Probability III: Assignment 1

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Problems to be discussed in Aug 4th class with a quiz on some problems at the end.

1. Suppose $\mathcal{A}$ is an algebra closed under the formation of countable disjoint unions. Is $\mathcal{A}$ a σ-algebra ?

2. ★ Find examples of the following : (1) An algebra that is not a σ-algebra ; (2) A λ-system that is not a σ-algebra ; (3) A π- system that is not an algebra ; (4) A π-system that is not a λ-system ; (5) A λ-system that is not a π-system.

3. *Atomic Algebras:* Let $A_\alpha, \alpha \in I$ be a disjoint partition of Ω. Define $\mathcal{F} := \{\cup_{\alpha \in J} A_\alpha : J \subset I\}$. Show that $\mathcal{F}$ is a σ-algebra.

4. Suppose that $\mathcal{F}_n$ are algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ (increasing class of algebras) then $\cup_n \mathcal{F}_n$ is an algebra. Are increasing classes of σ-algebras are closed under unions ?

5. Show that a σ-algebra cannot be countably infinite. Is the same true for an algebra ?

6. By interval I, we refer to an interval $(a, b]$ for $a \leq b$. Define $\mathcal{B}_0$ to be the collection of all finite unions of disjoint intervals in $\Omega = (0, 1]$. Show that $\mathcal{B}_0$ is an algebra but not a σ-algebra. Defining $\mu(a, b] := b - a$ and for $B \in \mathcal{B}_0$, define $\mu(B) := \sum_i \mu(I_i)$ where $B = \cup_i I_i$ for disjoint intervals I_i. Show that μ is well-defined on $\mathcal{B}_0$. Further show that μ is countably additive on $\mathcal{B}_0$.

7. Show that $\sigma(\mathcal{B}_0) = \sigma\{(a, b] : 0 \leq a \leq b \leq 1\} = \mathcal{B}((0, 1])$, the Borel σ-algebra of $(0, 1]$.

8. Let $\mathcal{C} = \{(a, b] : -\infty < a \leq b < \infty\}$ and F be a non-decreasing and right continuous function on $\mathbb{R}$. Define $\mathbb{P}_F$ on $\mathcal{C}$ by $\mathbb{P}_F((a, b]) = F(b) - F(a)$. Show that $\mathbb{P}_F$ is a countably additive function on $\mathcal{C}$.

9. Let $\mathbb{P} : \mathcal{A} \to [0, \infty]$ be a finitely additive and countably subadditive set function on an algebra $\mathcal{A}$. Is $\mathbb{P}$ countably additive ? Is it necessary that $\mathcal{A}$ be an algebra ?