Quiz 4A (Module 6)
Recurrence Relations

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ONLY THE ANSWERS IN THE ANSWER SHEET WILL BE GRADED.

Part A: Multiple choice questions (12 points)

Question 1-2: Let the recurrence relation a_n be the number of ternary strings (a string that contains only 0s, 1s, and 2s) of length n that **DO NOT** contain **three consecutive 0s**.

1. Given a_n is in the form of $a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$ Find A, B, and C respectively.
   a. 2, 2, 4
   b. 1, 2 ,4
   c. 1, 2, 3
   d. 2, 2, 2

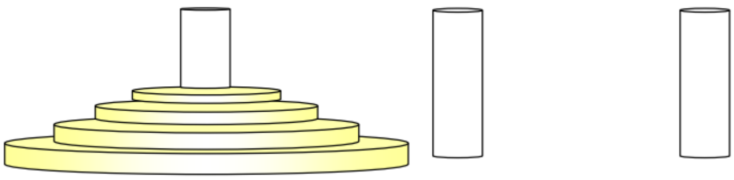
2. How many possible ternary strings of length 5?
   a. 111 b. 222 c. 333 d. 444

Question 3-4: Let the recurrence relation a_n be the number of ways to climb n stairs if the person climbing the stairs can take **one** or **three** stairs at a time ($n \geq 1$).

3. Given a_n is in the form of $a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3} + Da_{n-4}$
   Find A, B, C and D respectively.
   a. 0, 0, 1, 1
   b. 0, 1, 0, 1
   c. 1, 0, 1, 0
   d. 1, 1, 0, 0

4. How many possible ways to climb 6 stairs?
   a. 6 b. 5 c. 4 d. 3

Question 5-6: Let the recurrence relation a_n be **the minimum moves** needed to solve the **"Tower of Hanoi"** problem with n disks. ($n \geq 1$).



[Recalled the **rules of the game**, move a disk at a time from one peg to another, never place a disk on a smaller disk, and the goal is to have all disks on the second peg in order of size.]

5. Given a_n is in the form of $a_n = Aa_{n-1} + Ba_{n-2} + C$ Find A, B, and C respectively.
   a. 2, 0, 1
   b. 2, 1, 0
   c. 1, 0, 2
   d. 1, 2, 0

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6. Find **the minimum moves** needed to solve the **"Tower of Hanoi"** problem with 5 disks?
   a. 21 b. 31 c. 42 d. 63

<u>Question 7-8:</u> The recurrence relation satisfied by a_n where a_n is the number of intersection points from those n lines and b_n is the number of regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.

7. Given a_n is in the form of $a_n = Aa_{n-1} + Ba_{n-2} + Cn + K$ Find $A + B + C + K$
   a. -1 b. 0 c. 1 d. 2

8. Let $c_n = a_n + b_n$ and c_n is in the form of $c_n = Ac_{n-1} + Bc_{n-2} + Cn + K$
   Find $A + B + C + K$
   a. -1 b. 0 c. 1 d. 2

<u>Question 9-10:</u> The recurrence relation satisfied by a_n where a_n is the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. **[Hint:** Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.**]**

9. Given a_n is in the form of $a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3} + Da_{n-4}$
   Find A, B, C and D respectively.
   a. 1, 0, 0, 1
   b. 0, 1, 0, 1
   c. 1, 0, 1, 0
   d. 1, 1, 0, 0

10. From a_n got in the question 9, find a_7.
   a. 13 b. 14 c. 20 d. 21

<u>Question 11-12:</u> There are n lamp labelled $1, 2, \dots, n$. Lamp 1 can be switched on or off at any time. Lamp k, where $1 < k \leq n$ can only be switched (on or off) when the lamp $k - 1$ is the only lamp that is on out of lamps $1, 2, \dots, k - 1$. Given that all lamps are initially off, and let $P(n)$ be **the minimum moves to turn on the lamp n.**

11. Find $P(n)$.
   a. $1 + \frac{n(n-1)}{2}$
   b. $2^n - 1$
   c. 2^{n-1}
   d. $P(n-1) + 2P(n-2)\ when\ n > 2$

12. Find $P(3)$.
   a. 4 b. 5 c. 6 d. 7

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<u>Part B:</u> Numeric response questions (11 points)

13. **[4 points]** Given $a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = 1$ and $a_1 = 7$ for $n \geq 2$. Do the following tasks.
    a. Let r_1, r_2 be the answers of the characteristic equation where $r_1 < r_2$.
    Solve the characteristic equation to find r_1 and r_2.
    b. From a., given that the unique solution is in the form of $a_n = A(r_1)^n + B(r_2)^n$. Solve for the unique solution to find A and B

14. **[7 points]** The solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2n$ for $n \geq 2$ where $a_0 = 0$ and $a_1 = 2$. If general solution is in the form of $a_n = A3^n + B(C^n) + Dn^2 + En + F$
    a. Find the value of $\lfloor A \rfloor$, B, C, D, E and $\lfloor F \rfloor$
    b. Find a_5

15. **[Bonus 4 points]** A hunter, Taeyeon, and an invisible bunny play a game in the Euclidean plane. The bunny's starting point, A_0 and the Taeyeon's starting point, B_0 **are the same (Hint** : D_0 =?**).** After $n - 1$ rounds of the game, the bunny is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:
    i. The bunny moves invisibly away from Taeyeon to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
    ii. A tracking device reports a point P_n to Taeyeon. The guarantees provided by the tracking device to Taeyeon are that the distance between P_n and A_n is exactly 1 and the vector A_nP_n perpendicular to the vector A_nA_{n-1}.
    iii. Taeyeon moves visibly to a point B_n which is the direction to the tracking point such that the distance between B_{n-1} and B_n is exactly 1.

    Let D_n is distance between the Taeyeon and the bunny after n rounds and let's say that the recurrence relation of $D_n = \sqrt{(D_{n-1} + 1)^2 + 1} - 1$ by approximately after somehow ***"MAGIC"*** solving and got that $D_n = \sqrt{n + X - Y\sqrt{n + Z}}$. **Find D_0, X, Y and Z respectively.**
    [Hint: Maybe let $F_n = (D_n + 1)^2$ can help?, **FYI:** $X,\ Y,\ Z$ are positive integers.**]**

"Well, That Part Is A Little Dramatic." -Tenet-

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ANSWER SHEET for Quiz 4A

Part A: Choose the correct answer and provide the **X** mark.

No.	Choice				No.	Choice				No.	Choice			
	a.	b.	c.	d.		a.	b.	c.	d.		a.	b.	c.	d.
1.					5.					9.				
2.					6.					10.				
3.					7.					11.				
4.					8.					12.				

Part B: Provide an answer in terms of **INTEGER ONLY**.

No.	Variables	Answer	Variables	Answer
13.	r_1		A	
	r_2		B	
14.	$\lfloor A \rfloor$		D	
	B		E	
	C		$\lfloor F \rfloor$	
	a_5			
15.	D_0		Y	
	X		Z	

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