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The Steps Problem and the Fibonacci Sequence: an Interesting Connection

ANAND M. NARASIMHAMURTHY

The steps problem

There is a flight of n steps. A person standing at the bottom climbs either 1 or 2 steps at a time. In how many ways can he get to the top?

Let us consider the case of 10 steps. One way of climbing 10 steps is the sequence $1, 2, 2, 1, 1, 1, 2$. Other possible sequences are $2, 2, 2, 1, 2, 1$ and $1, 2, 1, 2, 2, 2$ and so on.

We provide two solutions to this problem and then link them.

The first solution

Let x be the number of times he climbs one step and y be the number of times he climbs two steps. The total number of steps is a constant (10 in our case, but n in general). However, the number of climbs can vary from 5 to 10. In general the number of climbs can vary from $(n+1)/2$ to n for odd n and $n/2$ to n for even n.

The possible values of x and y are:

$$\begin{array}{llllll} x=10 & x=8 & x=6 & x=4 & x=2 & x=0 \\ y=\ \ 0 & y=1 & y=2 & y=3 & y=4 & y=5 \end{array}$$

For each pair (x, y), we must find the number of ways of distributing the x single steps among the y double steps, namely

$$^{x+y}C_x = \frac{(x+y)!}{x!y!},$$

the familiar binomial coefficient.

The total number of possible ways is just the sum of the individual values. Hence the number of ways is

$$1 + 9 + 28 + 35 + 15 + 1 = 89.$$

The corresponding number for n steps is, for odd n,

$$^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + {}^{n-3}C_3 + \cdots + {}^{(n+1)/2}C_{(n-1)/2}$$

and, for even n,

$$^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + {}^{n-3}C_3 + \cdots + {}^{n/2}C_{n/2}.$$

A second solution

Let the number of ways in which he can climb n steps be $y(n)$. Since he can climb either 1 or 2 steps at a time, he could have come to the nth step from either the $n-1$th step or the $n-2$th step only. The total number of ways in which he can get to the nth step is thus the sum of the ways in which he can get to the $n-1$th and $n-2$th steps, i.e.

$$y(n) = y(n-1) + y(n-2), \qquad n > 2.$$

It is easily seen that $y(1) = 1$ since he can climb one step in only one way, and $y(2) = 2$ since he can climb two steps in two ways (sequence 1, 1 or 2).

Thus $y(n)$ is a sequence whose terms are $1, 2, 3, 5, 8 \ldots$, and $y(n)$ can be expressed as

$$\begin{aligned} &y(n) = y(n-1) + y(n-2) \qquad n > 2 \\ &y(1) = 1 \\ &y(2) = 2. \end{aligned}$$

It is evident that $y(n)$ is indeed the Fibonacci sequence displaced by a term. In particular

$$y(n) = F(n+1)$$

where $F(k)$ represents the kth term of the Fibonacci sequence.

Combining the two solutions we can find an expression for $F(n)$ (which is equal to $y(n-1)$). We have the identities

$$\begin{aligned} F(n) = {}&{}^{n-1}C_0 + {}^{n-2}C_1 + {}^{n-3}C_2 + {}^{n-4}C_3 \\ &+ \cdots + {}^{n/2}C_{(n/2)-1} \end{aligned}$$

when n is even, and

$$\begin{aligned} F(n) = {}&{}^{n-1}C_0 + {}^{n-2}C_1 + {}^{n-3}C_2 + {}^{n-4}C_3 \\ &+ \cdots + {}^{(n-1)/2}C_{(n-1)/2} \end{aligned}$$

when n is odd.

Anand M. Narasimhamurthy *is studying for a PhD degree in Computer Vision at Pennsylvania State University, USA. The 'steps problem' was given to a group of students at the Indian Institute of Science, Bangalore, India.*