Section 10.4

Graph Connectivity

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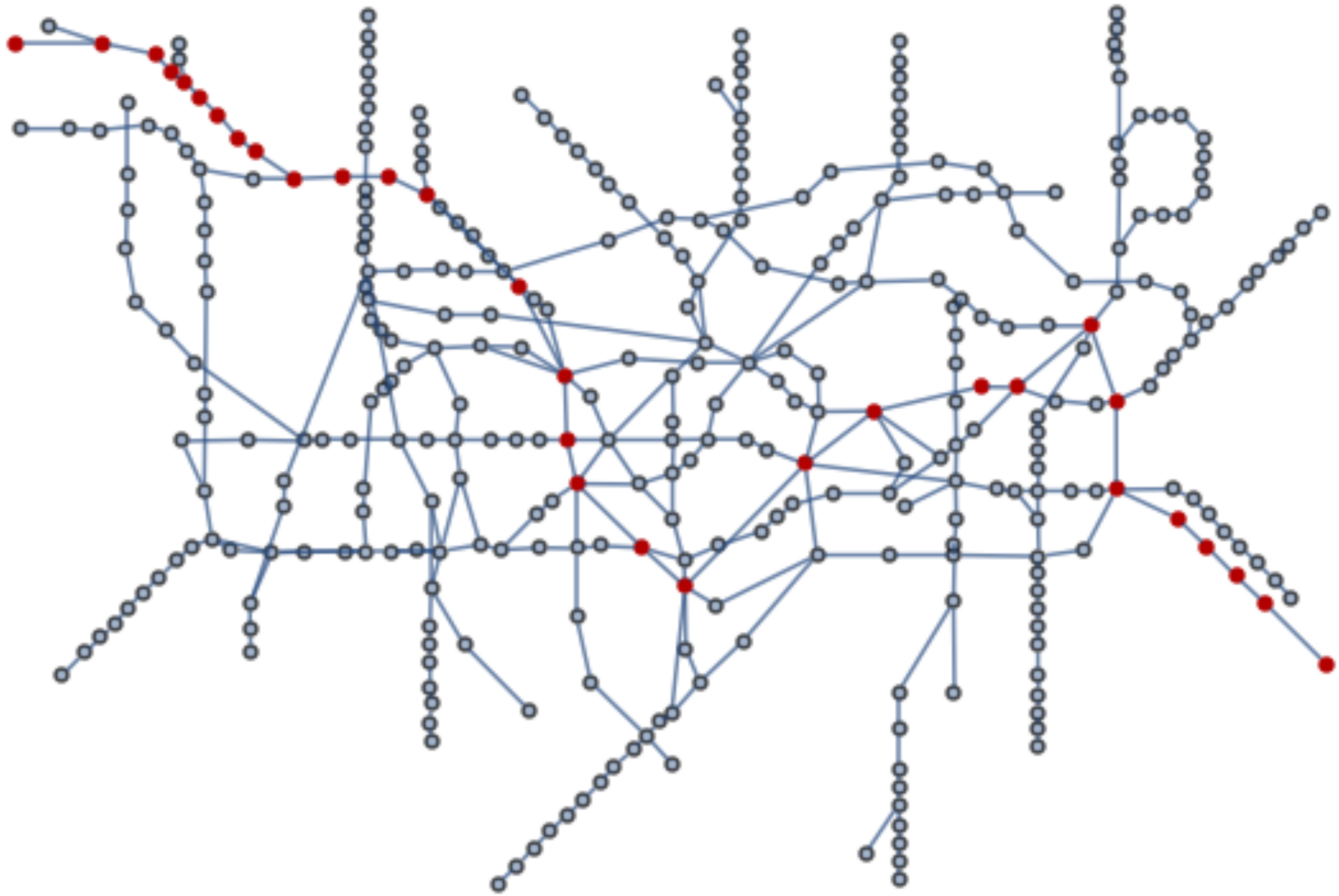
Kensal Rise
Brondesbury
Finchley Road
Swiss Cottage
St. John's Wood
Camden Town
Camden Road
Caledonian Road & Barnsbury
Canonbury
Hackney Central
Mornington Crescent
King's Cross St. Pancras
Highbury & Islington
Dalston Kingsland
Maida Vale
Paddington
Edgware Road
Baker Street
Great Portland Street
Euston
Angel
Old Street
Bethnal Green
Warren Street
Euston Square
Farringdon
Marylebone
Regent's Park
Liverpool Street
Bayswater
Barbican
Shoreditch
Russell Square
Moorgate
Goodge Street
Chancery Lane
Holland Park
Notting Hill Gate
Lancaster Gate
Bond Street
Oxford Circus
Holborn
Shepherd's Bush
Queensway
Marble Arch
Tottenham Court Road
St. Paul's
Aldgate East
Covent Garden
Bank
Aldgate
High Street Kensington
Hyde Park Corner
Green Park
Leicester Square
Cannon Street
Kensington (Olympia)
Piccadilly Circus
Mansion House
Monument
Tower Hill
Knightsbridge
Charing Cross
Tower Gateway
Barons Court
Gloucester Road
Sloane Square
St. James's Park
Blackfriars
River Thames
Temple
Rotherhithe
London Bridge
West Kensington
Earl's Court
South Kensington
Victoria
Westminster
Embankment
Bermondsey
Canada Water
Brompton
Waterloo
Broadway
Surrey Quays
Southwark
Pimlico
Borough
Lambeth North
Vauxhall
Elephant & Castle
Kennington



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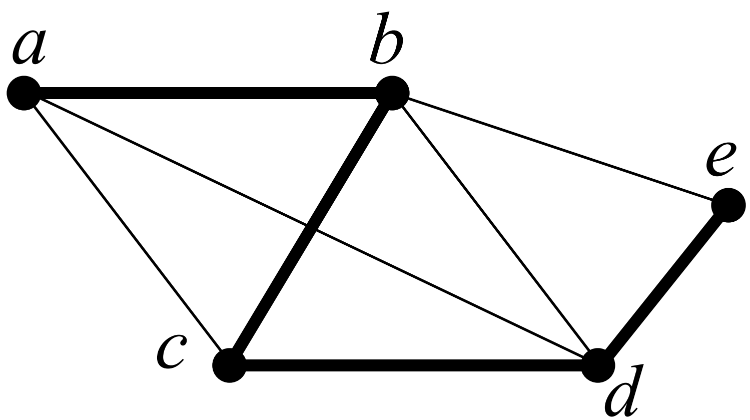
Paths

- A **path** is a sequence of edges that begins with a vertex of a graph and travels from vertex to vertex along edges of the graph.
- The **length** of a path is the number of edges in that path.

Paths



$(\{a,b\},\{b,c\},\{c,d\},\{d,e\})$ is a path of length 4.

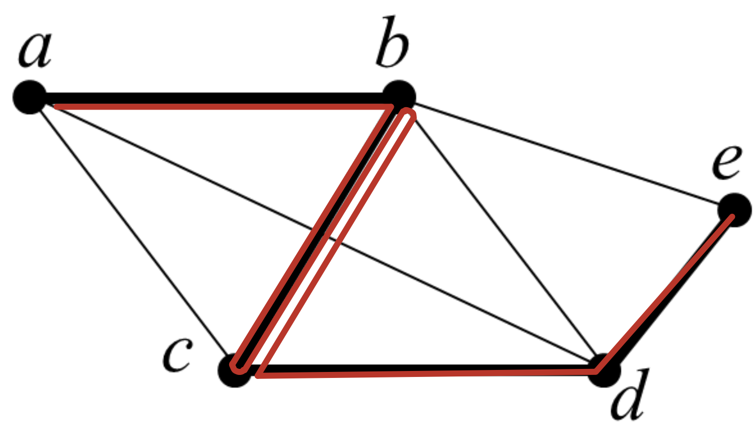
When there are no multiple edges (e.g., in simple graph), we can represent a path by the sequence of vertices it passes through.

$(\{a,b\},\{b,c\},\{c,d\},\{d,e\}) \Rightarrow (a,b,c,d,e)$

Paths /walk

trail

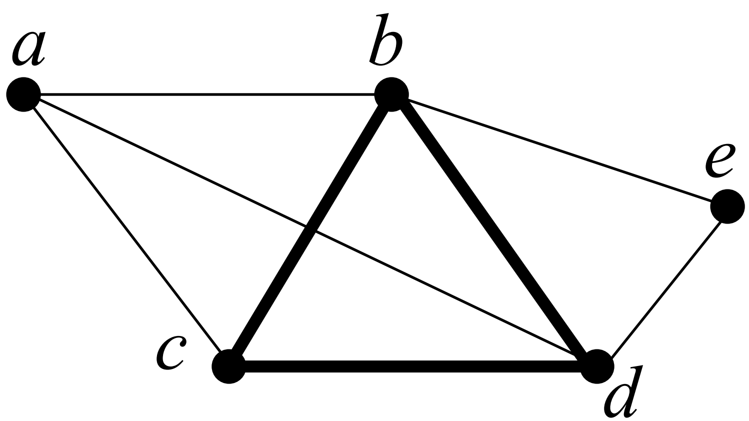
- A path is **simple** if it does not contain the same edge more than once.



Example of non-simple path: (a, b, c, b, c, d, e)

Paths

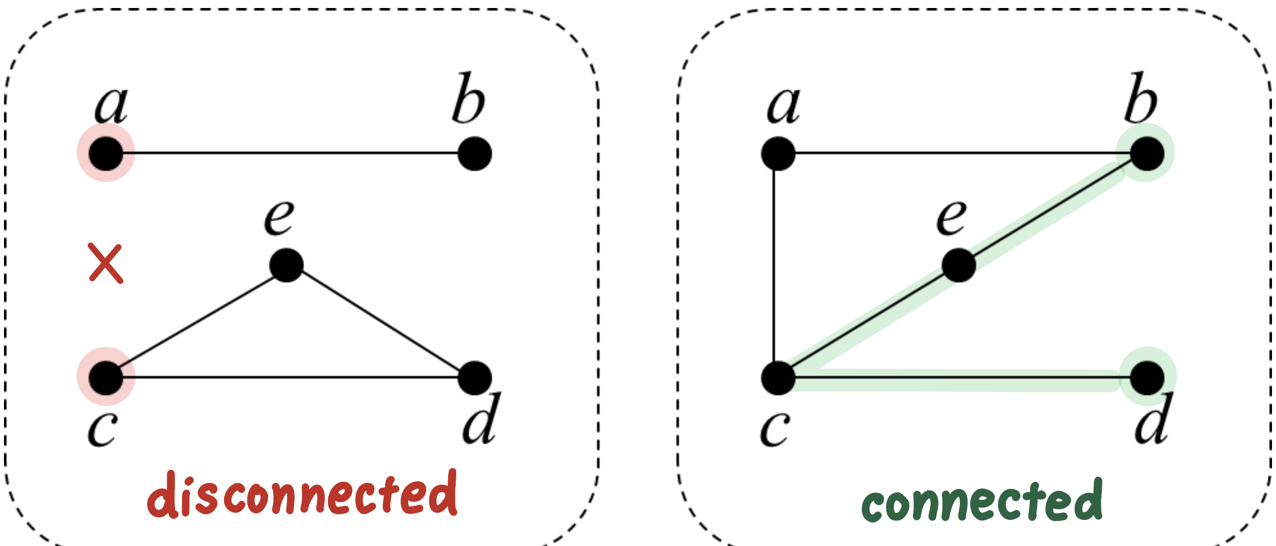
- A path is a **circuit** if it begins and ends at the same vertex and the length is not 0.



Circuit (b, c, d, b)

Connectedness

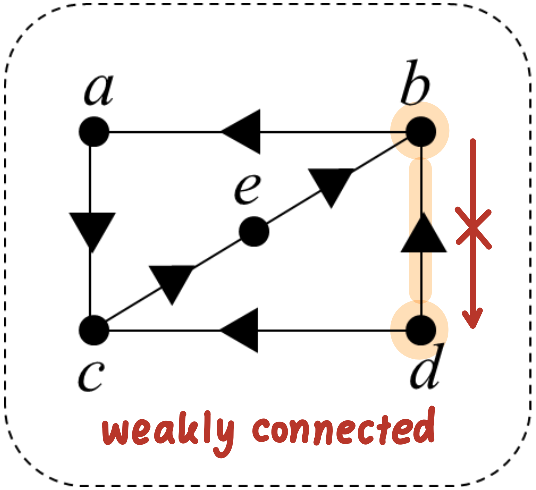
An <u>undirected graph</u> is called **connected** $\leftrightarrow$ There is a path between every pair of distinct vertices of the graph.

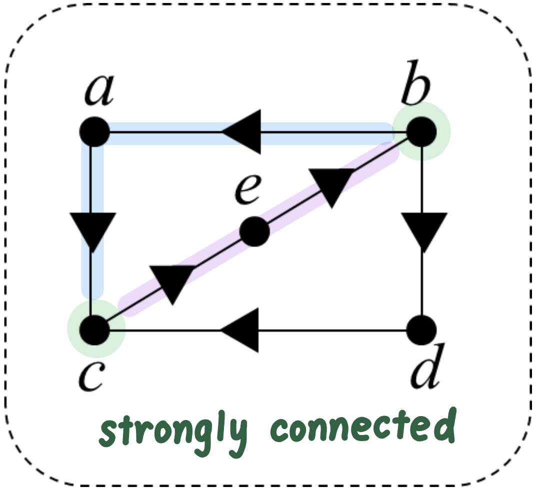


Connectedness

An <u>directed graph</u> is **strongly connected** $\leftrightarrow$ There is a path from a to b and from b to a whenever a and b are distinct vertices in the graph.

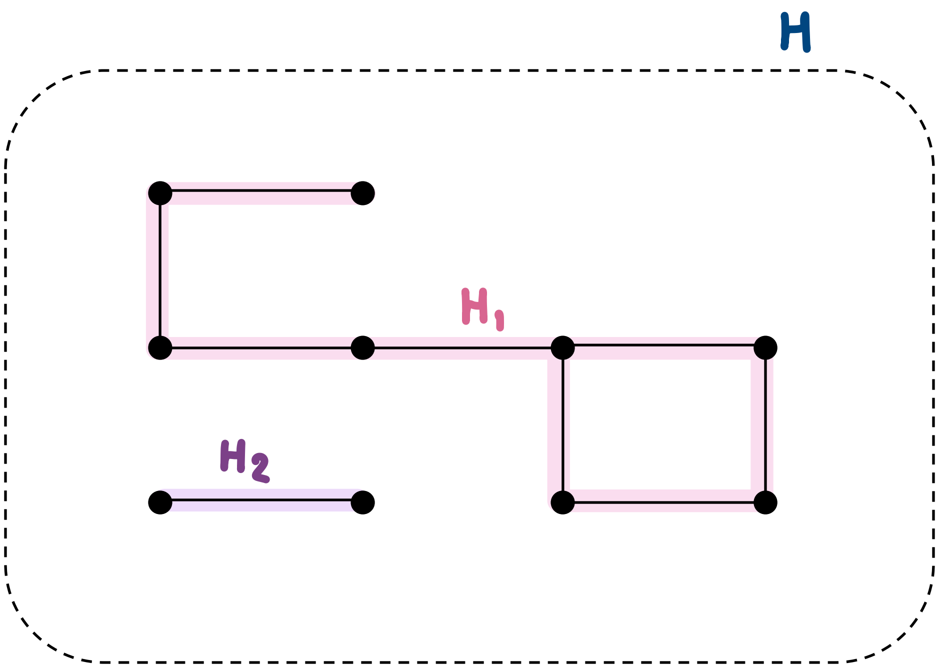
An <u>directed graph</u> is **weakly connected** $\leftrightarrow$ There is a path between every pair of distinct vertices of the graph. (Disregard the edge directions.)





Connected Components

- A **connected component** of a graph is maximal connected subgraph of that graph.



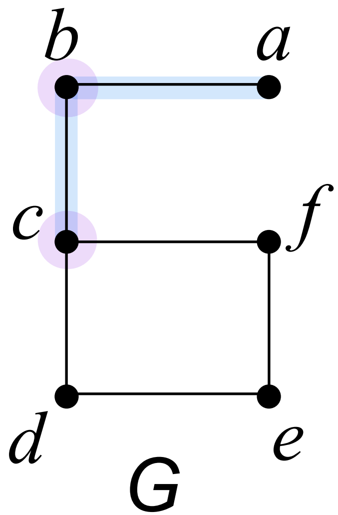
How many connected components are there in a simple graph with the following adjacency matrix?

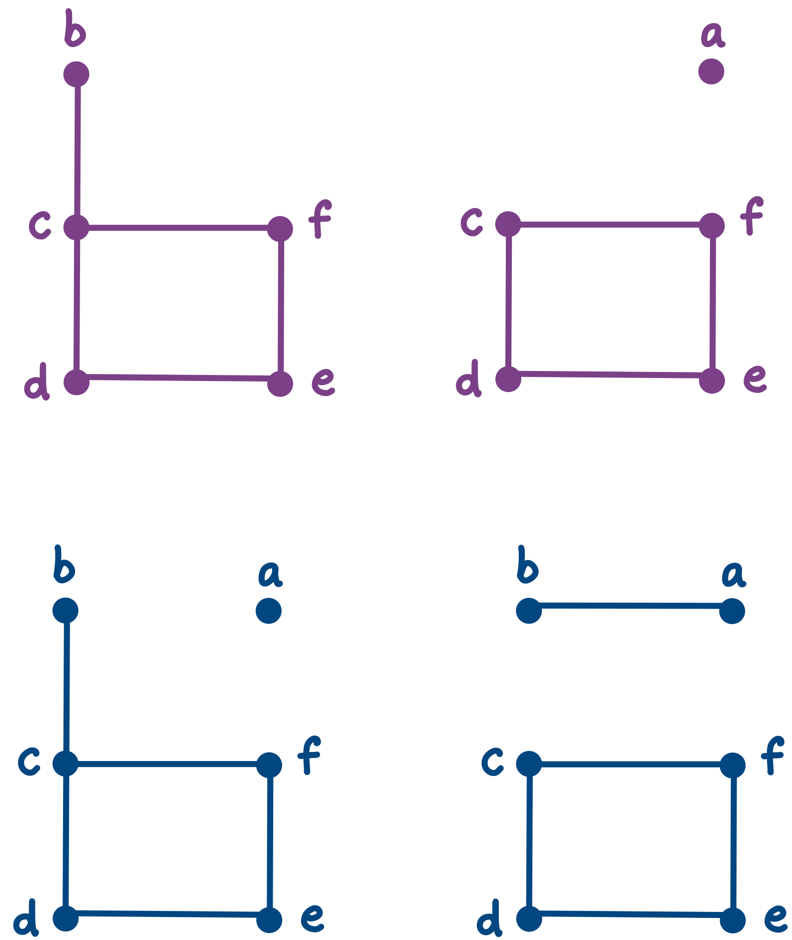
a	b	c	d	e	f	g	h	i	j
	1		1	1			1		1
1				1					1
					1	1			
1				1					1
1	1		1				1		1
		1				1			
		1			1				
1				1					1
1	1		1	1			1		

Cut Vertex / Cut Edge

- A vertex is called a **cut vertex** (or **articulation point**) if the removal of that vertex along with all edges incident with it produces a subgraph with more connected components.
- An edge whose removal produces a subgraph with more connected components is called a **cut edge** (or **bridge**).

Find all cut vertices and cut edges in G.





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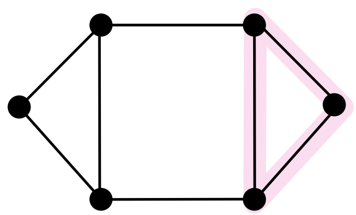
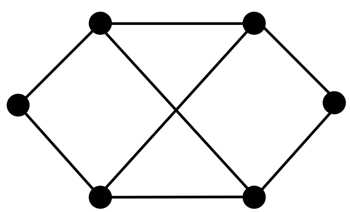
Paths and Isomorphism

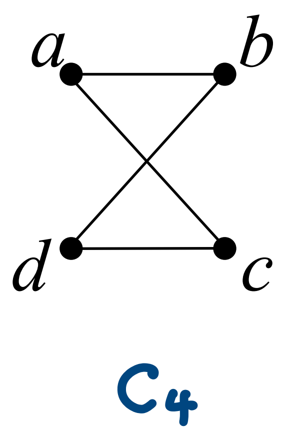
- The existence of a simple circuit of a particular length is a graph invariant.
- Paths are sometimes useful in finding an isomorphism.





Counting Paths Between Vertices

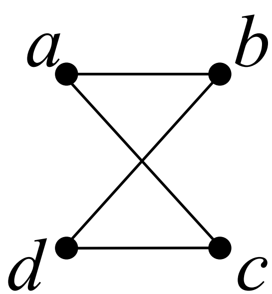
How many paths of length 4 are there from a to d in the following graph?



Counting Paths Between Vertices

How many paths of length 4 are there from a to d in the following graph?

8



$(a, b, a, b, d), (a, b, a, c, d),$
$(a, b, d, b, d), (a, b, d, c, d),$
$(a, c, a, c, d), (a, c, a, b, d),$
$(a, c, d, c, d), (a, c, d, b, d)$

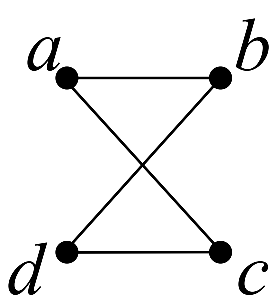
Counting Paths Between Vertices

The number of paths of a certain length between two vertices in a graph can also be determined from the graph's adjacency matrix.

Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

Counting Paths Between Vertices

How many paths of length 4 are there from a to d in the following graph?



$$A = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

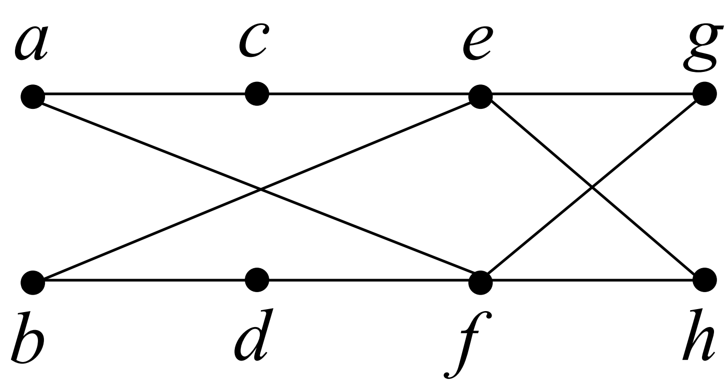
$$A^4 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 8 & 0 & 0 & 8 \\ b & 0 & 8 & 8 & 0 \\ c & 0 & 8 & 8 & 0 \\ d & 8 & 0 & 0 & 8 \end{array}$$

Section 10.5

- Euler Circuits

Euler Circuits

An **Euler circuit** in a graph is a *simple* circuit containing *every edges* of that graph.



An Euler circuit =

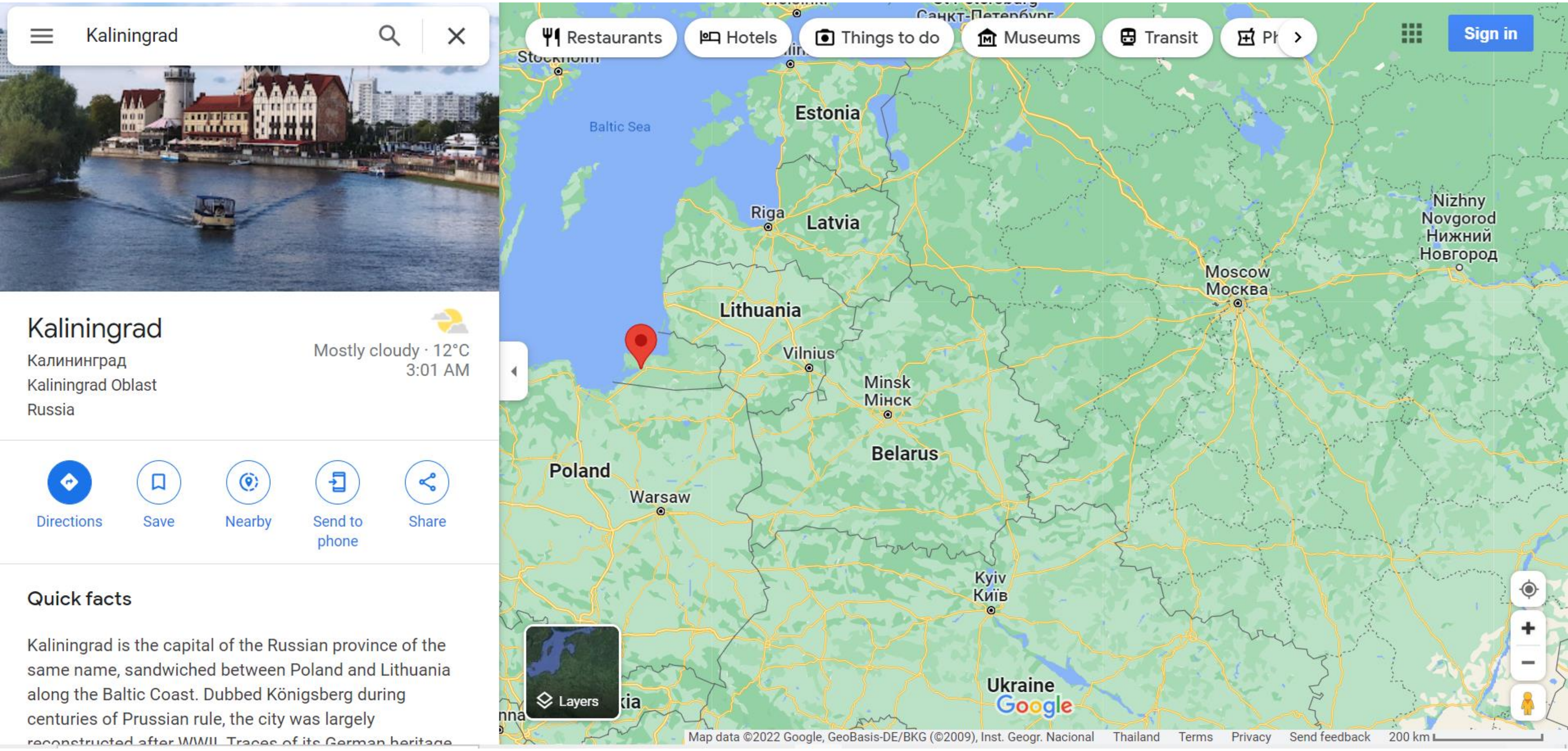
a c e g f h e b d f a

e g f h e b d f a c e

THE SEVEN BRIDGES OF

KÖNIGSBERG

IS IT POSSIBLE TO TOUR THE CITY, CROSSING EACH BRIDGE EXACTLY ONCE?


Kaliningrad
Restaurants
Hotels
Things to do
Museums
Transit
Sign in
Kaliningrad
Калининград
Kaliningrad Oblast
Russia
Mostly cloudy · 12°C
3:01 AM
Directions
Save
Nearby
Send to phone
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Quick facts
Kaliningrad is the capital of the Russian province of the same name, sandwiched between Poland and Lithuania along the Baltic Coast. Dubbed Königsberg during centuries of Prussian rule, the city was largely
Baltic Sea
Estonia
Riga
Latvia
Lithuania
Vilnius
Minsk
Минск
Belarus
Poland
Warsaw
Moscow
Москва
Nizhny
Novgorod
Нижний
Новгород
Kyiv
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Map data ©2022 Google, GeoBasis-DE/BKG (©2009), Inst. Geogr. Nacional
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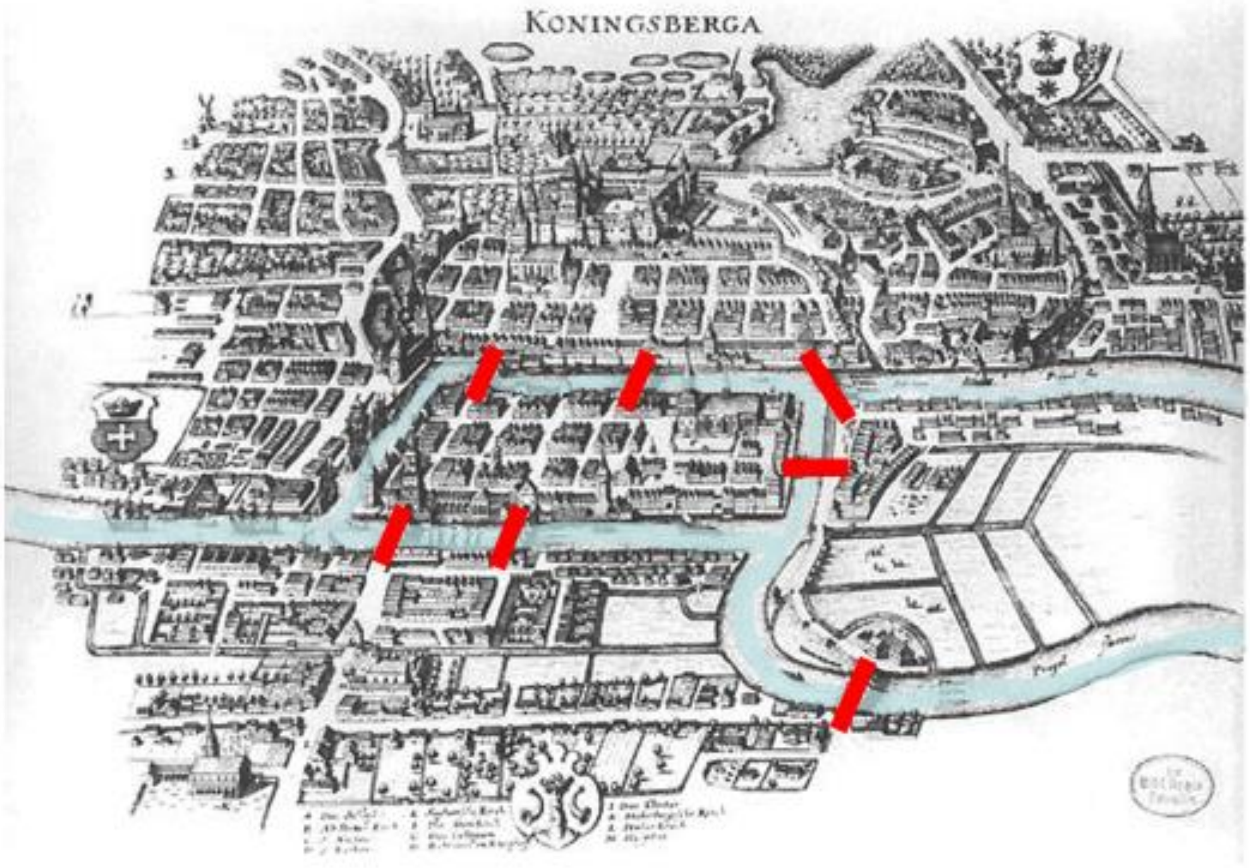
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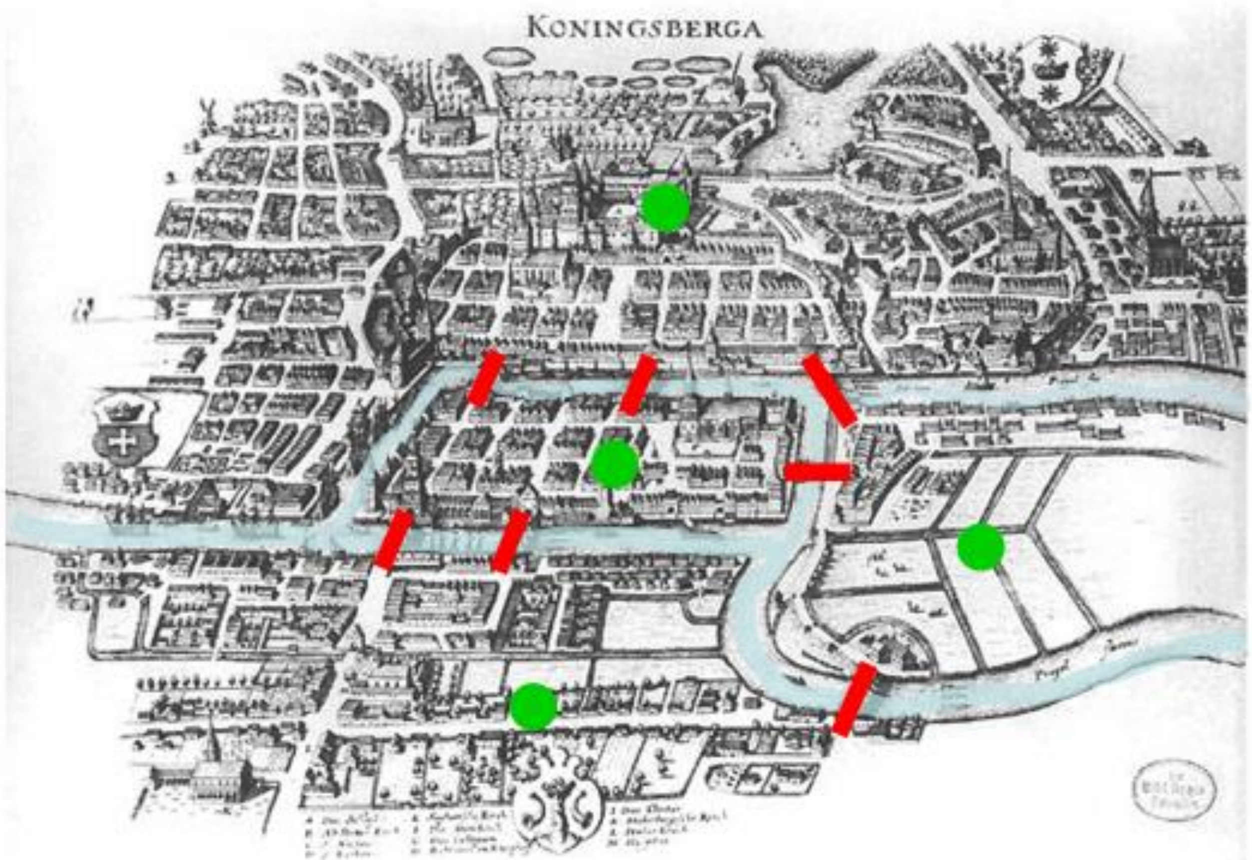
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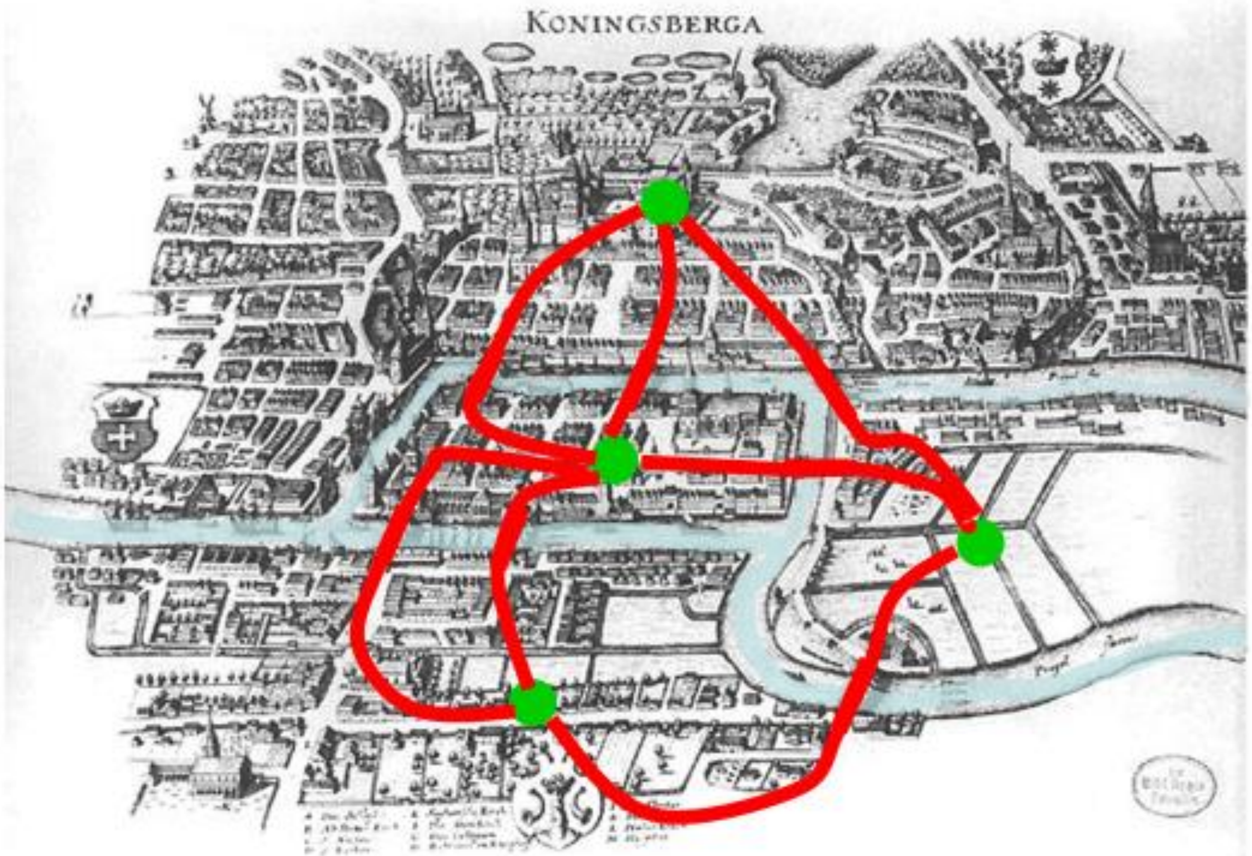
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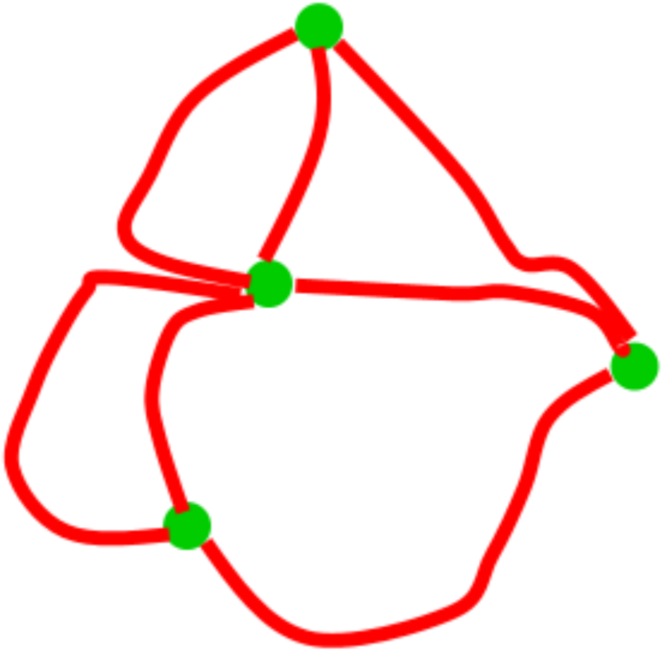

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LEONHARD EULER (1707-1783)

Conditions for Euler Circuits

A connected multigraph with at least two vertices has an Euler circuit $\leftrightarrow$ each of its vertices has even degree.

Proof:

Necessary condition

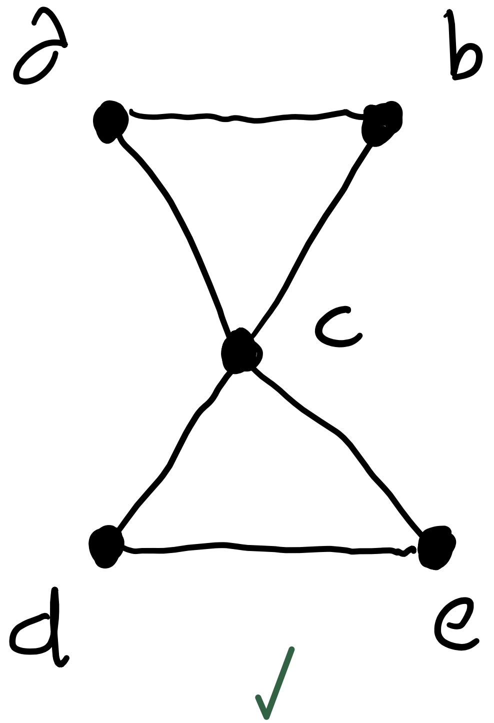
G has an Euler Circuit $\rightarrow$ each of its vertices must have even degree.

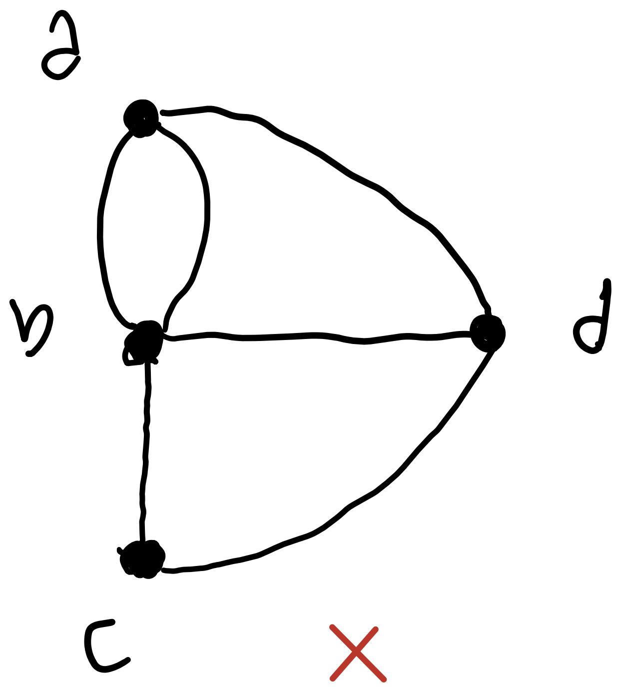
Sufficient condition

Each of the vertices in G has even degree $\rightarrow$ G has an Euler Circuit.

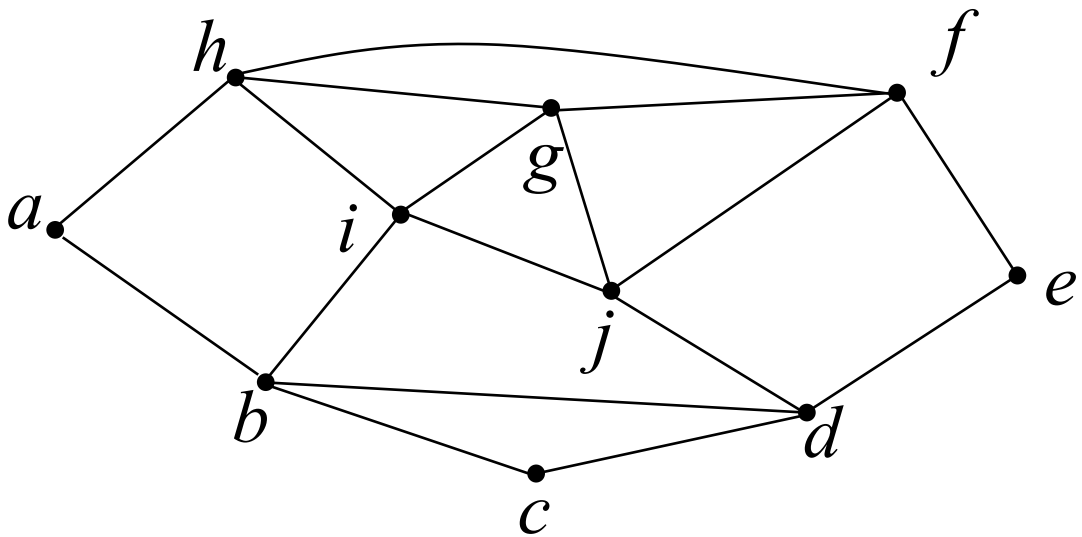
G has an Euler Circuit → each of its vertices must have even degree.

Some vertex has odd degree → G does not have Euler circuit

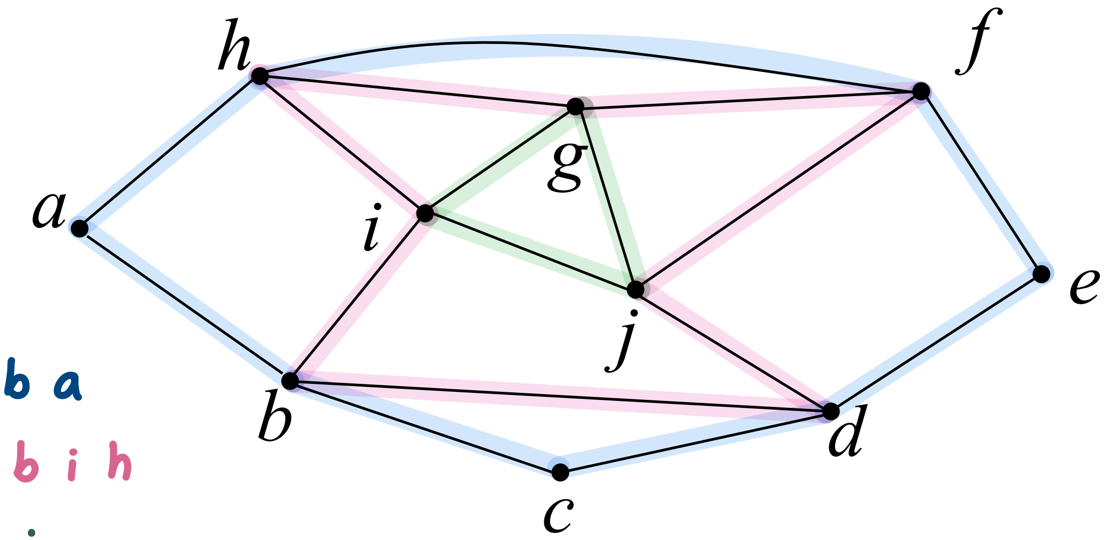




Each of the vertices in *G* has even degree $\rightarrow$ *G* has an Euler Circuit.



Finding an Euler Circuit

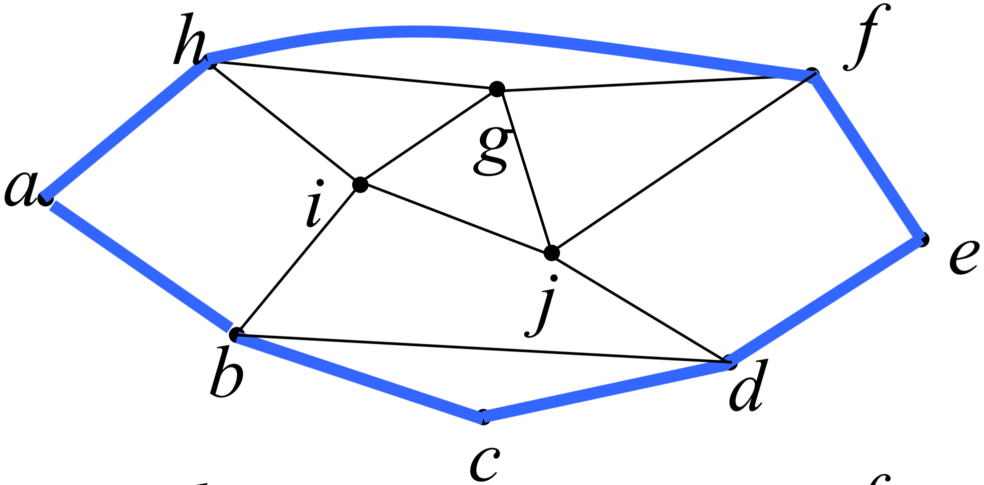


a f e d c b a

h g f d b i h

j i g j

a h g f j i g j d b i h f e d c b a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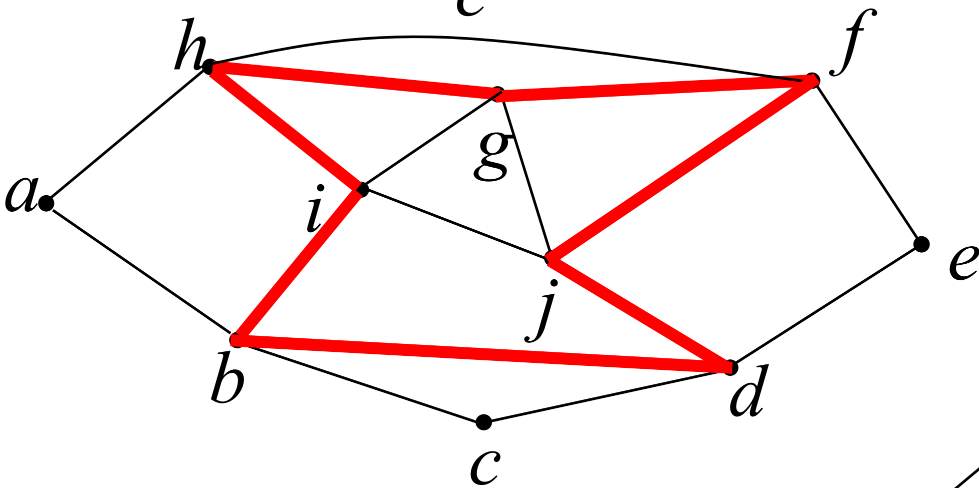
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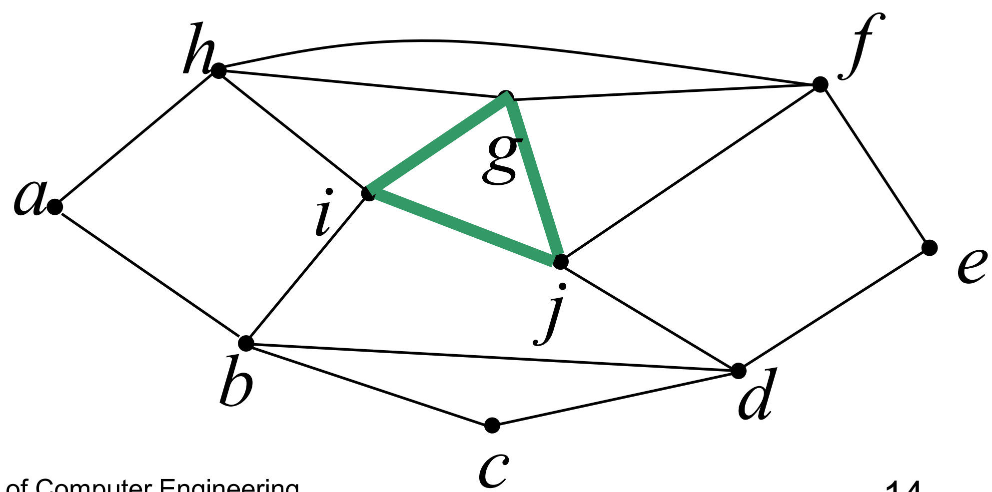
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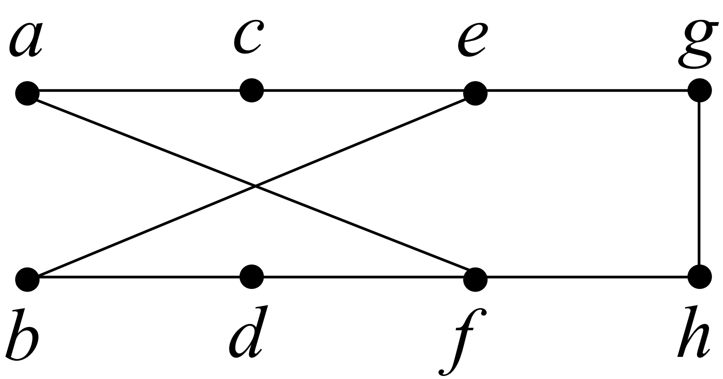
Section 10.5

- Euler Paths
- Hamilton Paths & Circuits

Euler Paths

An **Euler path** in a graph is a *simple* path containing *every edges* of that graph.

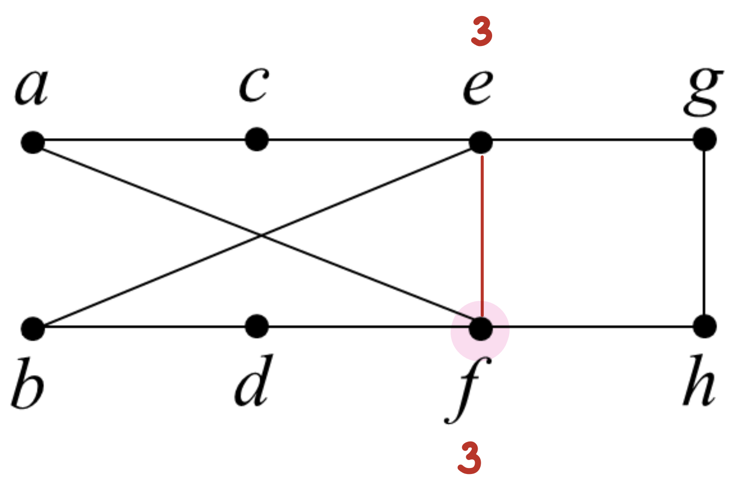


An Euler path =

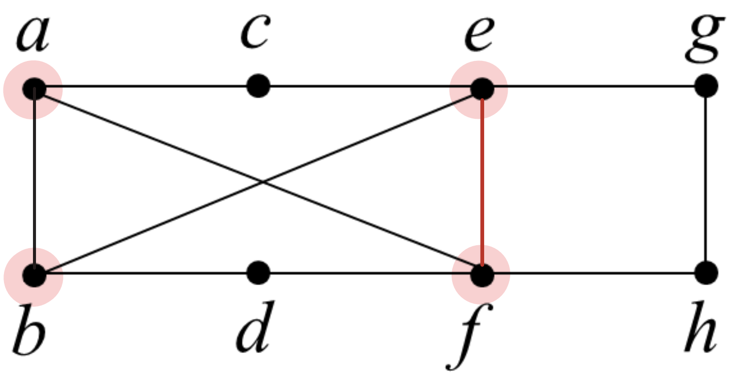
e b d f h g e c a f

Conditions for Euler Paths

A connected multigraph with at least two vertices has an Euler path but not an Euler circuit ↔ it has exactly 2 vertices with odd degree.



e b d f h g e c a f ~~e~~

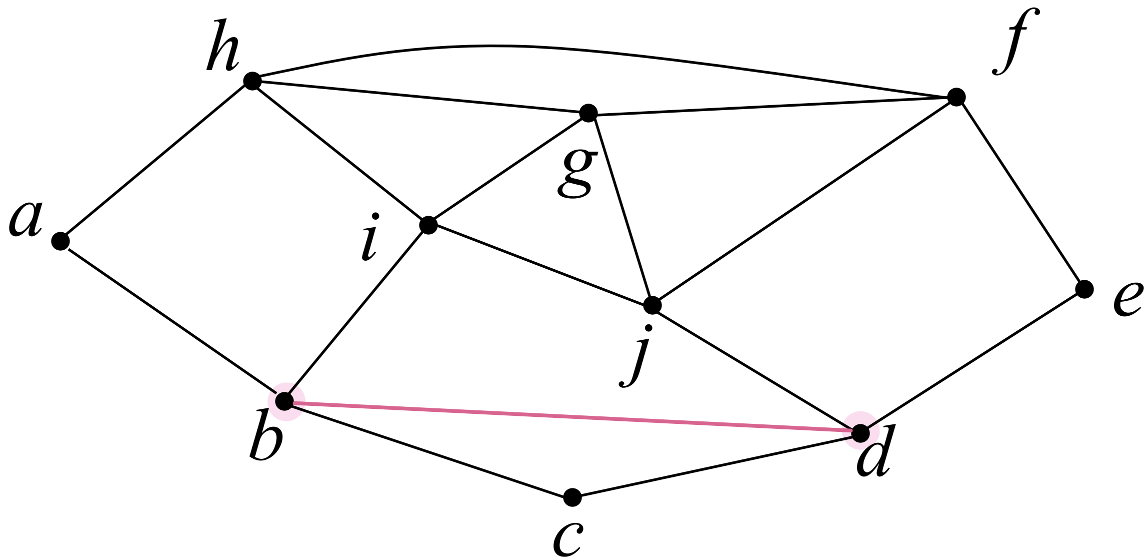


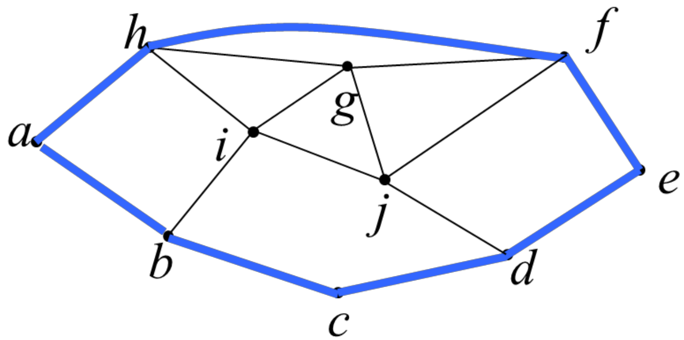
vertex degree คี่ มีได้ 2,4,6,8,... X

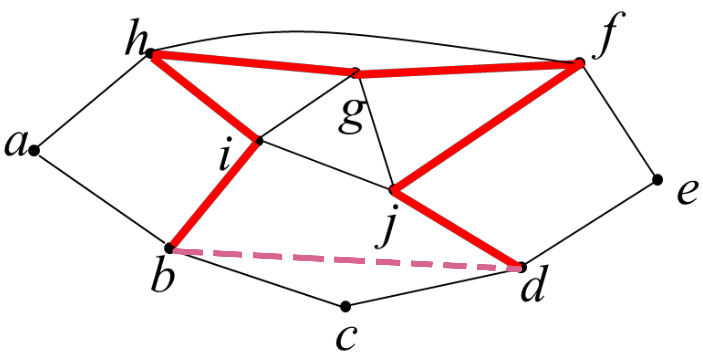
มีได้ 2 เท่านั้น

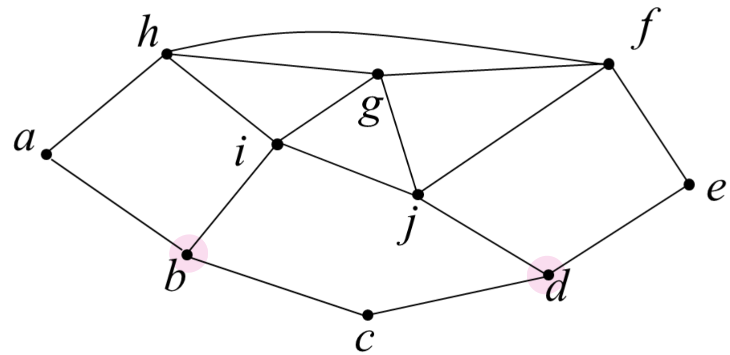
เติมเส้นเชื่อม ระหว่าง 2 จุด เป็นหัวท้าย

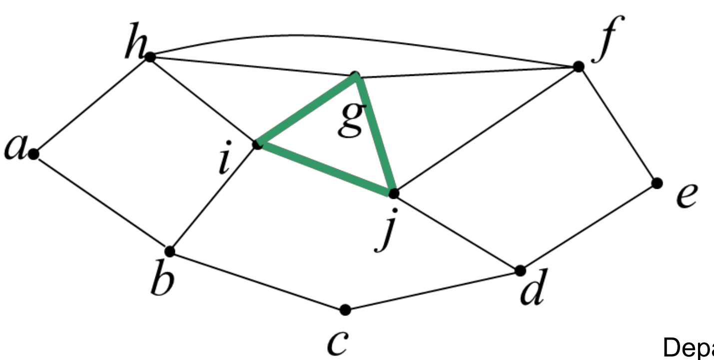
Finding an Euler Path









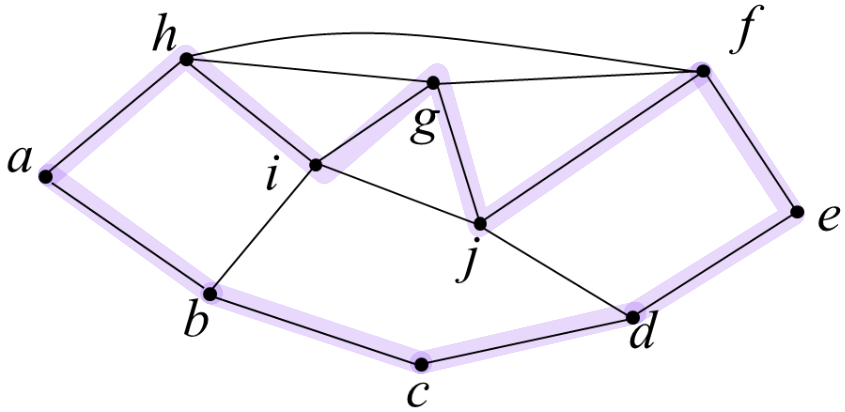


b a h f e d c b i g j i h g f j d

Hamilton Paths and Circuits

A **Hamilton path** in a graph is a *simple* path that passes through *every vertex* of the graph *exactly once*.

For $G=(V,E)$ and $V = \{v_1, v_2, \ldots, v_n\}$, the simple circuit $v_1,v_2,\ldots,v_n,v_1$ is a **Hamilton circuit** if $v_1,v_2,\ldots,v_n$ is a Hamilton path.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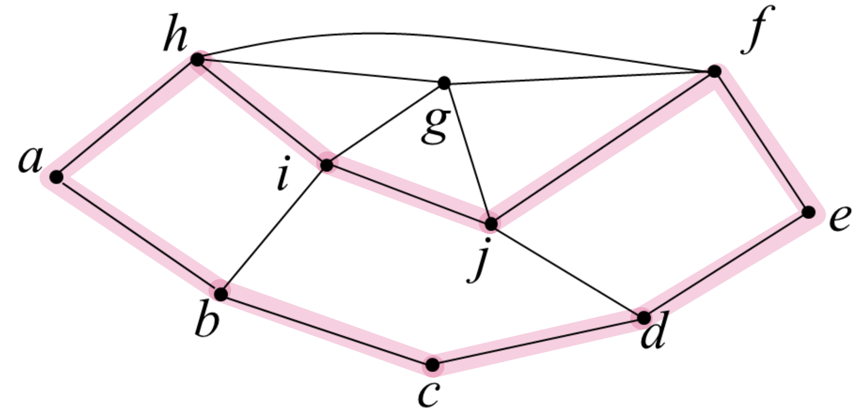
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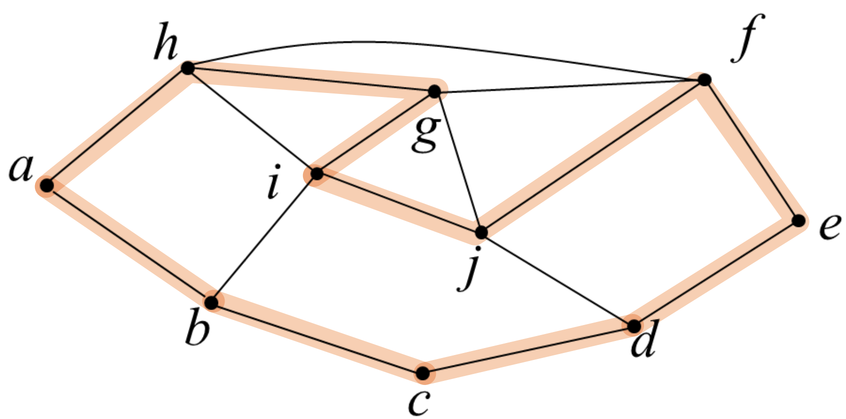
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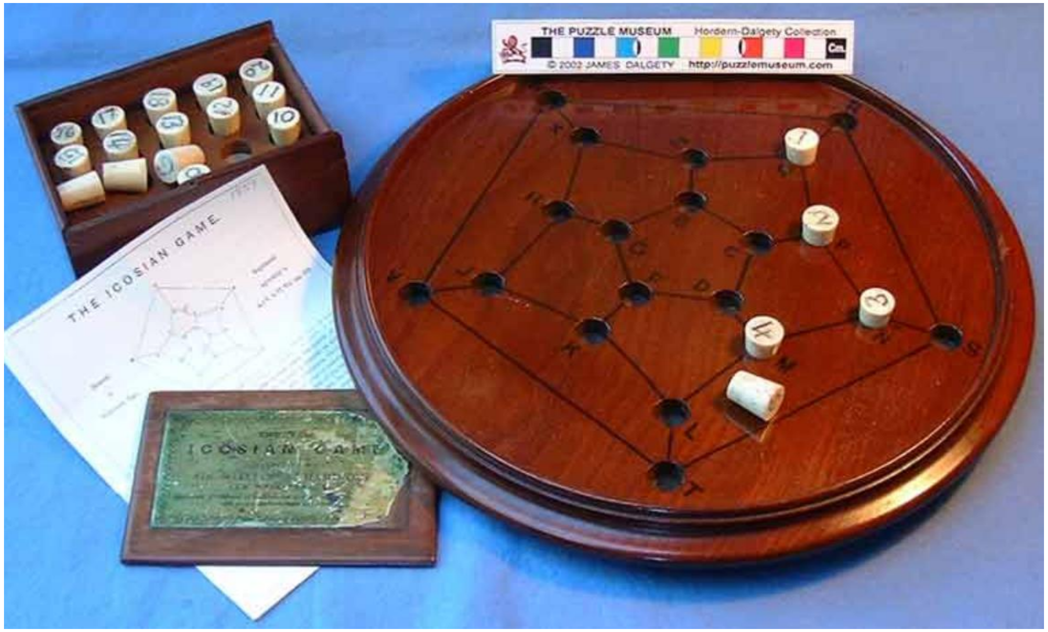
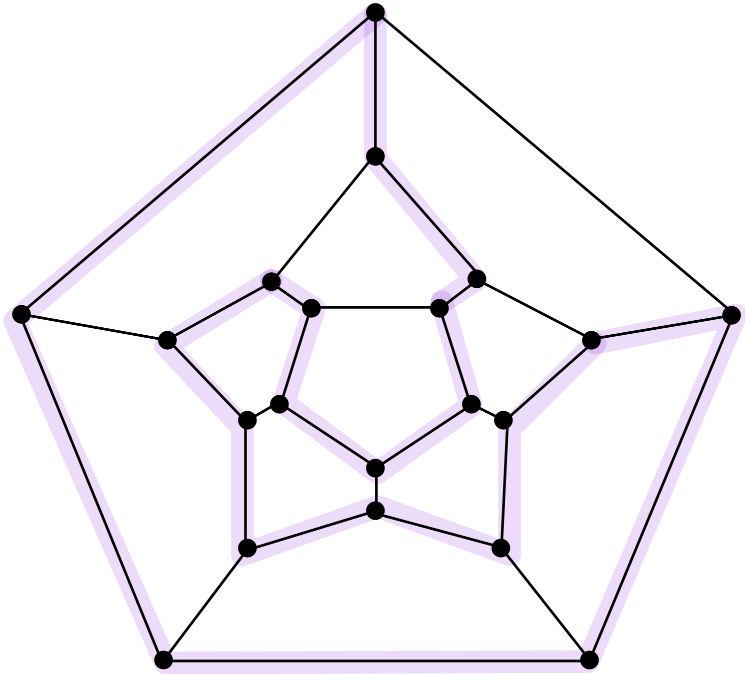
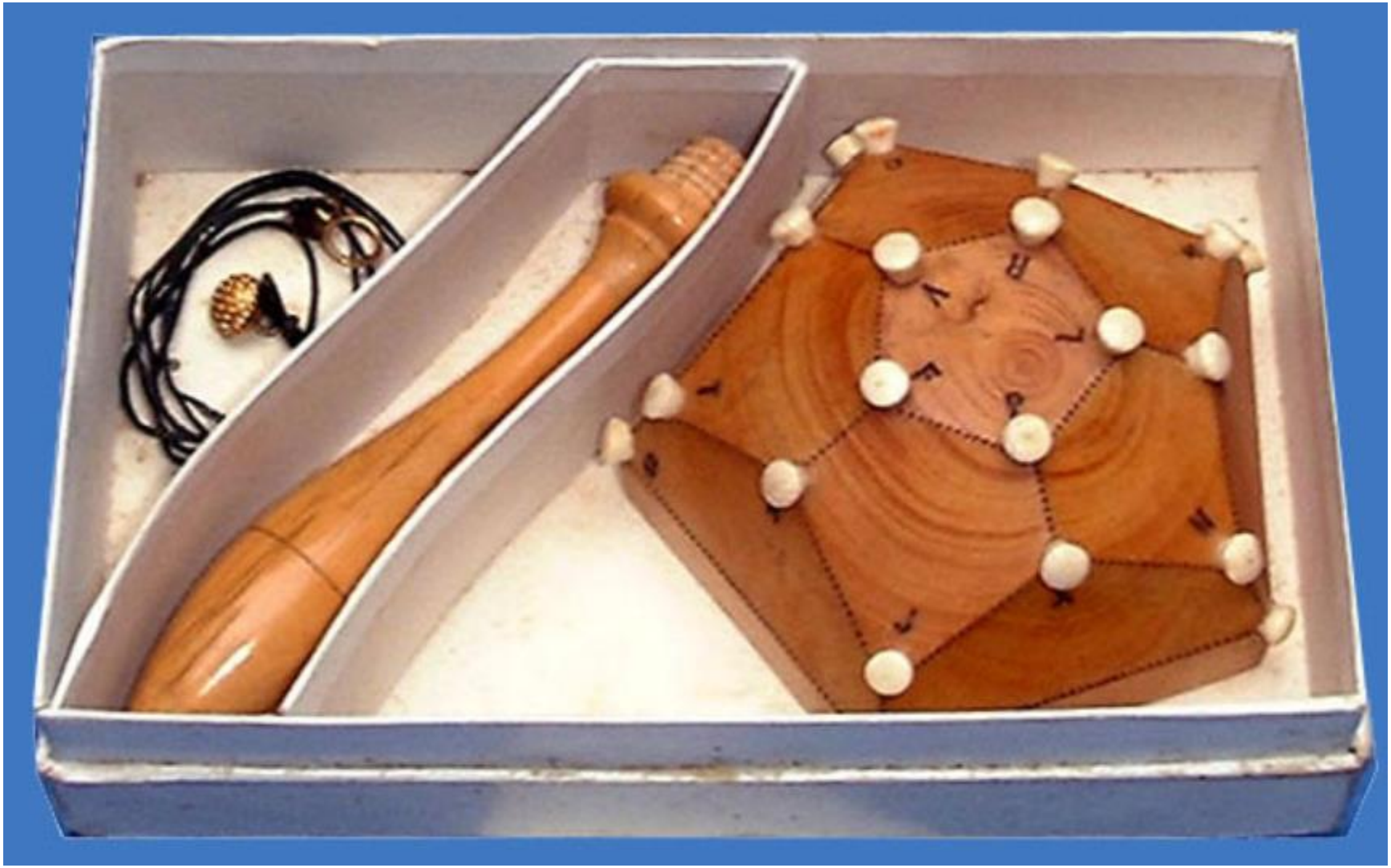
Iconian Puzzle



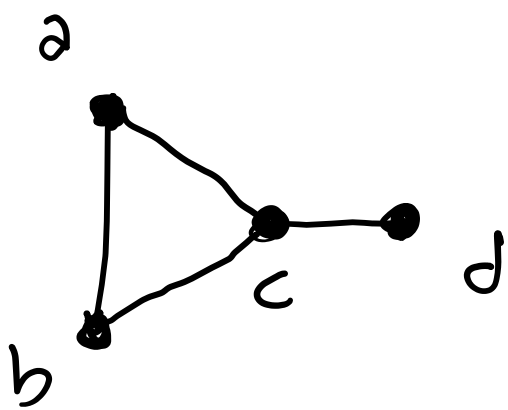


William Rowan Hamilton



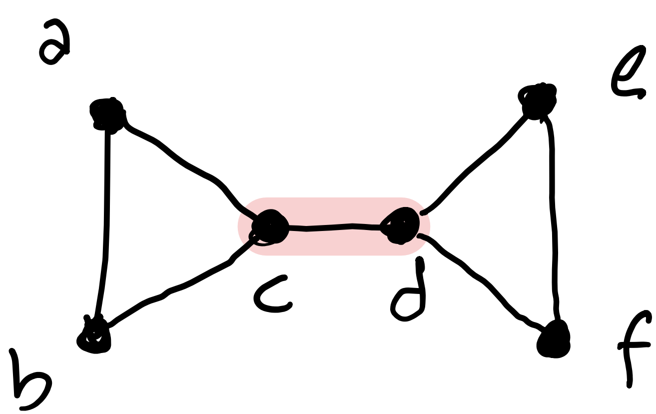
Conditions for Hamilton Circuits

- No 'necessary & sufficient' conditions exist.
- Certain properties can be used to show that no Hamilton circuits exist. E.g. degree one vertex.



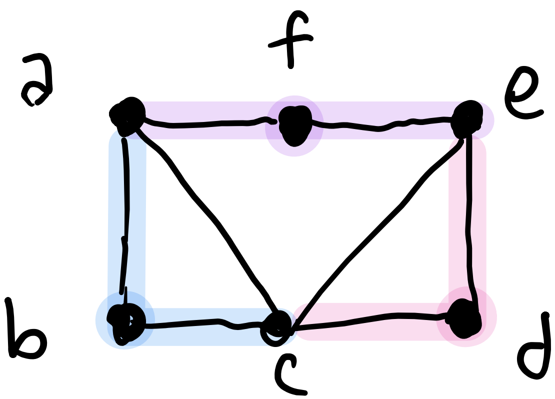
Conditions for Hamilton Circuits

- No Hamilton circuit exists if there is a cut vertex in the graph.



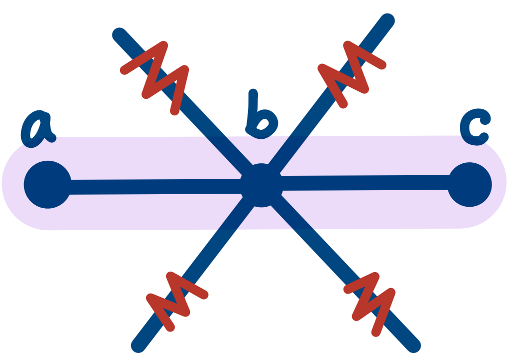
Conditions for Hamilton Circuits

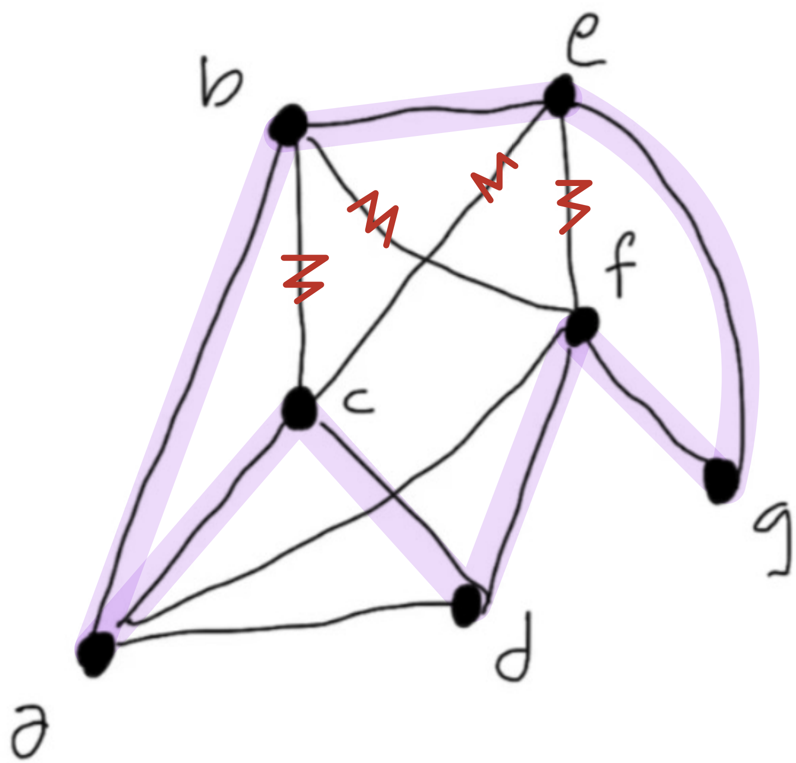
- Both edges incident of a vertex of degree two must be part of any Hamilton circuit.



Conditions for Hamilton Circuits

- While constructing a Hamilton circuit, if a vertex has already passes through, all remaining edges of that vertex can be removed from consideration.

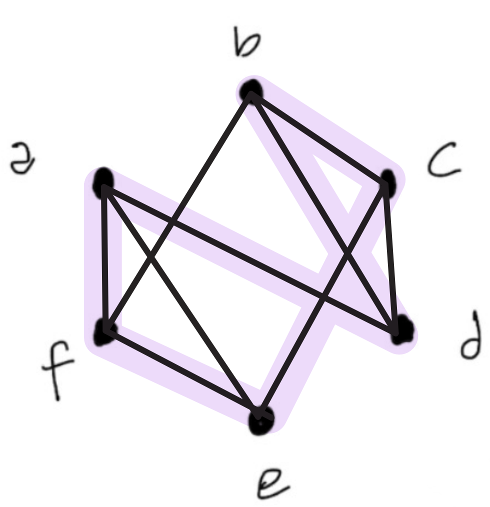


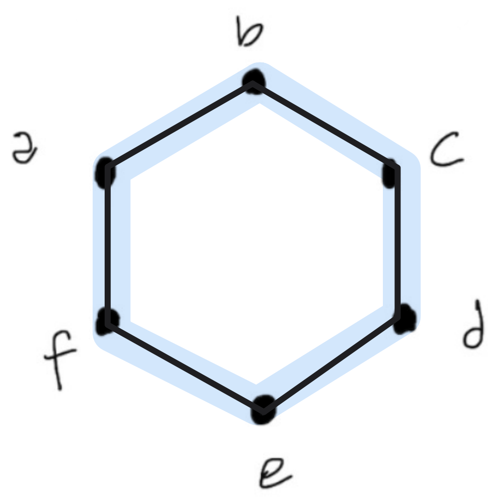

a
b
c
d
e
f
g

Some Sufficient Conditions

Dirac's Theorem

If G is a simple graph with n vertices ($n \geq 3$) such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.





Some Sufficient Conditions

Ore’s Theorem

If G is a simple graph with n vertices ($n \geq 3$) such that $\deg(u)+\deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.