin{cases} 0 \text{ if } k \neq \ell \\ \frac{T}{2} \text{ if } k = \ell \end{cases}$$

In order to compute the values of the coefficients $a_k$ and $b_k$ in Equation (3.1), a quadratic loss function $J$ is defined:

$$J(a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots) = \int\limits_{t_0}^{t_0+T} \underbrace{\left[\bar{x}(t) - \tilde{\bar{x}}(t)\right]^2}_{\text{"error"}} \mathrm{d}t$$

$$= \int\limits_{t_0}^{t_0+T} \left[\bar{x}(t) - \sum_{k=0}^{N-1} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)\right]^2 \mathrm{d}t \qquad (3.3)$$

Minimizing $J$ will yield the 'best fit', with necessary conditions:

$$\frac{\partial J}{\partial a_0} = \frac{\partial J}{\partial a_\ell} = \frac{\partial J}{\partial b_\ell} = 0,$$

for $\ell = 1, 2, \ldots, N$-1

Let us first consider parameter $a_0$. If $\dfrac{\partial J}{\partial a_0} = 0$ then:

$$\int_{t_0}^{t_0+T} 2\left[\bar{x}(t) - \left(a_0 + \sum_{k=1}^{N-1}\left[a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right]\right)\right] \cdot (-1)\ \mathrm{d}t = 0,$$

$$\begin{aligned}
\iff \int_{t_0}^{t_0+T} \bar{x}(t)\ \mathrm{d}t &= \int_{t_0}^{t_0+T} a_0\ \mathrm{d}t + \int_{t_0}^{t_0+T} \sum_{k=1}^{N-1}\left[a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right]\ \mathrm{d}t \\
&= Ta_0 + \sum_{k=1}^{N-1}\left(a_k \int_{t_0}^{t_0+T} \cos k\omega_0 t\ \mathrm{d}t + b_k \int_{t_0}^{t_0+T} \sin k\omega_0 t\ \mathrm{d}t\right)
\end{aligned}$$

Note that the two integral terms on the right-hand side are zero for all $k$, and we obtain:

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} \bar{x}(t)\ \mathrm{d}t} \tag{3.4}$$

Hence, $a_0$ can be considered the *average* of $\bar{x}(t)$ over the interval $[t_0, t_0 + T]$. We will come back to this property later.

We continue with deriving an expression for variables $a_k$. If $\dfrac{\partial J}{\partial a_\ell} = 0$ (for $\ell$ = 1, 2, ..., $N$-1) then:

$$\int_{t_0}^{t_0+T} 2\left[\bar{x}(t) - \left(a_0 + \sum_{k=1}^{N-1}\left[a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right]\right)\right] \cdot (-\cos \ell\omega_0 t)\ \mathrm{d}t \ = \ 0,$$

$$\begin{aligned}
\Longleftrightarrow \int_{t_0}^{t_0+T} \bar{x}(t)\cos \ell\omega_0 t\ \mathrm{d}t \ &= \ \int_{t_0}^{t_0+T} a_0 \cos \ell\omega_0 t\ \mathrm{d}t \\
&+ \int_{t_0}^{t_0+T}\sum_{k=1}^{N-1}\left[a_k \cos k\omega_0 t \cos \ell\omega_0 t + b_k \sin k\omega_0 t \cos \ell\omega_0 t\right]\ \mathrm{d}t \\
&= \ a_0 \int_{t_0}^{t_0+T} \cos \ell\omega_0 t\ \mathrm{d}t \\
&+ \sum_{k=1}^{N-1}\left(a_k \int_{t_0}^{t_0+T} \cos k\omega_0 t \cos \ell\omega_0 t\ \mathrm{d}t \ + \ b_k \int_{t_0}^{t_0+T} \sin k\omega_0 t \cos \ell\omega_0 t\ \mathrm{d}t\right)
\end{aligned}$$

Note that the first integral term on the right-hand side is zero for all $\ell$. Recall that all basic cosine and sine functions are orthogonal at the interval $[t_0, t_0 + T]$. Hence, we can obtain:

$$\int_{t_0}^{t_0+T} \bar{x}(t) \cos\ \ell\omega_0 t \, \mathrm{d}t \ = \ a_\ell \frac{T}{2}, \text{ or, substituting } k \text{ for } \ell:$$

$$\boxed{a_k = \frac{2}{T} \int_{t_0}^{t_0+T} \bar{x}(t) \cos\ k\omega_0 t \, \mathrm{d}t}, \tag{3.6}$$

for $k = 1, 2, \ldots, N$-1. Similarly, we can derive for $b_k$:

$$\boxed{b_k = \frac{2}{T} \int_{t_0}^{t_0+T} \bar{x}(t) \sin\ k\omega_0 t \, \mathrm{d}t}, \tag{3.7}$$

for $k = 1, 2, \ldots, N$-1.

Interestingly, we see that Eqs. (3.6) and (3.7) can be interpreted as the *average product function* ($R_{\bar{x}\bar{y}}$, see Chapter 2) of the original signal $\bar{x}(t)$ and the cosine and sine basis functions, resp., for frequencies that are an integer multiple of the fundamental frequency $\omega_0$.

For each frequency $k\omega_0$ the Fourier series coefficients can therefore be considered as the 'average' of what the signal has 'in common' with the orthogonal basis functions with that frequency $k\omega_0$.

The Fourier series coefficient $a_0$, the signal's average, can also be considered this way, namely as the average product of the original signal with a cosine signal that has "zero frequency". A cosine signal with zero frequency is the limit case of a cosine signal that fluctuates extremely slow, i.e., its value equals one for a very long time.

Summarizing, we obtain for the coefficients $a_k$ and $b_k$:

$$\boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} \bar{x}(t)\,\mathrm{d}t} \tag{3.4}$$

$$\boxed{a_k = \frac{2}{T} \int_{t_0}^{t_0+T} \bar{x}(t) \cos k\omega_0 t\,\mathrm{d}t}, \text{ and} \tag{3.6}$$

$$\boxed{b_k = \frac{2}{T} \int_{t_0}^{t_0+T} \bar{x}(t) \sin k\omega_0 t\,\mathrm{d}t}, \tag{3.7}$$

for $k = 1, 2, 3, \ldots$ Note that $b_0$ is always zero. Further note that $a_0$ can be considered the signal average, and $a_k$ and $b_k$ the average product of the signal $\bar{x}(t)$ with $\cos k\omega_0 t$ and $\sin k\omega_0 t$, respectively.

# $\boxed{\text{Complex form of the FS expansion}}$

It is custom to write the FS expansion in complex form.

Recall that: $\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$; and $\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$ (use rule of thumb: "sinus" - "minus")

Then:
$$\boxed{\tilde{\tilde{x}}(t) = \sum_{k=-N+1}^{N-1} c_k e^{jk\omega_0 t}}, \tag{3.9}$$

in which the original real coefficients $a_k$ and $b_k$ are substituted by the *complex* coefficients $c_k$, defined as:

$$\boxed{c_k = \begin{array}{ll} \frac{1}{2}(a_k - jb_k) & k = 1, 2, 3, ..., N-1 \\ a_0 & k = 0 \\ \frac{1}{2}(a_{|k|} + jb_{|k|}) & k = -N+1, ..., -3, -2, -1 \end{array}} \tag{3.10}$$

It is easy to prove that $c_k$ can be computed with:

$$\boxed{c_k = \frac{1}{T} \int_{t_0}^{t_0+T} \bar{x}(t) e^{-jk\omega_0 t} \mathrm{d}t} \tag{3.12}$$

for $k$ = -$N$+1, ..., -2, -1, 0, 1, 2, 3, ..., $N$-1

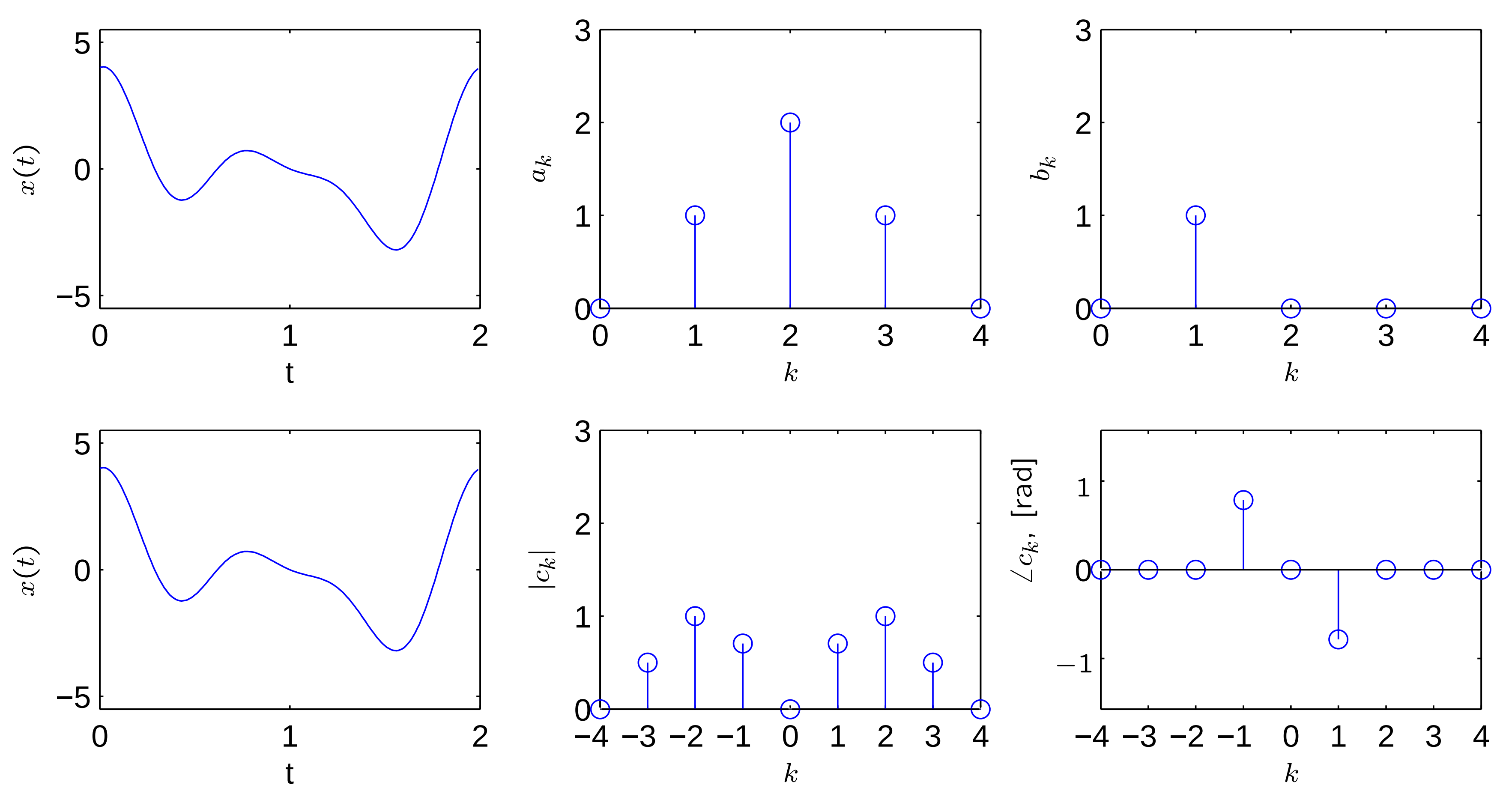
Hence, we obtain a so-called double-sided FS description, with “negative” frequencies. Note that (3.12) and (3.9) are each other’s inverse:

- Equation (3.12) is known as the analysis equation, because it allows us to analyze how the signal can be reconstructed using complex exponential basis functions.
- Equation (3.9) is known as the synthesis equation, because it constructs, or ‘synthesizes’, a signal using the complex exponential basis functions.

Some properties of the complex sequence $c_k$:

- $a_k = 2\text{Re}\{c_k\}$ the real part, related to the cosine
- $b_k = -2\text{Im}\{c_k\}$ the imaginary part, related to the sine
- $c_k^* = c_{-k}$ (read: $c$ for *minus* $k$ !) where $^*$ denotes the *complex conjugate*
- $|c_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$ the amplitude
- $\angle c_k = \arctan \frac{-b_k}{a_k}$ the phase

**Example** of a signal that equals the sum of 3 cosines and 1 sine: $x(t) = 1 \cdot \cos(\omega_0 t) + 2 \cdot \cos(2\omega_0 t) + 1 \cdot \cos(3\omega_0 t) + 1 \cdot \sin(\omega_0 t)$, with $\omega_0 = 2\pi/T_0$ and $T_0 = 2$ sec.



Some properties of the Fourier series expansion:

- When the number of Fourier series coefficients, $N$, increases, the approximation improves:
$$\lim_{N\to\infty} \tilde{\bar{x}}(t) = \bar{x}(t)$$

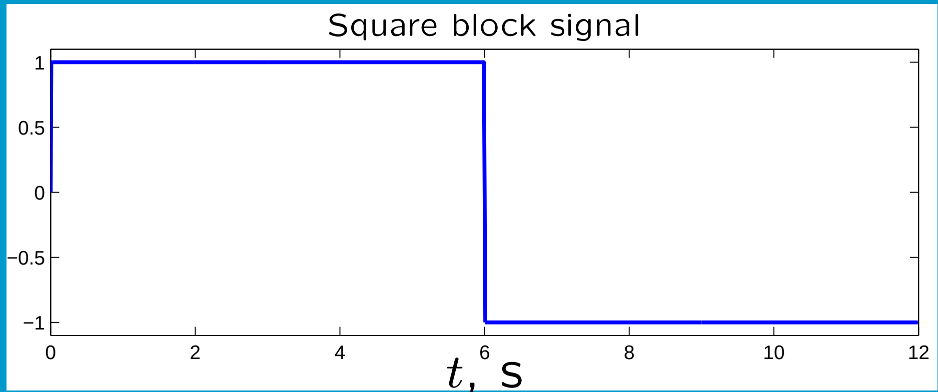
- Parseval's relation (power in time = power in frequency):
$$\frac{1}{T}\int_{t_0}^{t_0+T} \bar{x}^2(t)\mathrm{d}t = a_o^2 + \sum_{k=1}^{\infty} \frac{1}{2}\left(a_k^2 + b_k^2\right) = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- The FS expansion of the $n^{\mathrm{th}}$ derivative:
$$\frac{\mathrm{d}^n\bar{x}(t)}{\mathrm{d}t^n} = \sum_{k=-\infty}^{\infty} (jk\omega_0)^n\, c_k\, e^{jk\omega_0 t}$$

- <u>Even</u> functions result in FS series with <u>only cosine</u> terms

- <u>Odd</u> functions result in FS series with <u>only sine</u> terms

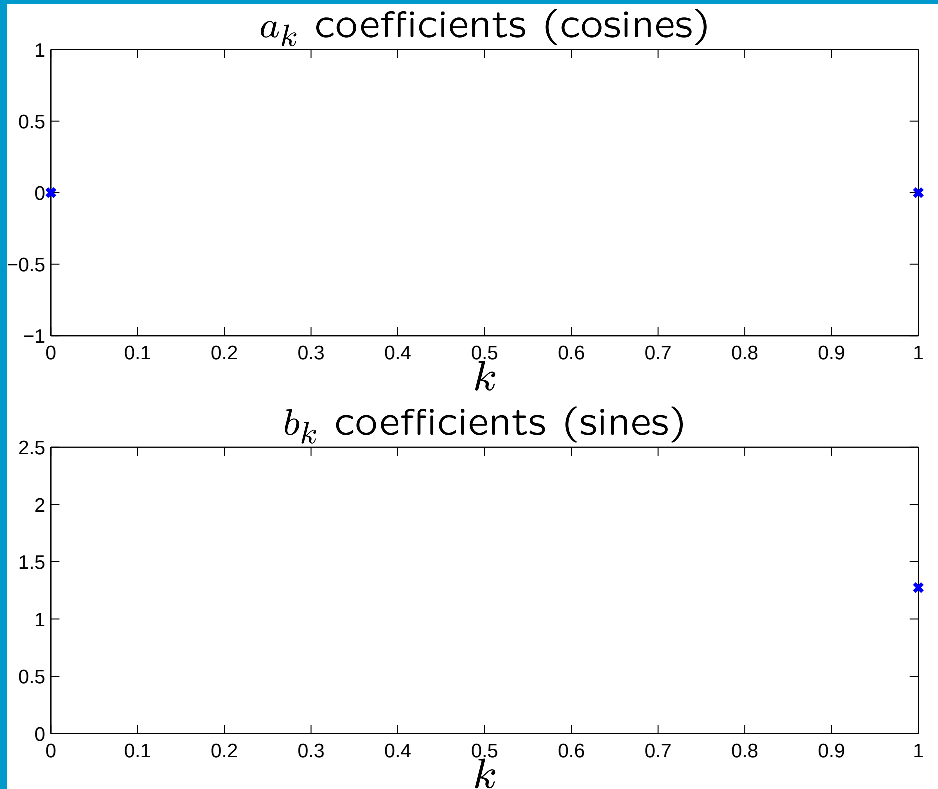
# **Example:** trigonometric FS of a square block wave




We see that $T_0$= 12 seconds, so $\omega_0 = 2\pi/12$ rad/s. We also see that it is an ODD function, and that the average equals 0. Hence, we expect only sinusoids (all $a_k$'s are zero), and also that $a_0$=0.

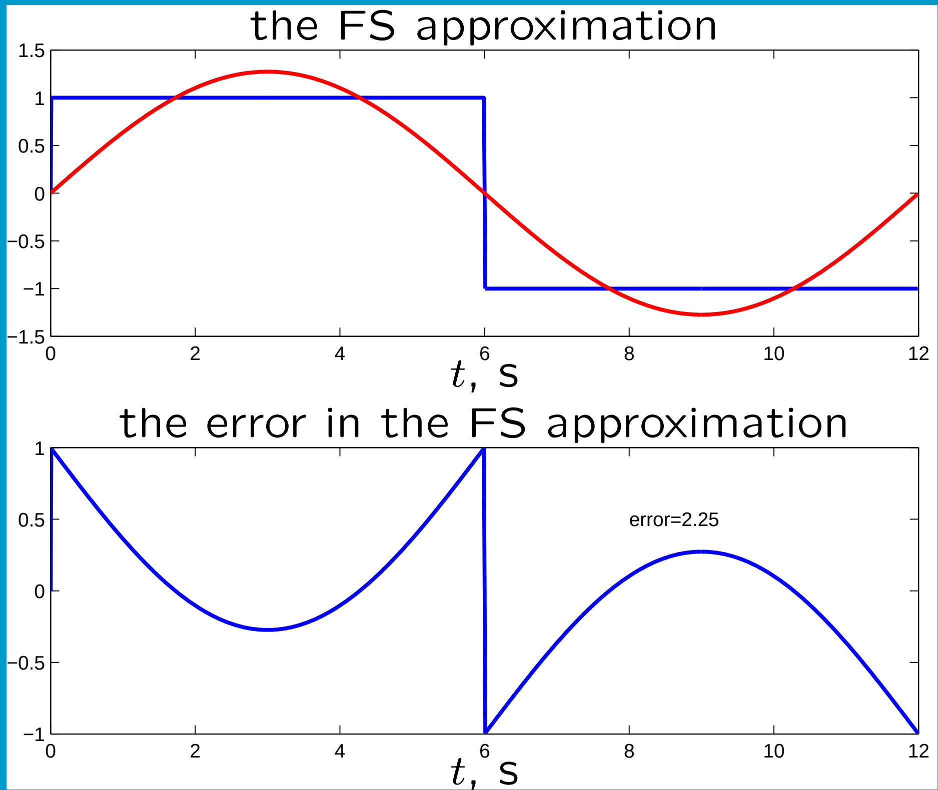
In addition, the signal is half-wave odd ($x(t + T_0/2) = -x(t)\ \forall\ t$), so we expect that all even-numbered FS coefficients will be zero.

## Fourier series coefficients with $N = 1$



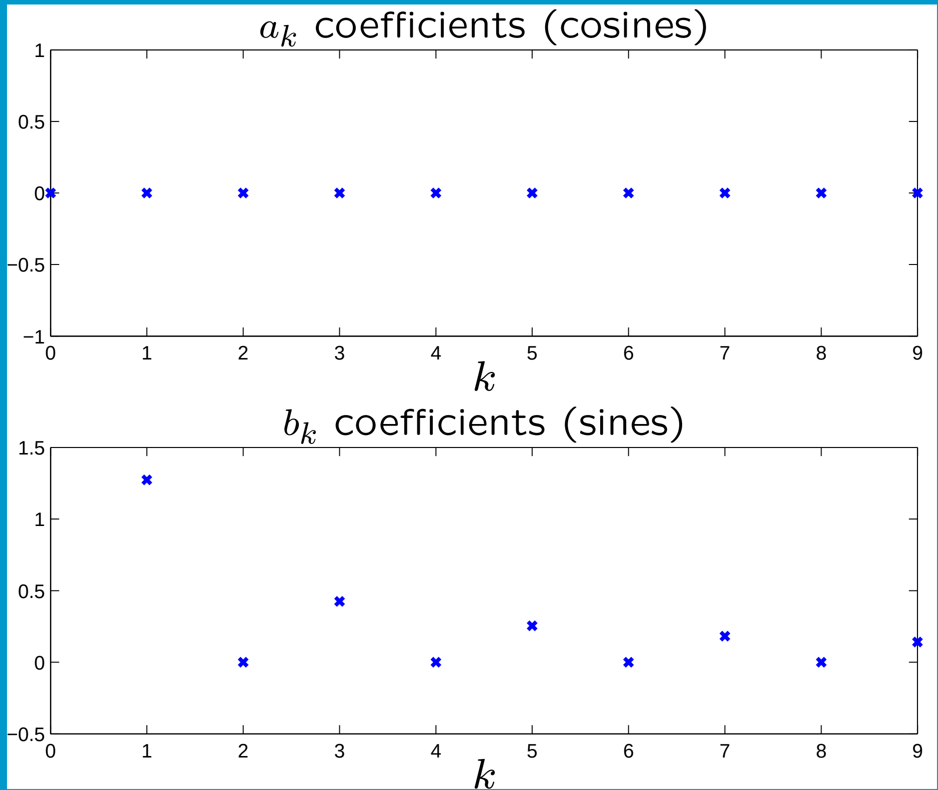

Indeed, all $a_k$'s so far are zero, $b_0 = 0$ by default, and also we see that $b_1 = 4/\pi$.

## Fourier series approximation and error with $N = 1$



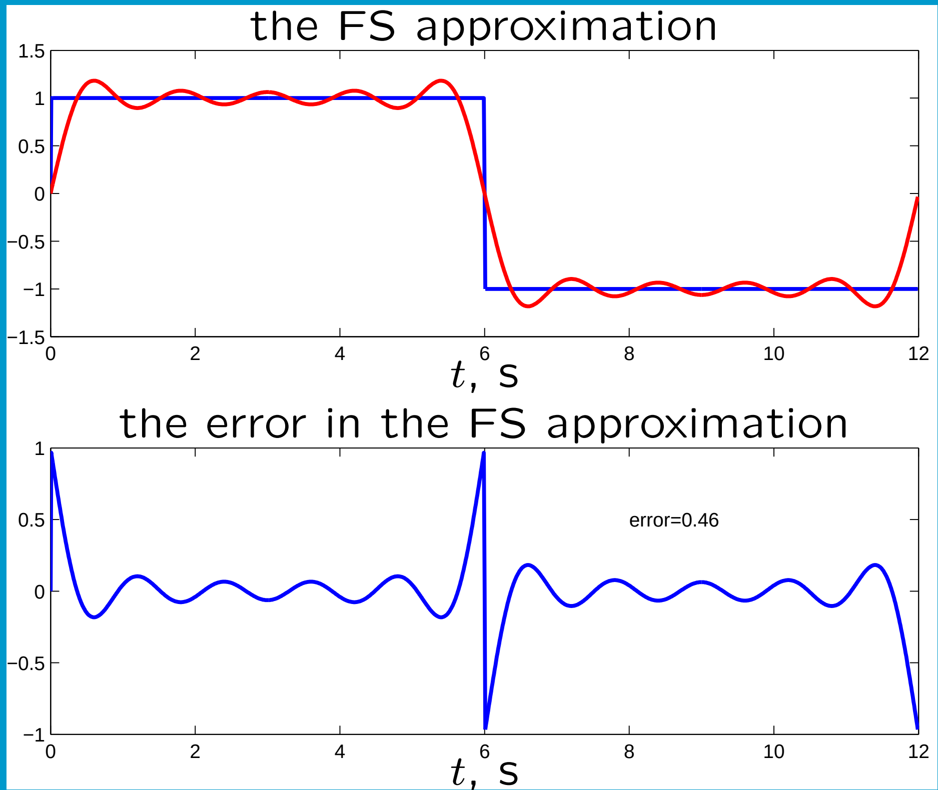

We basically use one sinusoid with the fundamental frequency to model the signal, and as a consequence the error is quite large. Note that the error is defined as $e = \int_{T_0} (x(t) - \tilde{x}(t))^2 \, dt$.

# Fourier series coefficients with $N = 10$



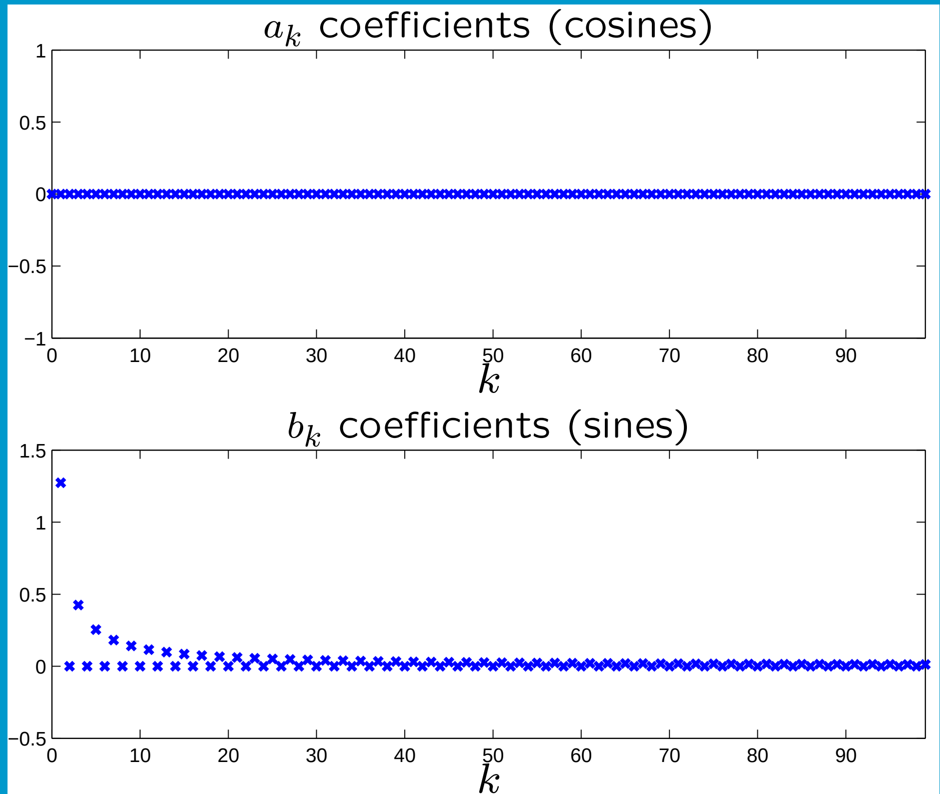

All $a_k$'s are zero (odd signal: no cosines are needed), all $b_k$'s are zero for even $k$ (half-wave odd signal). The $b_k$'s become smaller for larger $k$.

## Fourier series approximation and error with $N = 10$



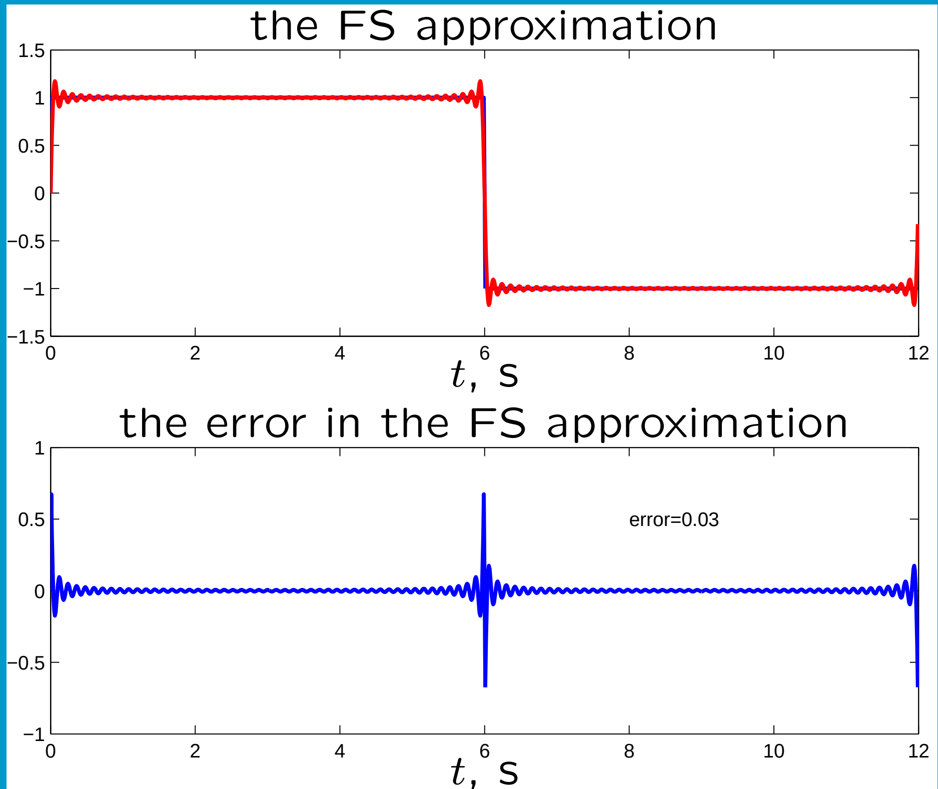

We now use five sinusoids, all harmonics (frequencies an integer multiple of the fundamental frequency) to model the signal. The approximation is a lot better.

## Fourier series coefficients with $N = 100$



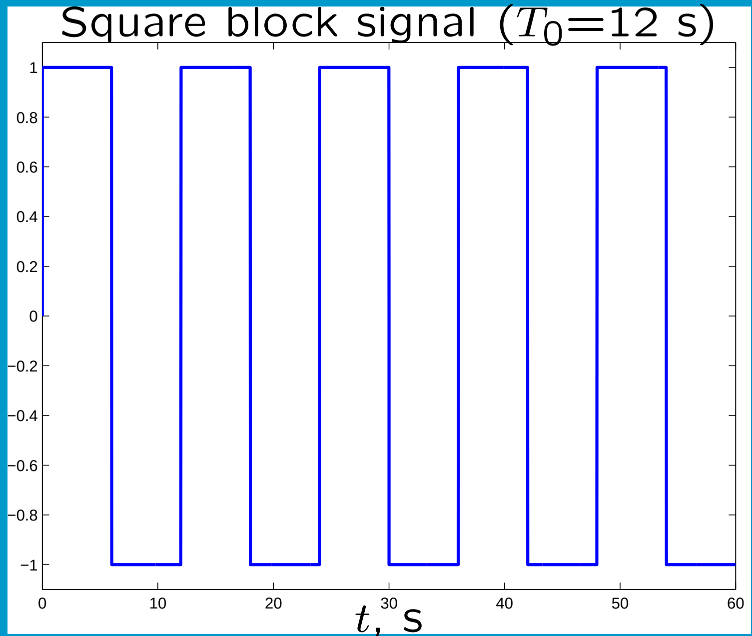

All $a_k$'s are zero, all $b_k$'s are zero for even $k$ (half-wave odd signal), and become smaller for larger $k$.

## Fourier series approximation and error with $N = 100$



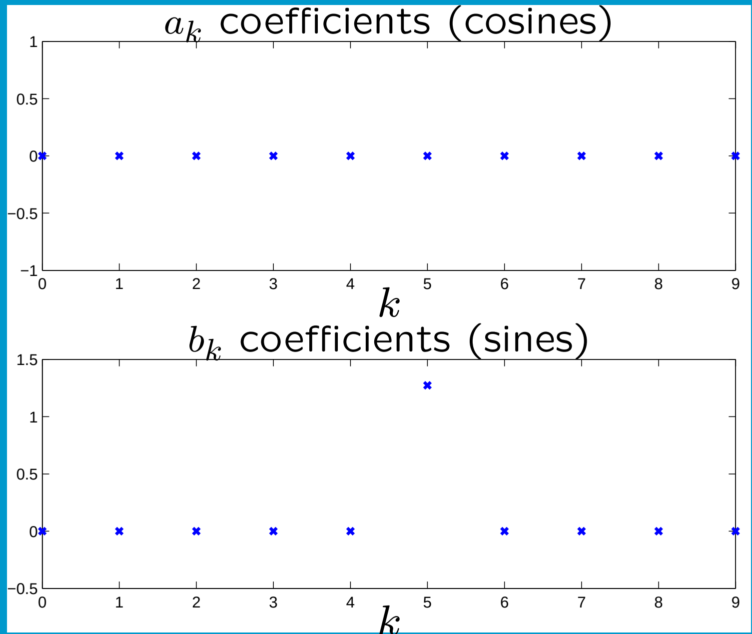

We now use fifty harmonics. The approximation is almost perfect, especially at those parts which are constant (the maxima), but poor at the large signal change around $T_0/2$ (the Gibbs' phenomenon).

**Example 2:** Take $T_0^* = 60$ seconds, and see what happens



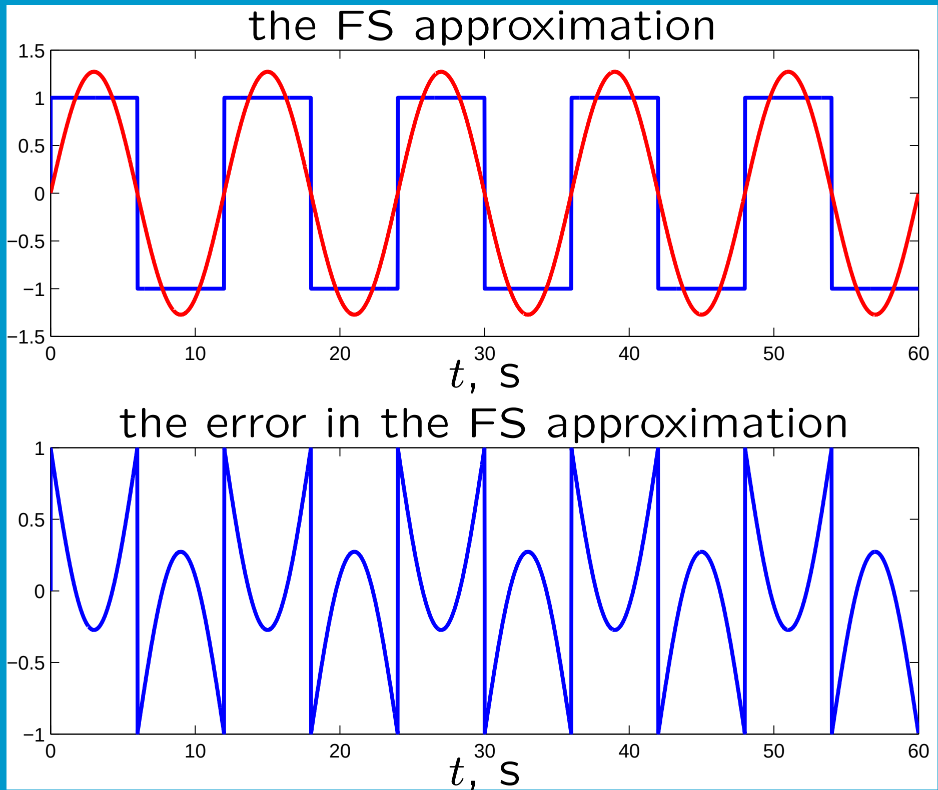

We will repeat the example, but now "accidentally" take the wrong period, $T_0^* = 5T_0$, and invert the block. The block is still ODD and half-wave odd, and the average is still 0.

## Fourier series coefficients with $N = 10$



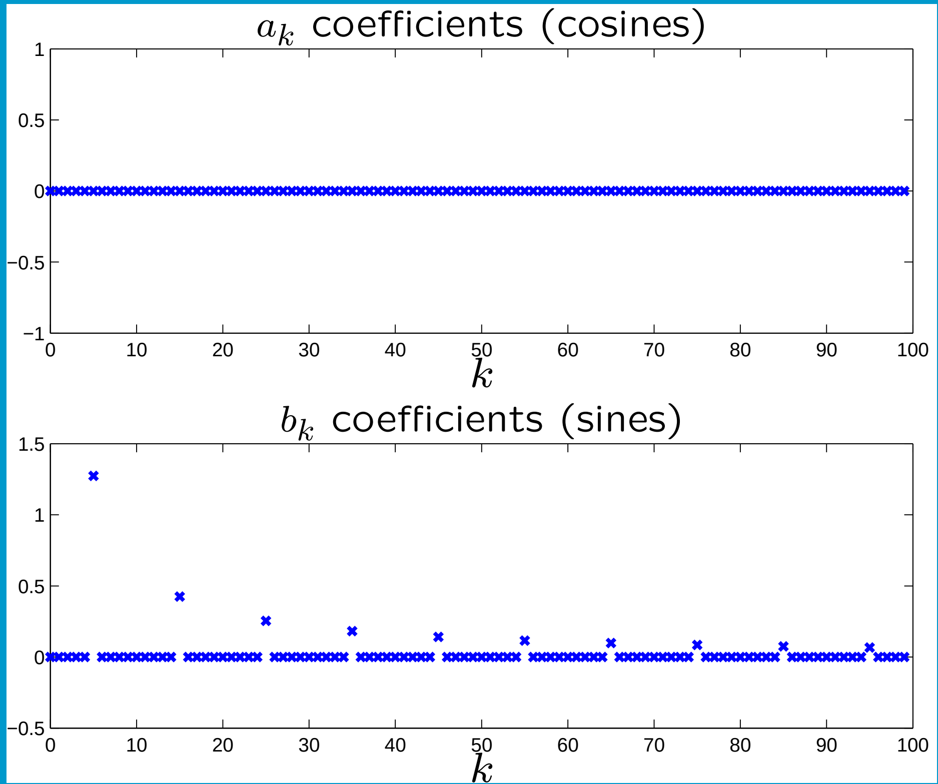

When computing the first 10 FS coefficients we see that the first $b_k$ that is non-zero, is $b_5$. Recall that $T_0^* = 5T_0$, so this makes sense! The first sinusoid that 'models' the five repetitions of the block has a frequency that is 5 times the 'fundamental frequency'.

## Fourier series approximation and error with $N = 10$



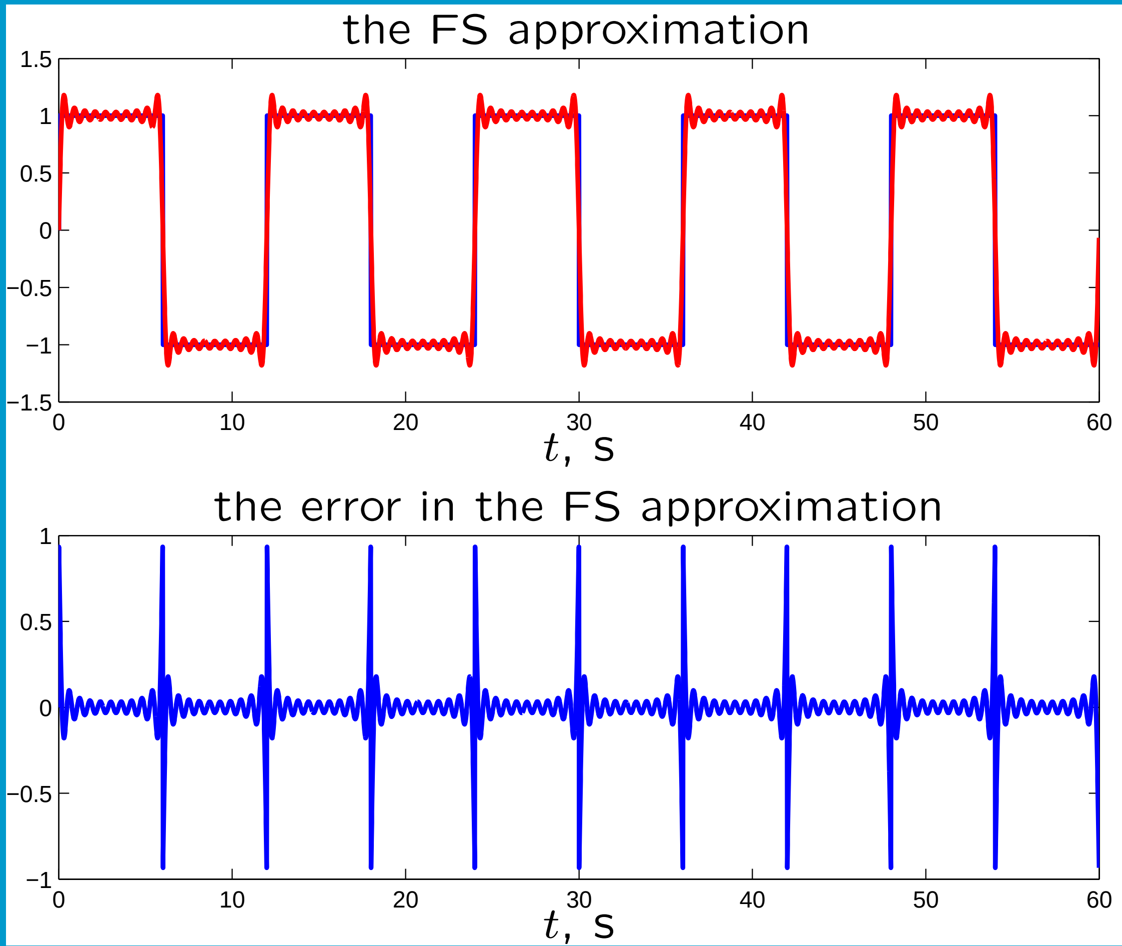

Indeed, the sinusoid that ‘models’ the block.

## Fourier series coefficients with $N = 100$




We see that we get non-zero values only for $b_5$ (1·5), $b_{15}$ (3·5), $b_{25}$ (5·5), etcetera. This is because (1) $T_0^* = 5T_0$, and (2) we have a half-wave odd signal (even $k$'s (for $T_0$!!) are zero.

## Fourier series approximation and error with $N = 100$



# $\boxed{\text{The Fourier integral}}$

for aperiodic functions!

Apply the FS expansion to the stochastic function $\bar{x}$(t) valid at an <u>infinite time interval</u> $-\infty < t < \infty$. Then, define the interval $[t_0, t_0+T]$ as $[-T/2, +T/2]$ and let $T \to \infty$, substitute $\frac{\omega_0}{2\pi}$ for $\frac{1}{T}$, and we obtain:

$$\tilde{\bar{x}}(t) = \lim_{T\to\infty}\left(\frac{\omega_0}{2\pi}\sum_{k=-\infty}^{k=\infty}\left[\int_{-T/2}^{+T/2}\bar{x}(t)e^{-jk\omega_0 t}\mathrm{d}t\right]e^{jk\omega_0 t}\right) \tag{3.13}$$

Then, when the fundamental period $T$ is extended to infinity, $\omega_0 \to$ d$\omega$, $k\omega_0 \to \omega$ and the sum in (3.13) becomes an integral with limits $\omega = -\infty$ to $\omega = +\infty$:

$$\tilde{\bar{x}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty}\bar{x}(t)e^{-j\omega t}\mathrm{d}t\right]}_{\equiv\text{the Fourier integral}}e^{j\omega t}\mathrm{d}\omega \qquad (3.14,\ 3.15)$$

# The continuous-time Fourier transform

The sequence of complex coefficients $c_k$ of the FS expansion, representing the contribution of the basic functions at the discrete frequencies $k\omega_0$ ($k$=...,-2, -1, 0, 1, 2, ...) now becomes a **complex function**, representing the contribution of the basic functions at the **continuous** frequency $\omega$.

We now define as the **Fourier transform** (CTFT) of a signal $\bar{x}(t)$:

$$\boxed{\bar{X}(\omega) = \mathcal{F}\{\bar{x}(t)\} = \int_{-\infty}^{\infty} \bar{x}(t) e^{-j\omega t}\,\mathrm{d}t}, \quad \text{ANALYSIS} \tag{3.16}$$

transforming the time-domain representation of a signal, $\bar{x}(t)$ to its complement in the frequency domain, $\bar{X}(\omega)$.

The **inverse Fourier transform** of $\bar{X}(\omega)$ is defined as:

$$\boxed{\bar{x}(t) = \mathcal{F}^{-1}\left\{\bar{X}(\omega)\right\} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \bar{X}(\omega)e^{j\omega t}\,\mathrm{d}\omega}, \quad \text{SYNTHESIS} \qquad (3.17)$$

transforming the frequency-domain representation of a signal, $\bar{X}(\omega)$ to its complement in the time domain, $\bar{x}(t)$.

Recall the remark of the introduction lecture about the placement of the $\frac{1}{2\pi}$-term: there is no universal convention about it.

Note: the $\frac{1}{2\pi}$-term is not needed when we use the frequency $f$ (in [Hz]) instead of the radial frequency $\omega$ (in [rad/s]). **Why?**

# Properties of the Fourier transform

| Property | Signal | Fourier Transform |
|---|---|---|
| Linearity | $a\,x(t) + b\,y(t)$ | $a\,X(\omega) + b\,Y(\omega)$ |
| Time delay | $x(t - t_0)$ | $\{e^{-j\omega t_0}\}\,X(\omega)$ |
| Multiplication | $x(t)\,y(t)$ | $\frac{1}{2\pi}X(\omega) * Y(\omega)$ |
| Convolution | $x(t) * y(t)$ | $X(\omega)\,Y(\omega)$ |
| Time derivative | $\dot{x}(t)$ | $j\omega\,X(\omega)$ |
| Integral of Parseval | $\int_{-\infty}^{+\infty} x^2(t)\,dt$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(\omega)\|^2\,d\omega$ |

Table 3.1: Properties of the Fourier transform.

*Can you prove these properties??*

**Some additional properties of the Fourier transform.**

- $x(0) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\omega)\,\mathrm{d}\omega$
- $X(0) = \int\limits_{-\infty}^{\infty} x(t)\,\mathrm{d}t$ the "zero frequency"
- $\mathcal{F}\left\{\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right\} = (j\omega)^n X(\omega)$
- Note that any function $x(t)$ can be written as the sum of an even function $x_e(t)$ and an odd function $x_o(t)$, i.e., through $x_e(t) = \frac{1}{2}(x(t) + x(-t))$ and $x_o(t) = \frac{1}{2}(x(t) - x(-t))$. Then:
  - Re $\{X(\omega)\}$ corresponds with the **even** part of $x(t)$, $x_e(t)$
  - Im $\{X(\omega)\}$ corresponds with the **odd** part of $x(t)$, $x_o(t)$
- $\mathrm{Re}\{X(\omega)\} = \int\limits_{-\infty}^{\infty} x(t)\cos\omega t\,\mathrm{d}t$; This is an even function in $\omega$
- $\mathrm{Im}\{X(\omega)\} = -\int\limits_{-\infty}^{\infty} x(t)\sin\omega t\,\mathrm{d}t$; This is an odd function in $\omega$
- $X(-\omega) = X^*(\omega)$, the complex conjugate

## Proof of the convolution property

Rule: a convolution $x(t) \star y(t)$ in the time domain becomes a multiplication $X(\omega)Y(\omega)$ in the frequency domain (and vice versa).

Remember, a **convolution** integral (Duhamel integral) is defined as:

$$x(t) \star y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,\mathrm{d}\tau$$

### Proof

$$\mathcal{F}\{x(t) \star y(t)\} = \int_{t=-\infty}^{\infty} \left[ \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau \right] e^{-j\omega t}\,\mathrm{d}t$$

bring $e^{-j\omega t}$ inside the $\tau$ integral

$$= \int_{t=-\infty}^{\infty} \left[ \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\omega(t-\tau)}e^{-j\omega\tau}\,\mathrm{d}\tau \right] \mathrm{d}t$$

now, change order of integration

$$= \int_{\tau=-\infty}^{\infty} \int_{t=-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\omega(t-\tau)}e^{-j\omega\tau}\,\mathrm{d}t\,\mathrm{d}\tau$$

inner integral is in $t$, get $\tau$'s out

$$= \int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau} \int_{t=-\infty}^{\infty} y(t-\tau)e^{-j\omega(t-\tau)}\,\mathrm{d}(t-\tau)\,\mathrm{d}\tau$$

because $dt = d(t-\tau)$ in inner integral

$$= \int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}Y(\omega)\,\mathrm{d}\tau \;=\; \left\{ \int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau \right\} \cdot Y(\omega) \;=\; X(\omega)Y(\omega) \text{ q.e.d.}$$

# Some basic Fourier transforms (1)

| Signal $\bar{x}(t)$ | Fourier transform $X(\omega)$ |
|---|---|
| $1$ | $2\pi\,\delta(\omega)$ |
| $\delta(t)$ | $1$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t-nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{T})$ |
| $\cos\ k\omega_0 t$ | $\pi[\delta(\omega - k\omega_0) + \delta(\omega + k\omega_0)]$ |
| $\sin\ k\omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - k\omega_0) - \delta(\omega + k\omega_0)]$ |

*Can you prove these basic transforms??*

# Some basic Fourier transforms (2)

| Signal $\bar{x}(t)$ | Fourier transform $X(\omega)$ |
|---|---|
| $\bar{x}(t) = \begin{cases} 1, & \lvert t\rvert < \frac{1}{2}T \\ 0, & \lvert t\rvert > \frac{1}{2}T \end{cases}$ | $T\ \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) = T\frac{\sin(\omega T/2)}{\omega T/2}$ |
| $\frac{W}{2\pi}\mathrm{sinc}\left(\frac{Wt}{2\pi}\right) = \frac{\sin(tW/2)}{\pi t}$ | $\bar{B}(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \frac{1}{2}W \\ 0, & \lvert\omega\rvert > \frac{1}{2}W \end{cases}$ |

*Can you prove these basic transforms??*

## Proof of the Fourier transform of function $x(t) = 1 \ \forall \ t$

Rule: $\mathcal{F}\{1\} = 2\pi\delta(\omega)$

Remember, the Dirac function (or delta function) $\delta(t)$ is a *generalized function*: it is not defined in terms of its values, but rather how it acts inside an integral when multiplied by a smooth function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,\mathrm{d}t = f(\tau) \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

### Proof

(using inverse proof)

$$\begin{aligned}
&Z(\omega) = 2\pi\delta(\omega) \\
&z(t) = \mathcal{F}^{-1}\{Z(\omega)\} \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\cdot e^{j\omega t}\,\mathrm{d}\omega \\
&\qquad\qquad = \ \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\,\mathrm{d}\omega \ = \ \int_{-\infty}^{\infty} \delta(0)e^{j0t}\,\mathrm{d}\omega \ = \ 1e^{0} = \ 1 \\
&z(t) = 1, \forall t
\end{aligned}$$

q.e.d.

## Proof of the Fourier transform of a cosine function

Rule: $\mathcal{F}\{\cos\omega_0 t\} = \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

Remember, $\mathcal{F}\{1\} = \int_{-\infty}^{\infty} 1\cdot e^{-j\omega t}\,\mathrm{d}t \;=\; 2\pi\delta(\omega)$

### Proof

$$
\begin{aligned}
\mathcal{F}\{\cos\omega_0 t\} &= \int_{-\infty}^{\infty} \cos\omega_0 t\cdot e^{-j\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2}\left(e^{j\omega_o t}+e^{-j\omega_o t}\right) e^{-j\omega t}\,\mathrm{d}t \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_o-\omega)t}+e^{-j(\omega_o+\omega)t}\,\mathrm{d}t \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_o-\omega)t}\,\mathrm{d}t+\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega_o+\omega)t}\,\mathrm{d}t \\
&= \tfrac{1}{2}2\pi\delta(\omega-\omega_0)+\tfrac{1}{2}2\pi\delta(\omega+\omega_0) \\
&= \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)
\end{aligned}
$$

q.e.d.

## Proof of the Fourier transform of a block function $b(t)$

Function $b(t)$ is defined as:

$$b(t) = \begin{cases} 1 & , \quad |t| < T/2 \\ 0.5 & , \quad |t| = T/2 \\ 0 & , \quad |t| > T/2 \end{cases}$$

Rule: $B(\omega) \;=\; \mathcal{F}\{b(t)\} \;=\; T \cdot \dfrac{\sin \omega T/2}{\omega T/2} \;=\; T \cdot \mathrm{sinc}(\frac{\omega T}{2\pi})$ 'sine cardinal' = sinc

**Proof**

$$B(\omega) = \int_{-\infty}^{\infty} b(t) \cdot e^{-j\omega t}\,\mathrm{d}t \;=\; \int_{-T/2}^{+T/2} e^{-j\omega t}\,\mathrm{d}t$$

$$= \int_{-T/2}^{+T/2} (\cos \omega t - j \sin \omega t)\,\mathrm{d}t \;=\; \int_{-T/2}^{+T/2} \cos \omega t\,\mathrm{d}t - j \int_{-T/2}^{+T/2} \sin \omega t\,\mathrm{d}t$$

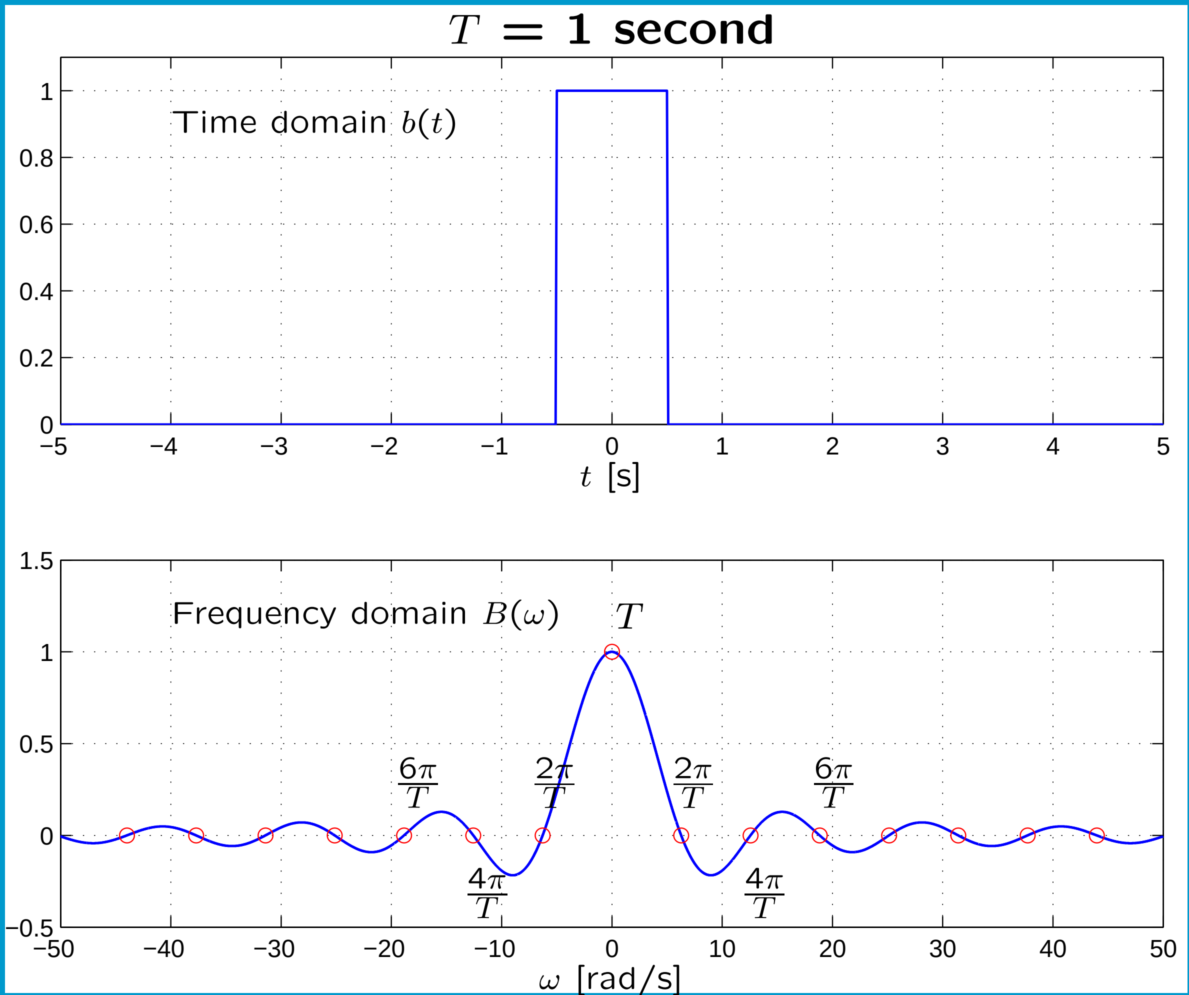
Note that a cosine is an even function and that a sine is an odd function.

Hence, the second integral is zero $\forall\ T$.

$$= 2 \int_{0}^{+T/2} \cos \omega t\,\mathrm{d}t \;=\; 2 \int_{0}^{+T/2} \mathrm{d}\tfrac{1}{\omega} \sin \omega t$$

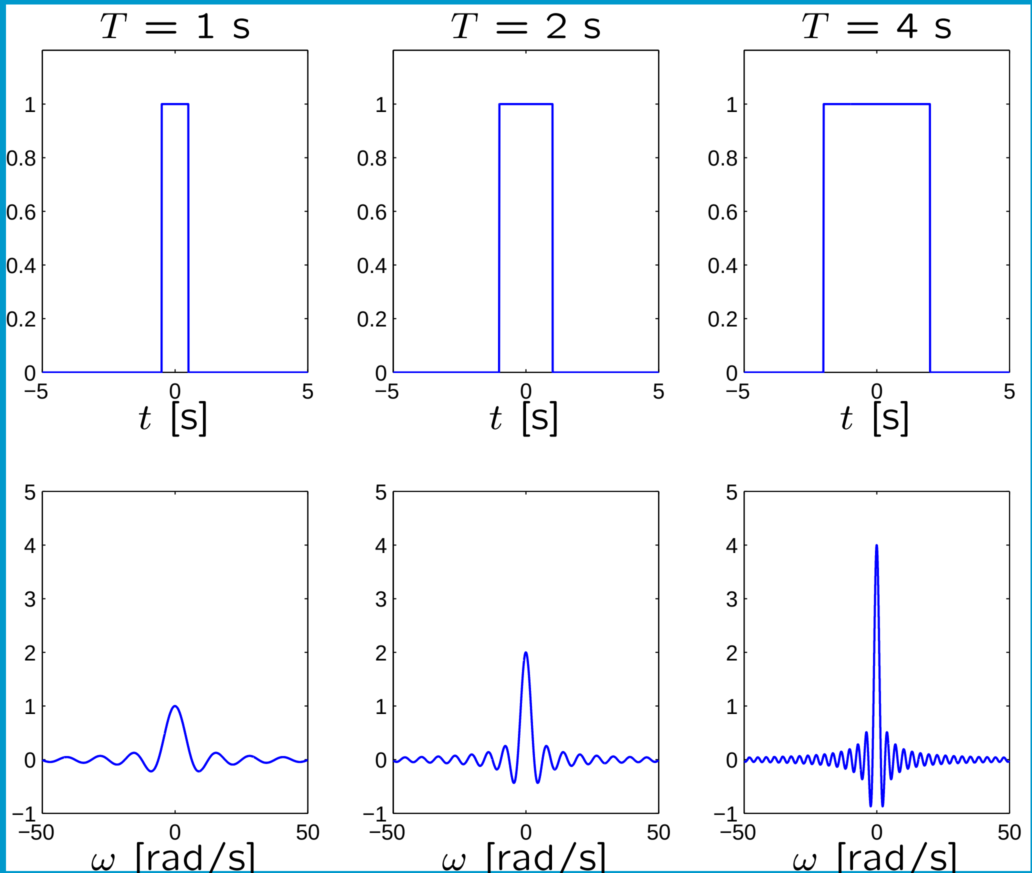
$$= 2 \left(\tfrac{1}{\omega} \sin \omega t\right)\Big|_{t=0}^{T/2} \;=\; 2\frac{\sin \omega T/2}{\omega} \;=\; T \cdot \frac{\sin \omega T/2}{\omega T/2}$$ q.e.d.

**The block function $b(t)$ and its Fourier transform $B(\omega)$, the sinc-function.**




**Note** that $B(0)$ equals area in time domain, and $x(0)$ equals the area in the $f$-domain multiplied with $1/(2\pi)$.

**The effect of changing $T$ in $b(t)$ on the Fourier transform $B(\omega)$.**




**Note:** The peak $B(0)$ equals $T$. When $T \to \infty$ (i.e., $b(t) = 1 \ \forall\ t$), $B(\omega)$ becomes a Dirac delta-function $\delta(\omega)$.

## Inverse Fourier transform of a block function: duality

Consider a block function $B(\omega)$ in the frequency domain:

$$B(\omega) = \begin{cases} 1 & , \ |\omega| < W/2 \\ 0.5 & , \ |\omega| = W/2 \\ 0 & , \ |\omega| > W/2 \end{cases}$$

It can be shown (see e.g., (Oppenheim, 1997)) that the inverse Fourier transform of this block function yields a sinc function in the time-domain:

$$b(t) = \mathcal{F}^{-1}\{B(\omega)\} = \frac{\sin W/2\, t}{\pi t} = \frac{W}{2\pi} \cdot \mathrm{sinc}(\frac{tW}{2\pi})$$

**Note** that $B(\omega)$ represents the transfer function of an 'ideal' low-pass filter. The impulse response function $b(t)$ is non-causal, however, and therefore the ideal filter is not practical.

Even more important to note is that Fourier transforming a time-domain block yields a sinc function in the frequency domain, and Fourier transforming a time-domain sinc function yields a block in the frequency domain.

This is a typical example of the duality property of the Fourier transform. It can be generalized as (see (Oppenheim, 1997)):

$$\text{If} \quad x(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(\omega) \quad \text{then} \quad X(t) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi x(-\omega)$$

Therefore, when we have one Fourier transform pair $\{x(t),\, X(\omega)\}$ (e.g., block in time domain, sinc in frequency domain) we can easily find the other pair $\{X(t),\, x(\omega)\}$ (e.g., block in frequency domain, sinc in time domain).

## Fourier transform of a periodic block function

As a final example of the Fourier transform, let us consider the block function introduced earlier, but now *periodic* with fundamental period $T_0$ and fundamental frequency $\omega_0 = 2\pi/T_0$:

$$b_p(t) = \begin{cases} 1 & , \quad |t| < T/2 + nT_0 \\ 0.5 & , \quad |t| = T/2 + nT_0 \\ 0 & , \quad |t| > T/2 + nT_0 \end{cases} \quad \text{for integer } n$$

Since $b_p(t)$ is a *periodic* function, the Fourier series expansion can be calculated. It can be shown that:

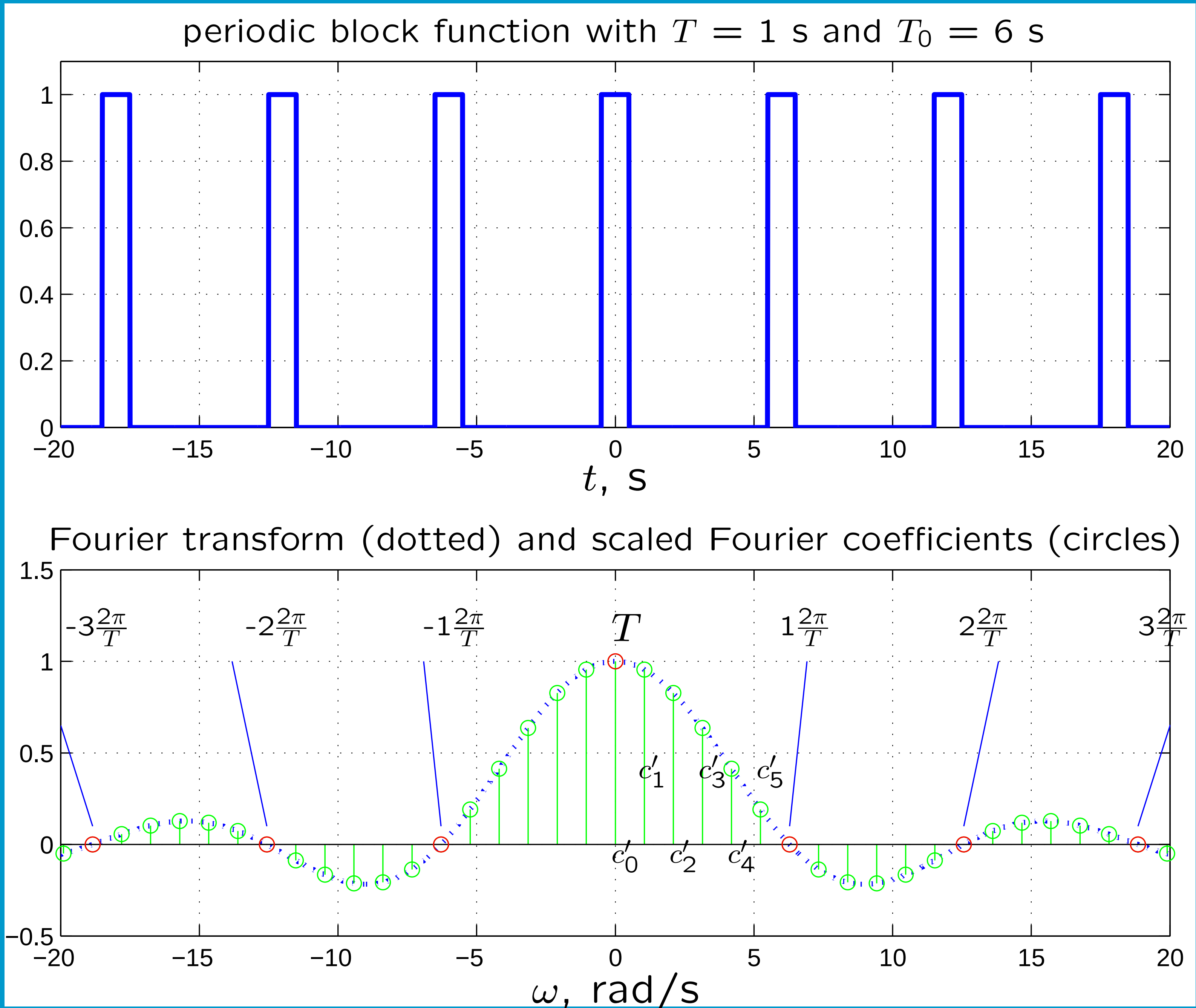
$$c_0 = \frac{T}{T_0}, \quad \text{and } c_k = 2\frac{\sin\ k\omega_0 T/2}{k\omega_0 T_0} = \frac{\sin\ k\omega_0 T/2}{k\pi} \quad \text{for } k \neq 0$$

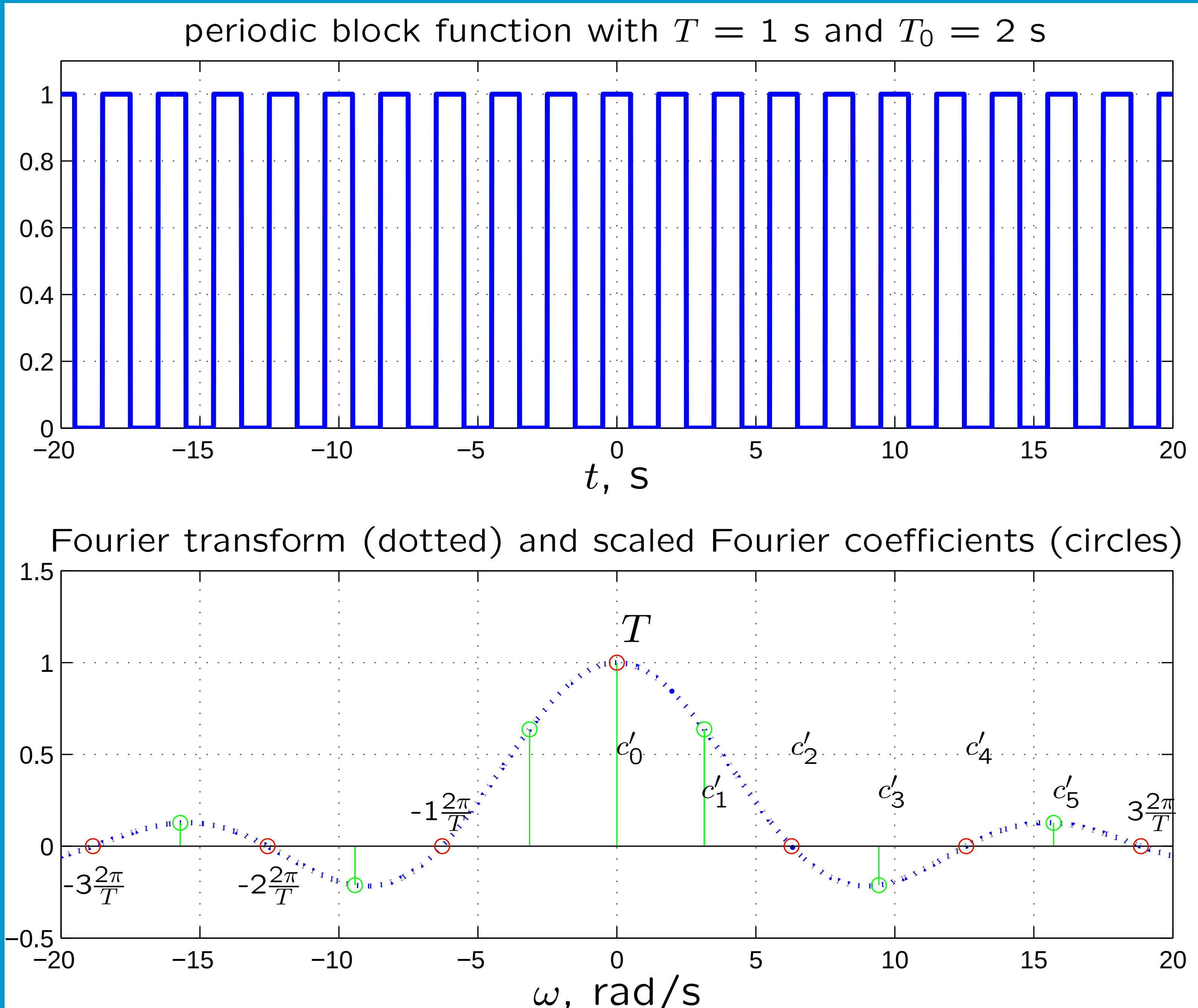
The following figure shows the Fourier series coefficients $c_k$ for the integer multiples of the fundamental frequency $\omega_0$, *scaled* with $T_0$.

It also shows the Fourier transform of the a-periodic block function $b(t)$, the sinc function $B(\omega)$ derived above.

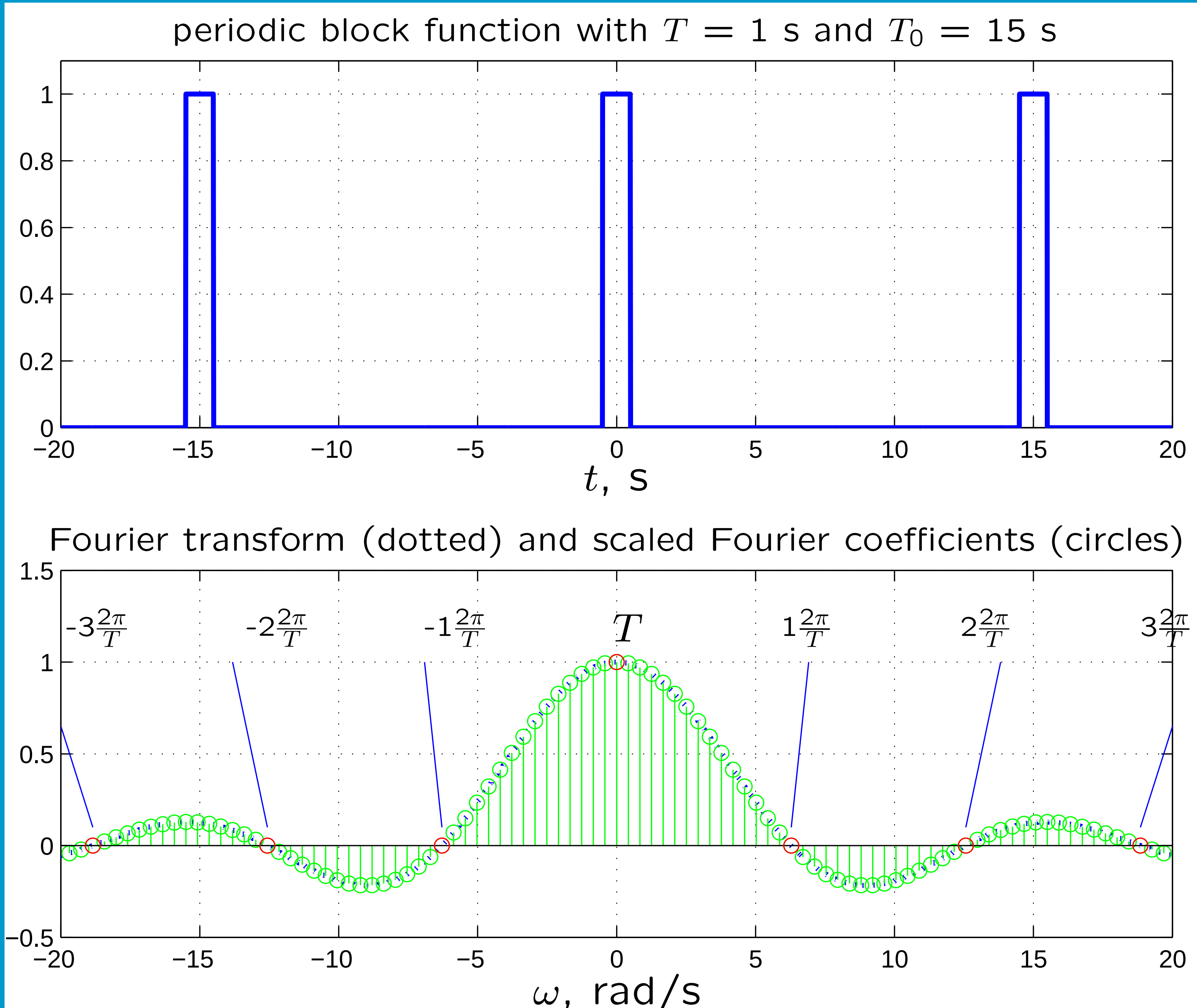
It can be seen that the *scaled* Fourier series components of the periodic block function correspond exactly with the Fourier-transformed aperiodical block.

**Note**: $B(k\omega_0) = T\frac{\sin k\omega_0 T/2}{k\omega_0 T/2} = T_0\frac{\sin k\omega_0 T/2}{k\pi} = T_0 c_k$


periodic block function with $T = 1$ s and $T_0 = 6$ s
1
0.8
0.6
0.4
0.2
0
−20
−15
−10
−5
0
5
10
15
20
$t$, s
Fourier transform (dotted) and scaled Fourier coefficients (circles)
1.5
1
0.5
0
−0.5
$-3\frac{2\pi}{T}$
$-2\frac{2\pi}{T}$
$-1\frac{2\pi}{T}$
$T$
$1\frac{2\pi}{T}$
$2\frac{2\pi}{T}$
$3\frac{2\pi}{T}$
$c'_1$
$c'_3$
$c'_5$
$c'_0$
$c'_2$
$c'_4$
−20
−15
−10
−5
0
5
10
15
20
$\omega$, rad/s



periodic block function with $T = 1$ s and $T_0 = 2$ s
$t$, s
Fourier transform (dotted) and scaled Fourier coefficients (circles)
$T$
$c'_0$
$c'_1$
$c'_2$
$c'_3$
$c'_4$
$c'_5$
$-3\frac{2\pi}{T}$
$-2\frac{2\pi}{T}$
$-1\frac{2\pi}{T}$
$3\frac{2\pi}{T}$
$\omega$, rad/s



periodic block function with $T = 1$ s and $T_0 = 15$ s
1
0.8
0.6
0.4
0.2
0
−20
−15
−10
−5
0
5
10
15
20
$t$, s
Fourier transform (dotted) and scaled Fourier coefficients (circles)
1.5
1
0.5
0
−0.5
$-3\frac{2\pi}{T}$
$-2\frac{2\pi}{T}$
$-1\frac{2\pi}{T}$
$T$
$1\frac{2\pi}{T}$
$2\frac{2\pi}{T}$
$3\frac{2\pi}{T}$
−20
−15
−10
−5
0
5
10
15
20
$\omega$, rad/s

# $\boxed{\text{The Power Spectral Density (PSD) function}}$

Our main interest in the frequency-domain is the *average amount of power per unit of frequency* by the product of two SVs $\bar{x}(t)$ and $\bar{y}(t)$.

For zero-mean signals, the **power spectral density** function (PSD) is defined as the Fourier transform of the covariance function:

$$\boxed{S_{\bar{x}\bar{y}}(\omega) = \mathcal{F}\left\{C_{\bar{x}\bar{y}}(\tau)\right\} = \int_{-\infty}^{\infty} C_{\bar{x}\bar{y}}(\tau) e^{-j\omega\tau}\, \mathrm{d}\tau} \tag{3.27}$$

and:

$$\boxed{C_{\bar{x}\bar{y}}(\tau) = \mathcal{F}^{-1}\left\{S_{\bar{x}\bar{y}}(\omega)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\bar{x}\bar{y}}(\omega) e^{j\omega\tau}\, \mathrm{d}\omega} \tag{3.28}$$

**Proof:**

Consider stochastic processes $\bar{x}(t)$ and $\bar{y}(t)$ which are approximated by signals $\bar{a}(t)$ and $\bar{b}(t)$:

$$\begin{array}{rll} \text{if } -\mathrm{T} \le \mathrm{t} \le +\mathrm{T}: & \bar{a}(t) = \bar{x}(t); & \bar{b}(t) = \bar{y}(t); \\ \text{else}: & \bar{a}(t) = 0; & \bar{b}(t) = 0; \end{array}$$

Then we obtain for the covariance function:

$$C_{\bar{a}\bar{b}}(\tau) = \frac{1}{2T - \tau} \int_{-T}^{T-\tau} (\bar{a}(t) - \mu_{\bar{a}})(\bar{b}(t+\tau) - \mu_{\bar{b}}) \, \mathrm{d}t \qquad (*)$$

Assume zero means: $\mu_{\bar{a}} = \mu_{\bar{b}} = 0$. Also: $\bar{b}(t+\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{B}(\omega) e^{j\omega(t+\tau)} \, \mathrm{d}\omega$, substitute into (*) yields

$$C_{\bar{a}\bar{b}}(\tau) = \frac{1}{2T - \tau} \int_{-T}^{T-\tau} \bar{a}(t) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{B}(\omega) e^{j\omega(t+\tau)} \, \mathrm{d}\omega \right) \mathrm{d}t$$

change order of integration

$$C_{\bar{a}\bar{b}}(\tau) = \frac{1}{2\pi} \frac{1}{2T - \tau} \int_{-\infty}^{\infty} \bar{B}(\omega) e^{j\omega\tau} \left( \int_{-T}^{T-\tau} \bar{a}(t) e^{j\omega t} \, \mathrm{d}t \right) \mathrm{d}\omega$$

between brackets: $\bar{A}(-\omega)$, when $T \to \infty$

When $T \to \infty$, $C_{\bar{a}\bar{b}}(\tau)$ equals $C_{\bar{x}\bar{y}}(\tau)$ : $C_{\bar{x}\bar{y}}(\tau) = \lim\limits_{T\to\infty} \frac{1}{2\pi} \frac{1}{2T} \int_{-\infty}^{\infty} \bar{B}(\omega) \bar{A}(-\omega) e^{j\omega\tau} \, \mathrm{d}\omega$

Then: $C_{\bar{x}\bar{y}}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left( \lim\limits_{T\to\infty} \frac{1}{2T} \bar{B}(\omega) \bar{A}(-\omega) \right)}_{\doteq S_{\bar{x}\bar{y}}(\omega)} e^{j\omega\tau} \, \mathrm{d}\omega$ **q.e.d.**

which explains the "time average amount of power per unit of frequency"

# Auto-functions and Cross-functions

Again, similarly as for the average product, covariance and correlation functions of two SVs one can investigate two different stochastic processes $\bar{x}(t)$ and $\bar{y}(t)$, yielding **cross-PSD** functions, or one and the same stochastic process, yielding the **auto-PSD** functions. Consider the auto-PSD function $S_{\bar{x}\bar{x}}(\omega)$:

$$\boxed{S_{\bar{x}\bar{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\bar{A}(\omega)\bar{A}(-\omega) \;=\; \lim_{T\to\infty} \frac{1}{2T}\left|\bar{A}(\omega)\right|^2} \tag{3.31}$$

This explains the interpretation 'time-average amount of power as a function of frequency'. Note that the last element on the right is an expectation $E\{\cdot\}$

Also note that since $\sigma_{\bar{x}}^2 = C_{\bar{x}\bar{x}}(\tau = 0)$ we get: ($C_{\bar{x}\bar{x}}(\tau)$ is even, and so is $S_{\bar{x}\bar{x}}(\omega)$)

$$\boxed{\sigma_{\bar{x}}^2 \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\bar{x}\bar{x}}(\omega)\,\mathrm{d}\omega \;=\; \frac{1}{\pi}\int_{0}^{\infty} S_{\bar{x}\bar{x}}(\omega)\,\mathrm{d}\omega} \tag{3.32}$$

Integrating the PSD of a certain signal yields its variance.

**Properties of the auto-PSD function (Table 3.3)**

$$\boxed{S_{\bar{x}\bar{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left| \bar{A}(\omega) \right|^2}$$

1. $S_{\bar{x}\bar{x}}(\omega)$ is a **real** function
2. $S_{\bar{x}\bar{x}}(\omega)$ is an **even** function (remember that $C_{\bar{x}\bar{x}}(\tau)$ is even)
3. So: $S_{\bar{x}\bar{x}}(\omega) = S_{\bar{x}\bar{x}}(-\omega)$
4. $\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} S_{\bar{x}\bar{x}}(\omega)\, \mathrm{d}\omega = C_{\bar{x}\bar{x}}(0) = \sigma_{\bar{x}}^2$

**Properties of the cross-PSD functions (Table 3.3)**

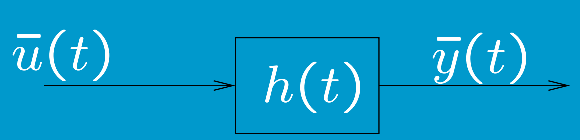
$$\boxed{S_{\bar{x}\bar{y}}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\bar{B}(\omega)\bar{A}(-\omega) = \mathrm{E}\left\{\bar{Y}(\omega)\bar{X}(-\omega)\right\}}$$

$$\boxed{S_{\bar{y}\bar{x}}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\bar{A}(\omega)\bar{B}(-\omega) = \mathrm{E}\left\{\bar{X}(\omega)\bar{Y}(-\omega)\right\}}$$

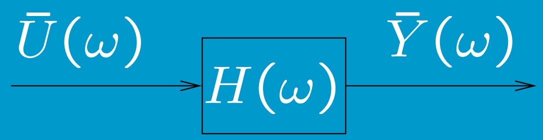
1. $S_{\bar{x}\bar{y}}(\omega)$ and $S_{\bar{y}\bar{x}}(\omega)$ are **complex** functions
2. $\mathrm{Re}\{S_{\bar{x}\bar{y}}(\omega)\} = \mathrm{Re}\{S_{\bar{x}\bar{y}}(-\omega)\}$, i.e., the <u>real</u> part of the cross-PSD is <u>even</u>
3. $\mathrm{Im}\{S_{\bar{x}\bar{y}}(\omega)\} = -\mathrm{Im}\{S_{\bar{x}\bar{y}}(-\omega)\}$, i.e., the <u>imaginary</u> part of the cross-PSD is <u>odd</u>
4. $S_{\bar{x}\bar{y}}(\omega) = S_{\bar{y}\bar{x}}(-\omega)$ (remember that $C_{\bar{x}\bar{y}}(\tau) = C_{\bar{y}\bar{x}}(-\tau)$)
5. $S_{\bar{x}\bar{y}}(-\omega) = S^*_{\bar{x}\bar{y}}(\omega)$, its complex conjugate (proof follows from items 2 and 3)
6. $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\bar{x}\bar{y}}(\omega)\,\mathrm{d}\omega = C_{\bar{x}\bar{y}}(0) = \mathrm{E}\{\bar{x}\bar{y}\}$

# Dynamic system analysis in the frequency domain

The dynamics of a system can be studied in the frequency domain by examining the PSD of the system output resulting from an input with a given PSD.




or




In the time domain, the relation between output signal $\bar{y}(t)$ and input signal $\bar{u}(t)$ is given by the system's **impulse response function** $h(t)$. The corresponding description in the frequency domain of linear time-invariant (LTI) systems is based on either the Laplace- or the Fourier transform:

Laplace : $\bar{Y}(s) = H(s) \cdot \bar{U}(s)$ **transfer function**

Fourier : $\bar{Y}(\omega) = H(\omega) \cdot \bar{U}(\omega)$ **frequency-response function**

More on the differences between Laplace- and Fourier-transforms later.

First calculate the expected value $\mu_{\bar{y}}$ of the system output signal $\bar{y}(t)$:

$$
\begin{aligned}
\mu_{\bar{y}} = \mathsf{E}\left\{\bar{y}(t)\right\} \;&=\; \mathsf{E}\left\{\bar{u}(t) \star h(t)\right\} \qquad \text{with } \star \text{ the convolution operator}\\
= \mathsf{E}\left\{\int_{-\infty}^{\infty} \bar{u}(t-\tau) h(\tau)\, \mathrm{d}\tau\right\} \;&=\; \int_{-\infty}^{\infty} \underbrace{\mathsf{E}\left\{\bar{u}(t-\tau)\right\}}_{=\mu_{\bar{u}}} h(\tau)\, \mathrm{d}\tau\\
= \mu_{\bar{u}} \int_{-\infty}^{\infty} h(\tau)\, \mathrm{d}\tau &
\end{aligned}
\tag{3.35}
$$

Then, remember that the frequency-response function $H(\omega)$ is the Fourier-transformed impulse response function $h(t)$:

$$\boxed{H(\omega) = \mathcal{F}\{h(t)\} = \int\limits_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau} \tag{3.36}$$

So, substitution of $\omega = 0$ into (3.36) yields an alternative expression for (3.35):

$$\boxed{\mu_{\bar{y}} = H(0)\mu_{\bar{u}}} \quad \text{the } \textit{static gain} \tag{3.38}$$

Next, we will search for a similar relation between the input PSD function $S_{\bar{u}\bar{u}}(\omega)$ and the output PSD function $S_{\bar{y}\bar{y}}(\omega)$. We assume <u>zero-mean</u> stationary random processes. First, we are considering the auto- and cross-covariance functions in the time-domain.

It can be shown that:

$$\boxed{C_{\bar{u}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau) \star h(\tau)} \tag{3.39}$$

**Proof:**

$$
\begin{aligned}
C_{\bar{u}\bar{y}}(\tau) &= \mathsf{E}\left\{\bar{u}(t)\bar{y}(t+\tau)\right\} \\
&= \mathsf{E}\left\{\bar{u}(t)\int\limits_{-\infty}^{\infty}\bar{u}(t+\tau-\theta)h(\theta)\,\mathrm{d}\theta\right\} \\
&= \int\limits_{-\infty}^{\infty}\mathsf{E}\left\{\bar{u}(t)\bar{u}(t+\tau-\theta)\right\}h(\theta)\,\mathrm{d}\theta \\
&= \int\limits_{-\infty}^{\infty}C_{\bar{u}\bar{u}}(\tau-\theta)h(\theta)\,\mathrm{d}\theta \\
&= C_{\bar{u}\bar{u}}(\tau)\star h(\tau) \qquad\qquad \text{q.e.d.}
\end{aligned}
$$

Similarly, it can be shown that:

$$\boxed{C_{\bar{y}\bar{u}}(\tau) = C_{\bar{u}\bar{u}}(\tau)\star h(-\tau)} \tag{3.40}$$

$$\boxed{C_{\bar{y}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau)\star h(-\tau)\star h(\tau)} \tag{3.41}$$

Transforming these equations to the frequency domain yields the expressions that we are looking for:

$$
\boxed{\begin{aligned}
S_{\bar{u}\bar{y}}(\omega) &= H(\omega)S_{\bar{u}\bar{u}}(\omega) && (3.42)\\
S_{\bar{y}\bar{u}}(\omega) &= H(-\omega)S_{\bar{u}\bar{u}}(\omega) && (3.43)\\
S_{\bar{y}\bar{y}}(\omega) &= H(-\omega)H(\omega)S_{\bar{u}\bar{u}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) && (3.44)
\end{aligned}}
$$

Note: an alternative way to prove (3.44) is to use the inverse dynamics:

$S_{\bar{u}\bar{y}}(\omega) = H(\omega)S_{\bar{u}\bar{u}}(\omega)$ "from $\bar{u}$ to $\bar{y}$"

$S_{\bar{y}\bar{u}}(\omega) = \frac{1}{H(\omega)}S_{\bar{y}\bar{y}}(\omega)$ "from $\bar{y}$ to $\bar{u}$"

$S_{\bar{y}\bar{y}}(\omega) = H(\omega)S_{\bar{y}\bar{u}}(\omega) = H(\omega)H(-\omega)S_{\bar{u}\bar{u}}(\omega)$ q.e.d.

Finally, the variance $\sigma^2_{\bar{y}}$ of an output signal $\bar{y}(t)$ can be calculated with:

$$
\boxed{\sigma^2_{\bar{y}} = \frac{1}{\pi}\int_0^\infty S_{\bar{y}\bar{y}}(\omega)\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^\infty |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega)\,\mathrm{d}\omega} \qquad (3.45)
$$

# White noise

The covariance function of white noise has the form of a **Dirac pulse**:

$$\boxed{C_{\bar{w}\bar{w}}(\tau) = W\delta(\tau)} \tag{2.38}$$

The parameter $W$ is called the **intensity** of the white noise.

Then the auto-PSD of the white noise process becomes:

$$S_{\bar{w}\bar{w}}(\omega) \;=\; \int_{-\infty}^{\infty} C_{\bar{w}\bar{w}}(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau \;=\; W, \tag{3.48}$$

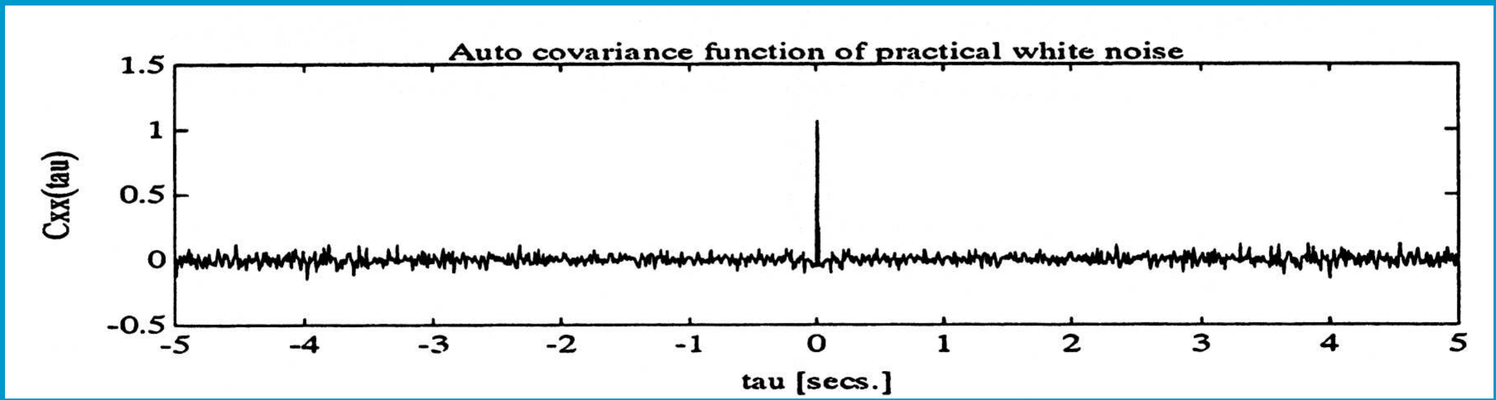
i.e., a constant over the total frequency range.

Again we see, by integrating the auto-PSD, that the variance of the white noise process is infinite, and can not physically exist.

**Note:** Qualitatively speaking, a very ‘broad’ shape in the frequency domain (the PSD) becomes a very ‘narrow’ shape in the time domain. Mutadis mutandis, the same is true when looking at a very ‘narrow’ shape in the frequency domain (like the Dirac pulses belonging to the FT of a cosine function) becomes a very ‘broad’ shape in the time domain (i.e., the auto-covariance function of the cosine function). (see Example 3.1).

In practice, a signal $\bar{w}$ is called ‘white’ if:

$$\boxed{S_{\bar{w}\bar{w}}(\omega) = W \text{ for } |\omega| < \omega_0}$$



# Colored noise, shaping filters

Consider an LTI system with input $\bar{u}(t)$ and output $\bar{y}(t)$. Assume that $\bar{u}(t)$ is a white noise process $\bar{w}(t)$: $S_{\bar{u}\bar{u}}(\omega) = W \; \forall \; \omega$.

$$\xrightarrow{\bar{u}(t)} \boxed{h(t)} \xrightarrow{\bar{y}(t)} \quad \text{or} \quad \xrightarrow{\bar{U}(\omega)} \boxed{H(\omega)} \xrightarrow{\bar{Y}(\omega)}$$

Then: $S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 \cdot W$ and

$$\boxed{\sigma_{\bar{y}}^2 = W \cdot \underbrace{\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, \mathrm{d}\omega \right\}}_{= \text{ integral } I \text{ in Table 3.6}}}$$

Hence, the output of the LTI system is 'filtered' white noise, also known as **colored noise**. The frequency response function $H(\omega)$ is known as the **shaping filter**.

| $H(\omega)$ | $I$ |
|---|---|
| $\frac{1}{1+j\omega\tau}$ | $\frac{1}{2\tau}$ |
| $\frac{1}{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2}$ | $\frac{\omega_0}{4\zeta}$ |
| $\frac{1}{(1+j\omega\tau_1)(1+j\omega\tau_2)}$ | $\frac{1}{2(\tau_1+\tau_2)}$ |
| $\frac{1+j\omega\tau}{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2}$ | $\frac{\omega_0}{4\zeta}(1+\omega_0^2\tau^2)$ |
| $\frac{1+j\omega\tau_1}{(1+j\omega\tau_2)(1+j\omega\tau_3)}$ | $\frac{1}{2(\tau_2+\tau_3)}\left(1+\frac{\tau_1^2}{\tau_2\tau_3}\right)$ |
| $\frac{1}{(1+j\omega\tau)\left\{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2\right\}}$ | $\frac{1}{2}\frac{\frac{\omega_0}{2\zeta}+\omega_0^2\tau}{1+2\zeta\omega_0\tau+\omega_0^2\tau^2}$ |

Table 3.6: Standard integrals for the calculation of the variance.

An alternative for table 3.5 is given in table 3.6, using the frequency response function $H(\omega)$ rather than the transfer function $H(s)$.

$$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 \, d\omega$$

## Relation between the Laplace and Fourier transforms

The Laplace transform is defined as:

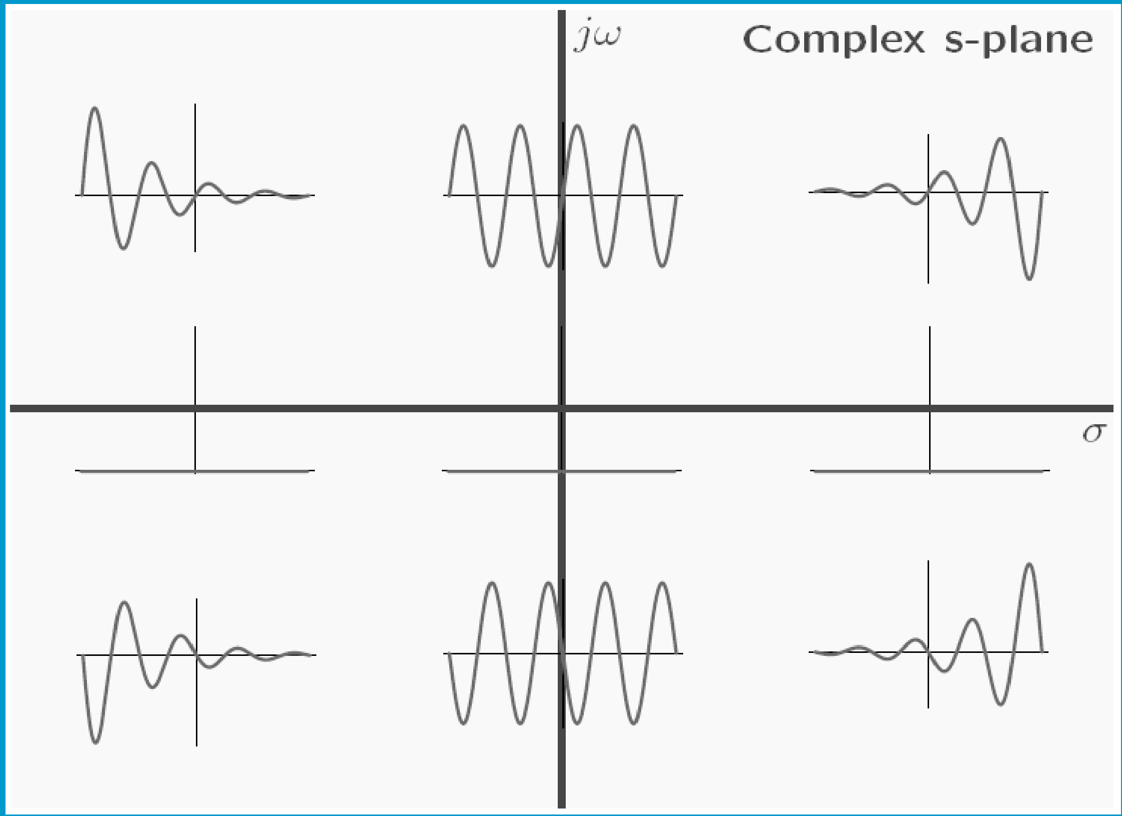
$$\boxed{X(s) = \mathcal{L}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t}, \tag{3.46}$$

with $s$ a complex variable: $s = \sigma + j\omega$.

The Fourier transform is defined as:

$$\boxed{X(\omega) = \mathcal{F}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t} \tag{3.16}$$

The Fourier transform equals the Laplace transform for values $\sigma$=0, i.e., for purely imaginary values of $s$.




The Fourier transform is limited as compared to the more general Laplace transform, in the sense that it is only to be used for the **time-invariant, or stationary part of a system's response**. This will be elaborated on using an example.

## **Example:** the response of a first order system to a sinusoid.

Consider a first order LTI system, described by the following ODE:

$$y(t) + \tau\dot{y}(t) = Gx(t), \quad \text{with initial conditions } y(t_0) = \dot{y}(t_0) = 0.$$

The input signal $x(t)$ is defined as: $x(t) = \sin(\omega_0 t) \cdot u(t)$, or $x(t) = \begin{cases} 0 & t < 0 \\ \sin\omega_0 t & t \geq 0 \end{cases}$

Use the Laplace transform to compute the system's response to this input signal:

$$Y(s) + \tau s Y(s) = GX(s),$$

where $X(s) = \mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$, so:

$$Y(s) = G\frac{1}{1+\tau s}\frac{\omega_0}{s^2+\omega_0^2},$$

three poles at $s = -j\omega_0$, $s = j\omega_0$ and $s = -\frac{1}{\tau}$, partial fraction:

$$Y(s) = G\left(-\frac{\omega_0\tau}{\omega_0^2\tau^2+1}\cdot\frac{s}{s^2+\omega_0^2} + \frac{1}{\omega_0^2\tau^2+1}\cdot\frac{\omega_0}{s^2+\omega_0^2} + \frac{\omega_0\tau}{\omega_0^2\tau^2+1}\cdot\frac{1}{s+\frac{1}{\tau}}\right)$$

inverse Laplace yields:

$$y(t) = G\cdot(\underbrace{-\frac{\omega_0\tau}{\omega_0^2\tau^2+1}\cos(\omega_0 t)u(t) + \frac{1}{\omega_0^2\tau^2+1}\sin(\omega_0 t)u(t)}_{\text{stationary response}} + \underbrace{\frac{\omega_0\tau}{\omega_0^2\tau^2+1}e^{-\frac{t}{\tau}}}_{\text{transient response}})$$

It is clear that when $t \to \infty$, the transient response (NL: 'inschakelverschijnsel') will become zero and only the *stationary response* will persist:

$$y(t) = \frac{G}{\omega_0^2\tau^2+1}(\sin\omega_0 t - \omega_0\tau\cos\omega_0 t)$$

It can be shown (*) that this equals: $y(t) = |H(\omega)|_{\omega=\omega_0}\sin(\omega_0 t + \angle\, H(\omega)|_{\omega=\omega_0})$,

with $H(\omega)$ the system's frequency response function: $H(\omega) = H(s)|_{s=j\omega} = G\frac{1}{1+j\omega\tau}$

Then: $|H(\omega)| = \frac{G}{\sqrt{1+\omega^2\tau^2}}$ and $\angle H(\omega) = \arctan(-\omega\tau)$.

This illustrates the fact that the Fourier transform only represents the stationary system's response. When analyzing the behavior of a dynamic system using spectral analysis one should be very careful to only use those (parts of) the system response where the transient response has 'died out'. Otherwise one will attribute phenomena that are caused by the systems transient response to its stationary response.

Proof of (*):

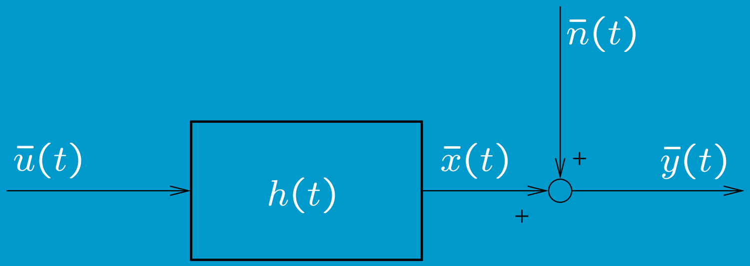
$$A\sin(\omega_0 t + \phi) = A\sin\omega_o t\cos\phi + A\cos\omega_o t\sin\phi \;=\; A\cos\phi\sin\omega_o t + A\sin\phi\cos\omega_o t$$

Assume: $A\cos\phi = \frac{G}{1+\omega_o^2\tau^2}$; and $A\sin\phi = \frac{-G\omega_o\tau}{1+\omega_o^2\tau^2}$; Then: $\tan\phi = \frac{A\sin\phi}{A\cos\phi} = -\omega_o\tau$

And because $A^2 = A^2\sin^2\phi + A^2\cos^2\phi$ we get: $A = \frac{G}{\sqrt{1+\omega_o\tau^2}}$.

# Estimation of the frequency response function

Consider an LTI system, characterized by its impulse response function $h(t)$, or, equivalently, its frequency response function $H(\omega)$. The system has a stochastic input signal $\bar{u}(t)$, and the response $\bar{x}(t)$ is perturbed by an unknown random noise $\bar{n}(t)$, i.e., the measured signal $\bar{y}(t)$ equals $\bar{x}(t)+\bar{n}(t)$.




Question 1: how do we obtain an estimate of the frequency response function?

Then: $\bar{y}(t) = \bar{x}(t) + \bar{n}(t) = \bar{u}(t) \star h(t) + \bar{n}(t)$

$$\bar{y}(t) = \int_{-\infty}^{\infty} h(\sigma)\bar{u}(t-\sigma)d\sigma + \bar{n}(t) \tag{3.49}$$

Shift this equation with time $\tau$:

$$\bar{y}(t+\tau) = \int_{-\infty}^{\infty} h(\sigma)\bar{u}(t+\tau-\sigma)\mathrm{d}\sigma + \bar{n}(t+\tau) \tag{3.50}$$

Multiply with a 'help signal', $\bar{z}(t)$, of which the properties will be selected later:

$$\bar{z}(t)\bar{y}(t+\tau) = \int_{-\infty}^{\infty} h(\sigma)\bar{z}(t)\bar{u}(t+\tau-\sigma)\mathrm{d}\sigma + \bar{z}(t)\bar{n}(t+\tau) \tag{3.51}$$

Take the expectation of this equation yields (for zero-mean signals):

$$C_{\bar{z}\bar{y}}(\tau) = \int_{-\infty}^{\infty} h(\sigma)C_{\bar{z}\bar{u}}(\tau-\sigma)\mathrm{d}\sigma + C_{\bar{z}\bar{n}}(\tau) \tag{3.52}$$

We can now eliminate the noise effect by choosing a signal $\bar{z}(t)$ that is uncorrelated with $\bar{n}(t)$.

Assume that $\bar{u}(t)$ is indeed uncorrelated with $\bar{n}(t)$ (which is often the case), we obtain (with $C_{\bar{u}\bar{n}}(\tau)$=0):

$$C_{\bar{u}\bar{y}}(\tau) = \int_{-\infty}^{\infty} h(\sigma)C_{\bar{u}\bar{u}}(\tau-\sigma)\mathrm{d}\sigma$$

In the frequency domain this becomes:

$$S_{\bar{u}\bar{y}}(\omega) = H(\omega)\cdot S_{\bar{u}\bar{u}}(\omega)$$

We can obtain the FRF using the quotient of two PSDs:

$$\boxed{H(\omega) = \frac{S_{\bar{u}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)}} \qquad \text{an "open loop" identification method} \tag{3.53}$$

What if $\bar{u}(t)$ and $\bar{n}(t)$ are correlated??

# Estimation of the noise spectrum PSDs

Consider the same system as in the previous case.

Question 2: how can we obtain an estimate of the PSD of the noise signal $\bar{n}(t)$?

Now:

$$\begin{aligned} \bar{y}(t) &= \bar{x}(t) + \bar{n}(t) \\ \bar{y}(t+\tau) &= \bar{x}(t+\tau) + \bar{n}(t+\tau), \end{aligned} \tag{3.54}$$

so:

$$\bar{y}(t)\bar{y}(t+\tau) = \bar{x}(t)\bar{x}(t+\tau) + \bar{n}(t)\bar{n}(t+\tau) + \bar{x}(t)\bar{n}(t+\tau) + \bar{n}(t)\bar{x}(t+\tau) \tag{3.55}$$

Take the expectation of this equation and assume that all means are zero:

$$C_{\bar{y}\bar{y}}(\tau) = C_{\bar{x}\bar{x}}(\tau) + C_{\bar{n}\bar{n}}(\tau) + C_{\bar{x}\bar{n}}(\tau) + C_{\bar{n}\bar{x}}(\tau) \tag{3.56}$$

When $\bar{u}(t)$ and $\bar{n}(t)$are uncorrelated, so are $\bar{x}(t)$ and $\bar{n}(t)$:

$$C_{\bar{y}\bar{y}}(\tau) = C_{\bar{x}\bar{x}}(\tau) + C_{\bar{n}\bar{n}}(\tau) \tag{3.57}$$

In the frequency domain this becomes:

$$S_{\bar{y}\bar{y}}(\omega) = S_{\bar{x}\bar{x}}(\omega) + S_{\bar{n}\bar{n}}(\omega), \quad \text{where:} \quad S_{\bar{x}\bar{x}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) \tag{3.58}$$

So we can obtain the PSD of the unknown noise using:

$$\boxed{S_{\bar{n}\bar{n}}(\omega) = S_{\bar{y}\bar{y}}(\omega) - |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega)}, \tag{3.59}$$

where $S_{\bar{y}\bar{y}}(\omega)$ and $S_{\bar{u}\bar{u}}(\omega)$ are measured, and $H(\omega)$ can be estimated from (3.53).

# Definition of the Coherence

Consider the same system as in the previous case.

Question 3: what is the contribution of the system's response to the measured output signal as compared to the noise contribution? remember: $\bar{y} = \bar{x} + \bar{n}$

This is expressed in the quotient of $S_{\bar{x}\bar{x}}(\omega)$ and $S_{\bar{y}\bar{y}}(\omega)$:

$$\frac{S_{\bar{x}\bar{x}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{\frac{|S_{\bar{u}\bar{y}}(\omega)|^2}{S^2_{\bar{u}\bar{u}}(\omega)} S_{\bar{u}\bar{u}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|S_{\bar{u}\bar{y}}(\omega)|^2}{S_{\bar{u}\bar{u}}(\omega) S_{\bar{y}\bar{y}}(\omega)} \tag{3.60}$$

The square root of this quotient is referred to as the **coherence** $\Gamma_{\bar{u}\bar{y}}(\omega)$ between the system input $\bar{u}(t)$ and its measured output $\bar{y}(t)$:

$$\boxed{\Gamma_{\bar{u}\bar{y}}(\omega) = \sqrt{\frac{|S_{\bar{u}\bar{y}}(\omega)|^2}{S_{\bar{u}\bar{u}}(\omega) S_{\bar{y}\bar{y}}(\omega)}}} \tag{3.61}$$

The coherence is a measure of the linear relationship between the input signal and the output signal of a system. It always has a value between 0 (no coherence) and 1 (full coherence).

convolution: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\mathrm{d}\tau$