Set, relation, function

SET

George Cantor (1845-1918)

Definition

A set is an unordered collection of objects.

The objects are called the elements or members of the set. The number of distinct elements in a set is the cardinality of the set.

จำนวนสมาชิก

Note:

An empty set (null set) is denoted by ∅

SET

George Cantor (1845-1918)

Definition

Two sets are equal if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write A = B if A = B if A and B are equal sets.

Definition

The set A is said to be a subset of B if and only if element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

สับเซตแท้

The set A is a proper subset of B if A is a subset of B and $A \neq B$.

Subset

Showing that A is a subset of B

To show that $A \subseteq B$, show that if x belongs to A then x is also belongs to B.

Showing that A is not a subset of B

To show that $A \not\subset B$, find a single $x \in A$ such that $x \notin B$.

SET

George Cantor (1845-1918)

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integers, we say that S is a finite set and that n is the cardinality (denoted by |S|) of S.

Definition

A set S is said to be infinite if it is not finite.

Theorem

For every set S, (a) $\emptyset \subseteq S$ (b) $S \subseteq S$.

Definition

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S).

$$n(P(S)) = 2^{n(S)}$$

SET

George Cantor (1845-1918)

Definition

ผลคูณคาร์ทีเซียน

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}.$$

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \}.$$

$(A \times B) \times C \neq A \times (B \times C)$

SET
George Cantor (1845-1918)

Set notation with quantifiers

We can restrict the domain of a quantified statement explicitly by making use of a particular notation. For examples,

$\forall x \in R\ (x^2 \geq 0)$	For any real x, $x^2 \geq 0$.
$\exists\ n \in Z\ (n^2 = 49)$	For some integer n, $n^2 = 49$.

$A = \{ x \in Z \mid 2x + 5 > 30 \}$

SET BUILDER NOTATION

Q = { x $\in$ R | x = p/q for some integers p and q }

R = { x | x is a real number }

U = { x | x is any of the objects under consideration }

EXERCISE

Let U be the universe described by

$$U = \{ x \mid 1000 \leq x \leq 9999 \}.$$

Let A_i be the set of all numbers in U such that the i^{th} position is i.

Find the cardinality of the union of A_1 A_2 A_3 and A_4

EXERCISE

Let S be the set of all x that x does not contain x.

$$S = \{ x \mid x \notin x \}$$

Note that x is also a set.

Does S contain S

set S ไม่มีจริง !

ถ้า $S \in S$ $\therefore S \notin S$ $T \rightarrow F$

$\therefore S \notin S \rightarrow$ มีสมบัติการเป็นสมาชิกใน S $\therefore S \in S$

Russell's paradox

Bertrand Russell (1872-1970)

SET

OPERATORS

Union

Intersection

Different

Disjoint $A \cap B = \phi$

Complement $A = U - A$

Power set

Pairwise disjoint

Mutually disjoint

Partition

MUTUALLY DISJOINT

Definition

Sets $A_1, A_2, A_3, \ldots, A_n$ are Mutually disjoint (pairwise or nonoverlapping)

Iff, any two sets A_i, A_j with distinct subscripts have not any elements in common, precisely $A_i \cap A_j$ = empty set $\varnothing$.

SET PARTITION

Definition

A collection of nonempty sets

$$A = \{A_1, A_2, A_3, \ldots, A_n\}$$

is a Partition of a set A Iff,

$A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ and

$A_1, A_2, A_3, \ldots, A_n$ are mutually disjoint.

SET

LOGIC

p	q	r	$\neg p$	$q \wedge r$	$\neg p \vee (q \wedge r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

SET

MEMBERSHIP

A	B	C	A^C	$B \cap C$	$A^C \cup (B \cap C)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

SET

Theorem Given sets A,B and C.

Commutative laws: $A \cap B = B \cap A$ $A \cup B = B \cup A$

Associative laws: $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Idempotent laws: $A \cap U = A$ $A \cup U = U$

De Morgan's laws: $(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$

Alternative representation for set difference $A - B = A \cap B'$

Absorption laws: $A \cup (A \cap B) = A$ $(A \cup B) \cap A = A$

SET

Question: Prove that $(A \cup B)' = A' \cap B'$.

Proof:

We show that $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

1) Show $(A \cup B)' \subseteq A' \cap B'$.
   $\forall x \in (A \cup B)'$, then $x \notin (A \cup B)$. Then $x \notin A$ and $x \notin B$.
   Since $x \notin A$, then $x \in A'$.
   Since $x \notin B$, then $x \in B'$.
   It is obtained that $x \in A' \cap B'$.
2) Show $A' \cap B' \subseteq (A \cup B)'$.
   ….
   ….

Q.E.D.

SET

Question: Prove that $(A \cup B)' = A' \cap B'$.

Proof:

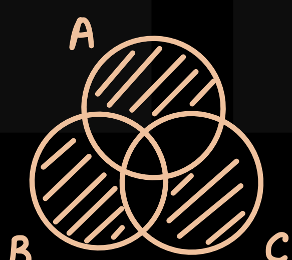
$(A \cup B)' = \{x \mid x \notin (A \cup B)\}$	Definition of complement
$= \{x \mid \neg(x \in (A \cup B))\}$	Does not belong symbol
$= \{x \mid \neg(x \in A \vee x \in B)\}$	Definition of union
$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$	De Morgan's law
$= \{x \mid x \notin A \wedge x \notin B\}$	Does not belong symbol
$= \{x \mid x \in A' \wedge x \in B'\}$	Definition of complement
$= \{x \mid x \in (A' \cap B')\}$	Definition of intersection
$= A' \cap B'$	Meaning of builder notation

Q.E.D.

EXERCISE

The symmetric difference of A and B, $(A \oplus B)$, is the set containing those elements in either A or B, but not in both A and B.

$$\boxed{(A \oplus (B \oplus C)) = ((A \oplus B) \oplus C)}$$



YES

EXERCISE

The symmetric difference of A and B, $(A \oplus B)$, is the set containing those elements in either A or B, but not in both A and B.

$$\boxed{(A \oplus (B \oplus C)) = ((A \oplus B) \oplus C)}$$

Given $A \oplus C = B \oplus C$

Must it be the case that $A = B$?

YES

MULTISETS

$A = \{1.a, 3.b, 5.c, 2.d\}$
$B = \{2.b, 6.c, 3.d\}$
$A \cup B = \{1.a, 3.b, 6.c, 3.d\}$ max
$A \cap B = \{0.a, 2.b, 5.c, 2.d\}$ min
$A - B = \{1.a, 1.b\}$
sum $A + B = \{1.a, 5.b, 11.c, 5.d\}$

Definition

Multisets are unordered collections of elements where an element can occur as a member more than once.

$$\{ m_1.a_1, m_2.a_2, m_3.a_3, \ldots, m_r.a_r \}$$

m_i are called the multiplicities of the elements a_i.

OPERATORS: UNION, INTERSECTION, DIFFERENCE, SUM

Fuzzy sets

Definition

Multisets are unordered collections of elements where an element can occur as a member more than once.

$0 \le m_i \le 1$

$\{ m_1.a_1, m_2.a_2, m_3.a_3, \ldots, m_r.a_r \}$

m_i are called th **Degree of membership** ents a_i.

OPERATORS: UNION, INTERSECTION, DIFFERENCE, SUM

THE END

ATS

>> 0 >> 1 >> 2 >> 3 >> 4 >>

Relations & functions

BINARY RELATION

Definition

Let A,B be sets. A binary relation R from A to B is a subset of the Cartesian product $A \times B$.
Given (x,y), ordered pair, in $A \times B$, x is related to y by R, written xRy, iff $(x,y) \in R$.

หาร 2 แล้วเหลือเศษเท่ากัน

Example (the congruence modulo 2 relation)

The relation R from Z to Z is defined as follows;
for all $(x,y) \in Z \times Z$, xRy iff x-y is even.

Example, 6R2, 120R36 etc.

FUNCTION

Definition

A function F from A to B is a relation from A to B, $F : A \rightarrow B$, that satisfies the following properties:

For every $x \in A$, there exists $y \in B$ such that $(x,y) \in F$.

For all $x \in A$, and $y, z \in B$, if $(x,y) \in F$ and $(x,z) \in F$ then $y=z$.

For $(x,y) \in F$, we usually write $y = F(x)$ = image of x under F, and x is called pre-image of y under F.

A is called domain of F.

B is called co-domain of F.

The set of all images of F is called range of F.

COMPOSITION OF FUNCTIONS

Definition

The composition of the functions f and g, denoted by $f \circ g$, is defined as

$$(f \circ g)(x) = f(g(x))$$

$f : B \to C$
$g : A \to B$

$(f \circ g) : A \to C$

$A \xrightarrow{g} B \xrightarrow{f} C$

$f(g(x)) \in C$
$x \in A$

ex. $f_1, f_2 : \mathbb{R} \to \mathbb{R}$

$f_1(x) = x^2 - 5$
$f_2(x) = x + 8$

$(f_1 \circ f_2)(x) = f_1(f_2(x))$
$= f_1(x+8)$
$= (x+8)^2 - 5$

Addition / Multiplication

$$f_1 : A \to B$$
$$f_2 : A \to B$$

Addition :

$$(f_1 + f_2)(a) = f_1(a) + f_2(a)$$

Multiplication:

$$(f_1 f_2)(a) = f_1(a) f_2(a)$$

ex. $f_1, f_2 : \mathbb{R} \to \mathbb{R}$

$f_1(x) = x^2 - 5$

$f_2(x) = x + 8$

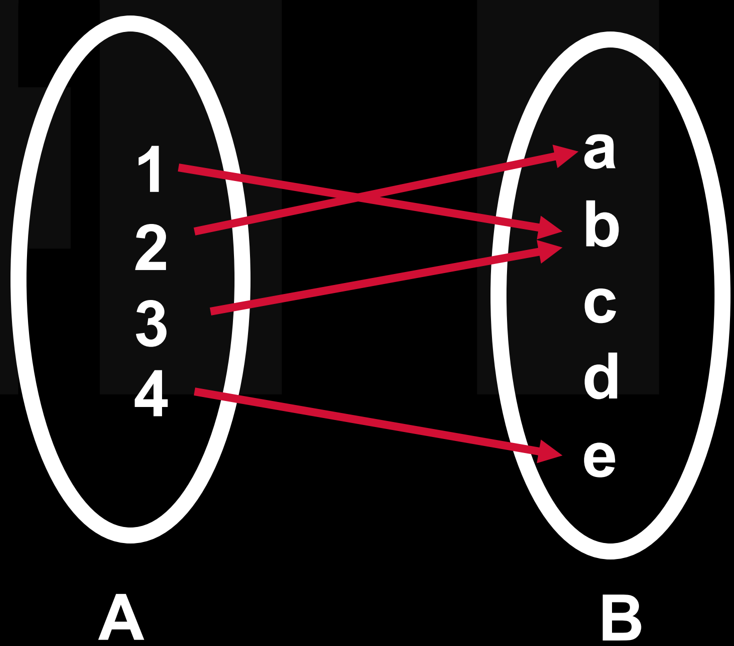
$(f_1 + f_2)(x) = x^2 + x + 3$

$f_1 f_2(x) = x^3 + 8x^2 - 5x - 40$

FUNCTION

Arrow diagram

A function F from A to B.



F(1) = b
F(2) = a
F(3) = b
F(4) = e

>> 0 >> 1 >> 2 >> 3 >> 4 >>

INJECTIVE FUNCTION

This function is not One-to-one.

Definition

1 ต่อ 1

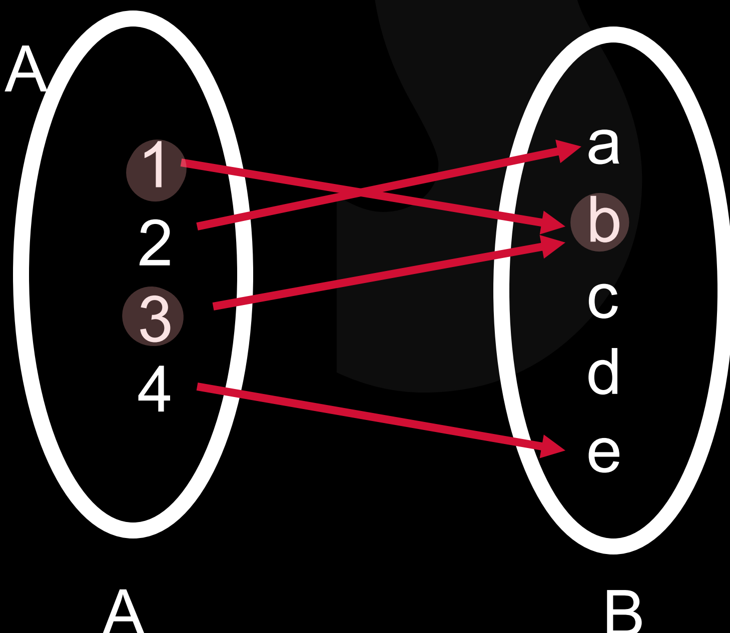
A function F from A to B is injective (or one-to-one)

iff for all elements x and y in A,

if $F(x) = F(y)$ then $x = y$.

Or, equivalently,

if $x \neq y$ then $F(x) \neq F(y)$.



INJECTIVE FUNCTION

Definition

A function F from A to B is injective (or one-to-one) **iff** for all elements x and y in A,

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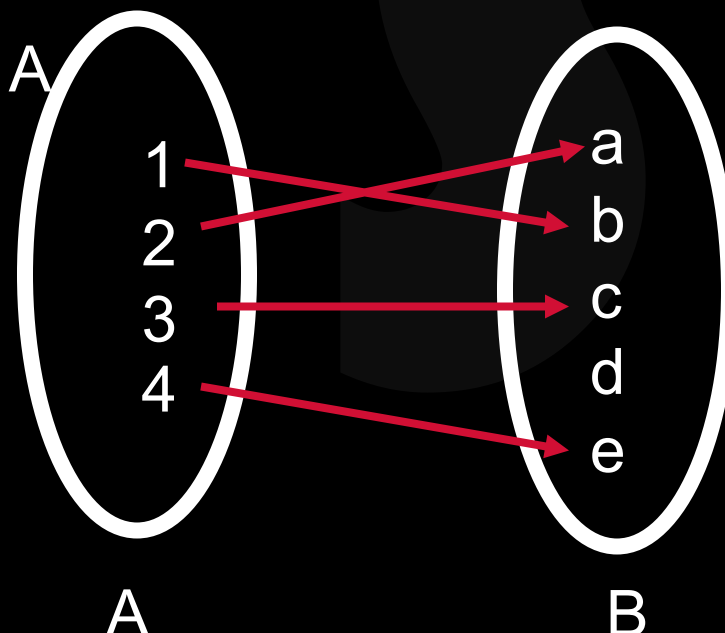


This function is an One-to-one.

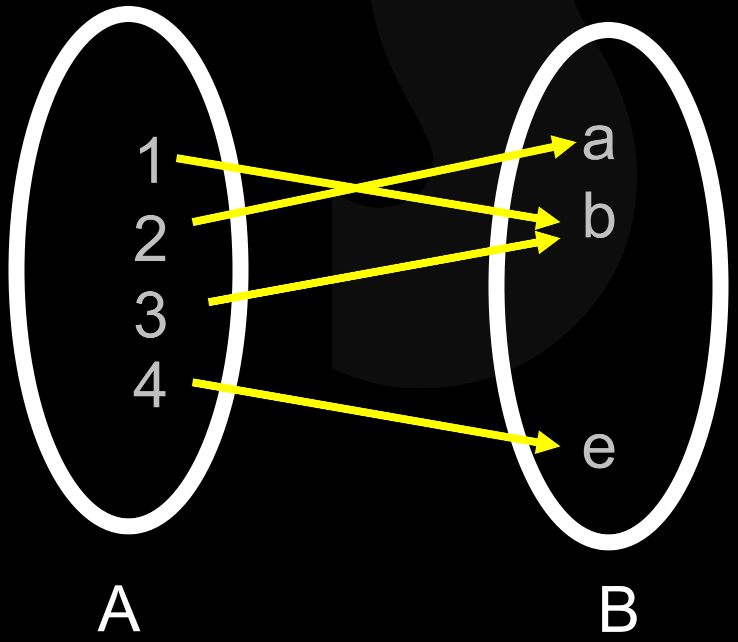
SURJECTIVE FUNCTION

Definition

ทั่วถึง

A function F from A to B is surjective (or onto)

iff for any element y in B, it is possible to find an element x in A such that y = F(x) .



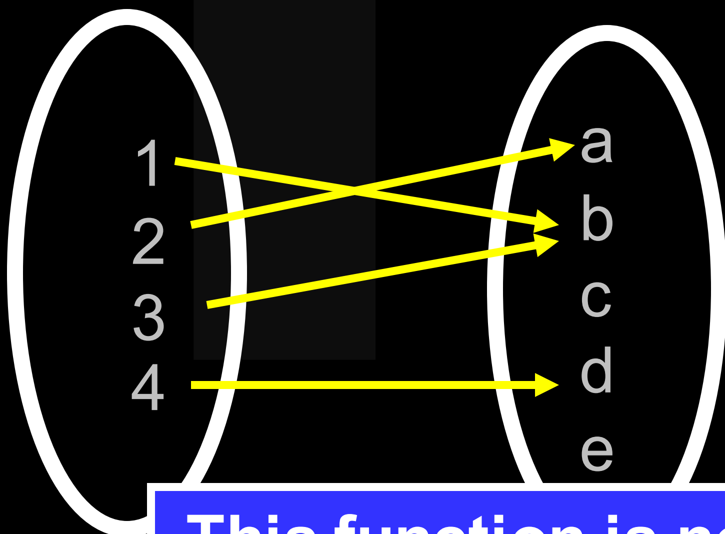
This function is Onto.

>> 0 >> 1 >> 2 >> 3 >> 4 >>

BIJECTIVE FUNCTION

Definition

A one-to-one correspondence (or bijection) F from A to B is a function that is both one-to-one and onto.



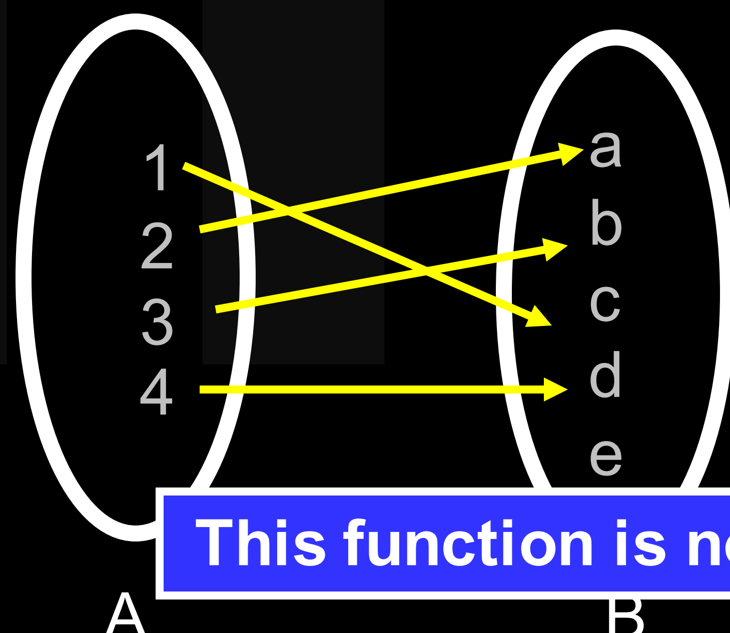
ไม่ one-to-one ไม่ onto

This function is not a bijection.

BIJECTIVE FUNCTION

Definition

A one-to-one correspondence (or bijection) F from A to B is a function that is both one-to-one and onto.



ไม่ onto

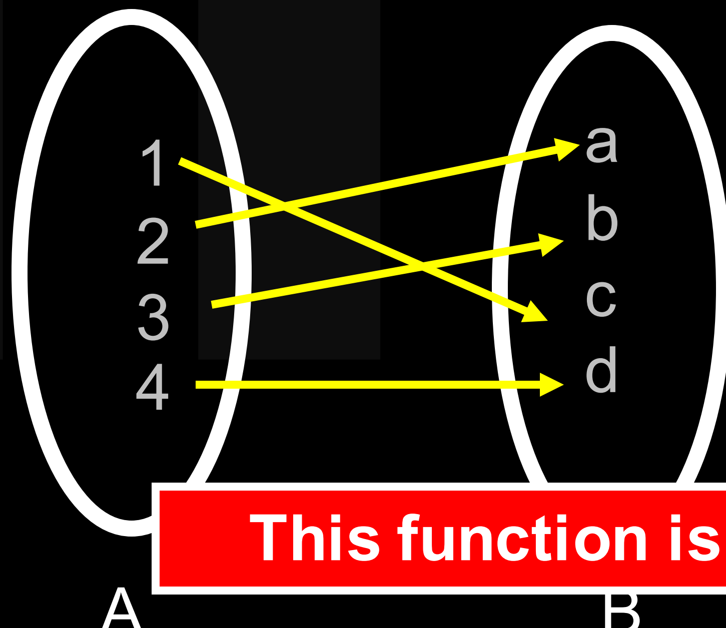
This function is not a bijection.

BIJECTIVE FUNCTION

Definition

A one-to-one correspondence (or bijection) F from A to B is a function that is both one-to-one and onto.

$|A| = |B|$



This function is a bijection.

ex. bijection function
→ even countable set

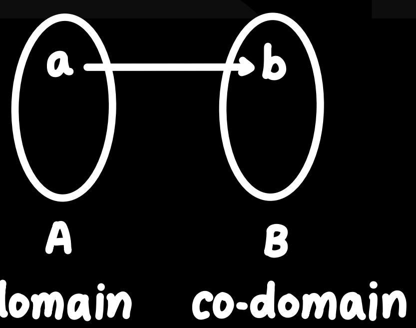
$\{0, 1, 2, 3, \dots\} = A$
$\downarrow \downarrow \downarrow \downarrow$
$\{0, 2, 4, 6, \dots\} = B$

$B \subsetneq A \qquad |A| = |B|$
สับเซตแท้

>> 0 >> 1 >> 2 >> 3 >> 4 >>

Function Recap

Function : $F : A \rightarrow B$ / $F : a \mapsto b$



$F \subseteq A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

$a \in A$ $F(a) = b$ $b \in B$

b = image of a

a = pre-image of b

$F(A)$ = Range of $F \subseteq B$

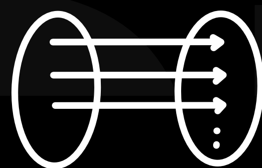
<u>สมบัติของ F</u>

1. ทุกสมาชิกใน Domain ต้องมี image
2. แต่ละสมาชิกต้องมี image แค่ตัวเดียว !

Inverse of Function

$f : A \rightarrow B$; $f^{-1} = B \rightarrow A$

1-1 :

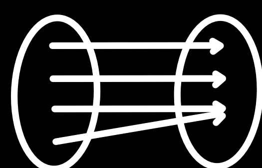


Given f : 1-1 but not onto

f^{-1} = ✗ function

* ทุกสมาชิกใน Domain ต้องมี image *

onto :



Given f : onto but not 1-1

f^{-1} : ✗ function

* แต่ละสมาชิก มี image แค่ตัวเดียว *

ex. $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$

$f_1(x) = x^2 - 5$

$f_2(x) = x + 8$

f_1^{-1}

$y = x^2 - 5$

$x^2 = y + 5$

$x = \pm\sqrt{y+5}$

$f_1^{-1}(x) = \pm\sqrt{x+5}$ ✗ function

$f_2^{-1}(x)$

$y = x + 8$

$x = y - 8$

$f_2^{-1}(x) = x - 8$

$(f_2^{-1} \circ f_2)(x) = x$

Floor & Ceiling Function

Floor Function $\lfloor\ \rfloor$ Ceiling Function $\lceil\ \rceil$

$\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ $\quad \forall r \in \mathbb{R}\ \exists! n \in \mathbb{Z}\ (\lfloor r \rfloor = n)$

$(n \leq r)$

n = จำนวนเต็มที่มากที่สุดที่มีค่าไม่เกิน r

น้อยกว่า หรือเท่ากับ

if and only if $(\leftrightarrow)$

$\lfloor r \rfloor = n$ iff 1) $n \leq r$

2) $\forall s \in \mathbb{Z}\ (s \leq r \to n \geq s)$

$\forall r \in \mathbb{R}\ \exists! n \in \mathbb{Z}\ (\lfloor r \rfloor = n)$ iff $(n \leq r) \land (\forall s \in \mathbb{Z}\ (s \leq r \to n \geq s))$

$\forall r \in \mathbb{R}\ \exists! m \in \mathbb{Z}\ (\lceil r \rceil = m)$ iff $(m \geq r) \land (\forall s \in \mathbb{Z}\ (s \geq r \to s \geq m))$

PROPERTIES

Definition

เป็นเซตเดียวกัน

Let R be a binary relation on A.

R is reflexive iff for all $x \in A$, $x \, R \, x$.

R is symmetric iff for all $x, y \in A$, if $x \, R \, y$ then $y \, R \, x$.

R is transitive iff for all $x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

ex. 1 $A = \{a, b, c\}$

$R_1 = \{(a,a), (a,b), (a,c)\}$

reflexive ✗

เพราะไม่มี $(b,b), (c,c)$

symmetric ✗

เพราะไม่มี $(b,a), (c,a)$

transitive ✓

ex. 2 let $n = 5$

reflexive $\forall a \in \mathbb{Z}$

จะได้ว่า $a \bmod \cancel{n}^{\,5} = a \bmod 5$

$\therefore \forall a \in \mathbb{Z} \quad aRa$

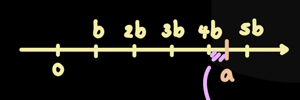
symmetric $\forall a, b \in \mathbb{Z}$

congruence modulo n $\quad a \equiv b \pmod{n}$

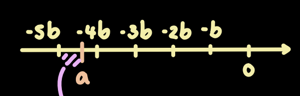
ถ้า aRb แปลว่า $a \bmod 5 = b \bmod 5$

$\because \quad b \bmod 5 = a \bmod 5$

$\therefore bRa$



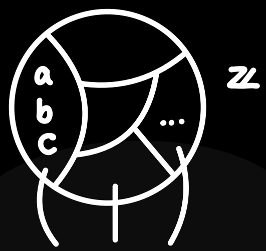
transitive



$a \bmod b = a - b\left\lfloor \frac{a}{b} \right\rfloor$

floor function

consider $a \equiv b \pmod{5}$ on $\mathbb{Z}$



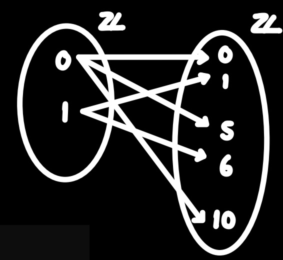
partition of $\mathbb{Z}$

let $A_i \subseteq \mathbb{Z}, \ i \geq 0$

$$A_i = \{ a \in \mathbb{Z} \mid a \equiv i \pmod{5} \}$$

show $\mathbb{Z} = \bigcup_{\text{all } i} A_i$; $A_i \cap A_j = \emptyset$

$(i \neq j \ ; \ i \not\equiv j \pmod{5})$



$(0,0), (0,5), (0,10), \ldots$

$(5,0)$

$A_0 = \{0, 5, -5, 10, -10, \ldots\}$

$A_1 = \{1, -4, 6, -9, \ldots\} = A_5$

$A_2 =$

$A_3 =$

$A_4 =$

$A_5 = \{0, 5, -5, \ldots\}$

$a \in A_i$, $b \in A_j$ $\quad A_i \cap A_j = \emptyset$

$\therefore a \not\equiv b \pmod{5}$

EQUIVALENCE RELATION

Definition

R is a equivalence relation on A iff

- R is a binary relation on A.
- R is reflexive.
- R is symmetric.
- R is transitive.

TRANSITIVE CLOSURE

Definition

Let R be a binary relation on A.

The transitive closure of R is the binary relation R^t on A

That satisfies the following three properties:

- R^t is transitive.
- $R \subseteq R^t$.
- S is any other transitive that contains R then $R^t \subseteq S$.

MUTUALLY DISJOINT

Definition

Sets $A_1, A_2, A_3, \ldots, A_n$ are Mutually disjoint (pairwise or nonoverlapping)

Iff, any two sets A_i, A_j with distinct subscripts have not any elements in common, precisely $A_i \cap A_j = $ empty set $\varnothing$.

SET PARTITION

Definition

A collection of nonempty sets

$$A = \{A_1, A_2, A_3, \dots, A_n\}$$

is a Partition of a set A Iff,

$A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ and

$A_1, A_2, A_3, \dots, A_n$ are mutually disjoint.

SET PARTITION

Definition

Given a partition of $A=\{A_1,A_2, A_3, \dots,A_n\}$.

The binary relation induced by the partition, R, is defined on A as follows:

for all $x,y \in A$, $x \, R \, y$ Iff,

there is a subset A_j of the partition such that both x and y are in A_j.

SET PARTITION

Theorem

Let A be a set with a partition and

Let R be the relation induced by the partition.

Then R is reflexive, symmetric and transitive.

How to prove this theorem?

EQUIVALENCE CLASS

Definition

Suppose A is a set and R is a equivalence relation on A.

For each a $\in$A, the equivalence class of a, denoted [a],

is the set of all elements x in A such that

x is related to a by R.

$$[\, a\,] = \{\, x \in A \mid x\, R\, a \,\}.$$

THEOREM

Lemma 1

Let R be an equivalence relation on A, a b $\in$ A.

If a R b then [a] = [b].

Lemma 2

Let R be an equivalence relation on A, a b $\in$ A, then

either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.

Theorem

A is a nonempty set and R is an equivalence relation on A,

then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

ANTISYMMETRIC

Definition

A relation R on a set A such that (a,b) and (b,a) are in R only if a=b, for all a,b in A, is called antisymmetric.

$A = \{1,2,3\}$

$R_1 = \{(1,1),(1,3),(2,3)\}$ ✓ ✓ ✓ Antisymmetric

* $R_2 = \{(1,1),(2,2),(3,3)\}$ symmetric Antisymmetric ✓

$R_3 = \{(1,1),(2,3),(3,2)\}$ symmetric Antisymmetric × $2 \neq 3$

$(a,b),(b,a)$ only if $a=b$

$a \neq b \longrightarrow \cancel{(a,b) \wedge (b,a)}$

EXERCISE

Let R be a relation on the set A

$$R = \{ (a,b) \mid a < b \}$$

Find the inverse relation R^{-1} and the complementary relation R.

IMPORTANT

SET is an unordered collection of objects.

DEFINITIONS

- ✓ Subset
- ✓ Proper subset
- ✓ Cardinality
- ✓ Power set
- ✓ Order n-tuple
- ✓ Cartesian product
- ✓ Union, intersection, difference, complement, and symmetric difference

FUNCTIONS

- Additive & multiplicative
- Injective function
- Surjective function
- Bijective function
- Strickly increasing and decreasing
- Inverse function
- Composite function
- Floor/ceiling functions
- Factorial function

RELATIONS

- ➢ Reflexive
- ➢ Symmetric
- ➢ Transitive
- ➢ Anti-symmetric
- ➢ Equivalence relations
- ➢ Mutually disjoint
- ➢ set partition

KEYWORDS

Venn diagram, membership table, congruence modulo relation, arrow diagram,.

You can find many more detail and examples on many websites.

THE END