Probability -II : Assignment 0

Yogeshwaran D.

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These are practice problems to revise analysis and probability. Solutions not to be submitted.

Let X be a set. For a function $a : X \to [0, \infty]$, we define

$$\sum_X a = \sum_{x \in X} a(x) := \sup\{\sum_{x \in F} a(x) : F \subset X, |F| < \infty\},$$

where $|F|$ denotes the cardinality of the set. Show the following :

1. If $X = \mathbb{N}$, then $\sum_X a = \sum_{n=1}^{\infty} a_n$ where the latter is the usual series summation.
2. If $\sum_X a \in \mathbb{R}$ then at most countably many $a(x)$'s are non-zero.
3. For $a, b : X \to [0, \infty]$, $\sum_X(\lambda a + \mu b) = \lambda \sum_X a + \mu \sum_X b$ for $\lambda, \mu \geq 0$.
4. For a subet $R \subset X \times Y$, set $R(x, .) = \{y : (x, y) \in R\}$ for $x \in X$ and similarly $R(., y)$ for $y \in Y$. For $a : X \times Y \to [-\infty, \infty]$, Show that

$$\sup_{(x,y) \in R} a(x, y) = \sup_{x \in X} \sup_{y \in R(x,.)} a(x, y) = \sup_{y \in Y} \sup_{x \in R(.,y)} a(x, y)$$

   and a similar identity for infimum.
5. If $f_n : X \to [0, \infty]$ is a sequence of functions such that $f_n(x)$ increases to $f(x)$ for all x, then prove that $\sum_X f_n \to \sum_X f$. Is this statement true if either $f_n(x) \to f(x)$ for all x or $f_n(x)$ decreases to $f(x)$ for all x ?
6. Show that for any sequence of functions $f_n : X \to [0, \infty]$, $\sum_X \liminf f_n \leq \liminf \sum_X f_n$. Is the same true for $\limsup$?

For a function $a : X \to \mathbb{R}$, we say that $\sum_X a = A \in \mathbb{R}$ if for all $\epsilon > 0$, there exists a finite set $F_\epsilon \subset X$ such that for all finite $F \subset X$ with $F_\epsilon \subset F$, we have that $|A - \sum_F a| < \epsilon$. We extend the definition to $A = \infty, -\infty$ in the usual fashion.

We say that a is summable if $\sum_X |a| < \infty$. Show the following :

7. Are exercises 1, 2 and 3 above also true for real-valued functions ?
8. Does the answer to above question change if we assume a is summable ?
9. $\sum_X a$ exists iff a is summable. If a is summable, then $|\sum_X a| \leq \sum_X |a|$.
10. Let $f_n : X \to \mathbb{R}$ be a sequence of functions such that $f_n(x) \to f(x)$. Further, there exists $g : X \to [0, \infty)$ such that $|f_n| \leq g$ for all n and g is summable. Then show that $\sum_X f_n \to \sum_X f$. Is this statement true if either $f_n(x) \to f(x)$ for all x ?
11. $a : X \times Y \to [0, \infty]$. Then

$$\sum_{X \times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

12. Show that the above conclusion holds for $a : X \times Y \to \mathbb{R}$ if a is summable.
13. Which of the above statements hold when the assumption of summability is removed ?

Assume that unless explicitly mentioned all probability spaces and random variables are discrete.

14. Suppose A and B are independently selected random subsets of $[n] = \{1, 2, , n\}$ (including the empty set and $[n]$ itself as possible subsets). Find $\mathcal{P}(A \subset B)$.

15. Suppose that the number of children N in a family satisfies

$$\mathcal{P}(N = n) = c(\frac{2}{5})^n, \; n = 0, 1, \ldots$$

for some constant $c > 0$. Now suppose that each child in a family is equally likely to be a boy or a girl. Let X be the number of girls in the family. Compute $\mathbb{E}[N]$ and $\mathbb{E}[X]$.

16. Let n balls be dropped independently at random into n bins. Let Y_n be the proportion of empty bins. Show that for all $\epsilon > 0$, $\mathcal{P}(|Y_n - e^{-1}| \geq \epsilon) \to 0$ as $n \to \infty$.

17. Let $X_i : (\Omega_i, \mathcal{F}_i, \mathcal{P}) \to \mathbb{R}, i = 1, 2$ be random variables with pmfs $p_i = p_{X_i}, i = 1, 2$ on $\mathbb{R}$.

    (a) Suppose that $p_1 = p_2$ and $f_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$ be continuous functions. Set $Y_i = f_i(X_i), i = 1, 2$. Prove or disprove that $p_{Y_1} = p_{Y_2}$.

    (b) Suppose that $p_1 \neq p_2$. and $f_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$ be continuous functions. Set $Y_i = f_i(X_i), i = 1, 2$. Prove or disprove that $p_{Y_1} = p_{Y_2}$.

18. Let there be n coupons and every day you get a coupon at random (possibly the same coupon again). Let T_n be the number of days required to get all n coupons. Find the sequence a_n such that $\mathbb{E}[|a_n T_n - 1|] \to 0$ as $n \to \infty$.

19. Let $k \geq 1$. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. uniform random variables in $[0, 1]$ and $f, g : [0, 1] \to \mathbb{R}$ be bounded continuous functions. Set $S_n = n^{-1} \sum_{i=1}^{n} f(X_i) g(X_{i+1})$. Show that $\mathbb{E}[|S_n - \int f \times \int g|] \to 0$ as $n \to \infty$ where $\int f$ denotes the Riemann integral $\int_0^1 f(x) \mathrm{d}x$ and similarly $\int g$.