Sections 5.4

- Binomial Coefficients
- Combinatorial Proof

The Binomial Theorem

- A binomial expression is the sum of two terms, such as $(x + y)$
- The binomial theorem concerns the expansion of powers of binomial expressions.
- E.g.: $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

The Binomial Theorem

$(x+y)^3 = (x+y)(x+y)(x+y)$

$x^3 \longrightarrow C(3,3)$

$x^2y \longrightarrow C(3,2)$

$xy^2 \longrightarrow C(3,1)$

$y^3 \longrightarrow C(3,0)$

- C(n,r) is also called a ***binomial coefficient***.

The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} c(n,j)x^{n-j}y^j$$

$$= c(n,0)x^n + c(n,1)x^{n-1}y + c(n,2)x^{n-2}y^2 + \ldots + c(n,n-1)xy^{n-1} + c(n,n)y^n$$

Example

$(x+y)^4 =$ $x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + y^4$

What is the coef. of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

$\binom{25}{12} = \binom{25}{13}$

- Example:

  What is the coef. of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$C(25,12) \times 2^{12} \times (-3)^{13}$$

- Show that:

  Let *n* be a positive integer. Then:

$$\sum_{k=0}^{n}(-1)^k c(n,k) = 0$$

$$(x+y)^n = \sum_{k=0}^{n} c(n,k)\, x^{n-k} y^k$$

$$0 = (1+(-1))^n = \sum_{k=0}^{n} c(n,k)\,(-1)^k$$

- Show that:

  Let *n* be a nonnegative integer. Then:

$$\sum_{k=0}^{n} 2^k c(n,k) = 3^n$$

$$3^n = (1+2)^n = \sum_{k=0}^{n} c(n,k)(2)^k$$

Combinatorial Proof

- A combinatorial proof is a proof that uses counting arguments to prove a theorem rather than some other method such as algebraic techniques.

$$A = B$$

\# วิธีในการ
ทำบางอย่าง

\# วิธีในการทำ
อย่างนั้นเหมือนกัน

- <u>Example</u> :

  Prove that when n and r are nonnegative integers with $n \leq r$, $C(n,r) = C(n,n-r)$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}$$

- <u>Example</u> :

Prove that $\boxed{\sum_{k=0}^{n}\binom{n}{k}}=2^n$

พิจารณาจำนวน bit string ความยาว n ที่แตกต่างกัน

① การสร้าง bit string ยาว n อาจเกิดจากขั้นตอนย่อยๆ n ขั้นตอน แต่ละขั้นตอน คือการเลือก bit ที่ i ($i = 1, 2, \dots, n$)

∴ จากกฎการคูณ จำนวน bit string ความยาว n ทั้งหมดจึงมีได้ 2^n แบบ

② แบ่งกรณีของ bit string ความยาว n เป็น n+1 กรณี โดยที่กรณีที่ i สอดคล้องกับ bit string ความยาว n ที่มี 0 ทั้งสิ้น i ตัว ($i = 0, 1, \dots, n$) ในแต่ละกรณี i ใดๆ # bit string 0 = $\binom{n}{i}$

∴ จากกฎการบวก # bit string ทั้งหมด = $\sum_{i=0}^{n}\binom{n}{i}$

③ จาก ① และ ② จะเห็นว่าทั้ง 2^n และ $\sum_{i=0}^{n}\binom{n}{i}$ คือ จำนวน bit string ความยาว n ที่แตกต่างกันทั้งหมดทั้งคู่

ดังนั้น $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ #

<u>Example</u> : Pascal's Identity

Let n and k be positive integers with $n \geq k$, Then

$$c(n+1,k) = c(n,k-1)+c(n,k)$$

<u>Pascal's Triangle</u>

Pascal's identity together with ***initial conditions*** $c(n,0) = c(n,n) = 1$ for all integers n can be used to *recursively define binomial coefficients*.

$c(0,0)$

$c(1,0)$ $c(1,1)$

$c(2,0)$ $c(2,1)$ $c(2,2)$

$c(3,0)$ $c(3,1)$ $c(3,3)$ $c(3,3)$

$c(4,0)$ $c(4,1)$ $c(4,2)$ $c(4,3)$ $c(4,4)$

Example :

Combinatorial Proof of Pascal's Identity:

$$c(n+1,k) = c(n,k-1)+c(n,k)$$

Proof: We will use a combinatorial proof. Suppose that T is a set containing $n+1$ elements. Let a be an element in T, and let $S = T - \{a\}$. Note that there are $\binom{n+1}{k}$ subsets of T containing k elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a. Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S, there are $\binom{n}{k-1}$ subsets of k elements of T that contain a. And there are $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S. Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

◁

<u>Example</u> : Vandermonde's Identity

- Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then:

$$\boxed{c(m+n,r)=\sum_{k=0}^{r} c(m,r-k)c(n,k)}$$

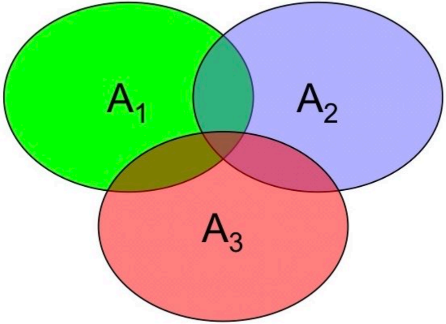
<u>Combinatorial Proof:</u>

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the second set and then $r-k$ elements from the first set, where k is an integer with $0 \le k \le r$. Because there are $\binom{n}{k}$ ways to choose k elements from the second set and $\binom{m}{r-k}$ ways to choose $r-k$ elements from the first set, the product rule tells us that this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence, the total number of ways to pick r elements from the union also equals $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$.

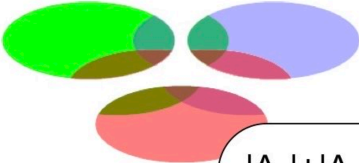
We have found two expressions for the number of ways to pick r elements from the union of a set with m items and a set with n items. Equating them gives us Vandermonde's identity. ◁

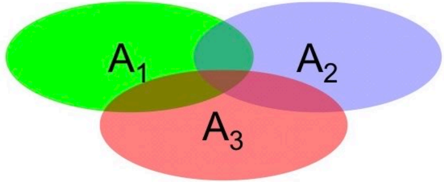
- How many elements are in the union of finite sets?



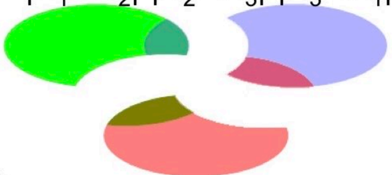
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

$|A_1|+|A_2|+|A_3|$

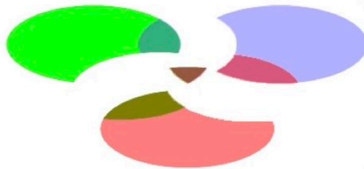




$|A_1|+|A_2|+|A_3|$
$-|A_1 \cap A_2|-|A_2 \cap A_3|-|A_3 \cap A_1|$



$|A_1|+|A_2|+|A_3|$
$-|A_1 \cap A_2|-|A_2 \cap A_3|-|A_3 \cap A_1|$
$+|A_1 \cap A_2 \cap A_3|$



The Principle of Inclusion-Exclusion

- Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\left|A_1 \cup A_2 \cup \cdots \cup A_n\right| = \sum_{1 \le i \le n} \left|A_i\right| - \sum_{1 \le i \le j \le n} \left|A_i \cap A_j\right|$$

$$+ \sum_{1 \le i \le j \le k \le n} \left|A_i \cap A_j \cap A_k\right|$$

$$- \cdots + (-1)^{n+1} \left|A_1 \cap A_2 \cap \cdots \cap A_n\right|$$

- Examples:

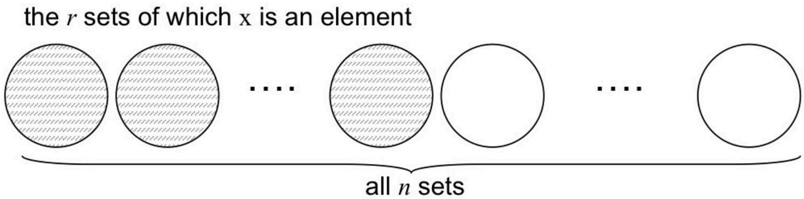
$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4|$$
$$- |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4|$$
$$+ |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Proof: Inclusion-Exclusion Principle

- Showing that an element in the union is counted exactly once.

Let x be an element of exactly r sets.

For example, x is an element of A_1, A_2, …,A_r,
But not of A_{r+1}, A_{r+2}, …,A_n.



$$\sum_{1 \le i \le n} |A_i| \longrightarrow r$$

$$- \sum_{1 \le i1 \le i2 \le n} |A_{i1} \cap A_{i2}| \longrightarrow C(r,2)$$

$$+ \sum_{1 \le i1 \le i2 \le i3 \le n} |A_{i1} \cap A_{i2} \cap A_{i3}| \longrightarrow C(r,3)$$

$$\ddots$$

$$- \cdots + (-1)^{r+1} \sum_{1 \le i1 \le i2 \le \cdots \le ir \le n} |A_{i1} \cap A_{i2} \cap \cdots \cap A_{ir}| \longrightarrow C(r,r)$$

$$+ \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

$$\boxed{\binom{r}{1}+\binom{r}{2}-\binom{r}{3}-\dots+(-1)^{r+1}\binom{r}{r}=1}$$

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

$$(1+(-1))^n=0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\dots+(-1)^n\binom{n}{n}$$

$$0=\binom{r}{0}-\binom{r}{1}+\binom{r}{2}-\dots+(-1)^r\binom{r}{r}$$

$$\binom{r}{1}+\binom{r}{2}-\dots+(-1)^{r+1}\binom{r}{r}=\binom{r}{0}=1$$

Using the Formula to find $|A_1 \cap A_2 \cap \ldots \cap A_n|$

$$
\begin{aligned}
|A_1 \cap A_2 \cap \ldots \cap A_n| &= |U| - |(A_1 \cap A_2 \cap \ldots \cap A_n)'| \\
&= |U| - |A'_1 \cup A'_2 \cup \ldots \cup A'_n| \\
&= |U| - \underbrace{|B_1 \cup B_2 \cup \ldots \cup B_n|}_{\text{Use the formula}}
\end{aligned}
$$

Let $B_i = A'_i$

Therefore, to find the number of elements in an intersection of sets

Another Notation

- To find elements with all properties $Q_1, Q_2, .., Q_n$
- Define properties $P_1, P_2, .., P_n$ *so that* P_i *is the opposite of* Q_i
- Let A_i be the subset of elements with property P_i.
- Let $N(P_1'P_2'\cdots P_n')$ denote the number of elements with none of the properties $P_1, P_2, .., P_n$

$$N(Q_1Q_2\cdots Q_n) = N(P_1'P_2'\cdots P_n') = N - \left|A_1 \cup A_2 \cup \cdots \cup A_n\right|$$

where N = the total number of elements.

$$N(P_1'P_2'\cdots P_n') = N - \left|A_1 \cup A_2 \cup \cdots \cup A_n\right|$$

$$N(P_1'P_2'\cdots P_n') = N - \Big(\sum_{1\le i\le n} \left|A_i\right| - \sum_{1\le i\le j\le n} \left|A_i \cap A_j\right|$$

$$+ \sum_{1\le i\le j\le k\le n} \left|A_i \cap A_j \cap A_k\right|$$

$$- \cdots + (-1)^{n+1} \left|A_1 \cap A_2 \cap \cdots \cap A_n\right| \Big)$$

$$\boxed{\begin{aligned} N(P_1'P_2'\cdots P_n') = N &- \sum_{1\le i\le n} N(P_i) + \sum_{1\le i\le j\le n} N(P_iP_j) \\ &- \sum_{1\le i\le j\le k\le n} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2\cdots P_n) \end{aligned}}$$

- <u>Example</u>:

How many solutions does $x_1+x_2+x_3=100$ have,

where x_1 is a non negative integer ≤ 50,

x_2 is a non negative integer ≤ 50,

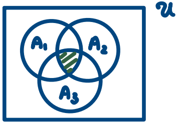
and x_3 is a non negative integer ≤ 50?

$\mathcal{U}$: เซตของ Sol^{n} ที่ $x_1, x_2, x_3 \in \mathbb{Z}^+ \cup \{0\}$

A_1 : ——"——"—— $x_1 \le 50$

A_2 : ——"——"—— $x_2 \le 50$

A_3 : ——"——"—— $x_3 \le 50$



ต้องการ $|A_1 \cap A_2 \cap A_3| = |\mathcal{U}| - |(A_1 \cap A_2 \cap A_3)'|$

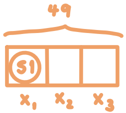
$= |\mathcal{U}| - |(A_1' \cup A_2' \cup A_3')|$

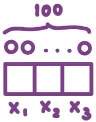
$= |\mathcal{U}| - |B_1 \cup B_2 \cup B_3|$

โดยที่ B_1 : ——"——"—— $x_1 \ge 51$

B_2 : ——"——"—— $x_2 \ge 51$

B_3 : ——"——"—— $x_3 \ge 51$





$C(102,2)$

$|B_1| + |B_2| + |B_3|$

$- |B_1 \cap B_2| - |B_1 \cap B_3| - |B_2 \cap B_3|$

$+ |B_1 \cap B_2 \cap B_3|$

$= C(51,2) + C(51,2) + C(51,2) - 0 - 0 - 0 + 0$

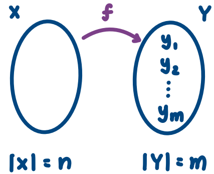
$\therefore C(102,2) - 3 \times C(51,2)$

The Number of Onto Functions

$\mathcal{U}$: เซตของ func จาก $X \to Y$

A_i : เซตของ $f_{X \to Y}$ ที่ y_i ต้องถูกใช้ $(i = 1, 2, \dots, m)$

B_i : ——"——"—— ที่ y_i ต้องไม่ถูกใช้



ต้องการหา $|A_1 \cap A_2 \cap \dots \cap A_m|$

$= |\mathcal{U}| - |(A_1 \cap A_2 \cap \dots \cap A_m)'|$

$= |\mathcal{U}| - |B_1 \cup B_2 \cup B_m|$

$= m^n - \sum_{i=1}^{m} |B_i| - \sum_{i<i_2}^{m} |B_{i_1} \cap B_{i_2}| + \sum_{i_1, i_2, i_3}^{m} |B_{i_1} \cap B_{i_2} \cap B_{i_3}| - \dots + (-1)^{m+1} |B_1 \cap B_2 \cap \dots \cap B_m|$

$m(m-1)^n - \binom{m}{2} \times (m-2)^n + \binom{m}{3}(m-3)^n + 0$

$\therefore = m^n - \left(\binom{m}{1}(m-1)^n - \binom{m}{2}(m-2)^n + \binom{m}{3}(m-3)^n - \dots + (-1)^m \binom{m}{m-1}(m-(m-1))^n \right)$

- Example:

  How many ways are there to assign five different jobs to four employees if every employee is assigned at least one job?

$$4^5 - C(4,1)3^5 + C(4,2)2^5 - C(4,3)1^5 = 1024 - 972 + 192 - 4 = 240$$

Derangements

- A ***derangement*** is a permutation of objects that leaves no object in its original position.
- <u>Example</u>:

  Consider a sequence 12345.

  21453 ✓

  43512 ✓

  42351 ✗

Derangements

- The number of derangements of a set with n elements, $D_n = ?$

$\mathcal{U}$: เซตของ permutation ทั้งหมด

A_i : —— " —— " —— ที่ x_i ห้ามอยู่ที่ i

ต้องการหา $|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|$

$$= |\mathcal{U}| - |(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)'|$$

$$= |\mathcal{U}| - |B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n|$$

โดยที่ $B_i = A_i'$: เซตของ permutation ที่ x_i อยู่ที่ i

$|\mathcal{U}| \rightarrow n!$

$$|B_1 \cup B_2 \cup \ldots \cup B_n| \rightarrow \sum|B_i| - \sum|B_{i_1} \cap B_{i_2}| + \sum|B_{i_1} \cap B_{i_2} \cap B_{i_3}| - \ldots + (-1)^{n+1}\cancel{|B_1 \cap B_2 \cap \ldots \cap B_n|} \rightarrow 1$$

$$n \times (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^n \binom{n}{n-1}(n-(n-1))! + (-1)^{n+1} 1$$

$$= n! - \left[n \times (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^n \binom{n}{n-1}(n-(n-1))! + (-1)^{n+1} 1\right]$$

$$= \cancel{n!} - \left[\cancel{n!} - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^n \frac{n!}{(n-1)!} + (-1)^{n+1}\frac{n!}{n!}\right]$$

$$\therefore D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} - \ldots + \frac{(-1)^n}{n!}\right]$$

- Example: "The Hatcheck Problem"

  An employee checks the hats of n people at a restaurant. He forgot to put claim check numbers on the hats. When customers return for their hats, this checker gives hats chosen at random to them.

  What is the probability that no one receives the correct hat?