## Section#1 Formulation Derivation

The purpose of this section is to develop every formulation we need in the finite element method code, but I won't go through the very beginning of the entire derivation, some details please refer section#1 in my assignment #3.

From first two equations in the differential form of Maxwell equations, which are:

$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

Separate the variable t, and assume the magnetic source to complete the symmetric duality and get the time-independent form of Maxwell equation as:

$$\nabla \times E(r) = -j\omega B(r) - M(r)$$
$$\nabla \times H(r) = j\omega D(r) + J(r)$$

And here we have isotropic materials in 1D problem with $\varepsilon = \varepsilon_0 \varepsilon_{re}, \mu = \mu_0 \mu_{re}$, we get:

$$\nabla \times E(r) = -j\omega \mu_0 \mu_{re} H(r) - M(r)$$
$$\nabla \times H(r) = j\omega \varepsilon_0 \varepsilon_{re} E(r) + J(r)$$

Expand the curl and ignore the terms does matter in the 1D problem, and assume the electric field exist only in x-direction, we got the simplified form for 1D problem:

$$\hat{x}: -\frac{\partial H_y}{\partial z} = j\omega \varepsilon_0 \varepsilon_{re} E_x + J$$
$$\hat{y}: \frac{\partial E_x}{\partial z} = -j\omega \mu_0 \mu_{re} H_y - M$$

Combining above two equation together by using differential operation on one of them, we have the equation regarding only E (or only H):

$$\frac{d}{dz}\{\frac{1}{\mu_{re}}\frac{dE_x(z)}{dz}\} + k_0^2 \varepsilon_{re} E_x(z) = j\omega \mu_0 J(z) - \frac{d}{dz}\left\{\frac{1}{\mu_{re}} M(z)\right\} \qquad (1\text{-}1)$$

Reference to Chapter 1 formula (1.4-1), it can also be expressed in general form of 1st order ordinary differential equation:

$$-\frac{d}{dz}\{p(z)\frac{df(z)}{dz}\}+k_0^2 q(z) f(z)=g(z)$$

Now for the 1D infinite periodic structure, there is no impressed source exists. So the source term $g(z)=0$, we have a wave equation now:

$$\begin{aligned} &-\frac{d}{dz}\{p(z)\frac{df(z)}{dz}\}+k_0^2 q(z) f(z)=0 \quad \Leftrightarrow \\ &\frac{d}{dz}\{\frac{1}{\mu_{re}}\frac{dE_x(z)}{dz}\}+k_0^2 \varepsilon_{re} E_x(z)=0 \end{aligned} \tag{1-2}$$

To deal with differential equation numerically, we need to solve the weak form of equation (Here after check the definition of weak form in mathematics, I still think the integrated weighted residue form is the weak form. Green's identity is just a identity transform, but the integrated weighted residue is not, so it called the 'weak' one), as:

$$\int_{z_L}^{z_U}\left[p(z)\frac{dw(z)}{dz}\frac{df(z)}{dz}+k_0^2 q(z)w(z)f(z)\right]dz-p(z)w(z)\frac{df(z)}{dz}\Big|_{z_L}^{z_U}=0 \tag{1-3}$$

For the above part the details can be found in my assignment#3.

**Periodic boundary condition:**

Now to deal with the infinite periodic structure, we need to use its periodic property. Select one of its period with length d, the Floquet's theorem (or called Bloch theorem in physics) can give us the periodic boundary conditions at the ends of the unit cell are:

$$\begin{aligned} E_x(z_U)&=E_x(z_L)e^{-j\beta d} \\ H_y(z_U)&=H_y(z_L)e^{-j\beta d} \end{aligned} \tag{1-4}$$

In general form (1-3) are just like:

$$f(z_U)=f(z_L)e^{-j\beta d} \tag{1-5}$$

Where the value of $\beta$ is still unknown and need to be solved. So it's obvious that $f(z)$ is not a periodic function since it has continuous ascending or descending phase. However, we can separate it as two part with Floquet's theorem as:

$$f(z)=f_P(z)e^{-j\beta z} \tag{1-6}$$

Where $f_P(z)$ is a periodic function with $f_P(z) = f_P(z+d)$ and for its derivatives.

Now the $f_P(z)$ can be solved with FEM with the periodic boundary condition without phase change term. To deal with that, we need to replace the $f(z)$ in the weak form equation with $f_P(z)$. We have:

$$\frac{df(z)}{dz} = e^{-j\beta z}\left\{\frac{df_P(z)}{dz} - j\beta f_P(z)\right\} \tag{1-7}$$

If we choose the weighting function to be same with $f(z)$, the phase term $e^{-j\beta z}$ may get annoying in the following calculation, so it's much convenient that we can get rid of the $\beta$ in most terms. So we select the weighting function with opposite phase term, as:

$$w(z) = w_P(z)e^{+j\beta z} \tag{1-8}$$

And its derivative like:

$$\frac{dw(z)}{dz} = e^{j\beta z}\left\{\frac{dw_P(z)}{dz} + j\beta w_P(z)\right\} \tag{1-9}$$

Take (1-6) to (1-9) into the weak form equation (1-3), we have:

$$\begin{aligned}\int_{z_L}^{z_U}&\left\{p(z)\frac{dw_P(z)}{dz}\frac{df_P(z)}{dz} + j\beta p(z)\left[w_P(z)\frac{df_P(z)}{dz} - f_P(z)\frac{dw_P(z)}{dz}\right]\right.\\&\left.+\left(\beta^2 p(z) + k_0^2 q(z)\right)w_P(z)f_P(z)\right\}dz\\&- p(z)w_P(z)\left\{\frac{df_P(z)}{dz} - j\beta f_P(z)\right\}\Big|_{z_L}^{z_U} = 0\end{aligned} \tag{1-10}$$

(1-10) is the final equation we need to solve with FEM, and here both $\beta$ and $f_P(z)$ are unknown.

**Expansion function:**

Here we select the first order expansion functions to represent $f_P(z)$ in each element, and the weighting function to be same with expansion function (Galerkin choice). We have:

$$f_P^e(z) = \sum_{i=1}^{2} f_i^e N_i^e(z) \quad w_P(z) = N_j^e(z) \tag{1-11}$$

And the expression for $N_i^e(z)$ is:

$$N_1^e(z)=\begin{cases}\dfrac{z_2^e-z}{h_e}, z_1^e<z<z_2^e\\ 0, Otherwise\end{cases}\quad N_2^e(z)=\begin{cases}\dfrac{z-z_1^e}{h_e}, z_1^e<z<z_2^e\\ 0, Otherwise\end{cases}$$

And the normalized first order expansion functions are:

$$\begin{aligned}&N_1^e(\tau)=\frac{1}{2}(1-\tau)\quad N_2^e(\tau)=\frac{1}{2}(1+\tau)\\ &\text{with } \tau=\frac{2z-(z_1^e+z_2^e)}{h_e}\Rightarrow dz=\frac{h_e}{2}d\tau\end{aligned}\qquad (1\text{-}12)$$

Separate the unit cell into multiple elements, and for each element, take (1-11) into (1-10), get:

$$\begin{aligned}&\int_{z_a^e}^{z_b^e}\{p(z)\frac{dN_j^e(z)}{dz}\frac{d}{dz}\left(\sum_{i=1}^{2}f_i^eN_i^e(z)\right)+j\beta p(z)\left[N_j^e(z)\frac{d}{dz}\left(\sum_{i=1}^{2}f_i^eN_i^e(z)\right)-\left(\sum_{i=1}^{2}f_i^eN_i^e(z)\right)\frac{dN_j^e(z)}{dz}\right]\\ &\quad+\left(\beta^2p(z)+k_0^2q(z)\right)N_j^e(z)\left(\sum_{i=1}^{2}f_i^eN_i^e(z)\right)\}dz\\ &\quad-p(z_b^e)N_j^e(z_b^e)\left\{\left.\frac{df_P(z)}{dz}\right|_{z=z_b^e}-j\beta f_P(z_b^e)\right\}+p(z_a^e)N_j^e(z_a^e)\left\{\left.\frac{df_P(z)}{dz}\right|_{z=z_a^e}-j\beta f_P(z_a^e)\right\}=0\end{aligned}$$

Above expression should be satisfied with both j=1 and j=2;

Rearrange them we have:

$$\begin{aligned}&\sum_{i=1}^{2}f_i^e\left(\int_{z_a^e}^{z_b^e}p_e\frac{dN_j^e(z)}{dz}\frac{dN_i^e(z)}{dz}+j\beta p_e\left[N_j^e(z)\frac{dN_i^e(z)}{dz}-N_i^e(z)\frac{dN_j^e(z)}{dz}\right]+\left(\beta^2p_e+k_0^2q_e\right)N_j^e(z)N_i^e(z)\right)\\ &\quad-p_eN_j^e(z_b^e)\left\{\left.\frac{df_P(z)}{dz}\right|_{z=z_b^e}-j\beta f_P(z_b^e)\right\}+p_eN_j^e(z_a^e)\left\{\left.\frac{df_P(z)}{dz}\right|_{z=z_a^e}-j\beta f_P(z_a^e)\right\}=0\quad(\text{for } j=1,2)\end{aligned}$$

…(1-13)

The expression (1-13) is the FEM expression we need to solve.

**Matrix equation:**

Above FEM equations (1-13) can also be rewritten as Matrix form, which fit better with computer numerical solution. The matrix equation is shown in (1-14):

$$\begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} \end{bmatrix}\begin{bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}=\begin{bmatrix} P_1^{(e)} \\ P_2^{(e)} \end{bmatrix} \Leftrightarrow \left[K^{(e)}\right]\left[f^{(e)}\right]=\left[P^{(e)}\right] \tag{1-14}$$

where expression for each part as:

$$K_{ij}^{(e)} = S_{ij}^{(e)} + k_0^2 B_{ij}^{(e)} \qquad S_{ij}^{(e)} = A_{ij}^{(e)} + T_{ij}^{(e)} + D_{ij}^{(e)}$$

$$A_{ij}^{(e)} = p_e \int_{z_a^e}^{z_b^e} \frac{dN_i^e(z)}{dz}\frac{dN_j^e(z)}{dz}dz \qquad B_{ij}^{(e)} = q_e \int_{z_a^e}^{z_b^e} N_i^e(z)N_j^e(z)dz$$

$$T_{ij}^{(e)} = \beta^2 p_e \int_{z_a^e}^{z_b^e} N_i^e(z)N_j^e(z)dz \qquad D_{ij}^{(e)} = j\beta p_e \int_{z_a^e}^{z_b^e} \left\{ N_j^e(z)\frac{dN_i^e(z)}{dz} - \frac{dN_j^e(z)}{dz}N_i^e(z) \right\} dz$$

$$P_1^{(e)} = -p(z_a^e)\left\{ \left.\frac{df(z)}{dz}\right|_{z=z_a^e} - j\beta f(z_a^e) \right\} \qquad P_2^{(e)} = p(z_b^e)\left\{ \left.\frac{df(z)}{dz}\right|_{z=z_b^e} - j\beta f(z_b^e) \right\} \tag{1-15}$$

There are simpler matrices form to calculate terms in (1-15), its will be shown later. Now the total matrix equation we want to solve is the linear combination of every element, like:

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & \cdots & 0 \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 & \cdots & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & \cdots & 0 \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)}+K_{11}^{(4)} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & K_{12}^{(n)} \\ 0 & 0 & 0 & \cdots & K_{21}^{(n)} & K_{22}^{(n)} \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{n+1} \end{bmatrix}=\begin{bmatrix} P_1^{(1)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ P_2^{(n)} \end{bmatrix} \tag{1-16}$$

We haven't applied the periodic boundary conditions to the equations yet, remember the solution $f_P(z)$ is periodic with period d, so at the two ends of a unit cell with length d, $f_P(z)$ and its higher-order derivatives must keep equal. Here we only need itself and its first order derivative keep same:

$$\begin{aligned} &f_P(z_L) = f_P(z_U) \Leftrightarrow f_1 = f_{n+1} \\ &\left.\frac{df_P(z)}{dz}\right|_{z=z_L} = \left.\frac{df_P(z)}{dz}\right|_{z=z_U} \end{aligned} \tag{1-17}$$

Take (1-17) into (1-15) and consider that when we do the segmentation, the materials at the two ends keep same, which means $p(z_a^1) = p(z_b^n)$.Now we get:

$$P_1^{(1)} + P_2^{(n)} = -p(z_a^1)\left\{ \left.\frac{df(z)}{dz}\right|_{z=z_a^1} - j\beta f(z_a^1) \right\} + p(z_b^n)\left\{ \left.\frac{df(z)}{dz}\right|_{z=z_b^n} - j\beta f(z_b^n) \right\} = 0 \tag{1-18}$$

Take first part in (1-17) and (1-18) into (1-16), now we can simplify the matrix equation further.

$$\begin{bmatrix} K_{11}^{(1)}+K_{22}^{(\mathrm{n})} & K_{12}^{(1)} & 0 & \cdots & 0 & K_{21}^{(\mathrm{n})} \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 & \cdots & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & \cdots & 0 \\ \vdots & 0 & K_{21}^{(3)} & K_{22}^{(3)}+K_{11}^{(4)} & \ddots & \vdots \\ 0 & \vdots & \vdots & \ddots & \ddots & K_{12}^{(n)} \\ K_{12}^{(\mathrm{n})} & 0 & 0 & \cdots & K_{21}^{(n)} & K_{22}^{(\mathrm{n}-1)}+K_{11}^{(\mathrm{n})} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} P_1^{(1)}+P_2^{(\mathrm{n})} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice that above equation has zeros on the right side, so we can separate the K matrix to two part with $K_{ij}^{(\mathrm{e})} = S_{ij}^{(\mathrm{e})} + k_0^2 B_{ij}^{(\mathrm{e})}$. Then rewrite them as a generalized eigenvalue problem:

$$\begin{bmatrix} S_{11}^{(1)}+S_{22}^{(\mathrm{n})} & S_{12}^{(1)} & 0 & S_{21}^{(\mathrm{n})} \\ S_{21}^{(1)} & S_{11}^{(2)}+S_{22}^{(1)} & \ddots & 0 \\ 0 & \ddots & \ddots & S_{12}^{(\mathrm{n}-1)} \\ S_{12}^{(\mathrm{n})} & 0 & S_{21}^{(\mathrm{n}-1)} & S_{11}^{(\mathrm{n})}+S_{22}^{(\mathrm{n}-1)} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = -k_0^2 \begin{bmatrix} B_{11}^{(1)}+B_{22}^{(\mathrm{n})} & B_{12}^{(1)} & 0 & B_{21}^{(\mathrm{n})} \\ B_{21}^{(1)} & B_{11}^{(2)}+B_{22}^{(1)} & \ddots & 0 \\ 0 & \ddots & \ddots & B_{12}^{(\mathrm{n}-1)} \\ B_{12}^{(\mathrm{n})} & 0 & B_{21}^{(\mathrm{n}-1)} & B_{11}^{(\mathrm{n})}+B_{22}^{(\mathrm{n}-1)} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\Leftrightarrow [S][f] = -k_0^2 [B][f]$$

…(1-19)

Now we have the numerical FEM solution for this infinite periodic structure problem. Here notice that we still have two independent unknowns in (1-19), $\beta$ and $f$ (of course $k_0$ is also an unknown but is a eigenvalue related to $f$). So we can only select $\beta$ at beginning and then we can calculated the $f$ and $k_0$, since above (1-19) is a close form about $f$ and $k_0$, but not for $\beta$. Another thing need to remember is that the FEM solution $f$ is the periodic part of the real field, so after all we need to get the real field also:

$$f_{real}(\mathrm{z}) = f_P(\mathrm{z}) e^{-j\beta z} \qquad (1\text{-}20)$$

**Matrix form for some terms:**

According to (1-12) or chapter 1.16 in course notes, the integrated terms in (1-15) have simpler numerical expression since we used low-order expansion functions (of course here I used the simplest one, 1st order). So we don't need Gauss-quadrature to calculate them numerically.

For matrix $A^{(e)}, B^{(e)}, T^{(e)}, D^{(e)}$, one of the four entries in one element is like:

$$A_{12}^{e} = p_e \int_{z_a^e}^{z_b^e} \frac{dN_1^e(z)}{dz}\frac{dN_2^e(z)}{dz}dz = p_e \int_{-1}^{1} \left(\frac{-1}{h_e}\right)\left(\frac{1}{h_e}\right)\frac{h_e}{2}d\tau = -\frac{p_e}{h_e}$$

$$B_{12}^{(e)} = q_e \int_{z_a^e}^{z_b^e} N_1^e(z)N_2^e(z)dz = q_e \int_{-1}^{1} \frac{1}{2}(1-\tau)\frac{1}{2}(1+\tau)\frac{h_e}{2}d\tau = \frac{q_e h_e}{6}$$

$$T_{12}^{(e)} = \beta^2 p_e \int_{z_a^e}^{z_b^e} N_1^e(z)N_2^e(z)dz = \beta^2 p_e \frac{h_e}{6} \quad \text{(similar to } B\text{)}$$

$$D_{12}^{(e)} = j\beta p_e \int_{z_a^e}^{z_b^e} \left\{ N_2^e(z)\frac{dN_1^e(z)}{dz} - \frac{dN_2^e(z)}{dz}N_1^e(z) \right\} dz$$

$$= j\beta p_e \int_{-1}^{1} \left\{ \frac{1}{2}(1-\tau)\left(\frac{-1}{h_e}\right) - \frac{1}{2}(1+\tau)\left(\frac{1}{h_e}\right) \right\} \frac{h_e}{2}d\tau = j\beta p_e$$

After same calculation for other three entries, we have:

$$[A^e] = \frac{p_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [B^e] = \frac{q_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[T^e] = \frac{\beta^2 p_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad [D^e] = j\beta p_e \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1-21}$$

Directly using (1-21) to calculate the entries value in (1-19) will reduce the computation complexity in a large degree, and in MATLAB we can use *kron()* function to do the Kronecker product and use vector operation rather than for-loop in the code, that will also shorten the run-time of FEM codes. So far we have got all the expressions we need in the FEM code using MATLAB.

## Section#2 Pre-processing and Post-processing

For this infinite periodic structure, there is no much pre-processing or post-processing step need to do comparing with assignment#3 and assignment#5, here I just mention some steps need to do.

### Meshing (pre-processing):

Remember that we assume in every element the field is linear, so there should be no discontinuity in an element. To do the meshing part, I still use the self-defined Matlab function `ass3_seg()` which I defined since assignment#3. The input

arguments are the z-coordinates of which interface or boundaries of materials located in a vector, and a scale input argument is the segmentation number in the smallest width slab. Then the function will get the closest segment length in every slab meanwhile keep the boundary or interface at end points.

**Post-processing**

First thing need to point out here is that the direct FEM solution is the periodic part of the real field, although the magnitude of them are same, we still need to add the phase term (with the corresponding $\beta$ value) back to get the true answer, as mentioned in (1-20).

In this assignment, what we need are only the field distribution in a unit cell, and the band-graph for the structure in a given range. So that when deal with the returned eigenvalues $-k_0^2$, I used the *sort()* and *find()* functions in MATLAB to rearrange the eigenvalues in a descending order and get the location which value satisfied the required range(e.g. $0 \le k_0 d \le 25, 0 \le \beta d \le \pi$ for the modeling problem).

Here in my code, I still get the FEM solution regarding both E and H field respectively. So I can get the eigenvalue from E field solution and from H field solution, I will show it later that for the group of eigenvalue with lower value, both solution return almost same result. The higher the difference between two solution get bigger, that may be caused by that the solution haven't get converged since for high frequency, more segments are need to precisely represent the field.

# Section#3 Code Validation and Convergence Check

## 1) Validation Check

To prove the validation of my code, I will check them by two ways. First, I will compare the analytical solution in the course notes, especially the band-graph in Fig.1-36, with my FEM solution. Secondly, I will set some arbitrary parameters in the 1D periodic structure with some complexity, and solve it, just like what I done in assignment#3 and #5.

### a) Analytical Solution and Comparison

In chapter 1.29 of course notes, we have the analytical solution of the structure with a matrix form equation for the analytical electric field, so it's not convenient to directly show a specific E field figure with (1.29-13) expression in the notes. However, the analytical solution give out the relation between Floquet wavenumber $\beta$ and the free space wavenumber $k_0$ in (1.29-15):

$$\beta(\omega)=\frac{1}{d}\cos^{-1}\left\{\cos(k_0 n_1 d_1)\cos(k_0 n_2 d_2)-\frac{n_1^2+n_2^2}{2n_1 n_2}\sin(k_0 n_1 d_1)\sin(k_0 n_2 d_2)\right\} \qquad (3\text{-}1)$$

Where $n_1=\sqrt{\mu_1\varepsilon_1}=\sqrt{\varepsilon_1}$ , $n_2=\sqrt{\mu_2\varepsilon_2}=\sqrt{\varepsilon_2}$ , and the $\beta-k_0$ plot should be like the Brillouin diagram for 1D periodic structure in Fig.1-36 in course notes when the parameters are set to $n_1=1,\ n_2=4,\ d_1/d=0.85, d_2/d=0.15$. And I selected the range of $0\le k_0 d\le 8$, $0\le \beta d\le \pi$.

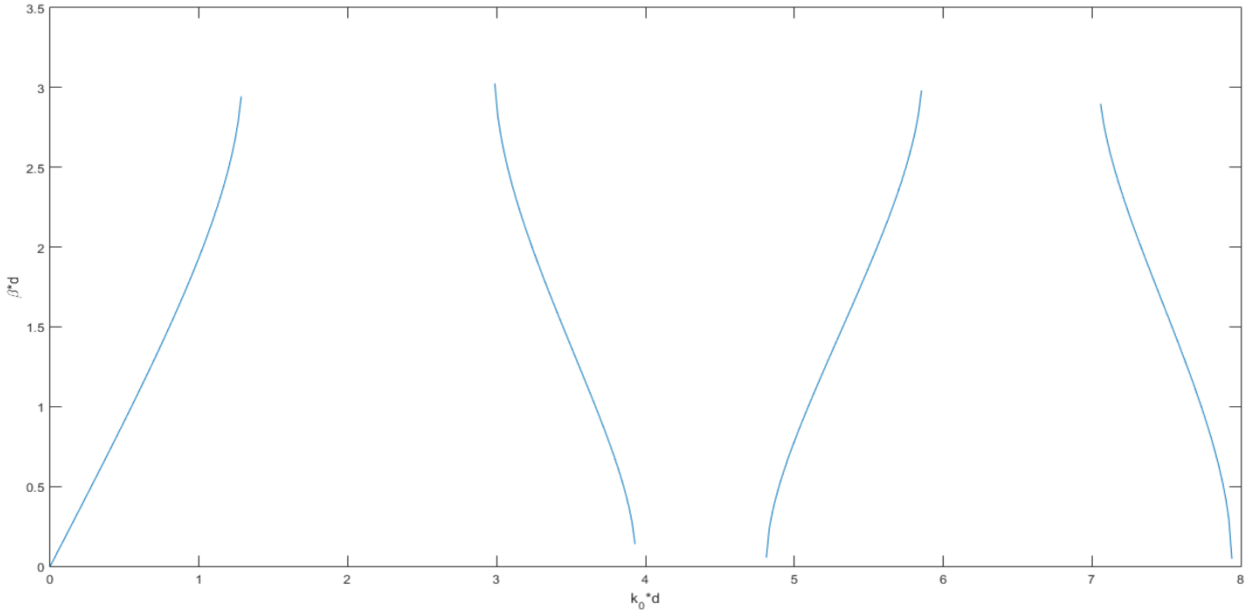



Figure 3-1. Analytical result of $\beta-k_0$ plot

The inverse cosine pattern can be seen from Fig. 3-1. Now I set the structure settings to be same with above parameters, as:

```
d1=85e-3;
d2=15e-3;
d=d1+d2;
z=[0 d1/2 d1/2+d2 d];          % first row: z;  second row: epsilon_r, mu_r
p_z=[1 16 1; 1 1 1];
```

Now I plotted the FEM solution together with above analytical solution, but with x-axis be $\beta d$ and y-axis be $k_0 d$. here the FEM solution region is $0 \le k_0 d \le 10$:

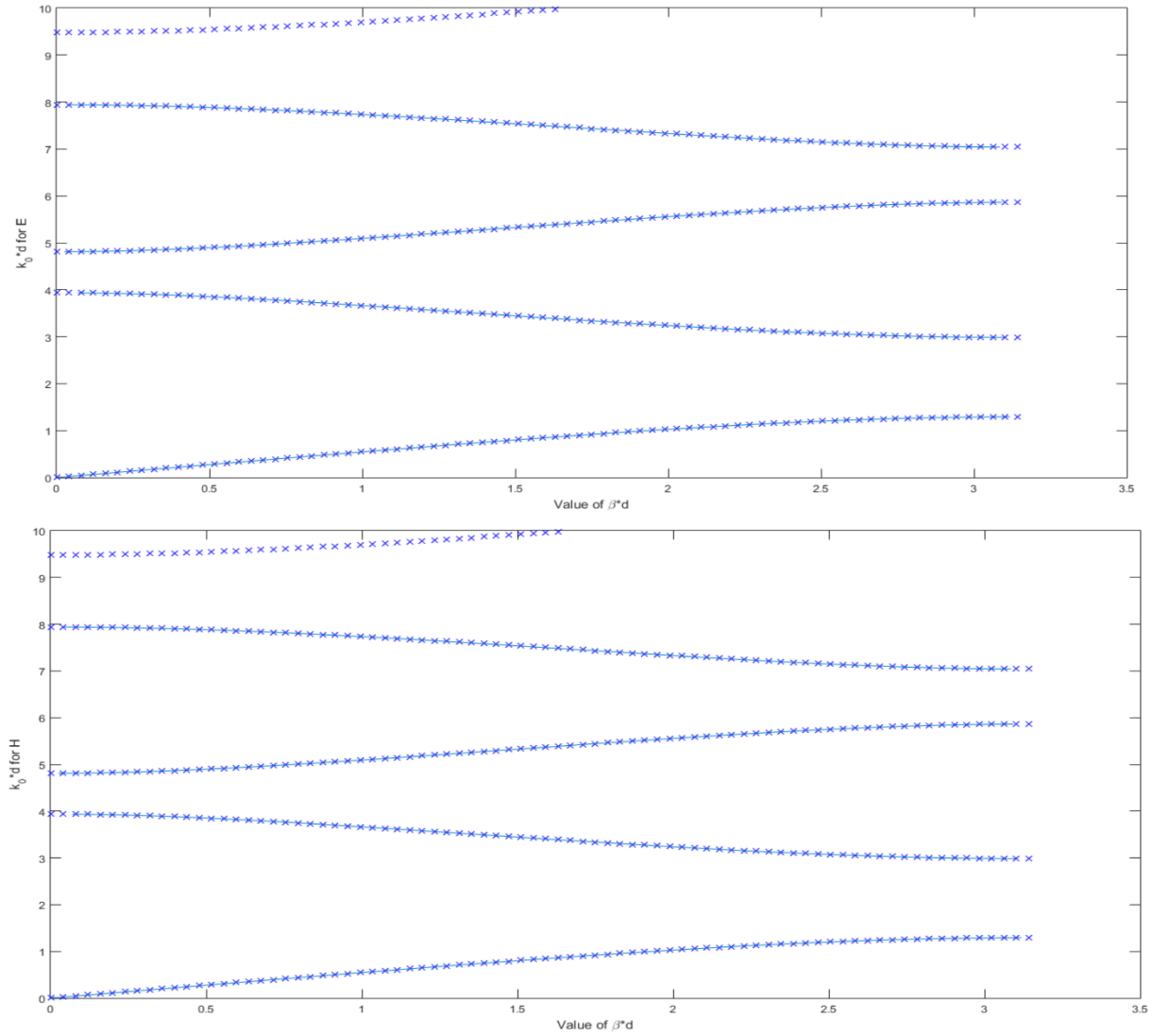



Figure 3-2. Analytical result and FEM result of $k_0 \sim \beta$ plot. Upper: eigenvalues got with electric field solution. Lower: eigenvalues with magnetic field solution.

From Fig.3-2 it's obvious that the FEM is precisely fit the analytical solution and for the low frequencies, the eigenvalues calculated with E field or H field are

almost same, the error ratio is about $10^{-5}$ magnitude order with only total 300 segmentations in the unit cell. Notice that this segmentation number is quite small, I set it in that level to reduce the run-time of the band-graph sweep process, when plot the field distribution I will add the segmentation number and thus the wavenumber calculated will get more precise.

According to above comparison, I show the precise correspondence between my code's result with the analytical one. I think it proved the validity of the code for the simple problem. Now I will move on to the complex 1D problem to prove it validity.

**b)** Complex modeling problem

This problem is similar with the one in my assignment 3 and assignment 5 but now be changed to an infinite periodic slabs problem, the values are selected arbitrarily. Here the unit cell begins with -3mm and ends at 47mm, and the most left and most right slabs has same parameters to satisfy the periodic condition.
The whole structure is set as following parameters: : The left end and the right end parts can be seen as one slab, with $\varepsilon_r = 3.97, \mu_r = 3.11$. From 0mm to 15mm there is free space, then there is an PMA slab located within 15mm to 22mm, followed by another slab of dielectric with small permeability; From 30mm to 40mm there is a 3 layers structure with one dielectric from 34mm to 37mm, then a high electric and magnetic loss material (34mm~37mm), them a special material with $\varepsilon_r = 1, \mu_r = 8$. As shown below:

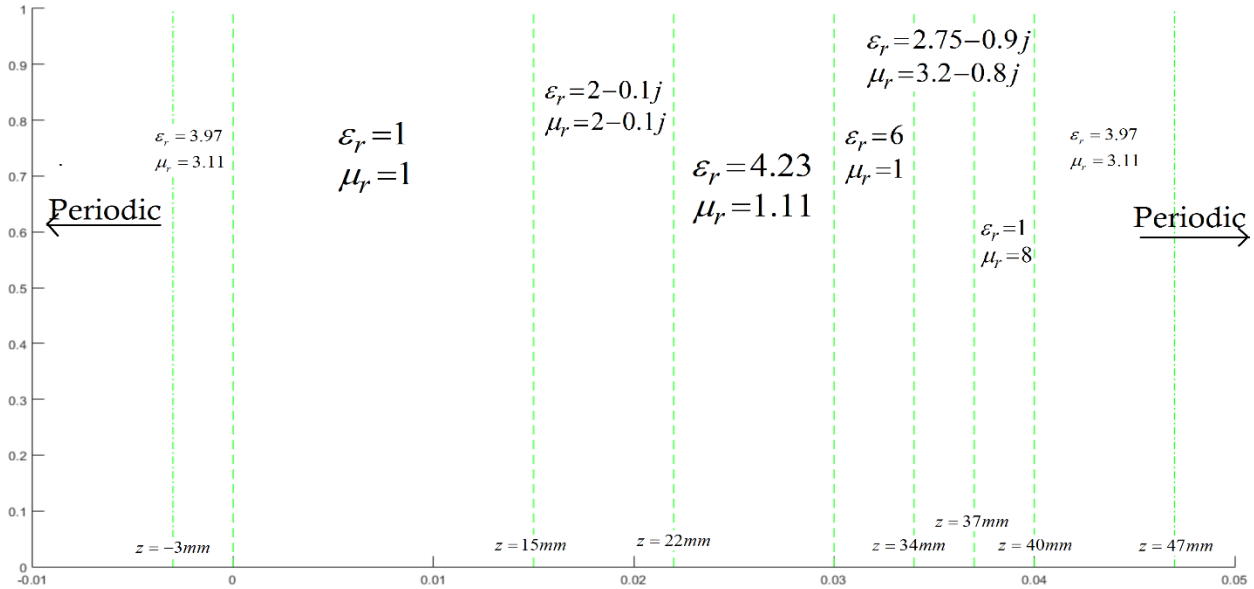


Figure 3-3. Arbitrary set complex structure for Demo

The settings above are expressed as vectors in Matlab code:

```
z=[-3e-3 0 15e-3 22e-3 30e-3 34e-3 37e-3 40e-3 47e-3];
p_z=[3.97 1 2-0.1j 4.23 6 2.75-0.9j 1 3.97; 3.11 1 2-0.1j 1.11 1 3.2-0.8j 8
3.11 ];
```

Now use my FEM code to get results of this problem with 40 segmentation for the smallest slab, and the Floquet wavenumber is select at $\beta = 10$, those results are listed as following:

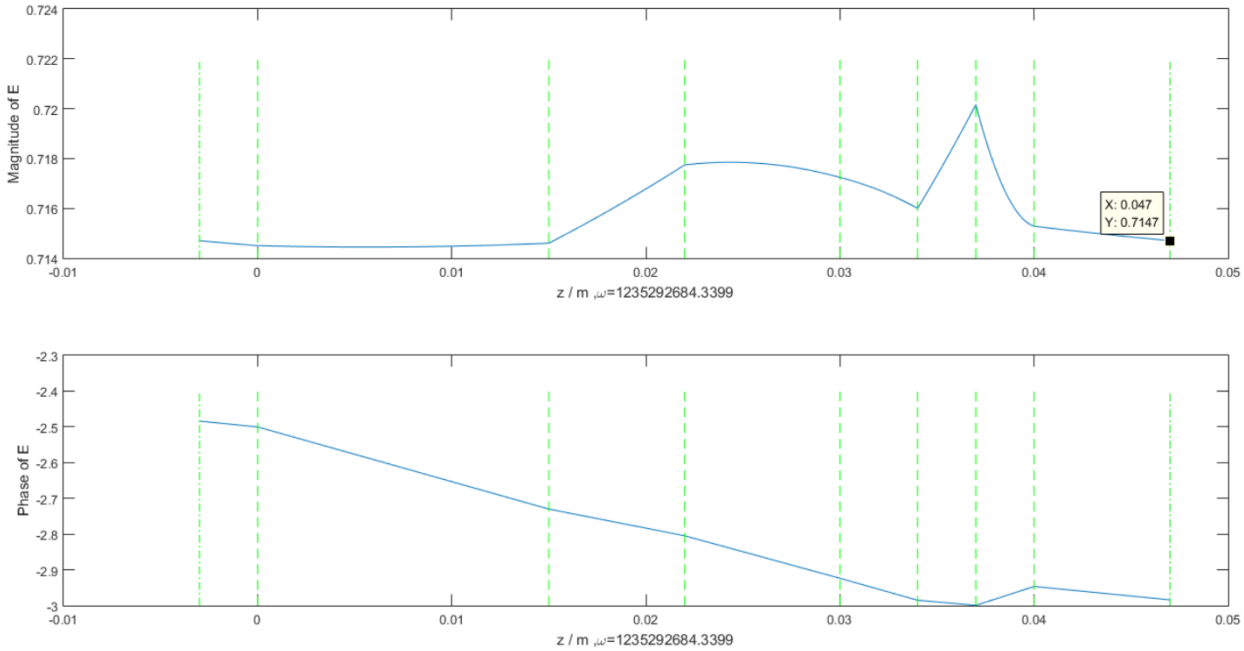



Figure 3-4. Magnitude and phase of electric field at lowest frequency

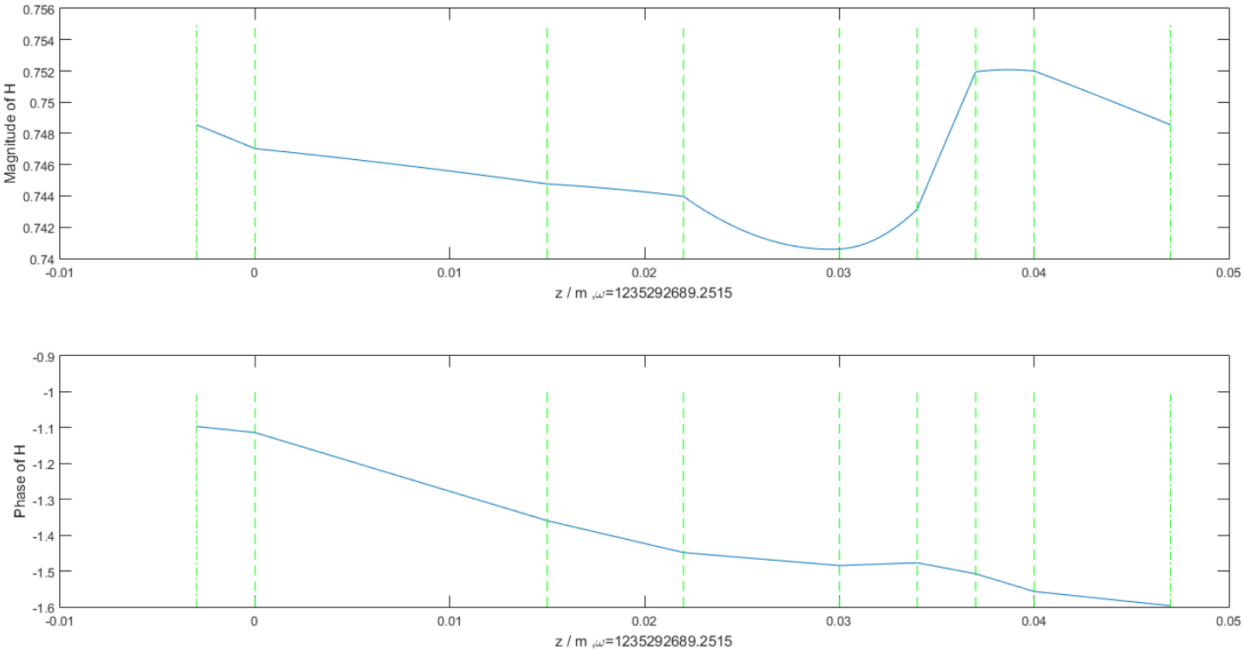



Figure 3-5. Magnitude and phase of magnetic field at lowest frequency

From Fig.3-4 and Fig.3-5 we can see the both electric field and magnetic field have continuous magnitude at two ends of the unit cell, and its derivatives are also continuous which can be seen from the trend of varying in the plots. And the phase also have derivative continuity, and the phase difference at two ends are constant (as $\Delta\phi = 0.5 = \beta d = 10 * 0.05$ ), and the difference keeps same in E field solution and H field solution because it decided by the $\beta$ value we selected.

Comparing the angular frequencies calculated from the E field eigenvalue and from H field eigenvalue, we will found that the error is very small for the lowest frequency, it's only about -10th magnitude order.

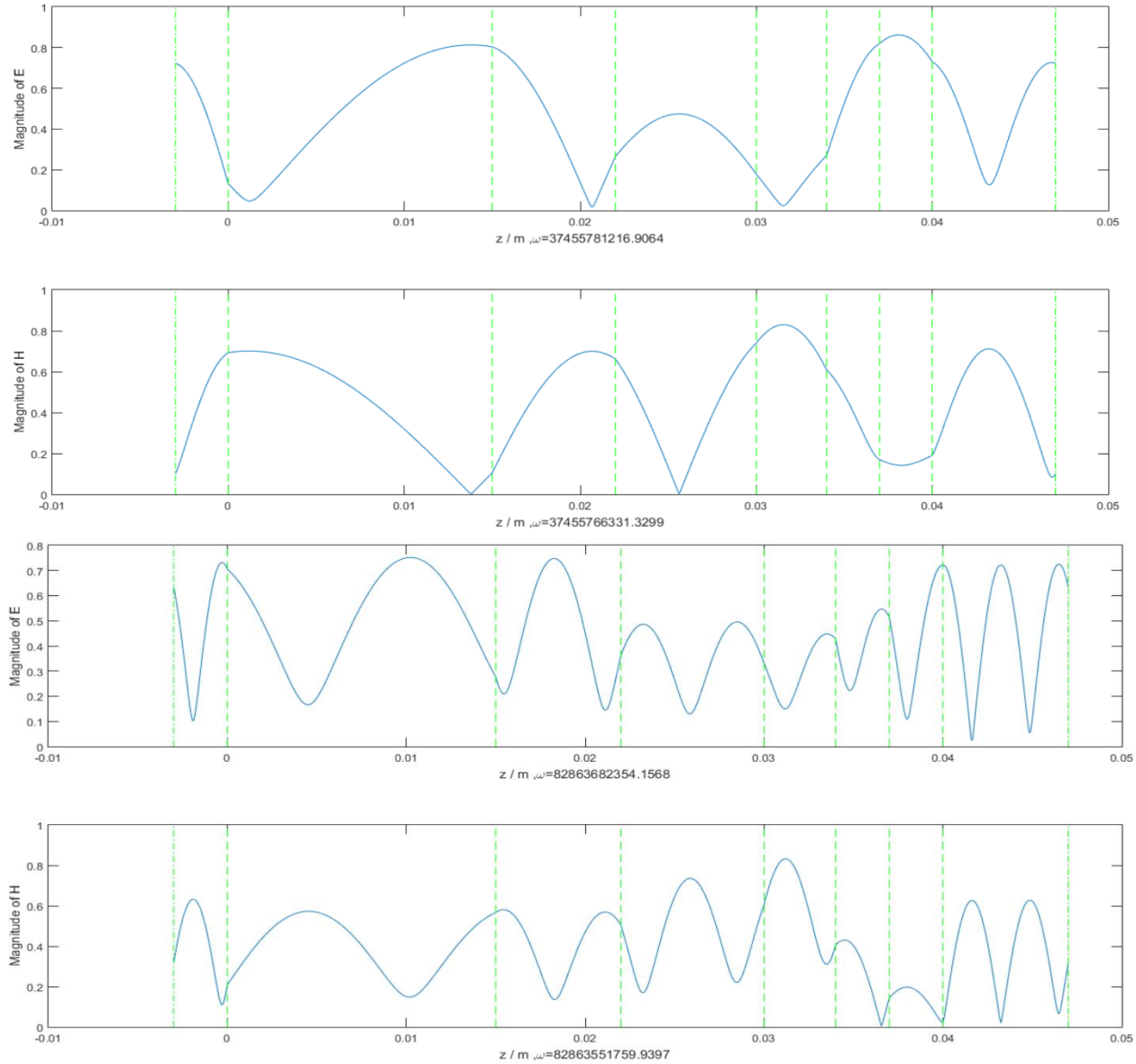



Figure 3-6. Magnitude of electric field and magnetic field at different frequencies. Upper: 5th lowest frequency. Lower: 10th lowest frequency

From Fig.3-6, we can see the periodic property of resulted E field and H field, also we can see the general $\pi/2$ phase shift between electric component and magnetic component which satisfies the first order differential relation between E and H in Maxwell equation. Have a look at the computed propagation frequencies, we can found that the order of error gets bigger with the order of frequency increases, reach $10^{-6}$ and $10^{-5}$ separately.

Now plot the band-graph for this structure with a rough resolution in Fig.3-7

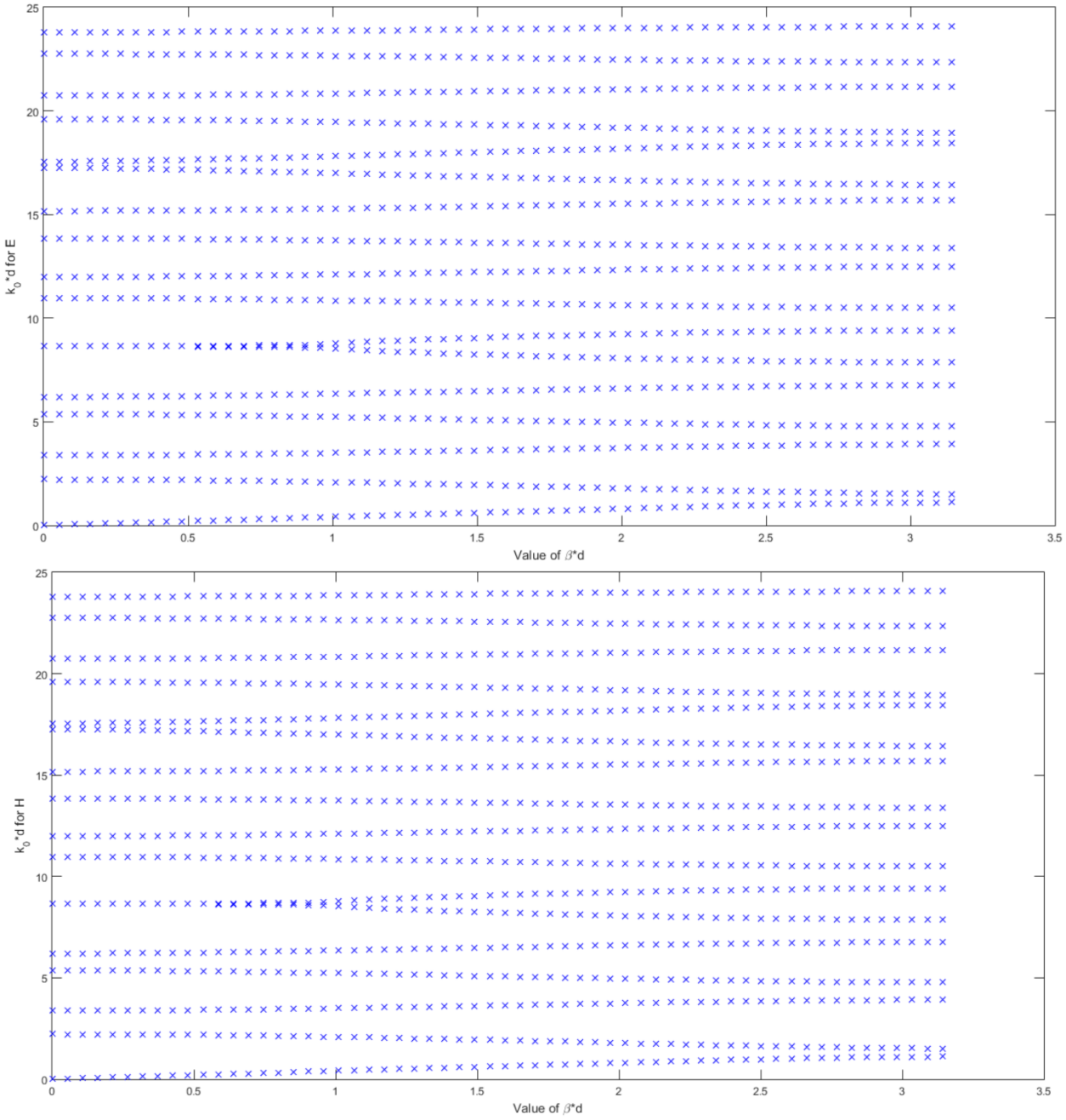



Figure 3-7. FEM result band-graph with eigenvalue of E field(upper) and H field(lower).

It can be seen form Fig.3-7, the band-graph got with E field solution or H field solution are almost same. Each of the cross-line has the trend of inverse cosine, but the analytical result in too complex for this arbitrarily set problem so I didn't compare it with an analytical pattern. But it looks right intuitively, near kd=10 we can seen the two band merge together to one.

Now I hope from above validity check part, my code can be proved to be robust. And you can change the settings to run another problem in my code submitted if you want.

## 2) Convergence check

To demonstrate the convergence of the code, one easy way is to keep all parameters fixed, the only varying argument is the segmentation N. But the result field is a vector rather than a number, so it's difficult to use whole field to judge whether it has been converged or not. Here for the periodic structure since we don't have many quantities in post-processing part to be indicators. Here I will only use the calculated frequencies (identity to eigenvalues) to indicate the convergence, since use a whole band-graph is too time-consuming although it can show the convergence state.

Here I selected the settings in the analytical result comparison of validity check part. Because if I still use the complex problem to demonstrate the convergence, it will spend a lot of time even if I only set segmentation number N from 10 to 40. So here I for the same setting with Fig.1-36 in course notes, I select N from 6 to 60 and focus on the lowest 6 frequencies in Fig.3-8:

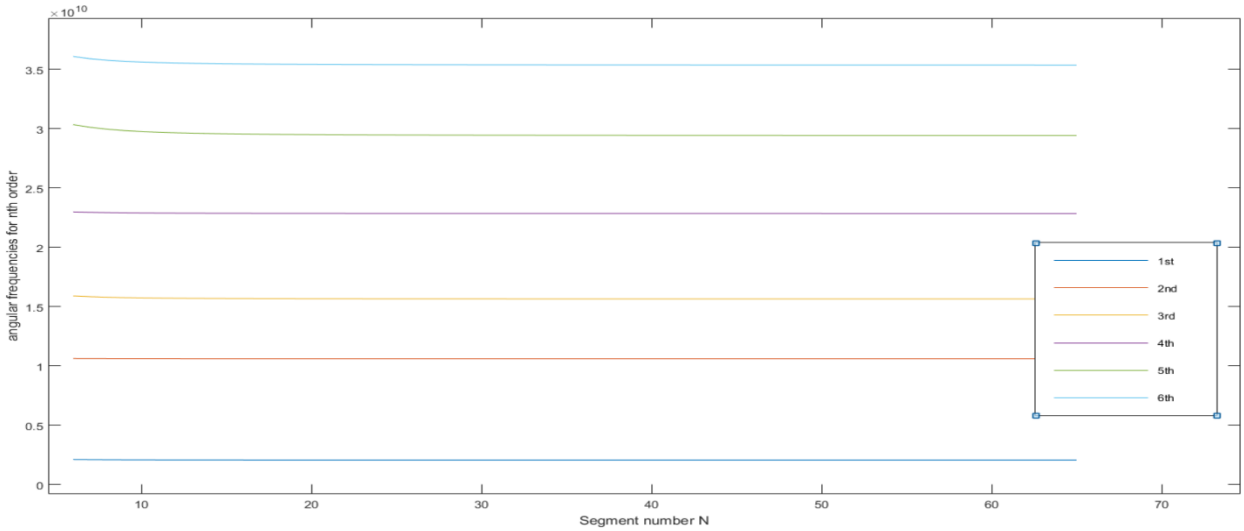



Figure 3-8. Lowest 6 frequencies of FEM result versus number of segments N

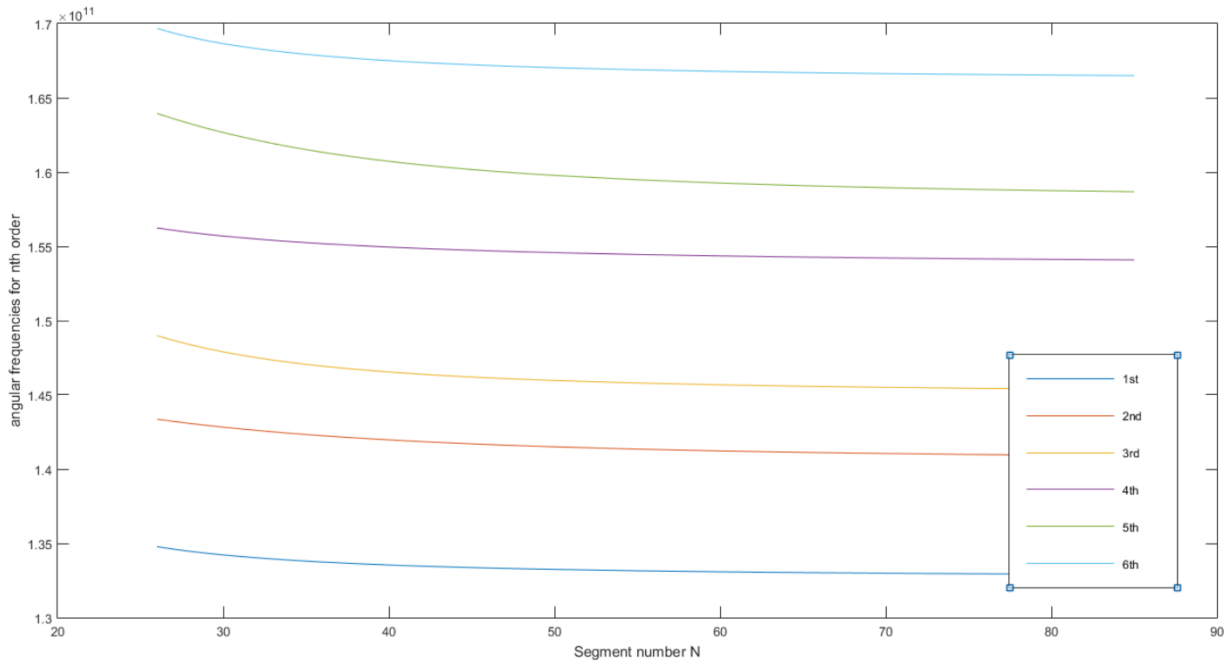
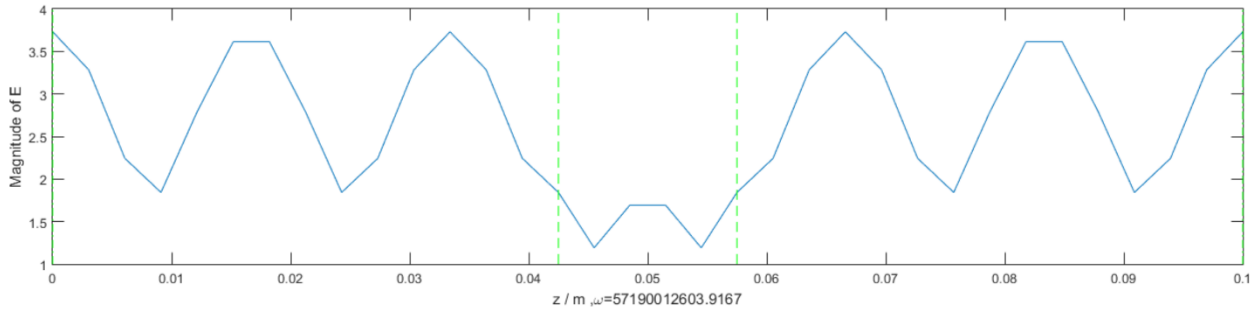



Figure 3-9. 20th to 26th Lowest frequencies of FEM result versus number of segments N

Fig.3-8 and Fig.3-9 shows the convergence situation for different order of frequency concerned. In Fig.3-9, I choose N from 26 to 80. Comparing above two figures, the lowest frequencies we want the less number we need to do the segmentation. For example, in this settings, around about N=15 all the six lowest frequencies get converged, but for 20th to 26th lowest frequencies, even N reach 80, the highest among them haven't get converged.
But in this assignment, we don't really need the frequencies high to the 20th, so for the same 2 slabs in an unit cell structure, I get at least 100 segments for this smallest slab, this resolution is enough for low frequencies we concerned to get converged. And the convergence plot for H field result are same, so I didn't post it here again.

The following figures are just some magnitude plots of E field with low segments number, at the 4th lowest resonance frequency.



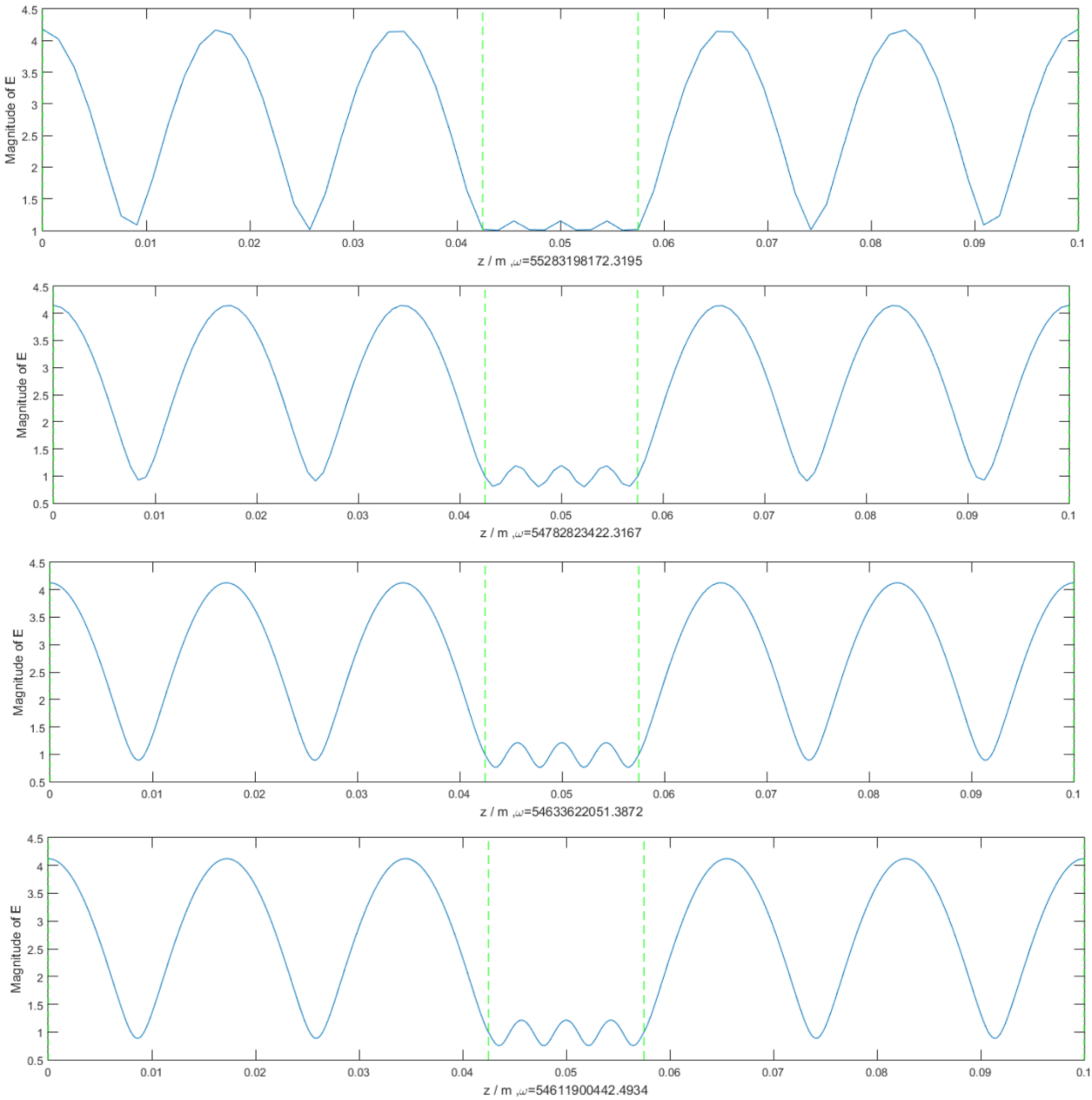



Figure 3-10. Total E field with different segmentation number N. Up to down: N=5,10,20 50,100

According to above checks, I believe I proved the validity of my code to complete the simulation for the given simple modeling problem.

# Section#4 Modelling Problem

For this specific problem, I think the selected range of $k_0$ is too small, so I choose $0 \le k_0 d \le 25$ instead of $0 \le k_0 \le 25$.

(a) $d_2 = d_1$, the setting of structure for condition (a) in code is:

```
d1=100e-3;
d2=d1;
d=d1+d2;
z=[0 d1/2 d1/2+d2 d];          % first row: z;  second row: epsilon_r, mu_r
p_z=[9 1 9; 1 1 1];
```

get the $k_0 d \sim \beta d$ relation shown in Fig.4-1. If we fill the region which exist a real Floquet wavenumber $\beta$ at a free-space wavenumber $k_0$, we got Fig.4-2, a more clear band-graph shows up:

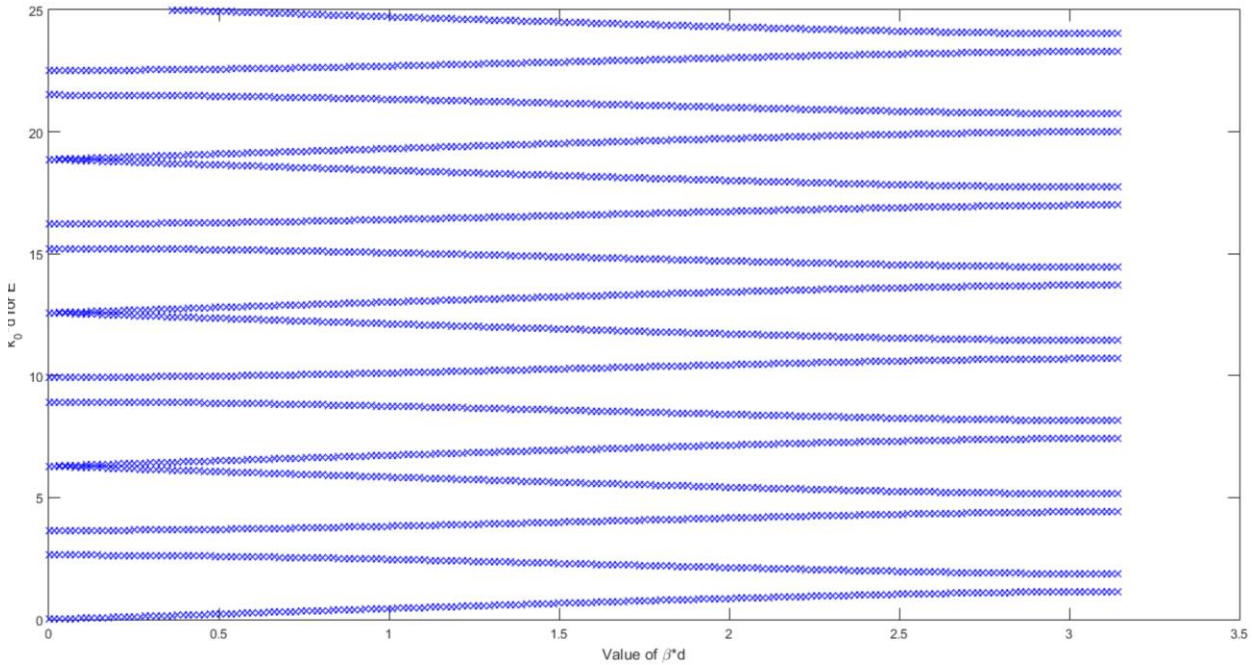



Figure 4-1. $k_0 d \sim \beta d$ plot for $d_2 = d_1$ in (a).

In Fig.4-2 below, the filled plotted gives a clear bands distribution, each band means there exist a wave can propagation in this structure for the wavenumber (or frequency) inside this band. Where the band doesn't exist means no wave at this frequencies can propagate in this structure.

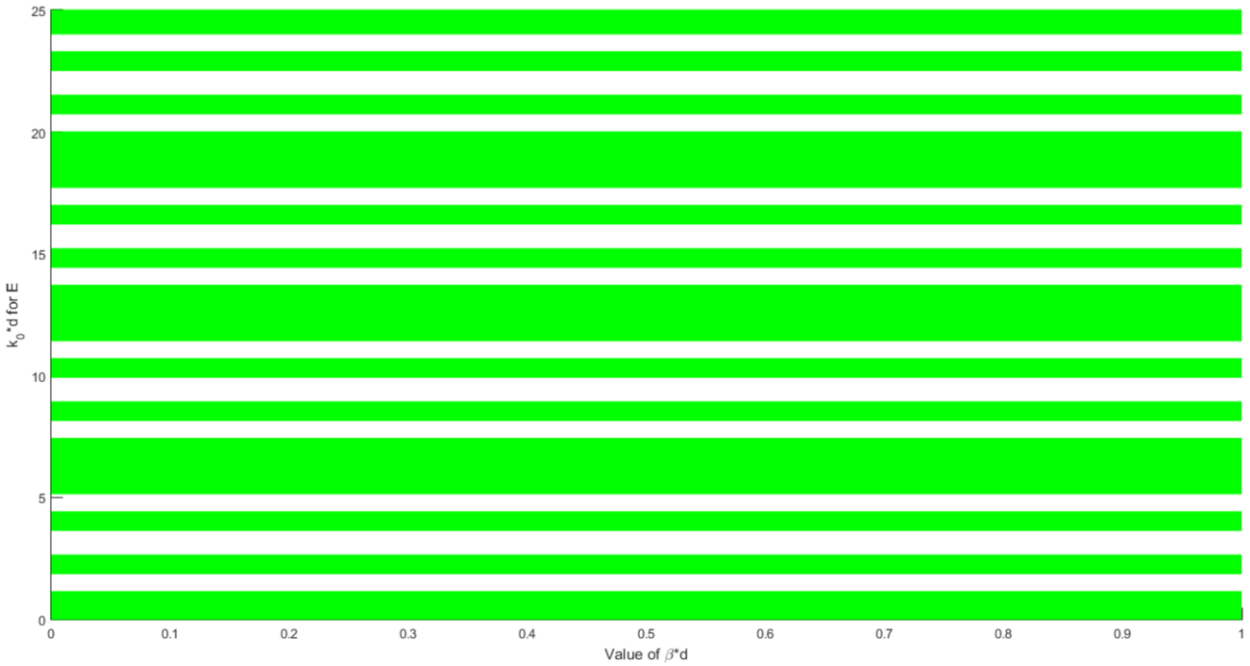



Figure 4-2.The 'band-graph' with $k_0 d \sim \beta d$ plot filled with horizontal lines, for $d_2 = d_1$.

(b) $d_2 = d_1 / 2$, the setting of structure for condition (b) is similar with (a):

```
d1=100e-3;
d2=d1/2;
d=d1+d2;
z=[0 d1/2 d1/2+d2 d];          % first row: z;  second row: epsilon_r, mu_r
p z=[9 1 9; 1 1 1];
```

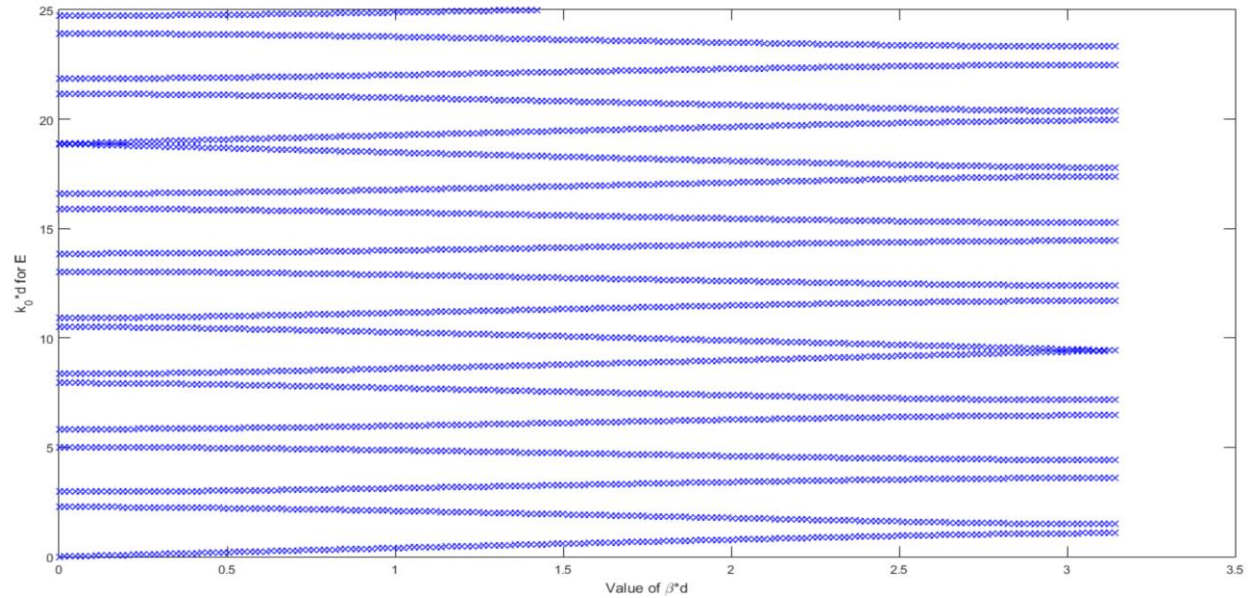



Figure 4-3. $k_0 d \sim \beta d$ plot for $d_2 = d_1 / 2$ in (b)

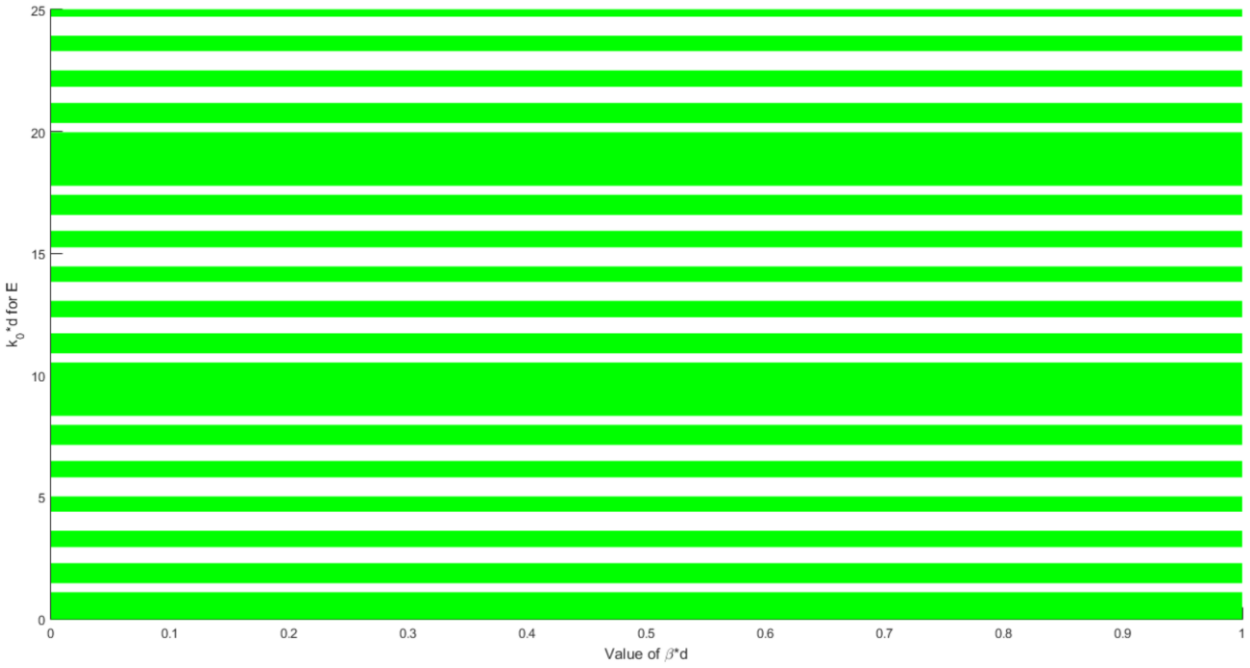



Figure 4-4. The 'band-graph' with $k_0 d \sim \beta d$ plot filled with horizontal lines, for $d_2 = d_1 / 2$.

The $k_0 d \sim \beta d$ relation and the filled band structure are plotted in Fig.4-3 and Fig.4-4 respectively. Comparing the band-graph gives us more intuitive idea that when change the ratio of $d_1 / d_2$, the major band with largest width seems only change the position but not change the width. But the minor bands between two major band change a lot. The number of minor bands increased with the ratio get larger, and the distance between two major band get larger also. The width of minor bands are not equal, its looks like an overlap pattern of sinc funtions. To demonstrate the trend, I plotted more band-graphs below:

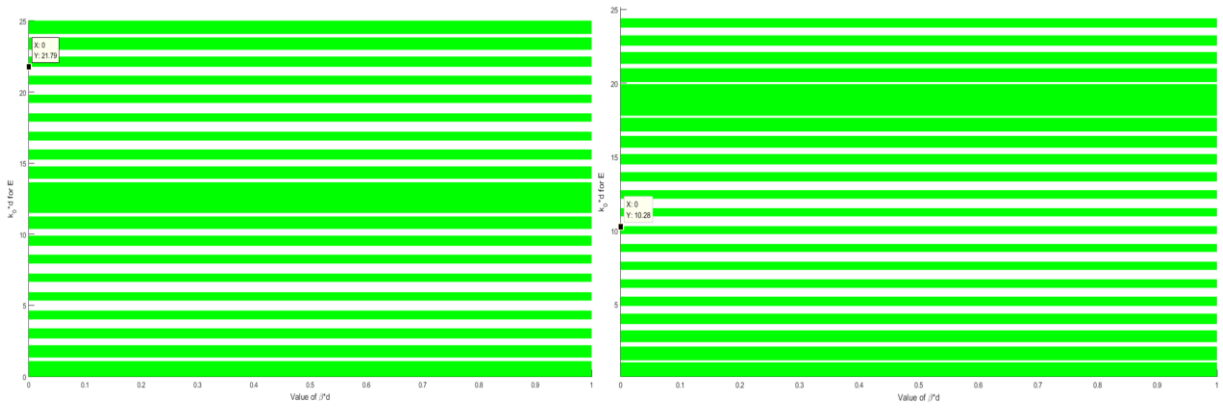


Figure 4-5. The 'band-graph' for $d_2 = d_1 / 3$ (left) and for $d_2 = d_1 / 5$ (right).

To validate my result, I compared my FEM results with the analytical solution in the notes for the situations in (a) and (b), separately in Fig. 4-6 and Fig.4-7.

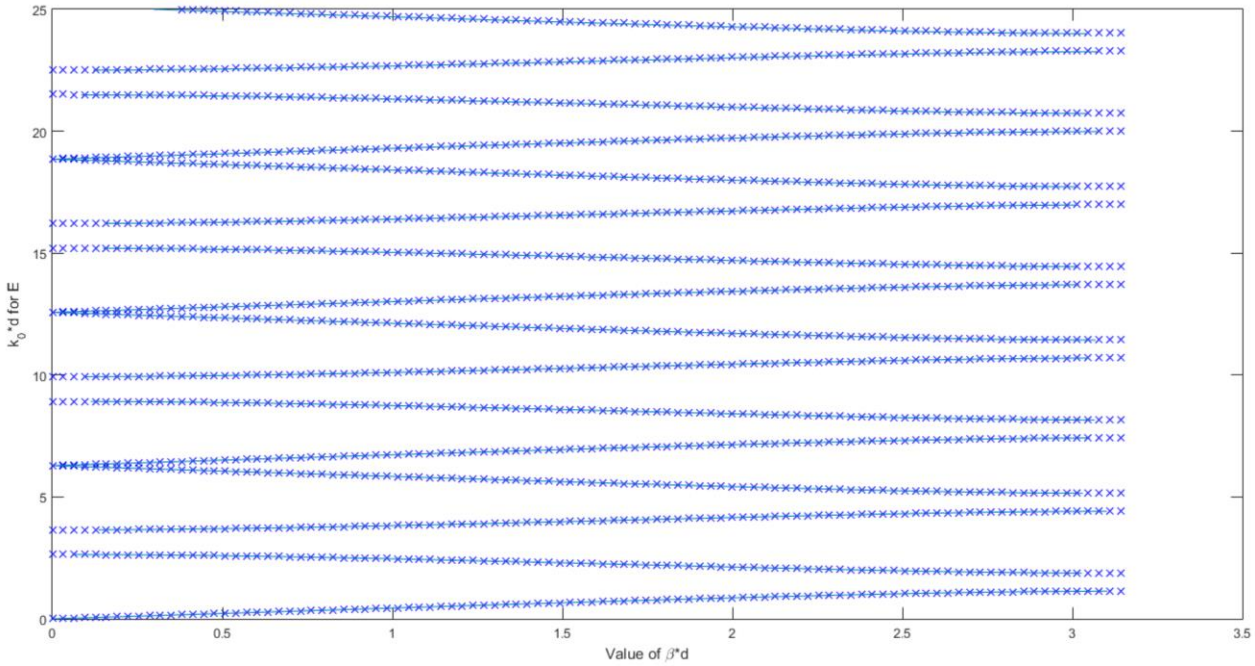



Figure 4-6. Comparison of FEM results (cross points)

and analytical result (lines), for situation in (a).

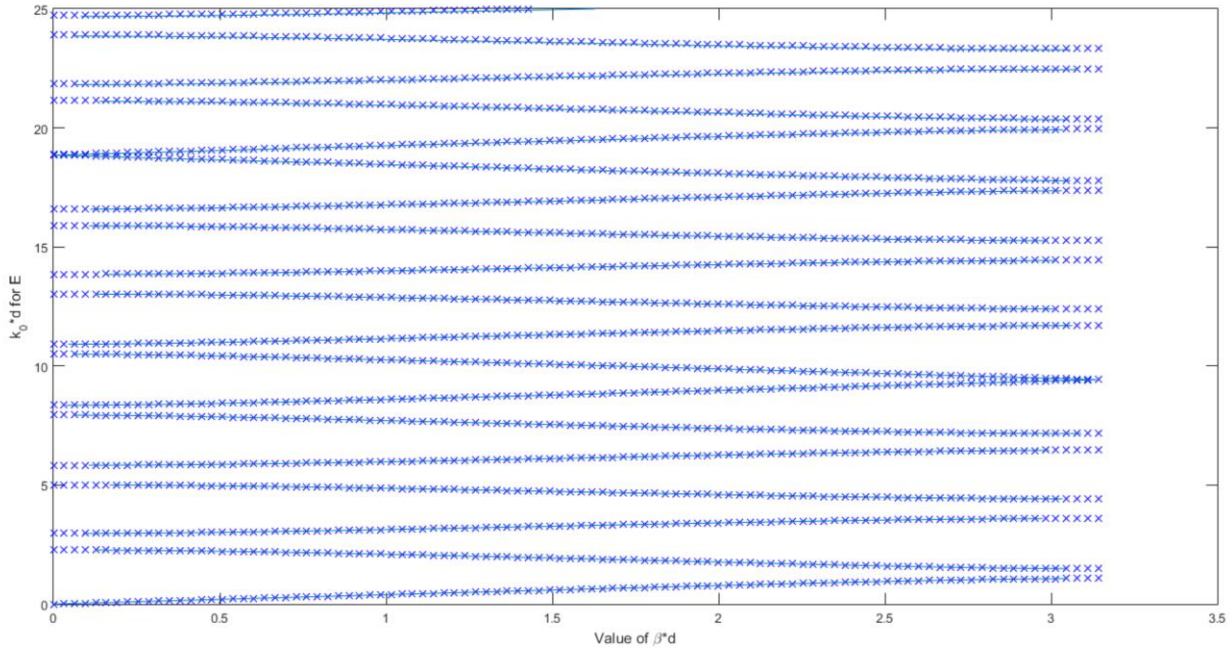



Figure 4-7. Comparison of FEM results (cross points)

and analytical result (lines), for situation in (b).

From Fig.4-6 and Fig.4-7, we can found the results of FEM overlaps with analytical results almost everywhere. Only when the $k_0$ value are too large, then due to the insufficient segmentation numbers in each element, there occurs some error between the analytical result with numerical one. When increasing the segment number N, those error will decrease rapidly. Those figures prove the validity of my solutions.

Then I will plot the magnitude and phase of the electric field and the magnetic field over the unit cell at some low frequencies, and for all the figures the $\beta$ is set to 10.

For $d_2 = d_1$ in (a), we have the E field and H field at lowest frequency as:

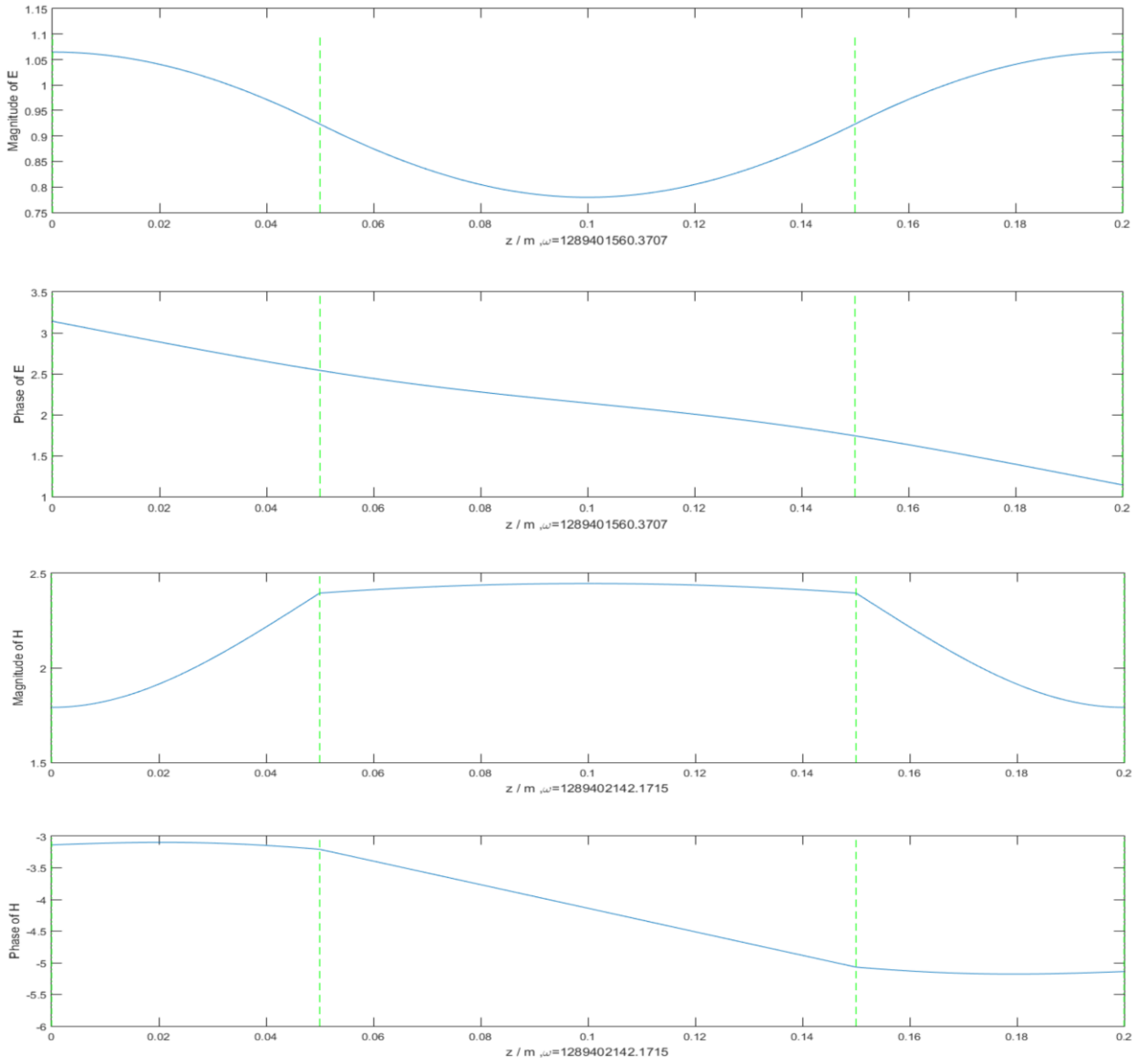



Figure 4-8. Magnitude and phase of E field and H field at lowest frequency, for (a).

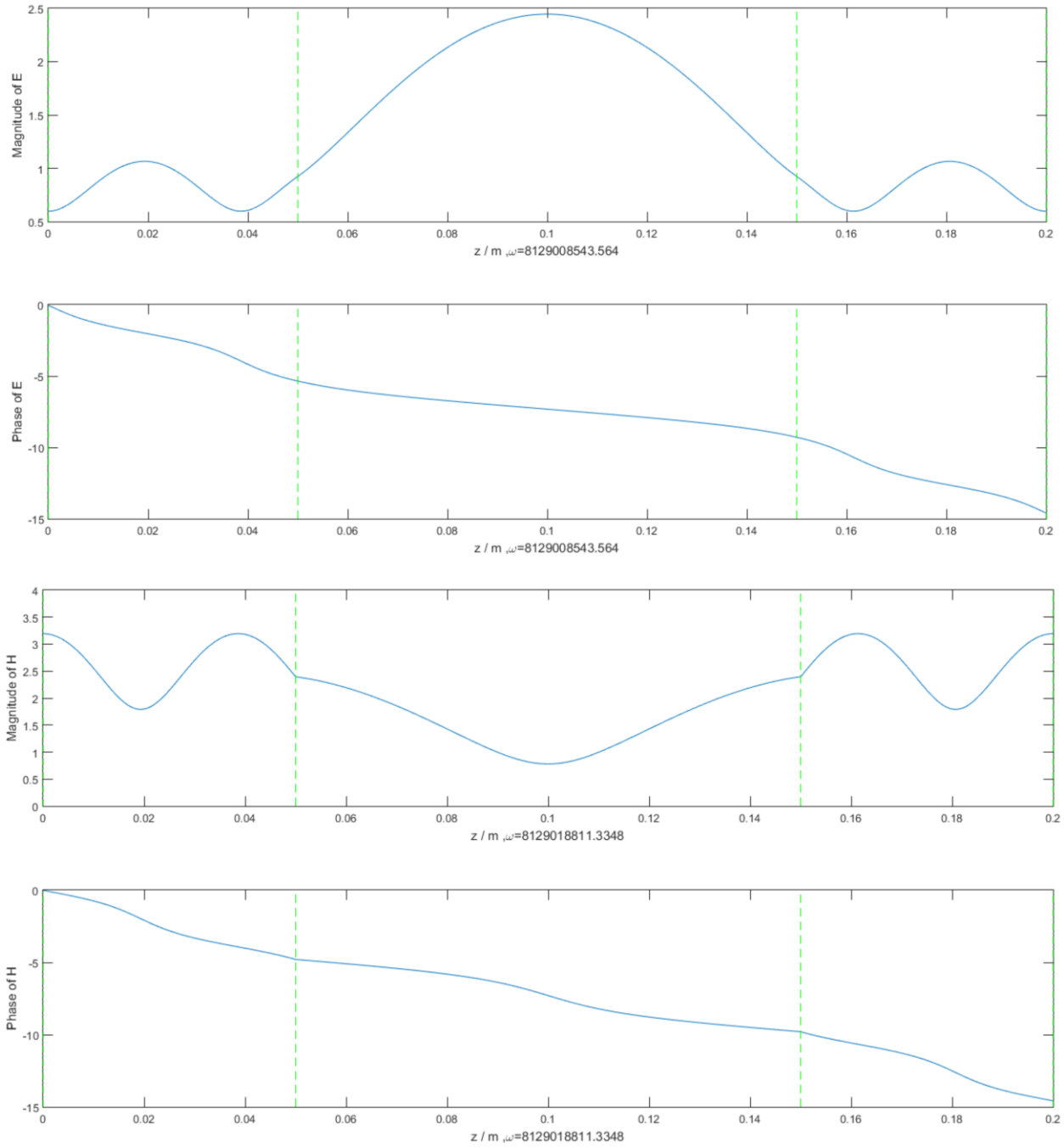



Figure 4-9. Magnitude and phase of E field and H field at the 4th lowest frequency, for (a)

You can see more result by run my code *Assignment_Finalv1.m* directly. Same for the situation in (b), we have the magnitude and phase for lowest frequency E field and H field in Fig.4-10 with d2=50mm now.

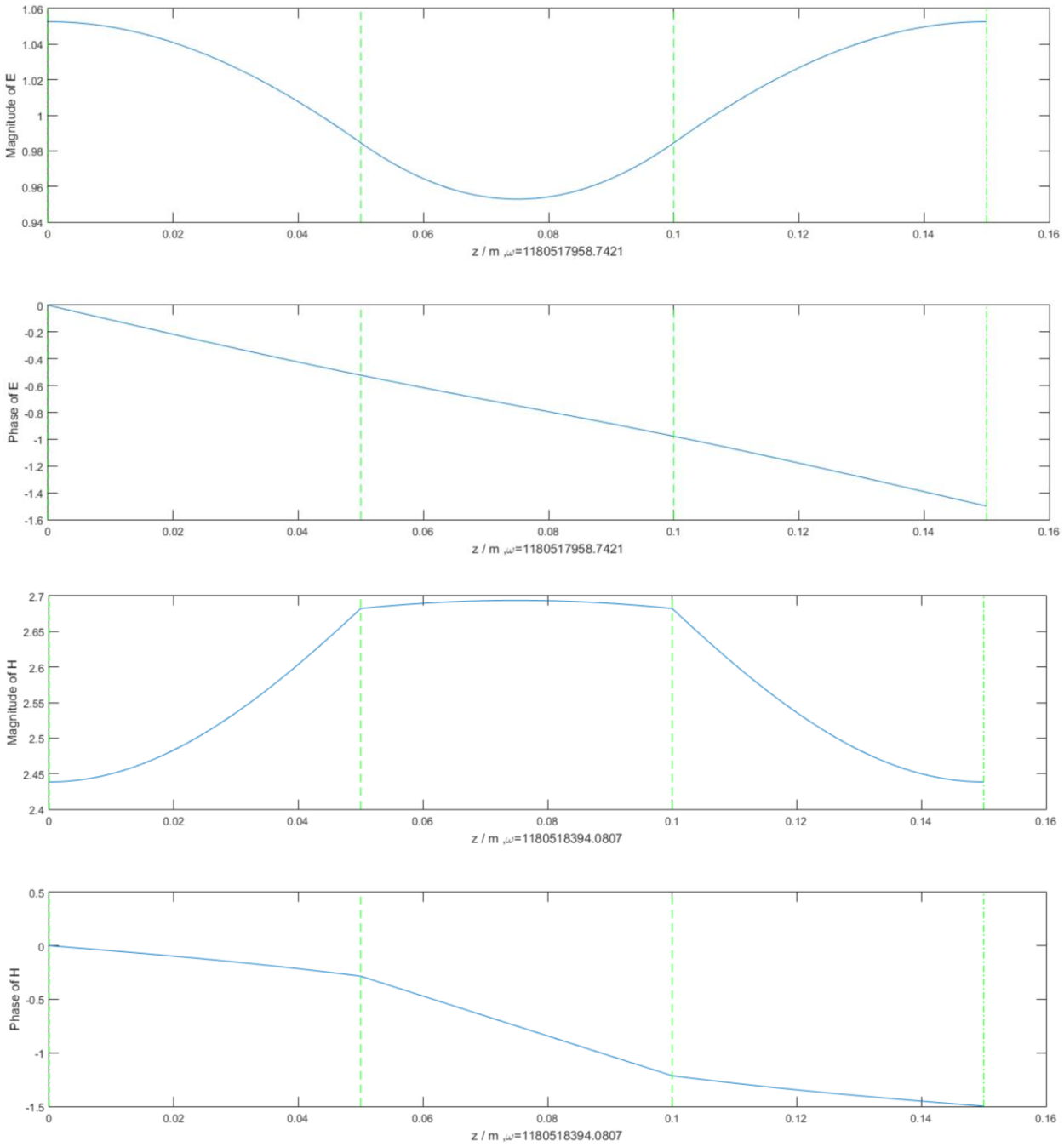



Figure 4-10. Magnitude and phase of E field and H field at the 1st lowest frequency, for (b)

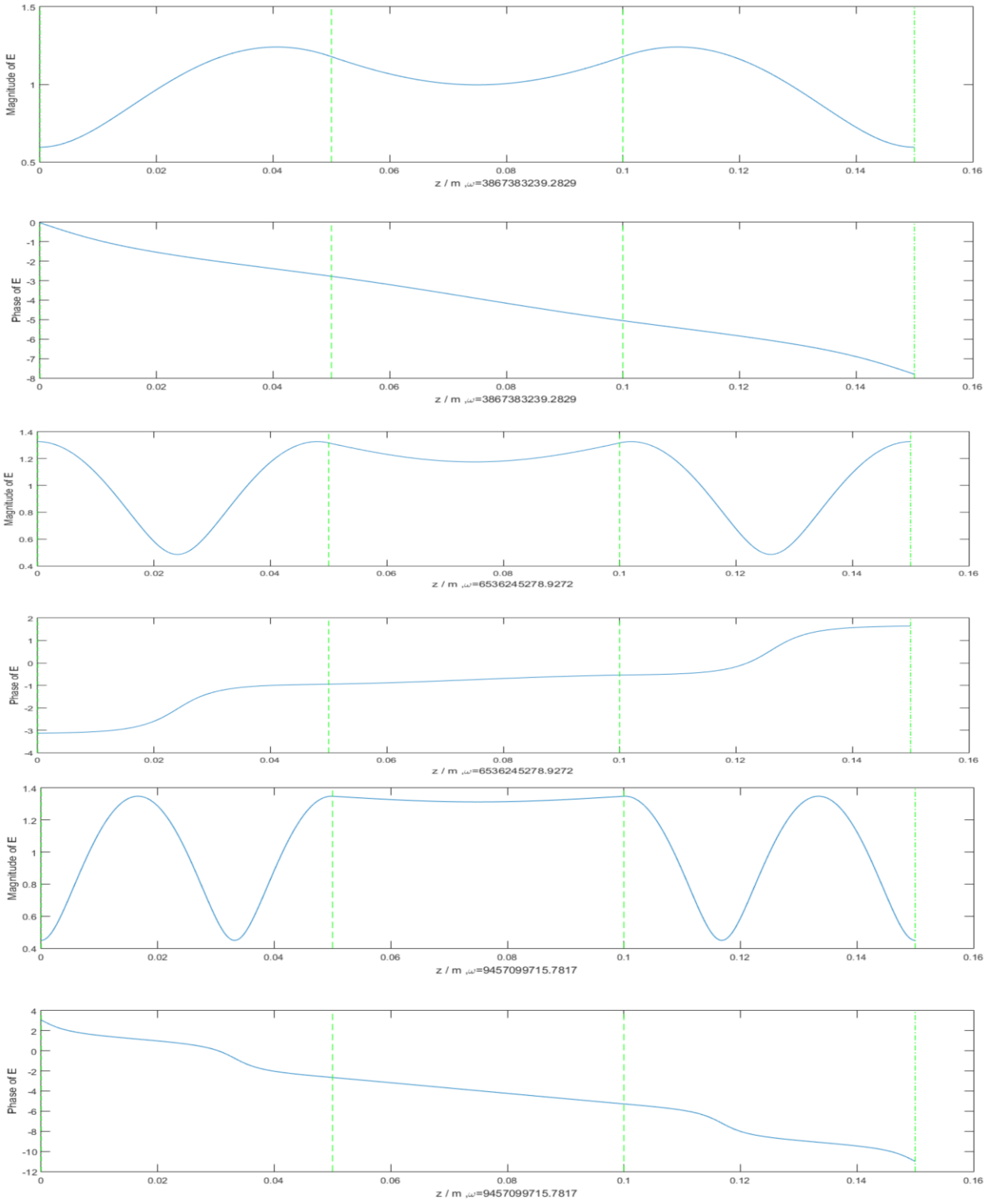



Figure 4-11. Magnitude and phase of E field and H field for (b), up to down: at the 2nd lowest frequency to the 4th one.

From Fig.4-8 to Fig.4-11, we notice that the similarity between the lowest E field for (a) and (b), the general shape are close. But for the $4^{th}$ lowest frequency component, the curve of magnitude are totally different with (a) and (b), corresponding the band-graph variation shown earlier in this section.

Now I believe I have finish all the requirements in the assignment sheet. And I think from above demonstrations and analysis, I proved my code works well for this 1D periodic structure problem, and it can deal with more complex 1D periodic problems than the modeling problems given.