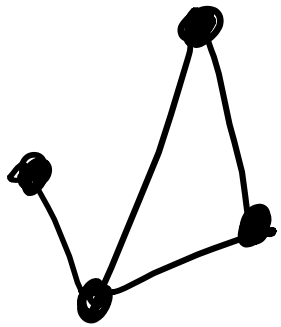
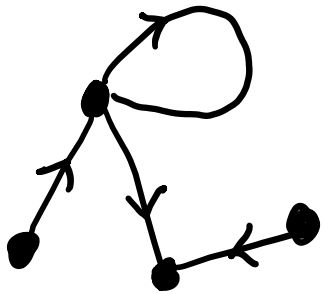
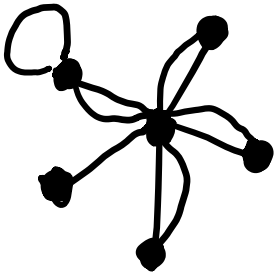
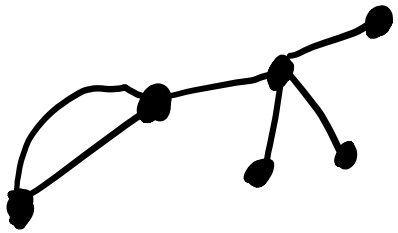
GRAPH THEORY

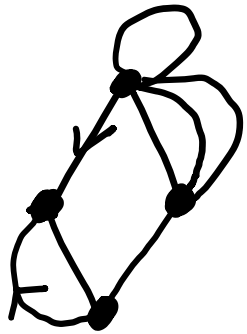
They are graphs.

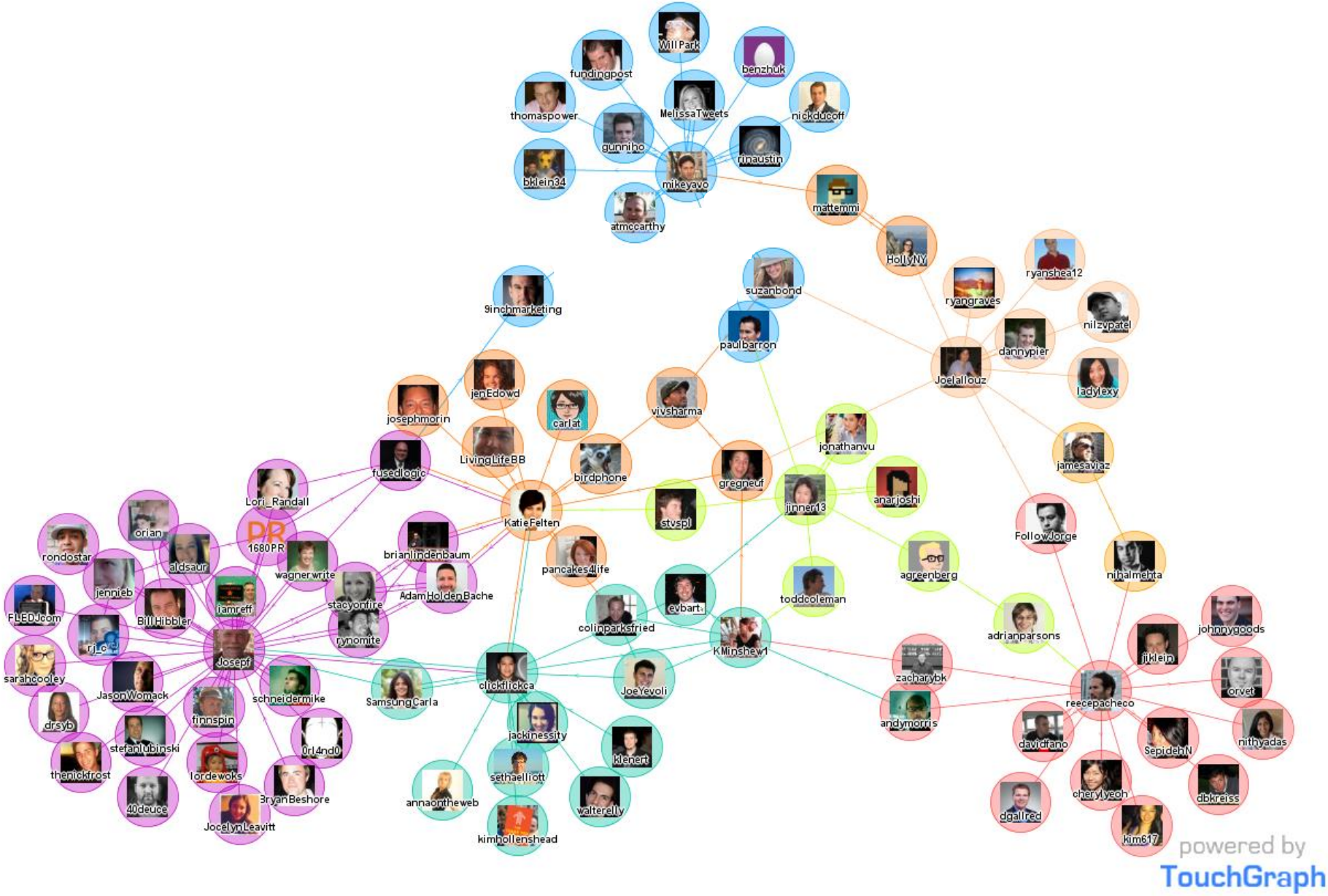




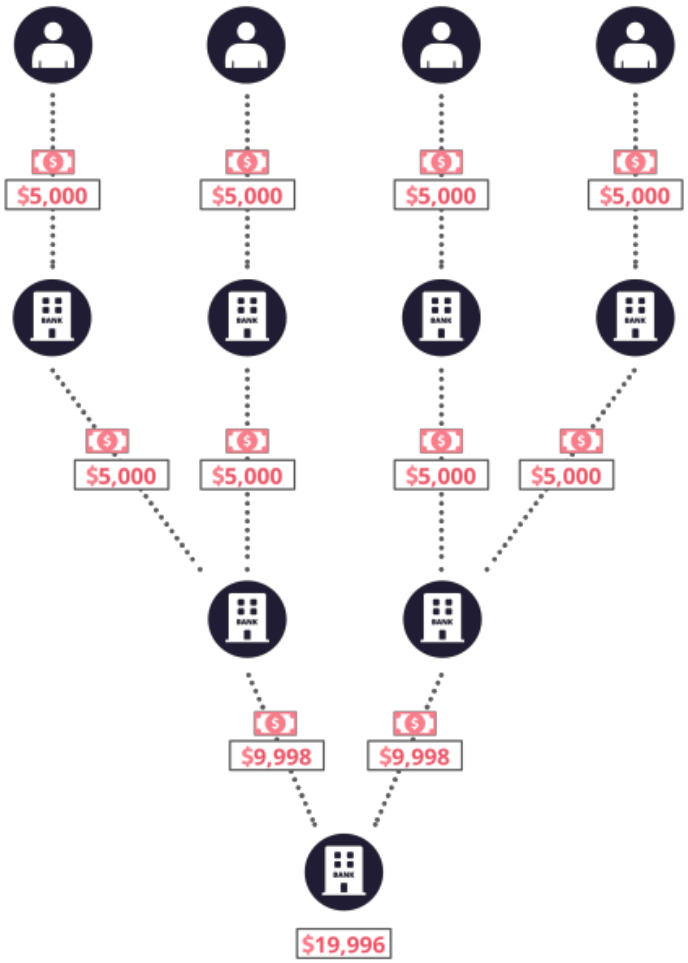








powered by TouchGraph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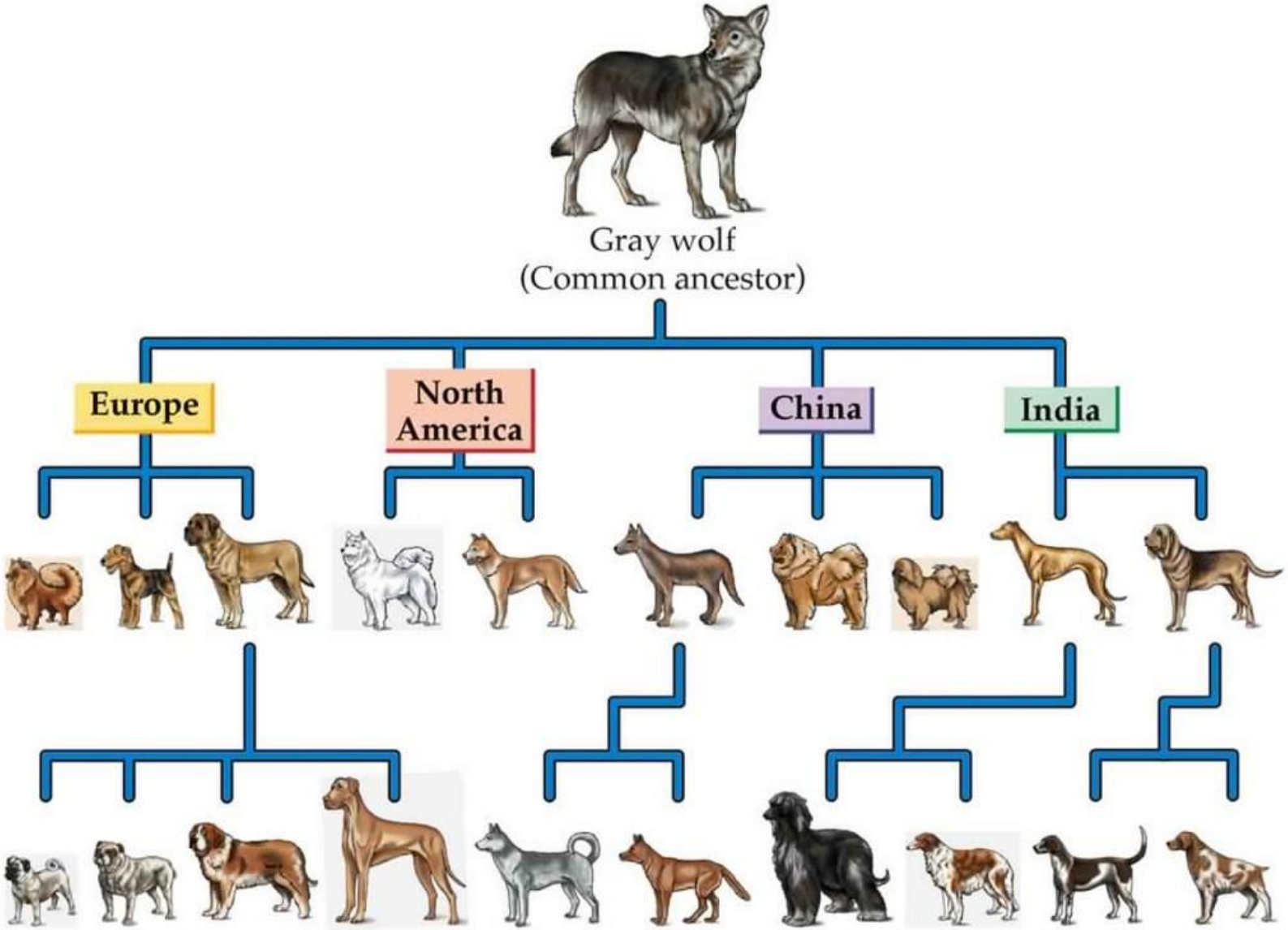
$9,998
$19,996



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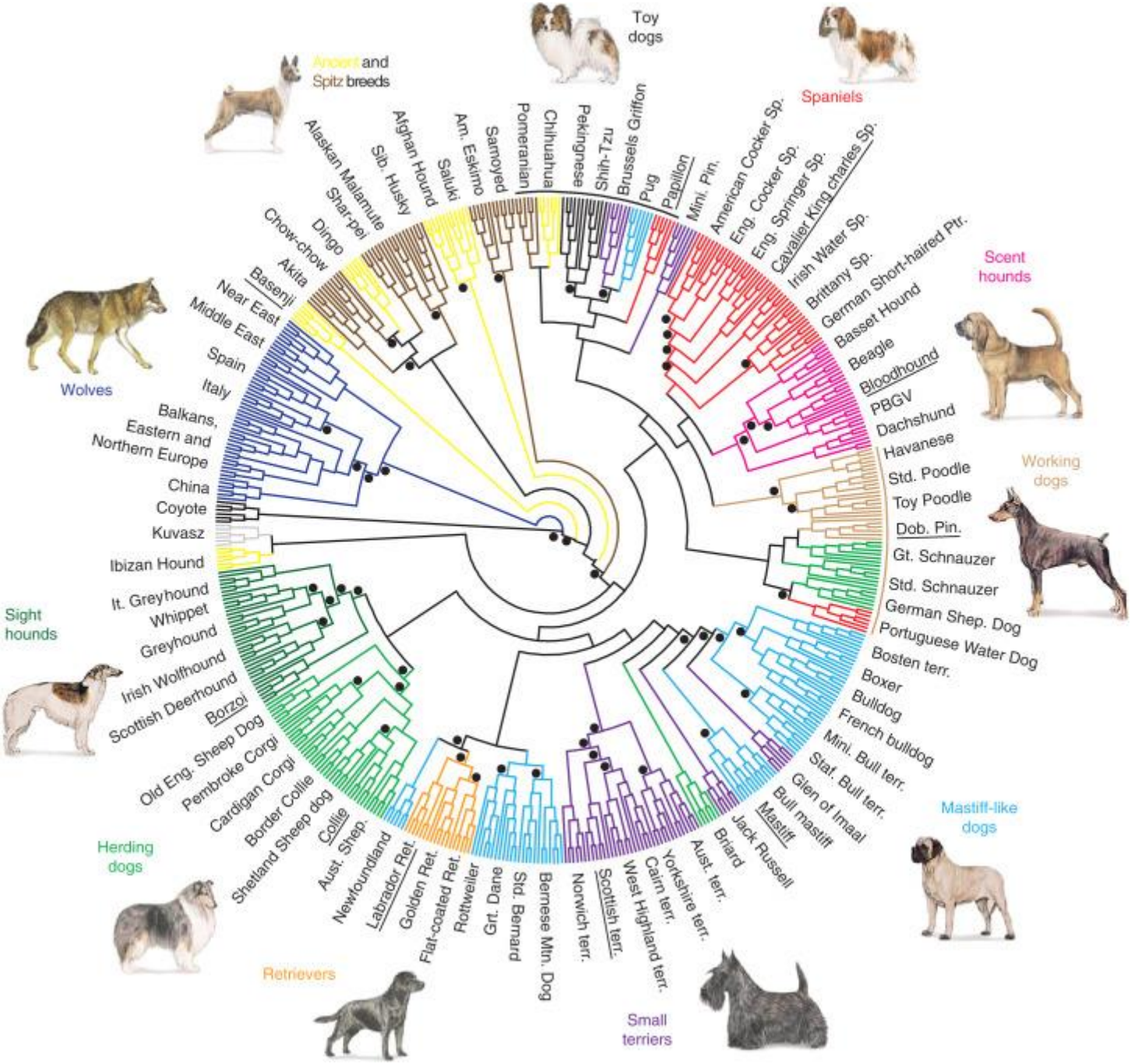

Toy dogs
Spaniels
Ancient and Spitz breeds
Scent hounds
Wolves
Working dogs
Sight hounds
Mastiff-like dogs
Herding dogs
Retrievers
Small terriers
Alaskan Malamute
Shar-pei
Dingo
Chow-chow
Akita
Basenji
Near East
Middle East
Spain
Italy
Balkans, Eastern and Northern Europe
China
Coyote
Kuvasz
Ibizan Hound
It. Greyhound
Whippet
Greyhound
Irish Wolfhound
Scottish Deerhound
Borzoi
Old Eng. Sheep Dog
Pembroke Corgi
Cardigan Corgi
Border Collie
Shetland Sheep dog
Collie
Aust. Shep.
Newfoundland
Labrador Ret.
Golden Ret.
Flat-coated Ret.
Rottweiler
Grt. Dane
Std. Bernard
Bernese Mtn. Dog
Norwich terr.
Scottish terr.
West Highland terr.
Cairn terr.
Yorkshire terr.
Aust. terr.
Briard
Jack Russell
Mastiff
Bull mastiff
Glen of Imaal
Staf. Bull terr.
Mini. Bull terr.
French bulldog
Bulldog
Boxer
Bosten terr.
Portuguese Water Dog
German Shep. Dog
Std. Schnauzer
Gt. Schnauzer
Dob. Pin.
Toy Poodle
Std. Poodle
Havanese
Dachshund
PBGV
Bloodhound
Beagle
Basset Hound
German Short-haired Ptr.
Brittany Sp.
Irish Water Sp.
Cavalier King charles Sp.
Eng. Springer Sp.
Eng. Cocker Sp.
American Cocker Sp.
Mini. Pin.
Papillon
Pug
Brussels Griffon
Shih-Tzu
Pekingnese
Chihuahua
Pomeranian
Samoyed
Am. Eskimo
Saluki
Afghan Hound
Sib. Husky

NONTHABURI
Yaek Tiwanon
Wong Sawang
Bang Son
Wat Samian Nari
Chatuchak
Ratchayothin
Phahon Yothin 24
Lat Phrao Intersection
Phahon Yothin
Chatuchak Park
Mo Chit
Lat Phrao
Ratchadaphisek
Bang Kruay
Taling Chan
Bang Bamru
Bang O
Bang Pho
Tao Poon
Bang Sue
Kamphaeng Phet
Saphan Kwai
Sutthisan
Ari
Huai Kwang
Chatuchak Market
Thailand Cultural Centre
Bang Phlat
Vimanmek Museum
Sirindhorn
Sanam Pao
Khaosan Road
Wat Saket
Sam Sen
Victory Monument
Phraram 9
Bang Yi Khan
Big Palace
Ratchaprarop
Makkasan
Phayathai
Bang Khun Non
Thonburi
Petchaburi
Ratchathewi
Siam
Chitlom
Phloen Chit
Nana
Wat Pho
National Stadium
Sukhumvit
Fai Chai
Jim Thompson Museum
Asok
Phromphong
Ratchadamri
Sanam Chai
Wang Burapha
Wat Mangkorn
Charan 13
Hua Lumpong
Sam Yan
Silom
Lumpini
Thong Lo
Tha Phra
Isaraphap
Bang Phai
China Town
Queen Sirikit Centre
Ekkamai
Khlong San
Pat Pong
Saladaeng
Klong Toei
Bang Wa
Wuttakat
Sky Bar
Chong Nonsi
Charoen Nakhon
Sathorn
Phetchakasem 48
Talat Phlu
Wongwian Yai
Saphan Taksin
Akhan Songkhro
Technic Krungthep
Po Nimitr
Krung Thonburi
Surasak
Thanon Chan
Asiatique

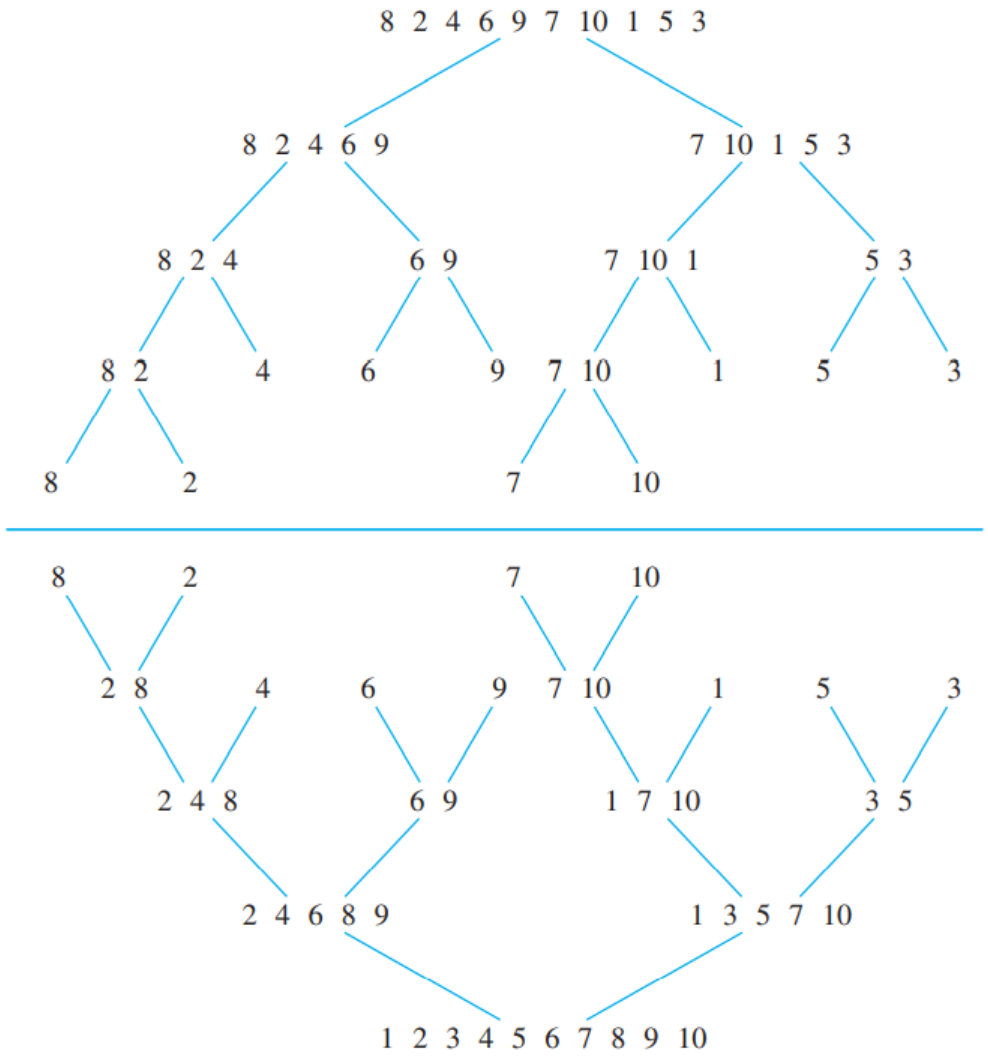
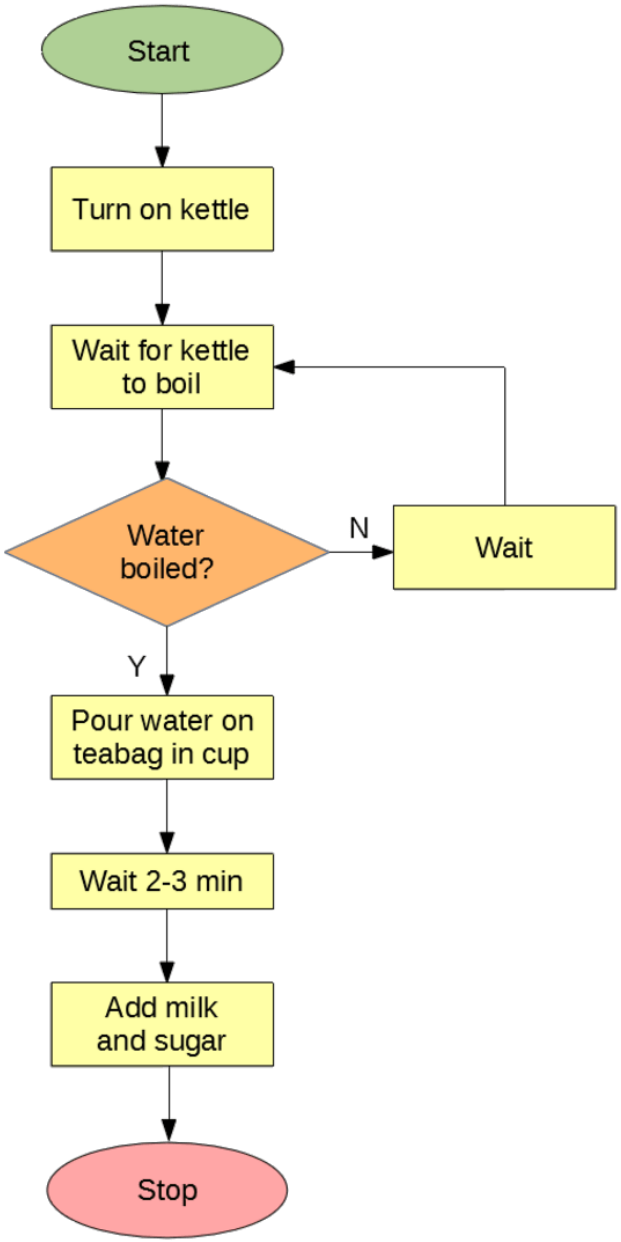
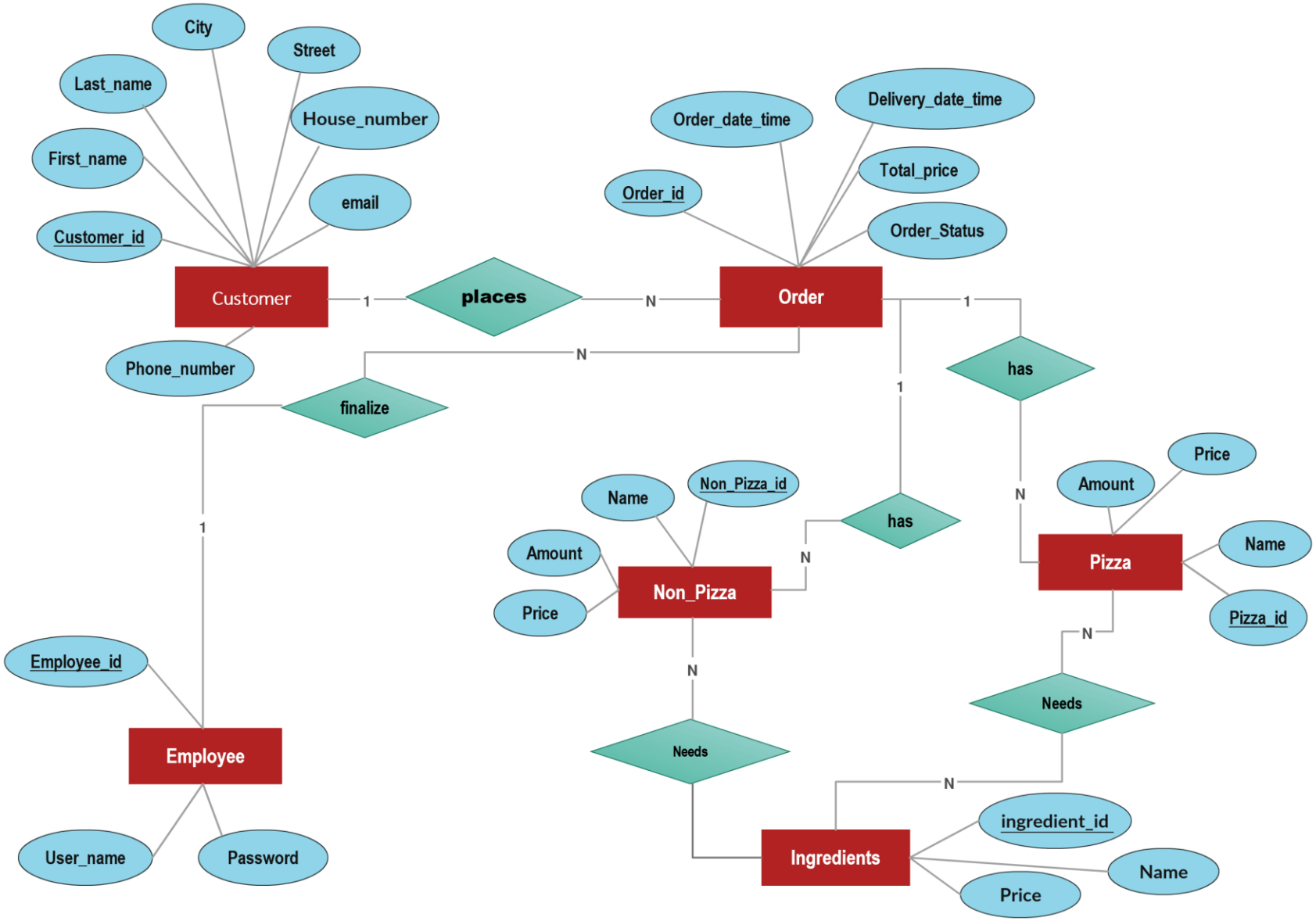


FIGURE 2 **The Merge Sort of 8, 2, 4, 6, 9, 7, 10, 1, 5, 3.**


Start
Turn on kettle
Wait for kettle to boil
Water boiled?
N
Wait
Y
Pour water on teabag in cup
Wait 2-3 min
Add milk and sugar
Stop



ER Diagram: Online Pizza Ordering System

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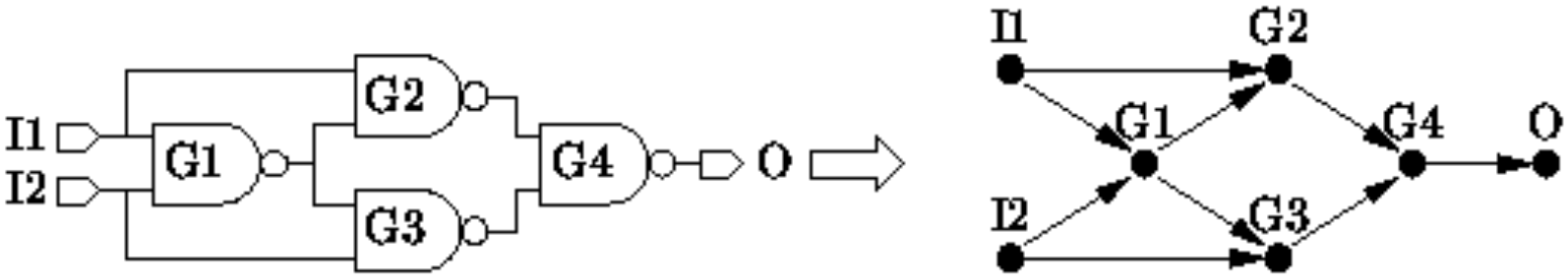

I1
I2
G1
G2
G3
G4
O
I1
I2
G1
G2
G3
G4
O

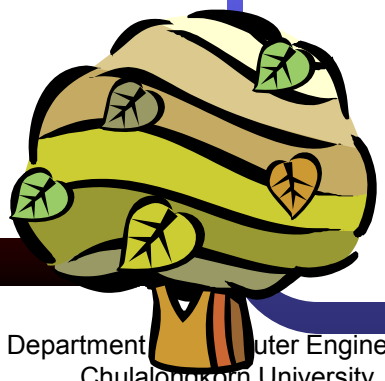
Graphs & Trees

Topics in Graphs

- Graphs and graph models
- Graph Terminology
- Representing graph / Isomorphism
- Connectivity
- Euler and Hamilton paths
- Planar graph
- Graph coloring

Topics in Trees

- Intro to trees
- Binary search trees
- Tree traversal
- Spanning trees



10 Graphs 641

10.1 Graphs and Graph Models 641
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10.3 Representing Graphs and Graph Isomorphism 668
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10.5 Euler and Hamilton Paths 693
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11.1 Introduction to Trees 745
11.2 Applications of Trees 757
11.3 Tree Traversal 772
11.4 Spanning Trees 785
~~11.5 Minimum Spanning Trees ..... 797~~
End-of-Chapter Material 803

Sections 10.1-10.2

Today's Topics

- Graph Definition
- Terminology
- Simple Graphs, Multigraphs, Pseudographs
- Directed Graphs
- Degrees
- Special Types of Graph
- Bipartite Graphs
- Subgraph
- Union of Graphs

Graphs

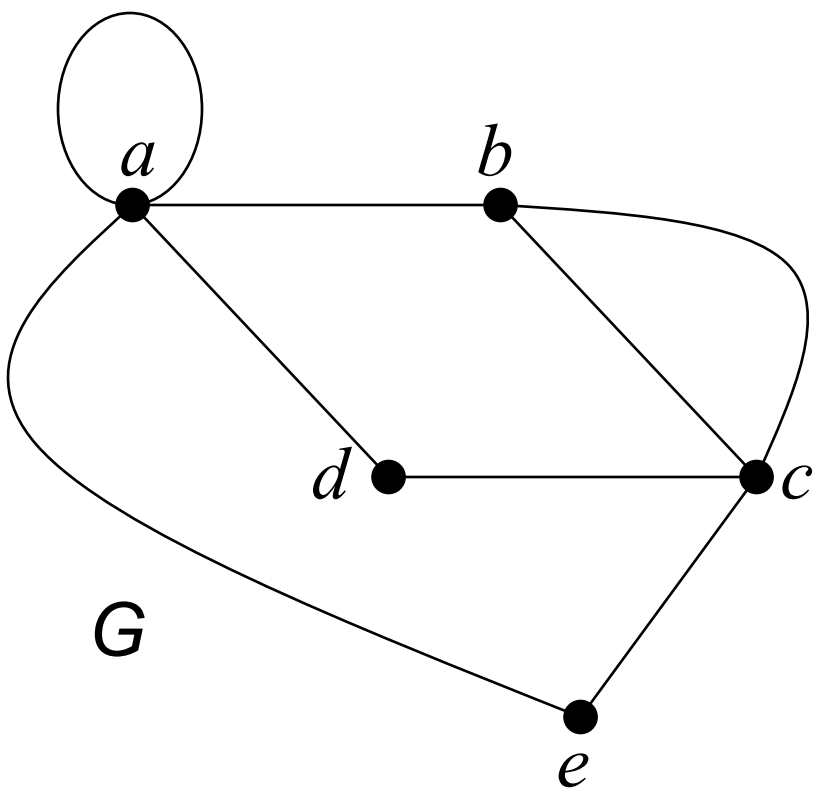
การเชื่อมจุดยอด ด้วยเส้นเชื่อม

vertex edge

$G = (V,E)$

$V = \{v_1, v_2, \ldots, v_n\}$ Set of Vertices

$E = \{e_1, e_2, \ldots, e_k\}$ Set of Edges

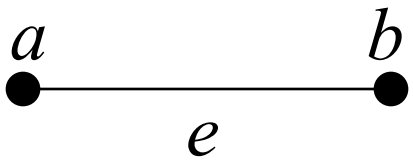


$V = \{a,b,c,d,e\}$

$E = \{\{a,a\},\{a,b\},\{a,d\},\{a,e\},\{b,c\},\{b,c\},\{c,e\},\{d,c\}\}$

G has 5 vertices and 8 edges.

Terminology



e is **incident** with a and b.
a is an **end point** of e.
b is the other end point of e.
a is **adjacent** to b.
b is also adjacent to a.

Simple Graph, Multigraph, Pseudograph

undirected graph

Simple Graph

No more than 1 edge between any pair of vertices. No loops.

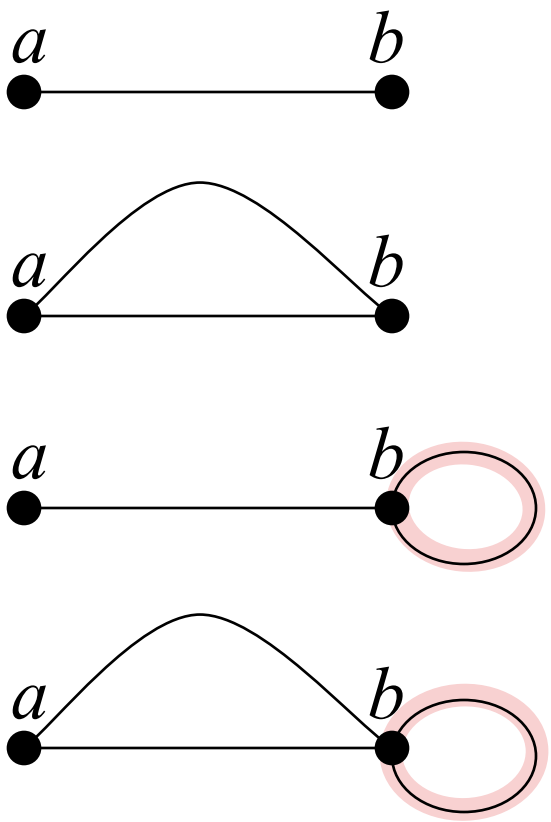
Multigraph

There can be more than 1 edge between any pair of vertices. No loops.

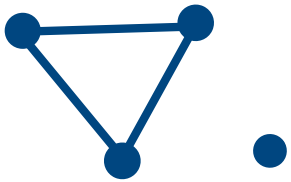
Pseudograph

Any graph.

Simple Graph, Multigraph, Pseudograph



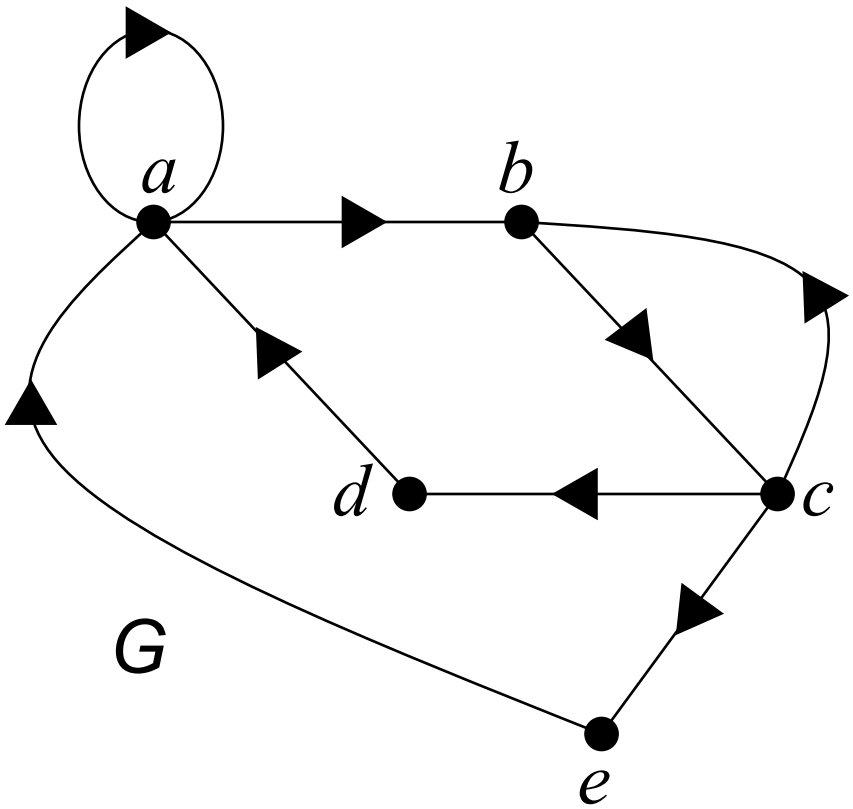
	Simple	Multi-	Pseudo-
	✓	✓	✓
		✓	✓
			✓
			✓



$$\binom{4}{2}$$

Directed Graphs

- Edges are described using “ordered pairs”.



$E = \{(a,a),(a,b),(d,a),(e,a),(b,c),(c,b),(c,e),(c,d)\}$

$V = \{a,b,c,d,e\}$

$G = (V,E)$

Directed Graphs

Simple Directed Graph

No repeated edges (order matters). No loops

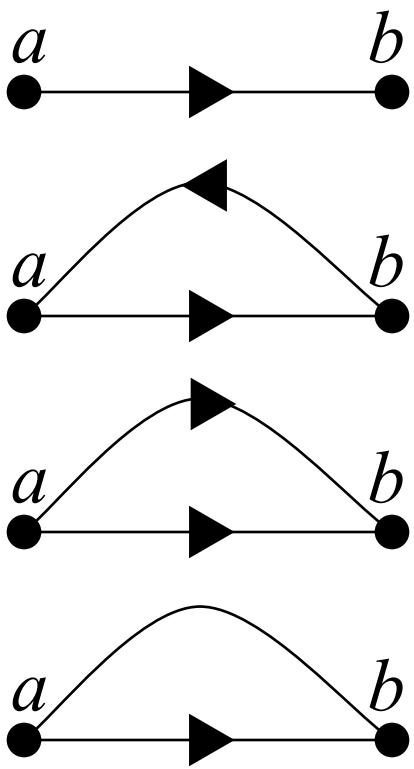
Directed Multigraph

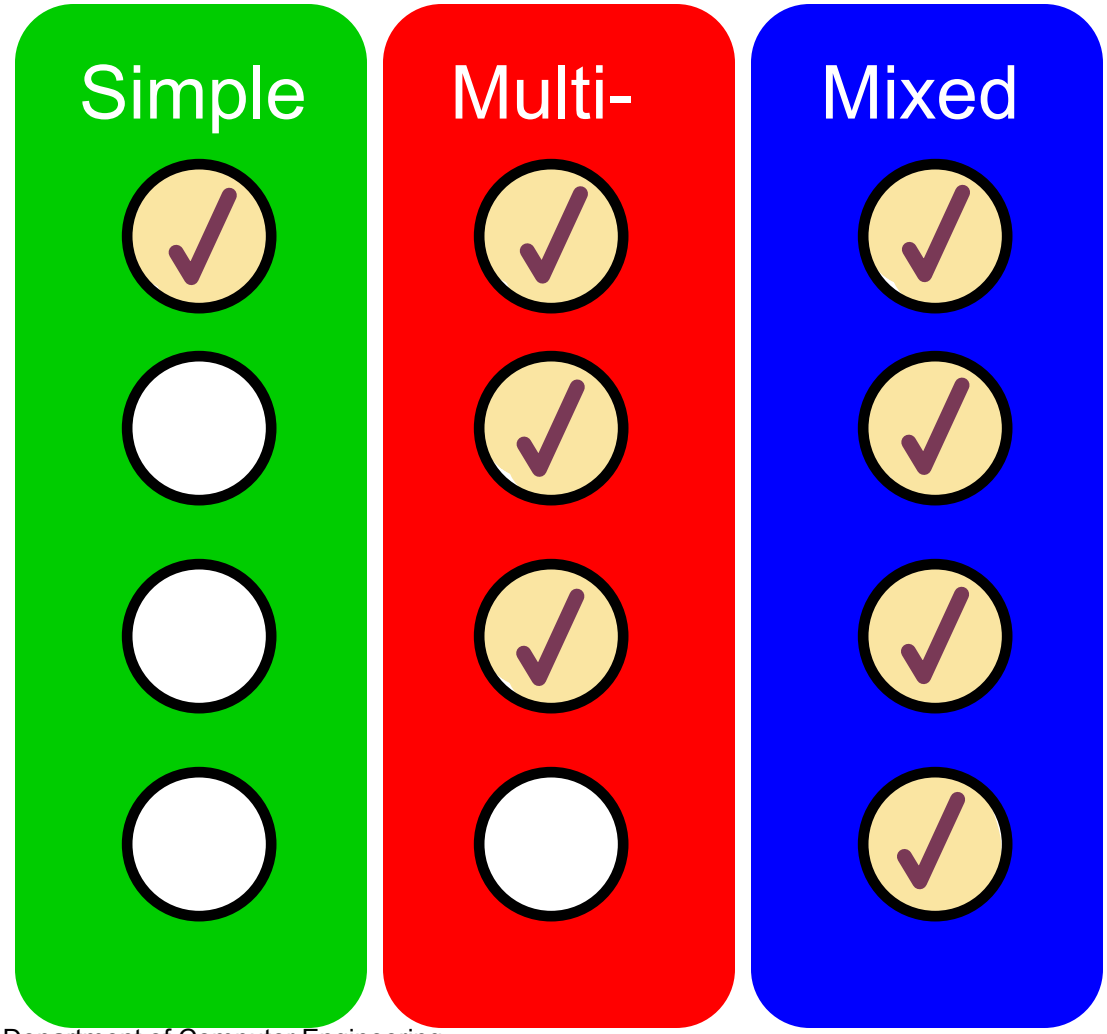
There can be repeated edges and/or loops.

Mixed Graph

There are both directed and undirected edges.

Directed Graphs





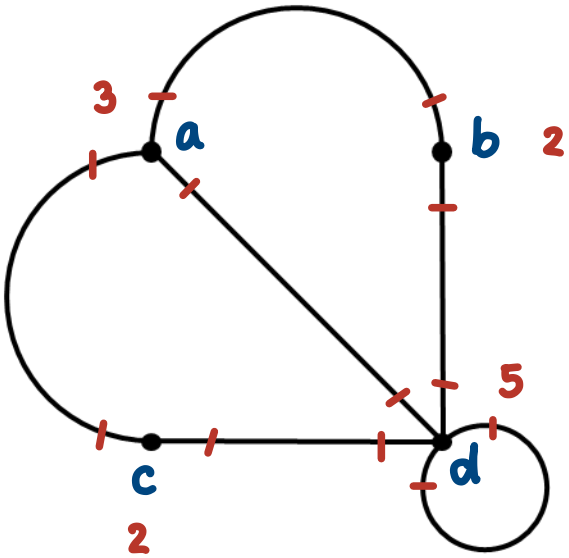
Graph Type Summary

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Degree of a Vertex

- Degree of v, deg(v) = number of edges incident with v



A loop contributes **twice** to the degree of a vertex.

How many edges are there in a graph with 10 vertices each of degree 6?

มี 10 vertex vertex ละ 6 degree

Σ degree $= 10 \times 6$

$\therefore$ เส้นเชื่อม $\frac{60}{2} = 30$ เส้นเชื่อม

The Handshaking Theorem

Let $G = (V,E)$ be an undirected graph with e edges, then

$$2e = \sum_{v \in V} \deg(v)$$

The Number of Vertices with Odd Degrees

In an undirected graph, there must be an even number of vertices with odd degree.

Proof:

ใช้ contradiction proof

โดยสมมติแย้งว่า vertex ที่มี degree เป็นคี่มีจำนวน n ตัว ซึ่งเป็นคี่

ดังนั้น $n = 2k+1$, $k \in \mathbb{Z}$

ผลรวมดีกรีของ vertex เหล่านี้คือ $\sum_{i=1}^{n}(2c_i+1) = \sum_{i=1}^{n}2c_i + \sum_{i=1}^{n}1 = 2\sum_{i=1}^{n}c_i + n$ คู่ + คี่ = คี่

ส่วน vertex ที่ดีกรีเป็นเลขคู่ให้มี m ตัว มีผลรวมดีกรีคือ $\sum_{i=1}^{m}2d_i = 2\sum_{i=1}^{m}d_i$ คู่

① ผลรวมดีกรีของทุก vertex จึงเป็นเลขคี่

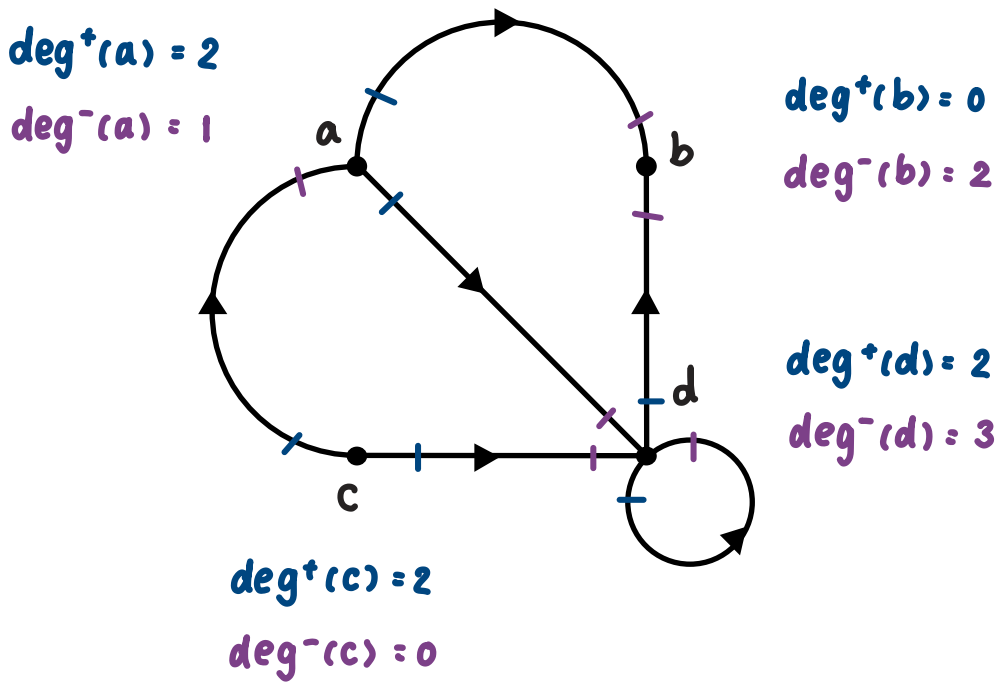
② จาก HST ผลรวม degree ทุก vertex ต้องเป็น คู่

Degree of a Vertex

- Degree of v, $\deg(v)$ = number of edges incident with v
- Out-degree of v, $\deg^+(v)$ = number of edges, each of which has v as their initial vertex. พุ่งออก
- In-degree of v, $\deg^-(v)$ = number of edges, each of which has v as their end vertex. พุ่งเข้า

A directed loop contributes **once** to the in-degree and **once** to the out-degree.

Degree of a Vertex



In-degree = Out-degree

Let $G = (V,E)$ be a directed graph, then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

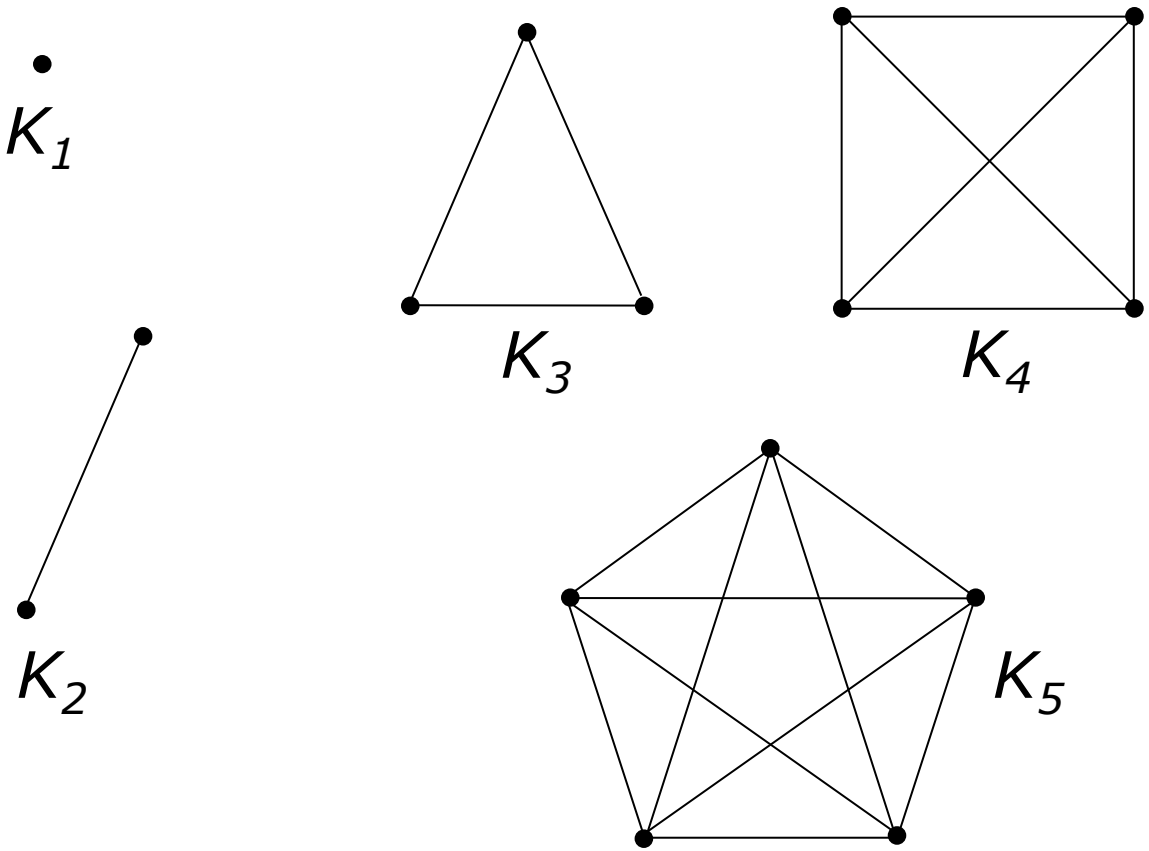
Special Simple Graphs

Complete Graphs

K_n – จำนวน vertex

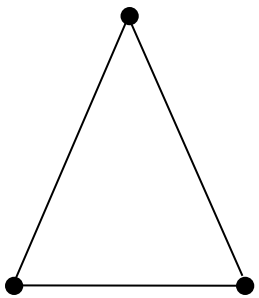
มีเส้นเชื่อมเชื่อมทุกคู่ของ vertex

จำนวนเส้นเชื่อม = $\binom{n}{2}$

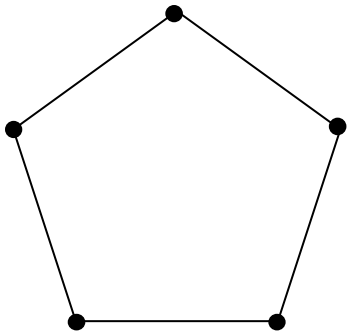


Cycles C_n

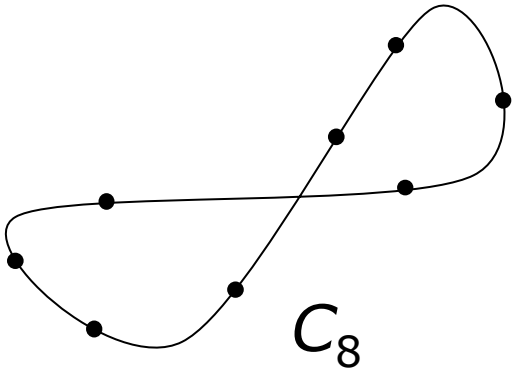
จำนวนเส้นเชื่อม = n



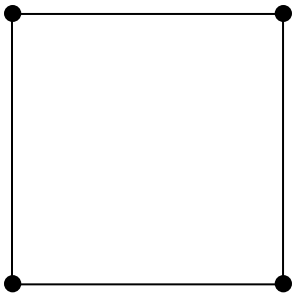
C_3



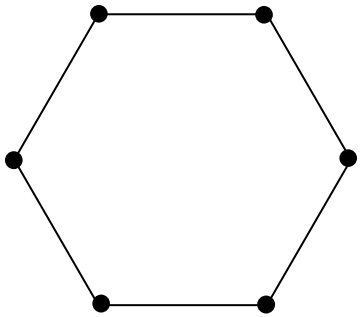
C_5



C_8



C_4



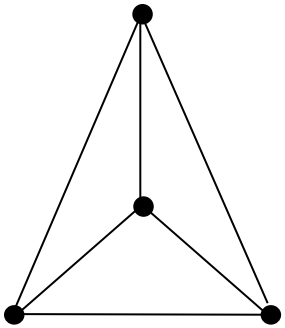
C_6

Wheels

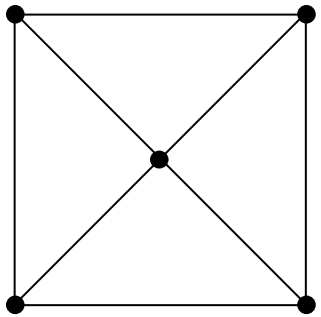
W_n – จำนวน vertex รอบๆ ไม่นับตรงกลาง

จำนวน vertex = n+1

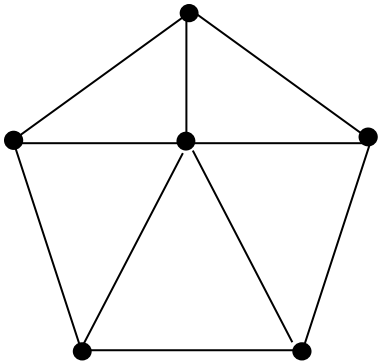
จำนวนเส้นเชื่อม = 2n



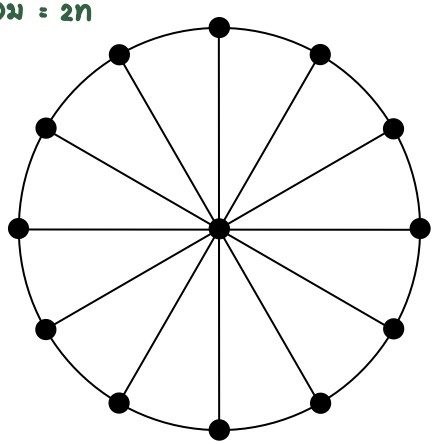
W_3



W_4



W_5



W_{12}

n-Dimensional Hypercube Q_n

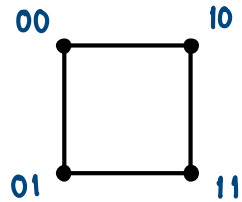
เส้นเชื่อม เชื่อมที่ต่างกัน 1 bit

จำนวน vertex = 2^n

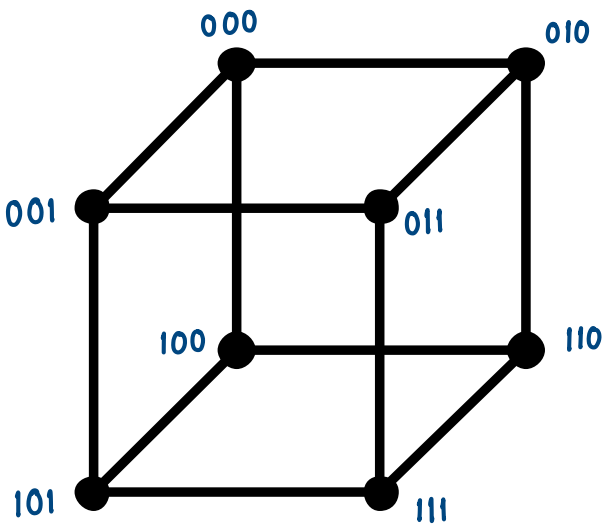
degree แต่ละ vertex = n

จำนวนเส้นเชื่อม = $n \times 2^{n-1}$

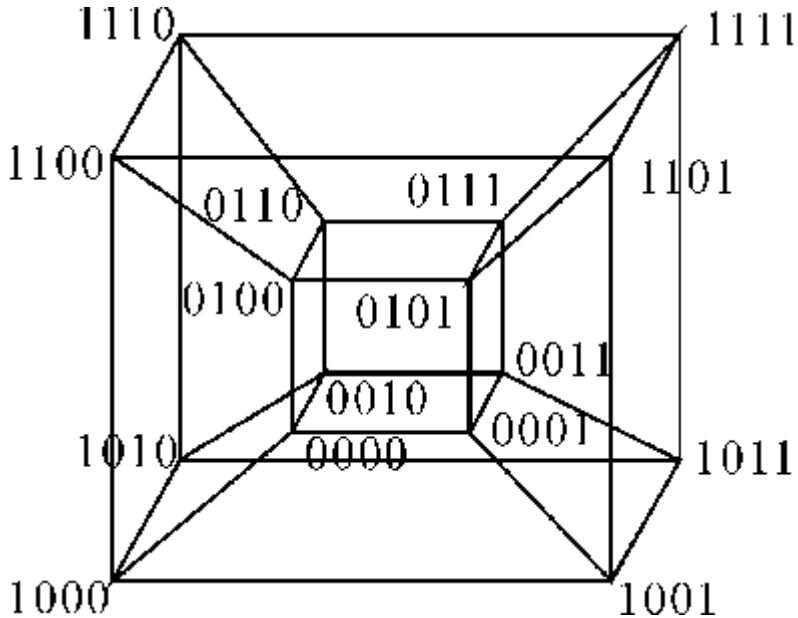
Q_1

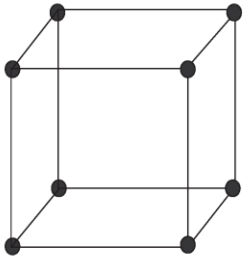


Q_2

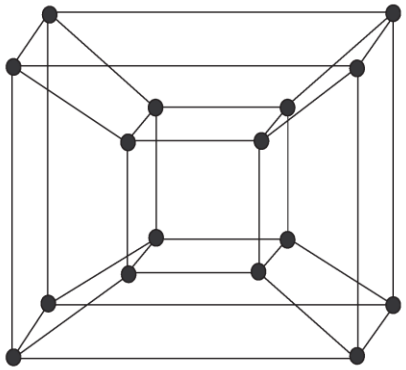


Q_3





Q_3

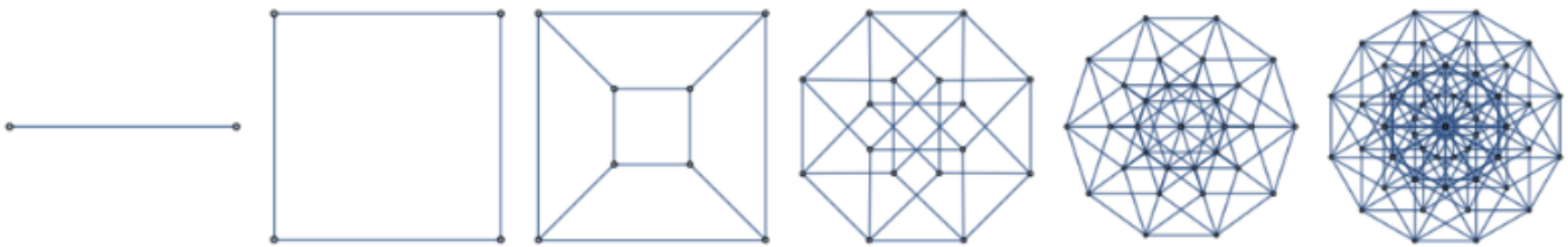


Q_4

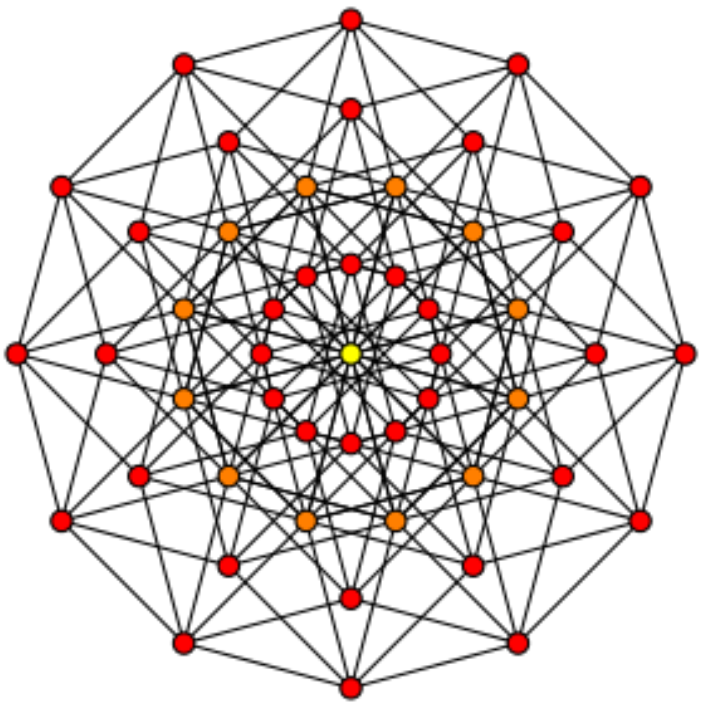
Tesseract

The theory was simple. It was important to build up the tension that which stone was which until finally bringing the third phase in. While also we find that the first stone to be mentioned was Tesseract in Captain America: The First Avenger. Actually, the look on Tony Stark's face in Avengers (2012) suggests something else. The reason is he has seen the Tesseract in his father's notes i.e. the space stone is first shown in Iron Man II.

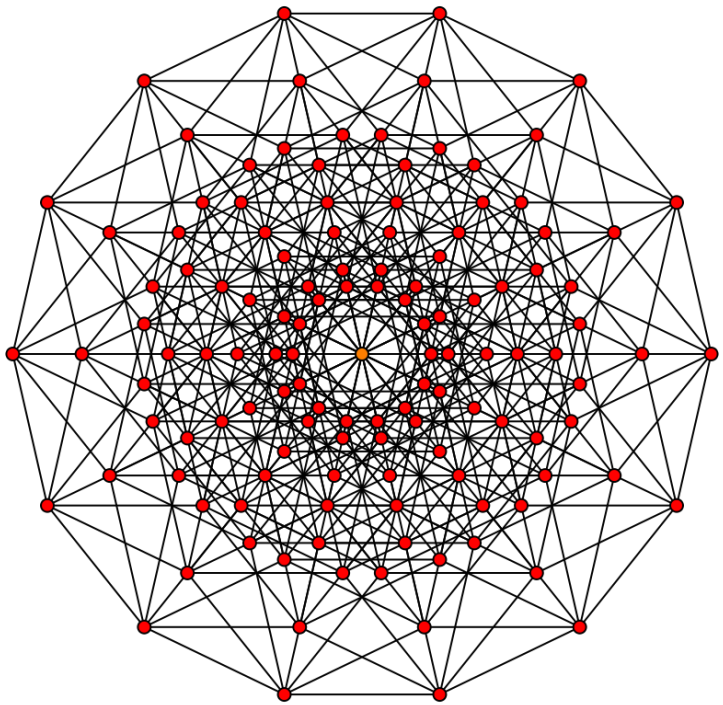
```
GraphicsRow@Table[GraphData[{"Hypercube", n}], {n, 6}]
```



Q_1 Q_2 Q_3 Q_4 Q_5 Q_6

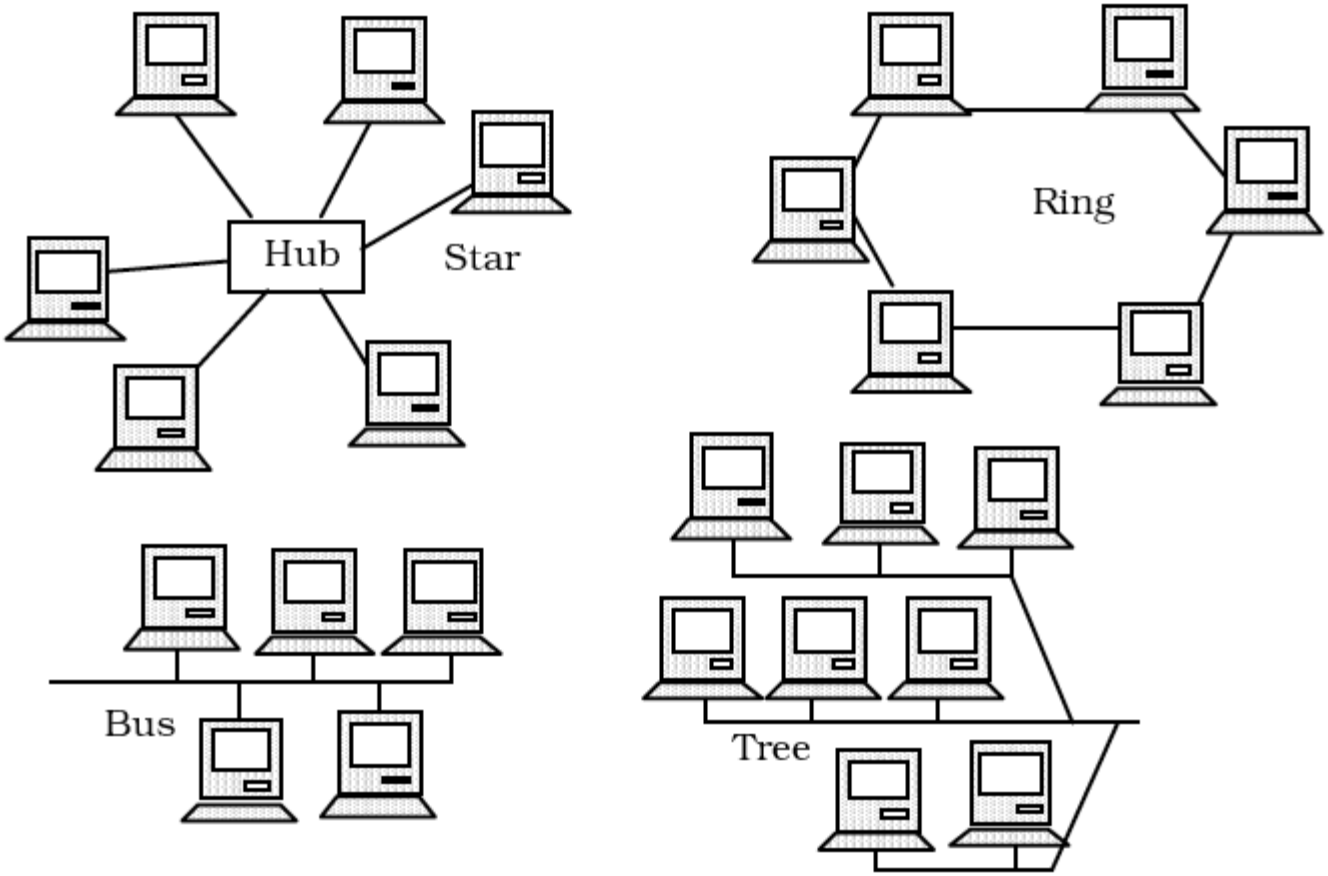


Q_6

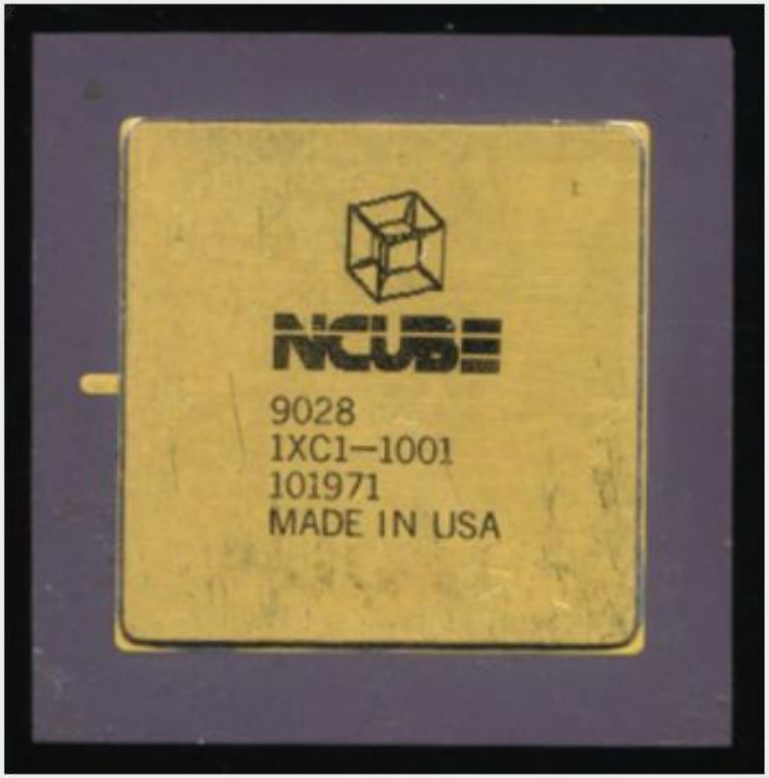


Q_7

LAN Topology



nCube and the Rise of the HyperCubes



nCube/2 Processor - 20MHz
The logo is a Tesseract - a 4-way Hypercube

In 1983 Stephen Colley, Dave Jurasek, John Palmer and 3 others from Intel's Systems Group left Intel, frustrated by Intel's seeming reluctance to enter the then emerging parallel computing market. They founded a company in Beaverton, Oregon known as nCube with the goal of producing MIMD (Multiple Instruction Multiple Data) parallel computers. In 1985 they released their first computer, known as the nCube/10. The nCube/10 was built using a custom 32-bit CMOS processor containing 160,000 transistors and running initially at 8MHz (later increased to 10). IEEE754 64-bit floating point support (including hardware sqrt) was included on chip. Each processor was on a module with its own 128KB of ECC DRAM memory (implemented as 6 64k x 4 bit DRAMs.) A full system, with 1024 processor nodes, had 128MB of usable memory (160MB of DRAM counting those used for ECC). From the outset the nCube systems were designed for reliability, with MTBFs of full systems running in the 6 month range, extremely good at the time.

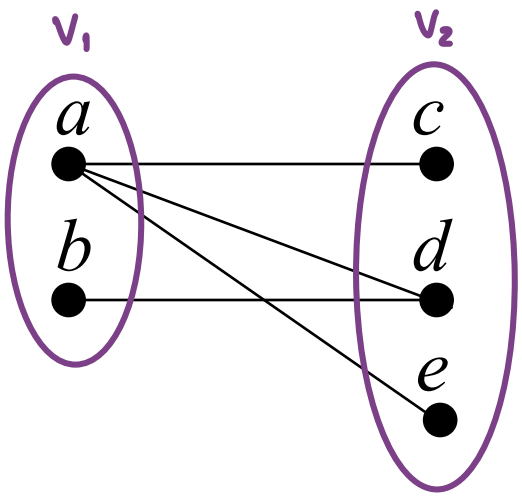
The nCube/10 system was organized in a Hypercube geometry, with the 10 signifying its ability to scale to a 10-way Hypercube, also known as a dekeract. This architecture allows for any processor to be a maximum of 10-hops from any other processor. The benefits are greatly reduced latency in cross processor communication. The downside is that expansion is restricted to powers of 2 (64, 128, 256, 512 etc) making upgrade costs a bit expensive as the size scaled up. Each processor

Bipartite Graphs

A simple graph $G = (V,E)$ is called **bipartite** $\leftrightarrow$

V can be partitioned into V_1 and V_2 so that

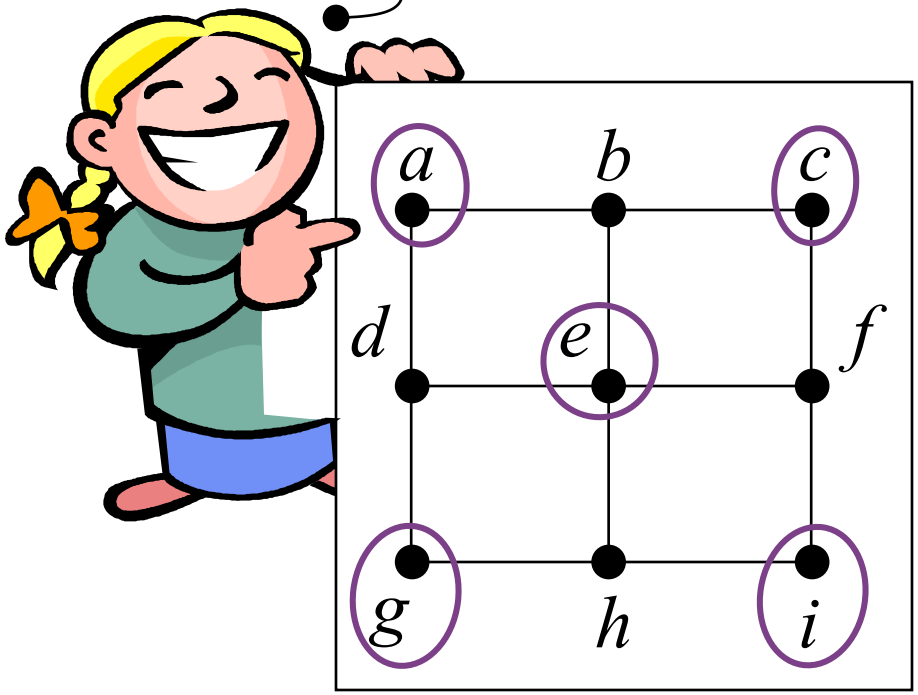
no edges connects vertices in the same partition.

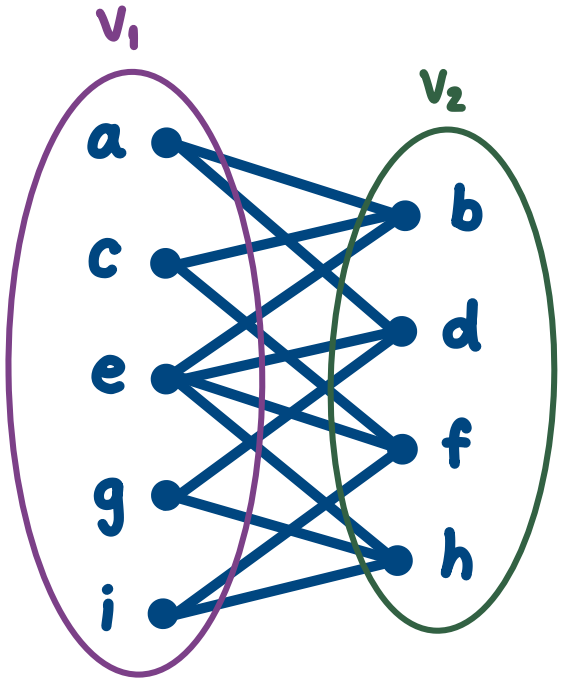


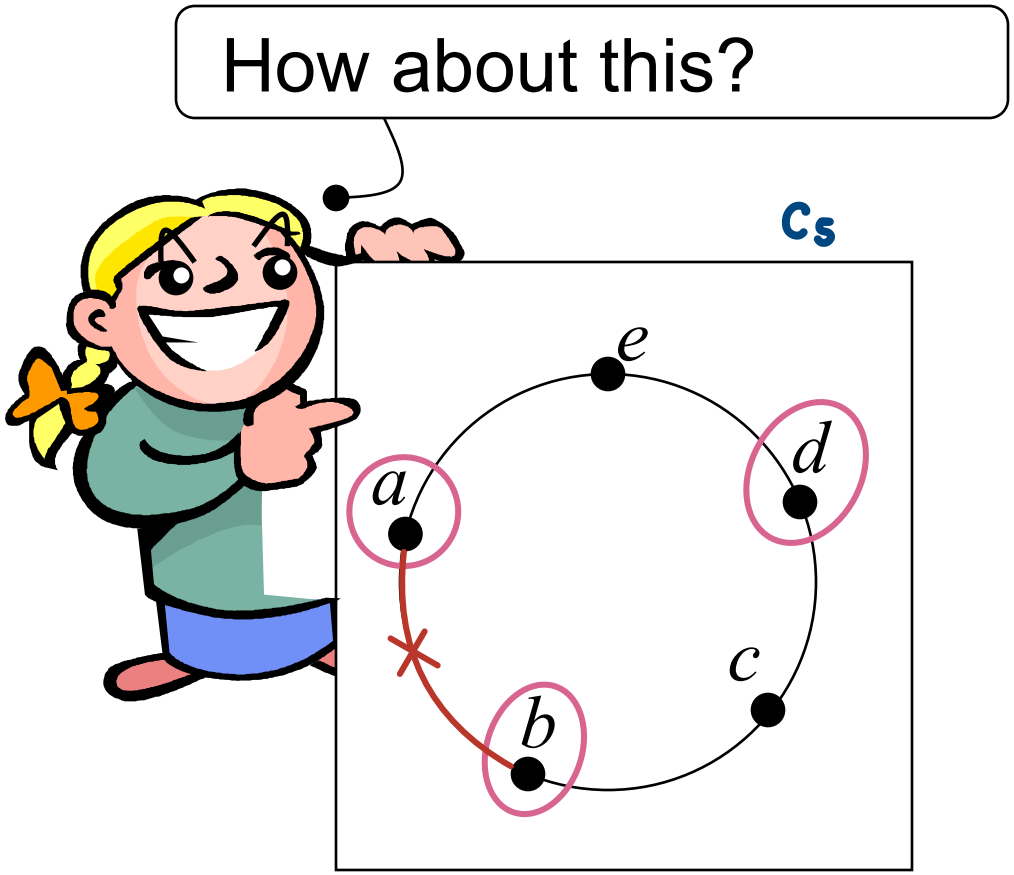
$$V = \{a,b,c,d,e\}$$

$$V_1 = \{a,b\} \qquad V_2 = \{c,d,e\}$$

Is this graph bipartite?





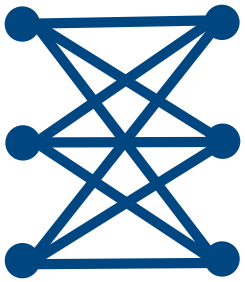

How about this?
C_S
e
d
a
c
b

Complete Bipartite Graphs

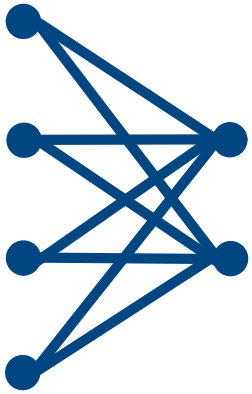
- Vertices are partitioned into a set of *m* vertices and a set of *n* vertices.
- There is an edge between two vertices ↔ one vertex is in the first subset and the other vertex is in the second subset.

Complete Bipartite Graph, $K_{m,n}$

Complete Bipartite Graphs



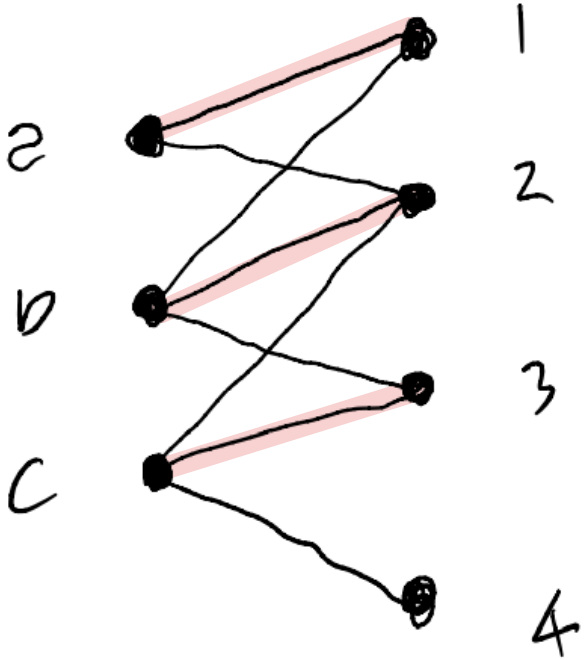
$K_{3,3}$

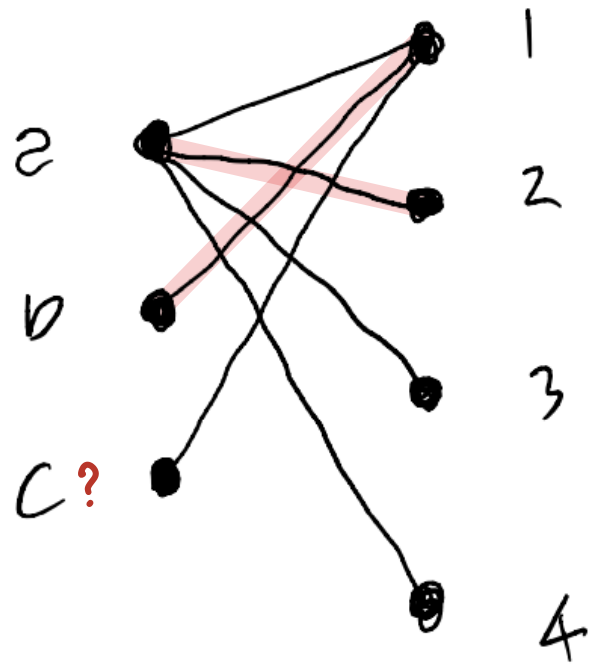


$K_{4,2}$

Complete Matching

เชื่อมไปยัง vertex ที่ต่างกัน ครบทุกตัว

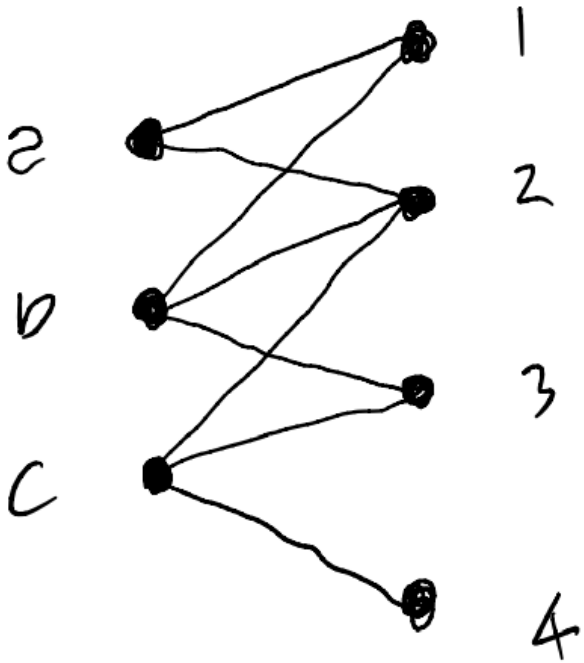




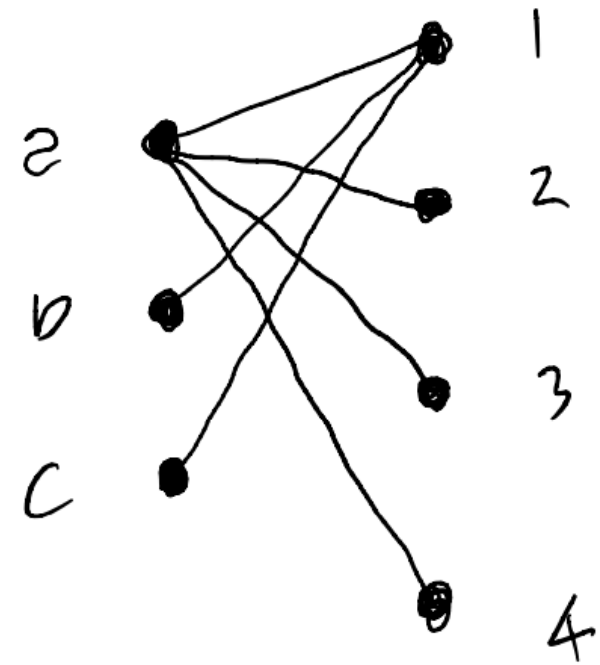
× complete matching

Complete Matching

$A = \{a, b\}$ $N(A) = \{1, 2, 3\}$



$A = \{B, C\}$ $N(A) = \{1\}$

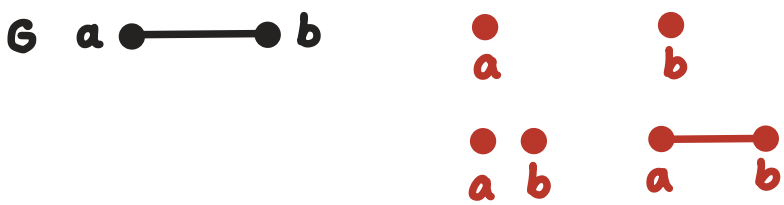


HALL'S MARRIAGE THEOREM The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

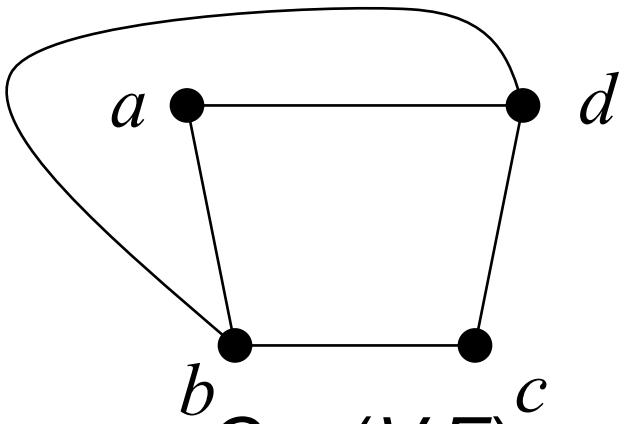
Subgraphs

A **subgraph** of $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$

A subgraph of H of G is a **proper subgraph** of G if $H \neq G$



Subgraphs



$G = (V,E)$

$V = \{a,b,c,d\}$

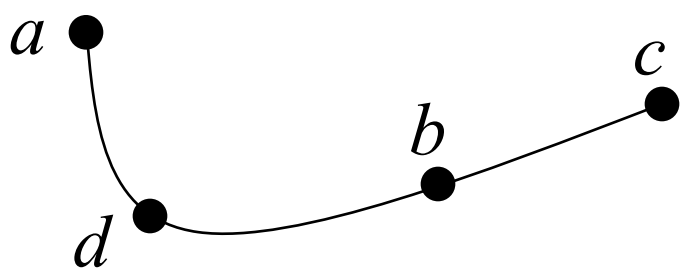
$E = \{\{a,b\},\{b,c\},\{c,d\},\{a,d\},\{b,d\}\}$

$W \subseteq V$

$F \subseteq E$

$W = \{a,b,c,d\}$

$F = \{\{b,c\},\{a,d\},\{b,d\}\}$



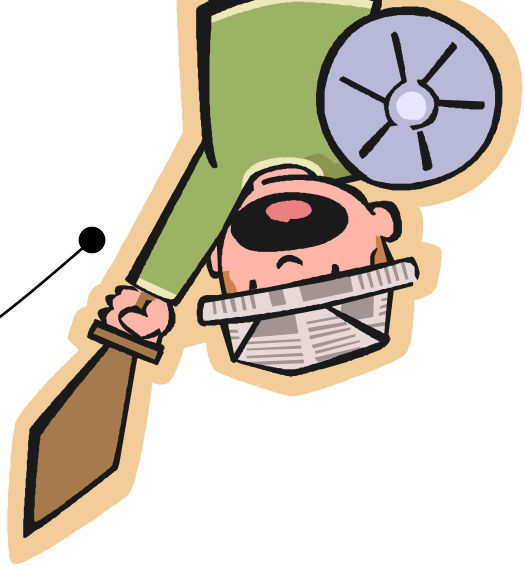
$H = (W,F)$

Union

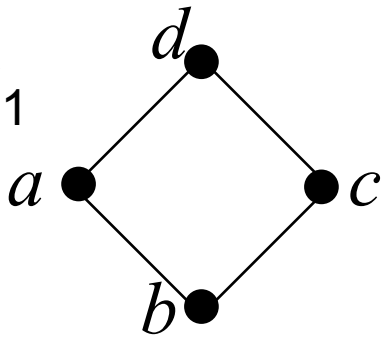
The **union** of two simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $H=(W,F)$ where $W = V_1 \cup V_2$ and $F = E_1 \cup E_2$

Denoted by $G_1 \cup G_2$

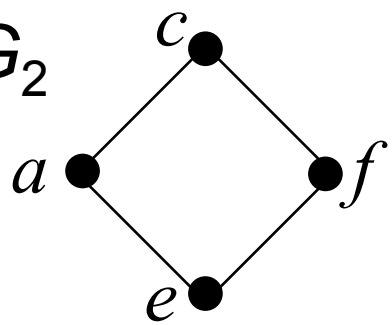
Find $G_1 \cup G_2$



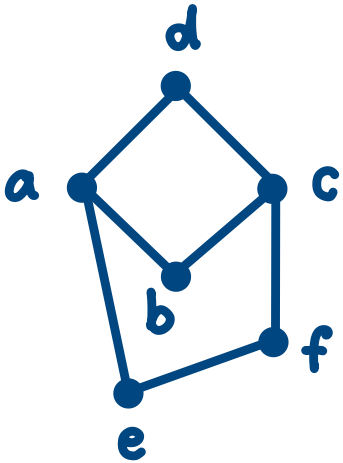
G_1



G_2



$G_1 \cup G_2 =$



- Edge Contraction

- Vertex Removal

Show that a simple graph with at least 2 vertices has at least 2 vertices with the same degree.

ใช้ contradiction proof โดยสมมุติว่าแต่ละ vertex มี degree ต่างกันทั้งสิ้น

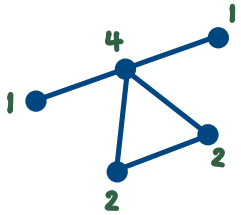
ให้กราฟที่พิจารณาเป็น simple graph ที่มี n vertex โดย $n \geq 2$

ดังนั้นสามารถให้ vertex i ; $1 \leq i \leq n$ มี degree ดังนี้คือ $\deg(v_i) = i-1$

แต่จะเห็นได้ว่ามี $\deg(v_1) = 0$ คือ v_1 ไม่ adjacent กับ vertex อื่นเลย

และมี $\deg(v_n) = n-1$ คือ v_n adjacent กับ vertex อื่นทั้งหมด

ทั้งสองไม่สามารถเกิดขึ้นพร้อมกันได้



Section 10.3

Topics

- Graph Representation
- Graph Isomorphism

Department of Computer Engineering
Chulalongkorn University

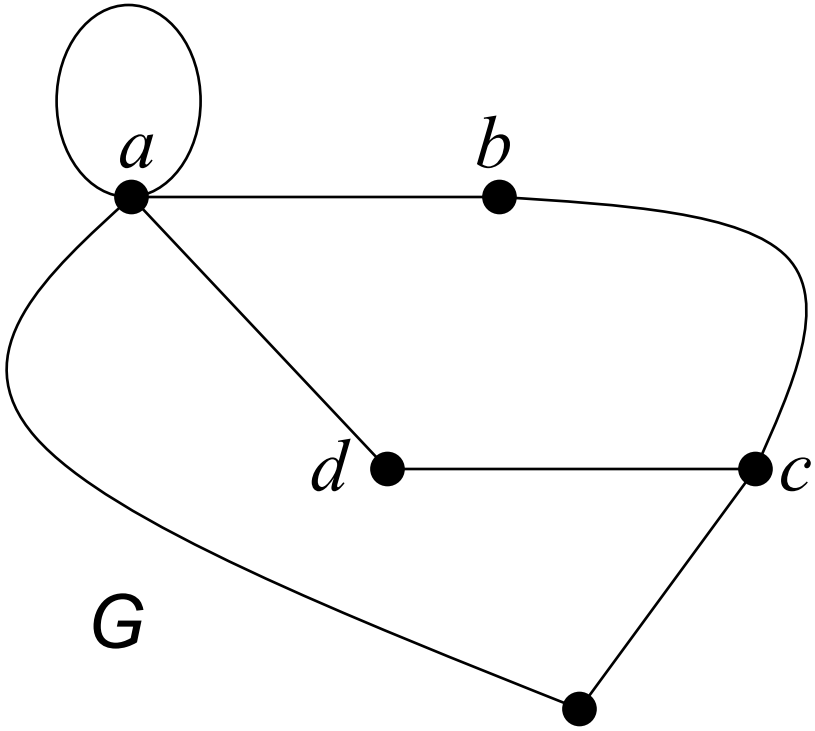
Representing Graphs

We can use:

- Adjacency Lists
- Adjacency Matrices
- Incidence Matrices

Adjacency Lists

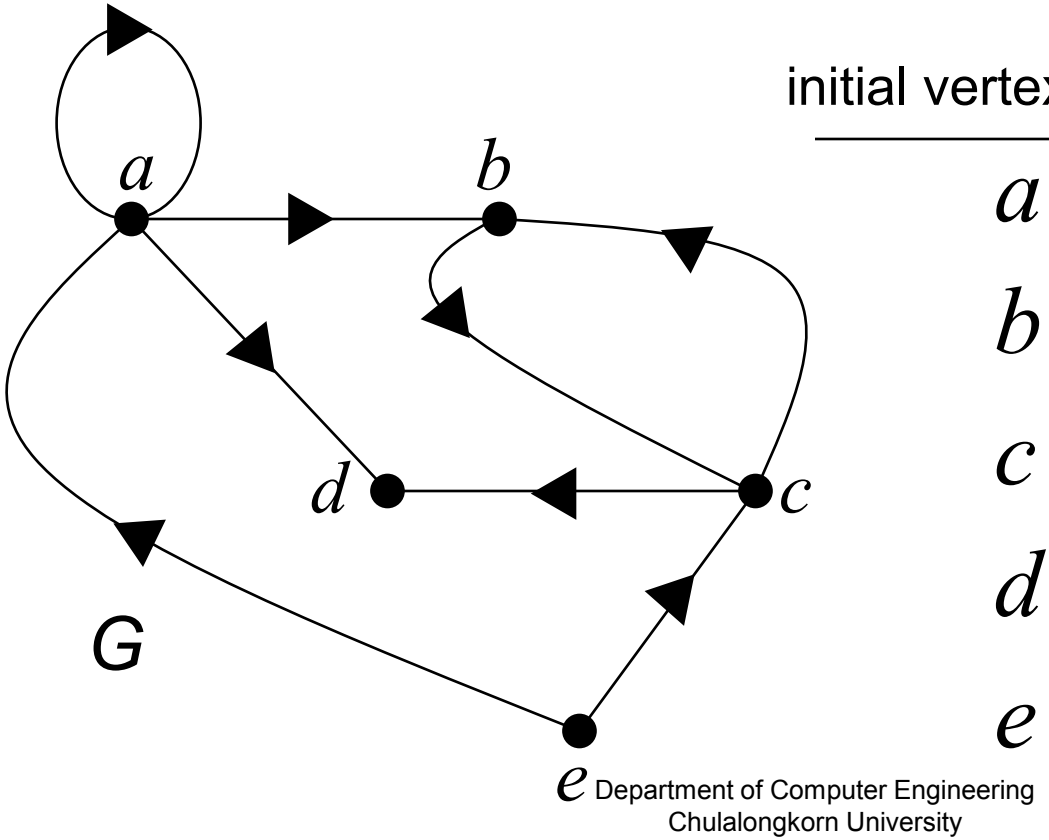
- represent graphs with no multiple edges
- specify vertices that are adjacent to each vertex



vertex	adjacent vertices
a	$a\ b\ d\ e$
b	$a\ c$
c	$b\ d\ e$
d	$a\ c$
e	$a\ c$

Adjacency Lists

- For directed graph, specify terminal vertices of each vertex



initial vertex	terminal vertices
a	$a\ b\ d$
b	c
c	$b\ d$
d	-
e	$a\ c$

Adjacency Matrices

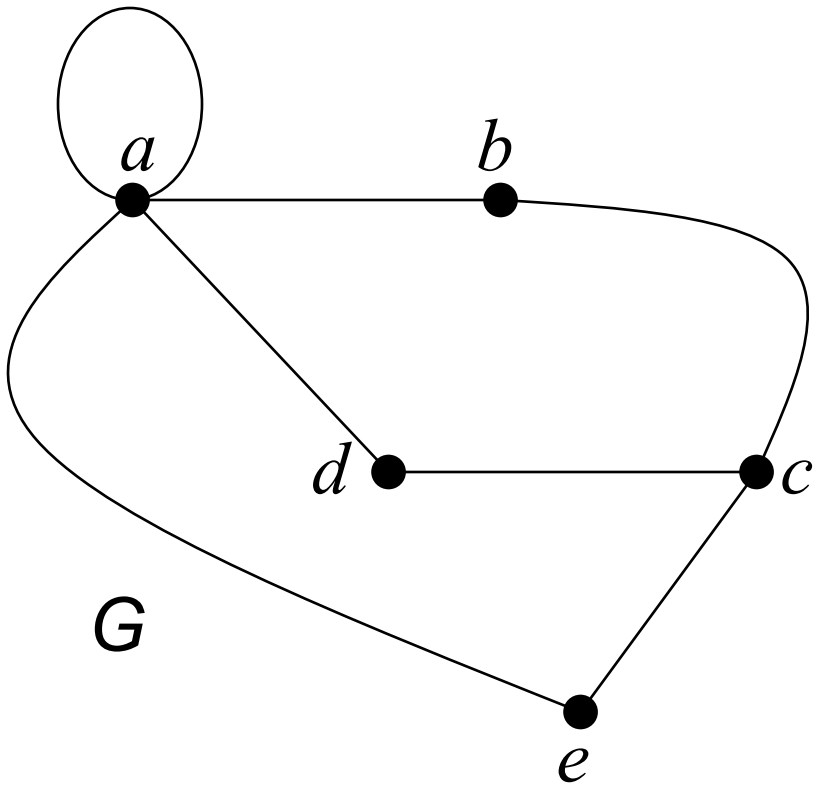
$$G = (V,E),\ V = \{v_1, v_2, \ldots, v_n\}$$

ขนาด $n \times n$ $\boxed{A = [a_{ij}]}$ $= [a_{ji}]$

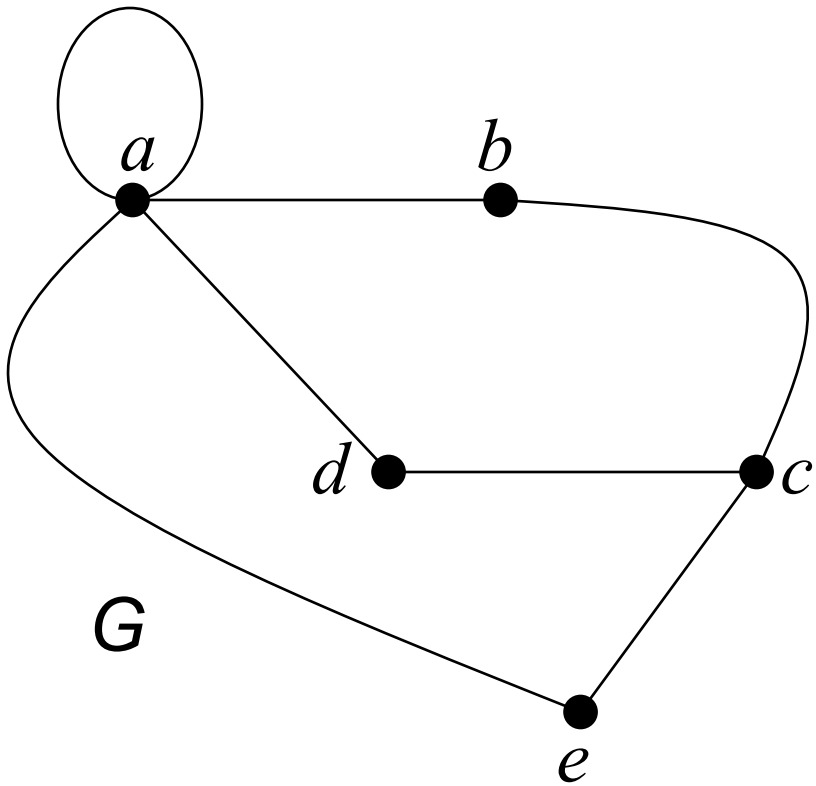
Undirected Graph

a_{ij} = Number of edges corresponding to $\{v_i, v_j\}$

Adjacency Matrices



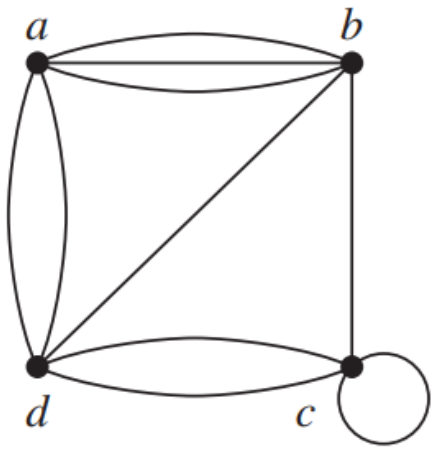
Adjacency Matrices



undirected graph – symmetric matric

$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & 1 & 1 & 0 & 1 & 1 \\
b & 1 & 0 & 1 & 0 & 0 \\
c & 0 & 1 & 0 & 1 & 1 \\
d & 1 & 0 & 1 & 0 & 0 \\
e & 1 & 0 & 1 & 0 & 0
\end{array}
$$

Adjacency Matrices



$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}
\end{array}.
$$

Adjacency Matrices

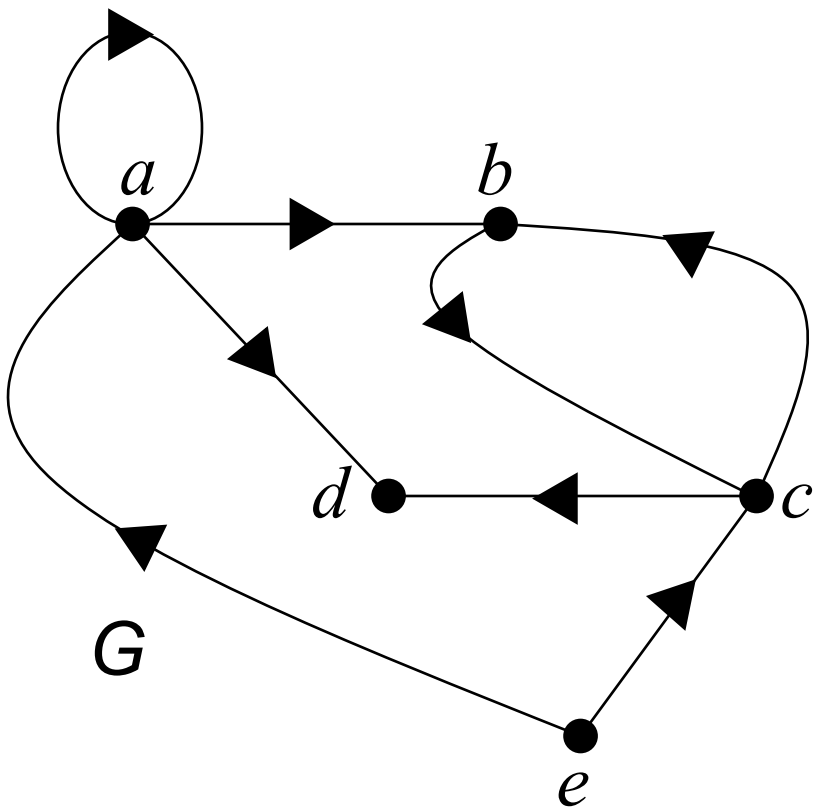
$G = (V,E),\ V = \{v_1, v_2, \ldots, v_n\}$

$$\boxed{A = [a_{ij}]}$$

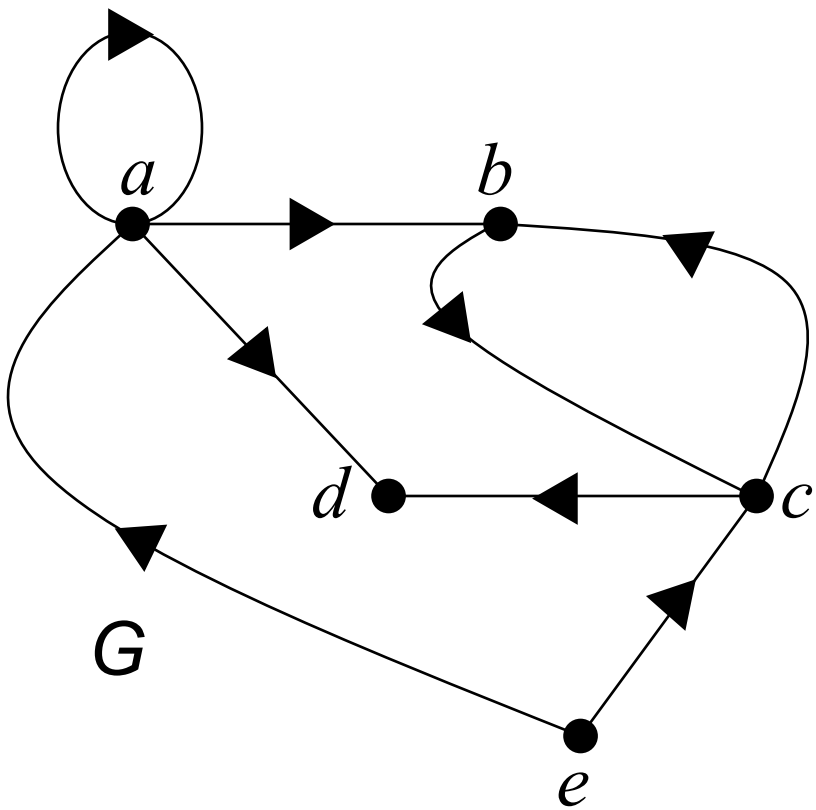
Directed Graph

a_{ij} = Number of edges corresponding to (v_i, v_j)

Adjacency Matrices



Adjacency Matrices



$$
\begin{array}{c|c}
 & \begin{matrix} a & b & c & d & e \end{matrix} \\
\begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

Incidence Matrices

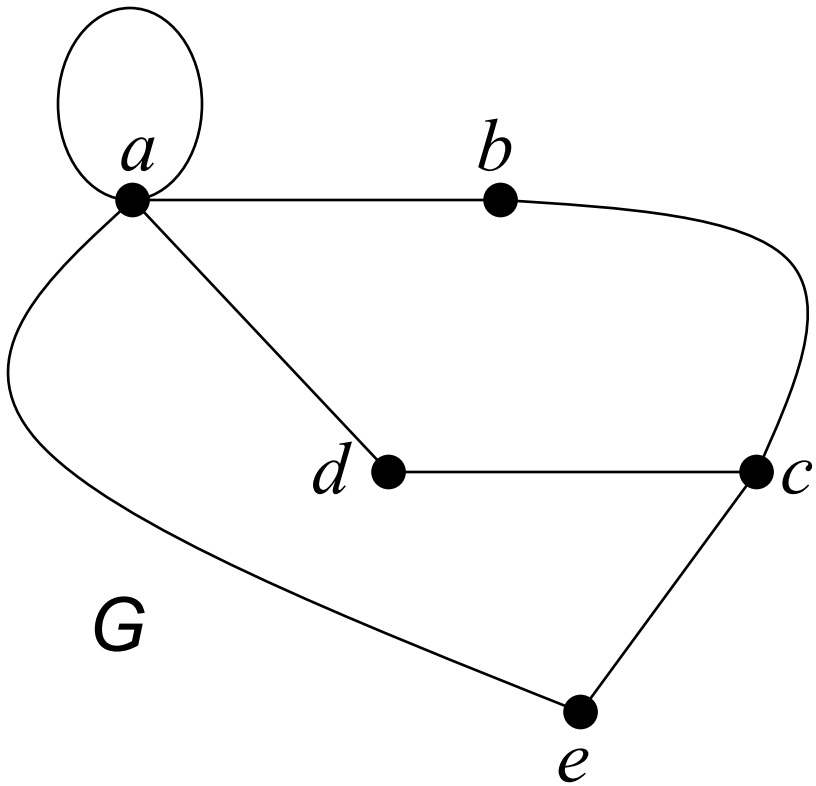
$G=(V,E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$

Undirected Graph

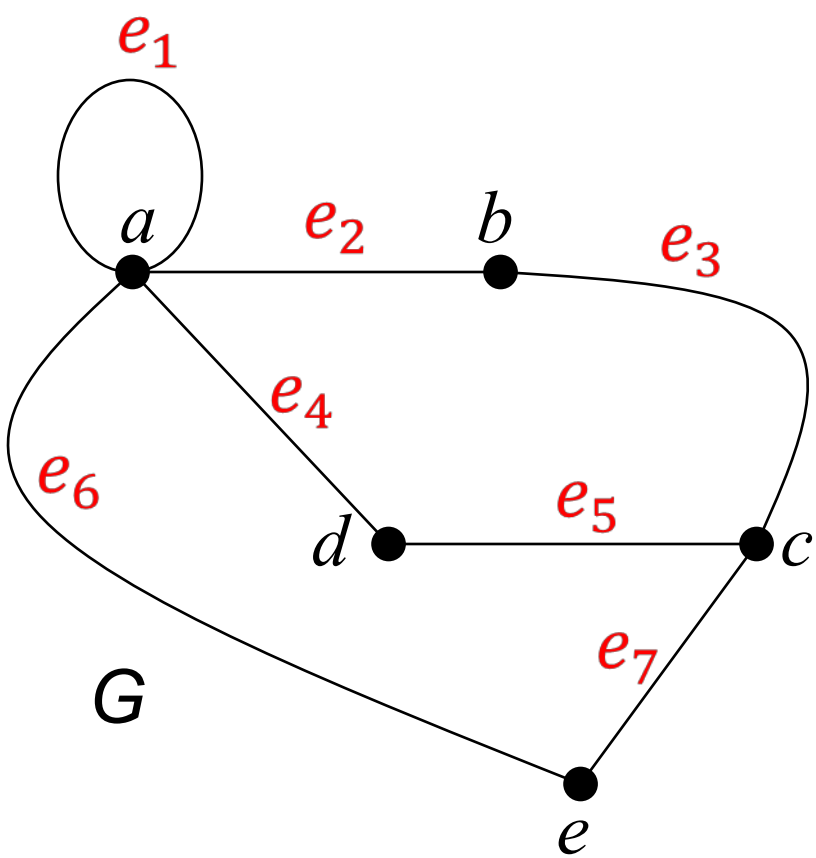
$$\boxed{M = [m_{ij}]}$$

$$m_{ij} = \begin{cases} 1 & \text{when } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

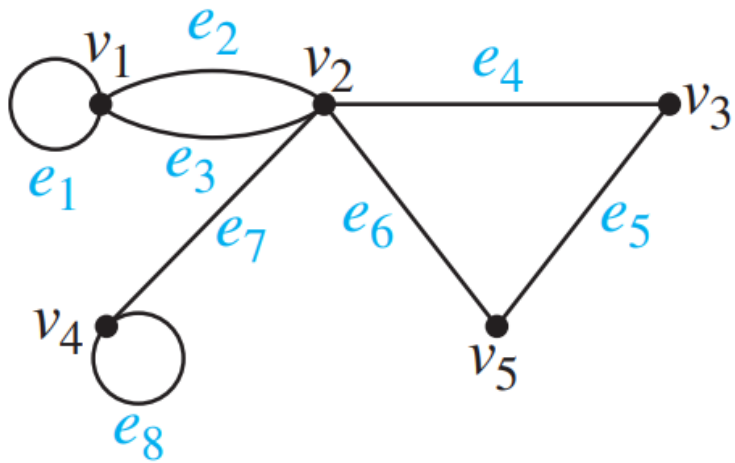
Incidence Matrices



Incidence Matrices



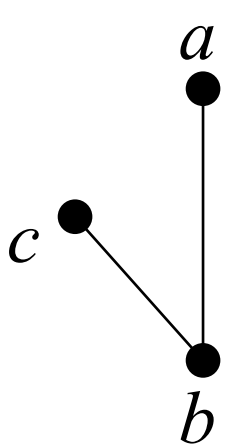
$$
\begin{array}{c}
 \\ a \\ b \\ c \\ d \\ e
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

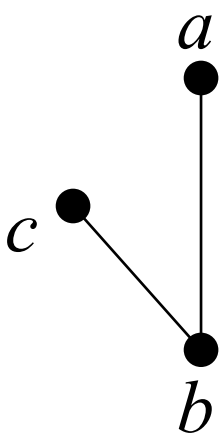


$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Isomorphism of Graphs

กราฟที่มีรูปร่างเหมือนกัน = สมสัณฐาน

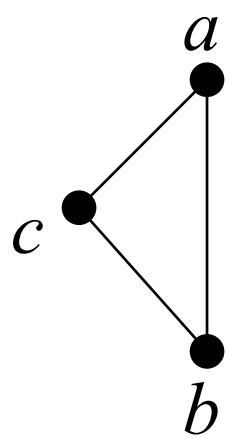




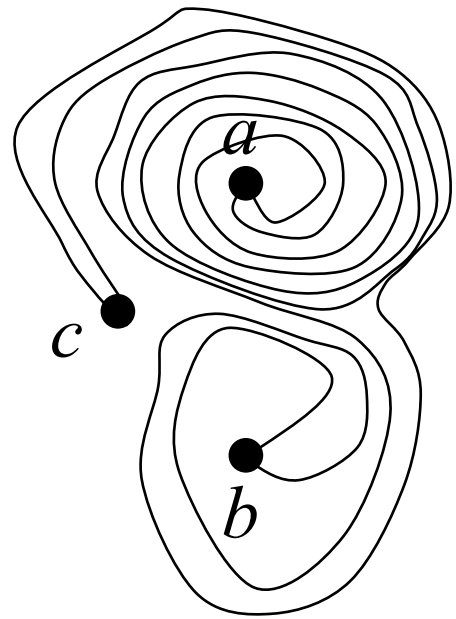
กราฟเดียวกัน

G_1		G_2
a-c	✗	✗
a-b	✓	✓
b-c	✓	✓

Isomorphism of Graphs



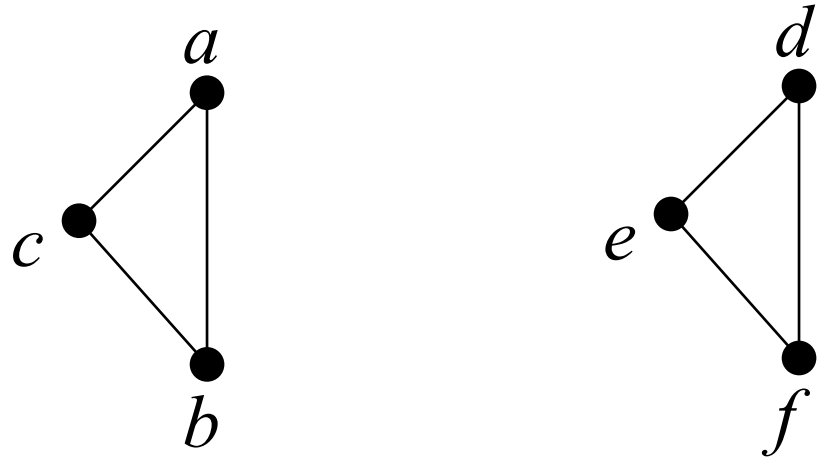
G_1



G_2

กราฟเดียวกัน

Isomorphism of Graphs



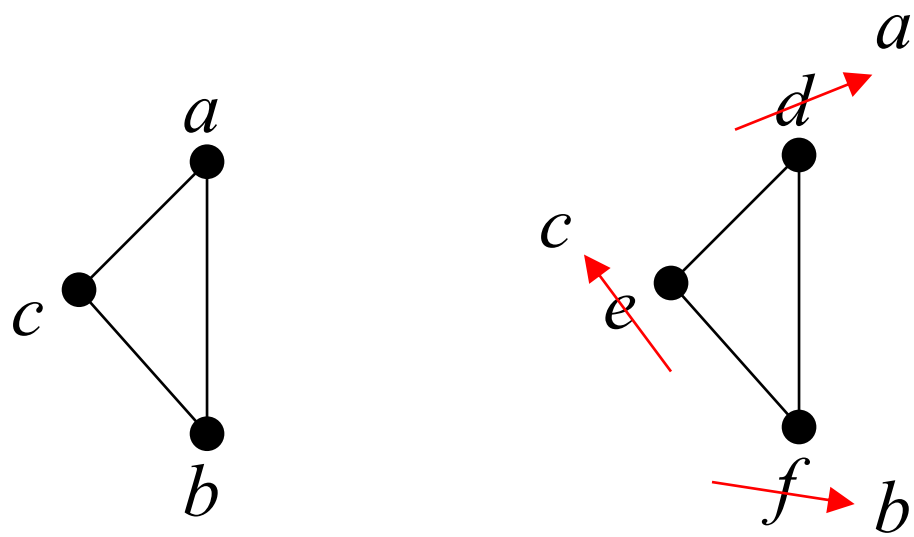
G_1 G_2

คนละกราฟ

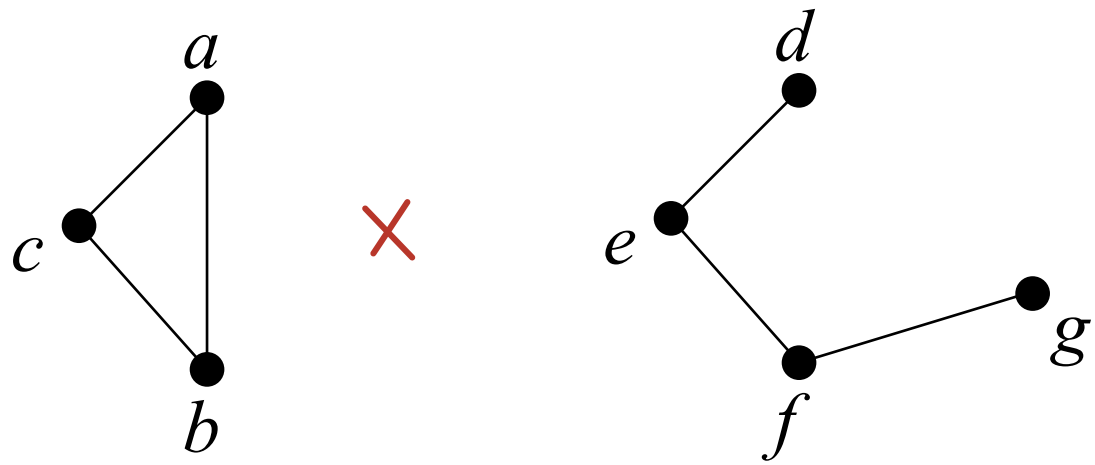
ชื่อ vertex ต่าง

Isomorphism

Isomorphism of Graphs



Isomorphism of Graphs



Isomorphism of Graphs

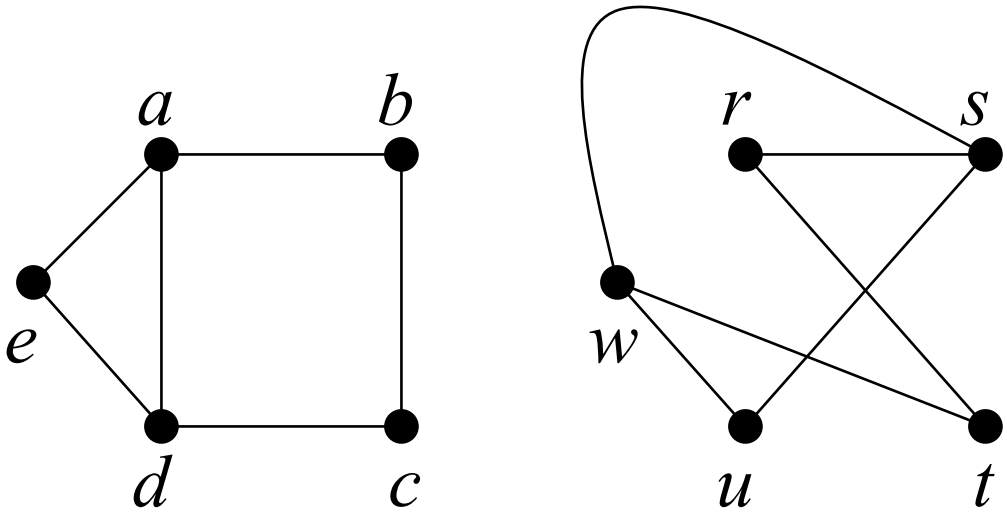
$G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are two simple graphs. G_1 and G_2 are **isomorphic** when:

There is a 1-to-1 and onto function f from V_1 to V_2 with the property that

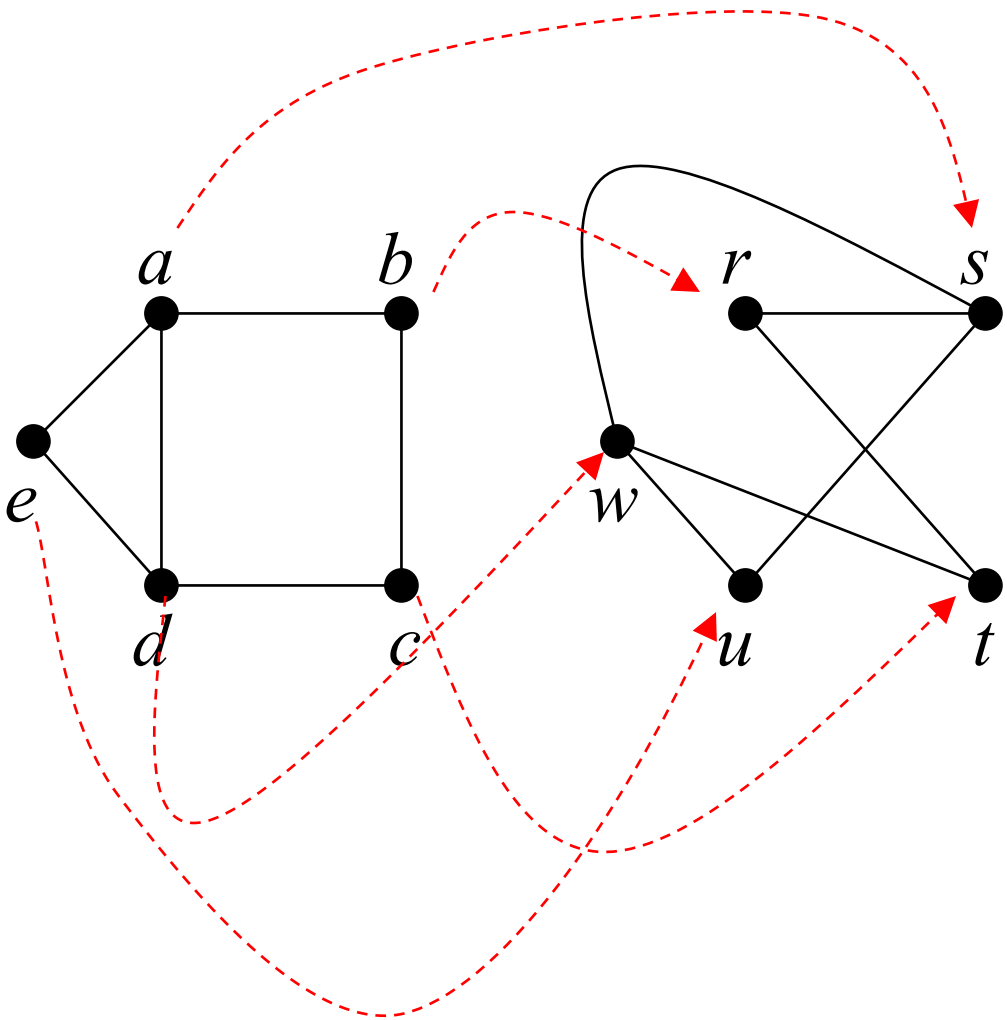
$$\forall_{a,b \in V_1} \quad a \text{ and } b \text{ are adjacent in G1} \leftrightarrow f(a) \text{ and } f(b) \text{ are adjacent in G2.}$$

f is called an **isomorphism**.

Isomorphism of Graphs



Isomorphism of Graphs



Isomorphism, f

$f(a) = s$, $f(b) = r$

$f(c) = t$, $f(d) = w$

$f(e) = u$

$\{a,b\} \rightarrow \{r,s\}$

$\{b,c\} \rightarrow \{r,t\}$

$\{c,d\} \rightarrow \{t,w\}$

$\{d,a\} \rightarrow \{u,w\}$

$\{d,e\} \rightarrow \{s,u\}$

$\{a,e\} \rightarrow \{s,w\}$

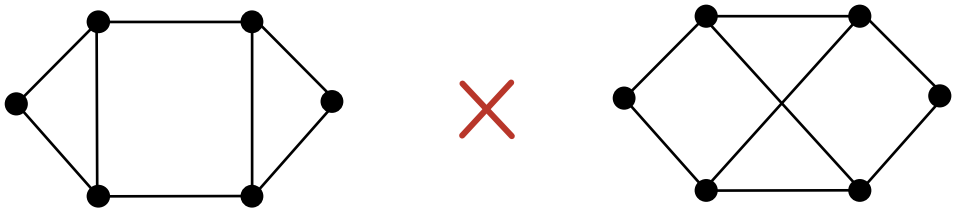
Isomorphism of Graphs

- To show that two graphs are isomorphic

  Find an isomorphism.

- To show that two graphs are NOT isomorphic

  Argue that no isomorphism exists.

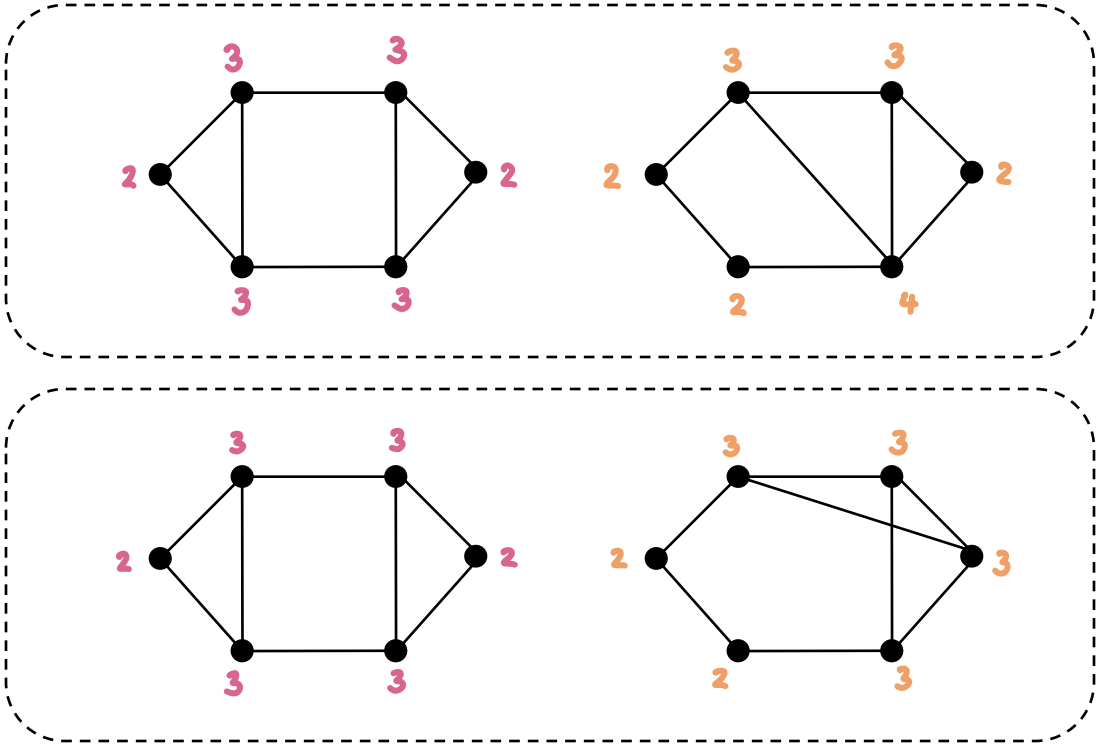


Graph Invariants

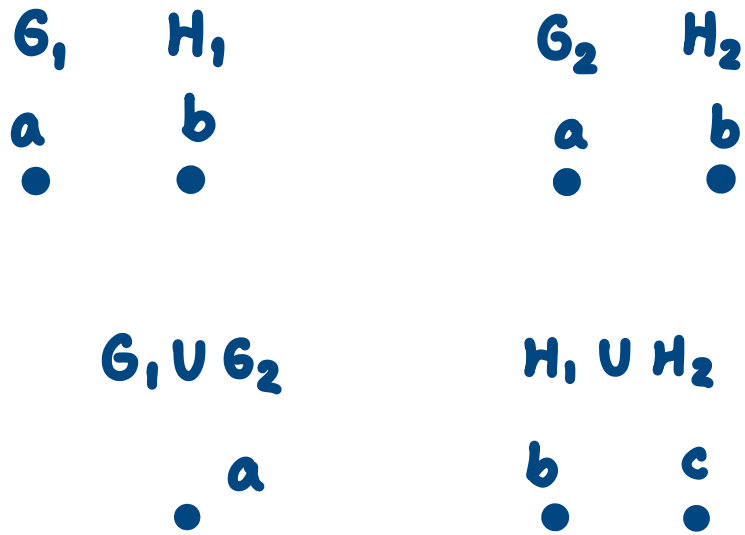
- A property preserved by isomorphism of graphs is called a **graph invariant**.

E.g.:

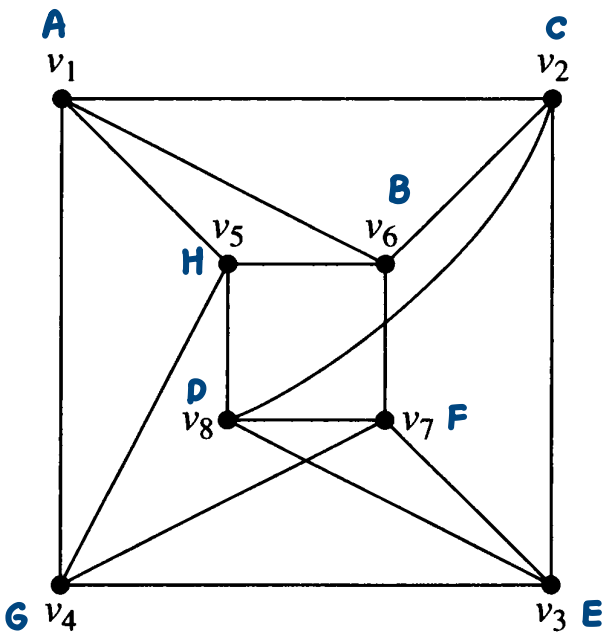
- Number of vertices
- Number of edges
- Degrees of vertices
- Adjacency between vertices with specified degrees

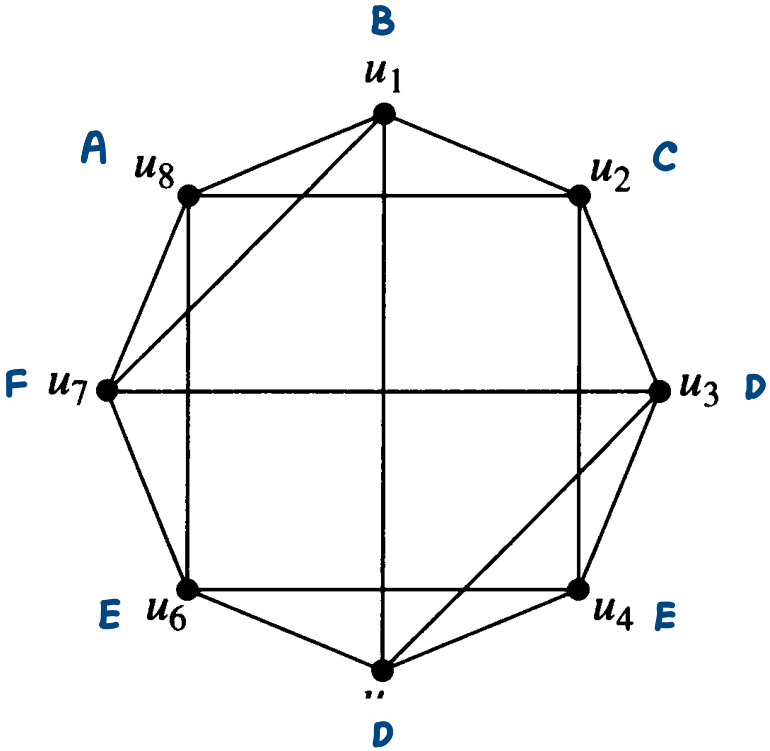


Suppose that G_1 and H_1 are isomorphic and that G_2 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.



Isomorphism of Graphs





tools : geogebra.org/geometry