Due: Wednesday, August 6th, 2025, 6:10pm
roblems to be turned in: to be announced at August 2nd, 5:00pm

1. **Set up:** Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$ given by
$\mathbb{P}(A) = \frac{|A|}{2^N}$, for any $A \subset \Omega_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

   (a) Let $a \in \mathbb{N}$, and $\sigma_a = \min\{k \geq 1 : S_k = a\}$. Assume that $\mathbb{P}(\sigma \leq n) = \mathbb{P}(S_n \notin [-a, a-1])$. Show that

$$\mathbb{P}(\sigma = n) = \frac{1}{2}\mathbb{P}(S_n = a - 1) - \frac{1}{2}\mathbb{P}(S_n = a + 1)$$

   (b) Let $L = \max\{0 \leq n \leq 2N : S_n = 0\}$.

      i. Show that for $n \leq N$,

$$\mathbb{P}(L = 2n) = \frac{1}{2^{2N}}\binom{2n}{n}\binom{2N-2n}{N-n}$$

      ii. Show that for $x \in (0, 1)$

$$\lim_{N\to\infty} \mathbb{P}(L \leq 2Nx) = \frac{2}{\pi}\arcsin(\sqrt{x})$$

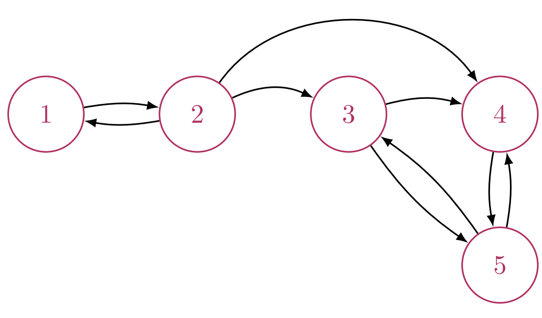
2. The examples given below can be modeled by a Markov chain. Determine the state space, initial distribution and the transition matrix for each.

   (a) Consider a frog moving along a river bank according to the following random mechanism. At each time step, it tosses a coin; if heads is the result, it jumps 1 unit up and if tails is the result, it jumps 1 unit down. Let X_n denote the position of the frog at time n.

   (b) Suppose N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step one ball is selected at random from each urn and the two balls interchange places. Let X_n be the number of white balls in the first urn after the n-th interchange.

   (c) Suppose a gambler starts out with a certain initial capital of N rupees and makes a series of 1 rupee bets against the gambling house until her captial runs out. Assume that she has probability p of winning each bet. Let X_n denote her capital at the n-th bet.

   (d) A particle is moving along the graph shown below. At each time step it moves along one of the incident edges to a neighbouring vertex, choosing the edge with equal probability and independently of all previous movements. Assume that it starts at a uniformly chosen point on the graph. Let X_n be the position of the particle at time n.



3. Meteorologist Chakrapani could not predict rainy days very well in the wet city of Rainpuram. So he decided to use the following prediction model for rain. If it had rained yesterday and today, then it will rain tomorrow with probability 0.5. If it rained today but not yesterday, then it will rain tomorrow with probability 0.3. If it did not rain today, then it will rain tomorrow with probability 0.1. regardless of yesterday's weather. Let $X_n = R$ if it rained on day $n \in \mathbb{N}$ and $X_n = D$ if it was a dry day (no rain). Suppose $X_0 = D$ (so it didn't rain on day 0).

   (a) Show that $\{X_n : n \geq 0\}$ is not a Markov chain. (Hint: Show that the Markov property is violated).

   (b) Let $Y_n = (X_n, X_{n+1})$ for each $n \geq 0$. Show that Y_n is a Markov chain by writing down the state space, the initial distribution, and the transition matrix for the chain Y_n.

4. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $S = \{A, B, C\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} .2 & .4 & .4 \\ .4 & .4 & .2 \\ .4 & .6 & 0 \end{pmatrix}.$$

   (a) What is the probability of going from state A to state B in one step ?

   (b) What is the probability of going from state B to state C in exactly two steps ?

   (c) What is the probability of going from state C to state A in exactly two steps ?

   (e) What is the probability of going from state C to state A in exactly three steps ?

   (e) Calculate the second, third and fourth power of this matrix. Do you have a guess for $\mathbf{P}^n$ for large n (It happens that $\lim_{n\to\infty} p_{ij}^{[n]} = \pi(j)$ for $i, j \in S$ for some vector $(\pi(j))_{j\in S}$ which is called the stationary distribution. This will be discussed later in this chapter.)

5. Let S be a countable set. Let $\mathbf{P}$ and $\mathbf{Q}$ be transition matrices for Markov chains on S. Show that the product $\mathbf{R} = \mathbf{PQ}$ (so the entries are $r_{ij} = \sum_{k\in S} p_{ik}q_{kj},\ \ i, j \in S$) is also a transition matrix for a Markov chain on S.

6. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $i \in S$ such that $P(X_0 = i) > 0$. For each $n \geq 0$ let $A_n \subset S$. Let $k \geq 1$ and suppose $i_j \in S$ for each j such that $0 \leq j \leq k-1$.

   (a) Prove the following equality holds whenever the conditional probabilities involved are well-defined.

$$P(X_{k+n} \in A_n \text{ for all } n \geq 0 \mid X_k = i, X_j = i_j \text{ for } 0 \leq j \leq k-1) = P(X_n \in A_n \text{ for all } n \geq 0 \mid X_0 = i)$$

   (b) Conclude that if $P(X_k = i) > 0$ then

$$P(X_{k+n} \in A_n \text{ for all } n \geq 0 \mid X_k = i) = P(X_n \in A_n \text{ for all } n \geq 0 \mid X_0 = i).$$