Recall :

Notation : $\mathbb{N}$ – natural numbers

$\mathbb{N}_0$ – $\mathbb{N} \cup \{0\}$.

Simple random walk on $\mathbb{Z}$

Fix $N \in \mathbb{N}$.

$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{-1, +1\} \} \equiv \{-1, 1\}^N$

Define for $1 \leq k \leq N$

$X_k : \Omega_N \to \{-1, 1\}$ by $X_k(\omega) = \omega_k$

(step of random walk at time k)

$S_0(\omega) = 0$; $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ $1 \leq n \leq N$

(position of walk at time n.)

$\mathcal{F}_N = \{ A \mid A \subseteq \Omega_N \}$

$\mathbb{P}_N : \mathcal{F}_N \to [0, 1]$ (Uniform Probability) ($p = \frac{1}{2}$ intuitive)

(1) $\mathbb{P}_N(A) = \dfrac{|A|}{2^N}$ $\forall A \subseteq \Omega_N$

(i.e. any sequence in $\{-1, 1\}^N$ has the same probability)

Definition : Sequence of random variable $\{S_n\}_{n=0}^{N}$ on Probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is called a simple symmetric random walk of length N starting at 0.

1.2 Stopping times :

Interpretation : • $S_n \equiv$ represent "amount of capital" of the player after n rounds

• $X_k \equiv$ amount a player wins in round k.

* Expected "amount of Capital" after n round

$= E[S_n] = 0$ $0 \leq n \leq N$

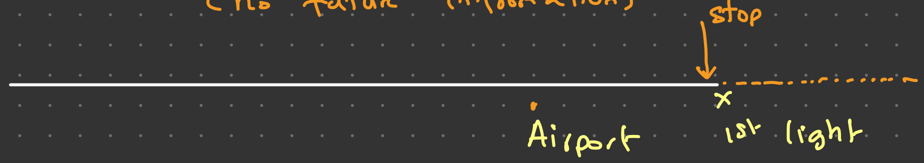
Question : Is it possible to stop the game in a favorable moment ?

(clever stopping strategy ... to a positive expected gain)

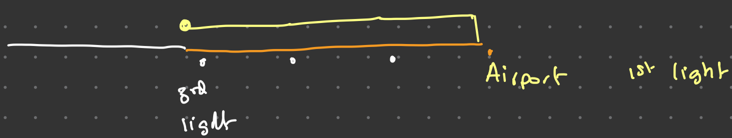
(no future). no -insider trading { decision to stop may

Intuition :

- stop at the first light after the airport
(no future information)



- stop at the 3rd last light before the airport
(use future information)



Definition : An event $A \subseteq \Omega$ is observable until time n when it can be written as a union of basic events of the form

$$\{\omega \in \Omega \mid \omega_1 = o_1, \ldots, \omega_n = o_n\} \quad o_i \in \{-1, 1\}, \; 1 \leq i \leq n$$

i.e. A - can be determined from the outcome of the 1st n tosses.

$\mathcal{A}_n :=$ class of event A that can be observed by time n. [include ϕ]

Notation: $A \subseteq \Omega$ (indicator of A)

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Definition: A map $T : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$ is called a stopping time if

$$\{T = n\} \equiv \{\omega \in \Omega \mid T(\omega) = n\} \in \mathcal{A}_n \quad 0 \leq n \leq N$$

i.e. $\{T = n\}$ is an event observable until time n.

Example: For $a \in \mathbb{Z}$ let

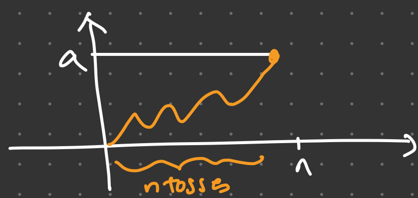
$$\sigma_a : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$$

$$\sigma_a(\omega) = \min\{k \in [0, N] \mid S_k(\omega) = a\}$$

$\min(\phi) = \infty$

Ex:- $\sigma_a(\cdot)$ is a stopping time

$\{\omega \in \Omega : \sigma_a(\omega) = n\} = \ldots$ "only depend on 1st n tosses"



Theorem 1 :: For any stopping time

[Impossibility of a favorite stopping] $T : \Omega \to \{0, 1, \ldots, N\}$

$$E[S_T] = 0$$

$S_T(\omega) \equiv S_{T(\omega)}(\omega) \equiv$ outcome of the trajectory ω at stopping time $T(\omega)$

Proof :- T is a stopping time.

$k = 0, 1, 2, \ldots N \qquad \{T \geq k\} \in \mathcal{A}_{k-1}$ — ①

$$\left[\because \{T \geq k\} = \big(\underbrace{\{T = 1\} \cup \cdots \{T = k-1\}}_{\in \mathcal{A}_{k-1}} \big)^c \right]$$

$\in \mathcal{A}_{k-1}$ & $\mathcal{A}_{k-1}$ is closed under compliments

$$S_T = \sum_{k=1}^{N} X_k \mathbb{1}(T \geq k) \quad \text{— ②}$$

$$\left[\because \quad S_T = \sum_{k=1}^{N} S_k \mathbb{1}(T = k) \right.$$

$$= \sum_{k=1}^{N-1} S_k \left[\mathbb{1}(T \geq k) - \mathbb{1}(T \geq k+1) \right] + S_N \mathbb{1}(T = N)$$

$$= \sum_{k=1}^{N-1} S_k \mathbb{1}(T \geq k) - \sum_{k=1}^{N-1} S_k \mathbb{1}(T \geq k+1) + S_N \mathbb{1}_{T=N}$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{k=2}^{N-1} S_k \mathbb{1}(T\geq k) + S_N \mathbb{1}_{T=N} - \sum_{k=1}^{N-1} S_k \mathbb{1}(T\geq k+1)$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{m=1}^{N-2} S_{m+1} \mathbb{1}(T\geq m+1) + S_N \mathbb{1}_{T=N} - \sum_{k=1}^{N-1} S_k \mathbb{1}(T\geq k+1)$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} (S_{l+1} - S_l) \mathbb{1}_{T\geq l+1} + (S_N - S_{N-1}) \mathbb{1}_{T=N}$$

$\{T=N\} \equiv \{T\geq N\}$

$$= X_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} X_l \mathbb{1}_{T\geq l+1} \quad X_N \mathbb{1}_{T=N} \,]$$

$$E[S_T] \overset{(2)}{=} E\left[\sum_{k=1}^{N} X_k \mathbb{1}_{T\geq k}\right]$$

linearity of E

$$= \sum_{k=1}^{N} E[X_k \underbrace{\mathbb{1}_{T\geq k}}_{\in A_{k-1}}]$$

depend on the kth toss $\rightarrow \omega_k$

" $\omega_1, \dots, \omega_{k-1}$ "

independence $\xleftarrow{Ex}$

$$= \sum_{k=1}^{N} E[X_k] \, E[\mathbb{1}(T\geq k)]$$

$$= \sum_{k=1}^{N} 0 \; \mathbb{P}(T \geq k) = 0$$

$\square$

Game System: $V_1, \dots, V_N : \Omega_N \to \mathbb{R}$

are random variables. such that

$$\{V_k = c\} \in \mathcal{A}_{k-1} \quad c \in \mathbb{R}, \; k = 1, 2, \dots N$$

- $V_k \equiv$ amount of money you will place as a bet in the k^{th} round
- Result in the k^{th} round $\equiv V_k X_k$

$$S_n^V = \sum_{k=1}^{N} V_k X_k$$

Theorem 2: For any $V_1, V_2, \dots, V_N$ game system

then

$$E[S_N^V] = 0$$

Proof: like before

$$E[S_N^V] = \sum_{k=1}^{N} E[V_k X_k] \; - \; \#$$

It is enough to show $E[V_k X_k] = 0$.

$\mathcal{A}_{k-1}$

depend on kth toss

$$\text{Range}(V_k) = \{c_1, c_2, \dots, c_m\}$$

$$\Rightarrow \quad X_k V_k = \sum_{i=1}^{m} c_i \, X_k \mathbb{1}(V_k = c_i)$$

$$E[X_k V_k] = \sum_{i=1}^{m} c_i \, E[X_k \mathbb{1}(V_k = c_i)]$$

"ω_k" "$\omega_1, \dots \omega_{k-1}$"

$$= \sum_{i=1}^{m} c_i \, E[X_k] \, \mathbb{P}(V_k = c_i)$$

$$= \quad \dots \quad 0 \quad \dots \quad = 0 .$$

$\hookrightarrow$ (*)

Place (*) into (#) $\Rightarrow$ to get result $\square$

Impact of Theorem 2:

$T : \Omega \to \{0, 1, \dots, N\}$ is a stopping time

$$V_k = \mathbb{1}_{\{T \geq k\}}$$

$$S_N^V = \sum_{k=1}^{N} X_k V_k = \sum_{k=1}^{N} X_k \mathbb{1}(T \geq k)$$

$$\overset{(2)}{=} S_T$$

[Theorem 2 $\Rightarrow$ Theorem 1]

- $V_k = \underbrace{\underbrace{S_{k-1}}\ \underbrace{1_{T \geq k}}_{①}}_{(\omega_1, \dots, \omega_{k-1})}$ (take this definition)

$\Rightarrow V_1, V_2, \dots, V_N$ is a game system.

$$V_k X_k = S_{k-1} X_k 1_{T \geq k}$$

Observe: $S_k^2 = (S_{k-1} + X_k)^2$

$$= S_{k-1}^2 + 2 \underline{S_{k-1} X_k} + X_k^2$$

$$\Rightarrow V_k X_k = \left(\frac{S_k^2 - S_{k-1}^2 - X_k^2}{2} \right) 1_{T \geq k}$$

$$S_N^V = \sum_{k=1}^{N} V_k X_k$$

$$= \sum_{k=1}^{N} \left(\frac{S_k^2 - S_{k-1}^2 - X_k^2}{2} \right) 1_{T \geq k}$$

$(X_k^2 = 1)$ $$= \sum_{k=1}^{N} \frac{S_k^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{S_{k-1}^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{1_{T \geq k}}{2}$$

$$= \sum_{k=1}^{N} \frac{S_k^2 1_{T = k}}{2} - \frac{T}{2}$$

$$S^{V}_{N} = \frac{(S^2_T - T)}{2} - xxx$$

Theorem 2 $\Rightarrow$ $E[S^V_N] = 0$

$\therefore$ By xxx for any $T : \Omega \to \{0, 1, \dots, N\}$

we have

$$0 = E[S^V_N] = E\left[\frac{S^2_T - T}{2}\right]$$

$$\Rightarrow \quad E[S^2_T] = E[T] - \#$$

Theorem 1:

$$E[S_T] = 0$$

$$Var[S_T] = E[T]$$

1.3 Ruin Problem:

Two players. Each with capital $\begin{cases} a - \mathrm{I} \\ b - \mathrm{II} \end{cases}$

Interpret

$S_n \equiv$ gain of player 1

Ruin (of player 1)

$$\{\sigma_{-a} < \sigma_b ; \sigma_{-a} \leq N\}$$

where, $\sigma_u = \min\{k : S_k = u\}$

<u>Define:</u> Ruin Probability

$$r'_N = \mathbb{P}(\sigma_{-a} \le \sigma_b, \ \sigma_{-a} \le N)$$

$$T_N = \min(\sigma_{-a}, \sigma_b, N)$$

Ex: T_N is a stopping time

$\therefore$ <u>Theorem 1</u> $\Rightarrow$ $E[S_{T_N}] = 0$ — (3)

$$T_N = \sigma_{-a} \Rightarrow S_{T_N} = -a$$

$$T_N = \sigma_b \Rightarrow S_{T_N} = b$$

$$T_N = N \Rightarrow S_{T_N} = S_N$$

— Use this in (3)

$$0 = -a\,\mathbb{P}(T_N = \sigma_{-a}) + b\,\mathbb{P}(T_N = \sigma_b) + E[S_N \mathbb{1}_{T_N = N})$$

As $T_N = \min(\sigma_{-a}, \sigma_b, N)$

$$0 = -a\,\mathbb{P}(\sigma_{-a} < \sigma_b \ \& \ \sigma_{-a} \le N)$$

$$+ \; b \; \mathbb{P}(\sigma_b < \sigma_{-a} \;\&\; \sigma_b \leq N)$$

$$+ \; \mathbb{E} \, S_N \left(\mathbb{1}_{\min(\sigma_{-a}, \sigma_b) \geq N} \right)$$

$$\Rightarrow \boxed{\begin{aligned} 0 = & - a \; \mathbb{P}_N(\text{Ruin of Player } 1) \\ & + b \; \mathbb{P}_N(\text{Ruin of Player } 2) \\ & + \mathbb{E}\left[S_N \, \mathbb{1}_{\min(\sigma_{-a}, \sigma_b) \geq N} \right] \quad (4) \end{aligned}}$$

- $\gamma^1 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player } 1)$ } Existence
- $\gamma^2 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player } 2)$

$$(4) \underset{\lim_{N\to\infty}}{\overset{\text{Ex}}{\Rightarrow}} \quad 0 = -a\gamma^1 + b\gamma^2 + 0$$

$$\Rightarrow \quad -a\gamma^1 + b\gamma^2 = 0 \quad \text{—} \; (5)$$

Ex

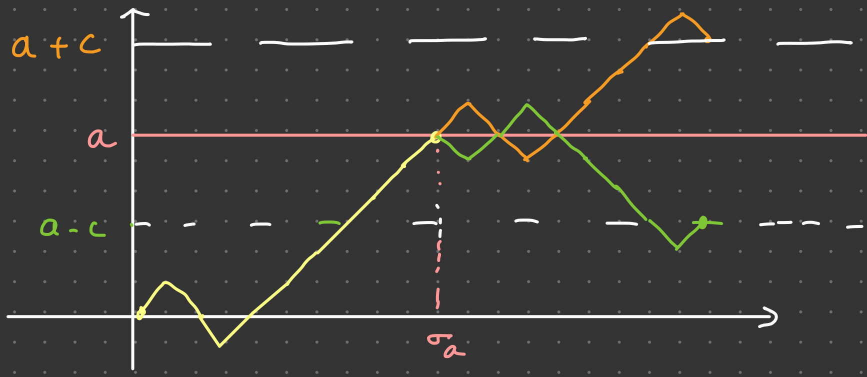
$$\gamma^1 + \gamma^2 = 1 \quad \text{—} \; (6)$$

$$(5) \;\&\; (6) \quad \gamma^1 = \frac{b}{a+b} \qquad \gamma^2 = \frac{a}{a+b} \; !$$

$\square$

1.4 Reflection Principle

Picture Concept: $a \in \mathbb{N}$ $\quad c \in \mathbb{N} \cup \{0\}$



Ex: For every path of the random walk after σ_a:

Each orange path that lands at $a+c$

there is a "Reflected" path across $y = a$

green path that lands at $a-c$

$$|\{\omega \in \Omega : S_n = a+c\}|$$
$$= |\{\omega \in \Omega : \sigma_a \le n,\ S_n = a+c\}|$$
$$= |\{\omega \in \Omega : \sigma_a \le n,\ S_n = a-c\}|$$

Lemma 1: $a, c \in \mathbb{N} \cup \{0\}$

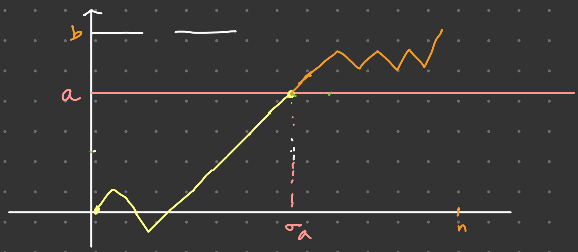
$$\mathbb{P}(S_n = a+c) = \mathbb{P}(\sigma_a \le n,\ S_n = a-c)$$

Theorem 2

(a) $\mathbb{P}(\sigma_a \le n) = \mathbb{P}(S_n \notin [-a, a-1])$

(b) $\mathbb{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbb{P}(S_n = a-1) - \mathbb{P}(S_{n+1} = a+1)\right]$

Proof:- $\mathbb{P}(\sigma_a \le n) = \mathbb{P}\left(\sigma_a \le n, \bigcup_{b \in \mathbb{Z}} \{S_n = b\}\right)$



i.e. $\mathbb{P}(\sigma_a \le n) = \sum_{b \in \mathbb{Z}} \mathbb{P}(\sigma_a \le n, S_n = b)$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(\sigma_a \le n, S_n = b)$$

Reflection Principle

Apply lemma 1.3.1
$b = a - c$
$(\Rightarrow a + c = 2a - b)$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(S_n = 2a - b)$$

(Ex)

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(S_n = b) + \mathbb{P}(S_n > a)$$

$\{\sigma_a \le n\} \subseteq \{S_n = b\}$

$$= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n > a)$$

$$\overset{\text{(Symmetry)}}{=} \mathbb{P}(S_n \geq a) + \mathbb{P}(S_n < -a)$$

$$= \mathbb{P}(S_n \notin [-a, a-1])$$

(b) $$\mathbb{P}(\sigma_a = n) = \mathbb{P}(\sigma_a \leq n) - \mathbb{P}(\sigma_a \leq n-1)$$

$$\overset{\text{Ex}}{=} \frac{1}{2}\left[\mathbb{P}(S_{n-1} = a-1) - \mathbb{P}(S_{n-1} = a+1)\right]$$

(Apply (a) + (Symmetry))

$\square$

Corollary 3:- $a, n \in \mathbb{N}$

$$\mathbb{P}(\sigma_a = n) = \frac{a}{n}\mathbb{P}(S_n = a)$$

Proof: Earlier class:

$$\mathbb{P}(S_{n-1} = a-1) = \frac{1}{2^{n-1}}\binom{n-1}{\frac{(n-1)+(a-1)}{2}}$$

$$\mathbb{P}(S_{n-1} = a+1) = \frac{1}{2^{n-1}}\binom{n-1}{\frac{n-1+a+1}{2}}$$

$$\mathbb{P}(S_n = a) = \frac{1}{2^n}\binom{n}{\frac{n+a}{2}}$$

(Do the Combinatorics)

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$\binom{n-1}{k} = \frac{n-k}{k}\binom{n}{k}$$

Thm 2 (b) $\Rightarrow$ $\mathbb{P}(\sigma_a = n) = \frac{a}{n} \mathbb{P}(S_n = a)$ $\square$

$n \in \mathbb{N}$ $\quad \sigma_a = \min\{n \geq 1 \mid S_n = a\}$

(hitting time of a) $a \in \mathbb{Z} / \{0\}$

(Return time) $\sigma_0 = \min\{n \geq 1 \mid S_n = 0\}$

lemma 4: (Escape from Origin)

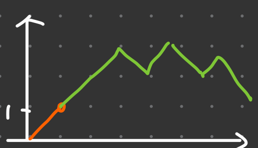
$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0)$$

Proof:-

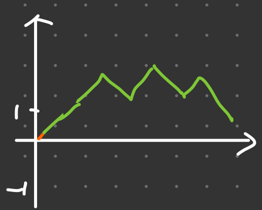
$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$$

(Reflection principle) E_r

$$= 2\,\mathbb{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$



$$= 2 \; \frac{\#\text{ paths start at 1 \& stay positive for } 2n-1 \text{ steps}}{2^{2n}}$$



$$= \frac{1}{2^{2n-1}} \; \#\text{ paths start at 0 \& stay above } -1 \text{ for } 2n-1 \text{ step}$$

(Reflection Principle)

$$= \frac{1}{2^{2n-1}} \; \#\text{ paths start at 0 \& stay below 1 for } 2n-1 \text{ steps}$$

$$= \mathbb{P}(\sigma_1 > 2n-1)$$

$$= 1 - \mathbb{P}(\sigma_1 \leq 2n-1)$$

Theorem 2 ⓐ with $a=1$

$$= 1 - \mathbb{P}(S_{2n-1} \notin [-1, 0])$$

$$\overset{\text{Ex}}{=} \mathbb{P}(S_{2n-1} = -1) = \mathbb{P}(S_{2n} = 0)$$

$\square$