Due: Wednesday, August 6th, 2025, 6:10pm
roblems to be turned in: to be announced at August 2nd, 5:00pm

Set up: Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$ given by

$\mathbb{P}(A) = \frac{|A|}{2^N}$, for any $A \subset \Omega_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

For $n \in \{0, 1, \ldots, N\}$. Let $\mathcal{A}_n$ be the events that are observable by time n.

1. Show that $\mathcal{A}_n$ is closed under complimentation and intersections.
2. For $n \in \{0, 1, \ldots, N-1\}$. Suppose $A_n \in \mathcal{A}_n$ Then show that
$$\mathbb{P}(A_n \cap \{X_{n+1} = 1\}) = \frac{1}{2}\mathbb{P}(A_n), \qquad \mathbb{E}[X_{n+1} 1_{A_n}] = 0.$$
3. For $1 \leq n \leq N$, show that the mode of S_n is $\{0, 1\}$ that is
$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k}\frac{1}{2^{2k}}$$
4. For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that
$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$
and conclude that $\lim_{N\to\infty} \mathbb{P}(a \leq S_N \leq b) = 0$.
5. Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},$,
$$\sigma_a = \min\{k \geq 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \geq 1 : S_k = b\}.$$
   (a) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.
   (b) Show that
   $$\mathbb{E}(S_{\tau_N}) = a\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N 1(\min\{\sigma_a \sigma_b\} > N))$$
   and
   $$\mathbb{E}(\tau_N) = a^2\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b^2\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N^2 1(\min\{\sigma_a, \sigma_b\} > N)).$$
   (c) $\mathbb{E}(\tau_N) \to -ab$ as $N \to \infty$.
6. (*Ballot Theorem*) Suppose L and W are candidates contesting the elections to be president of students union. In the ballot, suppose each student voter is equally likely to vote for L and W. THe results are announced and L receives l votes and W receives w votes with $w > l$.
   (a) Let $N = W + L$ and for $1 \leq k \leq N$ let S_k represents the number of votes received by W minus the number of votes received by L at the counting of k-th ballot.

i. Show that S_k is a simple symmetric random walk of length N.

ii. Describe the $\mathbb{P}$(that W was leading through out the counting and wins by $w - l$ votes)) in terms of Probabilities involving the random S_k

(b) Find the $\mathbb{P}(S_k > 0, 1 \leq k \leq N \mid S_N = w - l)$ using the following steps:

i. Show that the number of valid random walk paths from $(1, 1)$ to $(N, w - l)$ that touch the x-axis is equal to the number of paths valid random walk paths from $(1, -1)$ to $(N, w - l)$.

ii. Show that the number of valid random walk paths from $(1, 1)$ to $(N, w - l)$ is $\binom{w+1-1}{l}$.

iii. Show that the number of valid random walk paths from $(1, -1)$ to $(N, w - l)$ is $\binom{w+l-1}{l-1}$.

iv. Conclude that the number of random walk paths from $(1, 1)$ to $(N, w - l)$ that do not touch the x-axis is given by $\frac{w-l}{w+l}\binom{w+l}{l}$.

(c) Show that $\mathbb{P}$(that W was leading through out the counting and wins by $w - l$ votes)) $= \frac{w-l}{w+l}$

7. *Extra-credit:* The below R code produces a sample of a random walk of length 100 starting at 0 and produces a plot

```
rw = function (x ,N){
  y = sample(x,N, replace=TRUE)
 c = cumsum(y)
c
}
N=100
S= rw(c(-1,1),N)
S = c(0,S)
t=0:N
plot(S~t, type="l")
```

(a) Run the above code and examine the plot obtained.

(b) Write an R-code that, once specified an input N- length of the finite length random walk simulates the distribution (plots the histogram) of S_N from 100 trials.

(c) Write an R-code that, once specified an input N- length of the finite length random walk simulates a sample of a random walk that start at 0 of length N and computes the time spent at each site $\{l : -N \leq l \leq N\}$.

(d) Write an R-code that, once specified an input N- length of the finite length random walk and position x, simulates a sample of a random walk that start at 0 of length N and at time N are at position x.