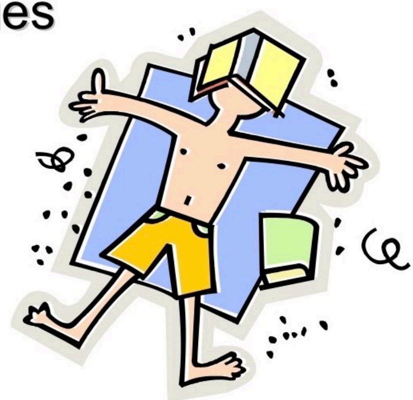
Sections 5.1-5.3

Counting Techniques

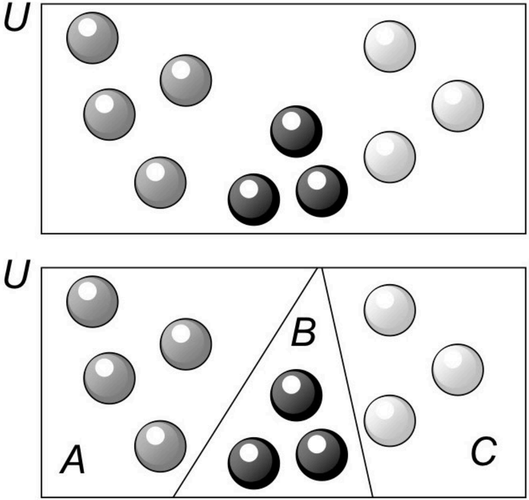


5.1 The Basics of Counting

5.2 The Pigeonhole Principle

5.3 Permutations and Combinations

Counting by Cases



The Sum Rule

$\# = N_A + N_B + N_C$

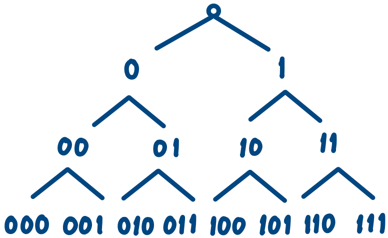
1) $(A \cap B = \emptyset) \wedge (A \cap C = \emptyset) \wedge (B \cap C = \emptyset)$

2) $A, B, C \in U$ แบ่งครบ ไม่มี D,E,F,... เพิ่ม

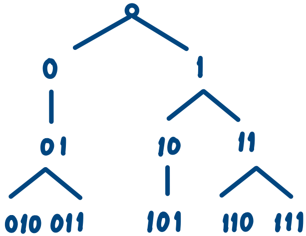
Tree Diagram

- help counting things composing of successive steps

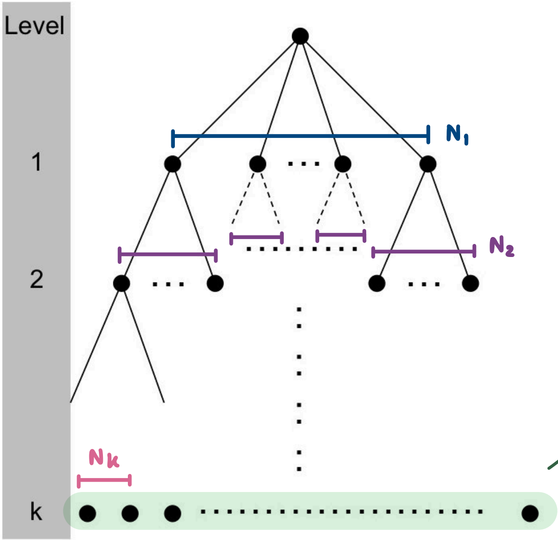
E.g.: List all bit strings w/ 3 bits.



E.g.: List all bit strings w/ 3 bits but without ’00’



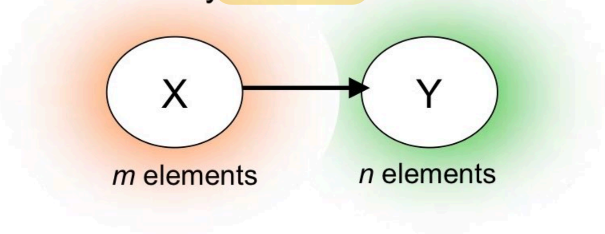
The Product Rule

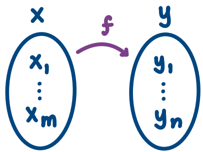


$$N = N_1 \times N_2 \times \ldots \times N_k$$

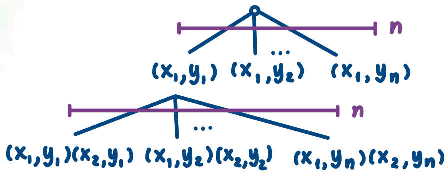
Example

How many functions?





$\{(x_1, \ldots), (x_2, \ldots), \ldots (x_m, \ldots)\}$



$\# = n^m$

How many one-to-one function?

$n \times (n-1) \times \ldots \times (n-(m-1))$

Basic Counting Principles

- <u>Example</u> : Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$

Solution: Let S be a finite set. List the elements of S in arbitrary order. Recall from Section 2.2 that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. Namely, a subset of S is associated with the bit string with a 1 in the ith position if the ith element in the list is in the subset, and a 0 in this position otherwise. By the product rule, there are $2^{|S|}$ bit strings of length $|S|$. Hence, $|P(S)| = 2^{|S|}$. (Recall that we used mathematical induction to prove this fact in Example 10 of Section 5.1.) ◀

Example:

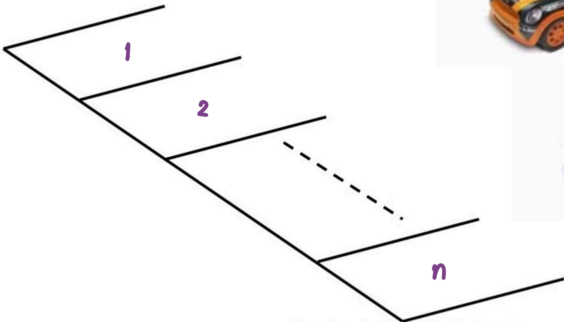
A password can contain 6 to 8 characters. Each character can be A-Z. How many possible passwords are there?

$26^6 + 26^7 + 26^8$

Example:

A parking lot consists of a single row of *n* parking spaces. Only two cars park in this parking lot. How many ways can they park?

$n \times (n-1)$



Example:

How many ways can they park if there can be at most one empty space between them?

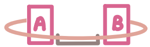
กรณีที่ 1 A กับ B จอดติดกัน

N_1

$2 \times (n-1)$

กรณีที่ 2 A กับ B จอดแล้วมี 1 ช่อง ระหว่างกัน

N_2

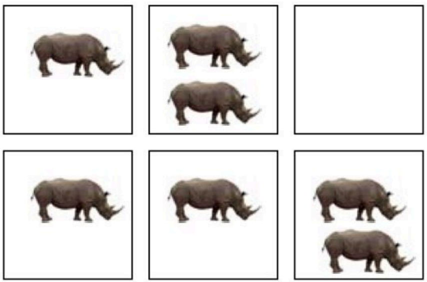


$2 \times (n-2)$

$\# = N_1 + N_2 = 2(n-1) + 2(n-2)$

The Pigeonhole Principle

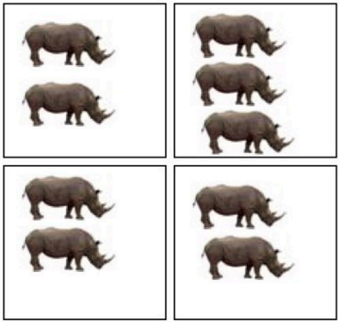
If *k+1* or more objects are placed into *k* boxes, then there are *at least one box containing two or more objects.*



6 boxes
7 objects

The Pigeonhole Principle

If N objects are placed into k boxes, then there is *at least one box containing at least* $\lceil N/k \rceil$ *objects.*



4 boxes
9 objects
$\lceil 9/4 \rceil = 3$

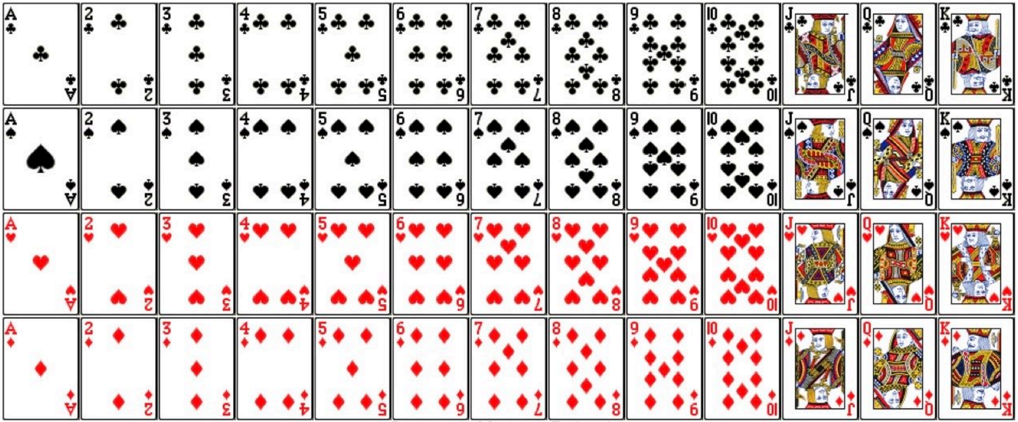
There is at least one box that contains at least 3 objects.

- Example:

  How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

$$\left\lceil \frac{N}{4} \right\rceil = 3$$

$$\therefore N = 9$$



Permutations การเรียงลำดับ

- An ordered arrangement of *r* elements of a set is called an ***r-permutation***
- E.g.: S = {1,2,3}

  1,2 is a 2-permutation of S

  2,1 is another 2-permutation of S

  3,2 is also another 2-permutation of S

  1,2,3 is a permutation of S

  2,1,3 is another permutation of S

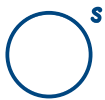
Permutations

The number of *r-permutations* of a set with *n* distinct elements is:

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

<u>Proof</u>:

$$\underbrace{\underset{\_\_\_\_}{n}\quad \underset{\_\_\_\_}{n-1}\quad \_\_\_\_\quad \cdots\quad \_\_\_\_}_{r}$$



$|S| = n$

- <u>Example</u>

  How many ways are there to select a 1^{st}-prize winner, a 2^{nd}-prize winner, and a 3^{rd}-prize winner from 100 people?

$$P(100,3) = \frac{100!}{97!} = 100 \times 99 \times 98$$

Combinations

- An ***r-combination*** of elements of a set is an <u>unordered</u> selection of r elements from the set.
- Or a subset, with r elements, of the set.

E.g.: S = {1,2,3,4}

{1,2,3} is a 3-combination of S

{3,2,1} is the same as {1,2,3}

Combinations

The number of *r-combinations* of a set with *n* distinct elements is:

$$C(n,r) = n! / r!(n-r)!$$

Proof:

$|S| = n$ → เลือกมา r → เอามาเรียงลำดับ

$$\underbrace{\_\_\ \_\_\ \_\_\ \cdots\ \_\_}_{r}$$

$$\frac{n!}{(n-r)!} = C(n,r) \times r!$$

$$\frac{n!}{(n-r)!\,r!} = C(n,r) = \binom{n}{r}$$

- <u>Example</u>:

  How many ways are there to select a 3 prize winners from 100 people (when the three prizes are identical)?

$$C(100,3) = \binom{100}{3} = \frac{100!}{3!\,97!} = \frac{100 \times 99 \times 98}{6}$$

- Example:

  How many bit strings of length *10* contain more than 2 ones?

$2^{10} - \overset{1 \text{ ตัว}}{10} - \overset{0 \text{ ตัว}}{1}$

Example:

How many subsets of three different integers between 1 to 90 (inclusive) are there whose sum is an even number?

<u>กรณีที่ 1</u> มี 2 คี่ 1 คู่

$C(45,2) \times C(45,1)$

<u>กรณีที่ 2</u> มี 3 คู่

$C(45,3)$

$\therefore\ C(45,2) \times C(45,1) + C(45,3)$ #

Permutations with Indistinguishable Objects

- <u>Example</u>:

  How many different strings can be made by reordering the string "*ABCDEFGHIJ*" ?

  $10!$

  How many different strings can be made by reordering the letters of the word PPPEERRCON

  "*PEPPERCORN*"

$$= \underset{P}{C(10,3)} \times \underset{E}{C(7,2)} \times \underset{R}{C(5,2)} \times \underset{C}{C(3,1)} \times \underset{O}{C(2,1)} \times \underset{N}{C(1,1)}$$

$$= \frac{10!}{3!\,\cancel{7!}} \times \frac{\cancel{7!}}{2!\,\cancel{5!}} \times \frac{\cancel{5!}}{2!\,\cancel{3!}} \times \frac{\cancel{3!}}{1!\,\cancel{2!}} \times \frac{\cancel{2!}}{1!\,1!} \times 1$$

$$= \frac{10!}{3!\,2!\,2!}$$

Permutations with Indistinguishable Objects

- The number of different ***permutations*** of ***n*** objects, where there are

  n_1 indistinguishable of type *1*,

  n_2 indistinguishable of type *2*,…, and

  n_k indistinguishable of type *k*,

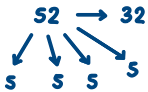
  is:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Distributing Objects into Boxes

- Example:

  How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52?



$$= C(52,5) \times C(47,5) \times C(42,5) \times C(37,5)$$

$$= \frac{52!}{5!\,\cancel{47!}} \times \frac{\cancel{47!}}{5!\,\cancel{42!}} \times \frac{\cancel{42!}}{5!\,\cancel{37!}} \times \frac{\cancel{37!}}{5!\,32!}$$

$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

Distributing Objects into Boxes

- The number of ways to distribute ***n*** distinguishable objects into k distinguishable boxes so that n_i objects are placed into box *i*, *i* =*1,2,…,k*, equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Section 5.5

Generalized Permutations / Combinations

Permutations with Repetition

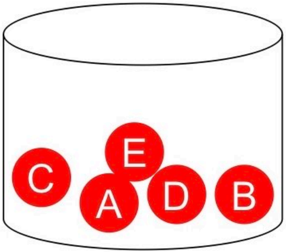
- Easily done using product rule.
- Example:

  How many strings of length n can be formed from the English alphabets, if each alphabet can be used no more than once?

  How many strings can be formed, if repetition is allowed?

- <u>Example</u>:

A bucket containing 5 different balls. We pick a series of 3 balls randomly from the bucket.

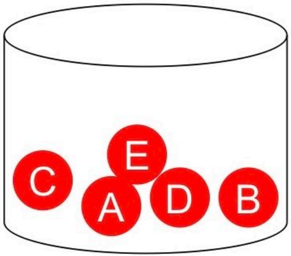


How many sequences of 3 balls are there that we can draw from the bucket? (without putting any balls back in) $5 \times 4 \times 3 = 60$

How many sequences if we put the drawn ball back in before we draw another ball? $5 \times 5 \times 5 = 75$

- Example:

A bucket containing 5 different balls. We pick 3 balls randomly from the bucket at the same time.

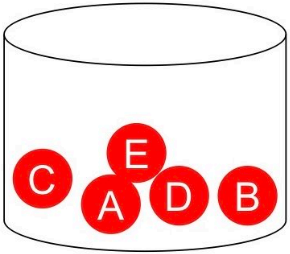


How many ways can the balls be selected?

$C(5,3) = \frac{5!}{3!2!} = 10$

- <u>Example</u>:

  A bucket containing 5 different balls. We pick 3 balls randomly from the bucket at the same time.



How many ways can the balls be selected, if repetitions are allowed?

กรณี 1 ไม่ซ้ำเลย $C(5,3) = \frac{5!}{3!2!} = 10$

กรณี 2 ซ้ำคู่หนึ่ง $5 \times 4 = 20$

กรณี 3 ซ้ำ 3 เลย 5

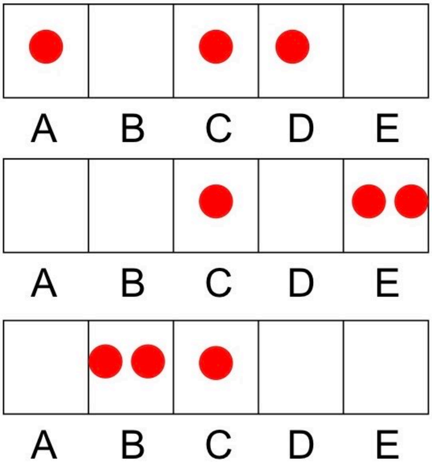
$\therefore 10 + 20 + 5 = 35$ #

i.e.: We draw one ball at a time and put the drawn ball back in before drawing another one, while we do not care about the order.

or

Instead of only five balls in the bucket, there are five types of balls where there are more than 3 balls for each type.

Combinations with Repetition



A,C,D → o||o|o|

C,E,E → ||o||oo

B,B,C → |oo|o||

$$C(7,3) = \frac{7 \times \cancel{6} \times 5}{\cancel{3} \times \cancel{2}} = 35$$

Combinations with Repetition

- There are ***c(n-1+r,r)*** *r-combinations* from a set with n elements when repetition of elements is allowed.
- Example:

  There are 4 types of cookies in a cookie shop. How many ways can 6 cookies be chosen?

$C(4-1+6, 4-1) = C(9,3) = C(9,6)$

- <u>Example</u>:

  How many solutions does the equation

  $x_1+x_2+x_3 = 11$

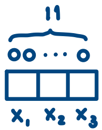
  have, where x_1, x_2, and x_3 are nonnegative integer?

<u>ตัวอย่างคำตอบ</u>

$x_1 = 2,\ x_2 = 3,\ x_3 = 6$

$x_1 = 2,\ x_2 = 6,\ x_3 = 3$

$x_1 = 0,\ x_2 = 0,\ x_3 = 11$



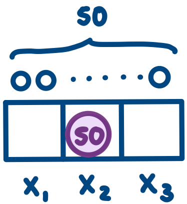
$C(11+3-1, 2)$ #

- <u>Example</u>:

  How many solutions does the equation

  $x_1+x_2+x_3 = 100$

  have, where x_1, x_2, and x_3 are nonnegative integer? $x_2 \geq 50$



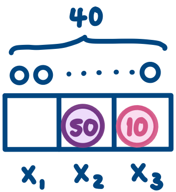
$C(50+3-1, 2)$ #

- <u>Example</u>:

  How many solutions does the equation

  $x_1+x_2+x_3 = 100$

  have, where x_1, x_2, and x_3 are nonnegative integer? $x_2 \geq 50$ $x_3 \geq 10$



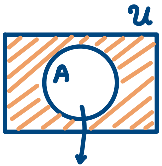
$C(40+3-1, 2)$ #

- <u>Example</u>:

  How many solutions does the equation

  $x_1+x_2+x_3 = 100$

  have, where x_1, x_2, and x_3 are nonnegative integer? $x_2 < 50$



$x_2 \geq 50$

$|U| = C(102,2)$

$|A| = C(52,2)$

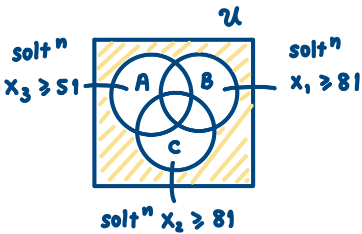
$\therefore \binom{102}{2} - \binom{52}{2}$ #

- Example:

  How many solutions does the equation

  $x_1+x_2+x_3 = 100$

  have, where x_1, x_2, and x_3 are nonnegative integer? $x_1 < 80,\ x_2 < 80,\ x_3 < 50$



$\mathcal{U}$ = set ของ solt^n

$|\mathcal{U}| = ?$

$|A \cup B \cup C| = ?$