Symmetric Aircraft Responses to Atmospheric Turbulence

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For this lecture the following material was used:

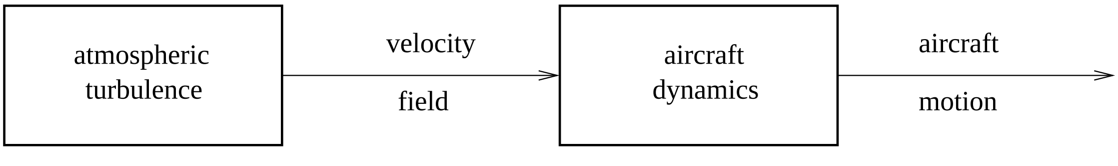
- Chapter 7 of Lecture notes *Aircraft Responses to Atmospheric Turbulence*.

Contents of this lecture

Introduction

Preceding lecture: general description of characteristics of atmospheric turbulence.
- meteorology
- statistics (stochastic theory)

$\rightarrow$ result: state-space description of turbulence velocity field.



Introduction

Aircraft model: has to be extended for effects of turbulence.
- this lecture: symmetric aircraft motions
- next lecture: asymmetric aircraft motions

From now on:
- stochastic variables will not be written with bar anymore
  e.g. u_g instead of $\bar{u}_g$.

Symmetric forces and moments: definitions

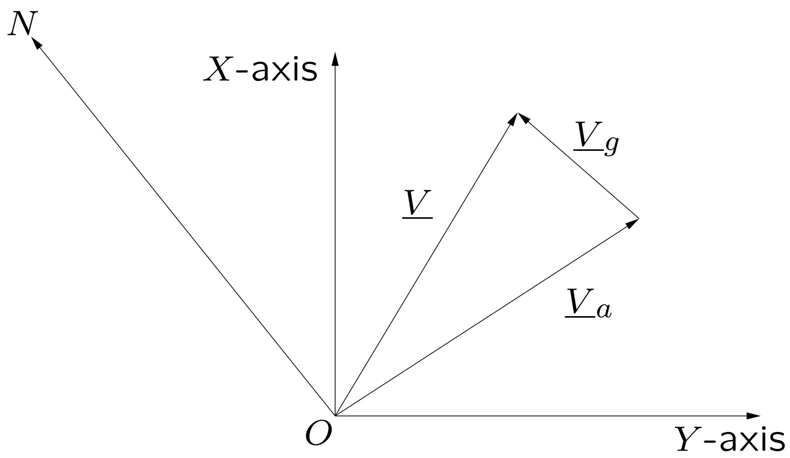
Concerning **motions of aircraft**:
atmospheric air is usually assumed to be in rest with respect to the earth.
→ only non-uniform air motions are considered

$$\underline{V} = \underline{V}_a + \underline{V}_g \tag{7.1}$$

where
- $\underline{V}$: aircraft cg relative to earth → stability and control
- $\underline{V}_a$: aircraft relative to air → forces and moments
- $\underline{V}_g$: air relative to earth (gust velocity)

with $\underline{V}_g = (u_g, v_g, w_g)^T$.

Symmetric forces and moments: definitions



Symmetric forces and moments: definitions

Gust velocity $\underline{V}_g$:

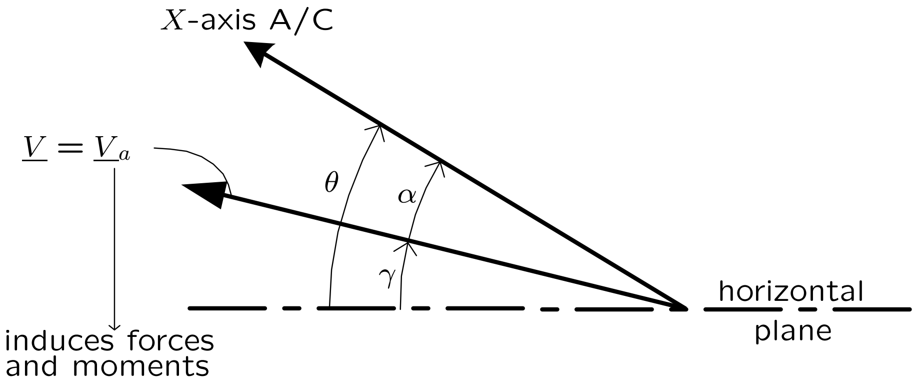
- Mean = 0 !!!
- Only variations (disturbances) are considered.
- Lateral components are not considered in this lecture.

$\Rightarrow$ only u_g and w_g are considered.

Note: '+' gust velocity directions are along the '-' stability frame axes.

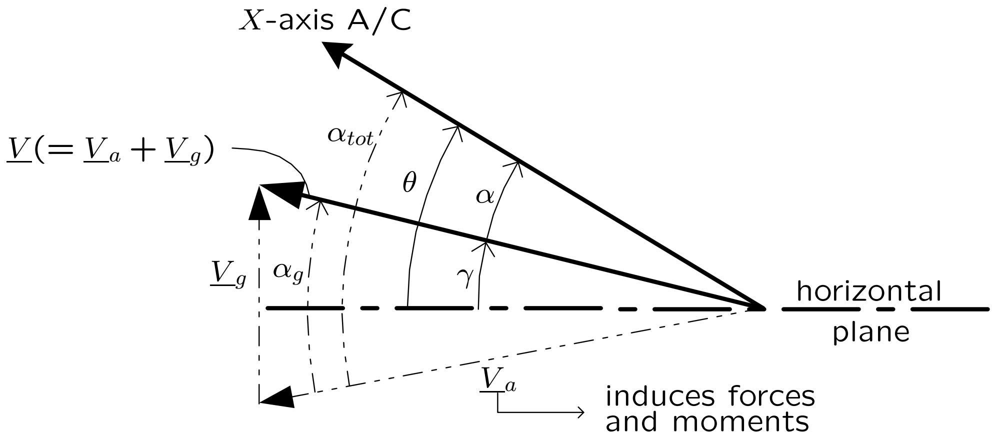
Symmetric forces and moments: definitions

Turbulence free air, without gust:



Symmetric forces and moments: definitions

Turbulent air, vertical gust wind:



Symmetric forces and moments: definitions

Without disturbances, $\underline{V}_a = \underline{V}$. This means that relative air flow around aircraft is only caused by aircraft velocity.

With disturbance, $\underline{V}_a = \underline{V} - \underline{V}_g$, an extra α_g arises ('gust angle of attack'). α_g is the angle between $\underline{V}$ and $\underline{V}_a$.

The following relations can be defined:

$$\alpha = \theta - \gamma \tag{7.2}$$

$$\alpha_g = \frac{w_g}{V}, \quad \text{for small } u_g \text{ and } w_g \text{ relative to } \underline{V} \tag{7.3}$$

$$\alpha_{tot} = \alpha + \alpha_g \tag{7.4}$$

α_{tot} is the angle between V_a and the aircraft X-axis ('total angle of attack').

Symmetric forces and moments: definitions

When α_g and γ are small, relation between **x-components**:

$$V_a = V + u_g \tag{7.5}$$

With the introduction of a **non-dimensional** gust velocity $\widehat{u}_g$,

$$\widehat{u}_g = \frac{u_g}{V} \tag{7.6}$$

leads to,

$$V_a = V\,(1 + \widehat{u}_g) \tag{7.7}$$

Symmetric forces and moments: expression

Turbulence → longitudinal **aerodynamic forces** and **moment**:

- Notation: X_g, Z_g and M_g.

- Function of:
  - u_g and w_g (non-dimensional: $\hat{u}_g$ and α_g)
  - time derivatives of $\hat{u}_g$ and α_g

$$\Rightarrow \quad \begin{array}{l} X_g(\hat{u}_g, \alpha_g; \frac{\dot{\hat{u}}_g \bar{c}}{V}, \frac{\dot{\alpha}_g \bar{c}}{V}; \frac{\ddot{\hat{u}}_g \bar{c}^2}{V^2}, \frac{\ddot{\alpha}_g \bar{c}^2}{V^2}; etc) \\ Z_g(\hat{u}_g, \alpha_g; \frac{\dot{\hat{u}}_g \bar{c}}{V}, \frac{\dot{\alpha}_g \bar{c}}{V}; \frac{\ddot{\hat{u}}_g \bar{c}^2}{V^2}, \frac{\ddot{\alpha}_g \bar{c}^2}{V^2}; etc) \\ M_g(\hat{u}_g, \alpha_g; \frac{\dot{\hat{u}}_g \bar{c}}{V}, \frac{\dot{\alpha}_g \bar{c}}{V}; \frac{\ddot{\hat{u}}_g \bar{c}^2}{V^2}, \frac{\ddot{\alpha}_g \bar{c}^2}{V^2}; etc) \end{array}$$

Symmetric forces and moments: expression

Using series expansion, X_g can be written as:

$$\begin{aligned} X_g \;=\; & \frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g \bar{c}}{V}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + \frac{\partial X_g}{\partial \frac{\ddot{\hat{u}}_g \bar{c}^2}{V^2}}\frac{\ddot{\hat{u}}_g \bar{c}^2}{V^2} + \cdots \\ + \;& \frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{\partial X_g}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}\frac{\dot{\alpha}_g \bar{c}}{V} + \frac{\partial X_g}{\partial \frac{\ddot{\alpha}_g \bar{c}^2}{V^2}}\frac{\ddot{\alpha}_g \bar{c}^2}{V^2} + \cdots \\ + \;& \frac{1}{2!}\left(\text{2nd order terms with respect to } \hat{u}_g, \alpha_g, \frac{\dot{\hat{u}}_g \bar{c}}{V}, \frac{\dot{\alpha}_g \bar{c}}{V}, \cdots\right) + \\ + \;& \frac{1}{3!}(\cdots\cdots) + \cdots\cdots \text{etc.} \end{aligned} \tag{7.8}$$

Analogue expressions for Z_g and M_g.

Symmetric forces and moments: expression

This expansion hold for any **reference state of equilibrium**.

Assumption:
$\hat{u}_g$ and α_g remain sufficiently small (same order as $\hat{u}$ and α).
$\rightarrow$ only linear terms remain in expression.

$$\Rightarrow X_g = \frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g \bar{c}}{V}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + \frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{\partial X_g}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}\frac{\dot{\alpha}_g \bar{c}}{V} \tag{7.9}$$

Symmetric forces and moments: expression

With the introduction of the non-dimensional coefficients:

$$\begin{aligned} C_{X_g} &= \frac{X_g}{\frac{1}{2}\rho V^2 S} \\ C_{Z_g} &= \frac{Z_g}{\frac{1}{2}\rho V^2 S} \\ C_{m_g} &= \frac{M_g}{\frac{1}{2}\rho V^2 S\bar{c}} \end{aligned} \tag{7.10}$$

C_{X_g} then reads:

$$\begin{aligned} C_{X_g} &= \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g \bar{c}}{V}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + \\ &+ \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}\frac{\dot{\alpha}_g \bar{c}}{V} \end{aligned} \tag{7.11}$$

Symmetric forces and moments: expression

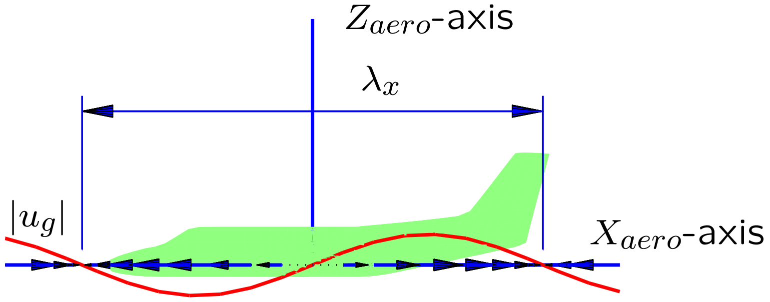
When **abbreviated**, the expressions for C_{X_g}, C_{Z_g} and C_{m_g} result in:

$$\boxed{\begin{aligned} C_{X_g} &= C_{X_{u_g}}\hat{u}_g + C_{X_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g + C_{X_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \\ C_{Z_g} &= C_{Z_{u_g}}\hat{u}_g + C_{Z_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{Z_{\alpha_g}}\alpha_g + C_{Z_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \\ C_{m_g} &= C_{m_{u_g}}\hat{u}_g + C_{m_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{m_{\alpha_g}}\alpha_g + C_{m_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \end{aligned}}\qquad \begin{aligned}(7.12)\\(7.13)\\(7.14)\end{aligned}$$

Partial derivatives $C_{X_{u_g}}$, $C_{X_{\dot{u}_g}}$, etc. will be called **'gust derivatives'**

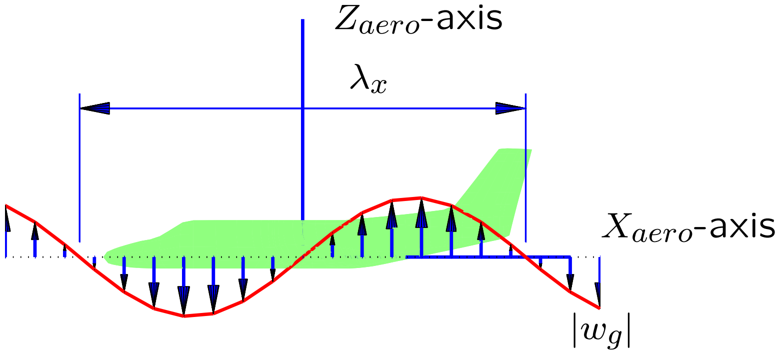
Gust derivatives: elementary fields

A **gust velocity field** can be thought of as a **superposition** of infinitely many **'elementary fields'**.
Longitudinal:



Gust derivatives: elementary fields

Vertical:



Gust derivatives: elementary fields

This can be expressed **mathematically**:

$$u_g = u_{g_{max}} e^{j\Omega x} = u_{g_{max}} e^{j\frac{\omega x}{V}} \tag{7.16}$$

$$w_g = w_{g_{max}} e^{j\Omega x} = w_{g_{max}} e^{j\frac{\omega x}{V}} \tag{7.17}$$

with
- Ω : spatial frequency
- ω : circular frequency

The **wavelength** in such an elementary field is given by,

$$\lambda = \frac{2\pi}{\Omega} \tag{7.18}$$

Note: a frozen field is assumed (Taylor's hypothesis).

Gust derivatives: elementary fields

Or, **non-dimensionally:**

$$\hat{u}_g = \hat{u}_{g_{max}} e^{j\Omega x} \tag{7.19}$$

With the following **definitions**:

$$s_c = \frac{x}{\bar{c}} = \frac{Vt}{\bar{c}} \tag{7.20}$$

$$k_c = \Omega \bar{c} = \frac{\omega \bar{c}}{V} \tag{7.21}$$

one can rewrite the expressions for $\hat{u}_g$:

$$\hat{u}_g = \hat{u}_{g_{max}} e^{jk_c s_c} \tag{7.22}$$

With k_c being the reduced frequency.

Gust derivatives: elementary fields

Hence, in complex form:

$$\frac{\dot{\hat{u}}_g \bar{c}}{V} \equiv jk_c \hat{u}_g \tag{7.23}$$

$$\frac{\dot{\alpha}_g \bar{c}}{V} \equiv jk_c \alpha_g \tag{7.24}$$

Hence, in an elementary field C_{X_g}, C_{Z_g} and C_{m_g} can be expressed as,

$$\boxed{\begin{aligned}
C_{X_g} &= \left(C_{X_{u_g}} + C_{X_{\dot{u}_g}}\, jk_c\right) \hat{u}_g + \left(C_{X_{\alpha_g}} + C_{X_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g && (7.25)\\
C_{Z_g} &= \left(C_{Z_{u_g}} + C_{Z_{\dot{u}_g}}\, jk_c\right) \hat{u}_g + \left(C_{Z_{\alpha_g}} + C_{Z_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g && (7.26)\\
C_{m_g} &= \left(C_{m_{u_g}} + C_{m_{\dot{u}_g}}\, jk_c\right) \hat{u}_g + \left(C_{m_{\alpha_g}} + C_{m_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g && (7.27)
\end{aligned}}$$

Gust derivatives: formulas

The problem will be split into **two parts**,

	$\widehat{u}_g$		α_g
$C_{X_g} \;=$	$\left(C_{X_{u_g}} + C_{X_{\dot{u}_g}}\, jk_c\right) \widehat{u}_g$	$+$	$\left(C_{X_{\alpha_g}} + C_{X_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g$
$C_{Z_g} \;=$	$\left(C_{Z_{u_g}} + C_{Z_{\dot{u}_g}}\, jk_c\right) \widehat{u}_g$	$+$	$\left(C_{Z_{\alpha_g}} + C_{Z_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g$
$C_{m_g} \;=$	$\left(C_{m_{u_g}} + C_{m_{\dot{u}_g}}\, jk_c\right) \widehat{u}_g$	$+$	$\left(C_{m_{\alpha_g}} + C_{m_{\dot{\alpha}_g}}\, jk_c\right) \alpha_g$

Assumption: aircraft is **wing/fuselage+horizontal tailplane**
$\rightarrow$ analytical expressions for gust derivatives can be found.

Gust derivatives: with respect to $\widehat{u}_g$

With a reduced frequency equal to zero ($k_c = 0$), the horizontal longitudinal gust field reduces to:

$$C_{X_g} = C_{X_{u_g}} \widehat{u}_g \tag{7.32}$$

This is a **constant gust-field**. There is no difference with a flight in turbulence free air.

Therefore:

$$C_{X_{u_g}} = C_{X_u} \tag{7.33}$$

$$C_{Z_{u_g}} = C_{Z_u} \tag{7.34}$$

$$C_{m_{u_g}} = C_{m_u} \tag{7.35}$$

which are the **'steady gust derivatives'**.

Gust derivatives: with respect to $\widehat{u}_g$

The other derivatives, with respect to the time derivative of $\widehat{u}_g$ are **'non-steady gust derivatives'**.

Variations in C_X:
- cause mainly changes in airspeed
- very low-frequent (phugoid motion)

$\rightarrow$ fast fluctuations of C_x are only of secondary importance
$\rightarrow$ $C_{X_{\dot{u}_g}}$ and $C_{X_{\dot{\alpha}_g}}$ are set to 0.

The remaining two unsteady gust derivatives $C_{Z_{\dot{u}_g}}$ and $C_{m_{\dot{u}_g}}$ will be derived next.

Consider: $\boxed{\begin{array}{l}\text{Aircraft = wing/fuselage + horizontal tailplane} \\ \rightarrow \text{gust hits wing first, tailplane later.}\end{array}}$

Gust derivatives: with respect to $\widehat{u}_g$

With preceding assumptions, in an elementary field of symmetric horizontal longitudinal turbulence, (____ are yet to be found)

$$C_{X_g} = \left(C_{X_u}\right) \widehat{u}_g \tag{7.32}$$

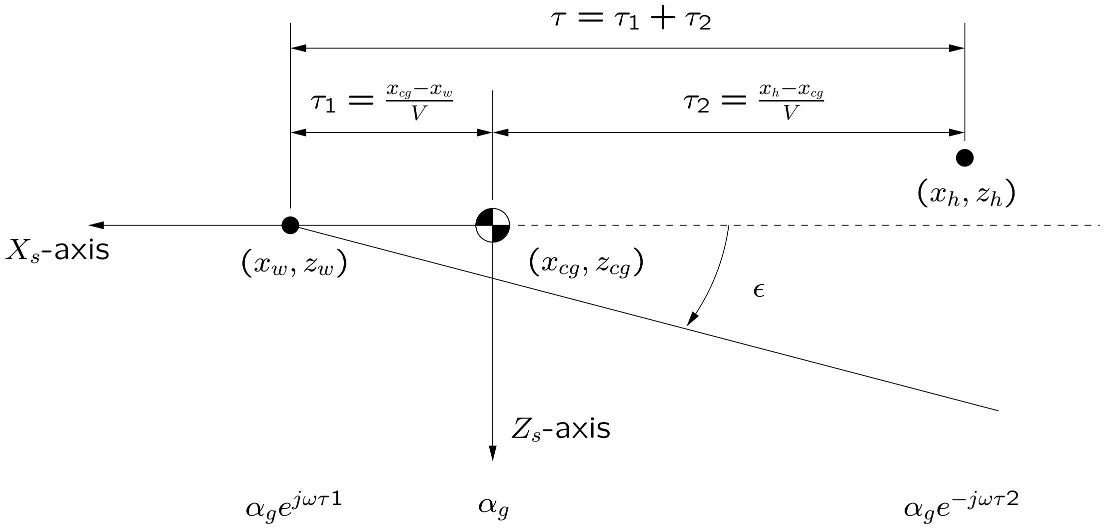
$$C_{Z_g} = \left(C_{Z_u} + \underline{C_{Z_{\dot{u}_g}}}\, jk_c\right) \widehat{u}_g \tag{7.30}$$

$$C_{m_g} = \left(C_{m_u} + \underline{C_{m_{\dot{u}_g}}}\, jk_c\right) \widehat{u}_g \tag{7.31}$$

Two methods are available to calculate $C_{Z_{\dot{u}_g}}$ and $C_{m_{\dot{u}_g}}$.
- **first method**: will be given here $\rightarrow$ see figure
- second method: see Chapter 7 of lecture notes

Gust derivatives: with respect to $\widehat{u}_g$

The gust velocity field is given at center of gravity **(cg)**, and required for **(w)ing** and **(h)orizontal tailplane**.



Gust derivatives: with respect to $\widehat{u}_g$

Time for gust **to travel**:
- from w to cg: $\tau_1 = \frac{x_{cg}-x_w}{V}$
- from cg to h: $\tau_2 = \frac{x_h-x_{cg}}{V}$

In gust field the **Horizontal** velocity of air at **wing** is:

$$V_{a_w} = V + u_{g_w} = V\,(1 + \widehat{u}_{g_w}) \qquad (7.36)$$

from which follows,

$$V_{a_w}^2 = V^2\,(1 + \widehat{u}_{g_w})^2 \approx V^2\,(1 + 2\,\widehat{u}_{g_w}) \qquad (7.37)$$

and in the same way at **horizontal tailplane**,

$$V_{a_h}^2 \approx V^2\,(1 + 2\,\widehat{u}_{g_h}) \qquad (7.38)$$

Gust derivatives: with respect to $\widehat{u}_g$

In **frequency domain**:

$$\widehat{u}_{g_w} = \widehat{u}_g \, e^{j\omega\tau_1} = \widehat{u}_g \, e^{j\omega\frac{x_{cg}-x_w}{V}} \tag{7.39}$$

$$\widehat{u}_{g_h} = \widehat{u}_g \, e^{-j\omega\tau_2} = \widehat{u}_g \, e^{j\omega\frac{x_{cg}-x_h}{V}} = \widehat{u}_g \, e^{-j\omega\frac{x_h-x_{cg}}{V}} \tag{7.40}$$

Z_g can be divided in a part contributed by the **wing** and a part by the **horizontal tailplane**:

$$Z_g = C_{Z_w} \, \tfrac{1}{2}\rho V^2 S \; 2\widehat{u}_{g_w} + C_{Z_h} \, \tfrac{1}{2}\rho V^2 S \; 2\widehat{u}_{g_h} \tag{7.42}$$

Using Eqs. (7.39), (7.40) in (7.42):

$$Z_g(\omega) = C_{Z_w} \, \tfrac{1}{2}\rho V^2 S \; 2\widehat{u}_g \, e^{j\omega\frac{x_{cg}-x_w}{V}} + C_{Z_h} \, \tfrac{1}{2}\rho V^2 S \; \widehat{u}_g \, e^{-j\omega\frac{x_h-x_{cg}}{V}} \tag{7.44}$$

Gust derivatives: with respect to $\widehat{u}_g$

In **non-dimensional** coefficients:

$$C_{Z_g}(\omega) = C_{Z_w} \, 2\widehat{u}_g \, e^{j\omega\tau_1} + C_{Z_h} \, 2\widehat{u}_g \, e^{-j\omega\tau_2} \tag{7.45}$$

For a **moderate frequency range**:

$$e^{-j\omega\tau} \approx 1 - j\omega\tau \tag{7.46}$$

This results in:

$$\begin{aligned} C_{Z_g}(\omega) &= C_{Z_w} \, 2\widehat{u}_g \left(1 + j\omega\frac{x_{cg}-x_w}{V}\right) + C_{Z_h} \, 2\widehat{u}_g \left(1 + j\omega\frac{x_{cg}-x_h}{V}\right) \\ &= 2\left(C_{Z_w} + C_{Z_h}\right)\widehat{u}_g + 2\left(C_{Z_w}\frac{x_{cg}-x_w}{V} + C_{Z_h}\frac{x_{cg}-x_h}{V}\right) j\omega \, \widehat{u}_g \\ &= 2\left(C_{Z_w} + C_{Z_h}\right)\widehat{u}_g + 2\left(C_{Z_w}\frac{x_{cg}-x_w}{\bar{c}} + C_{Z_h}\frac{x_{cg}-x_h}{\bar{c}}\right) \frac{j\omega\bar{c}}{V} \, \widehat{u}_g \end{aligned} \tag{7.47}$$

$\boxed{\text{Gust derivatives: with respect to } \widehat{u}_g}$

In the **time-domain** this becomes:

$$\begin{aligned} C_{Z_g} &= 2\left(C_{Z_w} + C_{Z_h}\right)\widehat{u}_g + 2\left(C_{Z_w}\frac{x_{cg}-x_w}{\bar{c}} + C_{Z_h}\frac{x_{cg}-x_h}{\bar{c}}\right)\frac{\dot{\widehat{u}}_g\bar{c}}{V} \\ &= C_{Z_{u_g}}\widehat{u}_g + C_{Z_{\dot{u}_g}}\frac{\dot{\widehat{u}}_g\bar{c}}{V} \end{aligned} \tag{7.48}$$

Hence,

$$C_{Z_{\dot{u}_g}} = 2\left(C_{Z_w}\tfrac{x_{cg}-x_w}{\bar{c}} + C_{Z_h}\tfrac{x_{cg}-x_h}{\bar{c}}\right) \tag{7.49}$$

In **steady flight**:

$$C_m = C_{m_{ac}} - C_{Z_w}\tfrac{x_{cg}-x_w}{\bar{c}} - C_{Z_h}\tfrac{x_{cg}-x_h}{\bar{c}} = 0 \tag{7.51}$$

This results in:

$$\boxed{C_{Z_{\dot{u}_g}} = 2\,C_{m_{ac}}} \tag{7.52}$$

$\boxed{\text{Gust derivatives: with respect to } \widehat{u}_g}$

In an analogous manner:

$$C_{m_{\dot{u}_g}} = 2\left(C_{m_w}\frac{x_{cg}-x_w}{\bar{c}} + C_{m_h}\frac{x_{cg}-x_h}{\bar{c}}\right) \tag{7.53}$$

In **steady flight** $C_m = 0$, hence,

$$C_{m_w} = -C_{m_h} \tag{7.54}$$

or,

$$C_{m_{\dot{u}_g}} = 2\left(-C_{m_h}\frac{x_{cg}-x_w}{\bar{c}} + C_{m_h}\frac{x_{cg}-x_h}{\bar{c}}\right) \tag{7.55}$$

Using the **taillength** $l_h = x_h - x_w$, the resulting **unsteady gust derivative** $C_{m_{\dot{u}_g}}$ becomes,

$$\boxed{C_{m_{\dot{u}_g}} = -2\,C_{m_h}\,\frac{l_h}{\bar{c}}} \tag{7.56}$$

Gust derivatives: with respect to α_g

Recapitulating:

		$\widehat{u}_g$ (OK)		α_g (?)
C_{X_g}	$=$	$\left(C_{X_u}\right)\widehat{u}_g$	$+$	$\left(C_{X_\alpha}\, jk_c\right)\alpha_g$
C_{Z_g}	$=$	$\left(C_{Z_u}+C_{Z_{\dot{u}_g}}\, jk_c\right)\widehat{u}_g$	$+$	$\left(C_{Z_\alpha}+C_{Z_{\dot{\alpha}_g}}\, jk_c\right)\alpha_g$
C_{m_g}	$=$	$\left(C_{m_u}+C_{m_{\dot{u}_g}}\, jk_c\right)\widehat{u}_g$	$+$	$\left(C_{m_\alpha}+C_{m_{\dot{\alpha}_g}}\, jk_c\right)\alpha_g$

Now gust derivatives with respect to α_g
Principle:
Symmetrical vertical gust reaches wing $\rightarrow$ change in downwash ϵ
$\rightarrow$ travels to horizontal tailplane ($\tau = \frac{l_h}{V}$)
$\rightarrow$ change in aerodynamic force.

Gust derivatives: with respect to α_g

Assumption: positions of aerodynamic center (wing+fuselage) coincides with center of gravity.

The **total aerodynamic force** along the Z-**axis** can be written as,

$$Z_g = C_{Z_{w\alpha}}\,\frac{1}{2}\rho V^2 S\,\alpha_g + C_{Z_{h\alpha}}\,\frac{1}{2}\rho {V_h}^2 S_h\,\alpha_{h_g} \qquad (7.77)$$

With, in the frequency domain,

$$\alpha_{h_g}(\omega) = \alpha_g\, e^{-j\omega\tau} - \frac{\partial\epsilon}{\partial\alpha}e^{-j\omega\tau}\alpha_g = \left(1-\frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\tau}\alpha_g = \left(1-\frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\frac{l_h}{V}}\alpha_g \qquad (7.78)$$

$\boxed{\text{Gust derivatives: with respect to } \alpha_g}$

The aerodynamic force along the Z-axis can now be written as,

$$C_{Z_g}(\omega) = C_{Z_{w\alpha}}\alpha_g + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\frac{l_h}{V}}\alpha_g \qquad (7.79)$$

or,

$$\frac{C_{Z_g}}{\alpha_g}(\omega) = C_{Z_{w\alpha}} + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\frac{l_h}{V}} \qquad (7.80)$$

In an analogue way,

$$\frac{C_{m_g}}{\alpha_g}(\omega) = C_{m_{w\alpha}} + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\frac{l_h}{V}} \qquad (7.81)$$

$$\frac{C_{X_g}}{\alpha_g}(\omega) = C_{X_{w\alpha}} + C_{X_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)e^{-j\omega\frac{l_h}{V}} \qquad (7.82)$$

$\boxed{\text{Gust derivatives: with respect to } \alpha_g}$

Expansion of exponential term in $\frac{C_{Z_g}}{\alpha_g}(\omega)$:

$$\begin{aligned}
\frac{C_{Z_g}}{\alpha_g}(\omega) &= C_{Z_{w\alpha}} + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\left(1 - j\omega\frac{l_h}{V}\right) \\
&= C_{Z_{w\alpha}} + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) - C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\frac{l_h}{V}j\omega \\
&= \left(C_{Z_{w\alpha}} + C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right) + \left(-C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)\frac{\bar{c}}{V}j\omega \\
&= C_{Z_{\alpha_g}} + C_{Z_{\dot{\alpha}_g}}\frac{\bar{c}}{V}j\omega \qquad (7.83)
\end{aligned}$$

$C_{Z_{\dot{\alpha}_g}}$ can be expanded and rewritten:

$$C_{Z_{\dot{\alpha}_g}} = C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}\frac{\partial\epsilon}{\partial\alpha} - C_{Z_{h\alpha}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = C_{Z_{\dot{\alpha}}} - C_{Z_q} \qquad (7.85)$$

Gust derivatives: with respect to α_g

With

$$C_{Z_{\dot{\alpha}}} = C_{Z_{h\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \frac{\partial \epsilon}{\partial \alpha} \tag{7.86}$$

$$C_{Z_q} = C_{Z_{h\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{7.87}$$

In a similar way,

$$C_{X_{\alpha_g}} = C_{X_\alpha} \tag{7.88}$$

$$C_{X_{\dot{\alpha}_g}} = C_{X_{\dot{\alpha}}} - C_{X_q} \tag{7.89}$$

$$C_{m_{\alpha_g}} = C_{m_\alpha} \tag{7.90}$$

$$C_{m_{\dot{\alpha}_g}} = C_{m_{\dot{\alpha}}} - C_{m_q} \tag{7.91}$$

Gust derivatives: summary

The following **table** can quickly and easily be used for **implementation**.

$C_{X_{u_g}} = C_{X_u}$ $C_{X_{\dot{u}_g}} = 0$	$C_{Z{u_g}} = C_{Z_u}$ $C_{Z_{\dot{u}_g}} = 2\ C_{m_{ac}}$	$C_{m_{u_g}} = C_{m_u}$ $C_{m_{\dot{u}_g}} = -2\ C_{m_h} \frac{l_h}{\bar{c}}$
$C_{X_{\alpha_g}} = C_{X_\alpha}$ $C_{X_{\dot{\alpha}_g}} = 0$	$C_{Z\alpha_g} = C_{Z_\alpha}$ $C_{Z_{\dot{\alpha}_g}} = C_{Z_{\dot{\alpha}}} - C_{Z_q}$	$C_{m_{\alpha_g}} = C_{m_\alpha}$ $C_{m_{\dot{\alpha}_g}} = C_{m_{\dot{\alpha}}} - C_{m_q}$

Symmetric equations of motion

The non-dimensional **symmetric forces and moment** on the aircraft due to **symmetric turbulence** are:

$$C_{X_g} = C_{X_{u_g}}\widehat{u}_g + C_{X_{\alpha_g}}\alpha_g \qquad (7.101)$$

$$C_{Z_g} = C_{Z_{u_g}}\widehat{u}_g + C_{Z_{\dot{u}_g}}\frac{\dot{\widehat{u}}_g\bar{c}}{V} + C_{Z_{\alpha_g}}\alpha_g + C_{Z_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \qquad (7.99)$$

$$C_{m_g} = C_{m_{u_g}}\widehat{u}_g + C_{m_{\dot{u}_g}}\frac{\dot{\widehat{u}}_g\bar{c}}{V} + C_{m_{\alpha_g}}\alpha_g + C_{m_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \qquad (7.100)$$

Symmetric equations of motion

Equations of motion for rigid aircraft flying in **turbulence** are:

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)\, D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} =$$

$$- \begin{bmatrix} C_{X_{\delta_e}} & C_{X_{u_g}} & 0 & C_{X_{\alpha_g}} & 0 \\ C_{Z_{\delta_e}} & C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} & C_{Z_{\alpha_g}} & C_{Z_{\dot{\alpha}_g}} \\ 0 & 0 & 0 & 0 & 0 \\ C_{m_{\delta_e}} & C_{m_{u_g}} & C_{m_{\dot{u}_g}} & C_{m_{\alpha_g}} & C_{m_{\dot{\alpha}_g}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \widehat{u}_g \\ D_c\widehat{u}_g \\ \alpha_g \\ D_c\alpha_g \end{bmatrix} \qquad (7.102)$$

In **state-space** representation (coefficients: see table 7-1):

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 \\ z_u & z_\alpha & z_\theta & z_q \\ 0 & 0 & 0 & \frac{V}{\bar{c}} \\ m_u & m_\alpha & m_\theta & m_q \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} x_{\delta_e} & x_{u_g} & 0 & x_{\alpha_g} & 0 \\ z_{\delta_e} & z_{u_g} & z_{\dot{u}_g} & z_{\alpha_g} & z_{\dot{\alpha}_g} \\ 0 & 0 & 0 & 0 & 0 \\ m_{\delta_e} & m_{u_g} & m_{\dot{u}_g} & m_{\alpha_g} & m_{\dot{\alpha}_g} \end{bmatrix} \begin{bmatrix} \delta_e \\ \widehat{u}_g \\ \frac{\dot{\hat{u}}_g\bar{c}}{V} \\ \alpha_g \\ \frac{\dot{\alpha}_g\bar{c}}{V} \end{bmatrix} \qquad (7.104)$$

Symmetric equations of motion

With the non-dimensional state-space model of atmospheric turbulence (Dryden):

$$\dot{\hat{u}}_g = \left[-\frac{V}{L_g}\right]\hat{u}_g + \left[\sigma_{\hat{u}_g}\sqrt{\frac{2V}{L_g}}\right] w_1 \tag{7.107}$$

$$\begin{bmatrix} \dot{\alpha}_g \\ \dot{\alpha}_g^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{V^2}{L_g^2} & -2\frac{V}{L_g} \end{bmatrix} \begin{bmatrix} \alpha_g \\ \alpha_g^* \end{bmatrix} + \begin{bmatrix} \sigma_{\alpha_g}\sqrt{\frac{3V}{L_g}} \\ (1-2\sqrt{3})\,\sigma_{\alpha_g}\sqrt{\left(\frac{V}{L_g}\right)^3} \end{bmatrix} w_3 \tag{7.108}$$

the symmetric aircraft equations of motion can be extended:

$$\begin{bmatrix} \dot{X} \\ \dot{X}_g \end{bmatrix} = \begin{bmatrix} A_{X\dot{X}} & A_{X_g\dot{X}} \\ A_{X\dot{X}_g} & A_{X_g\dot{X}_g} \end{bmatrix} \begin{bmatrix} X \\ X_g \end{bmatrix} + \begin{bmatrix} B_{\delta_e\dot{X}} & B_{N\dot{X}_g} \end{bmatrix} \begin{bmatrix} \delta_e \\ N \end{bmatrix}$$

with N being white noise.

Symmetric equations of motion

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{q\bar{c}}{V} \\ \dot{\hat{u}}_g \\ \dot{\alpha}_g \\ \dot{\alpha}_g^* \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 & x_{u_g} & x_{\alpha_g} & 0 \\ z_u & z_\alpha & z_\theta & z_q & z_{u_g} - z_{\dot{u}_g}\frac{V}{L_g}\frac{\bar{c}}{V} & z_{\alpha_g} & z_{\dot{\alpha}_g}\frac{\bar{c}}{V} \\ 0 & 0 & 0 & \frac{V}{\bar{c}} & 0 & 0 & 0 \\ m_u & m_\alpha & m_\theta & m_q & m_{u_g} - m_{\dot{u}_g}\frac{V}{L_g}\frac{\bar{c}}{V} & m_{\alpha_g} & m_{\dot{\alpha}_g}\frac{\bar{c}}{V} \\ 0 & 0 & 0 & 0 & -\frac{V}{L_g} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{V^2}{L_g^2} & -2\frac{V}{L_g} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \\ \hat{u}_g \\ \alpha_g \\ \alpha_g^* \end{bmatrix} +$$

$$+ \begin{bmatrix} x_{\delta_e} & 0 & 0 \\ z_{\delta_e} & z_{\dot{u}_g}\frac{\bar{c}}{V}\sigma_{\hat{u}_g}\sqrt{\frac{2V}{L_g}} & z_{\dot{\alpha}_g}\frac{\bar{c}}{V}\sigma_{\alpha_g}\sqrt{\frac{3V}{L_g}} \\ 0 & 0 & 0 \\ m_{\delta_e} & m_{\dot{u}_g}\frac{\bar{c}}{V}\sigma_{\hat{u}_g}\sqrt{\frac{2V}{L_g}} & m_{\dot{\alpha}_g}\frac{\bar{c}}{V}\sigma_{\alpha_g}\sqrt{\frac{3V}{L_g}} \\ 0 & \sigma_{\hat{u}_g}\sqrt{\frac{2V}{L_g}} & 0 \\ 0 & 0 & \sigma_{\alpha_g}\sqrt{\frac{3V}{L_g}} \\ 0 & 0 & (1-2\sqrt{3})\,\sigma_{\alpha_g}\sqrt{\left(\frac{V}{L_g}\right)^3} \end{bmatrix} \begin{bmatrix} \delta_e \\ w_1 \\ w_3 \end{bmatrix} \tag{7.109}$$

Examples

Responses of the Cessna Ce-500 'Citation' to turbulence

Investigate the influence of:

turb. velocity component: longitudinal u_g vs. vertical gust w_g

scale length: $L_g = 150$ m vs. $L_g = 1500$ m

flight condition: 'landing' $V = 59.9$ m/s vs. 'cruise' $V = 181.9$ m/s

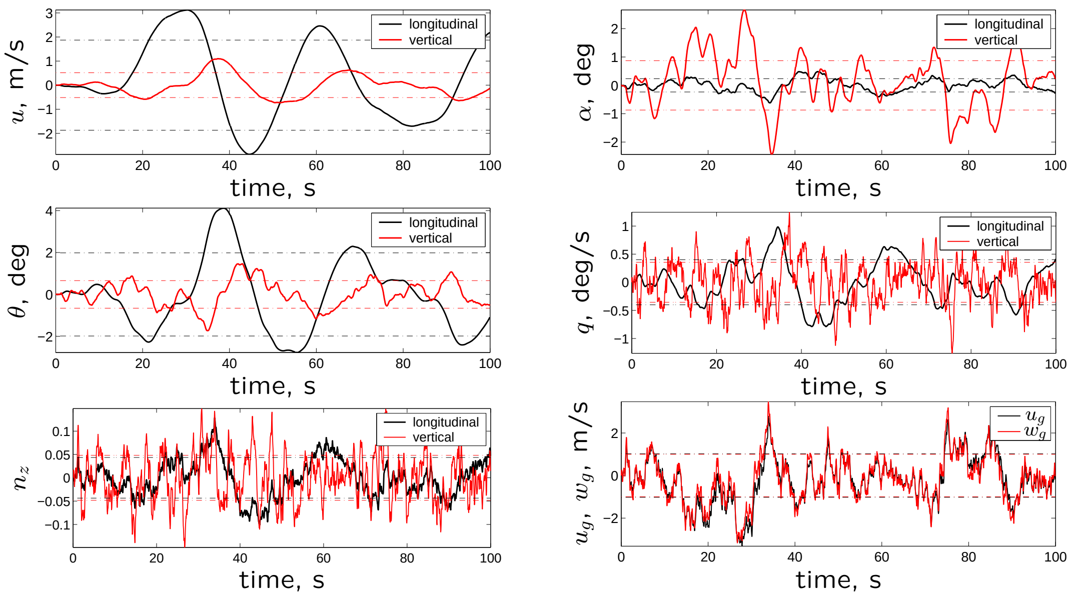
autopilot: no controller vs. pitch attitude hold

unsteady gust derivatives: $C_{Z_{\dot{\alpha}g}} = 0$ and $C_{Z_{\dot{\alpha}g}} = 0$

position in the a/c: front, c.g., and rear

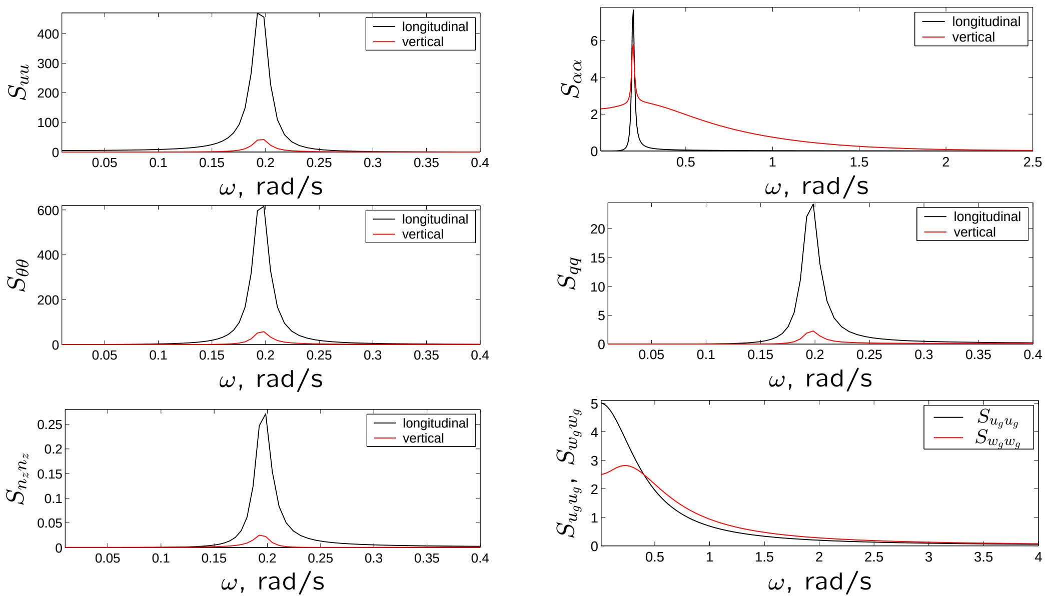
Examples

Longitudinal versus vertical turbulence



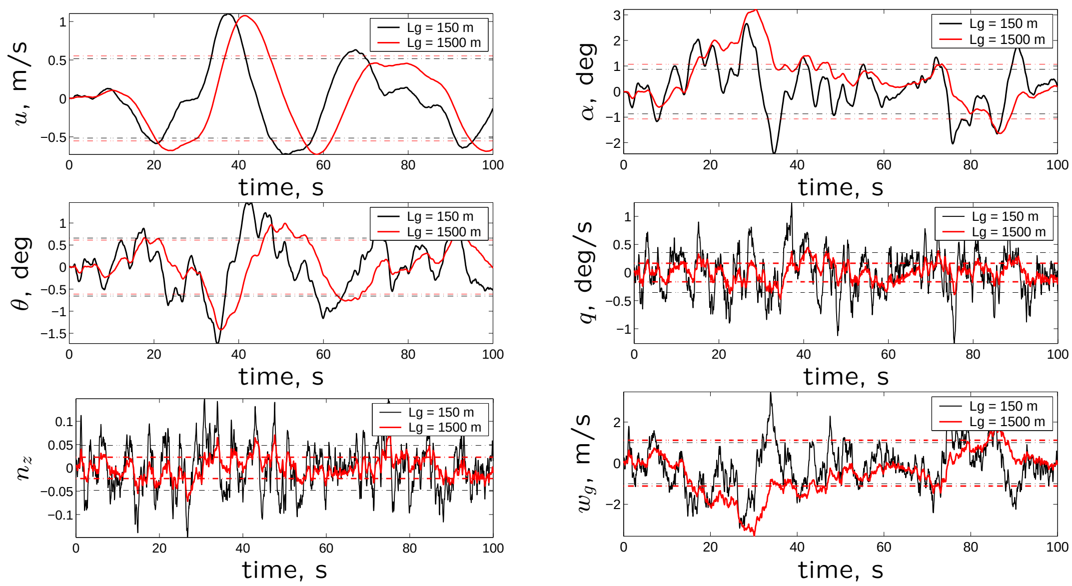
Examples

Longitudinal versus vertical turbulence (PSD)



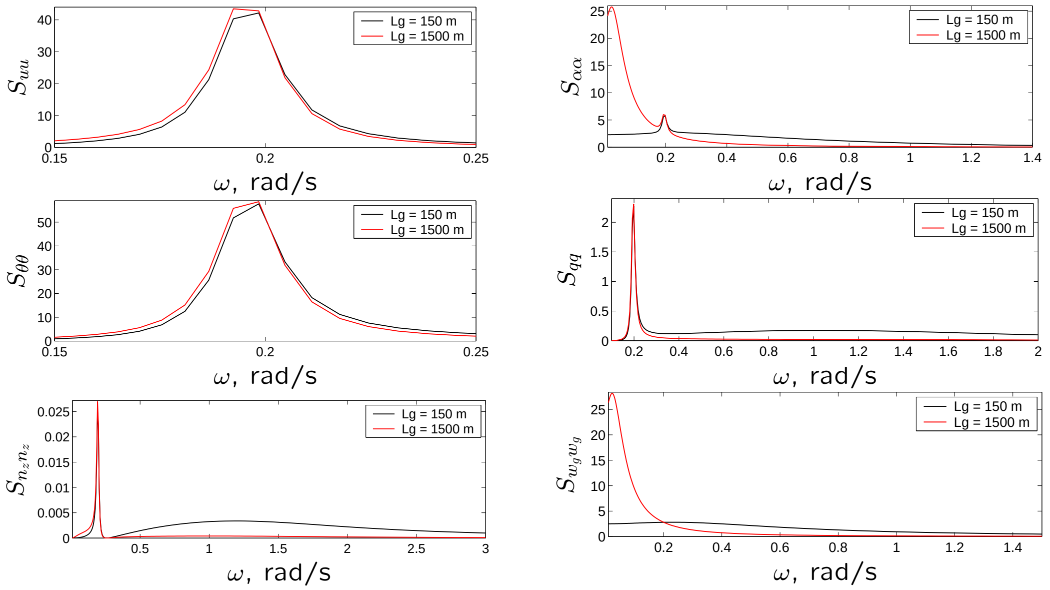
Examples

$L_g = 150$ m versus $L_g = 1500$ m with w_g



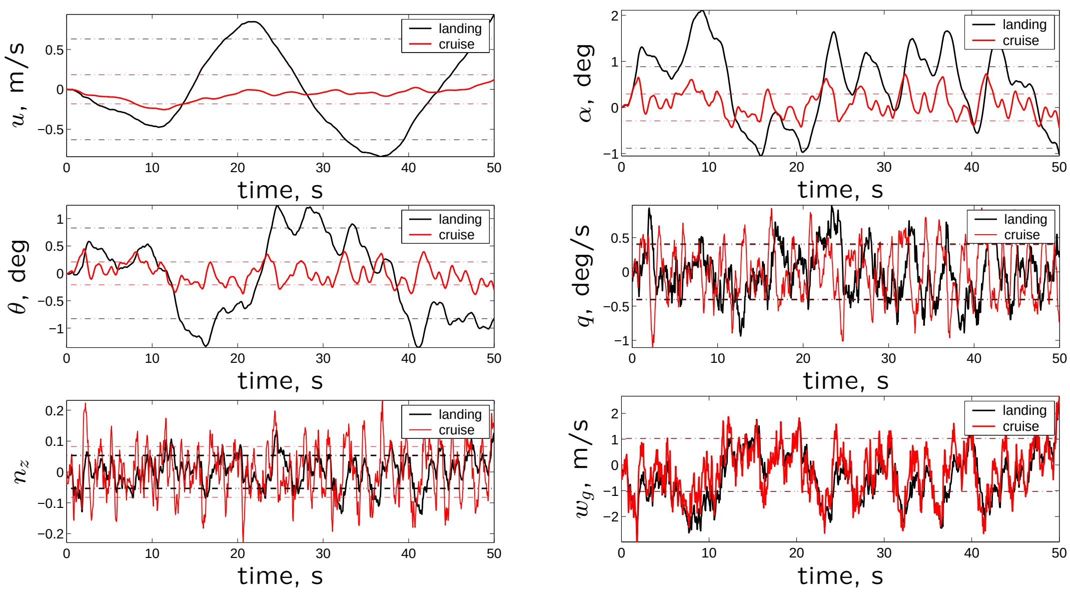
Examples

$L_g = 150$ m versus $L_g = 1500$ m with w_g (PSD)



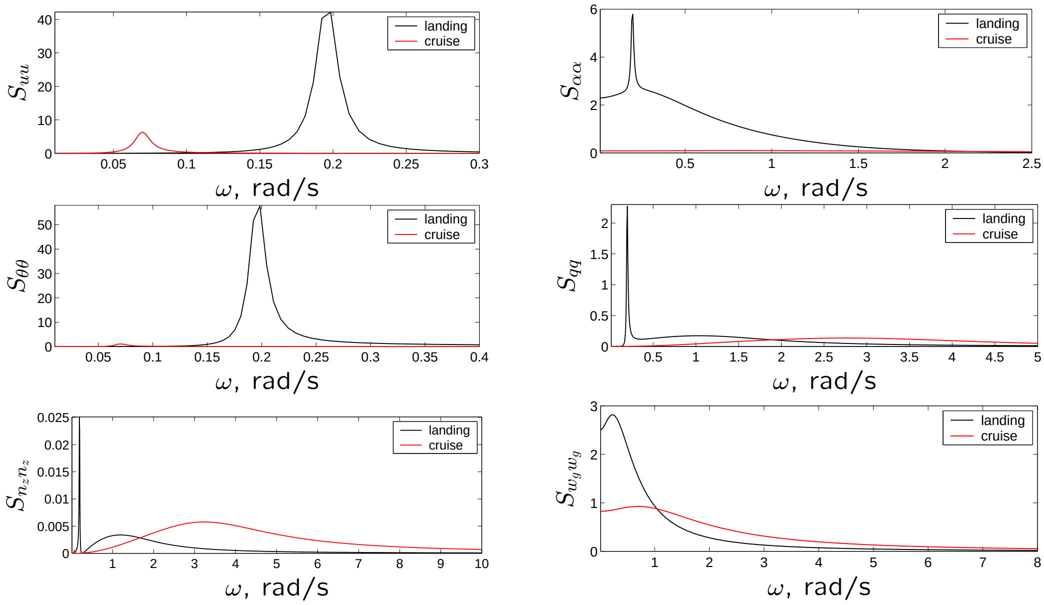
Examples

'Landing' ($V = 59.9$ m/s) vs. 'cruise' ($V = 181.9$ m/s) with w_g



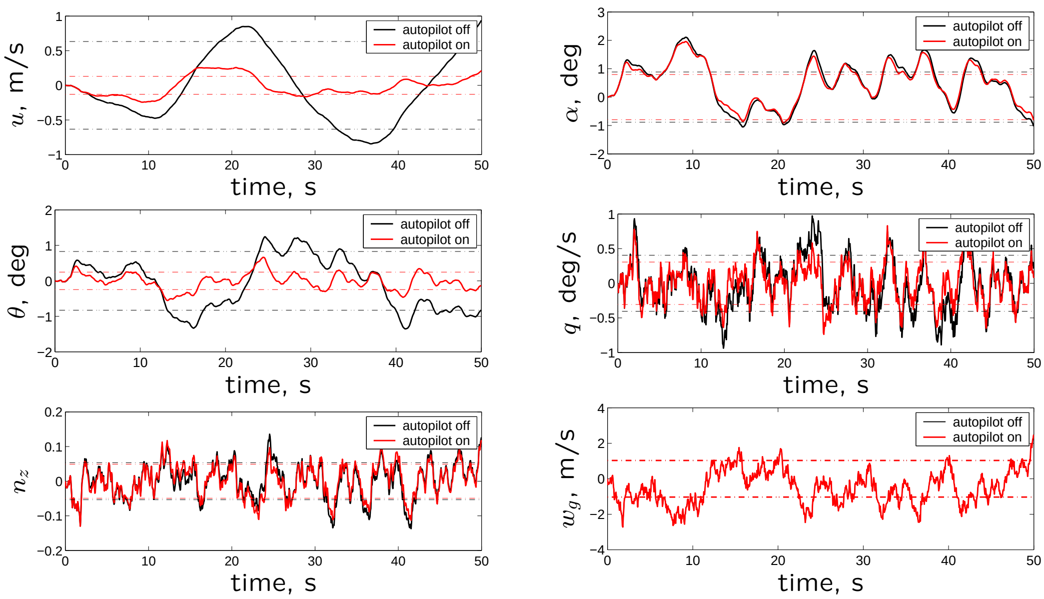
Examples

'Landing' (V = 59.9 m/s) vs. 'cruise' (V = 181.9 m/s) with w_g (PSD)



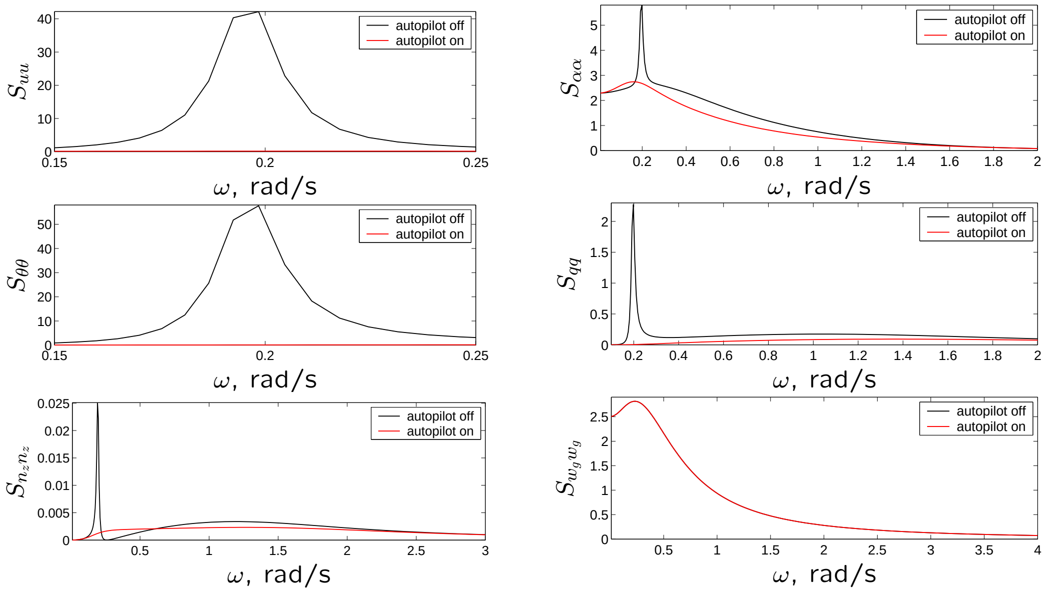
Examples

Pitch attitude hold controller vs. no controller with w_g



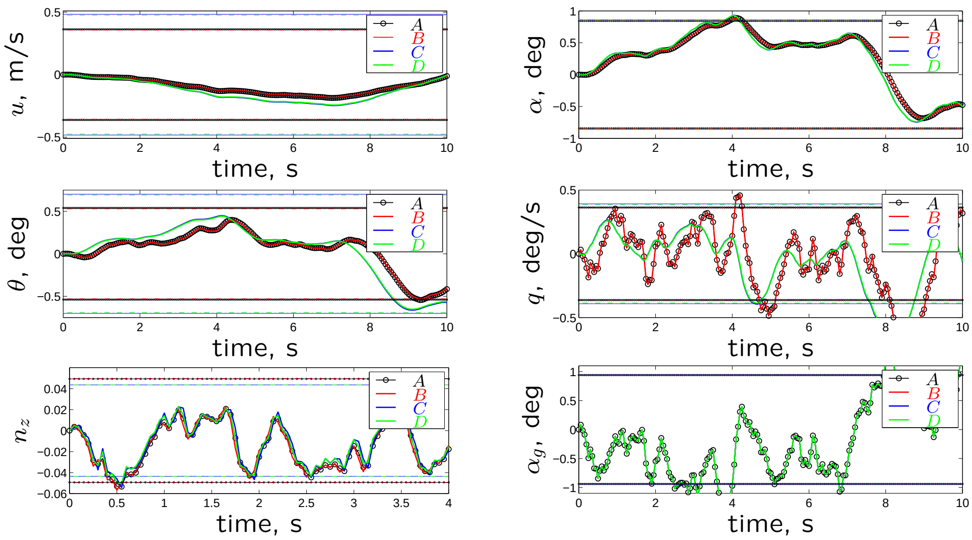
Examples

Pitch attitude hold controller vs. no controller (PSD)



Examples

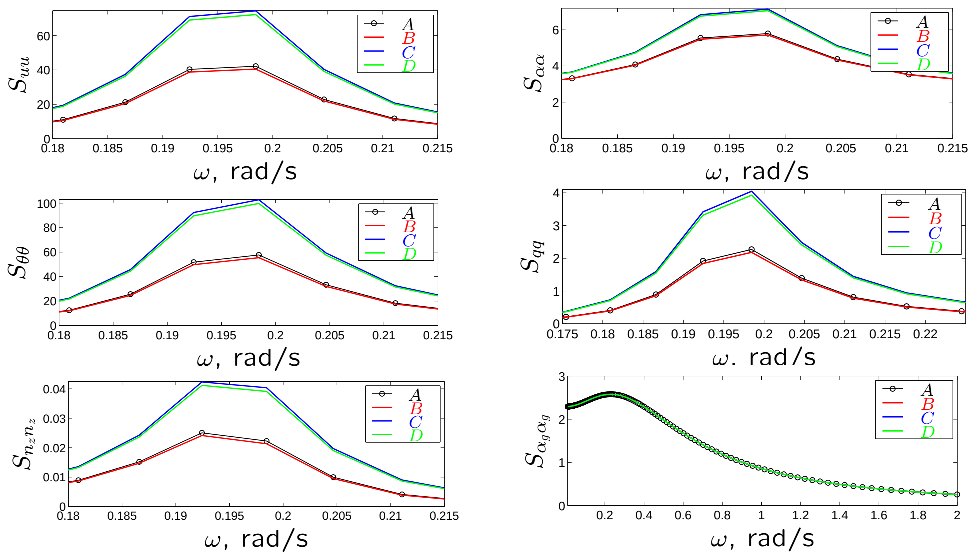
Unsteady gust derivatives with w_g



A: $C_{Z_{\dot\alpha_g}}$ and $C_{m_{\dot\alpha_g}}$ non-zero, B: $C_{Z_{\dot\alpha_g}} = 0$, C: $C_{m_{\dot\alpha_g}} = 0$, D: $C_{Z_{\dot\alpha_g}} = C_{m_{\dot\alpha_g}} = 0$.

Examples

Unsteady gust derivatives with w_g (PSD)



A: $C_{Z_{\dot\alpha_g}}$ and $C_{m_{\dot\alpha_g}}$ non-zero, B: $C_{Z_{\dot\alpha_g}} = 0$, C: $C_{m_{\dot\alpha_g}} = 0$, D: $C_{Z_{\dot\alpha_g}} = C_{m_{\dot\alpha_g}} = 0$.

Examples

Experience of n_z also depends on the position w.r.t. the c.g.