Quiz 2B: Methods of Proof Name:________________________

October 24, 2023 (11:15 - 12:15) ID:__________________No._______

<u>Quiz Instruction</u>:

1. The quiz contains 20 questions, each is worth 1 point, and is divided into 2 parts:
   - <u>Part A:</u> Multiple choice 15 questions
   - <u>Part B:</u> Short answer 5 questions
2. *****Only the answers in the answer sheet will be graded.*****

<u>Part A</u>: Multiple choice [15 points]
Instructions: Provide the **X** mark for the correct answer in the answer sheet.

<u>Questions 1-3:</u> Choose **the method** that can be used to prove or disprove each question.

1. Prove or Disprove that "The sum of two odd integers is odd."
   a. Contradiction proof
   b. Direct Proof
   c. Both a. and b. can be used
   d. Disprove by counterexample

2. Prove or Disprove that "If n^2 is odd, then n is odd." (For all integers n)
   a. Contradiction Proof
   b. Contraposition Proof
   c. Both a. and b. can be used
   d. Disprove by counterexample

3. Given $P(n)$ = "If $n < 1$, then $n^2 \leq n$", Prove or Disprove that $P(0)$ is true.
   a. Trivial Proof
   b. Vacuous Proof
   c. Both a. and b. can be used
   d. Disprove by counterexample

<u>Questions 4-6:</u> By using the given proof method, which of the following is **the assumption that should be used in the first step** for proving the given statement?

4. <u>Statement</u>: "If m and n are integers and mn is odd, then m is odd and n is odd."
   <u>Proof Method</u>: **CONTRAPOSITION** proof.
   a. Assume that "m is odd or n is odd" is true.
   b. Assume that "m is odd and n is odd" is true.
   c. Assume that "m is even or n is even" is true.
   d. Assume that "m is even and n is even" is true.

5. <u>Statement</u>: "For all integers n, if $3n + 2$ is even, then n is even."
   <u>Proof Method</u>: **CONTRADICTION** proof
   a. Assume that "For all integers n, if $3n + 2$ is even, then n is even" is true.
   b. Assume that "For all integers n, if $3n + 2$ is even, then n is odd" is true.
   c. Assume that "For all integers n, $3n + 2$ is even and n is even" is true.
   d. Assume that "For all integers n, $3n + 2$ is even and n is odd" is true.

6. Statement: “At least 10 of any 64 days chosen must fall on the same day of the week.”
   Proof Method: **CONTRADICTION** proof
   a. Assume that “At most 9 of any 64 days chosen must fall on the same day of the week” is true.
   b. Assume that “At least 7 of any 64 days chosen must fall on the same day of the week” is true.
   c. Assume that “There are more than 9 of any 64 days chosen that fall on the same day of the week” is true.
   d. Assume that “There are less than 7 of any 64 days chosen that fall on the same day of the week” is true.

Questions 7-8: Use the proof below to answer the question.

Prove that “for all $x > 0$, if $1/x$ is irrational, then x is irrational.”

Steps		Reason
(1)	$x = a/b$	Assume that “x is rational” is true, where a and b are integers that are both greater or both less than zero.
(2)	$1/x = b/a$	Inverting both sides of (1)
(3)	$1/x$ is rational.	From (2), $1/x$ can be written as a fraction of integers.

7. Which of the following is the method that this proof uses?
   a. Contradiction Proof
   b. Contraposition Proof
   c. Vacuous Proof
   d. None of the above.

8. Which of the following statements can be proven by using the proof above?
   a. For all $x < 0$, if $1/x$ is irrational, then x is irrational.
   b. For all $x < 0$, if x is rational, then $1/x$ is rational.
   c. For all $x > 0$, if $1/x$ is irrational, then x is irrational.
   d. For all $x > 0$, if x is rational, then $1/x$ is rational.

9. Suppose we want to prove that $\sum_{i=1}^{n} i^4$ is equal to $\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, by mathematical induction. Which pair of expressions **should be proved equal** by the end of the proof?

[1] $\sum_{i=1}^{n} i^4 + (n+1)^4$

[2] $\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + n^4$

[3] $\frac{(n+1)(n+2)(2n+3)(3n^2+9n+5)}{30}$

a. [1] and [2]
b. [1] and [3]
c. [2] and [3]
d. None of the above

10. Use mathematical induction to **find which operator** * satisfies

$$(1 - x)^n * (1 - nx)$$

for all positive integer n and real number x such that $0 < x < 1$.

a. $=$
b. $\neq$
c. $\geq$
d. $\leq$

11. If n is an integer, find the smallest positive integer k such that $3n^2 + k + 2$ is divisible by 3.

a. 0
b. 1
c. 2
d. 3

12. Consider the following statement:

"Any positive integer n, n^2 is either of the form $4m$ or $4m + 1$ for some non-negative integer m"

Which choice is the best to prove the statement above by **proof by cases**?

a. case 1: $n = 2k + 1$ for some non-negative integer k
   case 2: $n = 2k$ for some non-negative integer k
b. case 1: $n = 3k$ for some non-negative integer k
   case 2: $n = 3k + 1$ for some non-negative integer k
c. case 1: $n = 4k$ for some non-negative integer k
   case 2: $n = 4k + 1$ for some non-negative integer k
d. case 1: $n = 8k$ for some non-negative integer k
   case 2: $n = 8k + 1$ for some non-negative integer k

13. Which of the following best describes the statement

"There exists a triple (a, b, c) of positive integers such that $\frac{a+b+c}{3} = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$"

and the attempt to prove it by choosing $a = 3$, $b = 2$, $c = 2$?

a. The choice of $a = 3$, $b = 2$, $c = 2$ correctly proves the statement.
b. The choice of $a = 3$, $b = 2$, $c = 2$ is not a valid example to the statement but there exists some triple (a, b, c).
c. The statement cannot be proven using any values of a, b, c.
d. The statement is False.

14. There are n lamp labelled $1, 2, \ldots, n$. Lamp 1 can be switched on or off at any time. Lamp k, where $1 < k \leq n$ can only be switched (on or off) when the lamp $k - 1$ is the only lamp that is on out of lamps $1, 2, \ldots, k - 1$. If initially, all lamps are off, how many moves does it take to switch on the lamp n?

   Hint: - Let $P(n)$ is that it takes at least …. moves to turn on lamp n.
   - $P(1) = 1$, $P(2) = 2$, $P(3) = 4$
   - You can use **strong induction** to validate your answers.

   a. $1 + \frac{n(n-1)}{2}$
   b. $2^n - 1$
   c. 2^{n-1}
   d. $P(n - 1) + 2P(n - 2)\ when\ n > 2$

15. Consider the following proof:

Proof that "$x^2 + y^2 \geq 2xy$ for all real numbers x, y"

Steps		Reason
(1)	$x^2 + y^2 \geq 2xy$	Assumption
(2)	$x^2 + y^2 - 2xy \geq 0$	Subtract $2xy$ from both sides of (1)
(3)	$(x - y)^2 \geq 0$	Reduce the inequality in (2)
As (3) is a fact, this proof is finished.		

Which of the following statements is true?

   a. This proof is valid.
   b. This proof is only valid for all **integers** x, y.
   c. This proof is invalid because arriving at the fact after assuming that the statement is true does not prove the statement.
   d. The three steps mentioned above are the correct way to prove, but the proof is not finished yet.

<u>Part B</u>: Short answer [5 points]

Instructions for Part B: Provide **an answer in the INTEGER form** in the space provided for each question in the answer sheet.

16. From this proof by induction, the same one as in the exercise.

- <u>Theorem</u>: All Americans are the same age.
- Let $S(n)$: In any group of n Americans, everyone in that group is the same age.
- <u>Basis Step</u>: Since everyone in a group of one American has the same age, $S(1)$ is true.

- Inductive Step:

  Assume $S(n)$ is true for some n. We prove $S(n + 1)$.

  Let G be an arbitrary group of $n + 1$ Americans. We show that everyone in G has the same age by showing that any two members of G have the same age.

  Let $a, b \in G$, $a \neq b$, $G_a = G - \{a\}$, and $G_b = G - \{b\}$.

  Since G_a has n members, b (which is in G_a) has the same age as any other person in G_a. Similarly, a has the same age as any other person in G_b.

  Now, let c be any person in G other than a and b. Then, $c \in G_a$ and $c \in G_b$. So, a and b both have the same age as c.

  Hence, a and b have the same age.

  This proves $S(n + 1)$. **Q.E.D.**

Suppose we can change the basis step, according to some *'magic'* conditions, to:

"Since everyone in a group of n Americans has the same age"

Find the **minimum** number of n, in which the proof above will be valid.

17. Find the **minimum** positive integer n that serves for the statement below.

    "Any amount of postage of n cents or more can be formed using just 3-cent and 5-cent stamps."

    [Hint: Use strong induction to validate your answer.**]**

18. Consider the proof for the statement **"There are infinitely many primes"** below.

    Proof. We'll show that every finite list of primes is missing a prime number, so the list of all primes can't be finite.

    To begin, there are prime numbers such as 2. Suppose $p_1, p_2, \ldots, p_r$ is a finite list of prime numbers. We want to show this is not the full list of the primes.

    Consider the number $N = p_1 p_2 \ldots p_r + k$. Since N always has a remainder when divided by each p_i. Therefore, N is a prime which is not on our list.

    From the proof, find the **smallest positive integer** k to make the statement true.

Question 19-20: Answer an integer n that serves as a counterexample to the given statement.

19. For all positive integer n, $\frac{10 + n}{3} > \sqrt[3]{25n}$.

20. Every positive integer n can be written as the sum of two positive integers.

ANSWER SHEET for Quiz 2B

<u>Part A</u>: Choose the correct answer and provide the X mark.

No.	Choice				No.	Choice				No.	Choice			
	a.	b.	c.	d.		a.	b.	c.	d.		a.	b.	c.	d.
1.					6.					11.				
2.					7.					12.				
3.					8.					13.				
4.					9.					14.				
5.					10.					15.				

<u>Part B</u>: Provide an answer in the **INTEGER** form in the space provided.

No.	Answer
16.	
17.	
18.	
19.	
20.	

================ Wish you guys the best of luck krub. ================