Due: Wednesday, July 30th, 2025, 6:10pm
roblems to be turned in: to be announced at July 30th, 5:00pm

Set up: Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$ given by

$\mathbb{P}(A) = \frac{|A|}{2^N}$, for any $A \subset \Omega_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.

1. Show that $\{X_i\}_{1 \leq i \leq N}$ are indepdendent and identically distributed with distribution given by $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$.
2. Prove that an increment $S_m - S_k$ for $0 < k < m \leq N$ has the same distribution as S_{m-k}.
3. Prove that increment $S_m - S_k$ is independent of $S_n - S_l$ for $0 < l < n < k < m \leq N$
4. Assume for that $\{a_i\}_{i=1}^{n-1}$ are a set of integers such that $\mathbb{P}(S_{n-1} = a_{n-1}, \ldots, S_1 = a_1) > 0$. Show that
$$P(S_n = a_n \mid S_{n-1} = a_{n-1}, \ldots, S_1 = a_1) = \mathbb{P}(S_n = a_n \mid S_{n-1} = a_{n-1})$$
5. Let for $0 < k \leq N$, $a \in \mathbb{Z}, \mathbb{P}(S_k = a) > 0$. Prove that for $0 < k < m \leq N$,
$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$
for $b \in \mathbb{Z}$.
6. Let $a_n, b_n : \mathbb{N} \to \mathbb{R}_+$ be two sequences. We say $a_n \sim b_n$ if there exists $N_0 \in \mathbb{N}$ and $C_1, C_2 > 0$ such that $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \geq N_0$.
   (a) Show that $\mathbb{P}(S_{2n} = 0) \sim \frac{1}{\sqrt{n}}$
   (b) Show that $\mathbb{P}(S_{2n} = 0) - \mathbb{P}(S_{2n+2} = 0) \sim \frac{1}{n\sqrt{n}}$