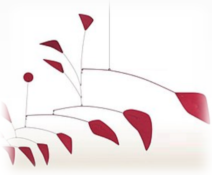
EECS 3101 - Design and Analysis of Algorithms



Shahin Kamali

Topic 1 - Introductions

York University

Picture is from the cover of the textbook CLRS.

Introduction

In a Glance . . .

- Algorithms are
  - Practical
  - Diverse
  - Fun (really!)

In a Glance . . .

- Algorithms are
  - Practical
  - Diverse
  - Fun (really!)
- Let's 'learn & play' algorithms and enjoy ...

Formalities

Textbook

- The main reference (optional):
  - Introduction to Algorithms, forth edition, by Cormen, Leiserson, Rivest, and Stein, MIT Press, 2024.
- Optional optional textbooks:
  - Algorithms and Data Structures, by Mehlhorn and Sanders, Springer, 2008.
  - The Algorithm Design Manual, second edition, by Skiena, Springer, 2008.
  - Advanced Data Structures, by Brass, Cambridge, 2008.

Grading

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  - Five assignments
  - Two quizzes
  - A midterm exam
  - A final exam

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The focus of this course is on learning, practising, and discovering.

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Corollary

Having fun in the process is important.

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    - 5 to 10 percent extra for bonus questions.
    - submit only pdf files (preferably use LaTeX) on Crowdmark (`https://www.crowdmark.com/`).

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  - 5 to 10 percent extra for bonus questions.
  - submit only pdf files (preferably use LaTeX) on Crowdmark (https://www.crowdmark.com/).
- Quizzes, Midterm & Final exams:
  - there will be extra for bonus questions in midterm and final.
  - all are closed-book.
  - sample exams will be provided for practice for midterm and final.

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- Basic sorting algorithms, e.g., quick sort and merge sort.

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  - Stacks, queues, dictionaries, binary search trees, hash tables, graphs.

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- Analysis techniques
  - E.g., how to analyse time complexity of a d&c algorithm?
  - Solving recursions, Master theorem, etc.

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- Transition from one step to another can be **deterministic** or **randomized**.
    - The algorithm is deterministic if it never uses randomization; otherwise, it is a randomized algorithm.
- Solving the problem requires the algorithm to **terminate**.
    - **Time complexity** concerns the number of steps that it takes for the algorithm to terminate (often on the worst-case input).

Abstract Data Type

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An abstract data type is formed by I) a set of values (data items) and II) a set of operations allowed on these items.

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An abstract data type is formed by I) a set of values (data items) and II) a set of operations allowed on these items.

- Stack is an ADT. Data items can be anything and operations are *push* and *pop*.
- An ADT is abstract way of looking at data (no implementation is prescribed).
- An ADT is the way data 'looks' from the view point of user.

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A data structure is a concrete representation of data, including how data is organized, stored, and accessed on a computer.

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- A linked-list is a data structure.
- Data structures are **implementations** of ADTs.
- A data structure is the way data 'looks' from the view point of implementer.

ADTs vs Data Structures

- ADTs: Stacks, queues, priority queues, dictionaries
- Data structures array, linked-list, binary-search-tree, binary-heap hash-table-using-probing, hash-table-using-chaining, adjacency list, adjacency matrix, etc.

Asymptotic Analysis

Algorithms (review)

- An **algorithm** is a step-by-step procedure carrying out a computation to solve an arbitrary instance of a problem.
  - E.g., sorting is a problem; a set of numbers form an instance of that and 'solving' involves creating a sorted output.

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- An **algorithm** is a step-by-step procedure carrying out a computation to solve an arbitrary instance of a problem.
  - E.g., sorting is a problem; a set of numbers form an instance of that and 'solving' involves creating a sorted output.
- A **program** is an implementation of an algorithm using a specific programming language.
- For a given problem (e.g., sorting) there can be several algorithms (e.g., Quicksort, Mergesort), and for a given algorithm (e.g., Quicksort) there can be several programs.
  - Our focus in this course is on algorithms (not programs).
  - How to implement a given algorithm relates to the art of **performance engineering** (writing a fast code)!

Algorithms Design & Analysis

- Given a problem P, we need to
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- Given a problem P, we need to
  - Design an algorithm A that solves P (**Algorithm Design**).
  - Verify **correctness** and **efficiency** of the algorithm (**Algorithm Analysis**).
  - If the algorithm is correct and efficient, **implement** it.
    - If you implement something that is not necessarily correct or efficient in all cases, that would be a **heuristic**.

Algorithm Evaluation

- How should we evaluate different algorithms for solving a problem?
  - In this course we are mainly concerned with amount of **time** it takes to solve a problem (this is called **running time**).
  - We can think of other measures such as the amount of **memory** that is required by the algorithm.
  - Other measures include amount of data movement, network traffic generated, etc.

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  - Other measures include amount of data movement, network traffic generated, etc.
- The amount of time/memory/traffic required by an algorithm depend on the **size** of the problem.
  - Sorting a larger set of numbers takes more time!

Running Time of Algorithms

- How to assess the running time of an algorithm?
- **Experimental analysis:**
  - Implement the algorithm in a program.
  - Run the program with inputs of different sizes.
  - Experimentally measure the actual running time (e.g., using *clock()* from time.h).

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- Shortcomings of experimental studies:
  - We need to implement the program (what if we are lazy and those engineers are hard to employ?)
  - We cannot test all input instances for the problem. What are the good samples? (remember the Morphy's law)
  - Many factors have impact on experimental timing, e.g., hardware (processor, memory), software environment (operating system, compiler, programming language), and human factors (how good was the programmer?)

Computational Models

- We need to assess time/memory requirement of algorithms using models that
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- We need to assess time/memory requirement of algorithms using models that
  - take into account all input instances.
  - do not require implementation of the algorithms.
  - are independent of hardware/software/programmer.
- In order to achieve this, we:
  - Express algorithms using **pseudo-codes** (don't worry about implementation).
  - Instead of measuring time in seconds, count the number of **primitive operations**.
    - This requires an abstract **model of computation**.

Random Access Machine (RAM) Model

- The **random access machine** (RAM):
  - Has a set of memory cells, each storing one ‘word’ of data.
  - Any **access to a memory location** takes constant time.
  - Any **primitive operation** takes constant time.
  - The **running time** of a program can be computed to be the number of memory accesses plus the number of primitive operations.
- Word-RAM is a RAM machine with the extra assumption that all values in our problem can ‘fit’ in a constant number of words (values are not too big).
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Observation

RAM is a simplified model which only provides an approximation of a 'real' computer.

Analysis of Insertion Sort under RAM

	INSERTION-SORT(A)	*cost*
1	**for** $j = 2$ **to** $A.length$	$c_1 = 2$
2	$key = A[j]$	$c_2 = 3$
3	// Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0
4	$i = j - 1$	$c_4 = 2$
5	**while** $i > 0$ and $A[i] > key$	$c_5 = 6$
6	$A[i+1] = A[i]$	$c_6 = 4$
7	$i = i - 1$	$c_7 = 2$
8	$A[i+1] = key$	$c_8 = 3$

- First, calculate the 'cost' (sum of memory accesses and primitive operations) for each line.
  - E.g., in line 5, there are 3 memory accesses and 3 primitive operations.

Analysis of Insertion Sort under RAM

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	$c_1 = 2$	n
$key = A[j]$	$c_2 = 3$	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1..j-1]$.	0	$n-1$
$i = j - 1$	$c_4 = 2$	$n-1$
while $i > 0$ and $A[i] > key$	$c_5 = 6$	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	$c_6 = 4$	$\sum_{j=2}^{n}(t_j - 1)$
$i = i - 1$	$c_7 = 2$	$\sum_{j=2}^{n}(t_j - 1)$
$A[i+1] = key$	$c_8 = 3$	$n-1$

- Next, find the number of times each line is executed.
  - This depends on the input, we may consider best or worst case input.
  - Let t_j be number of times the *while* loop is executed for inserting the j'th item.
    - In the best case, $t_j = 1$ and in the worst case $t_j = j$.
  - Summing up all costs, in the best case we have $T(n) = an + b$ for constant a and b.
  - In the worst case, we have $T_n = \alpha n^2 + \beta n + \gamma$ for constant α, β, γ.

Primitive Operations

- RAM model implicitly assumes primitive operations have fairly similar. running time
- Primitive operations:
  - basic integer arithmetic (addition, subtraction, multiplication, division, and modulo)
  - bitwise logic and bit shifts (logical AND, OR, exclusive-OR, negation, left shift, and right shift)
- Non-primitive operations:
  - exponentiation, radicals (square roots), logarithms, trigonometric, functions (sine, cosine, tangent), etc.

Asymptotic Notations

Statement

So, we can express the cost (running time) of an algorithm A for a problem of size n as a function $T_A(n)$.

- How do we compare two different algorithms? say $T_A(n) = \frac{1}{1000}n^3$ and $T_B(n) = 1000n^2 + 500n + 200$.
- Summarize the time complexity using asymptotic notations!
- Idea: assume the size of input grows to infinity; identify which component of $T_A(n)$ contributes most to the grow of $T_A(n)$.
- As n grows:
  - constants don't matter (e.g., $T_A(n) \approx n^3$).
  - low-order terms don't matter (e.g., $T_B(n) \approx 1000n^2$).

Asymptotic Notations

- Informally $T_B(n) = O(T_A(n))$ means T_B is **asymptotically smaller than or equal** to T_A.
- Is it sufficient to define O so that we have $T_B(n) < T_A(n)$?
  - No because the inequality might not hold for small values of n which we don't care about.
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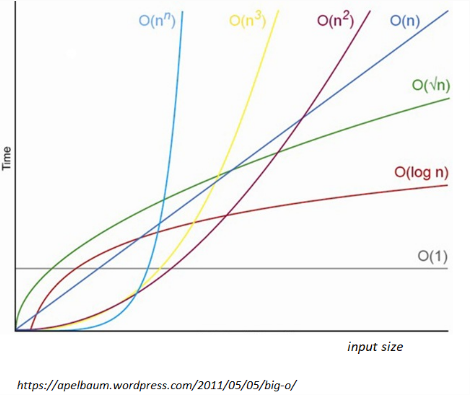
Definition

$$\mathrm{f(n)} \in \mathrm{O(g(n))} \Leftrightarrow$$

$$\exists M > 0, \exists n_0 > 0 \text{ s.t. } \underbrace{\forall n > n_0}_{\text{ignore low-order terms}}, \underbrace{f(n) \leq M \cdot g(n)}_{\text{ignore constants}}$$

Big Oh Illustration



- Let $f(n) = 1000n^2 + 1000n$ and $g(n) = n^3$. Prove $f(n) \in O(g(n))$

Review

- To analyze running time of an algorithm (under RAM model) we sum the number of primitive operations and memory accesses of the algorithm.
- The cost (running time) of algorithm A for a problem of size n would be a function $T_A(n)$.
- How do we compare two different algorithms? say $T_A(n) = \frac{1}{1000}n^3$ and $T_B(n) = 1000n^2 + 500n + 200$.
- Summarize the time complexity using asymptotic notations!
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    - Yes, $f(n)$ is asymptotically smaller than or equal (equal) to $g(n)$.
    - To prove, we should show $\exists M > 0, \exists n_0 > 0$ s.t.$\forall n > n_0, f(n) \leq M \cdot g(n)$
    - It suffices to define $n_0 = 1$ and $M = 3$, we have $\forall n > 1, 2n \leq 3n$.
    - M could be any number larger than or equal to 2, and n_0 could be any number.

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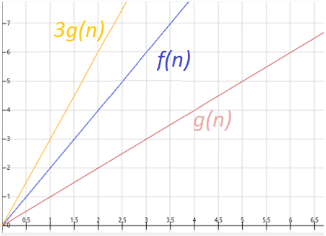
- We require specific values of M (not all choices for M work).



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  - To prove, we should show $\exists M > 0, \exists n_0 > 0 \text{ s.t.} \forall n > n_0, f(n) \leq M \cdot g(n)$.
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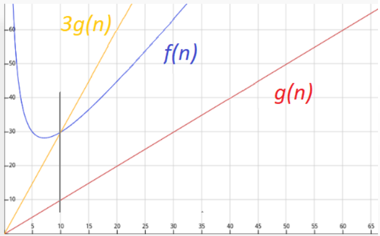
- We require specific values of M and n_0 (not all choices work).



Big O Notation

- Let $f(n) = 2023n^2 + 1402n$ and $g(n) = n^3$. Prove $f(n) \in O(g(n))$.

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- We should define M and n_0 s.t. $\forall n > n_0$ we have $2019n^2 + 1397n \leq Mn^3$. This is equivalent to $2023n + 1402 \leq Mn^2$.
- We have $2023n + 1402 \leq 2023n + 1402n = 3425n$. So, to prove $2023n + 1402 \leq Mn^2$, it suffices to prove $3425n \leq Mn^2$, i.e., $3425 \leq Mn$. This is always true assuming $M = 1$ and $n \geq 3425$ $(n_0 = 3425)$.
- Setting $M = 3426$ and $n_0 = 1$ also work!

Little o Notations

- Informally $f(n) = o(g(n))$ means f is **asymptotically smaller than** g.

Definition

$$f(n) \in o(g(n)) \Leftrightarrow$$
$$\forall M > 0, \exists n_0 > 0 \text{ s.t. } \underbrace{\forall n > n_0}_{\text{ignore low-order terms}}, f(n) < M \cdot g(n)$$

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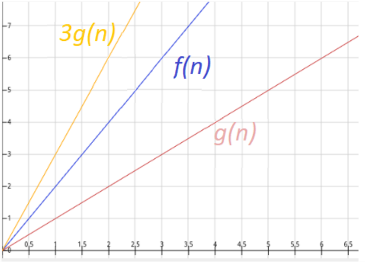
- E.g., $f(n) = 2n$, $g(n) = n$. Is it that $f(n) \in o(g(n))$?
  - No because for $M = 1$, it is not true that $f(n) < Mg(n)$ (i.e., $2n < n$) for large values of n.



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- Prove that $n^2 \sin(n) + 1984n + 2016 \in o(n^3)$.

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- Prove that $n^2 \sin(n) + 1984n + 2016 \in o(n^3)$.
  - We have to prove that for all values of M there is an n_0 so that for $n > n_0$ we have $n^2 \sin(n) + 1984n + 2016 < Mn^3$.
  - We know $n^2 \sin(n) \leq n^2$, $1984n \leq 1984n^2$ and $2016 \leq 2016n^2$. So, $n^2 \sin(n) + 1984n + 2016 \leq (1 + 1984 + 2016)n^2 = 4001n^2$.
  - So, to prove $n^2 \sin(n) + 1984n + 2016 < Mn^3$ it suffices to prove $4001n^2 < Mn^3$, i.e., $4001/M < n$, so, we can define n_0 to be any value larger than $4001/M$.

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  - We have to prove that for all values of M there is an n_0 so that for $n > n_0$ we have $n^2 \sin(n) + 1984n + 2016 < Mn^3$.
  - We know $n^2 \sin(n) \leq n^2$, $1984n \leq 1984n^2$ and $2016 \leq 2016n^2$. So, $n^2 \sin(n) + 1984n + 2016 \leq (1 + 1984 + 2016)n^2 = 4001n^2$.
  - So, to prove $n^2 \sin(n) + 1984n + 2016 < Mn^3$ it suffices to prove $4001n^2 < Mn^3$, i.e., $4001/M < n$, so, we can define n_0 to be any value larger than $4001/M$.
- For little o, n_0 is often defined as a function of M.

Big Ω Notation

- $f(n) = \Omega(g(n))$ means f is **asymptotically larger than or equal to** g.

Definition

$$\mathrm{f(n)} \in \Omega(g(n)) \Leftrightarrow \exists M > 0, \exists n_0 > 0 \text{ s.t.} \forall n > n_0, f(n) \geq M \cdot g(n)$$

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- Let $f(n) = n/2020$ and $g(n) = log(n)$. Prove $f(n) \in \Omega(g(n))$.

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  - We need to provide M and n_0 so that for all $n \geq n_0$ we have $n/2020 \geq M \log(n)$, i.e., $n \geq 2020M \log(n)$.
  - We know $\log(n) < n$ (assuming $n > 1$). So, in order to show $2020M \log(n) \leq n$, it suffices to have $2020M \leq 1$, i.e., M can be any value smaller than $1/2020$ (and n_0 can be 1 or any other positive integer).

Little ω Notation

- $f(n) = \omega(g(n))$ means f is **asymptotically larger than** g.

Definition

$$\mathrm{f(n)} \in \omega(\mathrm{g(n)}) \Leftrightarrow \forall M > 0, \exists n_0 > 0 \text{ s.t.} \forall n > n_0, f(n) > M \cdot g(n)$$

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  - For any constant M we need to provide n_0 so that for all $n \geq n_0$ we have $n/2020 > M \log(n)$, i.e., $n > 2020M \log(n)$.
  - We know $log(n) < \sqrt{n}$ (assuming $n > 16$). So, in order to show $2020M \log(n) < n$, it suffices to have $2020M\sqrt{n} < n$, i.e., $2020M < \sqrt{n}$. For that, it suffices to have $(2020M)^2 < n$, i.e., n_0 can be defined as $\max\{16, (2020M)^2\}$.

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- Similarly to little o, for ω, we often need to define n_0 as a function of M.

Θ Notation

- Informally $f(n) = \Theta(g(n))$ means f is **asymptotically equal to** g.

Definition

$$f(n) \in \Theta(g(n)) \Leftrightarrow$$
$$\exists M_1, M_2 > 0, \exists n_0 > 0 \text{ s.t.} \forall n > n_0, M_1 \cdot g(n) \leq f(n) \leq M_2 \cdot g(n)$$

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  - For the first inequality, we can have $M_1 = 1$ and for all n we have $n/2020 \leq n$.
  - For the second inequality, we let M_2 to be any constant larger than 2020 which gives $M_2/2020 \geq 1$.
  - n_0 can be any value, e.g., $n_0 = 1$.

Asymptotic Notations in a Nutshell

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- $\Theta(2^n) \rightarrow$ Exponential Complexity
  - The 'algorithm' terminates but the universe is likely to end much earlier even if $n \approx 1000$.

Techniques for Comparing Growth Rates

- Assume the running time of two algorithms are given by functions $f(n)$ and $g(n)$ and let

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)}.$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty \end{cases}$$

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  - If the limit is not defined, we need another method.

- Note that we cannot compare two algorithms using big O and Ω notations.
  - E.g., algorithm A can have complexity $O(n^2)$ and algorithm B has complexity $O(n^3)$. We **cannot** state that A is faster than B (why?)

Fun with Asymptotic Notations

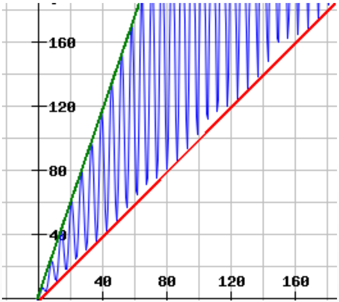
- Compare the grow-rate of $\log n$ and n^r where r is a positive real number.

Fun with Asymptotic Notations

- Prove that $n(sin(n) + 2)$ is $\Theta(n)$.

Fun with Asymptotic Notations

- Prove that $n(sin(n) + 2)$ is $\Theta(n)$.
- Use the definition since the limit does not exist.
  - Define n_0, M_1, M_2 so that $\forall n > n_0$ we have $M_1 n(sin(n) + 2) \leq n \leq qM_2 n(sin(n) + 2)$.
  - $M_1 = 1/3, M_2 = 1, n_0 = 1$ work!



Fun with Asymptotic Notations

- The same relationship that holds for relative values of numbers hold for asymptotic.
  - E.g., if $f(n) \in O(g(n))$ [$f(n)$ is asymptotically smaller than or equal to $g(n)$], then we have $g(n) \in \Omega(f(n))$ [$g(n)$ is asymptotically larger than or equal to $f(n)$].

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  suppose $\exists M_1, n_0'$ s.t., $f(n) \leq M_1 g(n)$ for $n \geq n_0'$. Also, $\exists M_2, n_0''$ s.t., $f(n) \geq M_2 g(n)$ for $n \geq n_0''$. Select, $n_0 = \max\{n_0', n_0''\}$ and we have $M_2 g(n) \leq f(n) \leq M_1 g(n)$.

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- We have **transitivity** in asymptotic notations: if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, we have $f(n) \in O(h(n))$.

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- **Max rule**: $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.
  - E.g., $2n^3 + 8n^2 + 16n \log n \in \Theta(\max\{2n^3, 8n^2, 16n \log n\}) = \Theta(n^3)$.

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  it holds that $\max\{f(n), g(n)\} \leq f(n) + g(n) \leq 2\max\{f(n), g(n)\}$ for $n \geq 1$. (select $n_0 = 1$, $M_1 = 1$ and $M_2 = 2$).

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    - $\sum_{i=0}^{n-1}(a + di)$

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  - $\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{1-r^n}{1-r} \in \Theta(1) & \text{if } 0 < r < 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a\frac{r^n-1}{r-1} \in \Theta(r^n) & \text{if } r > 1 \end{cases}$
- What about **Harmonic sequence**?
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- What about **Harmonic sequence**?
  - $H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln(n) + \gamma \in \Theta(\log n)$ (γ is a constant ≈ 0.577)

Loop Analysis

- Identify **elementary operations** that require constant time.
- The complexity of a loop is expressed as the **sum** of the complexities of each iteration of the loop.
- Analyse independent loops separately, and then **add** the results (use "maximum rules" and simplify when possible).
- If loops are nested, start with the innermost loop and proceed outwards.

Example of Loop Analysis

Algo1 (n)

1. $A \leftarrow 0$
2. **for** $i \leftarrow 1$ **to** n **do**
3.   **for** $j \leftarrow i$ **to** n **do**
4. $A \leftarrow A/(i-j)^2$
5. $A \leftarrow A^{100}$
6. **return** *sum*

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$$O(1) + \Sigma_{i=1}^{n} \Sigma_{j=i}^{n} c = O(1) + \Sigma_{i=1}^{n} (n - i + 1) c$$

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Example of Loop Analysis

```
Algo2 (A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            X ← 0
            for k ← i to j do
                X ← A[k]
                if X > max then
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    return max
```

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$$\sum_{i=1}^{n} \sum_{j=i}^{n} (O(1) + \sum_{k=i}^{j} c) = \Theta(n^3)$$

Example of Loop Analysis

Algo3 (n)

```
X ← 0
for i ← 1 to n^2 do
    j ← i
    while j ≥ 1 do
        X ← X + i/j
        j ← ⌊j/2⌋
return X
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$$= \Theta(n^2 \log(n^2)) = \Theta(2n^2 \log(n^2)) = \Theta(n^2 \log n)$$

MergeSort

Sorting an array A of n numbers

- **Step 1:** We split A into two subarrays: A_L consists of the first $\lceil \frac{n}{2} \rceil$ elements in A and A_R consists of the last $\lfloor \frac{n}{2} \rfloor$ elements in A.
- **Step 2:** Recursively run *MergeSort* on A_L and A_R.
- **Step 3:** After A_L and A_R have been sorted, use a function *Merge* to merge them into a single sorted array. This can be done in time $\Theta(n)$.

MergeSort

MergeSort(A, n)

if $n = 1$ **then**

  $S \leftarrow A$

else

  $n_L \leftarrow \lceil \frac{n}{2} \rceil$

  $n_R \leftarrow \lfloor \frac{n}{2} \rfloor$

  $A_L \leftarrow [A[1], \ldots, A[n_L]]$

  $A_R \leftarrow [A[n_L + 1], \ldots, A[n]]$

  $S_L \leftarrow \mathit{MergeSort}(A_L, n_L)$

  $S_R \leftarrow \mathit{MergeSort}(A_R, n_R)$

  $S \leftarrow \mathit{Merge}(S_L, n_L, S_R, n_R)$

return S

Analysis of MergeSort

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

- The exact and sloppy recurrences are identical when n is a power of 2.
- The recurrence can easily be solved by various methods when $n = 2^j$. The solution has growth rate $T(n) \in \Theta(n \log n)$.
- It is possible to show that $T(n) \in \Theta(n \log n)$ for all n by analyzing the exact recurrence.

Analysis of Recursions

- The sloppy recurrence for time complexity of merge sort:

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

- We can find the solution using **alternation method**:

$$\begin{aligned} T(n) &= 2T(n/2) + cn \\ &= 2(2T(n/4) + cn/2) + cn = 4T(n/4) + 2cn \\ &= 4(2T(n/8) + cn/4) + 2cn = 8T(n/8) + 3cn \\ &= \ldots \\ &= 2^k T(n/2^k) + kcn \\ &= 2^{\log n} T(1) + \log ncn = \Theta(n \log n) \end{aligned}$$

Substitution method

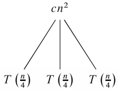
- **Guess** the growth function and prove an upper bound for it using induction.
  - For merge-sort, prove $T(n) < Mn \log n$ for some value of M (that we choose).
  - This holds for $n = 2$ since we have $T(2) = 2d + 2c$, which is less than $2M$ as long as $M \geq c + d$ (base of induction).
  - Fix a value of n and assume the inequality holds for smaller values. we have $T(n) = 2T(n/2) + cn \leq 2M(n/2(\log n/2)) + cn = Mn(\log n/2) + cn = Mn \log n - Mn + cn \leq Mn \log n$ as long as M is selected to be at least c (the inequality comes from the induction hypothesis).
- This shows $T(n) \in O(n \log n)$

Recursion Tree

- Suppose we want to solve the following recursion:

$$T(n) = \begin{cases} 3\,T\left(\frac{n}{4}\right) + cn^2 & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

- Let's form a **recursion tree**:



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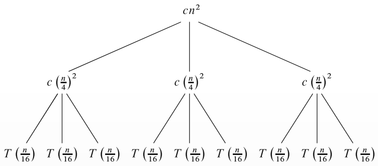

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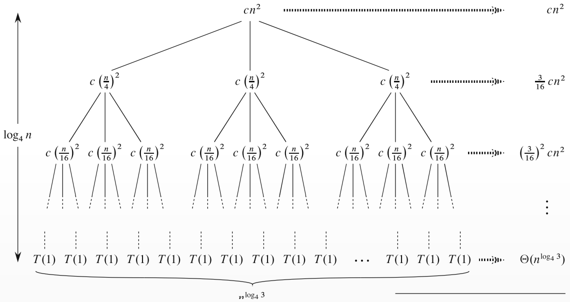

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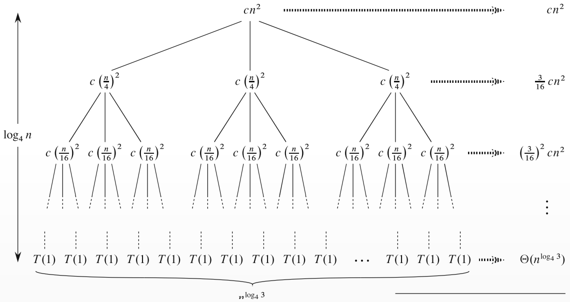

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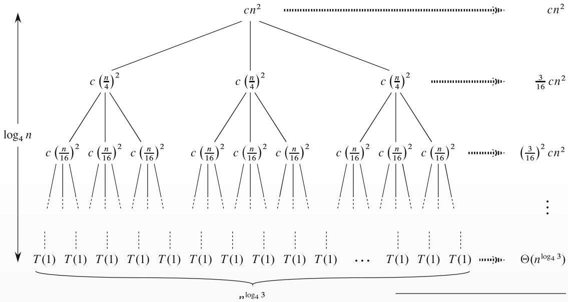

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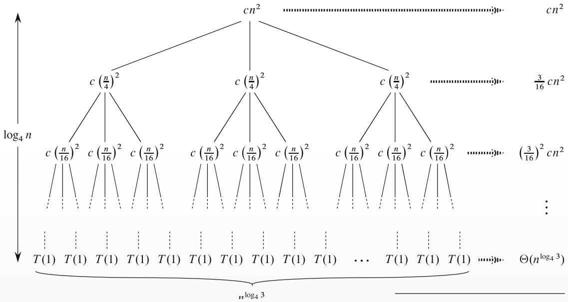

- The total work in **internal nodes** is $cn^2(1 + 3/16 + (3/16)^2 + \ldots) = \Theta(n^2)$.
- The total work in **leaves** is $n^{\log_3 4}$.
- The max rule indicates that $T(n) = \Theta(n^2)$.

Master theorem

$$T(n) = \begin{cases} a\,T\left(\frac{n}{b}\right) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

$(a \geq 1,\ b > 1,\ \text{and } f(n) > 0)$

- Compare $f(n)$ and $n^{\log_b a}$
- Case 1: if $f(n) \in O(n^{\log_b a - \epsilon})$, then $T(n) \in \Theta(n^{\log_b a})$
- Case 2: if $f(n) \in \Theta(n^{\log_b a} (\log n)^k)$ for some non-negative k then $T(n) \in \Theta(f(n) \log n) = \Theta(n^{\log_b a} (\log n)^{k+1})$
- Case 3: if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and if $af(n/b) \leq cf(n)$ for **some constant** $c < 1$ (regularity condition), then $T(n) \in \Theta(f(n))$

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- $T(n) = 3T(n/2) + n^2$?
  - Case 3, check whether regularity condition holds, i.e., whether $af(n/b) \leq cf(n)$ for some $c < 1$. Since we have $3(n/2)^2 = 3/4n^2$ the regularity condition holds (c can be any value in the range $(3/4, 1)$, i.e., $T(n) \in \Theta(n^2)$

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- $T(n) = T(n/2) + n(2 - cos(n))$?
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    - For $n = 2k\pi$, we have $cos(n/2) = -1$ and $cos(n) = 1$; we have $af(n/b) = n/2(2 - cos(n/2)) = 3n/2$, which is not within a factor $c < 1$ of $f(n) = n(2 - 1) = n$ [i.e., we cannot say $3n/2 \leq cn$ for any $c < 1$]. So we cannot get any conclusion from Master theorem.

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- $T(n) = 2T(n/2) + n(\log n)^3$? Case 2, we have $f(n) = \Theta(n^{\log_b a}(\log n)^k)$ for $k = 3$. We have $T(n) = \Theta(n(\log n)^4)$.