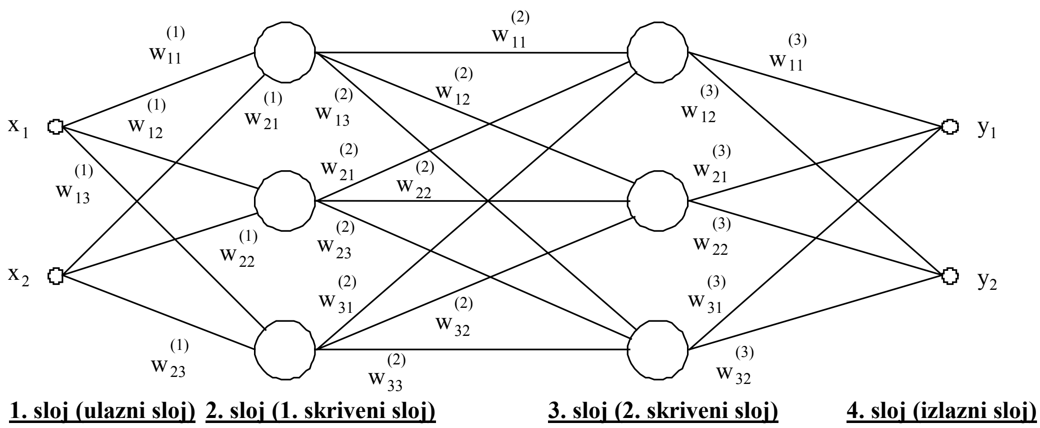
PRIMJER BACKPROPAGATION ALGORITMA

Za potrebe primjera odabrana je umjetna neuronska mreža (UNM) 2x3x3x2. Želi se istrenirati UNM tako da određuje vrijednost funkcije pripadnosti ulaznih podataka za dva neizrazita skupa. Ulazni podaci, i odgovarajuće vrijednosti funkcije pripadnosti dane su slijedećom tablicom:

Uzorak br.	1	2	3	4	5	6	7	8	9	10
x_1	.05	.09	.12	.15	.20	.75	.80	.82	.90	.95
x_2	.02	.11	.20	.22	.25	.75	.83	.80	.89	.89
$\mu_{R1}(x_1,x_2)$	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0
$\mu_{R2}(x_1,x_2)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Uzorak br.	11	12	13	14	15	16	17	18	19	20
x_1	.09	.10	.14	.18	.22	.77	.79	.84	.94	.98
x_2	.04	.10	.21	.24	.28	.78	.81	.82	.93	.99
$\mu_{R1}(x_1,x_2)$	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0
$\mu_{R2}(x_1,x_2)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

UNM prikazana je na slijedećoj slici:



Pretpostavimo da su početne vrijednosti težinskih faktora (koji su generirani slučajno) zadane:

$w_{11}^{(1)} = 0.5$ $w_{12}^{(1)} = 0.4$ $w_{13}^{(1)} = 0.1$ $w_{21}^{(1)} = 0.2$ $w_{22}^{(1)} = 0.6$ $w_{23}^{(1)} = 0.2$	$w_{11}^{(2)} = 0.10$ $w_{12}^{(2)} = 0.55$ $w_{13}^{(2)} = 0.35$ $w_{21}^{(2)} = 0.20$ $w_{22}^{(2)} = 0.45$ $w_{23}^{(2)} = 0.35$	$w_{31}^{(2)} = 0.25$ $w_{32}^{(2)} = 0.15$ $w_{33}^{(2)} = 0.60$	$w_{11}^{(3)} = 0.30$ $w_{12}^{(3)} = 0.35$ $w_{21}^{(3)} = 0.35$ $w_{22}^{(3)} = 0.25$ $w_{31}^{(3)} = 0.45$ $w_{32}^{(3)} = 0.30$

Neuroni koriste sigmoidalnu prijenosnu funkciju, te je njihov izlaz određen izrazom:

$$o = \frac{1}{1 + \exp\left[-\left(\sum x_i \cdot w_i\right)\right]} \qquad (1.1)$$

pri tome su x_i (i=0,1,2,…,n) ulazi u taj neuron, a w_i (i=0,1,2,…,n) pripadajući težinski faktori, pri čemu je $\theta=-w_0$ prag, a $x_0=1$ po definiciji. Radi što jednostavnijeg računanja u ovom primjeru svi će neuroni imati prag $w_0=0$.

Backpropagation algoritam provodi se kroz niz epoha, pri čemu se u svakoj epohi provodi niz iteracija u kojima se mreži predočuju uzorci iz skupa za učenje, a svaka iteracija sastavljena od slijedećih koraka:

- **Korak 1.** ▸ Računanje izlaznih vrijednosti neurona; smjer računanja je od prvog sloja prema zadnjem sloju.
- **Korak 2.** ▸ Računanje pogreški svakog neurona; smjer računanja je od zadnjeg sloja prema drugom sloju (ne računa se za prvi sloj).
- **Korak 3.** ▸ Korigiranje vrijednosti težinskih faktora; smjer provođenja korekcija je proizvoljan.

1. epoha, 1.iteracija, 1.uzorak

Uzimamo prvi uzorak iz skupa za učenje: $\vec{x}^T = [x_1 \quad x_2] = [0.05 \quad 0.02]$.

Korak 1. Računamo izlaze iz drugog sloja (prema relaciji 1.1):

$$\begin{aligned} o_1^{(2)} &= \frac{1}{1 + \exp\left[-\left(x_1 \cdot w_{11}^{(1)} + x_2 \cdot w_{21}^{(1)} + w_{01}^{(2)}\right)\right]} \\ &= \frac{1}{1 + \exp\left[-\left(0.05 \cdot 0.5 + 0.02 \cdot 0.2 + 0.0\right)\right]} = 0.507249 \end{aligned}$$

$$o_2^{(2)} = \frac{1}{1+\exp\left[-\left(x_1 \cdot w_{12}^{(1)} + x_2 \cdot w_{22}^{(1)} + w_{02}^{(2)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.05 \cdot 0.4 + 0.02 \cdot 0.6 + 0.0\right)\right]} = 0.507999$$

$$o_3^{(2)} = \frac{1}{1+\exp\left[-\left(x_1 \cdot w_{13}^{(1)} + x_2 \cdot w_{23}^{(1)} + w_{03}^{(2)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.05 \cdot 0.1 + 0.02 \cdot 0.2 + 0.0\right)\right]} = 0.502250$$

Računamo izlaze iz trećeg sloja (prema relaciji 1.1) pri čemu su ulazi u treći sloj zapravo izlazi iz drugog sloja:

$$o_1^{(3)} = \frac{1}{1+\exp\left[-\left(o_1^{(2)} \cdot w_{11}^{(2)} + o_2^{(2)} \cdot w_{21}^{(2)} + o_3^{(2)} \cdot w_{31}^{(2)} + w_{01}^{(3)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.507249 \cdot 0.10 + 0.507999 \cdot 0.20 + 0.502250 \cdot 0.25 + 0.0\right)\right]} = 0.569028$$

$$o_2^{(3)} = \frac{1}{1+\exp\left[-\left(o_1^{(2)} \cdot w_{12}^{(2)} + o_2^{(2)} \cdot w_{22}^{(2)} + o_3^{(2)} \cdot w_{32}^{(2)} + w_{02}^{(3)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.507249 \cdot 0.55 + 0.507999 \cdot 0.45 + 0.502250 \cdot 0.15 + 0.0\right)\right]} = 0.641740$$

$$o_3^{(3)} = \frac{1}{1+\exp\left[-\left(o_1^{(2)} \cdot w_{13}^{(2)} + o_2^{(2)} \cdot w_{23}^{(2)} + o_3^{(2)} \cdot w_{33}^{(2)} + w_{03}^{(3)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.507249 \cdot 0.35 + 0.507999 \cdot 0.35 + 0.502250 \cdot 0.60 + 0.0\right)\right]} = 0.658516$$

Računamo izlaze iz četvrtog sloja (prema relaciji 1.1) pri čemu su ulazi u četvrti sloj zapravo izlazi iz trećeg sloja:

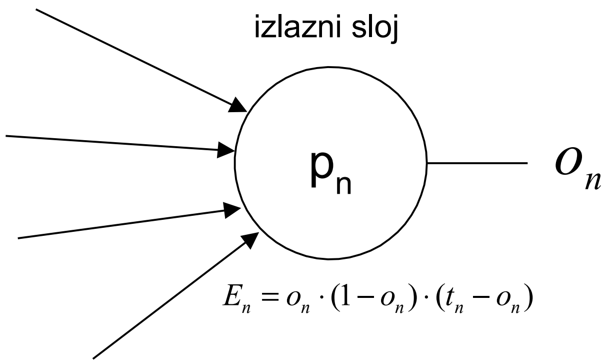
$$o_1^{(4)} = \frac{1}{1+\exp\left[-\left(o_1^{(3)} \cdot w_{11}^{(3)} + o_2^{(3)} \cdot w_{21}^{(3)} + o_3^{(3)} \cdot w_{31}^{(3)} + w_{01}^{(4)}\right)\right]}$$
$$= \frac{1}{1+\exp\left[-\left(0.569028 \cdot 0.30 + 0.641740 \cdot 0.35 + 0.658516 \cdot 0.45 + 0.0\right)\right]} = 0.666334$$

$$o_2^{(4)} = \frac{1}{1+\exp\left[-\left(o_1^{(3)}\cdot w_{12}^{(3)} + o_2^{(3)}\cdot w_{22}^{(3)} + o_3^{(3)}\cdot w_{32}^{(3)} + w_{01}^{(4)}\right)\right]}$$

$$= \frac{1}{1+\exp\left[-\left(0.569028\cdot 0.35 + 0.641740\cdot 0.25 + 0.658516\cdot 0.30 + 0.0\right)\right]} = 0.635793$$

Korak 2. Računamo pogreške neurona četvrtog sloja (izlazni sloj). Pogreška izlaznog sloja računa se prema relaciji:

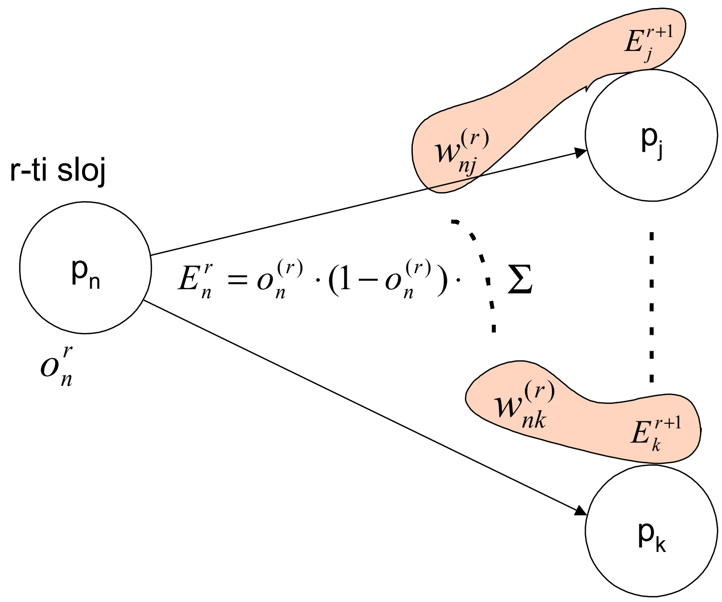
$$E_n = o_n \cdot \left(1 - o_n\right)\cdot\left(t_n - o_n\right)$$



gdje je o vrijednost izlaza neurona a t točna vrijednost koju želimo da neuron daje. Iz tablice možemo očitati da je za ulazni vektor $\vec{x}^T = [0.05 \quad 0.02]$ željeni izlaz $\vec{y}^T = [1.0 \quad 0.0]$. Slijedi:

$$E_1^{(4)} = o_1^{(4)}\cdot\left(1-o_1^{(4)}\right)\cdot\left(t_1 - o_1^{(4)}\right) = 0.666334\cdot\left(1-0.666334\right)\cdot\left(1.0-0.666334\right) = 0.0741850$$
$$E_2^{(4)} = o_2^{(4)}\cdot\left(1-o_2^{(4)}\right)\cdot\left(t_2 - o_2^{(4)}\right) = 0.635793\cdot\left(1-0.635793\right)\cdot\left(0.0-0.635793\right) = -0.147224$$

Sada se pogreška računa za skrivene slojeve. Najprije računamo za treći sloj. Pogreška za skrivene slojeve računa se prema relaciji:



$$E_n^{(r)} = o_n^{(r)} \cdot (1 - o_n^{(r)}) \cdot \sum_j w_{nj}^{(r)} \cdot E_j^{(r+1)}$$

gdje je:

$E_n^{(r)}$	pogreška n-tog neurona u r-tom sloju,
$o_n^{(r)}$	izlaz n-tog neurona u r-tom sloju,
$E_j^{(r+1)}$	pogreška j-tog neurona u r+1 sloju (koju već znamo jer smo to izračunali),
$w_{nj}^{(r)}$	težinski faktor koji se nalazi uz vezu izlaza n-tog neurona u r-tom sloju i ulaza j-tog neurona u r+1 sloju.

$$\begin{aligned}E_1^{(3)} &= o_1^{(3)} \cdot \left(1 - o_1^{(3)}\right) \cdot \left(w_{11}^{(3)} \cdot E_1^{(4)} + w_{12}^{(3)} \cdot E_2^{(4)}\right) \\ &= 0.569028 \cdot (1 - 0.569028) \cdot (0.30 \cdot 0.0741850 + 0.35 \cdot (-0.147224)) = -0.007179\end{aligned}$$

$$\begin{aligned}E_2^{(3)} &= o_2^{(3)} \cdot \left(1 - o_2^{(3)}\right) \cdot \left(w_{21}^{(3)} \cdot E_1^{(4)} + w_{22}^{(3)} \cdot E_2^{(4)}\right) \\ &= 0.641740 \cdot (1 - 0.641740) \cdot (0.35 \cdot 0.0741850 + 0.25 \cdot (-0.147224)) = -0.002493\end{aligned}$$

$$\begin{aligned}E_3^{(3)} &= o_3^{(3)} \cdot \left(1 - o_3^{(3)}\right) \cdot \left(w_{31}^{(3)} \cdot E_1^{(4)} + w_{32}^{(3)} \cdot E_2^{(4)}\right) \\ &= 0.658516 \cdot (1 - 0.658516) \cdot (0.45 \cdot 0.0741850 + 0.3 \cdot (-0.147224)) = -0.002425\end{aligned}$$

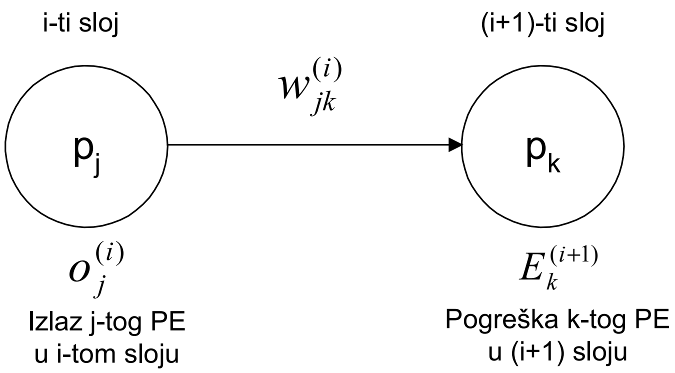
Zatim računamo pogrešku za drugi sloj:

$$
\begin{aligned}
E_1^{(2)} &= o_1^{(2)} \cdot \left(1 - o_1^{(2)}\right) \cdot \left(w_{11}^{(2)} \cdot E_1^{(3)} + w_{12}^{(2)} \cdot E_2^{(3)} + w_{13}^{(2)} \cdot E_3^{(3)}\right) \\
&= 0.507249 \cdot (1 - 0.507249) \cdot (0.1 \cdot (-0.007179) + 0.55 \cdot (-0.002493) + 0.35 \cdot (-0.002425)) \\
&= -0.000734 \\
E_2^{(2)} &= o_2^{(2)} \cdot \left(1 - o_2^{(2)}\right) \cdot \left(w_{21}^{(2)} \cdot E_1^{(3)} + w_{22}^{(2)} \cdot E_2^{(3)} + w_{23}^{(2)} \cdot E_3^{(3)}\right) \\
&= 0.507999 \cdot (1 - 0.507999) \cdot (0.2 \cdot (-0.007179) + 0.45 \cdot (-0.002493) + 0.35 \cdot (-0.002425)) \\
&= -0.000851 \\
E_3^{(2)} &= o_3^{(2)} \cdot \left(1 - o_3^{(2)}\right) \cdot \left(w_{31}^{(2)} \cdot E_1^{(3)} + w_{32}^{(2)} \cdot E_2^{(3)} + w_{33}^{(2)} \cdot E_3^{(3)}\right) \\
&= 0.502250 \cdot (1 - 0.502250) \cdot (0.25 \cdot (-0.007179) + 0.15 \cdot (-0.002493) + 0.6 \cdot (-0.002425)) \\
&= -0.000906
\end{aligned}
$$

Za prvi sloj pogreška se ne računa.

Korak 3. Potrebno je izvršiti korekciju faktora prema slijedećoj formuli:

$$w_{jk}^{(i)}(novi) = w_{jk}^{(i)} + \eta \cdot E_k^{(i+1)} \cdot o_j^{(i)}$$



Ovo je potrebno provesti za sve faktore; redoslijed nije bitan. Dobije se:

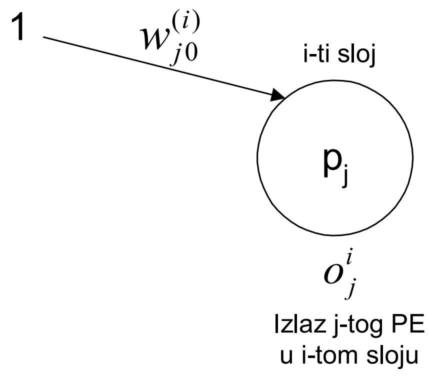
$$
\begin{aligned}
\mathrm{w}_{11}^{(1)} &= \mathrm{w}_{11}^{(1)} + \eta \cdot E_1^{(2)} \cdot o_1^{(1)} = 0.5 + 0.3 \cdot (\text{-}0.000734) \cdot 0.05 = 0.499989 \\
\mathrm{w}_{12}^{(1)} &= \mathrm{w}_{12}^{(1)} + \eta \cdot E_2^{(2)} \cdot o_1^{(1)} = 0.4 + 0.3 \cdot (\text{-}0.000851) \cdot 0.05 = 0.399987 \\
\mathrm{w}_{13}^{(1)} &= \mathrm{w}_{13}^{(1)} + \eta \cdot E_3^{(2)} \cdot o_1^{(1)} = 0.1 + 0.3 \cdot (\text{-}0.000906) \cdot 0.05 = 0.099986 \\
\mathrm{w}_{21}^{(1)} &= \mathrm{w}_{21}^{(1)} + \eta \cdot E_1^{(2)} \cdot o_2^{(1)} = 0.2 + 0.3 \cdot (\text{-}0.000734) \cdot 0.02 = 0.199996
\end{aligned}
$$

$$\mathrm{w}_{22}^{(1)} = \mathrm{w}_{22}^{(1)} + \eta \cdot E_2^{(2)} \cdot o_2^{(1)} = 0.6 + 0.3 \cdot (\text{-}0.000851) \cdot 0.02 = 0.599995$$
$$\mathrm{w}_{23}^{(1)} = \mathrm{w}_{23}^{(1)} + \eta \cdot E_3^{(2)} \cdot o_2^{(1)} = 0.2 + 0.3 \cdot (\text{-}0.000906) \cdot 0.02 = 0.199995$$

$$\mathrm{w}_{11}^{(2)} = \mathrm{w}_{11}^{(2)} + \eta \cdot E_1^{(3)} \cdot o_1^{(2)} = 0.1 + 0.3 \cdot (\text{-}0.007179) \cdot 0.507249 = 0.098908$$
$$\mathrm{w}_{12}^{(2)} = \mathrm{w}_{12}^{(2)} + \eta \cdot E_2^{(3)} \cdot o_1^{(2)} = 0.55 + 0.3 \cdot (\text{-}0.002493) \cdot 0.507249 = 0.549621$$
$$\mathrm{w}_{13}^{(2)} = \mathrm{w}_{13}^{(2)} + \eta \cdot E_3^{(3)} \cdot o_1^{(2)} = 0.35 + 0.3 \cdot (\text{-}0.002425) \cdot 0.507249 = 0.349631$$
$$\mathrm{w}_{21}^{(2)} = \mathrm{w}_{21}^{(2)} + \eta \cdot E_1^{(3)} \cdot o_2^{(2)} = 0.2 + 0.3 \cdot (\text{-}0.007179) \cdot 0.507999 = 0.198906$$
$$\mathrm{w}_{22}^{(2)} = \mathrm{w}_{22}^{(2)} + \eta \cdot E_2^{(3)} \cdot o_2^{(2)} = 0.45 + 0.3 \cdot (\text{-}0.002493) \cdot 0.507999 = 0.449620$$
$$\mathrm{w}_{23}^{(2)} = \mathrm{w}_{23}^{(2)} + \eta \cdot E_3^{(3)} \cdot o_2^{(2)} = 0.35 + 0.3 \cdot (\text{-}0.002425) \cdot 0.507999 = 0.349630$$
$$\mathrm{w}_{31}^{(2)} = \mathrm{w}_{31}^{(2)} + \eta \cdot E_1^{(3)} \cdot o_3^{(2)} = 0.25 + 0.3 \cdot (\text{-}0.007179) \cdot 0.502250 = 0.248918$$
$$\mathrm{w}_{32}^{(2)} = \mathrm{w}_{32}^{(2)} + \eta \cdot E_2^{(3)} \cdot o_3^{(2)} = 0.15 + 0.3 \cdot (\text{-}0.002493) \cdot 0.502250 = 0.149624$$
$$\mathrm{w}_{33}^{(2)} = \mathrm{w}_{33}^{(2)} + \eta \cdot E_3^{(3)} \cdot o_3^{(2)} = 0.6 + 0.3 \cdot (\text{-}0.002425) \cdot 0.502250 = 0.599635$$

$$\mathrm{w}_{11}^{(3)} = \mathrm{w}_{11}^{(3)} + \eta \cdot E_1^{(4)} \cdot o_1^{(3)} = 0.3 + 0.3 \cdot 0.074185 \cdot 0.569028 = 0.312664$$
$$\mathrm{w}_{12}^{(3)} = \mathrm{w}_{12}^{(3)} + \eta \cdot E_2^{(4)} \cdot o_1^{(3)} = 0.35 + 0.3 \cdot (-0.147224) \cdot 0.569028 = 0.324868$$
$$\mathrm{w}_{21}^{(3)} = \mathrm{w}_{21}^{(3)} + \eta \cdot E_1^{(4)} \cdot o_2^{(3)} = 0.35 + 0.3 \cdot 0.074185 \cdot 0.641740 = 0.364282$$
$$\mathrm{w}_{22}^{(3)} = \mathrm{w}_{22}^{(3)} + \eta \cdot E_2^{(4)} \cdot o_2^{(3)} = 0.25 + 0.3 \cdot (-0.147224) \cdot 0.641740 = 0.221656$$
$$\mathrm{w}_{31}^{(3)} = \mathrm{w}_{31}^{(3)} + \eta \cdot E_1^{(4)} \cdot o_3^{(3)} = 0.45 + 0.3 \cdot 0.074185 \cdot 0.658516 = 0.464656$$
$$\mathrm{w}_{32}^{(3)} = \mathrm{w}_{32}^{(3)} + \eta \cdot E_2^{(4)} \cdot o_3^{(3)} = 0.3 + 0.3 \cdot (-0.147224) \cdot 0.658516 = 0.270915$$

I zatim korekciju slobodnih faktora prema slijedećoj formuli:



$$w_{j0}^{(i)} = w_{j0}^{(i)} + \eta \cdot E_j^{(i)}$$

Dobije se:

$$w_{10}^{(2)} = w_{10}^{(2)} + \eta \cdot E_1^{(2)} = 0.0 + 0.3*(-7.34E\text{-}4)=-2.2E\text{-}4$$
$$w_{20}^{(2)} = w_{20}^{(2)} + \eta \cdot E_2^{(2)} = 0.0 + 0.3*(-8.51E\text{-}4)=-2.55E\text{-}4$$
$$w_{30}^{(2)} = w_{30}^{(2)} + \eta \cdot E_3^{(2)} = 0.0 + 0.3*(-9.06E\text{-}4)=-2.72E\text{-}4$$

$$w_{10}^{(3)} = w_{10}^{(3)} + \eta \cdot E_1^{(3)} = 0.0 + 0.3*(-0.007179)=-0.002154$$
$$w_{20}^{(3)} = w_{20}^{(3)} + \eta \cdot E_2^{(3)} = 0.0 + 0.3*(-0.002493)=-7.48E\text{-}4$$
$$w_{30}^{(3)} = w_{30}^{(3)} + \eta \cdot E_3^{(3)} = 0.0 + 0.3*(-0.002425)=-7.28E\text{-}4$$

$$w_{10}^{(4)} = w_{10}^{(4)} + \eta \cdot E_1^{(4)} = 0.0 + 0.3*0.074185=0.022256$$
$$w_{20}^{(4)} = w_{20}^{(4)} + \eta \cdot E_2^{(4)} = 0.0 + 0.3*(-0.147224)=-0.044167$$

Sada se može ići na slijedeću iteraciju. Mreži opet predočimo isti uzorak i ponovno obavimo sva tri koraka. I tako možemo ponavljati postupak sve dok ne budemo zadovoljni greškom koju mreža čini. Nakon toga mreži predočimo slijedeći uzorak iz skupa za učenje i opet ponovimo prethodnu proceduru. I tako radimo sve dok ne iscrpimo sve uzorke iz skupa za učenje. Time završavamo jednu epohu u učenju mreže. Kako je dobro mreža «naučila» gradivo, može se provjeriti testiranjem uzorcima iz skupa za provjeru. Ukoliko rezultatima nismo zadovoljni, možemo pokrenuti novu epohu učenja mreže, i ciklus ponavljati koliko je potrebno.

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