Scalar Stochastic Processes

ae4304 chapter # 2, edition 2017-2018

prof. dr ir Max Mulder

m.mulder@tudelft.nl

$\boxed{\text{Scalar Stochastic Processes}}$

For this lecture the following material was used:

- Chapter 2 of Lecture notes *Aircraft Responses to Atmospheric Turbulence*

- Chapter 8 from Papoulis, A. (1991). *Probability, Random Variables, and Stochastic Processes* (3rd edition). McGraw-Hill, Inc.

Deterministic and stochastic signals

The probability distribution function; The probability density function

Moments of the probability density function

The normal (Gaussian) distribution

The joint probability distribution and density functions

Moments of the joint probability density function

The product- covariance- and correlation functions

Uncorrelated, orthogonal and independent stochastic variables

Ergodicity

Miscellaneous

Deterministic and stochastic signals

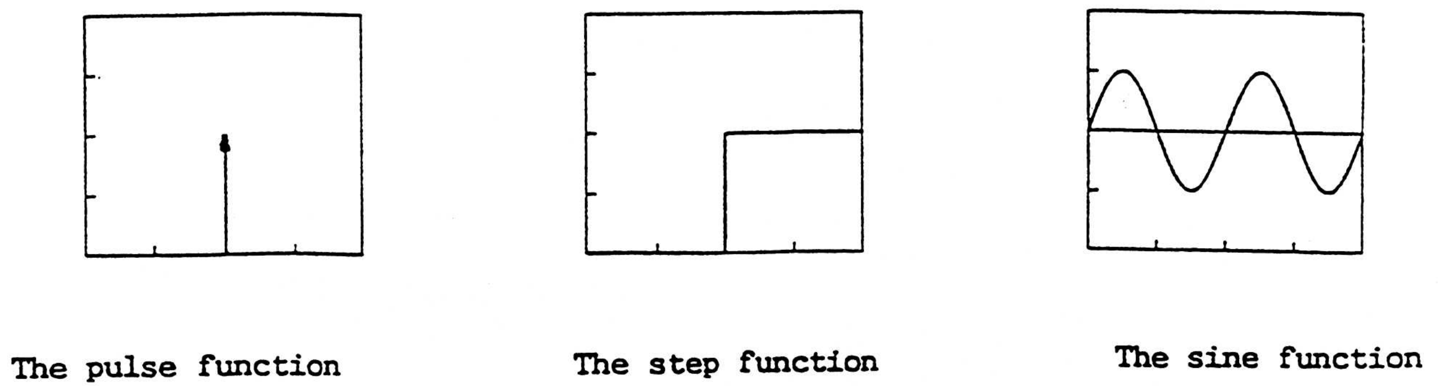


Figure 2.1: Some well known deterministic input signals used for system analysis.

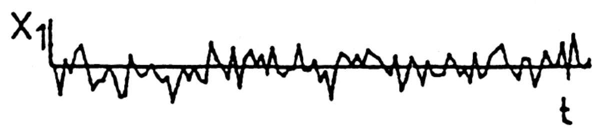


Figure 2.2: An example of a stochastic input signal.

Realization and ensemble

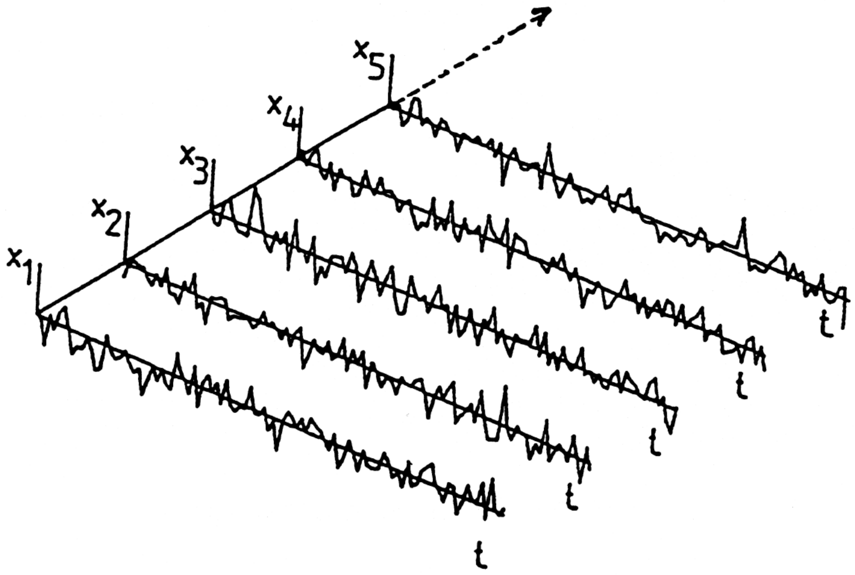
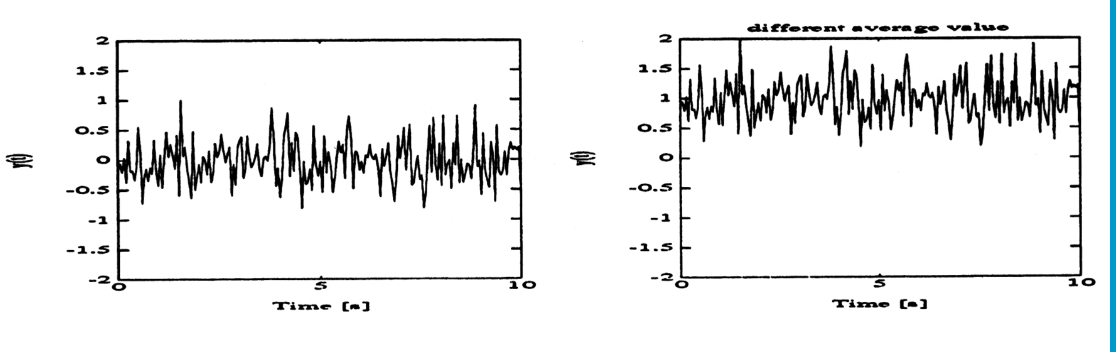


Figure 2.3: The concept of the ensemble of a stochastic process.

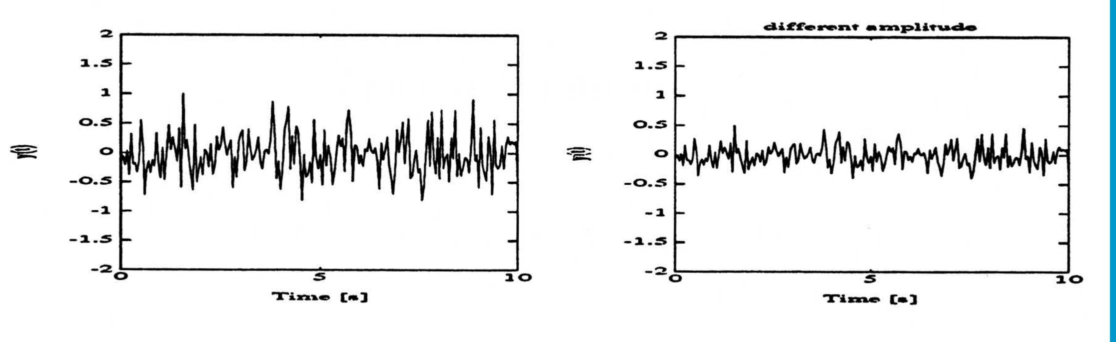
A single measurement is a *realization*. The set of all possible measurements is the *ensemble*.

Signals are characterized by their means (average):



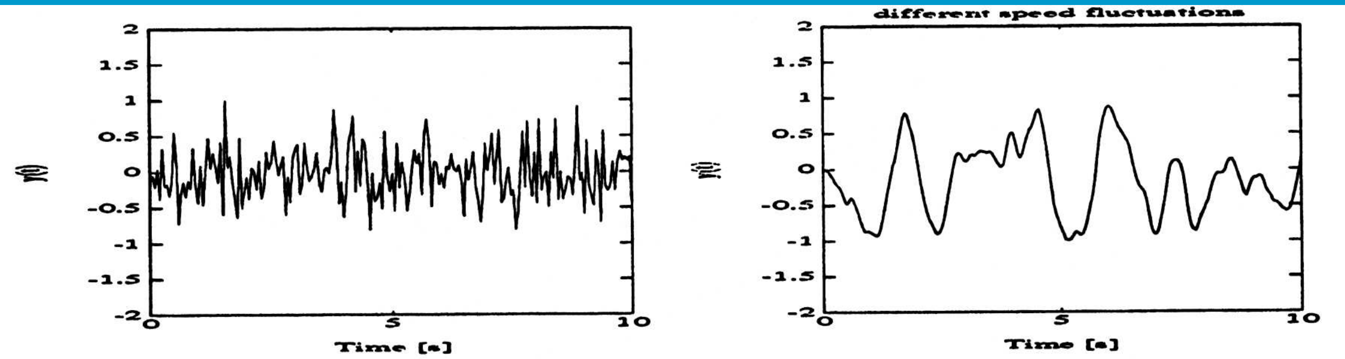
(a) Different average value

and their standard deviations (amplitude):



(b) Different amplitude

But the means and standard deviations do not tell everything:



(c) Different speed of fluctuations

The signals above have equal means and standard deviations, but ‘look’ completely different: they ‘appear’ “low-frequency” (right) and “high-frequency” (left).

$\boxed{\text{Stationary stochastic processes}}$

A stochastic process is **stationary** if the statistical parameters (like the mean and variance etc.) do not vary in time.

$\boxed{\text{The probability distribution function}}$

Stochastic functions can not be described exactly. Rather, they are described in **probabilistic** terms.

The **probability distribution function** $F_{\bar{x}}(x)$ of a stochastic variable (SV) $\bar{x}$ equals:

$$\boxed{F_{\bar{x}}(x) = \mathsf{Pr}\left\{\bar{x} \leq x\right\}}, \tag{2.1}$$

i.e., $F_{\bar{x}}(x)$ describes the probability that the SV $\bar{x}$ takes on smaller (or equal) values than a certain (deterministic) value x.

$$\begin{array}{lcll} F_{\bar{x}}(-\infty) & = & 0 & \text{‘impossibility’} \\ F_{\bar{x}}(+\infty) & = & 1 & \text{‘certainty’} \end{array}$$

And because $F_{\bar{x}}(a) \leq F_{\bar{x}}(b)$ when $a \leq b$, $F_{\bar{x}}(x)$ is a monotonically increasing function.

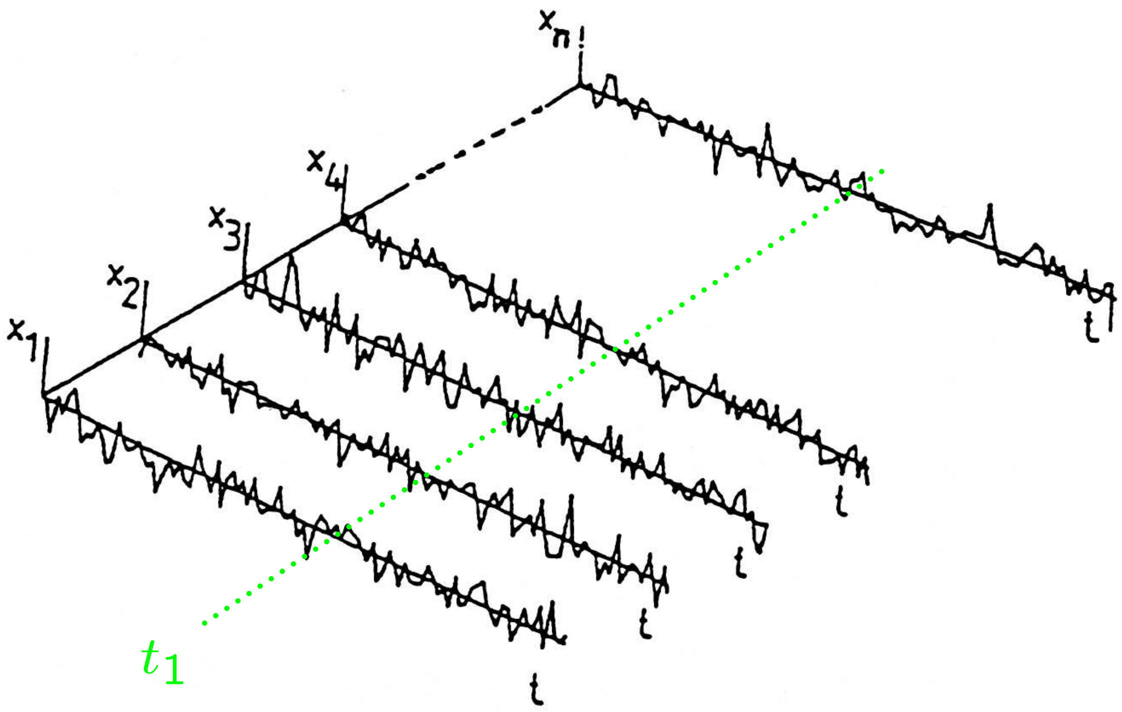


Figure 2.5: Part of the ensemble of a stochastic process.

When n is the total number of observations, and i is the number of observations where $\bar{x} \leq x$ then:

$$F_{\bar{x}}(x) = \lim_{n \to \infty} \frac{i}{n}$$

The probability density function (p.d.f.)

The **probability density function** (p.d.f.) $f_{\bar{x}}(x)$ of a stochastic variable (SV) $\bar{x}$ equals:

$$\boxed{\begin{aligned} f_{\bar{x}}(x) &= \frac{\mathrm{d}F_{\bar{x}}(x)}{\mathrm{d}x} \\ &= \lim_{\triangle x \to 0} \frac{\Pr\{x < \bar{x} \leq x + \triangle x\}}{\triangle x} \end{aligned} \qquad (2.5)}$$

Then $\int_a^b f_{\bar{x}}(x)\mathrm{d}x = F_{\bar{x}}(b) - F_{\bar{x}}(a)$, so the area under the p.d.f. equals '1' when the limits a, b are stretched towards $\pm\infty$. Also, because $F_{\bar{x}}(x)$ is a monotonically increasing function, $f_{\bar{x}}(x) \geq 0 \;\; \forall\; x$.

Moments of the p.d.f.

In most cases it is difficult to determine the functions $f_{\bar{x}}(x)$ and $F_{\bar{x}}(x)$ of a stochastic process. Instead, we concentrate on alternative quantities, such as the 'most likely' or 'expected value' of $\bar{x}$, $\bar{x}^2$, etc.

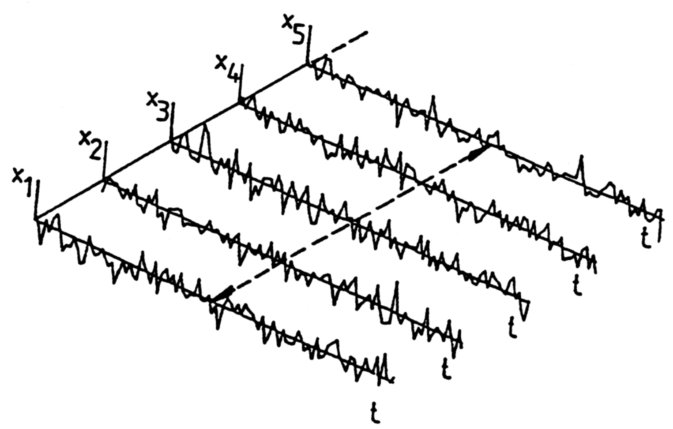


Figure 2.6: Expected value as an ensemble averaging.

The expected value is denoted by the **Expectation operator** $\mathsf{E}\{\cdot\}$. $\mathsf{E}\{\cdot\}$ is a **linear operator**, i.e.:

$$\mathsf{E}\{ag(\bar{x}) + bh(\bar{x})\} = a\mathsf{E}\{g(\bar{x})\} + b\mathsf{E}\{h(\bar{x})\},$$

for any constants a, b and functions $g(\bar{x})$, $h(\bar{x})$ of a SV $\bar{x}$.

Now, the i^{th} **moment** of the p.d.f. is defined as:

$$\boxed{m_i = \mathsf{E}\left\{\bar{x}^i\right\} = \int\limits_{-\infty}^{\infty} x^i f_{\bar{x}}(x)\mathrm{d}x} \tag{2.7}$$

The first order moment of the p.d.f. (m_1) is known as the **mean** (or, the 'average') of the SV $\bar{x}$: $\mathsf{E}\{\bar{x}\} = \mu_{\bar{x}}$. (compare with 'center-of-gravity')

The second order moment of the p.d.f. (m_2) is known as the **mean-square** of the SV $\bar{x}$: $\mathsf{E}\left\{\bar{x}^2\right\}$. (compare with 'moment of inertia')

The i^{th} **central moment** of the p.d.f. is defined as:

$$\boxed{m_i' = \mathsf{E}\left\{(\bar{x} - \mu_{\bar{x}})^i\right\} = \int\limits_{-\infty}^{\infty} (x - \mu_{\bar{x}})^i f_{\bar{x}}(x)\mathrm{d}x}, \tag{2.8}$$

i.e., we 'correct' for the mean, that is, we consider the variations 'around' the mean.

The first order central moment m_1' is zero.

The second order central moment m_2' is known as the **variance** $\sigma_{\bar{x}}^2$ of the SV $\bar{x}$. Its square root is known as the **standard deviation** $\sigma_{\bar{x}}$.

moments	$m_0 = \int_{-\infty}^{+\infty} f_{\bar{x}}(x)\,\mathrm{d}x = 1$ $m_1 = \int_{-\infty}^{+\infty} x\ f_{\bar{x}}(x)\,\mathrm{d}x = \mu_{\bar{x}}$ (average) $m_2 = \int_{-\infty}^{+\infty} x^2 f_{\bar{x}}(x)\,\mathrm{d}x = \sigma_{\bar{x}}^2 + \mu_{\bar{x}}^2$
central moments	$m'_0 = \int_{-\infty}^{+\infty} f_{\bar{x}}(x)\,\mathrm{d}x = 1$ $m'_1 = \int_{-\infty}^{+\infty} (x - \mu_{\bar{x}})\ f_{\bar{x}}(x)\,\mathrm{d}x = 0$ $m'_2 = \int_{-\infty}^{+\infty} (x - \mu_{\bar{x}})^2 f_{\bar{x}}(x)\,\mathrm{d}x = \sigma_{\bar{x}}^2$ (variance)

The normal (Gaussian) distribution

A very common (and useful) p.d.f. is the **normal distribution** (ND), also known as **Gaussian distribution**:

$$\boxed{f_{\bar{x}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}} \tag{2.9}$$

It can be shown that $\mu_{\bar{x}}=\mu$ and $\sigma_{\bar{x}}=\sigma$. When μ and σ are known, the p.d.f. is completely characterized.

According to the Central Limit Theorem (more about that later), the p.d.f. of a stochastic process originating from the *sum of a large number of random variables* tends towards the ND.

In the following, it is assumed that all SVs are normally distributed.

The classical 'bell' curve:

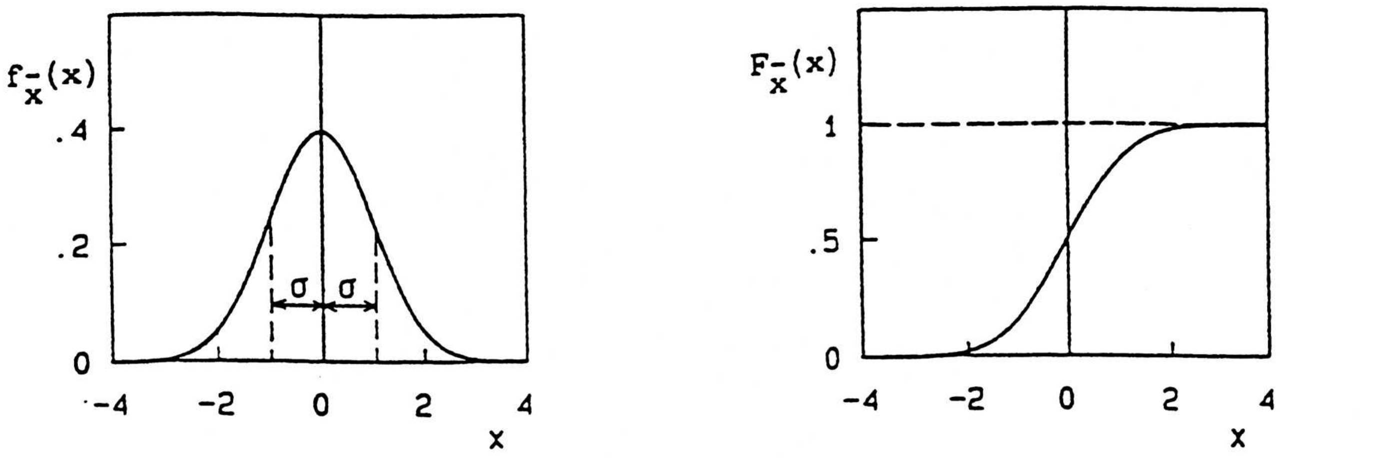


Figure 2.7: The Gaussian probability density- and distribution functions (plotted here with $\mu_{\bar{x}} = 0,\ \sigma_{\bar{x}} = 1$).

Note: $\int\limits_{-\sigma}^{+\sigma} f_{\bar{x}}(x)\mathrm{d}x = 0.6827$; $\int\limits_{-2\sigma}^{+2\sigma} f_{\bar{x}}(x)\mathrm{d}x = \mathbf{0.9545}$; and $\int\limits_{-3\sigma}^{+3\sigma} f_{\bar{x}}(x)\mathrm{d}x = 0.9973$.

So, 95% of the area enclosed by the bell curve is within the $\pm 2\sigma$ bounds.

$\boxed{\text{The joint probability distribution and density functions}}$

The two-dimensional **joint probability distribution function** $F_{\bar{x}\bar{y}}(x, y)$ of the SVs $\bar{x}$ and $\bar{y}$ is defined as:

$$\boxed{F_{\bar{x}\bar{y}}(x, y) = \Pr\{(\bar{x} \leq x) \wedge (\bar{y} \leq y)\}} \tag{2.10}$$

This function has the following characteristics:

$$\begin{aligned}
F_{\bar{x}\bar{y}}(-\infty, -\infty) &= 0 && \text{“impossibility”} \\
F_{\bar{x}\bar{y}}(+\infty, +\infty) &= 1 && \text{“certainty”} \\
F_{\bar{x}\bar{y}}(x, +\infty) &= F_{\bar{x}}(x) \\
F_{\bar{x}\bar{y}}(+\infty, y) &= F_{\bar{y}}(y)
\end{aligned} \tag{2.11}$$

Assuming that $F_{\bar{x}\bar{y}}(x, y)$ is twice differentiable (as is the ND) we get for the **joint probability density function** $f_{\bar{x}\bar{y}}(x, y)$:

$$\boxed{\begin{aligned} f_{\bar{x}\bar{y}}(x, y) &= \frac{\partial^2 F_{\bar{x}\bar{y}}(x, y)}{\partial x \partial y} \\ &= \lim_{\triangle x \to 0} \lim_{\triangle y \to 0} \frac{\text{Pr}\{(x < \bar{x} \leq x + \triangle x) \wedge (y < \bar{y} \leq y + \triangle y)\}}{\triangle x \triangle y} \end{aligned}} \quad (2.12)$$

So, $f_{\bar{x}\bar{y}}(x, y)\mathrm{d}x\mathrm{d}y = \text{Pr}\{(x < \bar{x} \leq x + \triangle x) \wedge (y < \bar{y} \leq y + \triangle y)\}$. This product is a *chance* (see Figure 2.8).

Because $F_{\bar{x}\bar{y}}(x, y)$ is a monotonically increasing function, $f_{\bar{x}\bar{y}}(x, y) \geq 0 \ \forall \ x, y$.

Some other characteristics:

$$\int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{\overline{x}\overline{y}}(x, y) \mathrm{d}y \mathrm{d}x = F_{\overline{x}\overline{y}}(x_1, y_1)$$

Then:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\overline{x}\overline{y}}(x, y) \mathrm{d}y \mathrm{d}x = F_{\overline{x}\overline{y}}(\infty, \infty) = 1,$$

so the joint p.d.f. $f_{\overline{x}\overline{y}}(x, y)$ may be represented as a three-dimensional surface, for which the volume underneath equals 1 (see Figure 2.8).

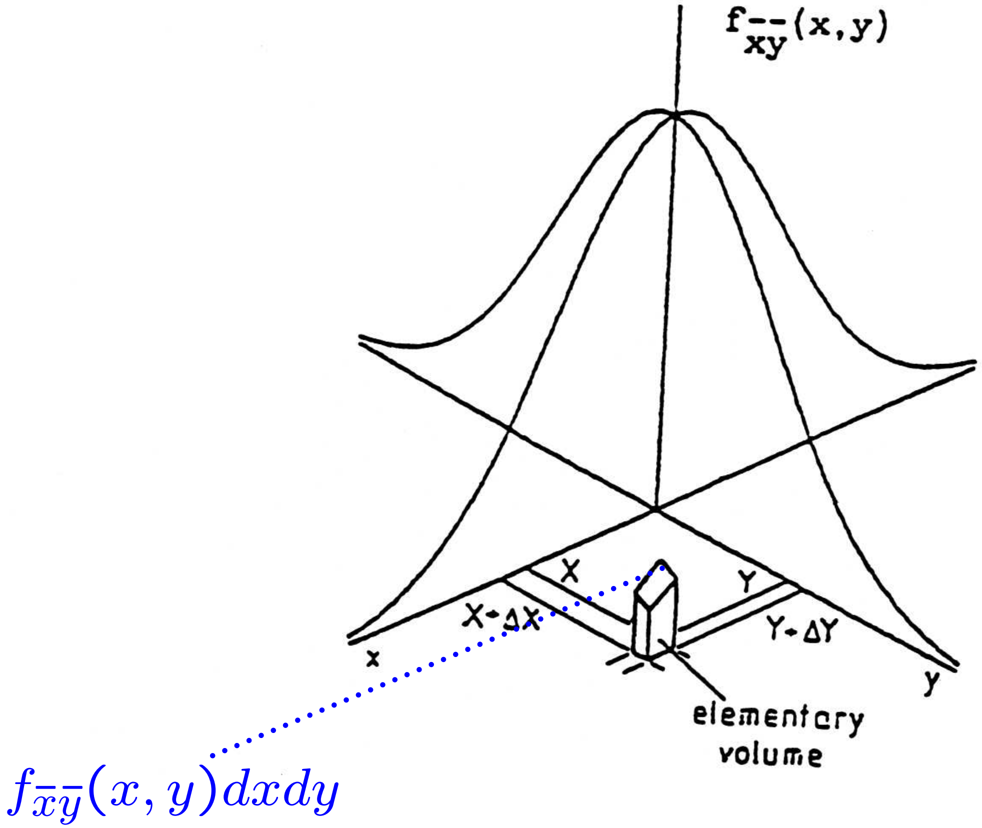


Figure 2.8: Graphical representation of the joint probability density function $f_{\bar{x}\bar{y}}(x, y)$.

Now:

$$F_{\overline{x}}(x) = \int\limits_{-\infty}^{x} \underbrace{f_{\overline{x}}(u)}_{*}\,\mathrm{d}u = F_{\overline{x}\overline{y}}(x,\infty) = \int\limits_{-\infty}^{x} \underbrace{\left[\int\limits_{-\infty}^{\infty} f_{\overline{x}\overline{y}}(u,v)\mathrm{d}v\right]}_{*}\mathrm{d}u,$$

so:

$$f_{\overline{x}}(x) = \int\limits_{-\infty}^{\infty} f_{\overline{x}\overline{y}}(x,y)\mathrm{d}y$$

and similarly:

$$f_{\overline{y}}(y) = \int\limits_{-\infty}^{\infty} f_{\overline{x}\overline{y}}(x,y)\mathrm{d}x$$

Moments of the joint p.d.f.

The **joint moment** m_{ij} of the two-dimensional p.d.f. is defined as:

$$\boxed{m_{ij} = \mathsf{E}\left\{\bar{x}^i \bar{y}^j\right\} = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x^i y^j f_{\bar{x}\bar{y}}(x, y)\mathrm{d}x\mathrm{d}y} \tag{2.15}$$

The *sum* $i + j$ is now defined as the *order* of the joint moment.

Then, $m_{10} = \mu_{\bar{x}}$ and $m_{01} = \mu_{\bar{y}}$, that is, the first order joint moments equal the averages of SVs $\bar{x}$ and $\bar{y}$.

The second order joint moment m_{11} is known as the **average product** $R_{\bar{x}\bar{y}}$:

$$R_{\bar{x}\bar{y}} = \mathsf{E}\left\{\bar{x}\bar{y}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{\bar{x}\bar{y}}(x, y)\mathrm{d}x\mathrm{d}y$$

Once again, we may 'correct' for the average values $\mu_{\bar{x}}$ and $\mu_{\bar{y}}$ and introduce the **joint central moments** $m_{ij}^{'}$:

$$\boxed{m_{ij}^{'} = \mathsf{E}\left\{(\bar{x} - \mu_{\bar{x}})^{i}(\bar{y} - \mu_{\bar{y}})^{j}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_{\bar{x}})^{i}(y - \mu_{\bar{y}})^{j} f_{\bar{x}\bar{y}}(x, y)\mathrm{d}x\mathrm{d}y} \quad (2.16)$$

The second order joint central moment $m_{11}^{'}$ is referred to as the **covariance** $C_{\bar{x}\bar{y}}$:

$$C_{\bar{x}\bar{y}} = \mathsf{E}\left\{(\bar{x} - \mu_{\bar{x}})(\bar{y} - \mu_{\bar{y}})\right\}$$

The covariance can be interpreted as a 'joint variance'. It is easy to show that $C_{\bar{x}\bar{y}} = R_{\bar{x}\bar{y}} - \mu_{\bar{x}}\mu_{\bar{y}}$ (Eq. (2.18)).

The covariance can be *normalized* by the product of the standard deviations of both SVs, yielding the **correlation** $K_{\bar{x}\bar{y}}$:

$$\boxed{K_{\bar{x}\bar{y}} = \frac{C_{\bar{x}\bar{y}}}{\sigma_{\bar{x}}\sigma_{\bar{y}}} = \mathsf{E}\left\{\frac{(\bar{x}-\mu_{\bar{x}})}{\sigma_{\bar{x}}} \cdot \frac{(\bar{y}-\mu_{\bar{y}})}{\sigma_{\bar{y}}}\right\}} \tag{2.19}$$

The SVs $\bar{x}$ and $\bar{y}$ are said to be **uncorrelated** if $K_{\bar{x}\bar{y}}$=0; they are fully **correlated** if $|K_{\bar{x}\bar{y}}| = 1$ (then $\bar{x}$=$\bar{y}$; when $\bar{x}$=-$\bar{y}$ then $K_{\bar{x}\bar{y}}$=-1).

Note that when $\bar{x}$ and $\bar{y}$ are uncorrelated, they have 'nothing in common'. That is, suppose you *know* that $\bar{x}$ will change with say $\triangle x$, then you can still make no prediction at all what will happen with $\bar{y}$.

(Note: make Example #2.1 at home.)

joint moments

$$m_{00} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = 1$$

$$m_{10} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \mu_{\bar{x}}$$

$$m_{01} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \mu_{\bar{y}}$$

$$m_{11} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = R_{\bar{x}\bar{y}} = C_{\bar{x}\bar{y}} + \mu_{\bar{x}}\mu_{\bar{y}}$$

$$m_{20} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \sigma_{\bar{x}}^2 + \mu_{\bar{x}}^2$$

$$m_{02} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \sigma_{\bar{y}}^2 + \mu_{\bar{y}}^2$$

joint central moments

$$m'_{00} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = 1$$

$$m'_{10} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_{\bar{x}})\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = 0$$

$$m'_{01} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (y-\mu_{\bar{y}})\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = 0$$

$$m'_{11} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_{\bar{x}})(y-\mu_{\bar{y}})\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = C_{\bar{x}\bar{y}}$$

$$m'_{20} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_{\bar{x}})^2\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \sigma_{\bar{x}}^2$$

$$m'_{02} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (y-\mu_{\bar{y}})^2\, f_{\bar{x}\bar{y}}(x,y)\,dx\,dy = \sigma_{\bar{y}}^2$$

Table 2.2: Definition of the moments and central moments of the joint probability density function.

The product- covariance- and correlation *functions*

Now consider the SVs $\bar{x}$ and $\bar{y}$ to be a **function of time**, i.e., they are **realizations** of a stochastic process. It is also possible to assign a probability distribution function and a probability density function to a stochastic process, but then these functions are *also a function of time*!

$$F_{\bar{x}}(x;t) = \mathrm{Pr}\left\{\bar{x}(t) \leq x\right\}, \tag{2.20}$$

$$f_{\bar{x}}(x;t) = \frac{\partial F_{\bar{x}}(x;t)}{\partial x} = \lim_{\triangle x \to 0} \frac{\mathrm{Pr}\left\{x < \bar{x}(t) \leq x + \triangle x\right\}}{\triangle x} \tag{2.21}$$

When considering the **joint distribution of two stochastic processes** $\bar{x}(t)$ and $\bar{y}(t)$, this becomes a function of four variables: x, y, t_1 and t_2:

$$F_{\bar{x}\bar{y}}(x,y;t_1,t_2) = \mathrm{Pr}\left\{(\bar{x}(t_1) \leq x) \wedge (\bar{y}(t_2) \leq y)\right\} \tag{2.22}$$

BUT, when $\bar{x}$ and $\bar{y}$ are assumed **stationary** (and this is the assumption that we make throughout this course), then it is only the *relative time* $\tau = t_2$ - t_1 that plays a role, and we obtain for the joint distribution function:

$$\boxed{F_{\bar{x}\bar{y}}(x,y;\tau) = \mathrm{Pr}\left\{(\bar{x}(t) \leq x) \wedge (\bar{y}(t+\tau) \leq y)\right\}} \tag{2.23}$$

and for the joint p.d.f.:

$$\boxed{f_{\bar{x}\bar{y}}(x,y;\tau) = \frac{\partial^2 F_{\bar{x}\bar{y}}(x,y;\tau)}{\partial x \partial y}} \tag{2.24}$$

The moments and central moments become now a function of τ, they become *(central) moment* <u>*functions*</u>:

moment functions

$$m_{ij}(\tau) \;=\; \mathsf{E}\left\{\bar{x}^i(t)\bar{y}^j(t+\tau)\right\}$$

central moment functions

$$m'_{ij}(\tau) \;=\; \mathsf{E}\left\{(\bar{x}(t)-\mu_{\bar{x}})^i(\bar{y}(t+\tau)-\mu_{\bar{y}})^j\right\}$$

When considering these moments for two different stochastic processes (i.e., $\bar{x}(t)$ and $\bar{y}(t+\tau)$) we refer to them as **cross**-functions.

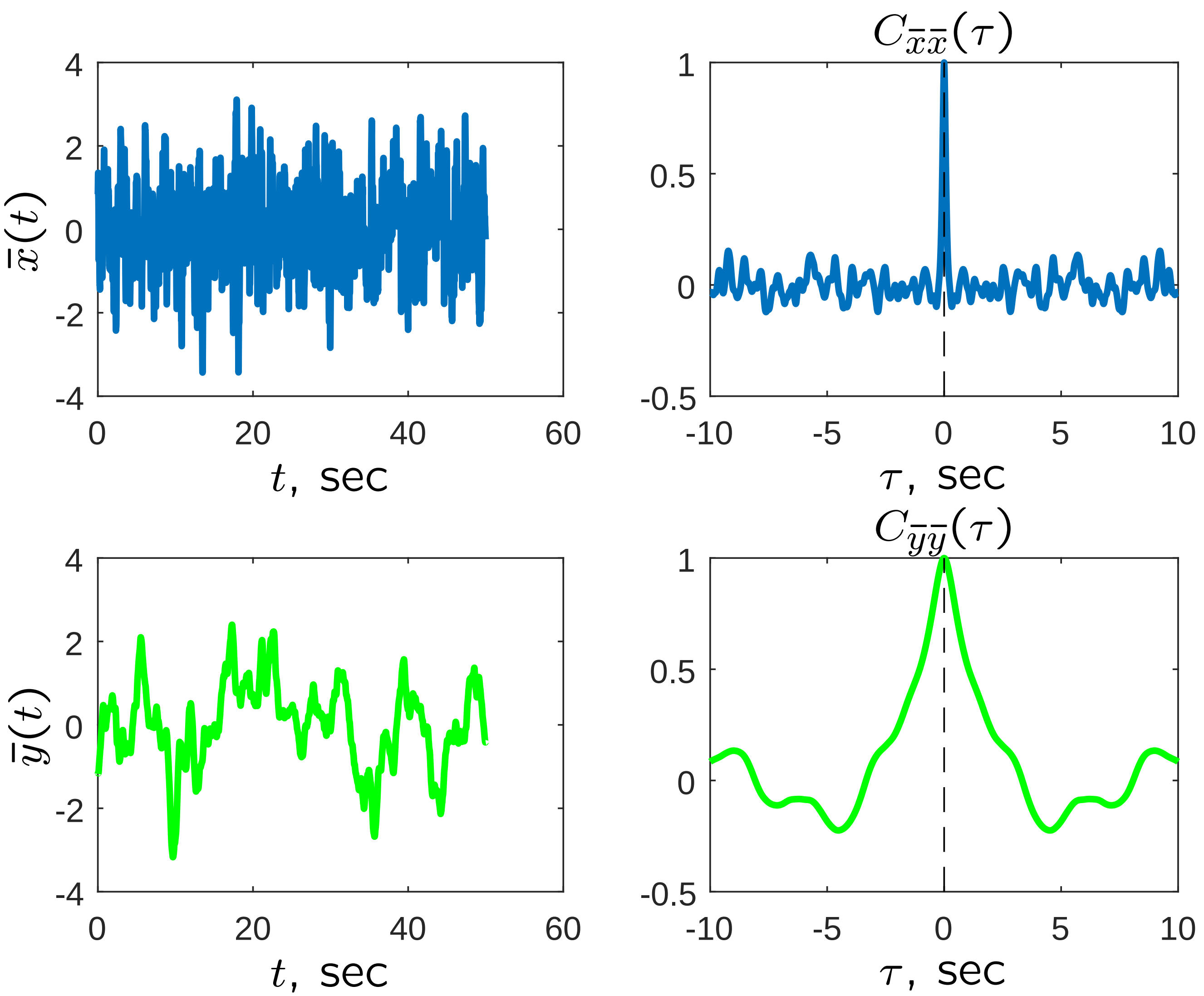
When considering them for the *same* stochastic process (i.e.,$\bar{x}(t)$ and $\bar{x}(t+\tau)$), we refer to them as **auto**-functions.

auto-product function	$R_{\bar{x}\bar{x}}(\tau)$	$=$	$\mathsf{E}\left\{\bar{x}(t)\bar{x}(t+\tau)\right\}$
auto-covariance function	$C_{\bar{x}\bar{x}}(\tau)$	$=$	$\mathsf{E}\left\{(\bar{x}(t)-\mu_{\bar{x}})(\bar{x}(t+\tau)-\mu_{\bar{x}})\right\}$
auto-correlation function	$K_{\bar{x}\bar{x}}(\tau)$	$=$	$C_{\bar{x}\bar{x}}(\tau)/\sigma_{\bar{x}}^2$
cross-product function	$R_{\bar{x}\bar{y}}(\tau)$	$=$	$\mathsf{E}\left\{\bar{x}(t)\bar{y}(t+\tau)\right\}$
cross-covariance function	$C_{\bar{x}\bar{y}}(\tau)$	$=$	$\mathsf{E}\left\{(\bar{x}(t)-\mu_{\bar{x}})(\bar{y}(t+\tau)-\mu_{\bar{y}})\right\}$
cross-correlation function	$K_{\bar{x}\bar{y}}(\tau)$	$=$	$C_{\bar{x}\bar{y}}(\tau)/(\sigma_{\bar{x}}\sigma_{\bar{y}})$

The auto-functions are a measure of the **coherence** between the values of the stochastic process $\bar{x}(t)$ at time t and at time $t + \tau$.

Hence, $\bar{x}(t)$ and $\bar{y}(t)$ may have the same mean and variance, from the $R_{\bar{x}\bar{x}}(\tau)$ and $C_{\bar{x}\bar{x}}(\tau)$ we may learn whether the stochastic process exhibits slow ($\bar{y}(t)$) or fast ($\bar{x}(t)$) fluctuations in time.

See the following figure, which also clearly shows that the auto-functions are <u>even</u> functions.

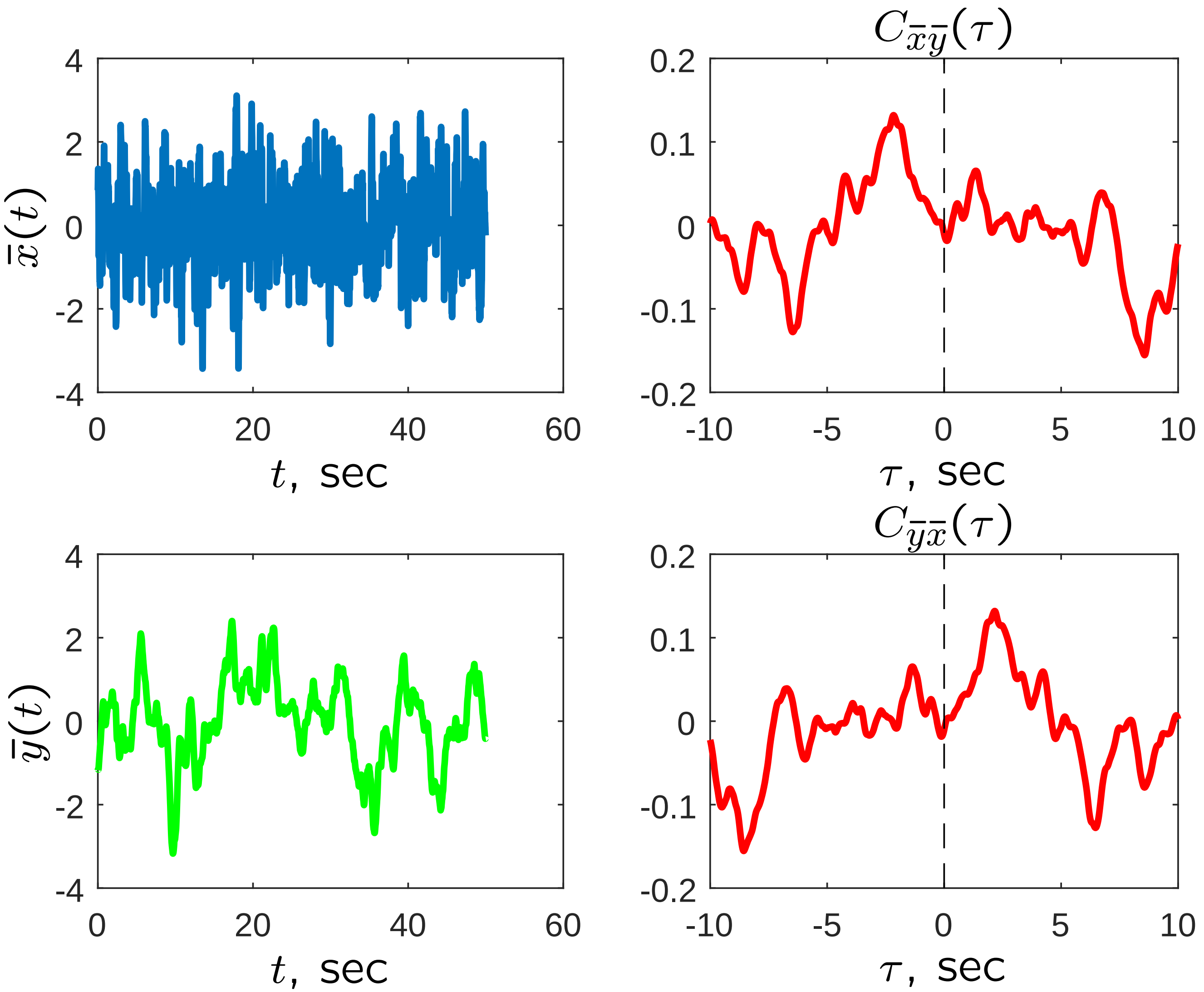


$C_{\bar{x}\bar{x}}(\tau)$
$C_{\bar{y}\bar{y}}(\tau)$
$\bar{x}(t)$
$\bar{y}(t)$
t, sec
τ, sec

The same holds for the cross-functions, which are a measure of the **coherence** between the values of the stochastic processes $\bar{x}$, at time t and $\bar{y}$, at time $t + \tau$.

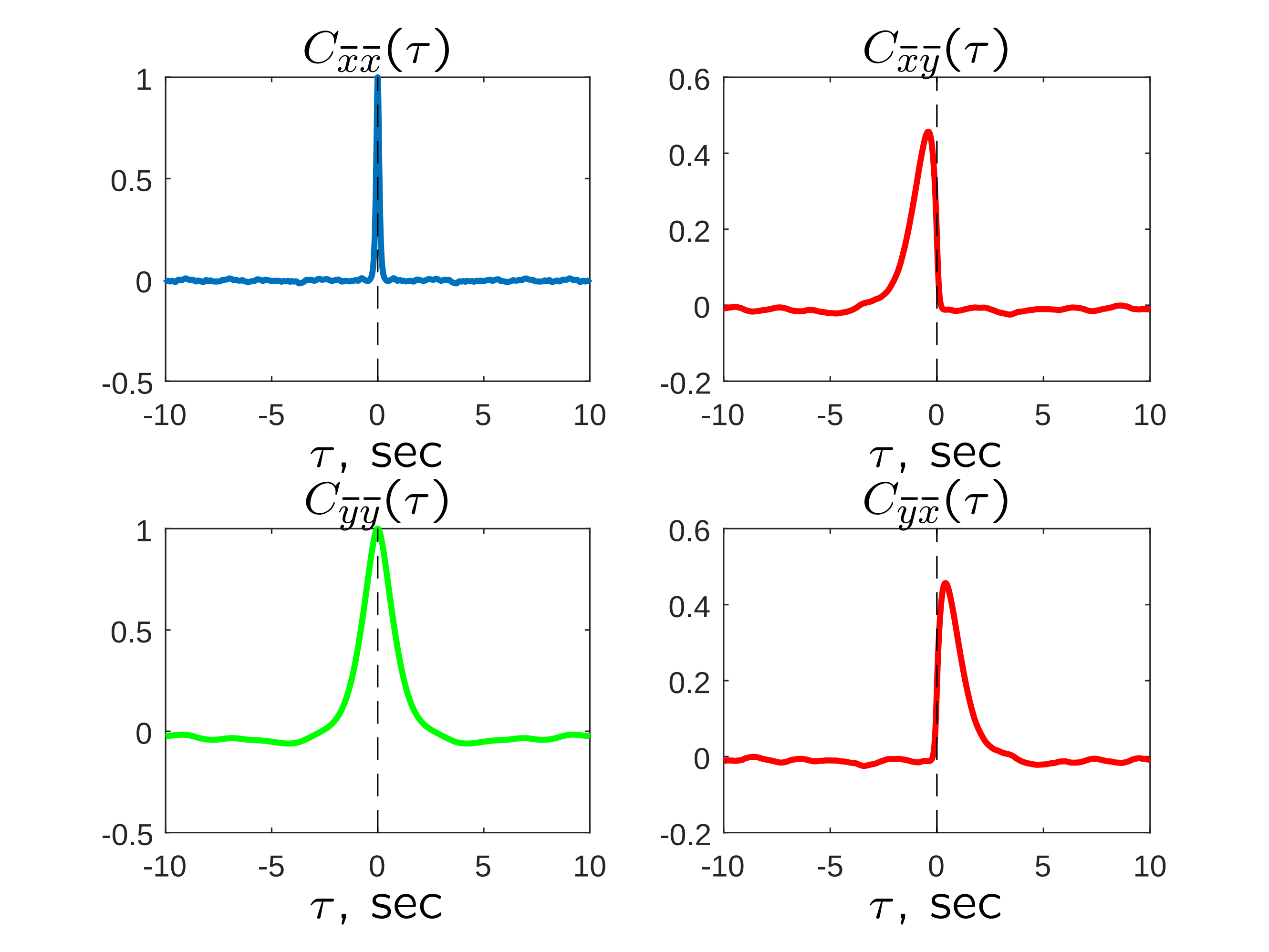
See the following figure, which also clearly shows that the cross-functions are neither <u>even</u> functions, nor <u>odd</u> functions.

The symmetry between $C_{\overline{x}\overline{y}}(\tau)$ and $C_{\overline{y}\overline{x}}(\tau)$ however, is also clear: $C_{\overline{x}\overline{y}}(\tau) = C_{\overline{y}\overline{x}}(-\tau)$.



$\bar{x}(t)$
t, sec
$C_{\bar{x}\bar{y}}(\tau)$
τ, sec
$\bar{y}(t)$
t, sec
$C_{\bar{y}\bar{x}}(\tau)$
τ, sec

Now look at the average covariance functions for $N=100$ repetitions:



<table>
<tr><th>Auto-functions</th><th>Cross-functions</th></tr>
<tr><td>$$\begin{aligned} R_{\bar{x}\bar{x}}(\tau) &= R_{\bar{x}\bar{x}}(-\tau) \\ C_{\bar{x}\bar{x}}(\tau) &= C_{\bar{x}\bar{x}}(-\tau) \\ K_{\bar{x}\bar{x}}(\tau) &= K_{\bar{x}\bar{x}}(-\tau) \end{aligned}$$</td><td>$$\begin{aligned} R_{\bar{x}\bar{y}}(\tau) &= R_{\bar{y}\bar{x}}(-\tau) \neq R_{\bar{x}\bar{y}}(-\tau) \\ C_{\bar{x}\bar{y}}(\tau) &= C_{\bar{y}\bar{x}}(-\tau) \neq C_{\bar{x}\bar{y}}(-\tau) \\ K_{\bar{x}\bar{y}}(\tau) &= K_{\bar{y}\bar{x}}(-\tau) \neq K_{\bar{x}\bar{y}}(-\tau) \end{aligned}$$</td></tr>
<tr><td colspan="2">$$\tau = 0$$</td></tr>
<tr><td>$$\begin{aligned} R_{\bar{x}\bar{x}}(0) &= \sigma_{\bar{x}}^2 + \mu_{\bar{x}}^2 \\ C_{\bar{x}\bar{x}}(0) &= \sigma_{\bar{x}}^2 \\ K_{\bar{x}\bar{x}}(0) &= 1 \end{aligned}$$</td><td>$$\begin{aligned} R_{\bar{x}\bar{y}}(0) &= R_{\bar{x}\bar{y}} = C_{\bar{x}\bar{y}} + \mu_{\bar{x}}\mu_{\bar{y}} \\ C_{\bar{x}\bar{y}}(0) &= C_{\bar{x}\bar{y}} \\ K_{\bar{x}\bar{y}}(0) &= K_{\bar{x}\bar{y}} \end{aligned}$$</td></tr>
<tr><td colspan="2">$$\tau \to \infty$$
here we assume that the signals have no periodic components!</td></tr>
<tr><td>$$\begin{aligned} \lim_{\tau\to\infty} R_{\bar{x}\bar{x}}(\tau) &= \mu_{\bar{x}}^2 \\ \lim_{\tau\to\infty} C_{\bar{x}\bar{x}}(\tau) &= 0 \\ \lim_{\tau\to\infty} K_{\bar{x}\bar{x}}(\tau) &= 0 \end{aligned}$$</td><td>$$\begin{aligned} \lim_{\tau\to\infty} R_{\bar{x}\bar{y}}(\tau) &= \mu_{\bar{x}}\mu_{\bar{y}} \\ \lim_{\tau\to\infty} C_{\bar{x}\bar{y}}(\tau) &= 0 \\ \lim_{\tau\to\infty} K_{\bar{x}\bar{y}}(\tau) &= 0 \end{aligned}$$</td></tr>
</table>

Proof.

$$R_{\bar{x}\bar{y}}(\tau) = \mathsf{E}\{\bar{x}(t)\cdot\bar{y}(t+\tau)\} \qquad *1$$

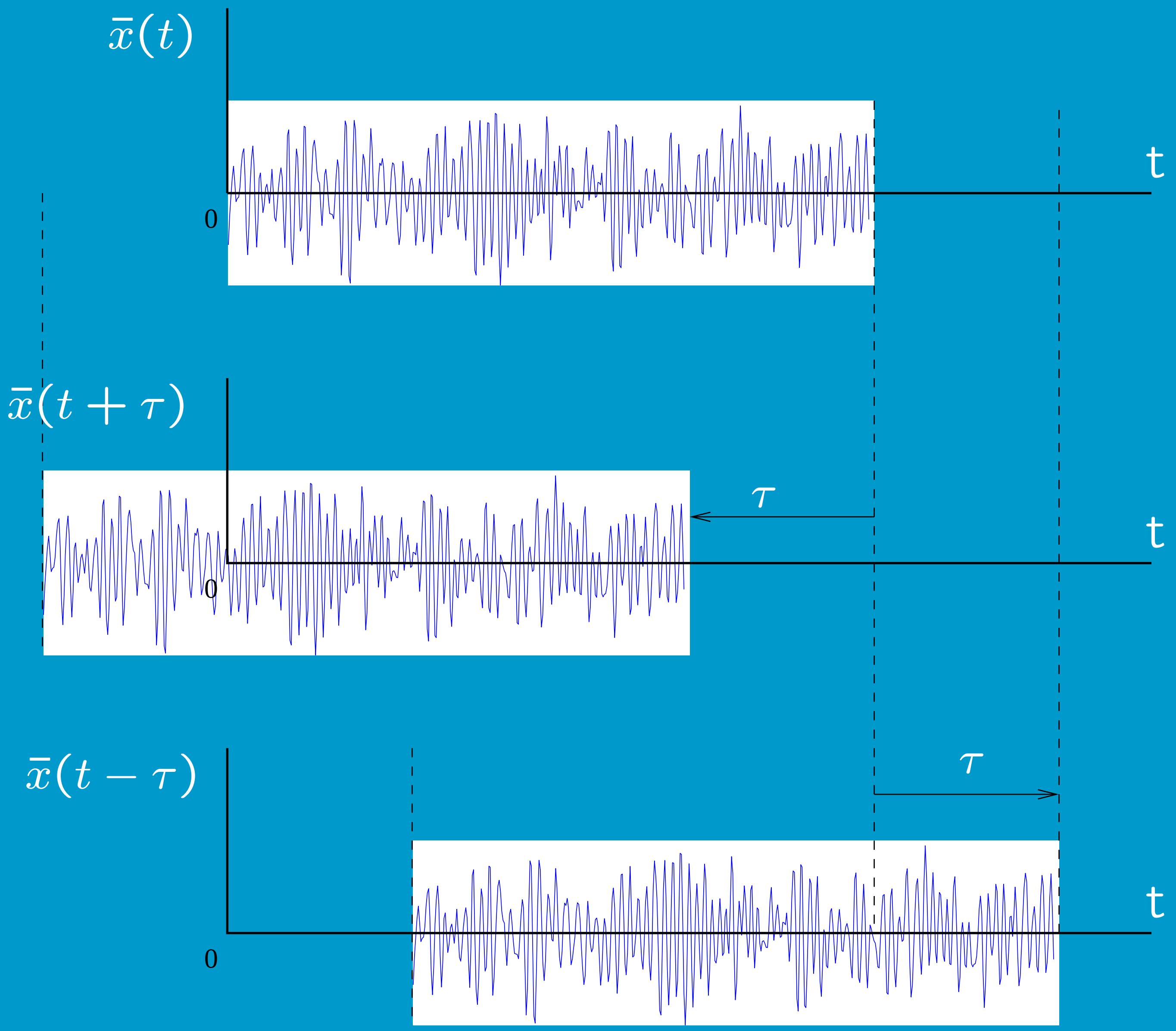
set $\nu = t+\tau$, then $t = \nu-\tau$, substitute in *1 yields:

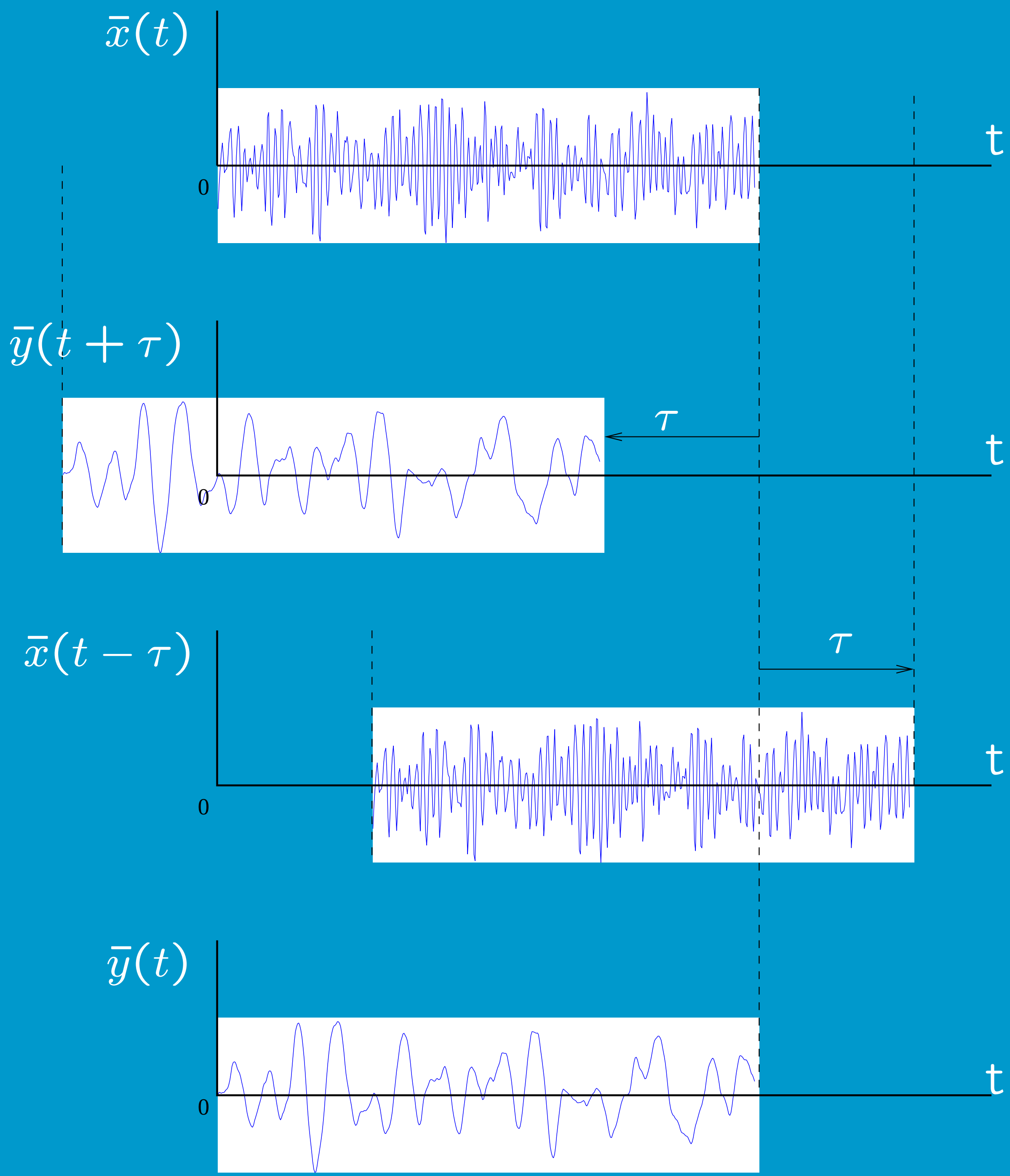
$$\begin{aligned} R_{\bar{x}\bar{y}}(\tau) &= \mathsf{E}\{\bar{x}(\nu-\tau)\cdot\bar{y}(\nu)\} \\ &= \mathsf{E}\{\bar{y}(\nu)\cdot\bar{x}(\nu-\tau)\} \end{aligned}$$

now, we are only considering stationary processes, so ν can be replaced with any other time, including t:

$$\begin{aligned} R_{\bar{x}\bar{y}}(\tau) &= \mathsf{E}\{\bar{y}(t)\cdot\bar{x}(t-\tau)\} \\ &= \mathsf{E}\{\bar{y}(t)\cdot\bar{x}(t+(-\tau))\} \\ &= R_{\bar{y}\bar{x}}(-\tau) \end{aligned}$$

Then, it follows that $C_{\bar{x}\bar{y}}(\tau) = C_{\bar{y}\bar{x}}(-\tau)$ and also that $K_{\bar{x}\bar{y}}(\tau) = K_{\bar{y}\bar{x}}(-\tau)$.


$\bar{x}(t)$
t
0
$\bar{x}(t+\tau)$
τ
t
0
$\bar{x}(t-\tau)$
τ
t
0


$\bar{x}(t)$
t
0
$\bar{y}(t+\tau)$
τ
t
0
$\bar{x}(t-\tau)$
τ
t
0
$\bar{y}(t)$
t
0

Uncorrelated, orthogonal and independent SVs

Two SVs $\bar{x}$ and $\bar{y}$ are **uncorrelated** if:

$$\boxed{\mathsf{E}\{\bar{x}\bar{y}\} = \mathsf{E}\{\bar{x}\}\,\mathsf{E}\{\bar{y}\}} \tag{2.30}$$

Two SVs $\bar{x}$ and $\bar{y}$ are **orthogonal** if:

$$\boxed{\mathsf{E}\{\bar{x}\bar{y}\} = 0} \tag{2.31}$$

Two SVs $\bar{x}$ and $\bar{y}$ are **independent** if:

$$\boxed{f_{\bar{x}\bar{y}}(x,y) = f_{\bar{x}}(x) f_{\bar{y}}(y)} \tag{2.32}$$

So, independence implies uncorrelation but not the other way around.

Ergodic processes

All expressions derived so far, are of *theoretical value only*, since they require either knowledge on the p.d.f. or *ensemble averages*. But in practice the p.d.f. is seldom known (although often a ND is assumed), and we only have one realization available for analysis. Hence, we introduce the concept of an **ergodic process**.

A stochastic process is called **ergodic** if its ensemble average can be replaced by its time average, i.e.:

$$\boxed{\mathrm{E}\left\{g(\bar{x}(t))\right\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} g(\bar{x}(t))\mathrm{d}t} \tag{2.33}$$

If a stochastic process is ergodic, it must also be stationary.

Note: Make Example 2.2 at home.

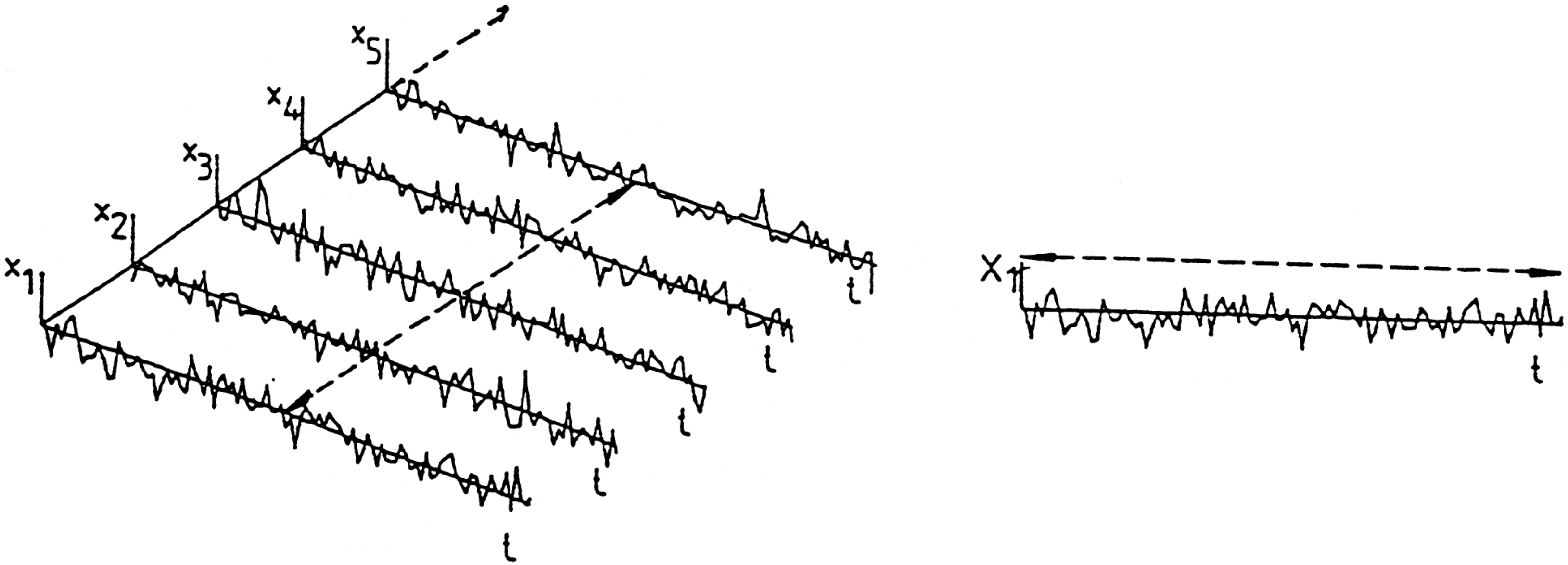


Figure 2.12: Difference between ensemble average and time average.

"One realisation tells all"

STATISTICAL PROPERTY		"THEORETICAL"		"PRACTICAL"
Name	**Symbol**	**Ensemble Average**	**Probability Density**	**Time Average**
Average	μ_x	$E\{x\}$	$\int_{-\infty}^{+\infty} x\, f_x(x)\, dx$	$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, dt$
Variance	σ_x^2	$E\{(x-\mu_x)^2\}$	$\int_{-\infty}^{+\infty} (x-\mu_x)^2\, f_x(x)\, dx$	$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)^2\, dt - \mu_x^2$
Auto Product	$R_{xx}(\tau)$	$E\{x(t)x(t+\tau)\}$		$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, x(t+\tau)\, dt$
Auto Covariance	$C_{xx}(\tau)$	$E\{(x(t)-\mu_x)(x(t+\tau)-\mu_x)\}$		$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, x(t+\tau)\, dt - \mu_x^2$
Cross Product	$R_{xy}(\tau)$	$E\{x(t)y(t+\tau)\}$		$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, y(t+\tau)\, dt$
Cross Covariance	$C_{xy}(\tau)$	$E\{(x(t)-\mu_x)(y(t+\tau)-\mu_y)\}$		$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, y(t+\tau)\, dt - \mu_x\mu_y$

Table 2.3: "Theoretical" versus "practical" calculation of the 6 most important statistical properties.

A practical approach: discrete-time covariance

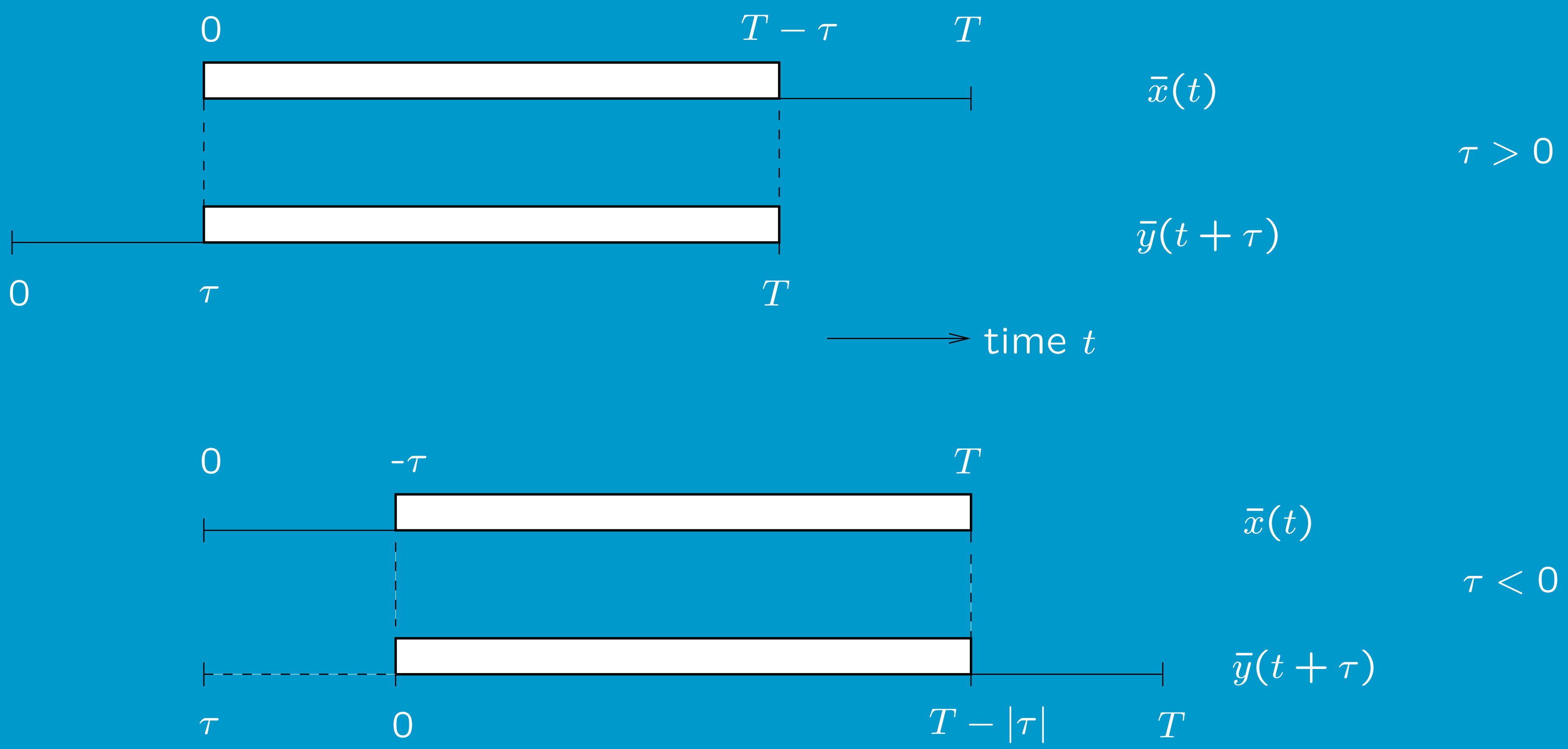
Suppose we have a realization of the SV $\bar{x}$, with N samples.

Then, τ becomes $k \bigtriangleup t$, with $\bigtriangleup t$ the sample time of the stochastic process.

$$\widehat{C}_{\bar{x}\bar{x}}[k] = \frac{1}{N-|k|-1} \sum_{i=1}^{N-|k|} (x[i] - \mu_{\bar{x}})(x[i+k] - \mu_{\bar{x}})$$

with: $\hat{\mu}_{\bar{x}} = \frac{1}{N} \sum_{i=1}^{N} x[i]$ Note that this is an *estimate* of $\mu_{\bar{x}}$!

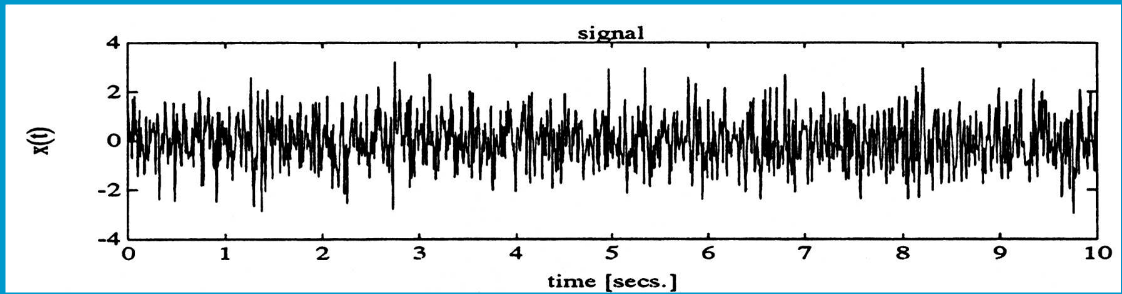
Note that when k increases, the term $\frac{1}{N-|k|}$ increases as well. But due to the fact that $\widehat{C}_{\bar{x}\bar{x}}[k]$ decreases for larger k for practically all stochastic processes (assume no periodic components) the computation is a feasible one. Again, note that when $N < \infty$, then $\widehat{C}_{\bar{x}\bar{x}}[k]$ is an *estimate* of the true autocovariance: $C_{\bar{x}\bar{x}}[k] = \lim_{N\to\infty} \widehat{C}_{\bar{x}\bar{x}}[k]$.



Note See Matlab examples: `showcorrelation.m` uses trailing zeros, and `showcorrelationcirc` uses the measurements in a 'circular' way. In the latter case, the correlation estimate becomes *periodic*! Why ?

White noise (Example 2.3)

Consider a realization of **white noise**, generated by Matlab's random generator (`rand.m`, `randn.m`).



When regarding this process $\bar{w}(t)$ at time t and time $t+\tau$, as the function values are totally random, the only value for τ where there exists *any* coherence between the function values is $\tau = 0$.

The white noise covariance function is a **Dirac pulse**:

$$\boxed{C_{\bar{w}\bar{w}}(\tau) = W\delta(\tau)} \tag{2.34}$$

The parameter W is called the **intensity** of the white noise.

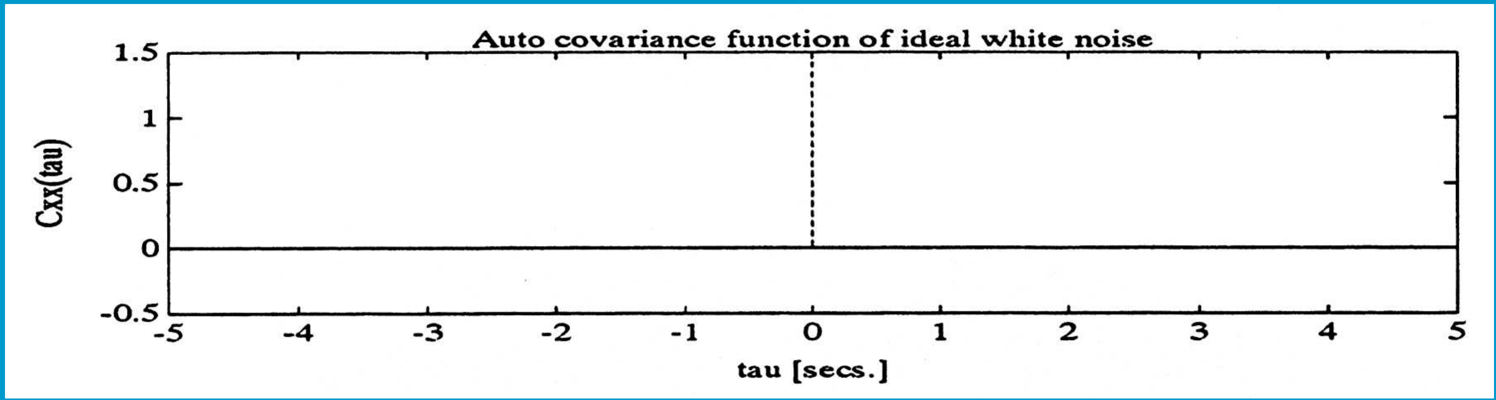
The Dirac pulse $\delta(\tau)$ is defined zero for all values of τ except for $\tau = 0$ where $\delta(\tau)$ is 'undefined' (infinite?). It has the following integral property:

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

NOTE
The Dirac delta function is a *generalized function*. It is not defined in terms of its values, but rather how it acts inside an integral when multiplied by a smooth function $f(t)$:

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\mathrm{d}t = f(\tau)}$$

This is known as the 'sifting property'.



Substitution of $\tau = 0$ in $C_{\bar{w}\bar{w}}(\tau)$ yields:

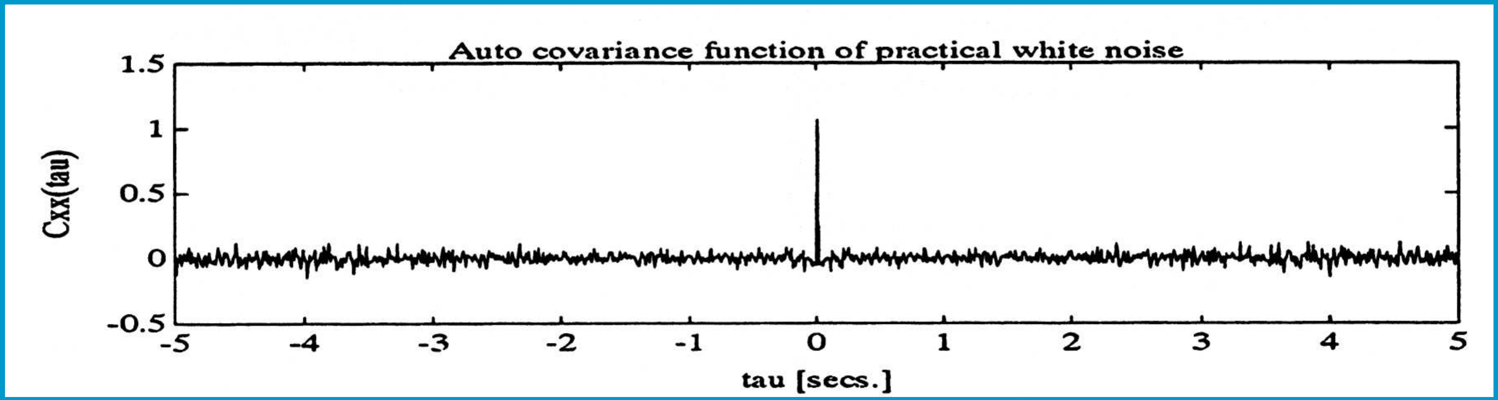
$$C_{\bar{w}\bar{w}}(0) = \sigma^2_{\bar{w}} = \infty,$$

implying an <u>infinite variance</u> for white noise. Also, the correlation function $K_{\bar{w}\bar{w}}(\tau)$ is not defined for white noise. It is clear that this stochastic process can not exist physically, like the Dirac pulse, but we will see that this abstraction is extremely useful in the mathematical treatment of systems and signals.

In practice, an SV $\bar{x}$ is called 'white' if:

$$\boxed{C_{\bar{w}\bar{w}}(\tau) \approx 0 \text{ for } |\tau| \ > \ \epsilon} \tag{2.35}$$

The auto-covariance of 'practical' white noise looks as follows:



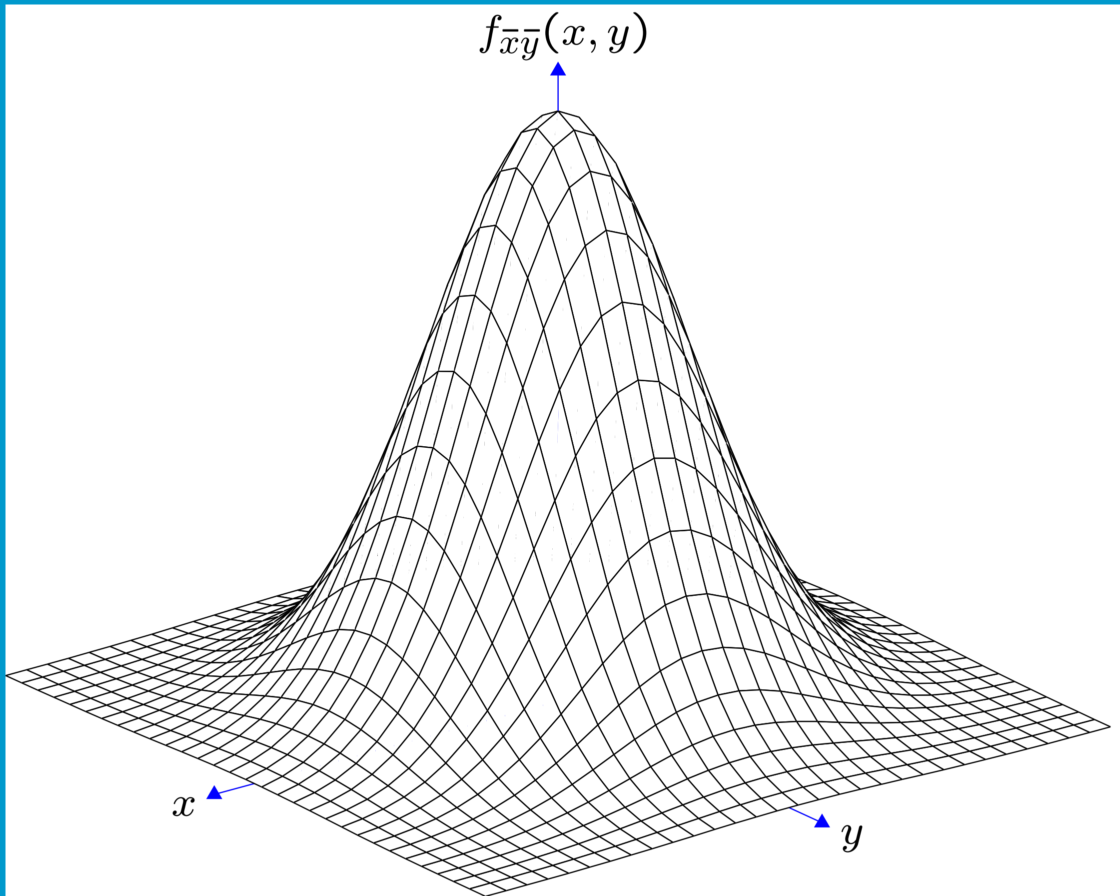
The two-dimensional ND

The two-dimensional normal p.d.f. is defined as:

$$\boxed{f_{\bar{x}\bar{y}}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-K_{\bar{x}\bar{y}}^2}}e^{-\frac{G}{2}}} \tag{2.40}$$

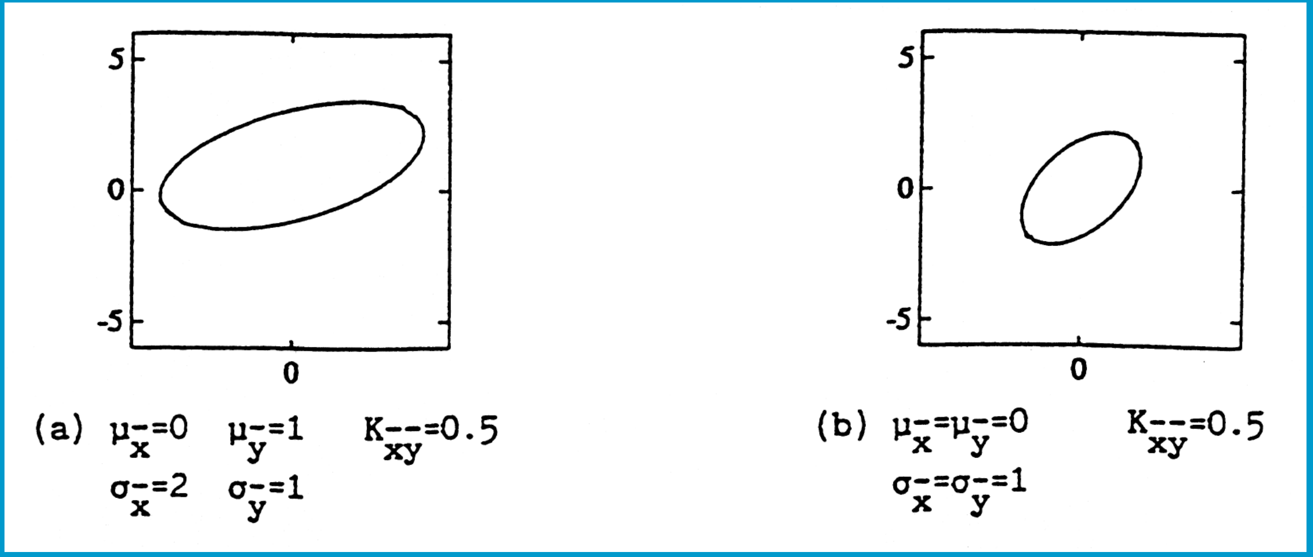
with:

$$\boxed{G = \frac{1}{1-K_{\bar{x}\bar{y}}^2}\left(\frac{(\bar{x}-\mu_{\bar{x}})^2}{\sigma_{\bar{x}}^2} - \frac{2K_{\bar{x}\bar{y}}(\bar{x}-\mu_{\bar{x}})(\bar{y}-\mu_{\bar{y}})}{\sigma_{\bar{x}}\sigma_{\bar{y}}} + \frac{(\bar{y}-\mu_{\bar{y}})^2}{\sigma_{\bar{y}}^2}\right)} \tag{2.41}$$


$f_{\bar{x}\bar{y}}(x, y)$
x
y

For the 2-D ND, when $K_{\bar{x}\bar{y}}$=0, that is, the SVs $\bar{x}$ and $\bar{y}$ are uncorrelated, these SVs are also independent: $f_{\bar{x}\bar{y}}(x,y) = f_{\bar{x}}(x)f_{\bar{y}}(y)$.

Cutting the surface in Fig. 2.15 with a plane $f_{\bar{x}\bar{y}}(x,y) = c$ yields a so-called equi-probability ellipse, with origin at ($\mu_{\bar{x}}$,$\mu_{\bar{y}}$) and (when $\sigma_{\bar{x}}$=$\sigma_{\bar{y}}$) the central axes under an angle of 45^o with the x and y axes.



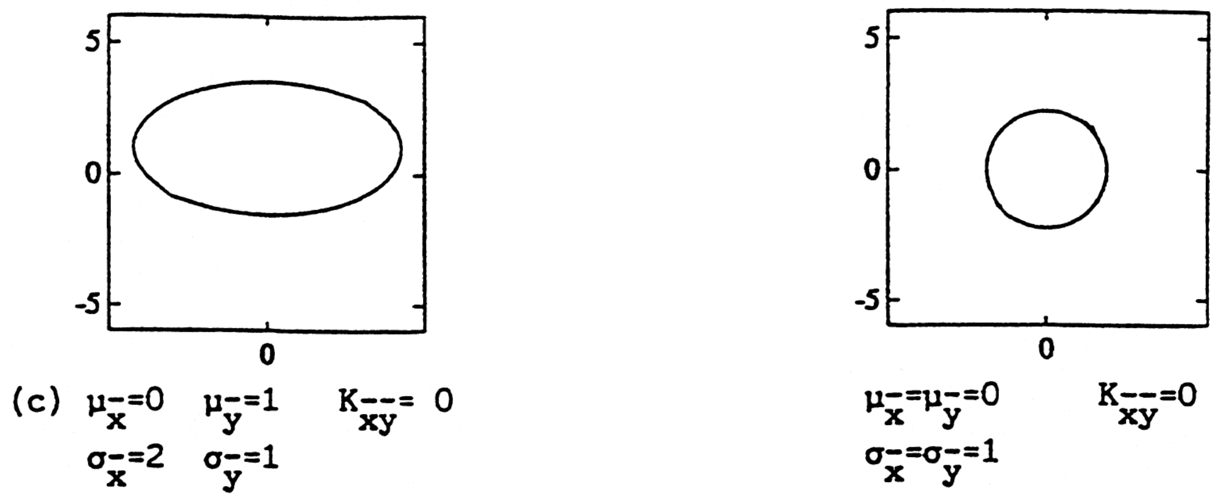


Figure 2.16: Equi-probability ellipses of the joint Gaussian distribution.

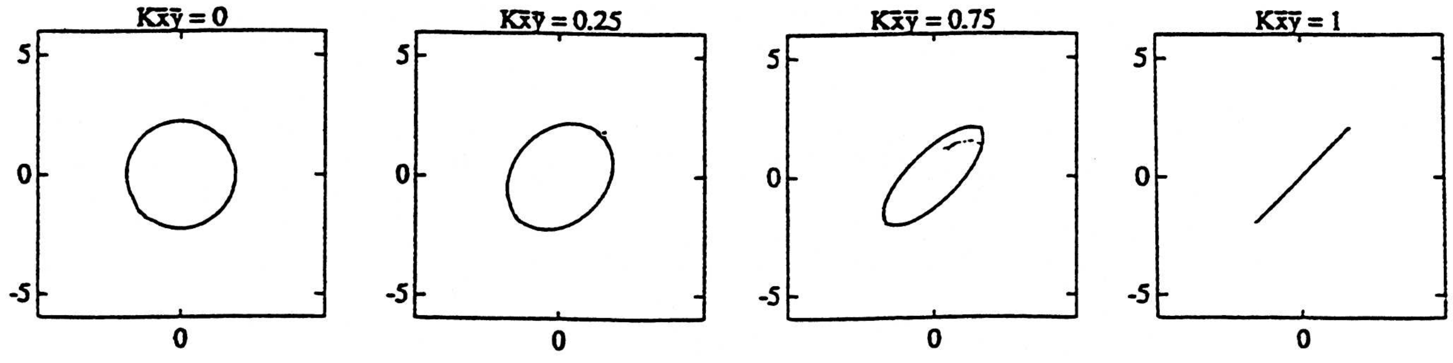


Figure 2.17: Equi-probability ellipses of the normalized joint Gaussian distribution.

The Central Limit Theorem (CLT)

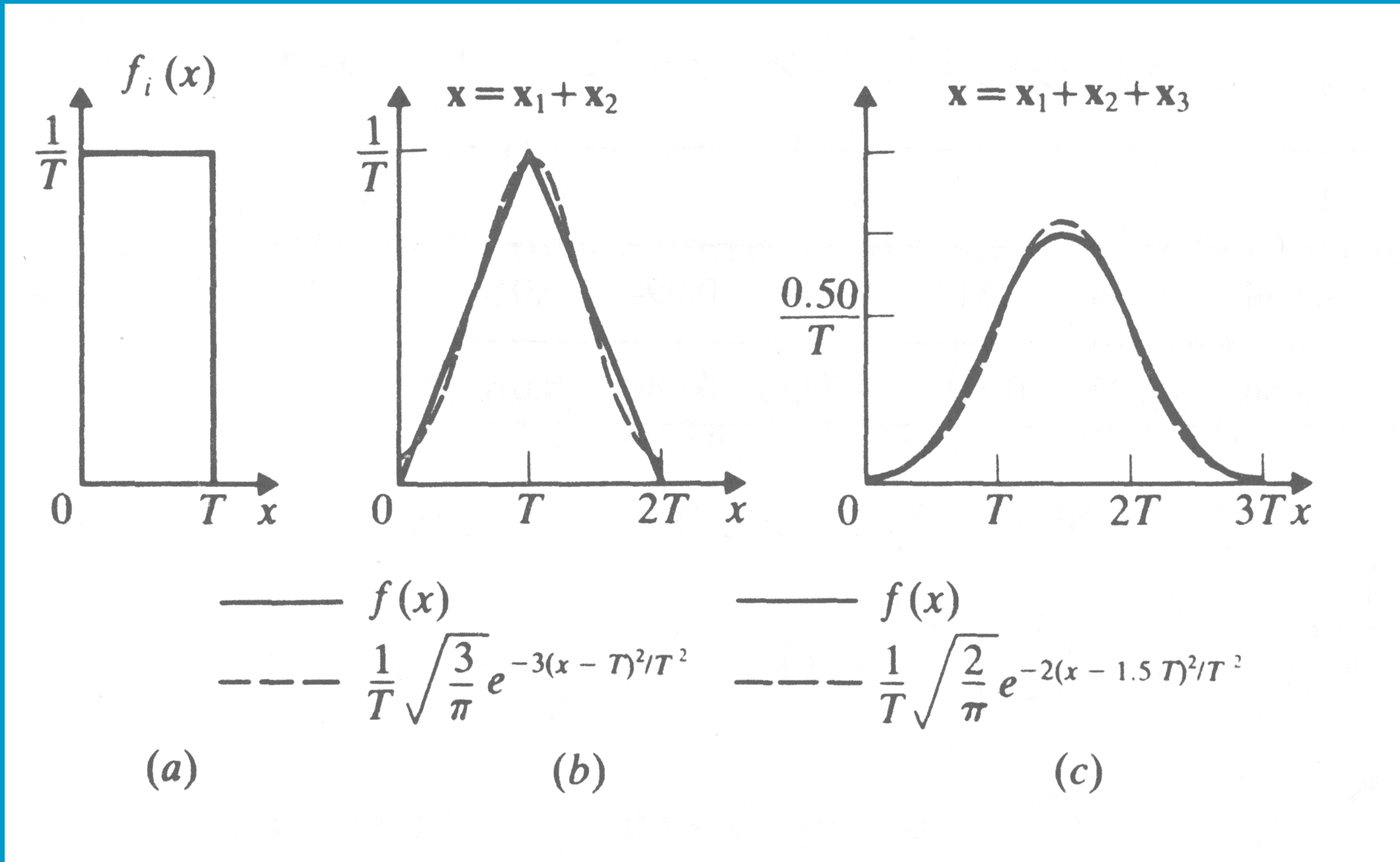
Given a sequence of n independent SVs $\bar{x}_i$, i=1,...n, that have the same p.d.f. of which the mean $\mu_{\bar{x}}$ and the variance $\sigma^2_{\bar{x}}$ are limited.

Consider a new random variable $\bar{y}$ that is defined as the *sum* of all SVs $\bar{x}_i$:

$$\bar{y}_n = \sum_{i=0}^{i=n} \bar{x}_i.$$

Then, when $n \to \infty$, the SV $\bar{y}_n$ has a normal distribution with $\mu_{\bar{y}}= n\mu_{\bar{x}}$ and $\sigma_{\bar{y}}= \sigma_{\bar{x}}/\sqrt{n}$.

Or, when defining $\bar{z}_n = \frac{\bar{y}_n - n\mu_{\bar{x}}}{\sigma_{\bar{x}}\sqrt{n}}$, the SV $\bar{z}_n$ will have a **standard normal distribution**, i.e., its mean $\mu_{\bar{z}}$ will be zero and its standard deviation $\sigma_{\bar{z}}$ will be one.



The **convolution** of the p.d.f. of a SV that results from summing up 2 (fig b) and 3 (fig c) SVs with the same **uniformly distributed** probability density function (fig a), yields a p.d.f. that approximates the normal distribution p.d.f.