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Discrete Structures

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Mathematical Induction

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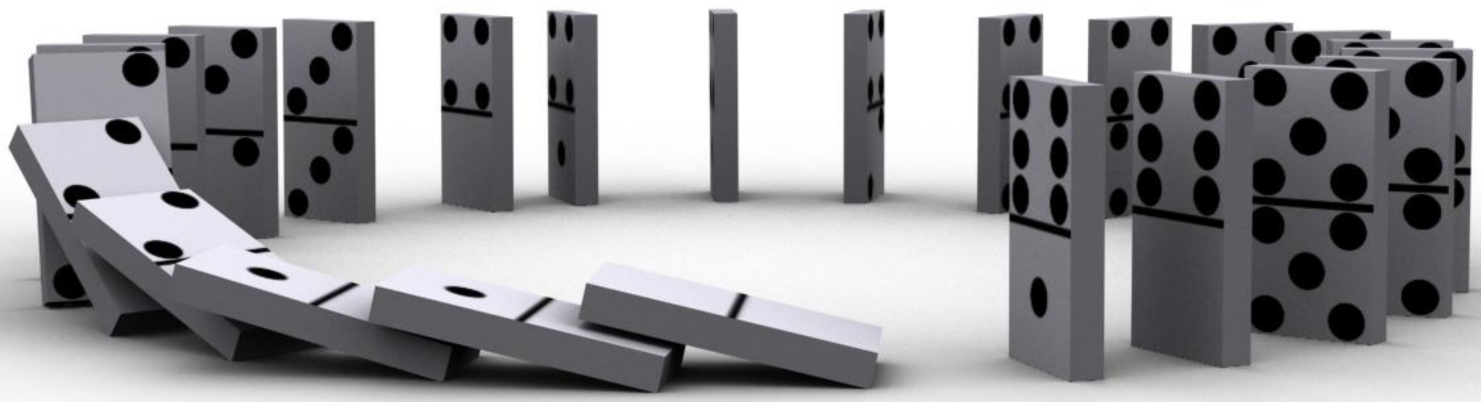

Mathematical Induction

A proof by induction that *P(n)* is true for every **positive integer** *n* consists of 2 steps:

BASIS STEP: Show that *P(1)* is true.

INDUCTIVE STEP:
Show that *P(k)*→*P(k+1)* is true for every positive integer *k* ≥ 1

$P(n) : n! > 0$

$n = 1, 2, 3, \ldots$

Mathematical Induction

Sometimes we want to prove that $P(n)$ is true for $n = b, b+1, b+2, \ldots$ where b is an integer other than 1.

BASIS STEP: Show that $P(b)$ is true.

INDUCTIVE STEP:
Show that $P(k) \rightarrow P(k+1)$ is true for every positive integer $k \geq b$

$Q(n)$ $n = 10, 11, 12$

Example :

Prove that the sum of the first n odd positive integers is n^2.

P(n): $1+3+5+\ldots+(2n-1) = n^2 \;;\; \forall n \geq 1$

Basis Step: $P(1) : 1 = 1^2$ true $\therefore P(1) \equiv T$

Inductive Step: $\forall k \geq 1$ Show $P(k) \rightarrow P(k+1)$

Assume $P(k)$ Show $P(k+1)$

Consider $1+3+\ldots+(2(\underbrace{k+1}_{n})-1) = \underbrace{1+3+\ldots+(2k+1)}_{P(k)} + (2(k+1)-1)$

$= k^2+2k+1$

$= (k+1)^2$

Q.E.D.

Example :

Prove that $n < 2^n$ for all positive integers n.

P(n): $\forall n \geq 1 \ (n < 2^n)$

Basis Step: Show $P(1)$ $\quad 1 < 2^1 = 2$ $\quad$ True $\quad \therefore P(1)$

Inductive Step: $\forall k \geq 1 \ (P(k) \rightarrow P(k+1))$

Assume $P(k) : k < 2^k$

Show $P(k+1)$

Consider $k+1 < 2^k + 1$

$< 2^k + 2^k$

$< 2^{k+1}$ $\quad$ Q.E.D.

Example :

Prove that n^3-n is divisible by 3 all positive integers n.

P(n): $\forall n \geq 1$ $(\exists m \in \mathbb{Z}\ (n^3 - n = 3m))$

$(n^3 - n \bmod 3 = 0)$

Basis Step: Show $P(1)$ $\quad 1^3 - 1 = 0 = 3 \times 0 \quad \therefore P(1)$

Inductive Step: $\forall k \geq 1\ (P(k) \rightarrow P(k+1))$

Assume $P(k)$ $\quad k^3 - k = 3m$

Show $P(k+1)$

Consider $\quad (k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$

$= (k^3 - k) + (3k^2 + 3k)$

$= (k^3 - k) + 3(k^2 + k) \qquad$ Q.E.D.

Proving Mathematical Induction

The well-ordering property

> Every nonempty set of nonnegative integers has a least element.

$S = \{ \ldots . \; 7 \; \ldots . \}$ infinite

$\downarrow$ non-neg int

$\{ \ldots \ldots \}$ $r \leq k$ ← k = non-neg int

มี least element

Proving Mathematical Induction

Show that *P(n)* must be true for all positive integers when *P(1)* and *P(k)*→*P(k+1)* are true.

Assume that *P(n)* is not true for at least a positive integer. Then, the set *S* for which *P(n)* is false is nonempty.

S has the least element, called *m*. ($m \neq 1$)

Since $m-1 < m$, then $m-1 \notin S$ (or *P(m-1)* is true)

But *P(m-1)*→*P(m)* is true. So, *P(m)* must be true.

This contradicts the choice of *m*.

ต้องการ $\forall n \geq b\,(P(n))$

ถ้า $P(b) \land \forall k \geq b\,(P(k) \rightarrow P(k+1))$

$$T \equiv (\underbrace{(P(b) \land \forall k \geq b\,(P(k) \rightarrow P(k+1)))}_{T} \rightarrow \forall n \geq b\,(P(n)))$$

Assume $P(b)$

$\forall k \geq b\,(P(k) \rightarrow P(k+1))$

Show $\forall n \geq b\,(P(n))$

Prove by contradiction,

Assume $\exists m \geq b\,(\neg P(m))$

Let $S = \{m \geq b \mid \neg P(m)\} \neq \emptyset$ (by assumption)

By well-ordering principle,

Let r be the smallest non-neg in S

$\therefore \neg P(r)$

We have $P(b)$ $\therefore b \notin S$ $\therefore r > b$

$r-1 \geq b$

$r-1 \notin S$

$\therefore P(r-1)$

$\therefore P(r)$

$\therefore r \notin S$

contradiction

Exercises

Use mathematical induction to show that

1) $1+2+2^2+2^3+2^4+\ldots+2^n = 2^{n+1} - 1$

   for all nonnegative integer n.

2) $2^n < n!$ for every positive integer n with $n \geq 4$.

① Proof : By M.I.

1) Basis step : $P(0)$

$$2^0 = 1 = 2^{0+1} - 1 \quad \therefore P(0)$$

2) $\forall k \geq 0\ (P(k) \rightarrow P(k+1))$

Assume $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$

Show $P(k+1)$

$$1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$\therefore P(k+1)$ Q.E.D.

② Proof: By M.I., $\forall n \geq 4\ (2^n < n!)$

1) Basic step : show $P(4)$ $\quad 2^4 = 16$ $\quad \therefore 2^4 < 4!$

$4! = 24$

$\therefore P(4)$

2) Inductive step : $\forall k \geq 4\ (P(k) \rightarrow P(k+1))$

Assume $2^k < k!$

Show $2^{k+1} < (k+1)!$

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &< 2 \times k! \\ &< (k+1) \times k! && \text{since } k \geq 4 \\ &< (k+1)! && k+1 \geq 5 > 2 \end{aligned}$$

Q.E.D

Strong Induction

A proof by induction that *P(n)* is true for every positive integer *n* consists of 2 steps:

Use a different induction step.

BASIS STEP: Show that *P(1)* is true.

INDUCTIVE STEP:

Show that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for every positive integer *k*

Example :

12 4 4 4
13 4 4 5 case 1 : มี 4 → 5
14 4 5 5 case 2 : ไม่มี 4 5 5 5 → 4 4 4 4
15 5 5 5
16 4 4 4 4

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction. Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true. That is, under this hypothesis, postage of k cents can be formed using 4-cent and 5-cent stamps. To complete the inductive step, we need to show that when we assume $P(k)$ is true, then $P(k+1)$ is also true where $k \geq 12$. That is, we need to show that if we can form postage of k cents, then we can form postage of $k+1$ cents. So, assume the inductive hypothesis is true; that is, assume that we can form postage of k cents using 4-cent and 5-cent stamps. We consider two cases, when at least one 4-cent stamp has been used and when no 4-cent stamps have been used. First, suppose that at least one 4-cent stamp was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of $k+1$ cents. But if no 4-cent stamps were used, we can form postage of k cents using only 5-cent stamps. Moreover, because $k \geq 12$, we needed at least three 5-cent stamps to form postage of k cents. So, we can replace three 5-cent stamps with four 4-cent stamps to form postage of $k+1$ cents. This completes the inductive step.

Because we have completed the basis step and the inductive step, we know that $P(n)$ is true for all $n \geq 12$. That is, we can form postage of n cents, where $n \geq 12$ using just 4-cent and 5-cent stamps. This completes the proof by mathematical induction.

Next, we will use strong induction to prove the same result. In this proof, in the basis step we show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true, that is, that postage of 12, 13, 14, or 15 cents can be formed using just 4-cent and 5-cent stamps. In the inductive step we show how to get postage of $k+1$ cents for $k \geq 15$ from postage of $k-3$ cents.

BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively. This shows that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. This completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$. To complete the inductive step, we assume that we can form postage of j cents, where $12 \leq j \leq k$. We need to show that under the assumption that $P(k+1)$ is true, we can also form postage of $k+1$ cents. Using the inductive hypothesis, we can assume that $P(k-3)$ is true because $k-3 \geq 12$, that is, we can form postage of $k-3$ cents using just 4-cent and 5-cent stamps. To form postage of $k+1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k-3$ cents. That is, we have shown that if the inductive hypothesis is true, then $P(k+1)$ is also true. This completes the inductive step.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that $P(n)$ is true for all integers n with $n \geq 12$. That is, we know that every postage of n cents, where n is at least 12, can be formed using 4-cent and 5-cent stamps. This finishes the proof by strong induction.

<u>Example</u> :

Show that if *n* is an integer greater than *1*, then *n* can be written as the product of primes.

P(n): $\forall n > 1\ (\exists p_1, p_2 \ldots p_s \in \text{Prime} \text{ such that } n = \prod_{i=1}^{s} p_i)$

<u>Basis Step</u>: $P(2)$ $2 = 2$ $\therefore P(2)$

<u>Inductive Step:</u>

Assume $\forall k \geq 2,\ P(2) \wedge P(3) \ldots \wedge P(k)$

Show $P(k+1)$

Consider $k+1 \geq 3$

Prove by cases

1) $k+1 \in \text{Prime}$ $\therefore P(k+1)$

2) $k+1 \notin \text{Prime}$

$\therefore \exists x_1, x_2 \in \mathbb{Z}\ ;\ x_1 \cdot x_2 = k+1$

$1 < x_1, x_2 < k+1$

$\therefore P(x_1) \wedge P(x_2)$ $x_1 = \prod_{i=1}^{r} p_i$ $x_2 = \prod_{j=1}^{s} p_j$

$\therefore k+1 = \left(\prod_{i=1}^{r} p_i\right)\left(\prod_{j=1}^{s} p_j\right)$

Q.E.D.