1. Random Walk on $\mathbb{Z}$ finite length N

Recall:

S - sample space , $|S| < \infty$

S - Countable .. $\mathbb{N}$, $\mathbb{Z}$

$(S, \mathcal{F}, P)$

Probability Space

$\mathcal{F}$ - Events $\mathcal{F} = \mathcal{P}(S)$

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

(A1) $\mathbb{P}(S) = 1$

(A2) $\{E_k\}_{k \geq 1}$ $E_\ell \cap E_j = \phi$ $i \neq j$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Independence

A finite collection of events $A_1, A_2, \dots, A_n$ is mutually independent if

(4) — $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cdots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i)$

where $E_i = A_i$ or A_i^c $i = 1, 2, \dots, n.$

An arbitrary collection of events A_t where $t \in I$ for some index set I is mutually independent if every finite subcollection is mutually independent.

- A, B are independent iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

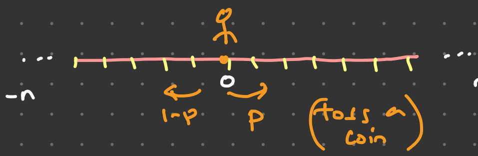
Conditional Probability

- A, B are two events $\mathbb{P}(B) > 0$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition of Random Walk

Intuitive : R.W.



- Toss a coin at each step
- If head occurs move right
- If tail occurs move left

S_n = position at time n

Q:- . Typical position at time n?

- $P(S_n = 0) = ?$
- Find $x \in [-n, n]$ s.t $P(S_n = x)$ is maximum?

Notation : $\mathbb{N}$ – natural numbers

$\mathbb{N}_0$ – $\mathbb{N} \cup \{0\}$.

Simple random walk on $\mathbb{Z}$

Fix $N \in \mathbb{N}$.

$$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{-1, +1\} \} \equiv \{-1, 1\}^N$$

Define for $1 \le k \le N$

$X_k : \Omega_N \to \{-1, 1\}$ by $X_k(\omega) = \omega_k$

(step of random walk at time k)

$$S_0(\omega) = 0 \ ; \quad S_n(\omega) = \sum_{k=1}^{n} X_k(\omega) \quad 1 \le n \le N$$

(position of walk at time n)

$\mathcal{F}_N = \{ A \mid A \subseteq \Omega_N \}$

$\mathbb{P}_N : \mathcal{F}_N \to [0,1]$ (p = 1/2 intuitive) (Uniform Probability)

(1) $$\mathbb{P}_N(A) = \frac{|A|}{2^N} \quad \forall A \subseteq \Omega_N$$

(i.e. any sequence in $\{-1,1\}^N$ has the same probability)

<u>Definition:</u> Sequence of random variable $\{S_n\}_{n=0}^{N}$ on Probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is called a simple symmetric random walk of length N starting at 0.

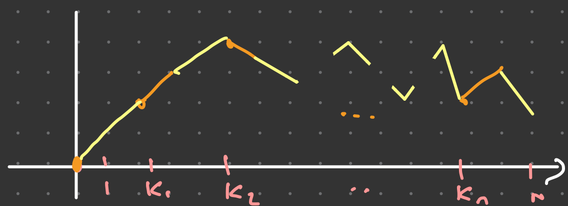
<u>Observations:</u>

<u>Given:-</u> $1 \le k_1 < k_2 \; .. \; < k_n \le N$

$x_1 \quad x_2 \quad .. \quad x_n$ $\qquad x_i \in \{-1,1\}$

steps n position

(*) $$\mathbb{P}(X_{k_1}=x_1, \; \dots \;, X_{k_n}=x_n) = \; ? \overset{1}{=} \frac{2^{N-n}}{2^N} = \frac{1}{2^n}$$



e.g. $x_1 = 1$

$x_2 = -1$

$\vdots$

$x_n = 1$

$$|\{X_{k_1}=x_1 \; .. \;, X_{k_n}=x_n\}| = |\{\omega \in \Omega_N \mid \omega_{k_1}=x_1 \; .. \; \omega_{k_n}=x_n\}| = 2^{N-n}$$

- (Ex.) $\mathbb{P}(X_k = 1) = \frac{1}{2} = \mathbb{P}(X_k = -1)$

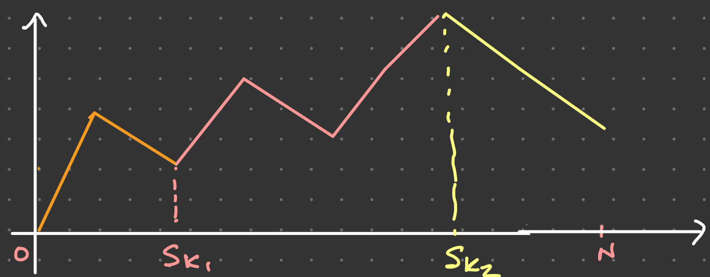
$\forall \; 1 \le k \le N$

[$\because$ $n=1$ in $*$]

$k \neq l$ $\quad \mathbb{P}(X_k = \pm 1, X_l = \pm 1) = \mathbb{P}(X_k = \pm 1)\,\mathbb{P}(X_l = \pm 1)$

(independence) $\quad k \neq l$

- (Ex)



Simple random walk has <u>independent increments</u>

$S_{k_1} - S_0, \quad S_{k_2} - S_{k_1}, \quad S_N - S_{k_2}$

are independent of each other.

<u>Note:</u>

- $S_{k_1} = \sum_{n=0}^{k_1} X_n$

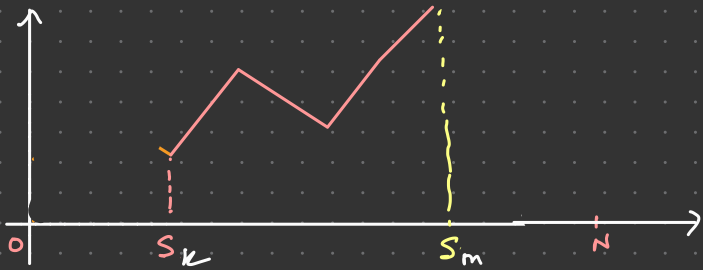
are <u>dependent</u>

- $S_{k_2} = S_{k_1} + \sum_{n=k_1+1}^{k_2} X_n$

- $0 < k < m \le N$

$\forall a \in \mathbb{Z}$

$$\mathbb{P}(S_m - S_k = a) = \mathbb{P}(S_{m-k} = a)$$



- $S_k = \sum_{n=0}^{k} X_n$
- $S_m = S_k + \sum_{n=k+1}^{m} X_n$

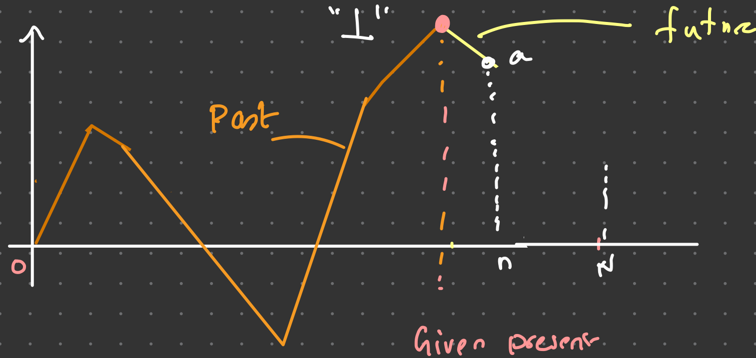
$$S_m - S_k = \sum_{n=k+1}^{m} X_n$$

(d) $\mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1}, \dots, S_1 = a_1)$ — (+)

$$= \mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1})$$

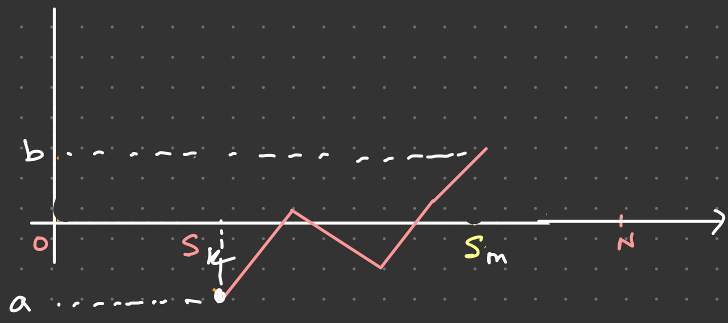
if $\mathbb{P}(S_{n-1} = a_{n-1}, \dots, S_1 = a_1) > 0$.

(+) (Markov Property)



(a) Suppose $\mathbb{P}(S_k = a) > 0$. for $k \geq 1$, $a \in \mathbb{Z}$.
$m > k$, $b \in \mathbb{Z}$

$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$



On an average where will the walk be:

Expectation $\quad E[S_n] = ? \quad Var[S_n] = ?$

$$E[X_k] = 1\,\mathbb{P}(X_k = 1) + (-1)\,\mathbb{P}(X_k = -1)$$

$$= 1 \cdot \frac{1}{2} - \frac{1}{2} = 0 \quad - \quad \#$$

$$S_n = \sum_{k=1}^{n} X_k + 0$$

$$\Rightarrow E[S_n] = \sum_{k=1}^{n} E[X_k] = 0$$

$$Var[S_n] = E[S_n^2] - (E[S_n])^2$$

$$= E[S_n^2] = E\left(\left(\sum_{k=1}^{n} X_k\right)^2\right)$$

$$= E\left[\sum_{k=1}^{n} X_k^2 + \sum_{\substack{i,j \\ i \neq j}} X_i X_j\right]$$

$$= \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i \neq j}} E[X_i X_j]$$

$\left(\begin{array}{c} i \neq j \\ X_i \perp X_j \\ \text{independent} \end{array} \right)$ $$= \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i \neq j}} E[X_i]E[X_j]$$

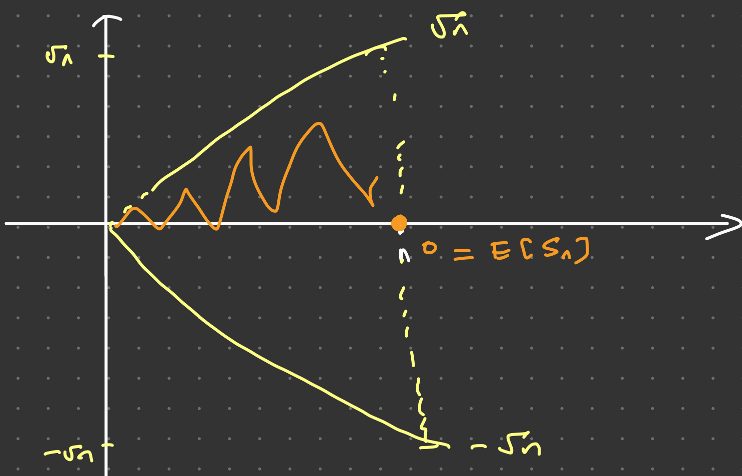
$\#$

0

Now $E[X_k^2] = 1^2 \, \mathbb{P}(X_k = 1) + (-1)^2 \, \mathbb{P}(X_k = -1)$

$= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1 \quad \forall\, 1 \leq k \leq n$ $\#\#$

$\Rightarrow \operatorname{Var}[S_n] = n$, $SD[S_n] = \sqrt{n}$



lemma 1.1: $x \in \{-n, -n+2, \dots, n-2, n\}$

$$\mathbb{P}(S_n = x) = \binom{n}{\frac{n+x}{2}} \frac{1}{2^n}$$

Proof

$\{S_n = x\} \overset{Ex}{\equiv}$ first n components of ω take precisely $\boxed{k = \frac{n+x}{2}}$ times value $+1$

i.e. $S_n = \underbrace{k}_{\text{\# up/right steps}}(+1) + \underbrace{(n-k)}_{\text{\# down/left steps}}(-1) = \boxed{2k-n \equiv x}$

\# of elements ω : $S_n(\omega) = x$

$$:= |\{\omega \in \Omega \mid S_n(\omega) = x\}| = \binom{n}{k} 2^{N-n}$$

we reach x in n steps

$$\mathbb{P}(S_n = x) = \frac{|\{\omega \in \Omega \mid S_n(\omega) = x\}|}{2^N}$$

$$= \frac{\binom{n}{k} 2^{N-n}}{2^N}$$

$$= \binom{n}{k} 2^{-n}$$

$$= \binom{n}{\frac{n+x}{2}} 2^{-n} \qquad \square$$

observations

- Distribution of S_n – symmetric around 0.

$$\mathbb{P}(S_n = x) = \frac{n!}{\frac{n+x}{2}!\;\frac{n-x}{2}!} = \mathbb{P}(S_n = -x)$$

- (maximal weight) $\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$

$$= \frac{2n!}{n!\,n!} 2^{-2n}$$

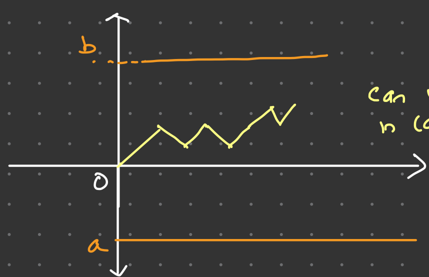
(Stirling's formula)

$$\boxed{k! \underset{k\to\infty}{\sim} k^k e^{-k} \sqrt{2\pi k}}$$

$$\sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi 2n}}{\left(n^n e^{-n} \sqrt{2\pi n}\right)^2} \cdot 2^{-2n}$$

$$\stackrel{\text{Ex.}}{=} \frac{1}{\sqrt{\pi n}} \quad (\text{as } n \to \infty)$$

Question



can walk S_n stay in (a,b) forever?

$$0 \le P(a \le S_n \le b) = \sum_{x \in [a,b]} \mathbb{P}(S_n = x)$$

$$\le \sum_{x \in [a,b]} \left[\mathbb{P}(S_n = 0) + \mathbb{P}(S_n = 1)\right]$$

for some $c_1 > 0$ $\quad \leq (b-a+1)\dfrac{c_1}{\sqrt{n}}$ — (3)

By (1) $\Rightarrow \mathbb{P}(a \leq S_n \leq b) \longrightarrow 0$ as $n \to \infty$ $\square$

Mathematical issue:

- our $\mathbb{P} \equiv \mathbb{P}_N$ N - fixed. (length of walk)
  So $n \to \infty$ need $N \to \infty$ as well.
- Understanding $\mathbb{P}_N$ when $N \equiv \infty$ [needs work] Come back to it later

1.2 Stopping times:

Interpretation: • $S_n \equiv$ represent "amount of capital" of the player after n rounds

• $X_k \equiv$ amount a player wins in round k.

* Expected "amount of Capital" after n round

$= E[S_n] = 0 \quad 0 \leq n \leq N$

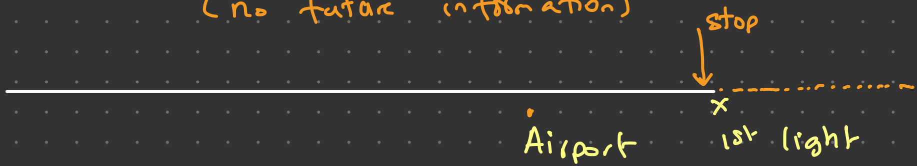
Question: Is it possible to stop the game in a favorable moment?

(clever stopping strategy ... to a positive expected gain)

(no future). no-insider trading { decision to stop may (information) only depend on toss till time n.

Intuition :

- stop at the first light after the air port
  (no future information)



- stop at the 3rd last light before the airport
  (use future information)