Lecture 6: Naive Bayes classifier

Introduction to Machine Learning

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Reminders on Bayes theorem

Question

Can anyone remind me of the Bayes theorem ?

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \times \mathbb{P}(A)}{\mathbb{P}(B)}$$

Main idea

The naïve Bayes classifier algorithm

The naïve Bayes classifier algorithm is a a probabilistic classifier which consists in applying to each record the class which is the most probable according to a probablity model.

Given a record $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and a set of labels $y_i \in \mathcal{Y}$, provide an estimation of $\mathbb{P}(y_i|\mathbf{x})$ $y_i \in \mathcal{Y}$ (and assign most likely label to $\mathbf{x}$).

Mathematical framework

We want to estimate for each label $y_i \in \mathcal{Y}$ $\mathbb{P}(y_i|\mathbf{x})$ (*what is the probability of being label y_i given the data records ?*) .
However, if n is large, the computation is infeasible.
Using the definition of conditional probabilities:

$$\mathbb{P}(y_i|\mathbf{x}) = \frac{\mathbb{P}(y_i, \mathbf{x})}{\mathbb{P}(\mathbf{x})}$$

$P(\mathbf{x})$ is a constant because $\mathbf{x}$ is given so we only need to find the value of $\mathbb{P}(y_i, \mathbf{x})$.

Mathematical framework

Using the definition of conditional probabilities iteratively,

$$
\begin{aligned}
\mathbb{P}(y_i, \mathbf{x}) =& \mathbb{P}(x_1, x_2, ..., x_n, y_i) \\
=& \mathbb{P}(x_1 | x_2, x_3, ..., y_i) \times \mathbb{P}(x_2, x_4, ..., y_i) \\
=& \mathbb{P}(x_1 | x_2, x_3, ..., y_i) \times \mathbb{P}(x_2 | x_3, x_4, ..., y_i) \times \mathbb{P}(x_3, x_4, ..., y_i) \\
=& \ ... \\
=& \mathbb{P}(x_1 | x_2, x_3, ..., y_i) \times \mathbb{P}(x_2 | x_3, x_4, ..., y_i) \times \mathbb{P}(x_n | y_i) \times \mathbb{P}(y_i)
\end{aligned}
$$

Mathematical framework

We now make the hypothesis that each feature x_i is independant (and only depends on the label y_i):

$$\mathbb{P}(x_1|x_2, x_3, ..., y_i) = \mathbb{P}(x_1|y_i)$$

We now have:

$$\mathbb{P}(y_i, \mathbf{x}) = \mathbb{P}(y_i) \prod_{j=1}^{n} \mathbb{P}(x_j|y_i)$$

and :

$$\mathbb{P}(y_i|\mathbf{x}) \propto \mathbb{P}(y_i) \prod_{j=1}^{n} \mathbb{P}(x_j|y_i)$$

We then select the most probable class

$$\hat{y} = argmax_{i=1,...,k}(\mathbb{P}(y_i) \prod_{j=1}^{n} \mathbb{P}(x_j|y_i))$$

Mathematical framework

Can you guess why this algorithm can be called naive ?

We have two terms to estimate:

- $\mathbb{P}(y_i)$: either assume class equiprobability or estimate using the frequency in training dataset
- $\mathbb{P}(x_j|y_i)$: we need to decide on a conditional law

Mathematical framework

Possible assumptions include:

- **If X_j is a continuous variable** ($\mathbf{x_j} \in \mathbb{R}$), the continuous values associated within class i are distributed according to a Gaussian distribution parametrized with mean μ_i and variance σ_i

$$f(v \mid y_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(v-\mu_i)^2}{2\sigma_i^2}}$$

- **If X_j is a binary variable** ($\mathbf{x_j} \in \{\mathbf{0}, \mathbf{1}\}^n$), the proportion of binary values observed within class y_i can be treated as a multivariate Bernouilli (p_{ij} being the frequency of event for variable x_j within class i):

$$\mathbb{P}(x_j \mid y_i) = \prod_{j=1}^{n} p_{ij}^{x_j}(1 - p_{ij})^{(1-x_j)}$$

Example

Training dataset:

Height	Weight	Tail	Label
45	30	0	Labradoodle
30	25	1	Labradoodle
40	35	1	Labradoodle
20	15	0	English cocker
22	18	1	English cocker
25	20	1	English cocker

Individual to classify

Height	Weight	Tail	Label
25	31	1	?

Example: solution

Estimate:

$$\begin{aligned}&\mathbb{P}(\text{labradoodle} \mid \text{height} = 25, \text{weight} = 31, \text{tail} = 1)\\ &\propto \mathbb{P}(\text{labradoodle}) \times \mathbb{P}(\text{height} = 25|\text{labradoodle})\\ &\times \mathbb{P}(\text{weight} = 31|\text{labradoodle})\\ &\times \mathbb{P}(\text{tail} = 1|\text{labradoodle})\end{aligned}$$

$\mathbb{P}(\text{labradoodle}) = \frac{1}{2}$

$\mathbb{P}(\text{height} = 25|\text{labradoodle}) = \frac{1}{\sqrt{2\pi \times 38.89}} e^{-\frac{(25-38.33)^2}{2\times 38.89}} = 0.006$

$\mathbb{P}(\text{weight} = 31|\text{labradoodle}) = \frac{1}{\sqrt{2\pi \times 16.67}} e^{-\frac{(31-30)^2}{2\times 16.67}} = 0.09$

$\mathbb{P}(\text{tail} = 1|\text{labradoodle}) = \frac{2}{3}$

$$\mathbb{P}(\text{labradoodle} \mid \text{height} = 25, \text{weight} = 31, \text{tail} = 1) = 0.00017$$

Example: solution

Estimate:

$$\begin{aligned}&\mathbb{P}(\text{cocker} \mid \text{height} = 25, \text{weight} = 31, \text{tail} = 1)\\ &\propto \mathbb{P}(\text{cocker}) \times \mathbb{P}(\text{height} = 25|\text{cocker})\\ &\times \mathbb{P}(\text{weight} = 31|\text{cocker})\\ &\times \mathbb{P}(\text{tail} = 1|\text{cocker})\end{aligned}$$

$\mathbb{P}(\text{cocker}) = \frac{1}{2}$

$\mathbb{P}(\text{height} = 25|\text{cocker}) = \frac{1}{\sqrt{2\pi \times 4.22}}\, e^{-\frac{(25-22.33)^2}{2\times 4.22}} = 0.08$

$\mathbb{P}(\text{weight} = 31|\text{cocker}) = \frac{1}{\sqrt{2\pi \times 16.67}}\, e^{-\frac{(31-30)^2}{2\times 16.67}} = 1.39e - 10$

$\mathbb{P}(\text{tail} = 1|\text{cocker}) = \frac{2}{3}$

$$\mathbb{P}(\text{cocker} \mid \text{height} = 25, \text{weight} = 31, \text{tail} = 1) = 0.00$$

Hyperparameters

Hyperparameters

What **hyperparameters*** do the naive Bayes classifier require ?

Advantages and limits

Limits:

- Strong independance hypothesis (but in practice, naive bayes behave rather well)
- Unable to classify unknown classes that do not show in training set (always sets it to 0, except in the case of artificial equiprobability)

Advantages:

- Extends naturally to multi-class
- Naturally deals with categorical variables

Questions

Questions ?