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Stocasthic Modeling and Simulation

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Preface

As a student of Scientific and Data Intensive Computing, I've created these notes while attending the **Stochastic Modeling and Simulation** course.
The course covers a wide range of topics, including Stochastic Nonlinear Models in different application fields (as physics, biomedicine, mathematics, ...), Stochastic Differential Equations, and Stochastic Simulation.

The topics covered in these notes include:

- Recap of Deterministic Models
- Stochastic Differential Equations and White Noise
- Fokker Planck Equation
- Noise-induced Transitions
- Colored noises
- Bounded Stochastic Processes
- Spatio-temporal Stochastic Processes
- Parameter Estimation form Data
- Stochasticity …
- …
- Continuous state space-discrete time Stochastic Processes
- Discrete Time Markov Chains
- Continuous Time Markov Chains
- Mean Field Approximation

While these notes were primarily created for my personal study, they may serve as a valuable resource for fellow students and professionals interested in Stochastic Modeling and Simulation.

Contents

1 Introduction

A diverse array of fields, ranging from epidemiology and systems biology to finance and renewable energy, are fundamentally concerned with understanding systems that evolve over time. While simple, deterministic models can offer initial insights, many real-world phenomena are characterized by inherent randomness, uncertainty, and intricate interconnections. To capture this richness, we turn to the powerful framework of **Stochastic Modeling and Simulation**. Many systems we encounter are ***complex***: disease spread modeled by the classic SIR (Susceptible, Infected, Recovered) framework, oscillating chemical reaction concentrations, or predator-prey ecosystem dynamics. In each case, the system's behavior emerges from interactions between countless individual components, that cannot be predicted from studying the parts in isolation.

Definition: ***Complex System***

A **complex system** is a system composed of interconnected parts that, as a whole, exhibit one or more properties which are not obvious from the properties of individual parts alone.

A key feature of complex systems is **emergent behavior**, where macroscopic patterns arise from the local interactions of microscopic agents. This is closely related to the concepts of adaptation and self-organization. **Adaptation** refers to the process by which a system achieves a better fit with its environment. **Self-organization** is the spontaneous process through which a system's internal structure changes, often to facilitate this adaptation, without external control. A classic example is the formation of a flock of birds: each bird follows simple rules based on its neighbors, yet the flock as a whole displays coordinated, complex flight patterns that no single bird orchestrates.

Observation: ***The Constructive Role of Noise***

When studying complex systems, we frequently discover that randomness, or *noise*, serves a purpose beyond merely interfering with deterministic signals. Surprisingly, the presence of noise combined with nonlinear dynamics can actually promote the formation of ordered structures and novel, persistent patterns.

1.1 The Modeling Framework

To build quantitative models of complex systems, we require a clear mathematical framework. This involves defining several key ingredients that form the foundation of any dynamical model:

- **Entities:** The fundamental components of the system. These can be modeled as *discrete* objects (e.g. individuals in a population) or as *continuous* quantities (e.g. the density of a population).
- **State:** A complete description of the system at a specific moment. The *state space* is the set of all possible states the system can occupy.
- **Time:** The independent variable against which the system's state evolves. Time can be treated as *discrete* (advancing in steps) or *continuous*.
- **Evolution Rules:** The laws or functions that dictate how the system's state changes from one moment to the next. These rules are often expressed using mathematical equations.

The Role of Differential Equations

For systems where the state and time are treated as continuous, the evolution rules are most naturally expressed using differential equations.

Definition: *Differential Equation*

A **differential equation** is a mathematical equation that relates one or more unknown functions to their derivatives. In modeling, it describes the instantaneous rate of change of a system's state:

$$\frac{dx}{dt} = f(x,t)$$

where x is the state variable and f is a function that describes how x changes over time.

In essence, differential equations are the language we use to describe a changing world. Perhaps the most famous and foundational example is Newton's Second Law of Motion, which relates the force on an object to its acceleration (the second derivative of its position).

$$F = ma \quad \Longrightarrow \quad m\frac{d^2x}{dt^2} = F(x(t),\dot{x}(t))$$

This single equation forms the basis of classical mechanics and is a prime example of a **dynamical system**: a system whose state evolves over time according to a deterministic rule.

1.2 Bridging Models and Data in the Modern Era

In our era of big data and machine learning, an important question emerges: will data-driven methods make traditional mathematical modeling redundant? The answer is both more complex and more promising: the path forward involves **hybrid approaches** that harness the complementary strengths of both methodologies. Although real-world datasets are frequently noisy and incomplete, while models necessarily simplify reality, their synergistic integration can unlock insights that neither approach could achieve alone.

Hybrid strategies are rapidly becoming central to scientific discovery:

1. **Physics-Informed Neural Networks (PINNs):**
   This is a cutting-edge deep learning framework for solving problems involving differential equations. A neural network is trained not only to fit observed data but also to obey the underlying physical laws of the system, encoded as differential equations. This helps the model generalize better from sparse or noisy data.
2. **Machine Learning for Parameter Estimation:**
   While a mathematical model may capture the structure of a system (e.g., the SIR model), the specific parameters (infection rate, recovery rate) must be estimated from real-world data. Machine learning and statistical inference techniques are essential tools for this task, especially when dealing with high-dimensional and complex models.
3. **Hybrid Modeling:**
   In some cases, parts of a system may be well-understood and easily described by equations, while other parts may be too complex. A hybrid model might use a traditional differential equation for the well-understood component and a neural network, trained on data, to represent the more complex, "black-box" component.

2 Stochastic Dynamics

In the previous chapter, we introduced the concept of modeling systems using differential equations. We begin our deeper exploration with **linear systems**. Many sophisticated systems, when analyzed locally around a point of equilibrium, can be effectively approximated by a linear model; this linearization is a cornerstone of dynamical systems theory.

2.1 Linear Systems: The Foundation of Dynamics

Linear Ordinary Differential Equations (ODEs) are characterized by equations where the dependent variable and its derivatives appear only to the first power. They are invaluable because they can often be solved analytically, providing clear insight into the system's behavior.

Consider a simple electrical **RL circuit** consisting of a resistor (R) and an inductor (L) in series. Kirchhoff's voltage law gives:

$$L\frac{di}{dt} + Ri = 0$$

This first-order linear homogeneous ODE has the solution:

$$i(t) = i_0 e^{-\frac{R}{L}t}$$

where $\tau = L/R$ is the *time constant* characterizing the exponential decay.

Adding a capacitor creates the **RLC circuit**, a second-order system. Using the state vector $\mathbf{y} = [i, Q]^T$ where Q is the charge, we obtain:

$$L\frac{di}{dt} + Ri + \frac{1}{C}Q = 0$$

This is a second-order differential equation. To analyze it as a system, we can define a state vector $\mathbf{y} = [i, Q]^T$. The system of first-order equations is:

$$\begin{cases} \frac{di}{dt} = -\frac{R}{L}i - \frac{1}{LC}Q \\ \frac{dQ}{dt} = i \end{cases} \qquad \Rightarrow \qquad A = \begin{bmatrix} -R/L & -1/(LC) \\ 1 & 0 \end{bmatrix}$$

The behavior of this system (e.g., oscillations, decay) is entirely determined by the eigenvalues of the matrix A.

Observation: ***The Ubiquity of the Linear Model***

The same mathematical structure appears in completely different physical domains. Consider a damped harmonic oscillator, such as a mass on a spring moving through a viscous fluid. Newton's second law gives:

$$m\ddot{x} = -kx - \gamma\dot{x}$$

where k is the spring constant and γ is the damping coefficient. Rearranging gives:

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

This equation shares the same mathematical structure as the RLC circuit equation, illustrating how mathematical models can uncover similarities across entirely different physical systems.

2.2 The Need for Stochasticity: introducing noise

The deterministic models discussed so far are powerful idealizations, however, they assume that all parameters and forces are known and constant. In reality, systems are constantly subjected to small, unpredictable influences from their environment. The aggregate effect of these influences is termed ***noise***. Modeling this randomness is crucial for creating realistic descriptions of the world.

A classic illustration of this is the phenomenon of **Brownian motion**. In 1827, botanist Robert Brown observed the erratic, random motion of pollen grains suspended in water. A simple deterministic model for a particle in a fluid, incorporating only a drag force ($m\dot{v} = -kv$), predicts that the particle's velocity should decay to zero almost instantly. This prediction is in contradiction with Brown's empirical observations of perpetual, jittery motion.

The solution to this paradox came from understanding what friction really is at the microscopic level. The drag force $-kv$ represents only the *average* effect of countless molecular collisions with the particle. Einstein and Langevin realized that each individual collision is random and unpredictable, creating fluctuating forces that keep the particle in perpetual motion.

In general, a particle moving through a fluid can experience three types of forces: the damping force we've already discussed, random collisions from molecules, and possibly some external deterministic force $F_d(x)$:

$$m\ddot{x} = -k\dot{x} + F_d(x) + F_s(t)$$

Let's consider the case where the particle is very light compared to the damping strength, meaning $m \ll k$. To make our notation cleaner, we can write $F_s(t) = k\xi(t)$ and $F_d(x) = kf(x)$:

$$m\ddot{x} = -k\dot{x} + kf(x) + k\xi(t)$$

When $m/k \ll 1$ (the so-called overdamped limit), the inertial term becomes negligible, and we arrive at:

$$\dot{x} = f(x) + \xi(t)$$

This is the famous **Langevin equation**. It represents a fundamental shift in how we think about physical systems—moving from purely deterministic descriptions to ones that embrace randomness. The equation captures the idea that what we observe as smooth, deterministic forces are often just the averaged effects of chaotic microscopic processes.

Example: *Noise in a RLC circuit*

In the previous example of the RLC circuit

$$Ri = -\frac{d}{dt}\Phi(B_{self}) = -\frac{d}{dt}(Li)$$

the derivation was incomplete!

Indeed, a circuit where a current i is flowing is embedded not only in the magnetic field B_{self} generated by $\Phi(Li)$ but also in other external random magnetic fields.

$$B_{total} = B_{self} + B_{external}$$

$$\Phi(B_{total}) = Li + \Phi(B_{external}) = Li + KB_{external}$$

with stochastic external disturbances reds as follows (by the current):

$$Ri = -L\frac{d}{dt}i - K\frac{d}{dt}B_{external}$$

which we may rewrite as

$$L\frac{d}{dt}i = -Ri + K\zeta(t)$$

2.2.1 Modeling Impulsive Events: The Dirac Delta

Before we can properly define the stochastic force $F_s(t)$, we must first develop a mathematical tool to describe events that are instantaneous and intense. Consider the impact of a baseball bat on a ball. The force is immense but acts over a very short duration. We care about the net effect, the change in the ball's velocity, rather than the precise evolution of the force during the infinitesimal impact time. Let's model this using Newton's law, $m\dot{v} = F(t)$. The total change in momentum is the integral of the force over the impact interval, say from $t = 0$ to $t = a$.
The velocity after the impact is:

$$v_{\text{after}} = \frac{1}{m}\int_0^a F(t)\,dt$$

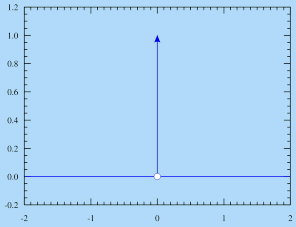
This integral, which represents the total impulse delivered, is finite. To model this, we introduce an idealized mathematical object known as the **Dirac delta function**, denoted $\delta(t)$. It is not a true function in the classical sense but a *distribution* defined by its effect under an integral.

Definition: ***The Dirac Delta Function***

The Dirac delta function, $\delta(t)$, is a generalized function that describes an idealized impulse. Rather than being a function in the classical sense, $\delta(t)$ is a mathematical object with the following properties:

1. There exists a "very small" interval $J = (-\varepsilon/2, \varepsilon/2)$ outside which $\delta(t) \approx 0$.
2. $\delta(t) > 0$ within this interval.
3. The integral of $\delta(t)$ over J is equal to one:

$$\int_J \delta(t)\,dt = 1$$



The crucial consequence of these properties is the ***sifting property***: for any function $f(t)$ that is continuous at $t = 0$, the Dirac delta "sifts out" its value at that point:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$$

To further illustrate the sifting property, consider a function $f(t)$ that is continuous and differentiable at $t = 0$, with $f(0) < \infty$ and $f'(0) < \infty$. Expanding $f(t)$ in a Taylor series around $t = 0$ gives:

$$f(t) = f(0) + f'(0)t + O(t^2)$$

Integrating $f(t)$ against the Dirac delta over the real line, we have:

$$\int_{-\infty}^{\infty} \delta(t)f(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\left[f(0) + f'(0)t\right]dt$$

which can be split as:

$$= \int_{-\infty}^{\infty} \delta(t)f(0)\,dt + \int_{-\infty}^{\infty} \delta(t)f'(0)t\,dt$$

Now, recall the two fundamental properties of the Dirac delta: $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ and $\int_{-\infty}^{\infty} \delta(t)t\,dt = 0$ (since $t\delta(t)$ is an odd function). Applying these, we find:

$$\int_{-\infty}^{\infty} \delta(t)f(t)\,dt = f(0)\cdot 1 + f'(0)\cdot 0 = f(0)$$

This demonstrates that the Dirac delta "picks out" the value of f at $t = 0$, regardless of the behavior of f elsewhere.

Dirac Delta Classes of Functions

The Dirac delta function, being a mathematical idealization, can be approximated by various families of functions that become increasingly "spike-like" as a parameter grows large.
For $N \gg 1$, the family of rectangular pulse functions

$$\delta_N(t) = \begin{cases} N, & \text{if } t \in \left(-\frac{1}{2N}, \frac{1}{2N}\right) \\ 0, & \text{otherwise} \end{cases}$$

provides an example of Dirac delta approximation. As N increases, the function becomes taller and narrower while maintaining unit area, converging to the delta function in the distributional sense.
Similarly, the exponential family

$$\delta_\gamma(t) = \frac{\gamma}{2} e^{-\gamma|t|}$$

for $\gamma \gg 1$ offers a smooth approximation that avoids the discontinuities of the rectangular pulse. This family has several advantages: it is infinitely differentiable, has exponentially decaying tails, and provides a more realistic model for physical phenomena.

Observation: *Verification of the Sifting Property*

Let's verify that the rectangular pulse function $\delta_N(t)$ satisfies the sifting property in the limit as $N \to \infty$. For any continuous function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\delta_N(t)dt = \int_{-1/(2N)}^{1/(2N)} f(t) \cdot N\, dt = N \int_{-1/(2N)}^{1/(2N)} f(t)\, dt$$

Since $f(t)$ is continuous at $t = 0$, we can use the mean value theorem.
There exists some $c \in [-1/(2N), 1/(2N)]$ such that:

$$N \int_{-1/(2N)}^{1/(2N)} f(t)\, dt = N \cdot f(c) \cdot \frac{1}{N} = f(c)$$

As $N \to \infty$, the interval shrinks to zero and $c \to 0$. By continuity of f, we have $f(c) \to f(0)$.
Therefore:

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} f(t)\delta_N(t)dt = f(0)$$

This confirms that $\delta_N(t)$ approaches the Dirac delta function in the distributional sense.

2.2.2 Defining the Stochastic Force: White Noise

The stochastic force in the Langevin equation, commonly denoted as $\xi(t)$ (or sometimes $F_s(t)$, after normalization by mass and other constants), models the cumulative effect of a huge number of independent, microscopic collisions. This motivates the following statistical properties, which define what is known as **Gaussian white noise**.

1. **White Noise (Temporal Uncorrelation):** The values of $\xi(t)$ at different times are uncorrelated, reflecting the physical assumption that molecular collisions at different instants are independent. This is mathematically expressed as:

   $$\langle \xi(t)\xi(q) \rangle = 0 \qquad \text{for } t \neq q$$

   More generally, the autocorrelation function is given by the Dirac delta:

   $$\langle \xi(t)\xi(q) \rangle = \delta(t-q)$$

where $\delta(t-q)$ is the Dirac delta function, indicating that the noise is only correlated with itself at the same instant.

2. **Gaussian Distribution:** By the central limit theorem, the sum of many independent random impacts leads to a Gaussian distribution for $\xi(t)$ at any fixed time t. This ensures that all finite-dimensional distributions of the process are Gaussian, making the process analytically tractable.
3. **Zero Mean:** The random molecular collisions are, on average, isotropic and unbiased, so the mean of the stochastic force vanishes:
$$\langle \xi(t) \rangle = 0$$
where $\langle \cdot \rangle$ denotes the ensemble average.
4. **Infinite Instantaneous Variance:** The idealized white noise process is so irregular that its variance at any fixed time is formally infinite:
$$\langle \xi^2(t) \rangle \gg 1$$
This reflects the mathematical abstraction of white noise, which is not a function in the usual sense but a generalized function (distribution).

Warning: *A Mathematical Abstraction*

The concept of white noise, with its delta-correlated structure, is a powerful but physically unrealizable abstraction. It implies infinite variance ($\langle \xi(t)^2 \rangle \to \infty$) and infinite power. In any real system, correlations exist over some small but non-zero timescale. However, if this correlation time is much shorter than any other characteristic timescale of the system, modeling the noise as "white" is an excellent and mathematically convenient approximation.

Multiplicative Noise: The Case of the SIS Epidemic Model

Multiplicative noise arises when the amplitude of the noise depends on the state of the system itself. In the context of stochastic differential equations, this means the noise term is not simply additive, but is multiplied by a function of the state variable:
$$\frac{dx}{dt} = f(x,t) + g(x,t)\,\xi(t)$$
where $g(x,t)$ is not constant.
A classic example is found in epidemiology, specifically in the **SIS epidemic model**. In this model, individuals can become infected, recover, and then become susceptible again (as opposed to the SIR model, where recovered individuals are removed from the susceptible pool). The deterministic SIS model is given by:
$$\frac{dS}{dt} = -\beta SI + \mu(N-S)$$
$$\frac{dI}{dt} = \beta SI - \mu I$$
where S and I are the numbers of susceptible and infected individuals, β is the infection rate, μ is the recovery rate, and N is the total population.

Incorporating Stochasticity

In reality, the contact rate β is not perfectly constant: it fluctuates due to random factors such as social behavior, environmental changes, or other sources of randomness. To model this, we

introduce a stochastic term:

$$\beta \to \beta + \sigma \xi(t)$$

where σ quantifies the strength of the noise and $\xi(t)$ is Gaussian white noise.
Substituting this into the SIS model, the equation for the number of infected individuals becomes:

$$\frac{dI}{dt} = [\beta SI - \mu I] + \sigma SI\,\xi(t)$$

Here, the noise term is ***multiplicative***: its amplitude depends on both S and I. This reflects the fact that random fluctuations in the infection rate have a larger effect when there are more susceptible and infected individuals interacting.

Observation: ***Multiplicative vs. Additive Noise***

Additive noise affects the system independently of its state, while multiplicative noise depends on the current state. In epidemic models, multiplicative noise is more realistic, as the impact of random fluctuations in transmission is naturally proportional to the number of possible contacts.

2.2.3 The Simplest Langevin Equation (SLAE)

To build a deeper intuition for the behavior of stochastic systems, we analyze the simplest possible Langevin equation, where the deterministic drift term is zero ($f(x) = 0$). This corresponds to a free particle subject only to random kicks from its environment:

$$\dot{x} = \omega \xi(t)$$

Here, ω is a constant representing the intensity of the noise $\xi(t)$. While this equation is ill-defined in its derivative form, we can formally integrate it to find the particle's position $x(t)$, assuming an initial position $x(0)$:

$$x(t) = x(0) + \omega \int_0^t \xi(s) ds$$

From this expression, we can derive the fundamental statistical properties of the process $x(t)$.

1. **Average Position**
   The average position, or the first moment of $x(t)$, is found by taking the ensemble average of the equation. Since the noise has zero mean, $\langle \xi(s) \rangle = 0$, the integral of the average noise vanishes:
   $$\langle x(t) \rangle = \langle x(0) \rangle + \omega \int_0^t \langle \xi(s) \rangle ds = \langle x(0) \rangle$$
   If the initial position is deterministic, $\langle x(0) \rangle = x_0$, the average position of the particle does not change over time, since the particle is equally likely to be pushed in any direction.
2. **Autocorrelation and "Memory"**
   To understand how the position at one time relates to the position at another, we compute the autocorrelation function, $\langle x(t)x(q) \rangle$.
   $$\langle x(t)x(q) \rangle = \left\langle \left(\omega \int_0^t \xi(s) ds \right) \left(\omega \int_0^q \xi(\theta) d\theta \right) \right\rangle = \omega^2 \int_0^t \int_0^q \langle \xi(s)\xi(\theta) \rangle ds d\theta$$
   Using the white noise property $\langle \xi(s)\xi(\theta) \rangle = \delta(s - \theta)$, we get:
   $$\langle x(t)x(q) \rangle = \omega^2 \int_0^t \left(\int_0^q \delta(s - \theta) d\theta \right) ds$$

The inner integral with respect to θ is 1 if s is within the interval $[0,q]$, and 0 otherwise. This simplifies the double integral. Assuming, without loss of generality, that $t \leq q$, the condition $s \in [0,q]$ is always met for the outer integral's range $s \in [0,t]$. The integral thus becomes:

$$\langle x(t)x(q)\rangle = \begin{cases} \omega^2(t-q) & \text{if } t > q \\ \omega^2(q-t) & \text{if } t < q \end{cases} \quad = \quad \omega^2|t-q|$$

We can combine these cases into a single elegant expression:

$$\langle x(t)x(q)\rangle = \omega^2 \min(t,q)$$

This result reveals that, unlike the driving noise $\xi(t)$, the position process $x(t)$ ***does*** have memory. Its position at time t is correlated with its position at all other times.

3. **Mean Squared Displacement and Variance**
   Setting $q = t$ gives the second moment, or ***mean squared displacement*** from the origin (assuming $x(0) = 0$):

$$\langle x^2(t)\rangle = \omega^2 t$$

   The ***variance*** of the process is then:

$$\mathrm{Var}[x(t)] = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = \omega^2 t$$

   The variance grows linearly with time, a hallmark of diffusive processes. The particle, on average, wanders further and further from its starting point.

4. **Mean Squared Increment and Non-Differentiability**
   Let's examine the behavior of the process over a small time increment. The mean squared change in position over an interval of length $|t-q|$ is:

$$\begin{aligned} \langle (x(t)-x(q))^2\rangle &= \langle x^2(t)\rangle + \langle x^2(q)\rangle - 2\langle x(t)x(q)\rangle \\ &= \omega^2 t + \omega^2 q - 2\omega^2 \min(t,q) \\ &= \omega^2|t-q| \end{aligned}$$

   Now consider the incremental ratio, which approximates the derivative. Let $q = t+h$:

$$\left\langle \left(\frac{x(t+h)-x(t)}{h} \right)^2 \right\rangle = \frac{\langle (x(t+h)-x(t))^2\rangle}{h^2} = \frac{\omega^2 h}{h^2} = \frac{\omega^2}{h}$$

   In the limit as the interval h shrinks to zero, this quantity diverges:

$$\lim_{h\to 0^+} \left\langle \left(\frac{x(t+h)-x(t)}{h} \right)^2 \right\rangle = \lim_{h\to 0^+} \frac{\omega^2}{h} = +\infty$$

   The mean squared value of the derivative is infinite. This is a profound result: it is the first formal evidence that the path $x(t)$, while continuous, is ***nowhere differentiable***. This highly irregular, "jagged" nature is a fundamental property of processes driven by white noise.

We said that the measure of the memory of the white noise is represented by the following average:

$$\langle\, \xi(t)\xi(q)\,\rangle = \delta(qt)$$

and we showed some heuristic reasoning. However, for the sake of precision, the above formula is derived by a statistical concept that is very useful to verify the degree of similarity between two generic random variables: the **covariance**, which is defined as follows:

$$\mathrm{COV}(x,y) = \langle (x - \langle x\rangle)(y - \langle y\rangle)\rangle = \langle xy\rangle - \langle x\rangle\langle y\rangle$$

For the sake of precision, in the cases where y is a temporal shift of x (as in our case), we say that we are computing the **autocovariance** of x.

2.2.4 The Wiener Process and its Properties

The mathematical pathologies of white noise $\xi(t)$ and the resulting non-differentiability of its integral motivated the development of a more rigorous framework. Instead of focusing on the ill-defined derivative form $\dot{x} = \xi(t)$, we work directly with the integrated process. This process, which formalizes the concept of Brownian motion, is named in honor of Norbert Wiener.

Definition: *The Wiener Process*

The **Wiener process** $W(t)$ is a continuous-time stochastic process that is the solution to the SDE:

$$\begin{cases} \dfrac{dW}{dt} = \xi(t) \\ W(0) = 0 \end{cases}$$

where $\xi(t)$ is Gaussian white noise with unit variance.

Note: While this differential form is intuitive, it should be understood that $\xi(t)$ is not a function in the classical sense, and the equation is interpreted in the sense of stochastic integration.

The Wiener process is a Gaussian process with the following key properties:

1. **Zero Mean:** The expected value of the process is zero:

   $$\langle W(t) \rangle = 0$$

   This property reflects the symmetric nature of the random fluctuations around the origin.

2. **Gaussian Distribution:** For any fixed time t, $W(t)$ is normally distributed with mean zero and variance t:

   $$W(t) \sim \mathcal{N}(0,t)$$

   This means the probability density function is given by:

   $$p(W,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{W^2}{2t}\right)$$

3. **Autocorrelation:** The autocorrelation function is given by

   $$\langle W(t)W(s) \rangle = \min(t,s)$$

   This property captures the ”memory” of the process.

4. **Independent Increments:** The increments of the process are independent. In particular, for any collection of non-overlapping time intervals $[t_1,s_1],[t_2,s_2],\ldots,[t_n,s_n]$, the corresponding increments $W(s_1) - W(t_1), W(s_2) - W(t_2), \ldots, W(s_n) - W(t_n)$ are independent random variables. The mean of each increment is:

   $$\langle W(s) - W(t) \rangle = 0$$

   and these increments are also Gaussian.

5. **Increment Variance:**
   The variance of the increment over the interval $[t,s]$ is proportional to the time difference:

   $$\langle (W(s) - W(t))^2 \rangle = |s-t|$$

   This scaling with time difference is a fundamental property that distinguishes diffusive processes from other types of motion.

6. **Increment Distribution:** More precisely, for $s > t$, the increment is distributed as

$$W(s) - W(t) \sim \mathcal{N}\Big(0, |s-t|\Big)$$

This means the increment has a normal distribution with zero mean and variance equal to the time difference. In particular, the distribution of $W(t)$ itself (starting from $W(0) = 0$) is:

$$W(t) \sim \mathcal{N}(0,t) \qquad \Rightarrow \qquad p(W,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{W^2}{2t}\right)$$

These properties completely characterize the Wiener process. Any continuous-time stochastic process satisfying these conditions is a Wiener process, making it a fundamental building block in the theory of stochastic processes and stochastic differential equations.

Increment Analysis and Non-Differentiability

The properties of the Wiener process's increments lead to one of its most counter-intuitive and important features: its paths are continuous but almost surely nowhere differentiable. Let's analyze an infinitesimal increment $dW_t = W(t+dt) - W(t)$ over a small time interval $dt > 0$. From the properties above, this increment is a Gaussian random variable with mean 0 and variance dt:

$$dW_t \sim \mathcal{N}(0, dt)$$

This implies that the standard deviation of the increment is $\sigma = \sqrt{dt}$. Now, consider the finite difference quotient used to define a derivative:

$$\frac{dW_t}{dt} = \frac{W(t+dt) - W(t)}{dt}$$

This new random variable is also Gaussian, but its variance is $\mathrm{Var}\left(\frac{dW_t}{dt}\right) = \frac{1}{dt^2}\mathrm{Var}(dW_t) = \frac{dt}{dt^2} = \frac{1}{dt}$. As $dt \to 0$, this variance diverges to infinity. The notion of a derivative in the classical sense is therefore problematic.

We can investigate this further by considering the probability that the magnitude of the incremental ratio exceeds some large, arbitrary value M. Let h be a small, finite time step. We are interested in:

$$\Pr\left(\left|\frac{W(t+h) - W(t)}{h}\right| > M\right)$$

Since the increment $W(t+h) - W(t)$ follows a $\mathcal{N}(0,h)$ distribution, we can standardize it by dividing by its standard deviation, $\sqrt{h}$. Let $Z = \frac{W(t+h)-W(t)}{\sqrt{h}}$, where $Z \sim \mathcal{N}(0,1)$. The probability becomes:

$$\Pr\left(\left|\frac{Z\sqrt{h}}{h}\right| > M\right) = \Pr\left(|Z| > M\sqrt{h}\right)$$

As $h \to 0$, the threshold $M\sqrt{h}$ also goes to zero. The probability that a standard normal variable Z has a magnitude greater than an infinitesimally small number approaches 1.

$$\lim_{h\to 0} \Pr\left(|Z| > M\sqrt{h}\right) = \Pr(|Z| > 0) = 1$$

This means that for any large M, as we make the time interval smaller, it becomes a near certainty that the slope of the secant line will exceed M. This confirms that the path of a Wiener process is not differentiable at any point.

3 Stochastic Numerical Methods

The non-differentiability of the Wiener process path fundamentally breaks the foundation of classical calculus, rendering traditional analytical techniques inadequate for stochastic differential equations. This mathematical obstacle necessitates the development of an entirely new framework—stochastic calculus, with its own integration theory and differentiation rules.

3.1 The Differential Form of SDEs: Itô Equation

Let's reconsider Newton's second law. In its most common form, it is written as $F = ma$. However, its more fundamental statement relates force to the change in momentum $p = mv$, expressed in differential form as:

$$dp = Fdt$$

This form is more general. For instance, consider the motion of a rocket, whose mass $m(t)$ changes as it consumes fuel. In this case, $F = ma$ is incorrect. The correct formulation is:

$$d(m(t)v(t)) = Fdt$$

This differential way of writing physical laws is powerful and provides the foundation for correctly interpreting stochastic equations. A Langevin equation written as $\dot{x} = f(x,t) + g(x,t)\xi(t)$ is mathematically problematic. The rigorous approach is to express it in its differential form using the Wiener process increment dW_t, which represents the integral of the white noise $\xi(t)$:

$$dx = f(x,t)dt + g(x,t)dW_t$$

This is a **Stochastic Differential Equation (SDE)**. Its solution is understood in an integral sense:

$$x(t) = x_0 + \int_0^t f(x(\tau),\tau)d\tau + \int_0^t g(x(\tau),\tau)dW_\tau$$

The second integral is a stochastic integral, an object whose properties are fundamentally different from the standard Riemann integral. The infinitesimal increment dW_t is defined as $dW_t = W(t + dt) - W(t)$. As we have established, it is a Gaussian random variable with mean 0 and variance dt, so $dW_t \sim \mathcal{N}(0, dt)$. This can be expressed as:

$$dW_t = G(t)\sqrt{dt}$$

where $G(t)$ is a random variable drawn from the standard normal distribution, $\mathcal{N}(0,1)$.
The stochastic integral $\int_0^t g(x(\tau),\tau)dW_\tau$ represents the cumulative effect of the random forcing over time. Unlike deterministic integrals, this integral cannot be evaluated using traditional calculus rules due to the irregular nature of the Wiener process paths. The integral must be understood in the sense of Itô or Stratonovich, with Itô integration being the more commonly used convention in stochastic differential equations.

Definition: ***Itô Equation***

Given an SLAE, we can always rewrite it as an **Itô equation**, by defining $dW = G(t)\sqrt{dt}$:

$$dx = f(x)dt + g(x)dW$$

3.1.1 The Euler-Maruyama Method

The Itô equation can be solved numerically using the Maruyama algorithm for stochastic differential equations. This is essentially the stochastic version of Euler's algorithm. Given a time interval $[0,T]$ and $h = T/N$, so that $t = jh$ for $j = 0,\ldots,N$. Suppose the Euler algorithm is:

$$dx = x(t_j + h) - x(t_j) = x(t_{j+1}) - x(t_j)$$

we can set $dt \approx h$ and write:

$$x(t_{j+1}) = x(t_j) + f(x(t_j))h$$

The Maruyama formula starts from this form by also considering the Gaussian effect, adding:

$$x(t_{j+1}) = x(t_j) + f(x(t_j))h + G_j\sqrt{h}$$

with $G_j \sim \mathcal{N}(0,1)$. Obviously, starting from this algorithm, more precise variants have been successively created.

Definition: ***The Euler-Maruyama Method***

For the stochastic differential equation $dX_t = a(X_t,t)dt + b(X_t,t)dW_t$ with initial condition $X(0) = x_0$ and uniform time step h, the Euler-Maruyama approximation is given by:

$$X_{j+1} = X_j + a(X_j,t_j)h + b(X_j,t_j)\sqrt{h}G_j$$

where $t_j = jh$ and $\{G_j\}_{j=0}^{N-1}$ is a sequence of independent standard normal random variables.

This numerical scheme provides a practical foundation for simulating stochastic processes, though more sophisticated methods have been developed to improve accuracy and stability for specific applications.

3.1.2 Itô's Lemma: The Stochastic Chain Rule

Having established the framework of Itô equations, we now turn our attention to the fundamental problem of change of variables in stochastic calculus. When dealing with deterministic differential equations, the chain rule provides a straightforward mechanism for transforming variables. However, in the stochastic setting, the irregular nature of Brownian motion necessitates a more sophisticated approach.

Consider an SDE in Itô form:

$$dx = a(x)dt + b(x)dW$$

Suppose we wish to perform a transformation from x to y defined by $y = \psi(x)$. Our objective is to derive an equation of similar form for the transformed variable y.

Following the classical approach, we expand dy in a Taylor series:

$$\begin{aligned} dy &= \psi'(x)dx + \tfrac{1}{2}\psi''(x)(dx)^2 + \ldots \\ &= \Psi'(x)[a(x)dt + b(x)dw] + \frac{1}{2}\Psi''(x)\left[b^2(x)(dw)^2 + \underbrace{\cancel{a^2(x)(dt)^2}}_{O(dt^2)} + \underbrace{\cancel{2a(x)b(x)dtdw}}_{O(dt^{3/2})}\right] \end{aligned}$$

To derive a formula consistent with the Itô framework, we must carefully consider the order of magnitude of each term. Since $dW = O(\sqrt{dt})$, we can eliminate terms of order $(dt)^{3/2}$ and higher, including the mixed term $dt\,dW$ and $(dt)^2$. The remaining term $(dW)^2$ is of order dt, and by the fundamental property of Brownian motion, we have $(dW)^2 = dt$.

Substituting this, we get:

$$dy = \psi'(x)a(x)dt + \psi'(x)b(x)dW + \frac{b(x)^2}{2}\psi''(x)dt$$

Realigning the terms, we get:

$$dy = \left[\psi'(x)a(x) + \frac{b(x)^2}{2}\psi''(x)\right]dt + \psi'(x)b(x)dW$$

Definition: ***Itô's Lemma***

Let X_t be an Itô process that satisfies the SDE $dX_t = a(X_t,t)dt + b(X_t,t)dW_t$. Let $\psi(x,t)$ be a twice-differentiable scalar function. Then the process $Y_t = \psi(X_t,t)$ is also an Itô process, and its differential dY_t is given by:

$$dY_t = \left[\psi'(x)a(x) + \psi''(x)\frac{b(x)^2}{2}\right]dt + \psi'(x)b(x)dW_t$$

This fundamental result encompasses the classical chain rule terms, namely $\psi' a$ and $\psi' b$, augmented by an additional stochastic correction term, $\psi''\frac{b^2}{2}$, known as the **Itô correction**. This correction term arises directly from the non-vanishing quadratic variation of the Wiener process and represents the fundamental distinction between stochastic and deterministic calculus.

3.2 The Stochastic Malthusian Model and its Paradox

The simplest model of population growth is the Malthusian model, which assumes an unlimited environment and a constant per capita growth rate, r. The dynamics are described by the ODE:

$$\dot{x} = rx$$

where $x(t)$ represents the population size. The solution is simple exponential growth or decay, $x(t) = x(0)e^{rt}$. A more realistic model acknowledges that the growth rate is not constant but fluctuates randomly due to environmental variations. We can model this by making the growth rate a stochastic process, $r \to r + \omega\xi(t)$. This leads to the ***stochastic Malthusian model***, an SDE with multiplicative noise:

$$dx = (r + \omega\xi(t))x$$

where ω denotes the multiplicative noise amplitude. We can obtain the Ito formula:

$$dx = rx\mathrm{d}t + \omega x\mathrm{d}W$$

The solution of this stochastic equation can be obtained by applying Itô's formula. Setting $y = \ln x$, we find the drift and diffusion coefficients for the transformed process. Using Itô's rule:

$$dy = \left[\frac{1}{x}rx - \frac{1}{2x^2}\omega^2x^2\right]\mathrm{d}t + \frac{1}{x}\omega x\mathrm{d}W \quad \xrightarrow{\text{simplifying}} \quad dy = \left[r - \frac{\omega^2}{2}\right]dt + \omega dW$$

This simplifies to a linear SDE which can be solved directly. Integrating from 0 to t:

$$y(t) = y_0 + \left(r - \frac{\omega^2}{2}\right)t + \omega W(t)$$

Supposing $W_0 = 0$, we can calculate the moments of $y(t)$:

$$\langle y(t)\rangle = \langle y_0\rangle + \left(r - \frac{\omega^2}{2}\right)t$$

Therefore, if we have $\omega^2/2 > r$, then $y(t) \to -\infty$. We notice that, intuitively considering what $y(t)$ represents, if ω is large the population will tend to extinction regardless of r. In other words, a population highly subject to events, whether negative or positive, will tend to extinction.
We can then find the second moment of $y(t)$:

$$\langle y(t)^2 \rangle = \left\langle \left[y_0 + \left(r - \frac{\omega^2}{2} \right) t \right]^2 + 2\omega \left[y_0 + \left(r - \frac{\omega^2}{2} \right) t \right] W(t) + \omega^2 W(t)^2 \right\rangle$$

which, upon solving, becomes:

$$... = \left[y_0 + \left(r - \frac{\omega^2}{2} \right) t \right]^2 + \omega^2 t$$

Assuming for convenience that y_0 is deterministic, then we have that the variance of $y(t)$ is:

$$\mathrm{Var}[y(t)] = \langle y(t)^2 \rangle - \langle y(t) \rangle^2 = \omega^2 t$$

Again, we can note that the variance tends to diverge over time. Returning now to $x(t)$, we have:

$$x(t) = e^{y(t)} = e^{y_0 + \left(r - \frac{\omega^2}{2}\right)t} e^{\omega W(t)} = x(t) = x_0 e^{\left(r - \frac{\omega^2}{2}\right)t} e^{\omega W(t)}$$

Then its mean value will be:

$$\langle x(t) \rangle = x_0 e^{\left(r - \frac{\omega^2}{2}\right)t} \langle e^{\omega W(t)} \rangle$$

So, we must compute the mean of $\exp(\omega W(t))$. We know that $W(t)$ is distributed with the distribution:

$$W(t) \sim N(0,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{W^2}{2t}}$$

so, to calculate the mean, we must compute:

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{\omega W} e^{-\frac{W^2}{2t}} dW$$

We can then combine the two exponentials and rewrite the resulting exponent as:

$$e^{\frac{-W^2 + 2t\omega W - t^2\omega^2 + t^2\omega^2}{2t}} = e^{-\frac{(W - t\omega)^2}{2t}} e^{\frac{\omega^2 t}{2}}$$

As a consequence, the integral rewritten this way becomes:

$$... = e^{\frac{\omega^2 t}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(W - t\omega)^2}{2t}} dW$$

This, however, is the integral of a translated Gaussian which is trivially equal to 1. Therefore:

$$\langle e^{\omega W(t)} \rangle = e^{\frac{\omega^2 t}{2}}$$

Thus,

$$\langle x(t) \rangle = \langle x_0 \rangle e^{\left(rt - \frac{\omega^2 t}{2}\right) + \frac{\omega^2 t}{2}}$$

In the Langevin hypothesis, it was the fact that the non-stochastic differential equation was nothing other than the differential equation of the mean, and here we find this result. Indeed, the two terms subtract, giving as a solution the solution of the non-stochastic case.

The Paradox

However, something strange emerges when we examine the long-term behavior. Returning to the definition of $y(t)$, we can rewrite it as:

$$y(t) = y_0 + \left(r - \frac{\omega^2}{2}\right)t + \omega\sqrt{t}G$$

As $t \to \infty$, the deterministic term $\left(r - \frac{\omega^2}{2}\right)t$ dominates, so:

$$\lim_{t\to+\infty} y(t) = \begin{cases} -\infty & \text{if } r < \frac{\omega^2}{2} \\ +\infty & \text{if } r > \frac{\omega^2}{2} \end{cases}$$

Since $x(t) = e^{y(t)}$, when $y(t) \to -\infty$, we have $x(t) \to 0$. This creates an apparent paradox: we showed that $\langle x(t)\rangle = x_0 e^{rt}$, which always grows exponentially regardless of ω, yet individual realizations $x(t)$ can tend to extinction when the noise is sufficiently large ($\omega^2/2 > r$).

The resolution lies in understanding the nature of the log-normal distribution. When $x(t) = e^{y(t)}$ where $y(t)$ is normally distributed, $x(t)$ follows a log-normal distribution, which is highly skewed. The mean and median can differ dramatically in such distributions.

Example: ***The Exponential Distribution***

Consider the simple exponential distribution with density ae^{-aU}. Its mean is $\langle U\rangle = 1/a$, but its median satisfies:

$$\int_0^{\text{MED}} ae^{-aU}dU = \frac{1}{2}$$

which gives $\text{MED} = \log(2)/a < 1/a$. The median is smaller than the mean because the distribution has a long tail that pulls the mean upward.

Similarly, in our stochastic population model, $x(t)$ has a log-normal distribution. While the mean grows exponentially due to rare but extremely large population bursts, the typical behavior (represented by the median) can show decline when noise is large. The mean becomes dominated by infrequent but massive population explosions, making it a poor representative of typical outcomes.

Calculation of the Median

To quantify the typical behavior of the population, we need to calculate the median of $x(t)$. Since $y(t) \sim \mathcal{N}(\mu, \omega^2)$ where $\mu = y_0 + (r - \omega^2/2)t$ and the variance parameter is $\omega^2 t$, the random variable $x(t) = e^{y(t)}$ follows a log-normal distribution with probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\omega\sqrt{t}\cdot x}\exp\left(-\frac{(\log x - \mu)^2}{2\omega^2 t}\right), \quad x > 0$$

The median m is defined as the value satisfying $P(X \le m) = 1/2$, which gives us the integral equation:

$$\frac{1}{2} = \int_0^m \frac{1}{\sqrt{2\pi}\omega\sqrt{t}\cdot x}\exp\left(-\frac{(\log x - \mu)^2}{2\omega^2 t}\right)dx$$

To solve this integral, we employ the substitution $w = \log x$, which transforms $dx = e^w dw = x\,dw$. The limits of integration become $w \in (-\infty, \log m]$, and our integral becomes:

$$\frac{1}{2} = \int_{-\infty}^{\log m} \frac{1}{\sqrt{2\pi}\omega\sqrt{t}}\exp\left(-\frac{(w - \mu)^2}{2\omega^2 t}\right)dw$$

This is precisely the cumulative distribution function of a normal random variable $\mathcal{N}(\mu, \omega^2 t)$ evaluated at $\log m$. Since the median of any normal distribution equals its mean, we have:

$$\log m = \mu = y_0 + \left(r - \frac{\omega^2}{2}\right) t$$

Therefore, the median of $x(t)$ is:

$$\mathrm{MED}[x(t)] = e^{\mu} = e^{y_0 + \left(r - \frac{\omega^2}{2}\right)t} = x_0 e^{\left(r - \frac{\omega^2}{2}\right)t}$$

This result elegantly resolves the paradox. While the mean $\langle x(t) \rangle = x_0 e^{rt}$ always grows exponentially, the median, which better represents typical population trajectories, follows the drift-corrected dynamics. The long-term behavior of the median is:

$$\lim_{t \to +\infty} \mathrm{MED}[x(t)] = \begin{cases} 0 & \text{if } r < \frac{\omega^2}{2} \quad \text{(extinction regime)} \\ x_0 & \text{if } r = \frac{\omega^2}{2} \quad \text{(critical regime)} \\ +\infty & \text{if } r > \frac{\omega^2}{2} \quad \text{(growth regime)} \end{cases}$$

The threshold $r = \omega^2/2$ represents a critical noise level: below this threshold, typical populations grow indefinitely, while above it, typical populations face extinction despite the exponentially growing mean. This dichotomy between mean and median behavior is a hallmark of log-normal processes and highlights the importance of choosing appropriate summary statistics for highly skewed distributions.

3.3 The Perturbed Logistic Equation

We have discussed the fact that the Malthus model is not realistic because it assumes infinite resources. A much better model for population dynamics is the **Logistic model**, which incorporates a density-dependent growth rate. The deterministic logistic model can be written as follows:

$$\dot{x} = r(x)x$$

where $r(x)$ is a decreasing function of the population size x. The simplest choice for this function is linear:

$$r(x) = r_0 - \alpha x$$

Here, r_0 is the intrinsic growth rate at low densities, and α is a coefficient representing the strength of density-dependent regulation (e.g., competition for resources).

Introducing Stochasticity

In a realistic scenario, the intrinsic growth rate r_0 is not constant but fluctuates due to environmental variability. We can model its fast fluctuations as:

$$r_0 \to r_0 + \omega\xi(t)$$

where $\xi(t)$ is Gaussian white noise. This transforms the deterministic ODE into the stochastically perturbed logistic model:

$$dx = (r_0 - \alpha x)x\,dt + \omega x\,dW_t$$

To analyze this equation, we can again use the logarithmic transformation $y = \ln(x)$. Applying Itô's formula yields:

$$dy = \left(r_0 - \frac{\omega^2}{2} - \alpha e^y\right)dt + \omega\,dW$$

This transformed SDE can be formally integrated to give:

$$y(t) = y(0) + \left(r_0 - \frac{\omega^2}{2}\right)t + \omega W(t) - \alpha\int_0^t e^{y(s)}ds$$

We can now analyze the long-term behavior of the system. We know two key facts:

1. If $\omega^2 > 2r_0$, then the non-integral part will tend to $-\infty$ as $t \to \infty$.
2. The integral of the exponential is always positive since $x = e^y$ must be positive.

Slightly more precisely, the fact that $\int_0^t e^{y(s)}ds > 0$ gives us an upper bound on the process $y(t)$:

$$y(t) \leq y(0) + \left(r_0 - \frac{\omega^2}{2}\right)t + \omega W(t)$$

This leads us to a **sufficient condition** for population extinction. If the right-hand side of the inequality goes to $-\infty$, then $y(t)$ must also go to $-\infty$. This happens when the drift of the bounding process is negative. Thus, the condition for extinction is:

$$\frac{\omega^2}{2} > r_0 \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0$$

This powerful result shows that if the environmental noise is sufficiently strong, the population will go extinct regardless of the density-dependent term. The noise effectively suppresses the intrinsic growth.

3.4 Foundations of Ito Calculus

3.4.1 Stochastic Equilibrium Points

The concept of an equilibrium point can be extended from deterministic to stochastic systems. For a deterministic system $\dot{x} = f(x)$, an equilibrium point x_e is defined by the condition $f(x_e) = 0$. For a stochastic differential equation (SDE) of the form

$$dx = f(x)dt + g(x)dW_t,$$

a **stochastic equilibrium point (SEP)**, denoted x_{se}, is a point where both the drift and diffusion terms vanish simultaneously:

$$f(x_{se}) = 0 \quad \text{and} \quad g(x_{se}) = 0.$$

This dual condition makes SEPs significantly rarer than their deterministic counterparts. As an example, consider the **stochastic Malthusian model**:

$$dx = r_0 x\,dt + \omega x dW_t.$$

In this case, $x_{se} = 0$ is a trivial SEP, as both $f(0) = r_0 \cdot 0$ and $g(0) = \omega \cdot 0$ are zero. The stability of this equilibrium is determined by the interplay between the growth rate r_0 and the noise intensity ω, as established previously:

- If $\omega^2/2 > r_0$, the median of the population size converges to zero, rendering the equilibrium point $x_{se} = 0$ **stochastically globally attractive**.
- If $\omega^2/2 < r_0$, the median grows exponentially, and the equilibrium point $x_{se} = 0$ acts as a **stochastic repulsor** (i.e., it is unstable).

This analysis extends to the **perturbed logistic model**, $dx = (r_0 x - \alpha x^2)dt + \omega x dW_t$, where $x_{se} = 0$ is also a SEP. The condition $\omega^2/2 > r_0$ remains sufficient for the global attractivity of the origin.

3.4.2 Derivation of Ito's Formula

We recall that during the analysis of the change of variable for Ito's formula, we substituted $(dW)^2 \to dt$ into the equation without formally proving its correctness. Let us now investigate this further, starting from the general SDE:

$$dx = a(x)dt + b(x)dW$$

We know that the Wiener increment can be expressed as $dW = \sqrt{dt}G(t)$, where $G(t) \sim \mathcal{N}(0,1)$. Our goal was to apply a variable transformation $y = \psi(x)$ to this equation. We found that this transformation resulted in:

$$dy \equiv d\psi = dt\left[\psi'(x)a(x) + \psi''(x)\frac{b(x)^2}{2}\right] + \psi'(x)b(x)dW$$

This result was achieved through the "magical" substitution mentioned earlier. The objective was to obtain a new SDE for y that has the same form as the original one:

$$dy = q(y)dt + r(y)dW$$

This required having one term of order $O(dt)$ and another of order $O(\sqrt{dt})$, and to achieve this, we discarded all terms of higher order than dt. To verify this substitution, we must revisit a concept from earlier. Let us consider the increment of the Wiener process:

$$z = W(t+h) - W(t)$$

If we now consider the random variable $q = z^2$, we have $\langle q \rangle = h$, which is the variance of z. This was the rationale for the substitution we made in the derivation of dW, but in doing so, we were neglecting potentially important elements. Let us now evaluate the variance of q.
To formally establish this, we compute the fourth moment of a Gaussian random variable $z \sim \mathcal{N}(0,h)$, which represents the Wiener increment dW_t over a time step $h = dt$.

$$\langle z^4 \rangle = \int_{-\infty}^{+\infty} z^4 \frac{1}{\sqrt{2\pi h}} e^{-\frac{z^2}{2h}} dz.$$

Integration by parts, with $u = z^3$ and $dv = z e^{-z^2/2h} dz / \sqrt{2\pi h}$, yields:

$$\langle z^4 \rangle = \left[z^3 \left(-\frac{h}{\sqrt{2\pi h}} e^{-z^2/2h} \right) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left(-\frac{h}{\sqrt{2\pi h}} e^{-z^2/2h} \right) 3z^2 dz.$$

The boundary term vanishes, leaving:

$$\langle z^4 \rangle = 3h \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi h}} e^{-\frac{z^2}{2h}} dz = 3h \langle z^2 \rangle = 3h(h) = 3h^2.$$

With $h = dt$, this gives $\langle (dW_t)^4 \rangle = 3(dt)^2$. The variance of $(dW_t)^2$ is therefore:

$$\mathrm{Var}[(dW_t)^2] = \langle (dW_t)^4 \rangle - \left(\langle (dW_t)^2 \rangle \right)^2 = 3(dt)^2 - (dt)^2 = 2(dt)^2.$$

Since the variance is of order $(dt)^2$, the fluctuations of $(dW_t)^2$ around its mean dt are of a higher order than dt itself. Consequently, in the limit $dt \to 0$, we can make the substitution $(dW_t)^2 = dt$. Substituting this result into the Taylor expansion for $d\Psi$ gives the final expression:

$$d\Psi = \left[\Psi'(x)a(x) + \frac{1}{2}\Psi''(x)b^2(x) \right] dt + \Psi'(x)b(x)dW,$$

which is the celebrated **Ito's formula**. The presence of the second-derivative term, arising from the non-zero quadratic variation of the Wiener process, is the fundamental feature that distinguishes stochastic from ordinary calculus.

3.4.3 Probability Density Function and its Evolution

Given a continuous-time, continuous-state stochastic process, $x(t)$, we often want to describe its behavior not by a single trajectory, but by the probability of finding the process in a certain state at a certain time. This is accomplished using the **probability density function (PDF)**, denoted $\rho(x,t)$. The PDF is defined such that the probability of the random variable $x(t)$ being in an infinitesimal interval $[\hat{x}, \hat{x} + d\hat{x}]$ is given by:

$$\mathrm{Prob}\left(x(t) \in [\hat{x}, \hat{x} + d\hat{x}]\right) = \rho(\hat{x}, t) d\hat{x}$$

This implies that the probability of finding $x(t)$ in a finite interval $[a,b]$ is the integral of the PDF over that interval:

$$\mathrm{Prob}\left(x(t) \in [a,b]\right) = \int_a^b \rho(x,t) dx$$

A central question in stochastic modeling is: if we know the initial distribution of the process, $\rho(x,0)$, how does this distribution evolve for $t > 0$?
In the most general case, the law of evolution for $\rho(x,t)$ could depend on the entire history of the process, often denoted as $\Omega(x(\theta), 0 \le \theta < t)$. This would mean that the future state depends on the full path taken to reach the present, leading to a complex law that could be described by an integro-differential equation.
However, for a large and very important class of processes, the situation is significantly simpler.

The Markov Property and Stochastic Differential Equations

Processes described by an Itô Stochastic Differential Equation (SDE) of the form

$$dx = f(x)dt + g(x)dW_t$$

have a special structure. The state of the system at an infinitesimal future time $t + dt$ is given by:

$$x(t + dt) = x(t) + f(x(t))dt + g(x(t))dW_t$$

Crucially, the statistical properties of $x(t + dt)$ depend only on the state $x(t)$ at the current time t, and not on the entire prior history. This is the hallmark of a **Markov process**.

Definition: ***Markov Process***

A stochastic process $x(t)$ is called a ***Markov process*** if its future probability distribution, conditioned on its past and present values, depends only on the present value. In other words, the past and the future are conditionally independent given the present.

For a process modeled by an Itô SDE, the following key properties hold:

1. **The process is Markovian.** The future state depends only on the present, not the path taken to get there.
2. **The process has continuous paths.** The Itô equation implies that increments are infinitesimal. Since the drift term $f(x)dt$ is of order $O(dt)$ and the diffusion term $g(x)dW_t$ is of order $O(\sqrt{dt})$, the process $x(t)$ does not have finite jumps.

These two properties together, being Markovian and having continuous paths, have a profound implication: the law governing the evolution of the PDF, $\rho(x,t)$, must be local in both time and space. It cannot depend on spatially distant values or past temporal values. This means the evolution operator must be a local differential operator. Therefore, the evolution equation for $\rho(x,t)$ must be a **Partial Differential Equation (PDE)**.

3.4.4 The Fokker-Planck Equation

While Ito's Lemma allows us to find the SDE for a transformed variable, its most powerful application is in deriving a deterministic equation for the evolution of the probability density function (PDF), $\rho(x,t)$. This bridge from the stochastic world of individual paths to the deterministic world of distributions is the celebrated **Fokker-Planck equation**.

The derivation is a beautiful piece of mathematical physics that relies on a "weak" formulation. Instead of tracking $\rho(x,t)$ directly, we analyze how the expected value of an arbitrary, well-behaved "test function" $\psi(x)$ evolves over time.

Derivation from Ito's Lemma

Let $x(t)$ be a process governed by the SDE $dx = a(x)dt + b(x)dW_t$. Let $\psi(x)$ be an arbitrary, twice-differentiable function that vanishes (along with its derivatives) at the boundaries of the domain (e.g., at $\pm\infty$).

1. **Apply Ito's Lemma to $\psi(x)$**
   From Ito's Lemma, we know the differential for $\psi(x(t))$ is:

$$d\psi(x) = \left[a(x)\psi'(x) + \frac{b(x)^2}{2}\psi''(x)\right]dt + b(x)\psi'(x)dW_t$$

2. **Take the Expectation**
   Next, we take the ensemble average (expectation) of this equation.

$$\langle d\psi(x)\rangle = \left\langle \left[a(x)\psi'(x) + \frac{b(x)^2}{2}\psi''(x) \right] dt \right\rangle + \langle b(x)\psi'(x)dW_t \rangle$$

   The expectation of the stochastic term vanishes, because dW_t has zero mean and is independent of the state $x(t)$ at the beginning of the infinitesimal step. This leaves us with an equation for the evolution of the mean of $\psi(x)$:

$$d\langle\psi(x)\rangle = \left\langle a(x)\psi'(x) + \frac{b(x)^2}{2}\psi''(x) \right\rangle dt$$

   Dividing by dt, we get the time derivative:

$$\frac{d}{dt}\langle\psi(x)\rangle = \left\langle a(x)\psi'(x) + \frac{b(x)^2}{2}\psi''(x) \right\rangle$$

3. **Introduce the PDF**
   The expectation of any function $h(x)$ can be written as an integral of that function against the PDF $\rho(x,t)$. Applying this to both sides of our equation:

$$\frac{d}{dt}\int_S \psi(x)\rho(x,t)dx = \int_S \left[a(x)\psi'(x) + \frac{b(x)^2}{2}\psi''(x) \right] \rho(x,t)dx$$

   where S is the state space. Since $\psi(x)$ does not depend on time, we can bring the time derivative inside the integral on the left:

$$\int_S \psi(x)\frac{\partial\rho(x,t)}{\partial t}dx = \int_S \psi'(x)a(x)\rho(x,t)dx + \int_S \psi''(x)\frac{b(x)^2}{2}\rho(x,t)dx$$

4. **Integration by Parts**
   The goal now is to remove the derivatives from the test function $\psi(x)$ and transfer them onto the other terms using integration by parts. The general formula is $\int udv = [uv] - \int vdu$.
   For the first term on the right, let $u = a(x)\rho(x,t)$ and $dv = \psi'(x)dx$.

$$\int_S \psi'(x)[a(x)\rho(x,t)]dx = \cancel{[\psi(x)a(x)\rho(x,t)]_{\partial S}} \quad \boxed{-\int_S \psi(x)\frac{\partial}{\partial x}[a(x)\rho(x,t)]dx}$$

   For the second term on the right, we must integrate by parts twice.

$$\int_S \psi''(x)\left[\frac{b(x)^2}{2}\rho(x,t)\right]dx = \left[\psi'(x)\frac{b^2\rho}{2}\right]_{\partial S} - \int_S \psi'(x)\frac{\partial}{\partial x}\left[\frac{b^2\rho}{2}\right]dx$$

   Due to our assumption that ψ and its derivatives are zero at the boundary ∂S, all the boundary terms vanish. We apply integration by parts again to the remaining integral:

$$-\int_S \psi'(x)\frac{\partial}{\partial x}\left[\frac{b^2\rho}{2}\right]dx = -\cancel{\left[\psi(x)\frac{\partial}{\partial x}\left[\frac{b^2\rho}{2}\right]\right]_{\partial S}} \quad \boxed{+\int_S \psi(x)\frac{\partial^2}{\partial x^2}\left[\frac{b(x)^2}{2}\rho(x,t)\right]dx}$$

   The boundary term again vanishes, leaving only the final integral.

5. **The Final Equation**
   Substituting these results back into our main equation, we have:

$$\int_S \psi(x)\frac{\partial \rho}{\partial t}dx = -\int_S \psi(x)\frac{\partial}{\partial x}[a\rho]dx + \int_S \psi(x)\frac{\partial^2}{\partial x^2}\left[\frac{b^2\rho}{2}\right]dx$$

   We can now group all terms under a single integral:

$$\int_S \psi(x)\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[a\rho] - \frac{\partial^2}{\partial x^2}\left[\frac{b^2\rho}{2}\right]\right]dx = 0$$

   This equation must hold for ***any*** valid choice of the test function $\psi(x)$. The only way an integral can be zero for every arbitrary test function is if the expression inside the square brackets is itself identically zero.

This gives us the celebrated Fokker-Planck equation.

Definition: ***Fokker-Planck Equation***

Given a Stochastic Differential Equation of the type:

$$dx = a(x)dt + b(x)dW$$

its associated Probability Density Function, $\rho(x,t)$, will solve the Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(a(x)\rho(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(\frac{b(x)^2}{2}P(x,t)\right)$$

This must, of course, be supplemented with boundary conditions (e.g., Dirichlet) and an initial condition $\rho(x,0)$ to obtain a unique solution. Note that imposing a specific initial state $x(0)$ actually yields the conditional PDF, $\rho(x,t|x(0))$.

In summary, the probability density function $P(x,t)$ associated with the stochastic process governed by the SDE above satisfies the celebrated **Fokker-Planck equation** (also known as the forward Kolmogorov equation):

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[a(x)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left(\frac{b^2(x)}{2}P(x,t)\right)$$

This partial differential equation describes the time evolution of the probability density $P(x,t)$ for the random variable x.
Since $P(x,t)$ is a probability density function, it must always satisfy the **normalization constraint**:

$$\int_{-\infty}^{\infty} P(x,t)\,dx = 1$$

This ensures that the total probability is conserved at all times.
To uniquely determine the solution, we must also specify the **initial distribution** of x:

$$P(x,0) = \theta(x)$$

where $\theta(x)$ is the given initial probability density function (for example, it could be a Dirac delta function if the initial state is known exactly, or a broader distribution if there is uncertainty).
Together, the Fokker-Planck equation, the normalization condition, and the initial condition fully characterize the time evolution of the probability density for the stochastic process.

Example: *Particle in a Potential Well with Additive Noise*

Let us describe the motion of a particle in a conservative potential field $U(x)$, subject to random thermal fluctuations. In the overdamped limit ($m \ll 1$), where inertial effects are negligible, the system's dynamics can be described by:

$$dx = f(x)dt + \omega dW_t$$

where $f(x) = -\frac{\partial U}{\partial x}$, and ω represents the constant intensity of the noise. Our goal is to find the **steady-state probability distribution**. The Fokker-Planck equation for this system is:

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)\rho(x,t)\right] + \frac{\omega^2}{2}\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

At steady state, the distribution no longer changes with time, so $\frac{\partial \rho}{\partial t} = 0$. We denote the stationary solution as $P(x)$. The equation simplifies to an ordinary differential equation:

$$\frac{d}{dx}\left[-f(x)P(x) + \frac{\omega^2}{2}\frac{dP(x)}{dx}\right] = 0$$

The term in the square brackets (the probability current $J(x)$) must be constant. For a system with reflecting or natural boundaries at infinity, this current must be zero everywhere:

$$-f(x)P(x) + \frac{\omega^2}{2}\frac{dP(x)}{dx} = 0$$

Substituting $f(x) = -U'(x)$, this becomes a separable first-order ODE:

$$\frac{\omega^2}{2}\frac{dP}{dx} = f(x)P = -U'(x)P \quad \Longrightarrow \quad \frac{dP}{P} = -\frac{2}{\omega^2}U'(x)dx$$

Integrating both sides gives $\ln P = -\frac{2}{\omega^2}U(x) + \text{const}$. Exponentiating yields the solution:

$$P(x) = C\exp\left(-\frac{2}{\omega^2}U(x)\right)$$

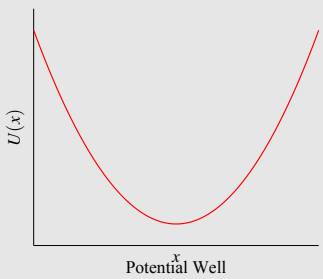
This is the famous **Boltzmann distribution**. C is determined by the condition $\int_{\mathbb{R}} P(x)dx = 1$:

$$C = \frac{1}{\displaystyle\int_{-\infty}^{+\infty} \exp\left(-\frac{2}{\omega^2}U(x)\right)dx}$$

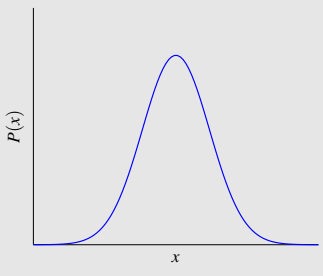
The particle is most likely to be found where $U(x)$ is lowest, though noise (ω) allows it to occasionally reach higher-energy states.
Let's consider the specific case of a harmonic potential well, $U(x) = \frac{1}{2}kx^2$. We have:

$$P(x) = C\exp\left(-\frac{k}{\omega^2}x^2\right)$$

This is a Gaussian distribution centered at $x = 0$; the variance is $\sigma^2 = \omega^2/(2k)$, showing that stronger noise (larger ω) leads to a wider distribution.



Potential Well



Stationary PDF

3.4.5 Special Cases of the Fokker-Planck Equation

Deterministic Systems with Random Initial Conditions

Suppose we have a deterministic system, $\dot{x} = a(x)$, but where the initial conditions are described by a probability distribution $\theta(x)$. Here, randomness comes only from the initial condition, not from noise in the dynamics: the diffusion coefficient is zero, $b(x) = 0$. The Fokker-Planck equation loses its second-order derivative term and becomes the **Liouville equation**:

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}\left(a(x)P(x,t)\right) \\ P(x,0) = \theta(x) \end{cases}$$

This type of problem appears, for example, when studying the evolution of a population's distribution over time under deterministic laws, but starting from a known initial distribution.

Purely Diffusive Systems

The opposite case occurs when $a(x) = 0$, and the system is governed only by noise:

$$dx = b(x)dW$$

In population dynamics, this is useful in cases where the baseline growth rate is zero ($r = 0$) or constant. In this situation, the Fokker-Planck equation becomes a pure diffusion equation:

$$\partial_t P = \partial_x^2 \left(\frac{b(x)^2}{2} P(x,t) \right)$$

This is, in fact, a generalization of the process for the Wiener process, which I recall is $\dot{w} = \xi(t)$ (i.e., $a = 0, b = 1$). Indeed, using the Fokker-Planck equation, we can find the expression for the PDF of the Wiener process, which will be:

$$\partial_t P = \frac{1}{2}\partial_w^2 P$$

Or, analogously, for $\dot{x} = \omega\xi(t)$, which corresponds to the overdamped Brownian motion:

$$\partial_t P = \frac{\omega^2}{2}\partial_x^2 P$$

The Probability Current

The Fokker-Planck equation can also be viewed in a different way. Given the formula:

$$\partial_t P = \partial_x \left[-a(x)P(x,t) - \partial_x \left(\frac{b(x)^2}{2} P(x,t) \right) \right]$$

If we pull the outer ∂_x out and name the term inside the square brackets J, we get:

$$\partial_t P + \partial_x J = 0$$

which is the continuity (or flux) equation; in the multi-dimensional case, it is typically written as:

$$\partial_t \eta + \nabla \cdot J = 0$$

This makes intuitive sense: if we think of probability as a "substance" (like a population density η), the change in its distribution is nothing more than a redistribution, so the total amount does not change. The 1D equation found above is easily reduced from the multi-D form of the flux equation. If we take Brownian motion as an example, then:

$$J = -K\nabla P$$

For this reason, J in these cases is also known as the **current**.

4
Raw Lecture Notes

Warning: *Raw Lecture Notes*

The following chapters contain unstructured notes taken during lectures and may require further organization and refinement.

5
Ornstein-Uhlenbeck Process

5.1 Liouville Equation for Systems with Uncertain Initial Conditions

Consider a physical system governed by the ordinary differential equation

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0, \end{cases}$$

where the initial condition x_0 is not known exactly. Instead, we assume that x_0 is drawn from a probability distribution $\theta(x_0)$, so that the system is defined by

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0 \sim \theta(x_0). \end{cases}$$

Suppose there exists an equilibrium point x_e such that $a(x_e) = 0$, and that x_e is globally asymptotically stable (G.A.S.). This implies that, regardless of the uncertainty in the initial condition, the state $x(t)$ converges to x_e as $t \to \infty$. Consequently, the probability density function (PDF) $\rho(x,t)$ of $x(t)$ evolves towards a Dirac delta distribution centered at x_e:

$$\lim_{t\to\infty} \rho(x,t) = \delta(x - x_e).$$

The time evolution of $\rho(x,t)$ is governed by the **Liouville equation**. For the deterministic dynamics

$$dx = a(x)\,dt,$$

the Liouville equation is given by

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\Big[a(x)\,\rho(x,t)\Big].$$

This partial differential equation expresses the conservation of probability along the flow of the system. The term $-\frac{\partial}{\partial x}\left[a(x)\,\rho(x,t)\right]$ represents the net flux of probability density in the state space due to the vector field $a(x)$. As time evolves and the system converges to the stable equilibrium x_e, the density $\rho(x,t)$ becomes increasingly concentrated around x_e, reflecting the loss of uncertainty in the long-term behavior of the system.

$$m\dot{v} = -\gamma v + F_s(t) \tag{I}$$
$$Ri = -L\frac{di}{dt} - K\frac{dB_{ext}}{dt} \tag{II}$$
$$m\ddot{x} = -\hat{k}x - \gamma\dot{x} + \hat{F}(t) \tag{III}$$

…

$$\dot{z} = -\gamma z + \omega\xi(t) \leftrightarrow dz = .\gamma z dt + \omega\xi(t)dt \quad \Rightarrow \quad z = e^{-\gamma t}Q$$

$$-\gamma e^{\gamma t}Q dt + e^{-\gamma t}dQ = -\gamma e^{-\gamma t}Q + \omega dW \quad \Rightarrow \quad e^{-\gamma t}dz = \omega dW$$

So we have:

$$dQ = e^{\gamma t}\omega dW \quad = \quad Q(t) = z(0) + \omega\int_0^t e^{\gamma s}dW(s) \quad \Rightarrow \quad \boxed{z(t) = z_0 e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)}$$

$$\langle z(t)\rangle = \langle z_0\rangle e^{-\gamma t} + \Phi$$

…

$$z(t) = z(0)e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)$$

$$\begin{aligned}
\langle z^2(t)\rangle &= \left\langle \left(z_0 e^{-\gamma t}dW(s) + \int_0^t e^{\gamma(s-t)}\xi(s)ds\right)\left(z_0 e^{-\gamma t} + \int_0^t e^{\gamma(\theta-t)}\xi(\theta)d\theta\right)\right\rangle \\
&= \left\langle \left(z_0^2 e^{-2\gamma t} + z_0 e^{-\gamma t}\int_0^t e^{\gamma(\theta-t)}\xi(\theta)d\theta + z_0 e^{-\gamma t}\int_0^t e^{\gamma(s-t)}\xi(s)ds + J(t)\right)\right\rangle \\
&= \dots
\end{aligned}$$

$$\begin{aligned}
\langle J(t)\rangle &= \omega^2\int_0^t\int_0^t e^{\gamma(\theta+s-2t)}\delta(\theta-s)d\theta ds \\
&= \omega^2 e^{-2\gamma t}\int_0^t\left\{\int_0^t e^{\gamma(\theta+s)}\delta(\theta-s)d\theta\right\}ds \\
&= \omega^2 e^{-2\gamma t}\int_0^t e^{2\gamma s}ds \\
&= \omega^2 e^{-2\gamma t}\left[\frac{e^{2\gamma t}-1}{2\gamma}\right] \\
&= \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)
\end{aligned}$$

$$Var[z(t)] = \underbrace{\langle z_0^2\rangle e^{-2\gamma t} - (\langle z_0\rangle)^2 e^{-2\gamma t}}_{= Var(z_0)e^{-2\gamma t}} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right) = Var(z_0)e^{-2\gamma t} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$$

$$Var(z(t)) \to \frac{\omega}{2\gamma} = \sigma^2$$

$$z(t) = e^{-\gamma t} + \int_0^t e^{\gamma(s-t)}\xi(s)dt$$

…

$$z(0) = 0$$

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t \int_0^q e^{\gamma(\theta+s)} \delta(\theta - s) d\theta ds$$

We have 2 cases:

- $t < q$:

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t e^{2\gamma s} ds = \frac{\omega^2}{2\gamma} e^{-\gamma(t-q)} \dots$$

$$\langle z(t)z(q)\rangle = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|q-t|} - e^{-\gamma(q+t)} \right)$$

—

$$z_0 = 0 \quad \begin{cases} \langle z(t)\rangle = 0 \\ \langle z(q)\rangle = 0 \end{cases} \qquad C[\alpha, \beta] = \langle (\alpha - \hat{\alpha})(\beta - \hat{\beta})\rangle$$

$$\to C[z(t), z(q)]; \quad q = t + h \quad X[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|h|} - e^{-\gamma h} e^{-2\gamma t} \right)$$

$$R_z(h) = \lim_{t\to\infty} C[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} e^{-\gamma|h|}$$

- $t > q$:

6
Lecture 24/03/2025

$$\dot{z} = -\gamma z + \omega \xi(t) \leftrightarrow dz = -\gamma z dt + \omega dW$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial z}(\gamma z \rho) + \frac{\omega^2}{2}\frac{\partial^2 \rho}{\partial z^2}$$

…
Consider a system with a quadratic potential given by

$$U(z) = \frac{\gamma}{2} z^2.$$

In the absence of noise, the deterministic dynamics drive the system toward the minimum of the potential, so that

$$z(t) \to 0.$$

When we add a stochastic perturbation, the dynamics can be modeled by the Langevin equation

$$\dot{z} = -\gamma z + \omega\, \xi(t),$$

where ω quantifies the noise strength and $\xi(t)$ is a white noise process. (Note that the negative sign in front of γ ensures stability around $z = 0$.) In the stationary regime, the fluctuations of z are characterized by the variance

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Assuming that the system reaches a steady state, its stationary probability density function (PDF) is given by the Boltzmann distribution,

$$P_s(z) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right).$$

Substituting $\sigma^2 = \frac{\omega^2}{2\gamma}$, we obtain

$$P_s(z) = \frac{1}{\sqrt{2\pi\,\frac{\omega^2}{2\gamma}}} \exp\left(-\frac{z^2}{\omega^2/\gamma}\right) = \sqrt{\frac{\gamma}{\pi\,\omega^2}} \exp\left(-\frac{\gamma z^2}{\omega^2}\right).$$

This stationary PDF describes how the probability of finding the system at a value z is distributed. In particular:

- **For weak noise** ($|\omega| \ll 1$): The variance $\sigma^2 = \omega^2/(2\gamma)$ is very small, so the distribution $P_s(z)$ becomes sharply peaked around $z = 0$. In the limit of vanishing noise, $P_s(z)$ approaches a Dirac delta function, indicating that the system is almost surely at the equilibrium $z = 0$.
- **For strong noise** ($|\omega| \gg 1$): The variance is large, which results in a broad stationary PDF. The probability spreads over a wider range of z values, reflecting significant fluctuations around the equilibrium.

In summary, the behavior of the stationary PDF,

$$P_s(z) = \sqrt{\frac{\gamma}{\pi\,\omega^2}}\exp\left(-\frac{\gamma z^2}{\omega^2}\right),$$

is controlled by the noise intensity ω and the potential curvature γ. For small ω, the distribution is narrowly concentrated (nearly a Dirac delta), while for large ω, it becomes broad, indicating more pronounced stochastic fluctuations.

6.1 Stochastic Relaxation to Equilibrium

Suppose we wish to study the differential equation

$$\dot{y} = \gamma(b - y) + \omega\,\xi(t),$$

which describes a system relaxing toward an equilibrium value b with rate γ, perturbed by a stochastic force $\omega\,\xi(t)$ (with $\xi(t)$ being a white noise process).
To analyze the fluctuations around the equilibrium, we introduce the variable

$$z = y - b \quad \Longrightarrow \quad y = b + z.$$

Substituting this into the original equation yields

$$\dot{z} = -\gamma z + \omega\,\xi(t).$$

In the deterministic case (i.e., when $\omega = 0$), the equation for the deterministic component y_d is

$$\dot{y}_d = \gamma(b - y_d).$$

This equation has the solution

$$y_d(t) \to b \quad \text{as} \quad t \to \infty,$$

indicating that, in the absence of noise, the system relaxes exponentially to the equilibrium b.
When the stochastic term is present, the fluctuations z about b are governed by

$$\dot{z} = -\gamma z + \omega\,\xi(t).$$

At long times, the system reaches a stationary state in which z is a Gaussian random variable with zero mean and variance

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Thus, for large t the full solution behaves as

$$y(t) \sim b + \mathcal{N}\left(0, \frac{\omega^2}{2\gamma}\right),$$

or equivalently, in terms of the deviation z,

$$z_{\text{cl}}(t) \sim \mathcal{N}\left(0, \frac{\omega^2}{2\gamma}\right) \quad \text{as} \quad t \to \infty.$$

This analysis shows that while the deterministic part drives the system to the equilibrium b, the stochastic fluctuations cause the state to be distributed around b according to a normal distribution with variance $\omega^2/(2\gamma)$.

6.2 Linearization around a Stable Equilibrium and Noise Scaling

Suppose x_e is an equilibrium point of the deterministic system

$$\dot{x} = F(x),$$

so that

$$F(x_e) = 0.$$

In the presence of stochastic perturbations, the dynamics are described by

$$\dot{x} = F(x) + \omega\,\xi(t),$$

where $\omega\,\xi(t)$ represents an additive noise term with $\xi(t)$ as white noise.
To study the behavior near the equilibrium, we expand $F(x)$ about x_e. Let y denote a variable so that

$$F(y) = F\big[x_e + (y - x_e)\big] = F(x_e) + F'(x_e)(y - x_e) + O((y - x_e)^2).$$

Since $F(x_e) = 0$, for small deviations we have

$$F(y) \approx F'(x_e)(y - x_e).$$

For a stable equilibrium we require

$$F'(x_e) < 0.$$

We define

$$F'(x_e) = -\gamma, \qquad \gamma > 0.$$

Thus, the linearized dynamics become

$$\dot{y} \simeq -\gamma(y - x_e) + \omega\,\xi(t).$$

It is convenient to introduce the deviation variable

$$z = y - x_e \quad \Longrightarrow \quad y = x_e + z.$$

Then the dynamics simplify to

$$\dot{z} = -\gamma z + \omega\,\xi(t), \qquad z(0) = 0.$$

The solution $z(t)$ is a Gaussian process with mean

$$\langle z(t) \rangle = 0,$$

and in the stationary state, its variance is given by

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Moreover, the autocorrelation function is

$$\langle z(t)\, z(t+h) \rangle = \frac{\omega^2}{2\gamma} e^{-\gamma |h|}.$$

This autocorrelation decays exponentially with a characteristic time $1/\gamma$. In the limit of rapid relaxation ($\gamma \to \infty$), the autocorrelation function becomes sharply peaked and tends toward a Dirac delta function.
To formalize this limit, we introduce a scaling relation between the noise intensity and the relaxation rate by setting

$$\boxed{\omega = c\,\gamma,}$$

with c a constant. Under this scaling, the SDE for z becomes

$$\dot{z} = -\gamma z + c\,\gamma\,\xi(t).$$

Alternatively, by defining $\delta = \frac{1}{\gamma}$, the equation can be written as

$$\dot{z} = -\frac{1}{\delta} z + \frac{c}{\delta}\,\xi(t).$$

The autocorrelation function for z then reads

$$R_z(h) = \frac{c^2}{2}\,\gamma e^{-\gamma|h|}, \qquad R_z(0) = \frac{c^2}{2}\,\gamma.$$

Its total area is given by

$$\int_{-\infty}^{+\infty} R_z(h)\,dh = \int_{-\infty}^{+\infty} \frac{c^2}{2}\,\gamma e^{-\gamma|h|}\,dh.$$

Since

$$\int_{-\infty}^{+\infty} e^{-\gamma|h|}\,dh = \frac{2}{\gamma},$$

it follows that

$$\int_{-\infty}^{+\infty} R_z(h)\,dh = \frac{c^2}{2}\,\gamma \cdot \frac{2}{\gamma} = c^2.$$

For small characteristic times ($\tau = 1/\gamma$ small), the autocorrelation function $R_z(h)$ approximates a white noise process:

$$\lim_{\gamma\to\infty} R_z(h;\gamma) = c^2\,\delta(h).$$

This derivation shows how, by choosing the scaling $\omega = c\,\gamma$, the fluctuations in the linearized dynamics around a stable equilibrium effectively become white noise in the fast relaxation limit.

The fourier transform

$$\frac{dz}{dt} = -\gamma z + \omega\xi(t)$$

let $f(t) : \mathbb{R} \to \mathbb{R}$ be a function s.t. $f^{(n)}(t)$ is continuous differentiable

$$\lim_{t\to\pm\infty} f(t) = 0$$

We have

$$\mathscr{F}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \hat{f}(\omega)$$

$$\mathscr{F}[f'(t)] = \int_{-\infty}^{+\infty} f'(t)e^{-i\omega t}dt = \underbrace{\cancel{[f(t)e^{-i\omega t}]_{-\infty}^{+\infty}}}_{\to 0} - \int_{-\infty}^{+\infty} f(t)(-\omega)e^{-i\omega t}dt = i\omega\hat{f}(\omega)$$

so we have:

$$\boxed{\mathscr{F}[f'(t)] = i\omega\mathscr{F}[f(t)]}$$

$$f'(t) + f(t) = y(t) \quad \to \quad i\omega\hat{f} + \alpha\hat{f} = \hat{y}(\omega)$$

so:

$$\hat{f}(\omega) = \frac{y(\omega)}{i\omega + \alpha}$$

$$\left|\hat{f}(\omega)\right|^2 = \frac{|g(\omega)|^2}{\omega^2 + \alpha^2}$$

This is the principle of low-pass filter

$$\frac{dz}{dt} + \gamma z = \kappa\xi(t)$$
$$(i\omega + \gamma)\hat{z}(\omega) = \kappa\mathscr{F}[\xi(t)]$$
$$\hat{z}(\omega) = \frac{\kappa}{i\omega + \gamma}\mathscr{F}[\xi(t)]$$

Suppose we are able to calculate thepower spectrum.

$$R_\xi(h) = \lim_{t\to\infty}\langle\xi(t)\xi(t+h)\rangle = \lim_{t\to\infty}\delta(h) = \delta(h)$$

The spectrum is:

$$\mathscr{F}[\delta(h)] = \int_{-\infty}^{+\infty}\delta(h)e^{-i\omega h}dh = 1$$

$$R_{out}(h) = \frac{\kappa^2}{2\gamma} e^{-\gamma|h|}$$

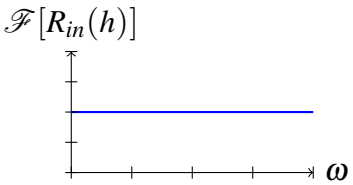
$$\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{2\gamma} \int_{-\infty}^{+\infty} e^{-\gamma|h|} e^{-i\omega|h|} dh = ...$$

We can split the integral in two parts:

$$\begin{cases} \frac{\kappa^2}{2\gamma} \int_0^{+\infty} e^{-(\gamma+i\omega)h} = \left(\frac{\kappa^2}{2\omega}\right) \left[\frac{e^{-(\gamma+i\omega)h}}{-(\gamma+i\omega)}\right] = \left(\frac{\kappa^2}{2\gamma}\right) \frac{1}{\gamma+i\omega} \\ \frac{\kappa^2}{2\gamma} \int_{-\infty}^{0} e^{-(\gamma+i\omega)h} = ... = \left(\frac{\kappa^2}{2\gamma}\right) \frac{1}{\gamma-i\omega} \end{cases}$$

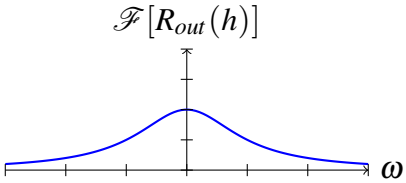
So the solution is:

$$\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{2\gamma} \left(\frac{1}{\gamma+i\omega} + \frac{1}{\gamma-i\omega}\right) = \frac{\kappa^2}{2\gamma} \frac{2\gamma}{\gamma^2+\omega^2} = \frac{\kappa^2}{\gamma^2+\omega^2}$$



Left:

$$\mathscr{F}[R_{in}(h)] = \kappa^2.$$



Right:

$$\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{\gamma^2+\omega^2}.$$

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…

$$\frac{di}{dt} = -\frac{R}{L}i + \omega\xi_1(t) + \frac{K}{L}i\xi_2(t)$$

Suppose to have an equation $dx = a(x)dt + b_1(X)dW_1 + b_2(x)dW_2$
Then

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[a(x)P] + \frac{\partial^2}{\partial x^2}\left[\frac{b_1^2(x)b_2^2(x)}{2}P(x,t)\right]$$

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(0) \sim \theta(X_0) \end{cases} \quad \rightarrow \quad \begin{cases} dx = f(x)dt + OdW \\ \rho(x,0) = \theta(x) \end{cases}$$

$$\begin{cases} \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho] \\ \rho(x,0) = \theta(x) \\ \int_\Omega \rho(x,t)dx = 1 \end{cases}$$

Example:

Spring Let's consider the following SDE describing the motion of a particle moved by a spring force.

$$\begin{cases} \frac{dx}{dt} = -kx \\ x(0) \sim \mathcal{N}(\mu = 1.8, \sigma^2 = 0.05) \end{cases}$$

The solution of the SDE is given by:

$$x(t) = x_0 e^{-kt} \quad \Rightarrow \quad x(t) \to 0$$

Example: *Stochastic Miultistability*

Let's consider the following SDE describing the motion of a particle in a double well potential.

$$\dot{x} = x - x^3 = x(1 - x^2)$$

This system has three equilibrium points:

$$\begin{cases} x_L = -1 & \Rightarrow LAS \\ x_C = 0 & \Rightarrow Unstable \\ x_R = +1 & \Rightarrow LAS \end{cases}$$

Where *LAS* means locally asymptotically stable.

$$\begin{cases} F = -\dfrac{dU}{dx} \\ U(x) = \dfrac{-x^2}{2} + \dfrac{x^4}{2} \end{cases}$$

We can calculate the limit of the probability density function $\rho(x,t)$ as $t \to \infty$.

$$\lim_{t\to\infty} \rho(x,t) = A_N \delta(x+1) + A_P \delta(x-1) + C\delta(x)$$

Where A_N and A_P are the normal and particular solutions of the system:

$$\begin{cases} A_P = \int_0^\infty \theta(x)dx = \Pr[x_0 > 0] \\ A_N = \int_{-\infty}^0 \theta(x)dx = \Pr[x_0 < 0] \\ C = \Pr[x_0 = 0] = 0 \end{cases}$$

...

$$\dot{x} = x - x^3 + \omega\xi(t)$$

$$\dot{x} = -\frac{dU}{dx} + \omega\xi(t)$$

$$P_s(x) = Ae^{-\frac{2}{\omega^2}U(x)}$$

Role of ω:

1. $\omega \ll 1$: we have small obscillations so we can use the linear approximation of the system. The system is stable and we have a single peak in the probability density function.
2. $\omega \gg 1$: we have large obscillations and in this case we can use the linear approximation of the system. The system is unstable and we have two peaks in the probability density function.

$$P_s(x) = A_s e^{-\frac{2}{\omega^2}} \approx 1$$

Observation:

Matematically it is possible to say that after a very long period of time, also the first case we could see that the system moves from one peak to the other. This is due to the fact that

the system is not stable and we have a non-zero probability of moving from one peak to the other.

$$\frac{di}{dt} = -\frac{Ri + k\xi_1(t)}{L} + \omega\xi_2(t) = -\frac{R}{L} - \frac{K}{L} i\xi_2(t) + \omega\xi_1(t)$$

multiplicative noise

The general form of a SDE with multiplicative noise is given by:

$$dx = fxdt + g(x)dW$$

We already saw a case of multiplicative noise:

$$\dot{x} = (r_1 + \omega\xi(t))x - r_2x^2 \quad \rightarrow \quad dx = \underbrace{(r_1x - r_2x^2)}_{f(x)}dt + \underbrace{\omega x}_{g(x)}dW$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho] + \frac{\partial^2}{\partial x^2}\left[\frac{g^2(x)}{2}\rho\right]$$

$$\frac{d^2}{dx^2}\left[\frac{g^2(x)}{2}\rho\right] = -\frac{d}{dx}[f(x)P_s]$$

$$\begin{cases} \dfrac{d}{dx}\left[\dfrac{g^2(x)}{2}\rho\right] = -f(x)P \\ Q(x) = \dfrac{g^2(x)}{2}\rho \quad \Rightarrow \quad P = \dfrac{2}{g^2(x)}Q \end{cases}$$

$$\frac{dQ}{dx} = -f(x)\frac{2}{g^2(x)}Q$$

$$\frac{dy}{dx} = a(x)y \quad \Rightarrow \quad \begin{cases} y(x) = Ce^{A(x)} \\ A(x) = \int_\alpha^x a(s)ds \end{cases}$$

$$Q(x) = C\exp\left\{\int_\alpha^x -\frac{2f(s)}{g^2(s)}ds\right\} \quad \Rightarrow \quad P(x) = \frac{2}{g^2(x)}\exp\left\{\int_\alpha^x \frac{2f(z)}{g^2(z)}dz\right\}$$

Now we have to derivate the probability density function, before doing this, let's rewrite it in a way that simplifies calculus.
Considering that:

$$\frac{1}{g^2(x)} = e^{-\ln g^2(x)} = e^{-2\ln g(x)}$$

We can rewrite the probability density function as:

$$P(x) = 2C\exp\left\{-2\log(g(x)) + \int_\alpha^x \frac{2f(s)}{g^2(s)}ds\right\}$$

Now we can derivate:

$$P'(x) = 2C\exp\left\{-2\frac{g'(x)}{g(x)} + \frac{2f(x)}{g^2(x)}\right\}$$

…

$$\boxed{f(x) \geq g'(x)g(x)}$$

…

$$g(x) = \omega \Rightarrow P_s'(x) \geq 0 \equiv f(x) \geq 0 \qquad -\frac{dU}{dx} \geq 0$$

As results we have that:

1. Extrema of P_s are different from extrema of $U(x)$ (= equoilibrium point of the deterministic system).
2. The number of extrema of P_s is different from the numnber of equilibrium points.

Something biological idk why

Proteins in our cells follows a self-assembly model. The model is given by the following SDE:

$$\frac{dx}{dt} = \Pi_R(x) - \delta x$$

where $\Pi_R(x)$ is the production rate of the protein.
The production rate is given by the following equation:

$$\Pi_R(x) = R + \frac{kx^2}{k_0 + x^2}$$

The equilibrium points are given by:

$$\Pi_R(x) = \delta x$$

Let's now consider $\delta = \delta + \alpha\xi(t)$, where α is a constant and $\xi(t)$ is a white noise. We can rewrite the equation as:

$$\frac{dx}{dt} = \left(R + k\frac{x^2}{k_0 + x^2} - \delta x\right) + \alpha x\xi(t)$$

In this case we have a multiplicative noise, so we can use the previous results, but now we do not have to intercept Π_R with δ, but we have to intercept it with $\delta + \alpha^2$.

$$\begin{cases} R + k\dfrac{x^2}{k_0 + x^2} - \delta x \geq \alpha^2 x \\ R + k\dfrac{x^2}{k_0 + x^2} \geq (\delta + \alpha^2)x \end{cases}$$

$$\dot{z} = r_1 z - r_2 z, \qquad x = r_2 z \to z = \frac{x}{r_2}$$

$$\begin{cases} \dot{x} = r_1 x - x^2 \\ r_1 \to r_1 + \omega \xi(t) \end{cases} \qquad \Rightarrow \quad dx = (r_1 x - x^2)dt + \omega x dW$$

$$f(x) = r_1 x - x^2, \qquad g(x) = \omega x$$

$$rx - x^2 \geq \omega^2 x \qquad \Rightarrow \qquad r_1 - x \geq \omega^2$$

in $\omega^2 = r_1$ we have a transition from an unimodal stationary distribution to a decreasing function.

$$P_s = \frac{2C}{\omega^2} x^{2r_1/\omega^2 - 2} \, e^{-(2/\omega^2)x}$$

which is not integrable

voglio anna a casa

$$dx = f(x)dt + g(x)dW$$

$$x(t+dt) = x(t) + f(x(t))dt + g(x(t))G_t\sqrt{dt}$$

We have that the probability of moving from s to a in a time dt is given by:

$$\Pr(x(t+dt)" = "a | x(t) = s) = \Omega(s,a)dt$$

While the probability of not moving is given by:

$$\Pr(x(t+dt)" = "s | x(t) = s) = 1 - \int \Omega(s,a)dt$$

8 Lecture: 31/03/2025

Last time we saw continuous state sspace and continuous time, but there are processes that are discrete in time and continuous in state space.

$$\Pr\left(x(t+dt)" = "a \mid x(t) = s\right) = \Omega(s,a)dt$$

where the probability of not moving is given by:

$$\Pr\left(x(x+dt)" = "s \mid x(t) = s\right) = 1 - dt \int \Omega(s,a)da$$

$$\Pr(x,t) = \Pr(x(t) = x)$$

$$\Pr(x(t+dt) = a \mid x(t) = s) = \Omega(s,a)dt$$

$$\Pr(x,t+dt) = \Pr(x,t)\left[1 - dt \int \Omega(x,a)da\right] + dt \int \Pr(s,t)\Omega(s,x)ds$$

$$\Pr(x,t+dt) = \Pr(x,t) = -dt\Pr(x,t) \int \Omega(x,a)da + dt \int \Pr(s,t)\Omega(s,x)ds$$

dividing by dt we get:

$$\frac{\partial P}{\partial t} = \int \Pr(s,t)\Omega(s,x)ds - \Pr(x,t) \int \Omega(x,a)da$$

This is called the **master equation** and it is also a *Fokker-Plank equation*. The first term is the probability of moving to x from s, while the second term is the probability of moving away from x to a.

Tip: ***Compact Master Equation***

We can write the master equation in a more compact form:

$$\frac{\partial P}{\partial t}(x,t) = \int \left[\Pr(y,t)\Omega(y,x) - \Pr(x,t)\Omega(x,y)\right] dy$$

but this is not the preferred form by the teacher.

$$\Omega(y,x) = C\text{Heaviside}(\varepsilon - |x-y|)$$

$$\frac{\partial P}{\partial t} = C\int_{x-\varepsilon}^{x+\varepsilon} \Pr(s,t) - \Pr(x,t) \cdot C \cdot 2 \cdot \varepsilon = \int_{x-\varepsilon}^{x+\varepsilon} \Pr(s,x)ds - 2\varepsilon C\Pr(x,t)$$

$$\frac{\partial P}{\partial t} = C\left\{\int_{x-\varepsilon}^{x+\varepsilon} \Pr(s,t)ds - 2\varepsilon \Pr(x,t)\right\} = C\left\{\int_{-\varepsilon}^{\varepsilon} \Pr(x+z,t)dz - 2\varepsilon \Pr(x,t)\right\}$$

where:

$$x-\varepsilon < s < x+\varepsilon; \quad z = s-x; \quad s = z+x; \quad -\varepsilon < z < \varepsilon$$

If we take the limit for $\varepsilon \to 0$ we can use the Taylor expansion:

$$\Pr(x+z,t) \approxeq \Pr(x,t) + \frac{\partial P}{\partial s}\bigg|_{s=x} z + \frac{1}{2}\frac{\partial^2 P}{\partial s^2}\bigg|_{s=x} z^2 = \Pr(x,t) + \frac{\partial P}{\partial x} z + \frac{1}{2}\frac{\partial^2 P}{\partial x^2} z^2$$

$$\int_{-\varepsilon}^{\varepsilon} \Pr(x,t) + \frac{\partial P}{\partial x} z + \frac{\partial^2 P}{\partial z^2}\frac{z^2}{2} dz = 2\varepsilon P + 0 + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\left|\frac{z^3}{3}\right| = 2\varepsilon P + \frac{\varepsilon^3}{3}\frac{\partial^2 P}{\partial x^2}$$

$$\frac{\partial P}{\partial t} = C\left\{2\varepsilon P + \frac{\varepsilon^3}{3}\frac{\partial^2 P}{\partial x^2} - 2\varepsilon P\right\} \quad \Rightarrow \quad \frac{\partial P}{\partial t} = \left[\frac{C\varepsilon^3}{2}\right]\frac{\partial^2 P}{\partial x^2}$$

—

Another example:

$$\Omega(y,x) = A\delta\big(|x-y| - \varepsilon\big) = A\big[\delta(x-(y+\varepsilon)) + \delta(y-(x+\varepsilon))\big]$$

$$\partial_t P = \int \Pr(s,t) A\delta\big(|x-s| - \varepsilon\big) ds - 2A\Pr(x,t)$$

$$\partial_t P = A\left\{\underbrace{\Pr(x-\varepsilon,t)}_{I} + \underbrace{\Pr(x+\varepsilon,t)}_{II} - 2\Pr(x,t)\right\}$$

$$\partial_t P = A\left\{\underbrace{\Pr(x,t) - \varepsilon\frac{\partial P}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2}}_{I} + \underbrace{\Pr(x,t) + \varepsilon\frac{\partial P}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2} - 2\Pr(x,t)}_{II}\right\}$$

$$\frac{\partial P}{\partial t} = \left(\frac{A\varepsilon^2}{2}\right)\frac{\partial^2 P}{\partial x^2} \qquad \text{where} \quad \begin{cases} A = O\left(\dfrac{1}{\varepsilon^2}\right) \\ \ldots \end{cases}$$

—

$$x(t) \in \mathbb{Z}$$

$$\Omega(s,a) = A\delta\big(|a-s| - 1\big)$$

$$\frac{\partial P}{\partial t} = \int \Pr(s,t) A\delta\big(|x-s| - 1\big) ds - 2A\Pr(x,t)$$

$$\frac{\partial P}{\partial t} = A\Pr(x-1,t) + A\Pr(x+1,t) - 2A\Pr(x,t)$$

Which is the jump from $(x-1)$ to x, plus the jump backward from $(x+1)$ to x, minus the jump from x to $(x-1)$ and $(x+1)$.
So this is a Discrete Space and Continuous Time Markov Process (CTMC), becouse we have $x(0) \in \mathbb{Z}$.

9 Lecture: 04/04/2025

…

$$\frac{\partial P}{\partial t}(x,t) = \int_{S\in s} P(s,t)\Omega(s,x)ds - P(x,t)\int_{a\in s}\Omega(x,a)da$$

…

$$\frac{\partial P}{\partial t}(x,t) = \underbrace{rP(x+1,t)}_{backward} + \underbrace{rP(x-1,t)}_{forward} - 2rP(x,t)$$

…

More in general we can write

$$\Omega(s,a) = \sum_{j\in\mathbb{Z}} K_{s,a}\delta(s-a-j)$$

$$\Rightarrow \quad \frac{\partial P}{\partial t}(x,t) = \sum_{j\in\mathbb{Z}}\int P(s,t)K_{s,x}\delta(s-x-j)ds - P(x,t)\sum_{j\in\mathbb{Z}}\int K_{x,a}\delta(x-a-j)da$$

$$\frac{\partial P(x,t)}{\partial t} = \left(\sum_{j\in\mathbb{Z}} K_{x,x+j}K_{x+j,x}P(x+j,t)\right) - P(x,t)\left(\sum_{j\in\mathbb{Z}} K_{x,x-j}\right)$$

…

$$\begin{cases} S' = -\beta\dfrac{I}{N}S \\ I' = \beta\dfrac{I}{N}S - \gamma I \end{cases}$$

This system represents the dynamics of a population of individuals that can be in one of two states: susceptible (S) or infected (I). The parameter β represents the rate at which susceptible individuals become infected, while γ represents the rate at which infected individuals recover.
We have $(R = N - S - I)$ and:

$$I(t) \geq 0; \quad S(t) \geq 0; \quad R(t) \geq 0$$

$$(t, t+dt)X(t) = (S(t), I(t)) \in \mathbb{R}^2$$

The ”removal” of individuals accounts for the recovery of infected individuals, its probability is given by:

$$\Pr\Big[\big(S(t+dt), I(t+dt)\big) = \big(S(t), I(t)-1\big) \;\Big|\; \big(s(t), I(t)\big)\Big] = \gamma dt$$

…

$$\underbrace{\begin{pmatrix} S \\ I \end{pmatrix}}_{t} \to \underbrace{\begin{pmatrix} S \\ I-1 \end{pmatrix} = \begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{t+dt}$$

The contagion process instead is given by:

$$\underbrace{\begin{pmatrix} S \\ I \end{pmatrix}}_{t} \to \underbrace{\begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{t+dt}$$

…

We can think of σ and α as the states of the system at time t and $t+dt$, respectively:

$$\sigma = \begin{pmatrix} S_\sigma \\ I_\sigma \end{pmatrix}, \qquad \alpha = \begin{pmatrix} S_\alpha \\ I_\alpha \end{pmatrix}$$

…

$$\Omega(\sigma,\alpha) = \gamma I_\sigma \delta\left(\sigma - \alpha - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) + \beta \frac{I_\sigma}{N} S_\sigma \Omega\left(\sigma - \alpha - \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$

…

$$\frac{\partial P(S,I,t)}{\partial t} = \beta \frac{(I-1)}{N}(S+1)P(S+1,I-1,t) + \gamma(I+1)P(S,I+1,t) - \left(\gamma I + \beta \frac{I}{N} S\right) P(S,I,t)$$

Your starting point is an integer position; whatever jump you take, x + dt must be another integer position. In fact, the probability sitribution is non-zero only for integer positions.

9.1 Discrete time, Discrete state space

State s is discrete
Time $t \subseteq \mathbb{N} \cup \{0\}$

$$P\{x(t+1) = \alpha \big| x(t) = \sigma\} = \theta_{\sigma\alpha} \in [0,1]$$

$$P\{x(t) = \omega\} = P_\omega(t)$$

The probability of beiing in a state α at time $t+1$ is given by the probability of being in state σ at time t multiplied by the transition probability from σ to α:

$$\boxed{P_\alpha(t+1) = \sum_{\sigma \in S} \Pr_\sigma(t)\theta_{\sigma\alpha}}$$

$$P_\alpha(t+1) = P_\alpha(t)\theta_{\alpha\alpha} + \sum_{\sigma \in S\setminus\{\alpha\}} P_\sigma(t)\theta_{\sigma\alpha}$$

$$\theta_{\alpha,\alpha} = 1 - \sum_{\beta \in S\setminus\{\alpha\}}$$

$$P_\alpha(t+1) = P_\alpha(t)\left[1 - \sum_{\beta \in S\setminus\{\alpha\}} \theta_{\alpha,\beta}\right] + \sum_{\sigma \in S\setminus\{\alpha\}} P_\sigma(t)\theta_{\sigma,\alpha}$$

$$P_\alpha(t+1) = P_\alpha(t) + \sum_\sigma P_\sigma(t)\theta_{\sigma,\alpha} - P_\alpha(t) \sum_{\beta \in S\setminus\{\alpha\}} \theta_{\alpha,\beta}$$

$$P_\alpha(t+1) - P_\alpha(t) = \sum_\sigma P_\sigma(t)\theta_{\sigma,\alpha} - P_\alpha(t) \sum_{\beta \in S\setminus\{\alpha\}} \theta_{\alpha,\beta}$$

$$\frac{P_\alpha(t+1) - P_\alpha(t)}{U} = \sum_\sigma P_\sigma(t)\frac{\theta_{\sigma,\alpha}}{U} - P_\alpha(t)\sum_\beta \frac{\theta_{\alpha,\beta}}{U}$$

…

9.2 SIS model

A SIS model is a simple model of disease spread in a population. In this model, individuals can be in one of two states: susceptible (S) or infected (I). The dynamics of the system are governed by two parameters: the infection rate β and the recovery rate γ. The model assumes that individuals can move between these two states, with susceptible individuals becoming infected at a rate proportional to the number of infected individuals they come into contact with, and infected individuals recovering at a constant rate.
Moreover, μ is the natural death rate of the population, which is assumed to be the same as the birth rate. This means that the population size remains constant over time, and the total number of individuals in the population is given by $N = S + I$.

$$\begin{cases} S' = \mu - \mu S - \beta IS \\ I' = \beta IS - (\mu + \gamma)I \end{cases}, \qquad \begin{cases} \mu \to \mu + \omega_\mu \xi_\mu \\ \beta \to \beta + \omega_\beta \xi_\beta \end{cases}, \qquad \xi(t) = \begin{pmatrix} \xi_\mu \\ \xi_\beta \end{pmatrix}.$$

This is actually a stochastic model of disease spread, where the parameters μ, β, are subject to random fluctuations.

…
[a lot of stuff missing]
…
Case whit indipendent noise and ...

$$x_j = x(jh)$$

$$x_{j+1} = x_j + \alpha(x_j)h + \beta(x_j)\begin{bmatrix} G_j \\ G_{j+1} \end{bmatrix}\sqrt{h}$$

…

9.3 ma che cazzo ne so

Suppose we are in a region of R^N and all the points follows the following law:

$$\dot{x}_i = f(x_i)$$

$$n(x,0) = \tilde{O}(x)$$

$$\inf n(x,t)dx = N$$

$$\int n(x,0)dx = \int \tilde{O}(x)dx = N$$

$$c = [a,b]$$

$$N_c(t) = \int_a^b n(x,t)dx$$

$$P[x(t) \in [a,b]] \simeq \frac{N_c(t)}{n}$$

…

9.4 ma che cazzo ne so pt.2

Let's consider an interval $[t, t+dt]$ and some particles that follows the law:

$$\dot{x} = f(x) \equiv v(x)$$

Then, the number of particles that enter and exit the interval $[x, x+dx]$ at time t is given by:

$$\begin{bmatrix} Enter: & n(x,t)v(x)dt \\ Exit: & n(x+dt,t)v(x+dt)dt \end{bmatrix}$$

…
[missing a lot of stuff]
…
the product $n \cdot v$ is called ***current*** density and is denoted by $J(x,t)$:

$$\begin{cases} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} J(x,t) = 0 \\ J(x,t) = n(x,t)v(x) \end{cases} \quad , \qquad \frac{\partial n}{\partial t} + \text{div} J(x,t) = 0$$

…
... probabiulity current ...

10 Lecture 11/04/2025

…
Ito:

$$dx = a(x)dt + b(x)dW \quad \Rightarrow \quad x(t+dt) = x(t) + a(x(t))dt + b(x(t))dW$$

Stratonovich:

$$dx = a(x)dt + b(x) \circ dW \quad \Rightarrow \quad x(t+dt) = x(t) + a(x(t))dt + b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW$$

$$x\left(t+\tfrac{dt}{2}\right) = x(t) + a(x(t))dt + b\left(x(t)\right)d\hat{W} \quad \text{where } d\hat{W} = W\left(t+\tfrac{dt}{2}\right) - W(t)$$

$$b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW = b(x(t))dW + b'(x(t))\left(a(x)dt + b\left(t+\tfrac{dt}{2}\right)d\hat{W}\right)dW + b'(x(t))b\left(x\left(t+\tfrac{dt}{2}\right)\right)d\hat{W}dW$$

$$\left\langle \left(W\left(t+\tfrac{dt}{2}\right) - W(t)\right)\left(W(t+dt) - W(t)\right)\right\rangle =$$
$$= \left\langle \left(W\left(t+\tfrac{dt}{2}\right)W(t+dt)\right) - W\left(t+\tfrac{dt}{2}\right)W(t) - W(t)W(t+dt) + W^2(t)\right\rangle$$

$$= t + \tfrac{dt}{2} - t - t + t = \tfrac{dt}{2} \quad \Rightarrow \quad b(x(t+dt))dW = \left(\tfrac{dt}{2}\right)b'(x(t))b\left(x\left(t+\tfrac{dt}{2}\right)\right) =$$

$$= \tfrac{dt}{2}b'(x(t))\left[b(x(t)) + O(\sqrt{dt})\right] = dt\frac{b'(x(t))b(x(t))}{2}$$

$$dx = a(x)dt + b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW = a(x)dt + b(x)dW + \frac{b'(x)b(x)}{2}dt$$

$$dx = a(x)dt + b(x) \circ dW \quad \Rightarrow \quad \left[a(x) + \frac{b'(x)b(x)}{2}\right]dt + b(x)dW = dx$$

So the Stratonovich formula is equivalent to the Ito where instead of $a(x)$ we have: $a(x) + \dfrac{b'(x)b(x)}{2}$
…

$$I = \sum_i f(x(t_i + \tfrac{\Delta}{2}))(W(t_i+\Delta) - W(t_i)) \quad \Rightarrow \quad I = \sum_i \frac{f(x(t_i)) + f(x(t_i+\Delta))}{2}(W(t_i+\Delta) - W(t_i))$$

$$b(x(t+dt))dW = [b(x(t))b'(x(t))[a(x)dt + b(x)dW]]\,dW \quad \Rightarrow \quad b(x)dW + \frac{1}{2}b'(x)b(x)dt$$

so

$$x(t+dt) = x(t) + a(x(t)) + dW\frac{b(x)+b(x)}{2} + \frac{b'(x)b(x)}{2}dt = \left\{a(x) + \frac{b'(x)b(x)}{2}\right\}dt + b(x)dW$$

—

$$dx = a(x)dt + b(x) \circ dW$$

$$dx = \left[a(x) + \frac{b'(x)b(x)}{2}\right]dt + b(x)dW$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[a(x)\rho + \frac{b'(x)b(x)}{2}\rho\right] + \frac{\partial^2}{\partial x^2}\left[b(x)^2\rho\right] = -\frac{\partial}{\partial x}\left[a(x)\rho\right] + \frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\partial}{\partial x}\left(b(x)^2\right)\rho\right] + \frac{\partial^2}{\partial x^2}\left[b(x)^2\rho\right]$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[a(x)\rho] + \frac{1}{2}\frac{\partial}{\partial x}\left[b(x)\frac{\partial}{\partial x}\{b(x)\rho\}\right]$$

$$dx = \left[a(x) + \frac{b'(x)b(x)}{2}\right]dt + b(x)dW$$

—

$$\dot{x} = \alpha(x) + ph(x) \qquad p \to p + \omega\xi(t)$$

$$\dot{x} = [\alpha(x) + ph(x)] + \begin{cases} \omega h(x)\xi(t) & \text{Ito} \\ \omega h(x) \circ \xi(t) & \text{Stratonovich} \end{cases}$$

If we choose the Ito interpretation, we have:

…

Else, if we choose the Stratonovich interpretation, we have:

$$dx = \left[\alpha(x) + \frac{b'(x)b(x)}{2}\right]dt + b(x)dW$$

…

$$a(x) + \frac{b'(x)b(x)}{2} = b'(x)b(x) \quad \Rightarrow \quad a(x) = \frac{b'(x)b(x)}{2}$$

—

Suppose we have a population

$$\dot{X} = Rx, \qquad R > 0$$

if we have $R \to R + \omega\xi(t)$

The Ito interpretation gives us:

$$R < \frac{\omega^2}{2} \to x(t) \to 0$$

Instead, the Stratonovich interpretation gives us:

$$dx = \underbrace{Rx}_{a(x)}dt + \underbrace{\omega x}_{b(x)} \circ dW \quad \Rightarrow \quad dx = \left[R + \frac{\omega^2}{2}\right]xdt + \omega x dW$$

let's consider the transformation $y = \ln x \quad \rightarrow \quad x = e^y$:

$$dy = \left[R + \frac{\omega^2}{2} - \frac{\omega^2}{2}\right] dt + \omega dW$$

so

$$dy = Rdt + \omega dW \quad \Rightarrow \quad y(t) = y_0 + Rt + \omega W(t)$$

—

$$\dot{x} = a(x) + b(x)\eta_h(t)$$

$$\langle \eta_h(t) \rangle = 0$$

$\mathbb{R}_h$ is a function with a peak at $|z| < h$ and $\mathbb{R}_h(z) = 0$ for $|z| > h$.

If the limit $\lim_{h\to 0^+} \mathbb{R}_h(z) = \delta(\tau)$, then:

$$\dot{x}_h = a(x_h) + b(x_h)\eta_h(t) \quad \rightarrow \quad \dot{x} = a(x) + b(x) \circ \xi(t)$$

—

Let's consider a population and two opinions:

$$x + y = 1$$

People can change their opinion with a rate R and the ratio of changing from x to y is θ, and from y to x is k.

$$\begin{cases} \dot{x} = +\theta xy - kyx - \varepsilon x + \varepsilon y \\ \dot{y} = -\theta xy + kyx + \varepsilon x - \varepsilon y \end{cases}$$

where ε is the rate of changing opinion. (?)
Substituting $y = 1 - x$ we can rewrite the first equation as:

$$\dot{x} = x(1-x)(\theta - k) + \varepsilon(1-x) - \varepsilon x$$

$$\frac{dx}{dt} = \lambda x(1-x) + 1 - 2x, \quad \lambda \to \lambda + \alpha\xi(t)$$

$$dx = \underbrace{\{\lambda x(1-x) + 1 - 2x\}}_{a(x)} dt + \underbrace{\alpha x(1-x)}_{b(x)} \circ dW$$

…

$$dx = \{1 - 2x\} dt + \alpha x(1-x) \circ dW$$

$$a(x) = \tfrac{1}{2} b'(x) b(x), \qquad b(x) = \alpha(x - x^2), \qquad b'(x) = \alpha(1 - 2x)$$

so

$$1-2x=\frac{\alpha^2}{2}x(1-x)(1-2x)$$

which has two equilibrium points:

$$x_1=\frac{1}{2},\qquad x_2:1=\frac{\alpha^2}{2}x(1-x)\ \ (?)$$

…

—

$$m\ddot{x}=-\gamma_T\dot{x}+F_T(x)+\omega_T\xi(t)$$

$$\begin{cases}\dot{x}=v\\ \dot{v}=-\dfrac{\gamma_T}{m}v+\dfrac{F_T(x)}{m}+\dfrac{\omega_T}{m}\xi(t)\end{cases}$$

to simplify the notation we can write:

$$\gamma=\frac{\gamma_T}{m},\qquad F=\frac{F_T(x)}{m},\qquad \omega=\frac{\omega_T}{m},\qquad \underbrace{U=\int F_T(x)dx,\qquad U'=\frac{dU}{dx}=F_T(x)}_{?}$$

we get:

$$\begin{cases}\dot{x}=v\\ \dot{v}=-\gamma v+F(x)+\omega\xi(t)\end{cases}$$

$$\frac{\partial p}{\partial t}=-v\frac{\partial p}{\partial x}-\frac{\partial}{\partial v}\left[(F(x)-\gamma v)p\right]+\frac{\partial^2}{\partial v^2}\frac{\omega^2}{2}$$

$$\frac{\partial p}{\partial t}=-v\frac{\partial p}{\partial x}-F(x)\frac{\partial p}{\partial v}+\gamma\frac{\partial}{\partial v}(vp)+\frac{\partial^2}{\partial v^2}\frac{\omega^2}{2}$$

$$0=-v\frac{\partial p}{\partial x}+U'(x)\frac{\partial p}{\partial v}+\gamma p+\gamma v\frac{\partial p}{\partial v}+\frac{\partial^2}{\partial v^2}\frac{\omega^2}{2}$$

$$p(x,v)=A(x)B(v)$$

$$-vA'(x)B(v)+U'(x)A(x)B'(v)+\gamma A(x)B(v)+\gamma vA(x)B'(v)+\frac{\omega^2}{2}B''(v)=0$$

$$\underbrace{\overbrace{-v\frac{A'(x)}{A(x)}+U'(x)\frac{B'(v)}{B(v)}}^{1^{st}\text{ term}}}_{=0}+1\overbrace{\gamma+\gamma v\frac{B'(v)}{B(v)}+\frac{\omega^2}{2}\frac{B''(v)}{B(v)}}^{2^{nd}\text{ term}}=0$$

We have to options:

1. set the second term to zero
2. boh

Let's consider the first option and let's define some "test" variables B_T and B'_T. We have:

$$\frac{B'_T(v)}{B_T(v)} = -\eta v \quad \Rightarrow \quad B'_T(v) = -\eta v B_T(v) \quad \Rightarrow \quad B(v) = Ce^{-\eta v^2/2}$$

we have

$$B'(v) = -\eta v B(v), \qquad B''(v) = -\eta v B'(v) = -\eta v(-\eta v B(v))$$

MISSING: boh

$$P_s = \frac{1}{z} e^{-\frac{\gamma}{\omega^2}v^2 - \frac{2\gamma}{\omega^2}U(x)} = \frac{1}{z} e^{-\frac{2\gamma}{\omega^2}\left[\frac{v^2}{2} + U(x)\right]}$$

Applying back the transformation we have:

$$p(x,v) = \frac{1}{z} e^{-\frac{2\gamma_T}{\omega_T}\left[\frac{mv^2}{2} + U_T(x)\right]}$$

so:

$$\iint p_s(x,v)dxdv = 1, \qquad \frac{1}{z}\iint e^{-\frac{2\gamma_T}{\omega_T^2}E_T(x,v)}dxdv = 1$$

TODO: check if this is correct
MISSING: fishes example ?

$$dx = f(x)dt - \underbrace{(cxdt + \omega x dW)}_{\text{\#fishes killed in } (t,\, t+dt)}$$

We want the number of fishes to be positive.
MISSING: end of the lecture

11 Lecture: 05/05/2025

…
... if there is no linearity, ...

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{(\theta \int zP(z,t)\mathrm{d}z + (1-\theta)x - x^3)P\right\} + \frac{\omega^2}{2}\frac{\partial^2 P}{\partial x^2} \qquad N \gg 1$$

$$M(t) = \int_{\mathbb{R}} xP(x,t)dx$$

$$P_s(x,M_s) = C(M_s)\exp\{\theta M_s x + ...\}$$

MISSING: end of the formula above
This solution is not actually so ”usable”

$$M_s = \int_{\mathbb{R}} xP_s(x;M_s)dx \quad \Rightarrow \quad M_s = \Psi(M_s)$$

$$M_s = \Psi(M_s) \quad \rightarrow \quad \text{”unique solution”}$$

There are more interesting cases, for instance when $\Psi(M_s)$ has more than one solution:
In this case, our system has more than one steady states. It means that we loose the unicity of the solution (so there is no more global actractiveness)

E.g:

$$\dot{X}_i = f(x_i, < x >) + g(x_i)\xi_i \quad N \gg 1$$

$$\dot{x} = f(x, M(t)) + g(x)\xi(t)$$

$$M(t) = \int zP(z,t)dt$$

The Fokker-Plank equation will be:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x,M(t))] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2(x)P]$$

The steady state solutions will be the solutions of the following equation:

$$\begin{cases} 0 = -\dfrac{\mathrm{d}}{\mathrm{d}x}[t(x,M_s)P] + \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\dfrac{g^2(x)}{2}P\right] \\ M_s = \int zP_s(z;M_s)dx \end{cases}$$

$$\boxed{M_s = \Psi(M_s)}$$

TODO: add linking sentence

$$P(x,M_s,\theta) = C(M,\theta)\exp\left[\frac{2}{\omega}\left(\theta M_s x + (1-\theta)\frac{x^2}{2} - \frac{x^4}{4}\right)\right]$$

$$\boxed{M_s = 0}$$

…(?)
$0 < \theta < \theta_c$
…(?)
So we have two solutions:

$$M_s = a$$
$$M_s = -a$$

Example:

$$M_s = \Psi(M_s;\theta)$$

$$\begin{cases} y = M_s \\ y = \Psi(M_s;\theta) \end{cases} \quad \Rightarrow \quad \begin{matrix} P_s(x;M_1) \\ P_s(x;M_2) \\ P_s(x;M_3) \end{matrix}$$

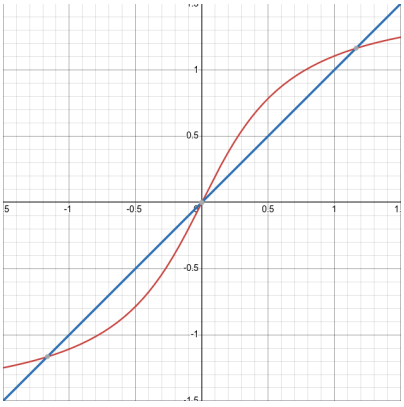
so for $\theta = \theta_1$ we have ***multistability***, while for $\theta = \theta_2$ we have ***monostability***.

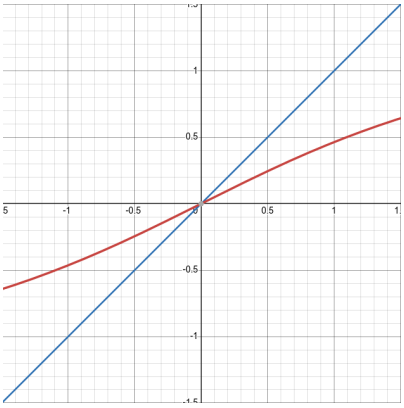
Theorem 1.

$$\left|\frac{d\Psi}{dM_s}\Big|_{M_s=M_c}\right| < 1 \quad \Rightarrow \quad P_s(x;M_1,\theta^1) \textit{ is locally stable}$$

$$\left|\frac{d\Psi}{dM_s}\Big|_{M_s=M_c}\right| > 1 \quad \Rightarrow \quad P_s(x;M_2,\theta^2) \textit{ is locally unstable}$$

…





$$\dot{x} = (ax + x^3 - x^5) - D(x - M(t)) + \alpha(1 + x^2) \odot \xi(t)$$

$$M = \Psi(M; D, \alpha)$$

we have that for small D and α we have a unique solution, while for large D and α we have 5 different solutions.

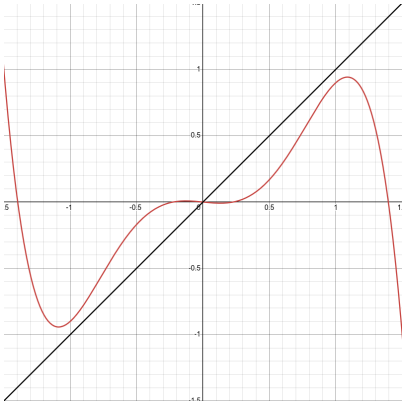


Figure 11.1: 1 solution

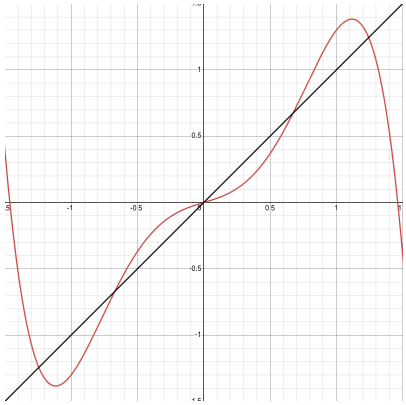


Figure 11.2: 5 solutions

So 0 is always a solution, and from a certain value of D we have 5 solutions, 3 of which are stable and 2 are unstable.

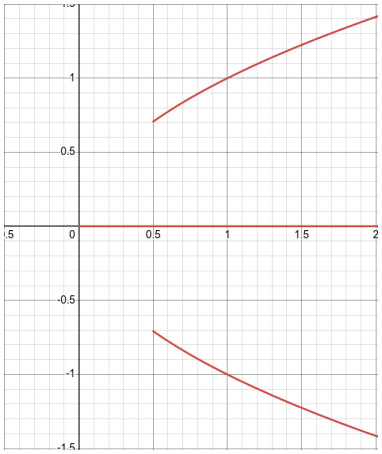


Figure 11.3: Stable solutions of the system

…

$$\dot{x} = F(x_i, < x >) + g(x_i, < x >)\xi_i(t)$$

An example iis the movement of a guitar string that vibrates.

$$m_i \ddot{x}_i = -\gamma \dot{x}_i$$

If the deviation is big, we have a function in the copmplex space, but if the deviation is small, we can use the linear approximation.

$$m_j \ddot{z}_j = -\gamma \dot{z}_j - k(z_j - z_{j-1}) - k(z_j - z_{j+1}) = -\gamma \dot{z}_j - k\,(z_{j-1} - 2z_j + z_{j+1})$$

we have now a discretization of the position of the string $z(t,x)$:

$$m_j \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}(t,x_j) = -\gamma \frac{\mathrm{d}z}{\mathrm{d}t}(t,x_j) + k[z(t,x_j+D) - 2z(t,x_j) + z(t,x_j-D)]$$

$$\mu \frac{\partial^2 z}{\partial t^2} = -\gamma \frac{\partial z}{\partial t} + c\frac{\partial^2 z}{\partial x^2} + \hat{\omega}\xi(x,t)$$

We can see the stochastic term as the wind that moves the string.

12 Lecture: 09/05/2025

12.1 Spatiotemporal noisy model

$$\frac{\partial \phi}{\partial t} = f(\phi) + g(\phi)\xi_m(r,t) + D\mathscr{L}[\phi] + h(\phi)F(t) + \xi_a(r,t)$$

- $f(\phi)$: deterministic part
- $g(\phi)$: multiplicative noise
- $D\mathscr{L}[\phi]$: linear part
- $h(\phi)$: additive noise

L[ϕ] is a Laplacian or a integral operator
Examples:

- $$\mathscr{L}[\phi] = \nabla^2\phi$$
- $$\mathscr{L}[\phi] = -a_0\nabla^2\phi - \nabla^4\phi$$
- $$\mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi = -(K_0^2 + 2k_0\nabla^2 + \nabla^4)\phi$$

Observation:

If we apply the fourier transform of:

$$\mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi = -(K_0^2 + 2k_0\nabla^2 + \nabla^4)\phi$$

we get:

$$...F(\phi)$$

$$\mathscr{L}[\phi(r)] = \int \phi(r')\omega(r-r')\mathrm{d}r'$$

How do we simulate this equation? We can simply obtain the domain and discretize it.
Lattice.based Approximation:

...

Field coupling approximation:

$$l(\phi_i, \phi_j) = w_i\phi_i + \sum_{j \in nn(i)} w_j\phi_j$$

For example:

$$\mathscr{L}[\phi] = \nabla^2\phi \approx l(\phi_i,\phi_j) = \frac{1}{\Delta^2}\sum_{j\in nn(i)}(\phi_j - \phi_i)$$

If we have a stochastic process which is discrete in time and space, we have:

$$\langle \xi(r,t)\xi(r',t')\rangle = sC\left(\frac{|r-r'|}{d},\frac{|t-t'|}{\tau_c}\right)$$

As in the purely temporal noise, τ_c is a measure of the temporal memory of the noise, d is the spatial memory of the noise.
The spatiotemporal brother of the ornstein-uhlenbeck noise is "Ojalvo e al" noise.

$$\frac{\partial\phi}{\partial t} = a\phi + D\nabla^2\phi + \xi_{gn}$$

Observation: ***Ojalvo e al and the Ornstein-Uhlenbeck process***

If we set $D = 0$ we have a series of Ornstein-Uhlenbeck processes at each point of the domain.

$$\frac{\partial\phi}{\partial t} = a\phi + \xi_{gn}$$

Noise induced pattenrs §

$$\frac{\partial\phi}{\partial t} = f(\phi) + g(\phi)\xi_m(r,t) + D\mathscr{L}[\phi] + ...$$

…
Perturbed Swift-Hohenberg model:

$$\frac{\partial\phi}{\partial t} = a\phi + D\mathscr{L}[\phi] + \xi_{gn}...$$

$$\frac{\partial\phi}{\partial t} = f(\phi) + D\mathscr{L}[\phi] = a\phi - D(\nabla^2 + k_0^2)^2\phi$$

Transitory pattern that disappear.
Additive noise generate pattenrs

$$\frac{\partial\phi}{\partial t} = a\phi + D\mathscr{L}[\phi] + \xi_{gn}, \qquad \mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi$$

Permanent patterning: details change in time.
we can distinguish two cases:

- $a < 0$
- $a > 0$

with multiplicative noise it can induce bimodality in the pdf of ϕ:

$$\frac{\partial\phi}{\partial t} = a\phi - phi^3 + \phi\xi_{gn} + D\mathscr{L}[\phi]$$

—

A bad model of glaciations

$$dx = [x(a - x^2) + A\cos\Omega t]dt$$

where

- x: is the (normalized) Earth's temperature
- $A\cos(\Omega t)$: small periodic variations of the solar irradiation.

We have that if A is small, $x(t)$ fluctuates around $+\sqrt{(a)}$.
The model fails.
Including stochastic noise:

$$dx = [x(a - x^2) + A\cos\Omega t]dt + \varepsilon dW$$

where ε is the noise intensity.
This time we have a white noise, according to:

- ε is small: the noise is negligible
- ε is large: the noise is dominant

they finally managed to model the glaciations.

—

Spatial Stochastic Resonance

$$\frac{\partial \phi}{\partial t} = a\phi - \phi^3 + D\frac{\partial^2 \phi}{\partial x^2} + F(t) + \varepsilon\xi_{gn}$$

TODO: check the formula

…

13 Discrete Time Markov Chains

We consider the case where the time is a subset of the integers ($t \subset \mathbb{Z}$ or $t \subset \mathbb{N}_0$)
The state space is a finite set $S = \{s_1, s_2, \ldots, s_N\}$

$$P\{x(t+1)|x(0),x(1),\cdots,x(t)\} = P\{x(t+1)|x(t)\}$$

The probability that the process in $t+1$ is σ is the sum of the probability that the process in t is δ and the probability that the process in $t+1$ is σ given that the process in t is δ.

$$P\{x(t+1)=\sigma\} = \sum_{\delta\in S} P\{x(t+1)=\sigma|x(t)=\delta\}P\{x(t)=\delta\} = \sum_{\delta\in S} P\{x(t)=\delta\}\theta_{\delta\sigma}$$

with $\theta_{\delta\sigma} \in [0,1]$.
So we have:

$$P_\sigma(t+1) = \sum_{\delta\in S} P_\delta(t)\theta_{\delta\sigma} \quad \Rightarrow \quad P(t+1) = P(t)\Theta(t)$$

Where $P(t) = [P_1(t), P_2(t), \cdots, P_N(t)]$ is the probability vector at time t and $\Theta(t)$ is the transition matrix at time t.
We have:

$$P(1) = P(0)\Theta, \quad P(2) = P(1)\Theta = P(0)\Theta^2, \quad P(3) = P(2)\Theta = P(0)\Theta^3, \quad \cdots$$

We can write:

$$P(t) = P(0)\Theta^t = P(0)\prod_{q=0}^{t-1}\Theta(q)$$

We have two properties:

$$\sum_{\sigma\in S}\theta_{\delta\sigma} = 1, \qquad \sum_{\delta\in S} P_\delta(t) = 1$$

So we have:

$$P_\sigma(t+1) = \sum_{\delta\in S} P_\delta(t)\theta_{\delta\sigma} \quad \Rightarrow \quad \sum_{\sigma\in S} P_\sigma(t+1) = \sum_\sigma\sum_\delta P_\delta(t)\theta_{\delta\sigma} = \sum_\delta P_\delta(t)\sum_\sigma\theta_{\delta\sigma} = 1$$

So
MISSING: something

—

Let's consider again the discrete time Markov chain.

$$P_\sigma(t+1) = P(t)\Theta \quad \Rightarrow \quad P(t) = P(0)\Theta^t$$

…

$$P^{\infty} = P^{\infty}\Theta$$

Let's study the eigenvalues of Θ:

$$\Theta v = \lambda v$$

$$\sum_{\sigma} \theta_{\delta\sigma} = 1$$

(... non ho capito perchè ma l'autovettore di Θ è formato da tutti 1 ...)
If the multiplicity of the eigenvalue is more than 1, we have multiple solutions to our system.

Tip:

Sometimes this equation is written as:

$$P_a(t+1) = \sum_{s \in S} W_{as} P_s(t)$$

where W_{as} ...

MISSING: something

—

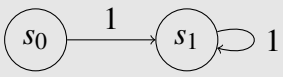
Example: ***Simple markov chain***

The simplest markov chain is the one with only one state, with a transition with probability 1 from the state to itself.

$$P(0) = [1], \qquad P(t) = [1]$$

—

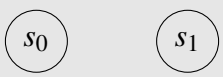
Another simple markov chain is the one with two states, with a transition with probability 1 from the state s_0 to the state s_1 and with probability 1 from the state s_1 to the state s_1 itself.



We have the situation below:

$$P(0) = [1,0], \qquad P(t) = [0,1]$$

A more complex example is the following:



TODO: add the transitions in the figure

$$P_0(t+1) = \theta_{00} P_0(t) + P_1(t)\theta_{10}$$

$$P_1(t) = 1 - P_0(t) \quad \Rightarrow \quad P_0(t+1) = \theta_{00}P_0(t) + \theta_{10}(1 - P_0(t))$$

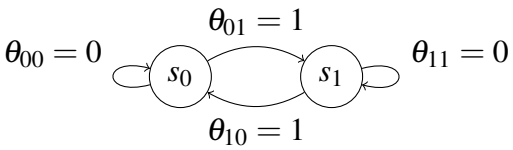
$$\boxed{P_0(t+1) = (\theta_{00} - \theta_{10})P_0(t) + \theta_{10}}$$

If we have a case like:

$$x(t+1) = ax + b$$

then we have a term b that don't allow us to use the resolution formula we are used to ($x(t) = x(0)a^t$), but ...

…



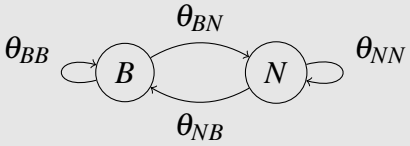
$$P(0) = [1,0] = x(0) = 0$$
$$P(1) = [0,1] = x(1) = 1$$
$$P(2) = [1,0] = x(2) = 0$$
$$\vdots$$

Which is periodic.

…

Example: ***Bora example***

Let's consider the case of Bora in Trieste. We have two states: B and N.

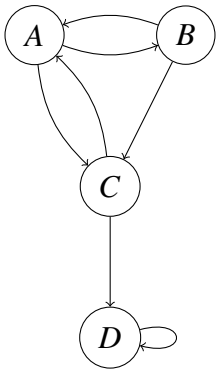


We have the following transition probabilities:

$$\theta_{BB} \approxeq \frac{NT^{BB}}{N_B} \qquad \theta_{BN} \approxeq \frac{NT^{BN}}{N_B}$$
$$\theta_{NB} \approxeq \frac{NT^{NB}}{N_N} \qquad \theta_{NN} \approxeq \frac{NT^{NN}}{N_N}$$

This model is actually too artificious, because the transition probabilities are not independent, and also depends on other parameters, like the temperature.

The most natural representatiin of Markov Chains are oriened graphs.
Let's consider a 4 states model:



...boh...

14 Lecture: 12/05/2025

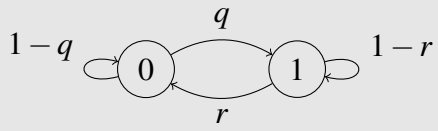
$$P(t+1) = P(t)\theta = ...$$

$$P(t) = P(0)\theta^t = P(0)...$$

???????

Example: *2 States Markov Chain*

Let's consider the following example (similar to the previous one):



We have two states (0 and 1), and a probability q of going from 0 to 1, and a probability r of going from 1 to 0. Let $P_0(t)$ be the probability that the system is in state 0 at time t, and $P_1(t)$ the probability of being in state 1 at time t. The evolution of these probabilities is given by:

$$\begin{cases} P_0(t+1) = (1-q)P_0(t) + rP_1(t) \\ P_1(t+1) = qP_0(t) + (1-r)P_1(t) \end{cases}$$

Since there are only two states, the probabilities must sum to 1:

$$P_1(t) + P_0(t) = 1$$

So, $P_1(t) = 1 - P_0(t)$.

Stationary (Equilibrium) Distribution: In the long run, the probabilities reach a steady state (stationary distribution), where $P_0(t+1) = P_0(t) = P_0^e$ and $P_1(t+1) = P_1(t) = P_1^e$. Setting the evolution equations to equilibrium, we get:

$$\begin{cases} P_0^e = (1-q)P_0^e + rP_1^e \\ P_1^e = qP_0^e + (1-r)P_1^e \end{cases}$$

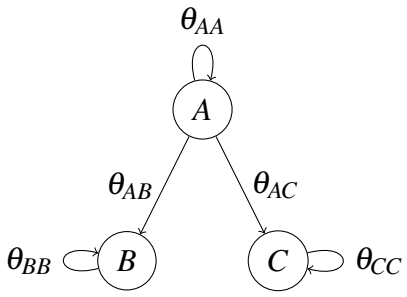
Using $P_1^e = 1 - P_0^e$, substitute into the first equation, we obtain:

$$P_0^e = \frac{r}{q+r}$$

Similarly:

$$P_1^e = \frac{q}{q+r} = 1 - P_0^e$$

Let's consider a Markov chain with three states: A, B, and C. The transitions are as follows:



Let $P_A(t)$, $P_B(t)$, $P_C(t)$ be the probabilities of being in states A, B, C at time t. The initial state is:

$$P(0) = (P_A(0), P_B(0), P_C(0))$$

Evolution Equations:

$$P_A(t+1) = \theta_{AA} P_A(t) ...$$

…

$$P_B(t+1) = P_B(t) + \theta_{AB} P_A(t) = P_B(t) + \theta_{AB} P_A(0) \theta^t$$

$$P_B(1) = P_B(0) + \theta_{AB} P_A(0)$$

$$P_B(2) = P_B(1) + \theta_{AB} P_A(1) = P_B(0) + \theta_{AB} P_A(0) + \theta_{AB} P_A(1)$$

$$P_B(t) = P_B(0) + \theta_{AB} P_A(0) \left[1 + \theta_{AA} + \theta_{AA}^2 + ... + \theta_{AA}^{t-1}\right]$$

…

$$P_B^{eq} = P_B(0) + P_A(0) \frac{\theta_{AB}}{1 - \theta_{AA}}$$

$$P_C^{eq} = P_C(0) + P_A(0) \frac{\theta_{AC}}{1 - \theta_{AA}}$$

$$det(\theta - \lambda z) = \begin{vmatrix} \theta_{AA} - \lambda & \theta_{AB} & \theta_{AC} \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

...

$$P^e = H \, \text{Diag}[1, 0, ..., 0] H^{-1} P(0)$$

and we have:

$$P^e = W P^e$$

Properties of W:

1. At least $\lambda_1 = 1$
2. $|\lambda_j| \leq 1$ for all j

3. $\sum_{j=1}^{n} V_j = 0$

where V_j is the eigenvector of W associated with λ_j.

…

$$WV_j = \lambda_j V_j$$

$$\sum_{C=1}^{n} W_{RC}(V_j)_C = \lambda_j (V_j)_R$$

$$\sum_{R=1}^{n}\sum_{C=1}^{n} W_{RC}(V_j)_C = \lambda_j \left(\sum_{R=1}^{n} (V_j)_R \right)$$

$$\underbrace{\sum_{C=1}^{n} (V_j)_C}_{S} = \lambda_j \underbrace{\left(\sum_{R=1}^{n} (V_j)_R \right)}_{S} \quad \Rightarrow \quad (1-\lambda_j)S = 0$$

…

$$P^{eq} = H \,\mathrm{Diag}[1,0,...]P(0)$$

$$P(0) = C_2 V_1 + \sum_{j=2}^{N} C_j V_j$$

…

$$P(0) = U_1 + \sum_{j=2}^{N} C_j V_j$$

Where U_1 is the normalized vector

…

$$P(0) = C_1 V_1 + C_2 V_2 + \sum_{j=3}^{N} C_j V_j$$

$$1 = C_1 \|V_1\|_1 + C_2 \|V_2\|_1 \quad \Rightarrow \quad C_2 = \frac{1 - C_1 \|V_1\|_1}{\|V_2\|_1}$$

$$P(0) = C_1 U_1 + (1-C_1)U_2 + \sum_{j=3}^{N} C_j U_j$$

$$P(t) = W^t P(0) = C_1 U_1 + (1-C_1)U_2 + \sum_{j=3}^{N} C_j \lambda_j^t V_j$$

So C_1 depends on the initial state $P(0)$:

$$\boxed{C_1 = P(P(0))}$$

…

$$P(t) = C_1 P(0) U_1 + (1 - C_1(P(0)))U_2 + \sum_j C_j \lambda_j^t V_j$$

$$P^e = C_1 U_1 + (1 - C_1) U_2$$

—

If we have a problem and we want to study the time of remaining in a state A, we can simply consider all the transitions from A to other states as a single transition from A to a new state B (which represents the rest of the world), and it is given by the sum of all the initial transitions.
So we have now only 2 transitions: the loop from A to itself, and the new AB transition.
We have:

$$x(0) = A, \qquad P(0) = (1,0)$$

$$P_A(t) = \theta_{AA}^t$$

…

$$P_A(T) = \theta_{AB} = \theta_{AB}\theta_{AA}^t = (1 - \theta_{AA})\theta_{AA}$$

So the average value of T is:

$$\langle T \rangle = \sum T \theta_{AB} \theta_{AA}^t = \theta_{AB} \sum_{t=0}^{\infty} t \theta_{AA}^t$$

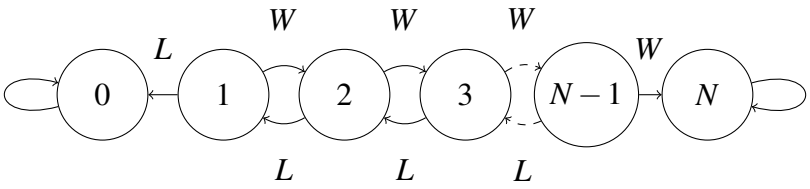
$$= \theta_{AB} \left[0 + \theta_{AA}^1 + 2\theta_{AA}^2 + 3\theta_{AA}^3\right] = \theta_{AB}\theta_{AA} \left[1 + 2\theta_{AA} + 3\theta_{AA}^2 + ...\right]$$

$$= \theta_{AB}\theta_{AA} \frac{\mathrm{d}}{\mathrm{d}\theta_{AA}} \left[-1 + 1 + \theta_{AA} + \theta_{AA}^2 + ...\right]$$

$$= \theta_{AB}\theta_{AA} \frac{\mathrm{d}}{\mathrm{d}\theta_{AA}} \left[\frac{1}{1 - \theta_{AA}}\right]$$

15 Lecture: 16/05/2025

Consider a game, you have a probability p of winning, and a probability $1-p$ of losing.
You can bet 1 euro each time, and if you win you gain 1 euro, if you lose you lose 1 euro, if you run out of money you stop playing.



We can find that $L < W$.

…

$$P(t) = W^t P(0)$$

…

$$(\theta^t)_{AB} > 0$$

…

0 and N are ***absorbing*** states.
We have that a state σ is ***transient*** if (for $t \gg 1$) $P_\sigma(t) = 0$.
Another important property is the ***ergodicity***: All the states are visited during the process lifetime, and there is no periodicity.

…

We can approximate the probability distribution of the transitions as follows:

$$P_A^{eq} = \frac{\#(x(t) = A)}{T}, \qquad P_B^{eq} = \frac{\#(x(t) = B)}{T}$$

Which is *the number of times the process is in state A at time t divided by the total number of transitions T* (Same for B).

…

The ***return time*** is the time it takes for a process to return to a given state. For a state σ, we can define the return time T_σ as:

$$T_\sigma = \min\{t > 0 : x(t) = \sigma | x(0) = \sigma\}$$

The ***mean return time*** $\langle T_\sigma \rangle$ is the average time it takes for the process to return to state σ after leaving it. For an ergodic Markov chain, the mean return time is related to the equilibrium probability by:

$$\langle T_\sigma \rangle = \frac{1}{P_\sigma^\infty}$$

This makes intuitive sense: if a state has a high equilibrium probability, it will be visited frequently, leading to a short mean return time. Conversely, states with low equilibrium probabilities will have longer mean return times.

All the problems continuous in time but that involves a discrete state space can be solved using a Contionuous Time Markov Chain (CTMC), which are a subset of all the stochastic processes with discrete state space and continuous time.

$$\Pr\{x(t) = \sigma \mid x(\theta), \quad \theta \in [0,t]\}$$

If we consider an infinitesimal time interval $\mathrm{d}t$, we can write:

$$\Pr\{x(t+\mathrm{d}t) = \alpha \mid x(t) = \sigma\}$$

So the probability of eving more than one transition in the interval is:

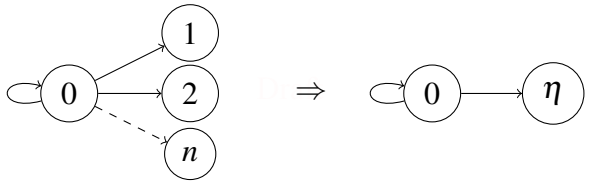
$$\begin{array}{ll} \Pr\{\geq 2 \text{ events} \quad (t,t+\mathrm{d}t)\} & = 0 \\ \Pr\{1 \text{ event} \quad (t,t+\mathrm{d}t)\} & = O(\mathrm{d}t) \\ \Pr\{0 \text{ events} \quad (t,t+\mathrm{d}t)\} & = 1 - O(\mathrm{d}t) \simeq 1 \end{array}$$

We can define the ***transition rate*** as:

$$\Pr\{x(t+\mathrm{d}t) = \alpha \mid x(t) = \sigma\} = W_{\alpha\sigma}\mathrm{d}t$$

$W_{\alpha\sigma}$ is always positive and can be $>$ or $<$ than 1.

…



To compute the probability of remaining in the state 0, we can collapse all the other states into a single state η.

…

$$x(t) = \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$$

With: $(S,I) \to (S-1, I+1)$,

$$S(t+\mathrm{d}t) = S(t) - 1, I(t+\mathrm{d}t) = I(t) + 1$$

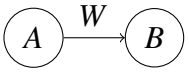
$$x(t+\mathrm{d}t) = x(t) + (-1,1)$$

…

$$\begin{cases} \Pr(Cont) = \Pr\left\{x(t+\mathrm{d}t) = x(t) + \binom{-1}{1}\right\} = \beta\dfrac{I(t)}{N}S(t)\mathrm{d}t \\ \Pr(Rec) = \Pr\left\{x(t+\mathrm{d}t) = x(t) + \binom{0}{-1}\right\} = \gamma I(t)\mathrm{d}t \end{cases} \quad \Rightarrow \quad \begin{cases} W_{Cont} = \beta\dfrac{I(t)}{N}S(t) \\ W_{Rec} = \gamma I(t) \end{cases}$$

…

$$W_{\eta\sigma} = \sum_{\alpha \in \mathscr{S} \setminus \{\sigma\}} W_{\alpha\sigma}$$



$$\mathscr{S} = \{A, B\}$$

We have that:

$$\begin{cases} \Pr\{x(t+\mathrm{d}t) = B | x(t) = A\} = W_{AB}\mathrm{d}t \\ \Pr\{x(t+\mathrm{d}t) = B | x(t) = B\} = 1 \end{cases}$$

$$P_A(t+\mathrm{d}t) = (1 - W_{AB}\mathrm{d}t)P_A(t) \quad \Rightarrow \quad P_A(t+\mathrm{d}t) = P_A(t) - WP_A(t)\mathrm{d}t$$

$$P'_A(t) = -WP_A(t)$$

$$\boxed{P_A(t) = P_A(0)e^{-Wt}}$$

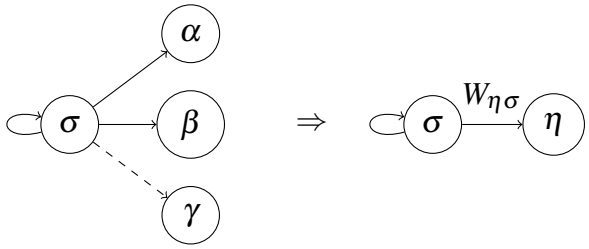
…
Let's define T the time of remaining in the state A:
…

$$\mathscr{P}_I(T) = We^{-WT}$$

Since this is an exponential distribution, we can compute the mean return time as:

$$\langle T \rangle = \frac{1}{W}$$

…
—



If we want to compute the probability of remaining in the state σ for a time T, we can write:

$$P\{x(T+\mathrm{d}t) \in \mathscr{S} \setminus \{\sigma\} | x(T) = \sigma\}$$

Also in this case we can collapse the states into a single state η.

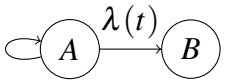
$$W_{\eta\sigma} = \sum_{\alpha \neq \sigma} W_{\alpha\sigma} = W_{sum}$$

So we can say that:

$$\begin{cases} t_\eta \quad x(t_\eta) = \sigma \\ T \sim \exp(\lambda = W_{sum}) \end{cases} \qquad \Rightarrow \qquad t_{next} = t_\eta + T$$

$$\Pr\{x(t_\eta + T + \mathrm{d}t) = \alpha | x(t+T) = \sigma\} = W_{\alpha\sigma} \cdot 1 \cdot \mathrm{d}t,$$
$$\Pr\{x(t_\eta + T + \mathrm{d}t) = \beta | x(t+T) = \sigma\} = W_{\beta\sigma} \cdot 1 \cdot \mathrm{d}t,$$
$$\dots$$

$\dots$



$$P_A(t+\mathrm{d}t) = (1-\lambda(t)\mathrm{d}t)P_A(t)$$

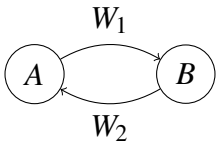
$$P'_A(t) = -\lambda(t)P_A(t) = P_A(t) = P_A(0)e^{-L(t)}, \qquad L(t) = \int_0^t \lambda(s)\mathrm{d}s$$

$$\mathscr{P}(T) = \lambda(T)e^{-L(t)}$$

$$T \sim \mathscr{P}(T) = \lambda(T)e^{-L(t)}$$

$$P(\sigma \to \alpha) = \frac{\lambda_\sigma(T)}{L_{sum}(t)}$$

—



$$\begin{cases} \dot{P}_1(t) = W_2P_B(t) - W_1 + P_A(t) \\ \dot{P}_2(t) = W_1P_A(t) - W_2P_B(t) \end{cases} \quad \Rightarrow \quad \begin{array}{l} \dot{P}_A + \dot{P}_B = 0 \\ P_A + P_B = 1 \end{array}$$

$$\begin{cases} \dot{P}_A = \sum(W_{Ay}P_y - W_{yA}P_A) \\ \sum_{A\in\mathscr{S}} P_A(t) = 1 \end{cases}$$

And we have that:

$$\dot{P}_A = -W_1P_A + W_2(1-P_A) = W_2 - (W_1+W_2)P_A$$

So, for an equilibrium we have that:

$$0 = W_2 - (W_1+W_2)P_A^{eq}$$

$$P_A^{eq} = \frac{W_2}{W_1+W_2}, \qquad P_B^{eq} = \frac{W_1}{W_1+W_2}$$

We can write our transition matrix:

$$W = \begin{bmatrix} -W_1 & W_2 \\ W_1 & -W_2 \end{bmatrix}$$

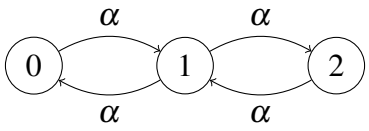
Which is singular, so one of the eigenvalue is 0.

…

$$\begin{cases} \dot{P}_\sigma = \sum (W_{\sigma y} P_y - W_{y\sigma} P_\sigma) \\ \dot{\underline{P}} = A\underline{P} \\ \sum_\sigma \dot{P}_\sigma(t) = 0 \end{cases}$$

$$0 = Ap^{eq}$$

…



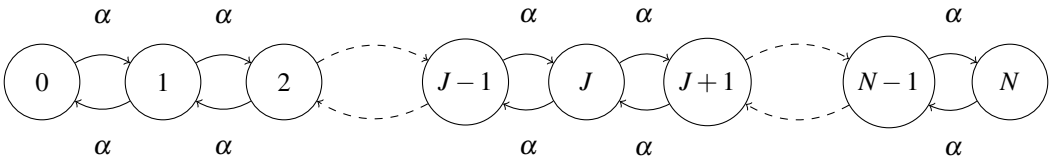
The steady state is given by:

$$\begin{aligned} \dot{P}_0 = &\quad -\alpha P_0 &&+\alpha P_1 \\ \dot{P}_1 = &\quad \alpha P_0 &&-2\alpha P_1 + \quad \alpha P_2 \\ \dot{P}_2 = &\quad\quad\quad &&\alpha P_1 - \quad\quad \alpha P_2 \end{aligned}$$

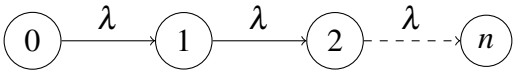
…

$$3\phi = 1 \quad \Rightarrow \quad \boxed{\phi = \frac{1}{3}}$$

we can now generalize this to a system with N states:



…
Let's consider an unidirectional chain, with a single transition rate λ:



$$\begin{aligned} &\dot{P}_0 = -\lambda P_0 \\ &\dot{P}_1 = \lambda P_0 - \lambda P_1 \\ &\dots \\ &\dot{P}_\sigma = \lambda P_\sigma - \lambda P_{\sigma-1} \end{aligned}$$

…
The probability distribution $P_n(t) = \Pr\{N(t) = n\}$ has a simmetry property:

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

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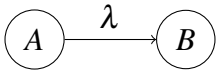
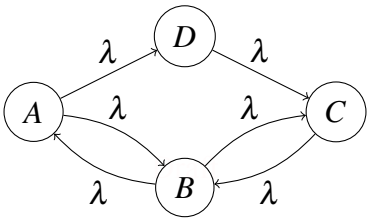


We have two properties:

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

$$\lim_{t\to\infty} P_n(t) = 0 \quad \forall n \geq 0$$

—



To solve a general case, we need to solve the following system of equations:

$$\dot{P} = A(t)P$$

we have:

$$A_{ij} = \lambda(t)\alpha_{ij}$$

$$\mathrm{d}\theta = \lambda(t)\mathrm{d}t$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(t)$$

$$\theta = \int_0^t \lambda(s)\mathrm{d}s = L(t)$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda(t)\alpha P$$

So:

$$P(t) = e^{\lambda(t)\alpha}P(0)$$

—

Let's consider now a case where we have a periodic

$$A(t) = \begin{cases} A_1 & 0 < \mathrm{mod}(t,T) < Q \\ A_2 & Q < \mathrm{mod}(t,T) < T \end{cases}$$

$$t = 0 \quad \Rightarrow \quad P(0) = P_0$$

$$0 < t < Q \quad \Rightarrow \quad \dot{P} = A_1 P \quad \Rightarrow \quad P(t) = e^{A_1 t} P_0$$

$$Q < t < T \quad \Rightarrow \quad \dot{P} = A_2 P \quad \Rightarrow \quad P(t) = e^{A_2 (t-Q)} P(Q)$$

$$T < t < T + Q \quad \Rightarrow \quad \dot{P} = A_1 P \quad \Rightarrow \quad P(t) = \underbrace{e^{A_2 (T-Q)} e^{A_1 Q}}_{B(T,Q)} P(0)$$

so

$$P(T) = B(T,Q)P(0)$$
$$P(2T) = B^2(T,Q)P(0)$$
$$P(nT) = B^n(T,Q)P(0)$$

—

CTMC model:

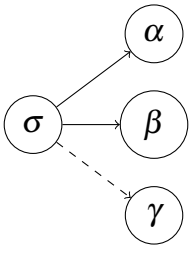
$$f = \{\sigma_1, \ldots, \sigma_n\}$$

$$\dot{P} = A(t)P$$

$$P(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = e_y$$

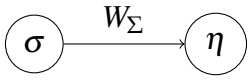
—

Gillespie algorithm:



$$X(0) = \sigma_0$$

We want to find when where will be the jump and which will be the next state.
Also in this case we can collapse the states in a single one; the transition rate is given by the sum of the transition rates of the original states.



So we can write:

$$\mathscr{P}(T) = W_\Sigma e^{-W_\Sigma T}$$

And we have:

$$t_{n+1} = t_n + T_n, \quad \text{where} \quad T_n \sim \mathscr{P}(T)$$

$$P(\sigma_i) = \Pr(Event(\sigma \to \sigma_i) ara\ t_n + T_n | oneE \sim t_n + T_n)$$

TODO: Che cazzo ha scritto il prof? ↑↑↑↑↑

$$\Pr\{\sigma \to \sigma_i \ \ (T, T+\mathrm{d}t)\} \quad = \quad (W_x \mathrm{d}t) e^{-W_\Sigma T} \quad = \quad \boxed{P(\sigma_i)(W_\Sigma e^{W_\Sigma T} \mathrm{d}t)}$$

MISSING: qualcosa

…

1. $$T_n \sim W_\Sigma e^{-W_\Sigma T} \quad \Rightarrow \quad t_{n+1} = t_n + T_n$$
2. $$P(\sigma_i) = \frac{W_{\sigma_i}}{W_\Sigma} \qquad t_n \to n$$

A first application of the Gillespie algorithm is the following:

Example: *Gillespie Algorithm: Contagion and Recovery (SIR Model)*

Consider a system with two possible events:

- **Contagion**: $(S,I) \to (S-1, I+1)$, with rate $W_{con} = \beta \frac{I}{N} S$
- **Recovery**: $(S,I) \to (S, I-1)$, with rate $W_{rec} = \gamma I$

The Gillespie algorithm proceeds as follows:

1. **Initialize**: $t_0 = 0$, $x_0 = (S_0, I_0)$
2. **Compute the rates**:
   - $W_{con} = \beta \frac{I_n}{N} S_n$
   - $W_{rec} = \gamma I_n$
   - $W_\Sigma = W_{con} + W_{rec}$
3. **Draw two random numbers** $U_1, U_2 \sim \mathscr{U}[0,1]$
4. **Determine the time to the next event**:
   The waiting time T_n is exponentially distributed:

$$T_n = \frac{-\ln U_1}{W_\Sigma}$$

Update the time: $t_{n+1} = t_n + T_n$

5. **Determine which event occurs**:
   Compute the probabilities:

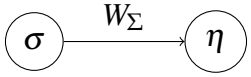
$$\Pr(\text{contagion}) = \frac{W_{con}}{W_\Sigma}, \qquad \Pr(\text{recovery}) = \frac{W_{rec}}{W_\Sigma}$$

   If $U_2 < \Pr(\text{contagion})$, a contagion event occurs; otherwise, a recovery event occurs.
6. **Update the state**:
   - If contagion: $(S_{n+1}, I_{n+1}) = (S_n - 1, I_n + 1)$
   - If recovery: $(S_{n+1}, I_{n+1}) = (S_n, I_n - 1)$
7. **Repeat** from step 2 until $I_n = 0$ or another stopping criterion is met.

Note: The formula for T_n comes from inverting the CDF of the exponential distribution:

$$CDF(T) = 1 - e^{-W_\Sigma T} \implies T = \frac{-\ln(1-U_1)}{W_\Sigma}$$

Since U_1 is uniformly distributed, so is $1 - U_1$, and it is common to write $T = \frac{-\ln U_1}{W_\Sigma}$.



$X(t_n) = \sigma$

$$\mathscr{P}(T_n) = W_\Sigma(t_n + T) e^{-\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z}$$

$$CDF(T_n) = 1 - e^{-\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z}$$

…
calling $\Psi(T) = \int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z$

$$1 - e^{-\Psi(T_n)} = U_n \quad \Rightarrow \quad e^{-\Psi(T_n)} = 1 - U_n \quad \Rightarrow \quad \Psi(T_n) = -\ln(1-U_n)$$

…

$$\beta(t) = \beta_n(1 + \delta\cos(\omega t))$$

$$x(t_n) = (S_n, I_n)$$

$$\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z = -\ln(1-U_n)$$

$$\int_{t_n}^{t_n+T_n} \left[\gamma I_n + \beta_n \frac{I_n}{N} S_n (1 + \delta\cos(\omega z))\right] \mathrm{d}z = -\ln(1-U_n)$$

$$\left[\gamma I_n + \beta_n \frac{I_n}{N} S_n\right] T_n + \beta_n \frac{I_n}{N} S_n \frac{1}{\omega} \left[\sin(t_n + T_n) - \sin(t_n)\right] = -\ln(1-U_n)$$

17 Lecture 26/05/2025

MISSING: everything

18 Lecture 30/05/2025

18.1 Random Walks

Let's consider a particle that moves on a line.

$$\rho(x,t) = \Pr[\text{Particle is at location } (x, x+\mathrm{d}x) \text{ at time } t]$$

we have:

$$\frac{\partial \rho}{\partial t} = \int [\rho(y,t)\Omega(y,x) - \rho(x,t)\Omega(x,y)]\,\mathrm{d}y$$

Let's consider now the probability that the particle just jumped in the location $(x, x+\mathrm{d}x)$ at time t:

$$p(x,t) = \Pr[\text{Particle just jumped in the location } (x, x+\mathrm{d}x) \text{ at time } t]$$

We define a jump weight as:

$$\eta(y)$$

so:

$$p(x,t) = \int_0^t \int_{-\infty}^{+\infty} \mathrm{d}y \eta(y) p(x-y, t-T)\omega(t)\mathrm{d}t + \delta(x)\delta(t)$$

Definition: ***Fourier and Laplace Transforms***

- **Fourier transform**
  Let's recall the Fourier transform:

  $$\mathscr{F}_x[f(x)] = \int_{-\infty}^{+\infty} f(x) e^{-ikx} \mathrm{d}x$$

  we have:

  $$\mathscr{F}_x[f(x) \star g(x)] = \hat{f}(k)\hat{g}(k)$$

- **Laplace transform**
  Let's recall the Laplace transform:

  $$\mathscr{L}_x[f(x)] = \int_0^{\infty} f(x) e^{-st} \mathrm{d}t$$

  we have:

  $$\mathscr{L}_x[f(t) \star g(t)] = \tilde{f}(s)\tilde{g}(s)$$

$$p(x,t) = [p(x,t) \star_s \eta(x)] \star_t \omega(t)$$

Let's call $\bar{p}(k,s) = \hat{\tilde{p}}(x,t)$, then we can write:

$$\bar{p}(k,s) = \hat{\eta}(k)\tilde{\omega}(s)\bar{p}(k,s) + 1 \qquad \Rightarrow \qquad \bar{p}(k,s)\{1 - \hat{\eta}(k)\tilde{\omega}(s)\} = 1$$

$$\bar{p}(k,s) = \frac{1}{1 - \hat{\eta}(k)\tilde{\omega}(s)}$$

$$\Psi(T) = \int_T^{+\infty} \omega(q)\mathrm{d}q = 1 - \int_0^T \omega(q)\mathrm{d}q$$

and we can rewrite $\rho(x,t)$ as:

$$\rho(x,t) = \int_0^t p(x,t-T)\Psi(T)\mathrm{d}T$$

The Laplace transform of $\Psi(T)$ is:

$$\mathscr{L}[\Psi(T)] = \mathscr{L}[1 - \int_0^T \omega(q)\mathrm{d}q] = \frac{1}{s} - \frac{1}{s}\tilde{\omega}(s)$$

$$\bar{\rho}(k,s) = \left(\frac{1 - \tilde{\omega}(s)}{s}\right)\left(\frac{1}{1 - \hat{\eta}(k)\tilde{\omega}(s)}\right)$$

Cauchy distribution (???)

$$\omega(T) \sim \frac{1}{T^{1+\alpha}} \qquad T \gg 1$$

$$\omega(T) = \frac{A}{1 + B\,T^2} \qquad T \gg 1 \qquad \Rightarrow \qquad \omega(T) \sim \frac{A}{B}\frac{1}{T^2}$$

Then its integral is infinite (cause compares a logarithm).

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \qquad \dot{x} = c\xi(t)$$

$$\langle x^2(t)\rangle \equiv t^{\alpha}$$

for $0 < \alpha < 1$ we have subdiffusion, for $1 < \alpha < 2$ we have superdiffusion.