

# The nCopula package

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August 2, 2017

## Abstract

The *nCopula* package aims to simplify the construction process and usage of hierarchical Archimedean copulas through compound distributions in R. It is possible to build structures with clear representations, obtain expressions for Archimedean copulas, whether they are hierarchical or not, as well as other important functions (i.e. Laplace Stieltjes Transform, pgf, etc.), given a certain path and structure. Furthermore, the generation of random vectors is possible from any given structure.

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## 1 Introduction

(...)

## 2 Archimedean copulas

### 2.1 Definition

Archimedean copulas are a very interesting family of copulas. Bla bla bla (...).

Let the copula's *generator* be defined as a decreasing function  $\psi : [0, \infty) \rightarrow [0, 1]$ , where  $\psi(0) = 1$  and  $\lim_{t \rightarrow \infty} \psi(t) = 0$ . In the same manner,  $\psi^{-1} : [0, 1] \rightarrow [0, \infty)$ , for which  $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$ , where

$\psi^{-1}$  is the inverse of the generator. The set of all such functions is denoted by  $\psi_{\infty}$ . Then, we can defined an Archimedean copulas  $C$  as

$$C(u_1, \dots, u_d) = \psi(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_d)). \quad (1)$$

## 2.2 Sampling

(...)

## 2.3 Symbolical derivatives

To obtain symbolical expressions of Archimedean copulas, one only needs to know the derivatives of the corresponding distribution's LST and inverse LST. Indeed, an Archimedean copula can be written as follows:

$$C(u_1, \dots, u_d) = \psi\left(\sum_{i=1}^d \psi^{-1}(u_i)\right), \quad (2)$$

where  $\psi$  is the generator of the copula. To compute its corresponding density, one should take the derivatives with respect to  $u_1$  up to  $u_d$  of (2). Doing so, we obtain that

$$c(u_1, \dots, u_d) = \prod_{i=1}^d (\psi^{-1})'(u_i) \psi^{(d)}(v), \quad (3)$$

where  $v = \sum_{i=1}^d \psi^{-1}(u_i)$ . This expression is often easily computable, since it only requires the  $d^{\text{th}}$  derivative of a Laplace Stieltjes Transform (LST) and the first derivative of its inverse.

**Remark 1.** Here,  $\psi^{(d)}(v)$  implies that we take the  $d^{\text{th}}$  derivative of the **univariate** function  $\psi$ , and then evaluate it at  $v$ .

In the case where the corresponding distribution is discrete, it is often easier to find the derivatives of the pgf. Thus, the following relation can be useful:

$$\begin{aligned} \mathcal{L}^{(d)}(t) &= \frac{\partial^d}{\partial t^d} \mathcal{P}(e^{-t}) \\ &= \sum_{r=1}^d \mathcal{P}^{(r)}(e^{-t}) e^{-td} \sum_{s=1}^r \frac{(-1)^{r-s}}{s!(r-s)!} (-s)^d. \end{aligned} \quad (4)$$

In the package, (4) is used for GEO and LOG.

## 2.4 Comparison with the copula package

```
library(nCopula)

## Loading required package: copula
## Warning: package 'copula' was built under R version 3.3.2
```

```

## Copula package ##
cop <- claytonCopula(5, 10)
res1 <- rCopula(10000, cop)

logv <- function(x, data)
{
  cop2 <- claytonCopula(x, 10)
  -sum(log(dCopula(data, cop2)))
}

suppressWarnings(optimize(logv, c(1e-5, 10), data = res1))

## $minimum
## [1] 5.012968
##
## $objective
## [1] -104552.6

## nCopula package ##

cop <- Clayton(5, 10, density = TRUE)
res1 <- rCop(10000, cop)

dens <- cop@dens

logv <- function(x, data)
{
  alpha <- x
  for (i in 1:10)
    eval(parse(text = paste("u", i, " <- data[,", i, "]", sep = "")))
  -sum(log(eval(parse(text = dens))))
}

suppressWarnings(optimize(logv, c(1e-5, 10), data = res1))

## $minimum
## [1] 4.997334
##
## $objective
## [1] -105620.7

```

## A Discrete distributions

### A.1 Geometric

Let  $M \sim \text{Geo}(p)$ , such that

$$\mathcal{P}_M(t) = \frac{pt}{1 - (1-p)t}.$$

Then,

$$\mathcal{P}_M^{(k)}(t) = \frac{k!}{t^{k-1}} \frac{1}{p} \left( \frac{\mathcal{P}_M(t)}{t} \right)^2 \left( \frac{\mathcal{P}_M(t)}{tp} - 1 \right)^{k-1}.$$

Furthermore,

$$(\mathcal{P}_M^{-1})'(t) = \frac{p}{(p + t(1-p))^2}.$$

### A.2 Logarithmic

Let  $M \sim \text{Log}(p)$ , such that

$$\mathcal{P}_M(t) = \frac{\ln(1-pt)}{\ln(1-p)}.$$

Then,

$$\mathcal{P}_M^{(k)}(t) = -\frac{(k-1)!}{\ln(1-p)} \left( \frac{p}{1-pt} \right)^k.$$

Furthermore,

$$(\mathcal{P}_M^{-1})'(t) = -\frac{\ln(1-p)}{p} (1-p)^t.$$

## B Continuous distributions

### B.1 Gamma

Let  $B \sim \text{Gamma}\left(\frac{1}{\alpha}, 1\right)$ , such that

$$\mathcal{L}_B(t) = (1+t)^{-\frac{1}{\alpha}}. \tag{5}$$

Then,

$$\mathcal{L}_B^{(k)}(t) = (-1)^k \prod_{j=0}^{k-1} \left( j + \frac{1}{\alpha} \right) (1+t)^{-\frac{1}{\alpha}-k}.$$

Furthermore,

$$(\mathcal{L}_B^{-1})'(t) = -\alpha t^{-\alpha-1}.$$

## B.2 Stable

Let  $B \sim \text{Stable}(\dots, \dots)$ , such that

$$\mathcal{L}_B(t) = e^{-t^\alpha}. \quad (6)$$

Then,

$$\begin{aligned} \mathcal{L}_B^{(k)}(t) &= \sum_{r=1}^k \sum_{s=1}^r \mathcal{L}_B(t) (-1)^s \frac{t^{(r-s)\alpha}}{s!(r-s)!} \frac{\Gamma(\alpha s + 1)}{\Gamma(\alpha s + 1 - k)} t^{\alpha s - k} \mathbb{1}_{\{k - \alpha s \notin \mathbb{N}^+\}} \\ &= \sum_{s=1}^k (-1)^s \left( \sum_{r=0}^{k-s} \frac{(t^\alpha)^r}{r!} \mathcal{L}_B(t) \right) \frac{\Gamma(\alpha s + 1)}{s! \Gamma(\alpha s + 1 - k)} t^{\alpha s - k} \mathbb{1}_{\{k - \alpha s \notin \mathbb{N}^+\}} \\ &= \sum_{s=1}^k (-1)^s \frac{\Gamma(\alpha s + 1)}{s! \Gamma(\alpha s + 1 - k)} t^{\alpha s - k} \mathbb{P}(N \leq k - s) \mathbb{1}_{\{k - \alpha s \notin \mathbb{N}^+\}}, \quad N \sim \text{Poisson}(t^\alpha) \\ &= \mathbb{E} \left[ \sum_{s=1}^{k-N} (-1)^s \frac{\Gamma(\alpha s + 1)}{s! \Gamma(\alpha s + 1 - k)} t^{\alpha s - k} \times \mathbb{1}_{\{N \leq k-1 \cap k - \alpha s \notin \mathbb{N}^+\}} \right] \\ &= \mathbb{E}[g(N)], \end{aligned}$$

where  $g(t) = \sum_{s=1}^{k-t} (-1)^s \frac{\Gamma(\alpha s + 1)}{s! \Gamma(\alpha s + 1 - k)} t^{\alpha s - k} \times \mathbb{1}_{\{t \leq k-1 \cap k - \alpha s \notin \mathbb{N}^+\}}$ . Furthermore,

$$(\mathcal{L}_B^{-1})'(t) = -\frac{1}{\alpha} \frac{(-\ln(t))^{\frac{1}{\alpha}-1}}{t}.$$

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