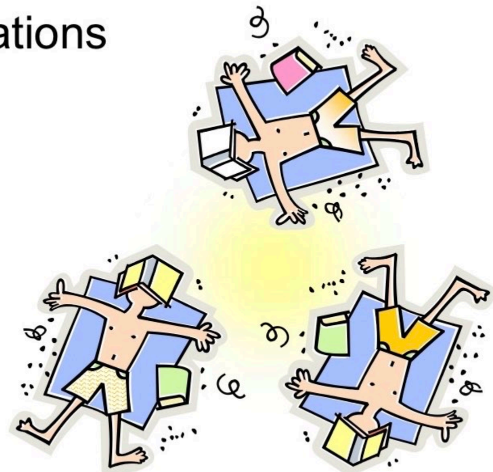
Sections 7.1-7.2

Recurrence Relations



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Recurrence Relations

- A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms, $a_0, a_1, \ldots, a_{n-1}$.

$$a_n = 5a_{n-1}$$

$$b_n = b_{n-1} - 2\, b_{n-2} + 100$$

$$c_n = c_{n-3} + c_{n-4} + \log(n) + e^n$$

- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

- Example

  $0, 3, 6, 9, \ldots$

  Determine whether $a_n = 3n$, for every nonnegative integer n, is a solution of

$$a_n = 2a_{n-1} - a_{n-2}; \quad n = 2, 3, \ldots$$

$$3n = 2 \times 3(n-1) - 3(n-2)$$

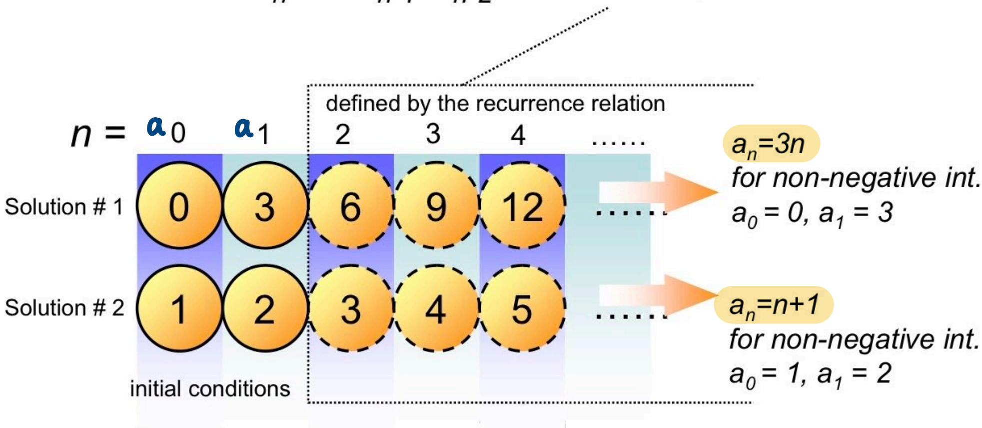
$$3n = 6n - 2 - 3n + 6$$

$$3n = 3n$$

Initial Conditions

The ***initial conditions*** specify the terms that precede the first term where the recurrence relation takes effect.

$$a_n = 2a_{n-1} - a_{n-2}\ ; \quad n = 2, 3, \ldots$$



Initial Conditions

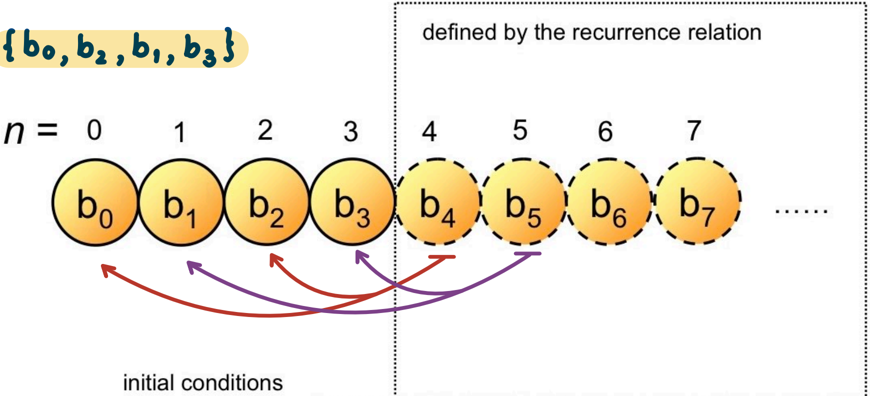
In order to find a *unique solution* for every non-negative integers to:

$$b_n = b_{n-2} + b_{n-4}\ ; \quad n = 4,\ 5,\ \ldots$$

degree 4

how many terms of b_n needed to be given in the initial conditions?

$\{b_0, b_2, b_1, b_3\}$



Modeling with Recurrence Relations

To find solutions for doing a task of a size *n*

<u>Find a way to:</u>

Construct the solution at the size n from the solution of the same tasks at smaller sizes.

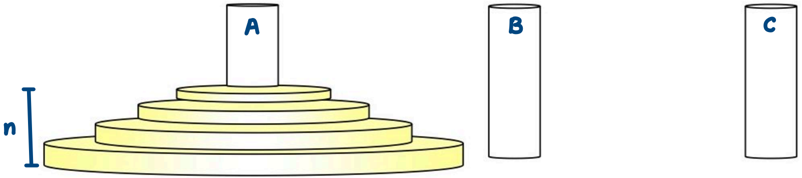
Example: The Tower of Hanoi

Rules:

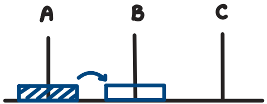
Move a disk at a time from one peg to another.

Never place a disk on a smaller disk.

The goal is to have all disk on the 2nd peg in order of size.

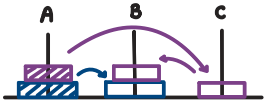


Find H_n, the number of moves needed to solve the problem with n disks.



$n = 1$ # 1

$A \rightarrow B$

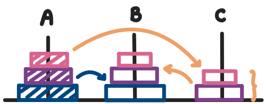


$n = 2$ # = 3

$A \rightarrow C$

$A \rightarrow B$

$C \rightarrow B$



$n = 3$ # 7

ย้าย 2 แผ่น จาก $A \rightarrow C$

$A \rightarrow B$

ย้าย 2 แผ่น จาก $C \rightarrow B$

solve(n,A,B,C)

solve(n-1, A,C,B)

$A \rightarrow B$

solve(n-1,C,B,A)

H_n : # move ที่จะ solve TOH n แผ่น

$H_1 = 1$

$H_2 = 3$

$H_3 = 7$

⋮

$H_n = H_{n-1} + 1 + H_{n-1}$

สูตรปิด

$\boxed{H_n = 2^n - 1 \quad n = 2, 3, \ldots}$

$\boxed{H_n = 2H_{n-1} + 1 \quad ; \quad n = 2, 3, \ldots}$

$\boxed{H_1 = 1}$

or

$\boxed{H_n = 2H_{n-1} + 1 \; ; \; n = 1, 2, 3, \ldots}$

$\boxed{H_0 = 0}$

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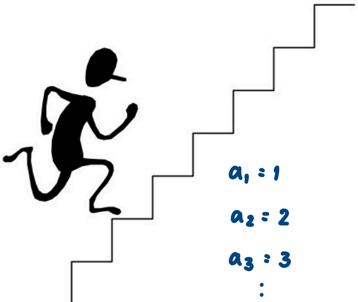
Example:

A man running up a staircase of *n* stairs. Each step he takes can cover either 1 or 2 stairs. How many different ways for him to ascend this staircase?



$a_1 = 1$

$a_2 = 2$

$a_3 = 3$

$\vdots$

a_n : # วิธีในการขึ้นบันได n ขั้น

$a_n = b_n + c_n = a_{n-1} + a_{n-2}$

$$\boxed{\begin{array}{l} a_n = a_{n-1} + a_{n-2} \ ; \ n = 3, 4, \ldots \\ a_1 = 1, \ a_2 = 2 \end{array}}$$

นับโดยแบ่งกรณีตาม "ก้าว"สุดท้าย

กรณีที่ 1 ก้าวสุดท้าย "สั้น"

$b_n = a_{n-1} \times 1 = a_{n-1}$

กรณีที่ 2 ก้าวสุดท้าย "ยาว"

$c_n = a_{n-2} \times 1 = a_{n-2}$

Linear Recurrence Relations with Constant Coefficients

Homogeneous

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

Non-homogeneous

100
e^n
$2n$

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers
and $c_k \neq 0$

Degree = k

Linear Homogeneous Recurrence Rel.

- $a_n = a_{n-1} + a^2_{n-2}$

- $H_n = 2H_{n-1} + 1$

- $B_n = nB_{n-1}$

- $f_n = f_{n-1} + f_{n-2}$

Solving Recurrence Relations

- A ***linear homogeneous*** recurrence relation can be solved in a systematic way.

Some Solutions

Show that:

$$a_n = K_1 r^n$$

where:

K_1 can be any real number and,
we can choose the value of r to be anything.

is a solution of:

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}}$$

degree = k

$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots c_k$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$$a_n = K_1 r_1^n + K_2 r_2^n + \dots + K_k r_k^n$$

$$\begin{aligned} K_1 r_1^n + K_2 r_2^n + \dots + K_k r_k^2 = {} & c_1 K_1 r_1^{n-1} + c_1 K_2 r_2^{n-1} + \dots + c_1 K_k r_k^{n-1} \\ & + c_2 K_1 r_1^{n-2} + c_2 K_2 r_2^{n-2} + \dots + c_2 K_k r_k^{n-2} \\ & \vdots \\ & + c_k K_1 r_1^{n-k} + c_k K_2 r_2^{n-k} + \dots + c_k K_k r_k^{n-k} \end{aligned}$$

$$\frac{r_1^k}{r_1^{k-n}} = \frac{c_1 r_1^{k-1} + c_2 r_1^{k-2} + \dots + c_k}{r_1^{k-n}}$$

$$r_1^n = c_1 r_1^{n-1} + c_2 r_1^{n-2} + \dots c_k r_1^{n-k}$$

$$K_1 r_1^n = c_1 K_1 r_1^{n-1} + c_2 K_1 r_1^{n-2} + \dots + c_k K_1 r_1^{n-k}$$

Some Solutions

Show that:

$$a_n = K_1 r_1^n + K_2 r_2^n + \ldots + K_k r_k^n$$

where:

K_i can be any real number,
each r_i is a root of $r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$
and there are k distinct r_i's.

is a solution of:

$$a_n = c_1 a_{n-1} + c_1 a_{n-2} + \ldots + c_k a_{n-k}$$

So far, we have found that ...

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

has "a" solution in the form of

$$\boxed{a_n = K_1 r_1^n + K_2 r_2^n + \dots + K_k r_k^n}$$

where all r_i's are the distinct roots of $r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$

Characteristic Equation

? Do all solutions have to be in this form?

Prove later!

Unique Solution

$$a_n = K_1 r_1^n + K_2 r_2^n + \ldots + K_k r_k^n$$

Without fixing the constants,
they are not unique.

E.g.: $a_n = K_1(2)^n + K_2(3)^n$

$a_n = 2(2)^n + 1(3)^n \longrightarrow$ 3, 7, 17, …

$a_n = 1(2)^n + 2(3)^n \longrightarrow$ 3, 8, 22, …

$a_n = 3(2)^n + 3(3)^n \longrightarrow$ 6, 15, 39, …

Must fix the initial conditions

Find the solution of

$k=2$

$$a_n = -5a_{n-1} - 6a_{n-2} \qquad n = 2,3,4,\ldots$$

if $a_0 = 3, a_1 = 7$

<u>Char Eq</u> $r^2 = -5r - 6$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -2$

$r_2 = -3$

$a_n = A(-2)^n + B(-3)^n$

$a_0 = 3 = A + B$ — (1)

$a_1 = 7 = -2A - 3B$ — (2)

$\therefore A = 16$

$B = -13$

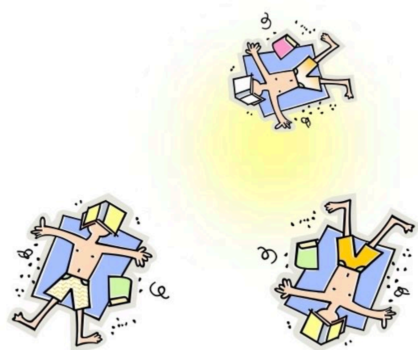
$\therefore a_n = 16(-2)^n - 13(-3)^n$

Next Step

- Prove that all solutions must be in the form we have just shown in this lecture.
- What if the characteristic equation of the k order have less than k distinct roots?
- How to solve a linear non-homogeneous recurrence relation?

Sections 7.1-7.2

Recurrence Relations (Continue)



Solutions to Linear Homogeneous Recurrence Relations

Sufficient Condition

Any sequences in the form: $a_n = K_1 r_1^n + K_2 r_2^n + ... + K_k r_k^n$

is a solution to: $a_n = c_1 a_{n-1} + c_1 a_{n-2} + ... + c_k a_{n-k}$

Necessary Condition

All solutions to: $a_n = c_1 a_{n-1} + c_1 a_{n-2} + ... + c_k a_{n-k}$

must be in the form: $a_n = K_1 r_1^n + K_2 r_2^n + ... + K_k r_k^n$

when the characteristic equation of the recurrence relation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + ... + c_k$$

has *k* distinct roots which are $r_1, r_2, ..., r_k$

Proof of the Necessary Condition

Solve Recurrence Relation

$R: a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k}$

degree = k

Char Eq

$\underbrace{r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k}$

มี k รากไม่ซ้ำกัน $r_1, r_2, \dots, r_k$

$R: a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$
Char Eq มีราก $r_1, r_2, \dots, r_k$
พิสูจน์ว่า ทุกๆ ค่าตอบต้องเขียนได้ในรูป
$a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n$

ข้อสรุป 1 ลำดับ $a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n$

เป็นค่าตอบหนึ่งของ R

แต่ ทุกๆ ลำดับที่เป็นค่าตอบของ R

ต้องอยู่ในรูป

$a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n$

หรือไม่

ต้อง **

ใช้ Proove by Contradiction

① สมมติว่ามี a_n ที่เป็นคำตอบของ R
แต่ไม่สามารถเขียนใส่ในรูป $a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n$

② กำหนด $b_n = E_1 r_1^n + E_2 r_2^n + \dots + E_k r_k^n$
เราบอกได้ว่า b_n ต้องเป็นคำตอบของ R แน่นอน

③ เราสามารถเลือกค่า $E_1, E_2, \dots, E_k$ ที่ทำให้

$$b_0 = E_1 + E_2 + \dots + E_k = a_0 \quad \text{—(1)}$$

$$b_1 = E_1 r_1^2 + E_2 r_1^2 + \dots + E_k r_1^2 = a_1 \quad \text{—(2)}$$

$$\vdots$$

$$b_{k-1} = E_1 r_1^{k-1} + E_2 r_1^{k-1} + \dots + E_k r_1^{k-1} = a_{k-1} \quad \text{—(k)}$$

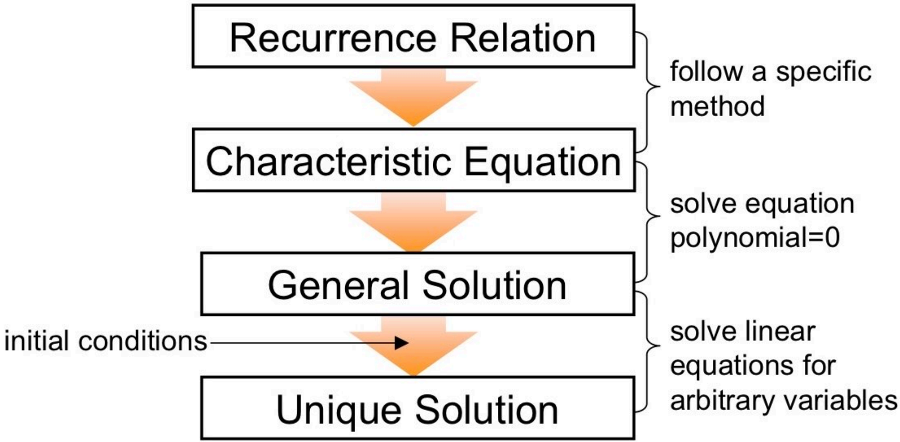
ได้

④ เนื่องจากทั้ง a_n และ b_n เป็นคำตอบของ R (ซึ่งมี degree = k)
และ k-1 พจน์แรกของทั้งสองลำดับเท่ากัน

⑤ จาก ④ เราแสดงให้เห็นว่า a_n และ b_n เท่ากันได้
ข้อความที่ว่า a_n ไม่สามารถเขียนได้ในรูป

$a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n$ **ไม่เป็นจริง**

Solving: Linear Homogeneous Recurrence Relations



- Example:

What is the solution of the recurrence relation:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$?

Char Eq $r^3 = 6r^2 - 11r + 6$

$r^3 - 6r^2 + 11r - 6 = 0$

$(r-2)(r^2 - 4r + 3) = 0$

$(r-2)(r-1)(r-3) = 0$

$\underline{r}_1 = 1,\ \underline{r}_2 = 2,\ \underline{r}_3 = 3$

$\therefore a_n = A(1)^n + B(2)^n + C(3)^n$

$a_0 = 2 = A + B + C$ — 1

$a_1 = 5 = A + 2B + 3C$ — 2

$a_2 = 15 = A + 4B + 9C$ — 3

$\therefore A = 1,\ B = -1,\ C = 2$

$\therefore a_n = 1^n - 2^n + 2(3^n)$

$= -2^n + 2(3^n)$

Repeated Roots

- Suppose the characteristic equation has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$.
- Solution:

$$\boxed{\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m1-1}n^{m1-1})r_1^n \\ & +(\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m2-1}n^{m2-1})r_2^n \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +(\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,mt-1}n^{mt-1})r_t^n \end{aligned}}$$

- <u>Example</u> :

What is the solution of the recurrence relation:

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$ $\rightarrow$ degree = 3

with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?

<u>Char Eq</u> $r^3 = -3r^2 - 3r - 1$

$r^3 + 3r^2 + 3r + 1 = 0$

$(r+1)(r+1)(r+1) = 0$

$r_1 = -1$

$m_1 = 3$

$\therefore a_n = (A + Bn + Cn^2)(-1)^n$

$a_0 = 1 = A$ — 1

$a_1 = -2 = -A - B - C$ — 2

$a_2 = -1 = A + 2B + 4C$ — 3

$\therefore A = 1, B = 3, C = -2$

$\therefore a_n = (1 + 3n - 2n^2)(-1)^n$

Solving: Linear <u>Nonhomogeneous</u> Recurrence Relations

$$\underbrace{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}}_{\text{Associated homogeneous recurrence relation}} + F(n)$$

$\{a_n^h\}$ $\{a_n^p\}$

$$\boxed{\{a_n\} = \{a_n^h\} + \{a_n^p\}}$$

$$\underbrace{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}}_{\text{Homogeneous Part}} + F(n)$$

ให้คำตอบ $a_n = a_n^h + a_n^p$

$a_n^h \rightarrow$ Homogeneous Solution

$a_n^p \rightarrow$ Particular Solution

เป็นคำตอบของ Homogeneous Part

$$\cancel{a_n^h} + a_n^p = \cancel{c_1 a_{n-1}^h} + c_1 a_{n-1}^p + \cancel{c_2 a_{n-2}^h} + c_2 a_{n-2}^p + \dots + \cancel{c_k a_{n-k}^h} + c_k a_{n-k}^p + F(n)$$

$$a_n^p = c_1 a_{n-1}^p + c_2 a_{n-2}^p + \dots + c_k^p a_{n-k} + F(n)$$

$$\underline{\underline{F(n)}} = a_n^p - c_1 a_{n-1}^p - c_2 a_{n-2}^p - \dots - c_k^p a_{n-k}$$

เช่น $F(n) = 7^n$

$a_n^p = A$ ✗

$a_n^p = An + B$ ✗

$a_n^p = \underline{\underline{A}} \times 7^n$

Solving: Linear Nonhomogeneous Recurrence Relations

- Key:

  1 – Solve for a solution of the associated homogeneous part.

  2 – Find a particular solution.

  3 – Sum the solutions in 1 and 2
- There is no general method for finding the particular solution for every *F(n)*
- There are general techniques for some *F(n)* such as *polynomials* and *powers of constants*.

Particular Solutions

$$F(n) = (\, b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0 \,)\, s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

<u>When ***s*** is **not** a root of the characteristic equation:</u>

The particular solution is of the form:

$$(\, p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0 \,)\, s^n$$

<u>When ***s*** is a root of multiplicity ***m***:</u>

The particular solution is of the form:

$$n^m (\, p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0 \,)\, s^n$$

- Example:

Find the solutions of $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$

$F(n) = 2n$

$a_n = a_n^h + a_n^p$

หา a_n^h

char Eq $r = 3$

$\therefore a_n^h = A3^n$

หา a_n^p

เนื่องจาก $F(n) = 2n$

ดังนั้น $a_n^p = Bn + C$

$Bn + C = 3(B(n-1) + C) + 2n$

$Bn + C = 3Bn - 3B + 3C + 2n$

$Bn = 3Bn + 2n$

$-2Bn = 2n$

$\therefore B = -1$

$C = -3B + 3C$

$C = 3 + 3C$

$-2C = 3$

$\therefore C = -3/2$

$\therefore a_n = a_n^h + a_n^p = A3^n - n - \frac{3}{2}$

$a_1 = 3 = A3 - 1 - \frac{3}{2}$

$\therefore A = \frac{11}{6}$

$\therefore a_n = \frac{11}{6}(3^n) - n - \frac{3}{2}$

- <u>Example</u>:

  Find the solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$a_n = a_n^h + a_n^p$

<u>หา a_n^h</u> <u>char Eq</u>

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r_1 = 2$

$r_2 = 3$

$\therefore a_n^h = A2^n + B3^n$

<u>หา a_n^p</u>

$\because F(n) = 7^n$

$\because a_n^p = C7^n$

แทนค่า $a_n^p = C7^n$ ลงใน R

ได้ $C7^n = \frac{5}{7} C7^n - \frac{6C}{49} 7^n + 7^n$

$C = \frac{5}{7} C - \frac{6C}{49} + 1$

$\left(1 - \frac{5}{7} + \frac{6}{49}\right) C = 1$

$\left(\frac{49 - 35 - 6}{49}\right) C = 1$

$\frac{8}{49} C = 1$

$\therefore C = 49/8$

$\therefore a_n^p = \frac{49}{8} \times 7^n$

$= \frac{7^{n+2}}{8}$

$\therefore a_n = A2^n + B3^n + \frac{7^{n+2}}{8}$

- <u>Example</u>:

  What form does a particular solution of

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

  have when:

  $F(n) = 3n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, $F(n) = (n^2+1)3^n$?

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = 3$

$m_1 = 2$

$a_n^h = (An + B)3^n$

$F(n) = (3n+0)(1)^n$

$a_n^p = p_1 n + p_0$

$a_n^p = n^2(p_1 n + p_0)3^n$

$= (p_1 n^3 + p_0 n^2)3^n$

$a_n^p = (p_2 n^2 + p_1 n + p_0)2^n$

$a_n^p = n^2(p_2 n^2 + p_1 n + p_0)3^n$

$= (p_2 n^4 + p_1 n^3 + p_0)3^n$

Summary

Linear Recurr.Rel. w/ Const. Coeff.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

Solution

$$a_n = a_n^h + a_n^p$$

Find the homogeneous solution

Consider only the homogeneous part

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Write characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

Root: $r_1, r_2, \ldots, r_t$

$$\begin{aligned} a_n^h = \; & (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m1-1} n^{m1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m2-1} n^{m2-1}) r_2^n \\ & + \ldots \\ & + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,mt-1} n^{mt-1}) r_t^n \end{aligned}$$

$F(n)=0$?

Yes: $a_n^p = 0$

No: a_n^p depends on $F(n)$

Find the particular solution

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

$$a_n^p = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

s equal a root w/ multiplicity ***m*** (***m***=0 if N/A)

Find all variables that makes a_n^p a solution of the recurr. rel.

General form of all solutions

$$a_n = a_n^h + a_n^p$$

Use Initial Conditions → The unique solution

- <u>Example</u> :

  Find the solution of $a_n = \sum_{k=1}^{n} k$