UniTs - University of Trieste

Faculty of Scientific and Data Intensive Computing

Department of mathematics informatics and geosciences

Stocasthic Modeling and Simulation

Lecturer:
Prof. ...

Author:
Andrea Spinelli

March 21, 2025

github.com/Spina02 andreaspinelli2002@gmail.com

Preface

As a student of Scientific and Data Intensive Computing, I've created these notes while attending the **Stochastic Modeling and Simulation** course.
The course covers a wide range of topics, including Stochastic Nonlinear Models in different application fields (as physics, biomedicine, mathematics, ...), Stochastic Differential Equations, and Stochastic Simulation.

The topics covered in these notes include:
- Recap of Deterministic Models
- Stochastic Differential Equations and White Noise
- Fokker Planck Equation
- Noise-induced Transitions
- Colored noises
- Bounded Stochastic Processes
- Spatio-temporal Stochastic Processes
- Parameter Estimation form Data
- Stochasticity …
- …
- Continuous state space-discrete time Stochastic Processes
- Discrete Time Markov Chains
- Continuous Time Markov Chains
- Mean Field Approximation

While these notes were primarily created for my personal study, they may serve as a valuable resource for fellow students and professionals interested in Stochastic Modeling and Simulation.

Contents

1 Introduction **1**
- 1.1 Modelling complex systems . . . 2

2 lecture 10/03/2025 **3**
- 2.1 Tumor Size over Time . . . 3
- 2.2 Stabily and Eq. Points . . . 5
  - 2.2.1 Equilibrium Points . . . 5
  - 2.2.2 Stability . . . 5
- 2.3 Local Analysis Near the Equilibrium Point . . . 6
- 2.4 Non-scalar Systems . . . 8
- 2.5 Exponential of a Matrix . . . 9

3 Lecture 10/03/2024 **11**
- 3.1 Dirak Delta . . . 12
- 3.2 Random Processes . . . 12

4 Lecture 14/03/2024 **14**

5 Lecture 17/03/2025 **22**
- 5.1 Linear Logistic Perturbed Model . . . 23
- 5.2 Ito's Formula (Physical) Demonstration . . . 25
- 5.3 Probability Density Function and Markov Processes . . . 26

6 Lecture 21/03/2025 **27**
- 6.1 Evolution of the Probability Density Function: The Fokker-Planck Equation . . . 27
- 6.2 Liouville Equation for Systems with Uncertain Initial Conditions . . . 30

1
Introduction

Different fields as Epidemics spreading, Cancer growth, and many others, can be modeled using Stochastic Differential Equations (SDEs).
An example is SIR model for epidemics spreading, formulized during a huge cholera infection in early 19th century. The model is based on three compartments: Susceptible, Infected, and Recovered individuals. This is modeled using a system of SDEs.
Often stochastic models represents well the problem, but real data are noisy and chaotic. For instance often we have to deal with data varying spatially and temporally, and we have to deal with the problem of parameter estimation.
Example: Oscillating chemical system

$$A + Y \xrightarrow{k_1} X + P, \quad X \xrightarrow{k_2} 2P$$
$$A + X \xrightarrow{k_3} 2Z, \quad X + Z \xrightarrow{k_4} 2A$$

Other examples are preys and predators models,
…
Renewable energies introduces a high volatility and unpredictability in energy production
If the demand of energy exceedes the production, we need to activate standard plans, to reduce consumption by switching off devices (e.g. water boilers remotely controlled), or to activate additional production plants. All these scenarios are complex systems.

Definition: ***Complex System***

A **complex system** is a system composed of interconnected parts that as a whole exhibit one or more properties (behavior among the possible properties) not obvious from the properties of the individual parts.

Emergent behavior is a property of complex systems, and it is not predictable from the behavior of the individual parts.

Definition: ***Adaptivity and self-organization***

- **Adaption** meand achieving a fit between the system and its environment.
- **Self-organization** is the process where a system changes its structure spontaneously, in order to adapt to the environment.

An instance of self-organization is the formation of a flock of birds, where each bird follows simple rules, but the flock as a whole exhibits a complex behavior.

Observation: ***Noise and Nonlinearities***

Noise and nonlinearities can (sometimes) favor the emergence of ”order”.

1.1 Modelling complex systems

Math for quantitatve models

We will be interersted in the temporal behavior of the system, and we will use some key ingredients for the maths:

- **Entities** can be modelled as *discrete* objects or *continuous* quantities.
- **...** ...
- **Time** can be *discrete* or *continuous*.

…

Data-Based vs Model-Based Approaches

Will data approaches make the kind of modelling obsolete?
Hybrid approacehs are possible:

1. Math models can be joined/hcybrized with machine learning models.
2. Deep Network to learn modules aor whole math models
   An exampleis *Physics-informed neural networks*: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations
3. ...

Definition: ***Dynamical System***

A **dynamical system** is a system whose state evolves over time according to a rule that depends on the current state.

[definition of differential equations]

Definition: ***Differential Equation***

A **differential equation** is an equation that relates one or more unknown functions and their derivatives.

In practice, differential equations are mathemathical instruments that describves the world around us. A note differential equation (maybe the first ever invented) is the Newton law:

$$F = ma \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = F(x(t))$$

2
lecture 10/03/2025

2.1 Tumor Size over Time

Let's consider an example case of a tumor growth. Let's assume that $X(t)$ is the size of a tumor at time t.
The differential equation that describes the growth of the tumor is:

$$X(t+dt) = X(t) + \Phi X(t) - MX(t)$$

where Φ is the growth rate and M is the decay rate.
We can rewrite the equation as:

$$\frac{X(t+dt) - X(t)}{dt} = X(t)\frac{(\Phi - M)}{dt}$$

Let's rewrite Φ and M as:

$$\begin{aligned} \Phi &= bdt + \cancel{O(dt^2)} \\ M &= mdt + \cancel{O(dt^2)} \end{aligned} \quad \text{(We neglect the higher order terms)}$$

Draft

We obtain the following differential equation:

$$\frac{X(t+dt) - X(t)}{dt} = X(t)(b-m) \quad \Rightarrow \quad \frac{dX}{dt} = X(b-m)$$

Often we have a starting condition $X(0) = X_0$. Defining $a = b - m$, the sistem becomes:

$$\begin{cases} \dfrac{dX}{dt} = aX \\ X(0) = X_0 \end{cases}$$

Since makes no sense to have a negative time or tumor size, we have the constraints:

$$\begin{cases} t \in \mathbb{R}^+ \cup \{0\} \\ X \in \mathbb{R}^+ \cup \{0\} \end{cases}$$

The solution is given by:

$$X(t) = X_0 e^{at}$$

$$X(t+dt) = X(t) + bdtX - mdtX - \theta dtX$$

$$\begin{cases} \dot{X} = (a-\theta)XX(0) = X_0 \end{cases}$$

dunque la soluzione è:

$$X(t) = X_0 e^{(a-\theta)t}$$

In questo caso, per $a > \theta$, il tumore cresce esponenzialmente, mentre per $a < \theta$, il tumore decresce esponenzialmente.
Nella realtà però il valore θ non è costante nel tempo, ma varia nel tempo. In tal caso il sistema diventa:

$$\begin{cases} \dot{X} = (a - \theta(t))X \\ X(0) = X_0 \end{cases}$$

e la soluzione è:

$$X(t) = X_0 e^{\int_0^t (a-\theta(s))ds}$$

Più in generale, un sistema del tipo:

$$z(y) = e^{G(y)} b$$

si ha

$$\frac{dz}{dy} = \frac{d}{dy} e^{G(y)} b = e^{G(y)} b \frac{dG}{dy} = G'(y) z(y)$$

Esempio:

$$\begin{cases} Z'(t) = \sin(t) Z(t) \\ Z(0) = Z_0 \end{cases}$$

Si ha:

$$\begin{cases} Z(t) = e^{-\cos(t)} B \\ Z_0 = e^{-1} B \end{cases} \qquad \Rightarrow \qquad Z(t) = e^{1-\cos(t)} z_0$$

2.2 Stabily and Eq. Points

2.2.1 Equilibrium Points

Given the system:

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}$$

An equilibrium point is a point x^* such that $f(x^*) = 0$.

Tip: ***Eq. points***

A system can have multiple equilibrium points.

We have three kinds of equilibrium:

- **Stable equilibrium**: if the system is in the neighborhood of the equilibrium point, it will remain there.
- **Neutral equilibrium**: if the system is in the neighborhood of the equilibrium point, it will remain there, but it will not return to it.
- **Unstable equilibrium**: if the system is in the neighborhood of the equilibrium point, it will move away from it.

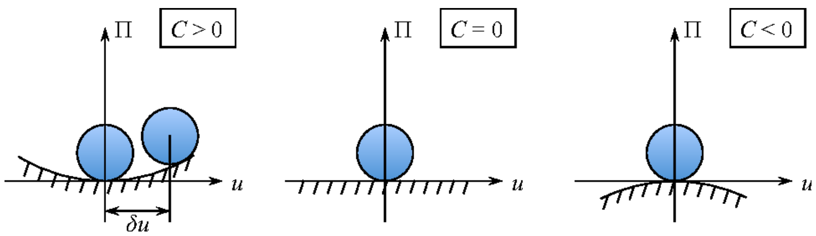


Figure 2.1: Stable, Natural and Unstable Equilibrium Points [1]

2.2.2 Stability

We say that a system is ***Globally Asymptotically Stable*** (or *Globally Attractive*) if it is stable and if it converges to the equilibrium point from any initial condition.

2.3 Local Analysis Near the Equilibrium Point

In this section, we study the system in the neighborhood of the equilibrium point. Let the initial condition be a small perturbation around the equilibrium:

$$X(0) = X_e + \varepsilon.$$

We introduce a deviation function $U(t)$ defined by

$$X(t) = X_e + U(t),$$

with the initial condition

$$U(0) = \varepsilon.$$

Thus, the evolution of the perturbation is governed by

$$\begin{cases} \dot{U} = f(X_e + U), \\ U(0) = \varepsilon. \end{cases}$$

Assume that the dynamics of the system are given by

$$\dot{X} = X(b(X) - m(X)).$$

Then the perturbed system becomes

$$\dot{U} = (b(X_e + U) - m(X_e + U))(X_e + U).$$

Expanding $b(X_e + U)$ and $m(X_e + U)$ in a Taylor series around X_e, we have:

$$b(X_e + U) \approx b(X_e) + b'(X_e)U,$$
$$m(X_e + U) \approx m(X_e) + m'(X_e)U.$$

Substituting these into the equation for $\dot{U}$, we get:

$$\begin{aligned} \dot{U} &= \big[b(X_e) + b'(X_e)U\big] - \big[m(X_e) + m'(X_e)U\big](X_e + U) \\ &= \big[b(X_e) + b'(X_e)U\big] - \big[m(X_e)X_e + m(X_e)U + m'(X_e)X_eU + m'(X_e)U^2\big]. \end{aligned}$$

Since the equilibrium condition implies that

$$(b(X_e) - m(X_e))X_e = 0,$$

the above expression simplifies (neglecting the higher-order term $m'(X_e)U^2$) to:

$$\dot{U} \approx [b'(X_e) - m(X_e) - m'(X_e)X_e]\,U.$$

Under the assumption that $b'(X_e)$ is negative, we can express this as

$$\dot{U} \approx -X_e\Big(|b'(X_e)| + m'(X_e)\Big)U.$$

The solution of this linearized differential equation is given by:

$$U(t) = U(0)\,e^{-X_e(|b'(X_e)|+m'(X_e))t}.$$

Hence, we identify the decay rate (or the inverse of the characteristic time constant) as

$$X_e\Big(|b'(X_e)| + m'(X_e)\Big),$$

and the characteristic time τ is:

$$\tau = \frac{1}{X_e\left(|b'(X_e)| + m'(X_e)\right)}.$$

This time constant represents the rate at which perturbations decay in the vicinity of the equilibrium point.

$$X(t) = X_e + U(t) \Rightarrow \dot{U} = f(X_e + U) = \underbrace{f(X_e)}_{=0} + f'(X_e)U + O(U^2)$$

$$\dot{U} = f'(X_e)U \Rightarrow U(t) = U(0)e^{f'(X_e)t}$$

So we have:

$$\begin{cases} f'(X_e) < 0 & \Rightarrow \quad X_e \text{ is Locally Asintotically stable} \\ f'(X_e) > 0 & \Rightarrow \quad X_e \text{ is Unstable} \end{cases}$$

2.4 Non-scalar Systems

Consider the system:

$$\begin{cases} \dot{x} = f(x) \\ x \in \{ \subseteq \mathbb{R}^n \end{cases}$$

As in the scalar case, we can linearize the system around the equilibrium point x_e:

$$f(X_e) = 0$$

$$X = X_e + U, \qquad |U| \ll 1$$

We have to consider the Jacobian matrix of f:

…

2.5 Exponential of a Matrix

Let's consider a linear system of the form:

$$\dot{x} = Ax$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix. The solution of this system is given by:

$$x(t) = e^{At}x(0)$$

where e^{At} is the exponential of the matrix A.

Definition: ***Exponential of a Matrix***

Given a matrix $A \in \mathbb{R}^{n\times n}$, the exponential of A is defined as:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

This comes from the Taylor series expansion of the exponential function.

$$\frac{d}{dt}e^{At} = \sum_{m=1}^{\infty} A^m m \frac{t^{m-1}}{m(m-1)!} = \sum_{k=0}^{\infty} AA^k \frac{t^k}{k!} = Ae^{At}$$

$$\begin{array}{lcl} A & = & H \cdot \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \cdot H^{-1} \\ A^2 & = & H \cdot \mathrm{Diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2) \cdot H^{-1} \\ A^m & = & H \cdot \mathrm{Diag}(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m) \cdot H^{-1} \end{array}$$

So we have:

$$e^{At} = \sum_{m=0}^{\infty} \frac{A^m t^m}{m!} =$$

An example of a matrix exponential is given by the Newton's law:

$$m\ddot{x} = F$$

Let's consider a more complex case with air resistance γ:

$$m\ddot{x} = -\gamma\dot{x} - F(x)$$

We can rewrite this system as:

3 Lecture 10/03/2024

Last lecture we saw that we can approximate a non-linear system with a linear one, locally in an equilibrium point.
Let's consider an electric circuit, if the elecrical field is not static, it generates a variable magnetic field and viceversa.

$$\begin{cases} \Phi = Li \\ Ri = -\dfrac{d\Phi(\vec{B})}{dt} = -\dfrac{d(Li)}{dt} \end{cases}$$

We can write this system in the cauchy form:

$$\begin{cases} \dfrac{di}{dt} = -\dfrac{R}{L}i \\ i(0) = i_0 \end{cases}$$

The solution is:

$$i(t) = i_0 e^{-\frac{R}{t}}$$

$$L\frac{di}{dt} + Ri + V_c = 0$$

$$VC = Q$$

$$y = \begin{bmatrix} i \\ q \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{LC} \\ 1 & 0 \end{bmatrix}$$

$$m\ddot{x} = \gamma\dot{x} + kx$$

3.1 Dirak Delta

If ypu consider a foorball player that kicks a ball, the force is not constant, and it is not possible to model it with a constant force. We can model it with a Dirak Delta function.

$$\begin{cases} m\ddot{x} = m\dot{v} = F(t) \\ v(0) = 0 \end{cases} \quad \Rightarrow \quad mv_{after} = \int_0^a F(t)dt \quad \Rightarrow \quad \boxed{v_{after} = \frac{1}{m}\int_0^a F(t)dt}$$

So the Dirak Delta function is a function that is zero everywhere except in zero, where it is infinite. It is used to model impulses.

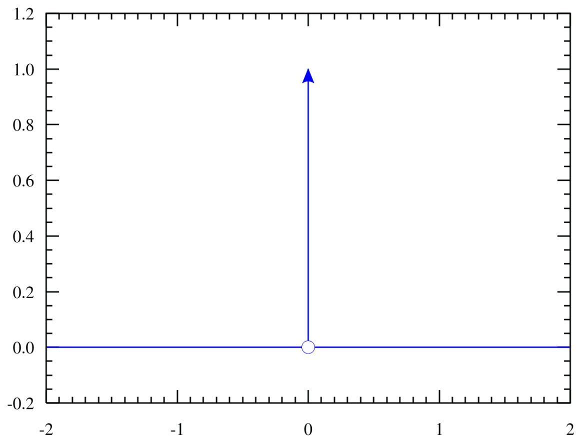


Figure 3.1: Dirak Delta Function

...

Let's consider a function $f(t)$ such that $f(0) < \infty$ and $f'(0) < \infty$. We can calculate:

$$\int_{\mathbb{R}} \delta(t)f(t)dt = \int_{\mathbb{R}} \delta(t)[f(0) + f'(0)t]dt = \int_{\mathbb{R}} \delta(t)f(0)dt + \int_{\mathbb{R}} \delta(t)f'(0)tdt$$

Now, we use two key properties of the Dirac delta function:

1. **Sifting Property:**
$$\int_{\mathbb{R}} \delta(t)\,dt = 1.$$
2. **First Moment:**
$$\int_{\mathbb{R}} \delta(t)t\,dt = 0,$$
   which follows because $t\delta(t)$ is an odd function.

Substituting these results, we obtain:

$$\int_{\mathbb{R}} \delta(t)f(t)\,dt = f(0)\cdot 1 + f'(0)\cdot 0 = f(0).$$

...

3.2 Random Processes

$$< x(t) >= \frac{1}{N}\sum_{i=1}^{N} x_{Ri}(t)$$

$$m_i\ddot{x}_i = -\gamma\dot{x}_i \quad \Rightarrow \quad m_i\dot{v}_i = -\gamma v_i \quad \Rightarrow \quad \dot{v}_i = -\frac{\gamma}{m_i}v_i \quad \Rightarrow \quad \boxed{v_i(t) = v_i(0)e^{-(\gamma/m_i)t}}$$

$$\boxed{m\dot{v} = -\gamma v + \hat{F}_s(t)}$$

$$\begin{aligned}
m\ddot{x} &= -k\dot{x} + \hat{F}_p(x) + \hat{F}_s(t) \\
m\ddot{x} &= -k\dot{x} + kf(x) + kf_s(t) \\
\frac{m}{k}\ddot{x} &= -\dot{x} + f(x) + f_s(t) \\
\frac{m}{k}\ddot{x} \ll 1 \quad &\Rightarrow \quad \frac{m}{k}\ddot{x} \approx 0 \\
\dot{x} \simeq f(x) + f_s(t) &= f(x) + \omega\xi(t)
\end{aligned}$$

4 Lecture 14/03/2024

$$m\dot{v} = -\gamma v + F_s(t)$$

if $m \ll 1$ then $\dfrac{m}{\gamma} \approx 0$

$$\dot{x} = f(x) + \omega\xi(t)$$

$$\frac{dx}{dt} = f(x) + g(x)\xi(t)$$

SI model:

$$\begin{cases} \dot{S} = -\beta SI + \theta I \\ \dot{I} = \beta SI - \theta I \end{cases}$$

where β is the infection rate and θ is the recovery rate.

$$\frac{dI}{dt} = \beta(1-I)I - \theta I$$

$\beta$is a stochastic variable, we can write it as:

$$\beta \to \beta + \omega\xi(t)$$

where ω is the amplitude of the noise and $\xi(t)$ is a white noise.
We can write the equation as:

$$\frac{dI}{dt} = (\beta + \omega\xi(t))(1-I)I - \theta I$$

Properties of $\xi(t)$

The noise process $\xi(t)$ is characterized by the following properties:

1. **White Noise:**
   $\xi(t)$ is a white noise process, meaning its values at different time instants are uncorrelated.
2. **Gaussian Noise:**
   $\xi(t)$ follows a Gaussian distribution, so all its finite-dimensional distributions are Gaussian.
3. **Zero Mean:**
   The expected value of the process is zero:
   $$\langle \xi(t) \rangle = 0.$$
4. **Temporal Uncorrelation:**
   For any two distinct time instants $t \neq q$, the noise is uncorrelated:
   $$\langle \xi(t)\xi(q) \rangle = 0.$$

5. **Delta-Correlated:**
   The autocorrelation function is given by the Dirac delta function:
   $$\langle \xi(t)\xi(q)\rangle = \delta(t-q).$$

6. **Infinite Instantaneous Variance:**
   The variance at any fixed time is formally divergent:
   $$\langle \xi^2(t)\rangle \gg 1,$$

   reflecting the idealized nature of white noise.

…

$$f(x) = 0$$
$$g(x) = \omega \neq 0$$
$$\dot{x} = \omega\xi(t)$$
$$x(t) = x(0) + \omega\int_0^t \xi(s)ds$$
$$< x(t) >= x(0) + \omega\int_0^t \underbrace{< \xi(s) >}_{=\,0} ds = x(0)$$
$$x(t)x(q) = \omega^2\int_0^t \xi(s)ds\int_0^q \xi(\theta)d\theta = \omega^2\int_0^t\int_0^q \xi(s)\xi(\theta)dsd\theta$$
$$< x(t)x(q) >= \omega^2\int_0^t\int_0^q \underbrace{< \xi(s)\xi(\theta) >}_{=\,\delta(s-\theta)} dsd\theta = \omega^2\int_0^t\int_0^q \delta(s-\theta)d\theta ds$$

We have $< x(t)x(q) >= \omega^2\min(t,q)$.

If $q \geq t$:

$$< x(t)x(q) >= \omega^2\int_0^t\int_0^t \delta(\theta-s)d\theta = \omega^2\int_0^t ds = \omega^2 t$$

Else if $0 < q < t$:

$$< x(t)x(q) >= \omega^2\int_0^q\left\{\int_0^q \delta(\theta-s)d\theta\right\}ds = \omega^2\int_0^q ds = \omega^2 q$$

…

$$\begin{aligned}
\langle (x(t)-x(q))^2\rangle &= \langle x^2(t)\rangle + \langle x^2(q)\rangle - 2\langle x(t)x(q)\rangle = \omega^2(t+q-2\min(t,q)) \\
&= \begin{cases} 0 & \text{if } t=q \\ \omega^2(t-q) & \text{if } t>q \\ \omega^2(q-t) & \text{if } t<q \end{cases} = \omega^2|t-q|
\end{aligned}$$

…

$$\left\langle \left(\frac{x(t+h) - x(t)}{h} \right)^2 \right\rangle = \frac{\omega^2}{h}$$

This is an incremental ratio, so, if we take the limit as $h \to 0^+$ we get:

$$\lim_{h \to 0^+} \left\langle \left(\frac{x(t+h) - x(t)}{h} \right)^2 \right\rangle = +\infty$$

Wiener Process

The **Wiener Process** (also known as Brownian motion) is a continuous-time stochastic process widely used in physics and finance to model random behavior. It is named after Norbert Wiener, who introduced it in the 1920s. The process is defined by the stochastic differential equation

$$\begin{cases} \dfrac{dw}{dt} = \xi(t), \\ w(0) = 0, \end{cases}$$

where $\xi(t)$ represents a white Gaussian noise.

Properties of the Wiener Process:
The Wiener process is a Gaussian process with the following key properties:

1. **Zero Mean:**
   The expected value of the process is zero:
   $$\langle w(t) \rangle = 0.$$
2. **Gaussian Distribution:**
   For any fixed time t, $w(t)$ is normally distributed.
3. **Autocorrelation:**
   The autocorrelation function is given by
   $$\langle w(t)w(q) \rangle = \min(t, q).$$

4. **Independent Increments:**
   The increments of the process are independent. In particular,
   $$\langle w(q) - w(t) \rangle = 0,$$
   and these increments are also Gaussian.
5. **Increment Variance:**
   The variance of the increment over the interval $[t, q]$ is proportional to the time difference:
   $$\langle (w(q) - w(t))^2 \rangle = |q - t|.$$
6. **Increment Distribution:**
   More precisely, for $q > t$, the increment is distributed as
   $$w(q) - w(t) \sim \mathcal{N}\Big(0, |q - t|\Big),$$
   and in particular,
   $$w(t) - w(0) \sim \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{w^2}{2t}\Big).$$

Increment Analysis and the Derivative of the Wiener Process

Consider a small time increment defined as

$$q = t + dt, \qquad dt > 0.$$

The increment of the Wiener process over this interval is given by

$$dw = w(t+dt) - w(t).$$

Since the process is Gaussian, the increment satisfies

$$dw \sim G(\mu = 0, \sigma = dt),$$

which implies that the finite difference quotient behaves as

$$\frac{dw}{dt} \sim G\left(\mu = 0, \sigma = \frac{1}{dt}\right).$$

This relation highlights that, in the limit as $dt \to 0$, the notion of a derivative for the Wiener process becomes problematic due to the divergence in the standard deviation.
Furthermore, for a finite interval h, we consider the probability

$$\Pr\left(\left|\frac{w(t+h) - w(t)}{h}\right| > M\right) \quad \Rightarrow \quad \Pr\Big(|w(t+h) - w(t)| > hM\Big).$$

This formulation reinforces the scaling behavior of the process's increments and underscores the fact that the Wiener process has almost surely nowhere differentiable paths.

Euler-Maruyama Method

In stochastic differential equations, the dynamics of a system are often described by equations of the form

$$\boxed{dp = F dt},$$

where dp represents the infinitesimal change in momentum and F is the force.
Similarly, the evolution of a state variable x is given by

$$dx = f(x,t)\,dt + g(x,t)\,dw,$$

with $f(x,t)$ denoting the drift term, $g(x,t)$ the diffusion coefficient, and dw the stochastic increment. The stochastic increment is defined as

$$dw = G(t)\sqrt{dt},$$

so that the update of x over a small time interval dt can be written as

$$x(t+dt) = x(t) + f(x,t)\,dt + g(x,t)G(t)\sqrt{dt}.$$

When discretizing time, let t_j denote the j-th time step, and define

$$G_j = G(t_j) \sim \mathcal{N}(0,1).$$

An increment over a discrete time-step is then approximated by

$$dx \simeq x_{j+1} - x_j.$$

Thus, the discretized form of the stochastic differential equation becomes

$$\begin{cases} x_{j+1} = x_j + f(x_j,t_j)h + g(x_j,t_j)G_j\sqrt{h}, \\ x_0 = x(0), \end{cases}$$

where h is the time-step size.
This numerical scheme is known as the ***Euler-Maruyama method***.

$$\dot{x} = x(1-x)$$

$$dx = x(1-x)dt$$

$$\frac{dx}{x(1-x)} = dt$$

$$\frac{dx}{x} + \frac{dx}{1-x} = dt$$

$$d(\ln|x| - \ln|1-x|) = dt \quad \Rightarrow \quad \ln\frac{x}{1-x} = \ln\frac{x_0}{1-x_0}$$

...

$$dx = a(x)dt + b(x)dw$$

$$y = \Psi(x)$$

$$dy = \Psi(x+dx) - \Psi(x)$$

$$dy = \Psi(x + a(x)dt + b(x)dw) - \Psi(x) =$$

...

$$dy = \Psi'(x)[a(x)dt + b(x)dw] + \frac{1}{2}\Psi''(x)\left[b^2(x)dt^2 + \underbrace{\cancel{a^2(x)dt^2}}_{O(dt^2)} + \underbrace{\cancel{2a(x)b(x)dtdw}}_{O(dt^{2/3})}\right]$$

$$dy = \Psi'(x)a(x)dt + \Psi'(x)b(x)dw + \Psi''(x)\frac{b^2(x)}{2}(dw)^2$$

$$(dw)^2 = (dt + \Omega(t))$$
$$dw \sim \mathcal{N}(0, dt)$$

$$dy = \left[\frac{\partial\Psi}{\partial x}a(x) + \frac{b^2(x)}{2}\Psi''(x)\right] + \Psi'(x)b(x)dw + \cancel{O(dt^{2/3})}$$

...

Malthus model

$$\dot{x} = bx - mx = (b-m)x = rx$$

$$\begin{cases} b \to b + \text{fluctuations}(t) \\ m \to m + \text{fluctuations}(t) \end{cases}$$

$$\frac{dx}{dt} = (r + \omega\xi(t))x$$

$$dx = rxdt + \omega xdw$$

$$y = \Psi(x) = \ln x$$

$$\begin{cases} a(x) = rx \\ b(x) = \omega x \end{cases}$$

$$\Psi'(x) = \frac{1}{x}$$

$$\Psi''(x) = -\frac{1}{x^2}$$

$$dy = \left[\frac{1}{\cancel{x}} r\cancel{x} + \frac{1}{2}\omega^2 \cancel{x^2}\left(-\frac{1}{\cancel{x^2}}\right)\right] dt + \frac{1}{\cancel{x}}\omega\cancel{x}dw \quad = \quad \boxed{\left(r - \frac{\omega^2}{2}\right) dt + \omega dw}$$

$$y(t) = \underbrace{y(0)}_{= e^{x_0}} + \left(r - \frac{\omega^2}{2}\right) t + \omega^2 w(t)$$

...

$$\boxed{x(t) = e^{y(t)}} \quad \Rightarrow \quad x(t) \to 0$$

5
Lecture 17/03/2025

In this lecture we analyze a stochastic version of the Malthusian law. In the deterministic case, the Malthusian growth is given by

$$\dot{x} = rx,$$

which has the solution $x(t) = x(0)e^{rt}$. Here, we introduce a multiplicative noise term to account for random fluctuations, leading to the stochastic differential equation

$$\dot{x} = \Big(r + \omega\xi(t)\Big)x,$$

where $\xi(t)$ is a white noise process. By definition of white noise, we have

$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t').$$

Since the noise has zero mean, the average growth rate remains r, so that

$$\langle r + \omega\xi(t) \rangle = r,$$

which implies

$$\langle x(t) \rangle = x(0)e^{rt}.$$

However, a paradox arises when comparing the average behavior with the typical (or almost sure) behavior of the system:

$$\begin{cases} x(t) \to 0 \quad \text{(typical behavior)}, \\ \langle x(t) \rangle = \infty \quad \text{(ensemble average)}. \end{cases}$$

This paradox is a consequence of the strong fluctuations induced by the multiplicative noise, which make the mean value unrepresentative of a typical realization.
To further analyze the dynamics, we perform a logarithmic transformation by setting

$$y(t) = \ln x(t).$$

Using Itô's calculus, the transformed variable satisfies

$$y(t) = y_0 + \left(r - \frac{\omega^2}{2}\right)t + \omega W(t),$$

where $W(t)$ is the Wiener process and $y_0 = \ln x(0)$. As $W(t)$ is normally distributed with mean 0 and variance t, it follows that

$$y(t) \sim \mathcal{N}\left(y_0 + \left(r - \frac{\omega^2}{2}\right)t,\, \omega^2 t\right).$$

Since $x(t) = e^{y(t)}$, the variable $x(t)$ is log-normally distributed. For a log-normal random variable, the mean is given by

$$\mu_{\log_N} = e^{\mu_G + \frac{\mathrm{Var}_G}{2}},$$

where μ_G and Var_G are the mean and variance of the corresponding Gaussian variable $y(t)$.
A useful side note is that if we define a new variable u with the density

$$\rho(u) = a\,e^{-au}H(u),$$

(where $H(u)$ is the Heaviside function) then its expected value is

$$\langle u \rangle = \frac{1}{a}.$$

This relation, although coming from a different context, similarly illustrates how averages may differ significantly from the most probable (median) value.
In fact, the median of a log-normally distributed variable is simply the exponential of the median of the underlying Gaussian distribution. Therefore, we have

$$\begin{cases} \text{Median}[x] = e^{\mu_{Gauss}}, \\ \text{Median}[x(t)] = e^{y_0 + \left(r - \frac{\omega^2}{2}\right)t}. \end{cases}$$

Notably, if $r - \frac{\omega^2}{2}$ is negative, the median of $x(t)$ decays to zero, even though the mean diverges. This discrepancy between the typical outcome and the ensemble average is a key feature of systems driven by multiplicative noise.

5.1 Linear Logistic Perturbed Model

We begin with the deterministic version of the logistic model in its linearized form:

$$\dot{x} = (b - m)x = rx, \qquad r > 0,$$

which implies that in the absence of density-dependent regulation, the solution grows exponentially and diverges as $x \to \infty$.
To incorporate environmental fluctuations, we introduce a stochastic perturbation into the model. The perturbed model is written as

$$\dot{x} = \big(r - \alpha x + \omega \xi(t)\big)x,$$

or equivalently, in differential form,

$$dx = (r_0 - \alpha x)x\,dt + \omega x \xi(t),$$

where r_0 represents the intrinsic growth rate, $\alpha > 0$ is the density-dependent regulation coefficient, ω quantifies the intensity of the noise, and $\xi(t)$ denotes a white noise process.
To simplify the analysis, it is useful to perform a logarithmic transformation by defining

$$y = \ln(x) \quad \Longleftrightarrow \quad x = e^y.$$

Applying Itô's formula to $y = \ln(x)$ (with the usual correction term due to the stochastic calculus), we obtain

$$dy = \left(r_0 - \frac{\omega^2}{2} - \alpha e^y\right)dt + \omega dW,$$

where dW is the Wiener process corresponding to $\xi(t)$.
This transformed stochastic differential equation can be formally integrated to yield

$$y(t) = y_0 + \left(r_0 - \frac{\omega^2}{2}\right)t + \omega W(t) - \alpha \int_0^t e^{y(s)} ds.$$

Notice that the first three terms,

$$y_0 + \left(r_0 - \frac{\omega^2}{2}\right)t + \omega W(t),$$

represent the contribution of the intrinsic growth and the noise, while the integral term

$$\alpha \int_0^t e^{y(s)} ds$$

captures the effect of density-dependent regulation.

If the noise intensity is sufficiently strong, specifically when

$$\frac{\omega^2}{2} > r_0,$$

then the combined effect of the noise and the regulation term drives $y(t)$ to $-\infty$ as $t \to \infty$. Consequently,

$$x(t) = e^{y(t)} \to 0^+,$$

which indicates that the population eventually goes extinct.

Let $\rho(x,t)$ denote the probability density function (PDF) of $x(t)$. As time progresses, the dynamics force the distribution to concentrate at $x = 0$, and one can show that

$$\lim_{t\to\infty} \rho(x,t) = \delta(x),$$

where $\delta(x)$ is the Dirac delta distribution. This result confirms that extinction is the almost sure outcome under strong stochastic perturbations.

It is also instructive to discuss the notion of an equilibrium in this stochastic context. For a deterministic system described by

$$\frac{dx}{dt} = f(x),$$

an equilibrium point x_e satisfies $f(x_e) = 0$, so that if $x(0) = x_e$, then $x(t) = x_e$ for all t. In contrast, for a stochastic differential equation of the form

$$dx = f(x)\,dt + g(x)\,dW,$$

a stochastic equilibrium (or steady state) x_{ES} is defined by the conditions

$$f(x_{ES}) = 0 \quad \text{and} \quad g(x_{ES}) = 0.$$

When these conditions hold, small deviations from equilibrium can be analyzed by setting

$$x = x_{ES} + U,$$

which leads to a linearized equation for the perturbation U:

$$dU = aU\,dt + bU\,dW.$$

In our logistic perturbed model, extinction ($x = 0$) acts as a stochastic equilibrium point. Linearizing the dynamics around $x = 0$, we find

$$dU = r_0 U\,dt + \omega U\,dW,$$

which describes the evolution of small perturbations near the extinct state.

In summary, the introduction of multiplicative noise in the logistic model not only modifies the dynamics but, under strong noise conditions, leads to extinction—even when the deterministic model predicts unbounded growth. The interplay between the intrinsic growth rate, the density-dependent term, and the noise intensity determines the long-term fate of the system.

5.2 Ito's Formula (Physical) Demonstration

We start by considering a stochastic differential equation (SDE) for a variable x:

$$dx = \underbrace{a(x)\,dt}_{O(dt)} + \underbrace{b(x)\,dW}_{O(\sqrt{dt})}.$$

Our goal is to derive the differential of a function $\Psi(x)$ using Ito's formula. Recall that if $\Psi(x)$ is twice differentiable, then

$$d\Psi = \Psi'(x)\,dx + \frac{1}{2}\Psi''(x)\,(dx)^2 + \dots.$$

Because dx contains a term of order $\sqrt{dt}$, the term $(dx)^2$ is of order dt. In particular, the properties of the Wiener process imply that

$$\boxed{\langle (dW)^2 \rangle = dt}.$$

To analyze the fluctuations in $(dW)^2$, we decompose it as follows:

$$(dW)^2 = dt + \left[(dW)^2 - dt\right] = dt + d\Omega,$$

where we define the random variable

$$y \equiv (dW)^2 - dt,$$

which satisfies $\langle y \rangle = 0$. Its variance is computed by

$$\mathrm{Var}(y) = \langle y^2 \rangle = \left\langle \left[(dW)^2 - dt\right]^2 \right\rangle.$$

Expanding the square, we have

$$\langle (dW)^4 - 2dt\,(dW)^2 + (dt)^2 \rangle.$$

Using the moment properties of the Wiener process:

$$\langle (dW)^2 \rangle = dt \quad \text{and} \quad \langle (dW)^4 \rangle = 3(dt)^2,$$

we obtain

$$3(dt)^2 - 2dt\,(dt) + (dt)^2 = 3(dt)^2 - 2(dt)^2 + (dt)^2 = 2(dt)^2.$$

Returning to the expansion for $d\Psi$, and substituting $dx = a(x)\,dt + b(x)\,dW$, we identify:

- The term $\Psi'(x)\,dx$ contributes a drift component and a stochastic component of order $O(\sqrt{dt})$.
- The term $\frac{1}{2}\Psi''(x)(dx)^2$ contributes an extra drift term of order $O(dt)$ due to the quadratic variation of dW.

Thus, the full expression for the differential of $\Psi(x)$ is given by

$$d\Psi = \left[\Psi'(x)a(x) + \frac{1}{2}\Psi''(x)b^2(x)\right]dt + \Psi'(x)b(x)dW.$$

This result is the celebrated Ito's formula. It shows that, unlike in ordinary calculus, the second derivative term multiplied by $\frac{1}{2}b^2(x)$ appears as a correction due to the non-negligible quadratic variation of the Wiener process. This additional term is what distinguishes stochastic calculus from its deterministic counterpart.

5.3 Probability Density Function and Markov Processes

Consider a stochastic process governed by the stochastic differential equation

$$dx = a(x)\,dt + b(x)\,dW.$$

Let $\rho(x,t)$ denote the probability density function (PDF) of $x(t)$, so that the probability of finding $x(t)$ in the interval $[\hat{x}, \hat{x}+d\hat{x}]$ is given by

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x}+d\hat{x}]\Big] = \rho(\hat{x},t)\,d\hat{x}.$$

In this way, the state of the system $x(t)$ is fully characterized by its PDF, $\rho(x,t)$.
More generally, if we consider

$$x \in \mathbb{R}, \quad t \in \mathbb{R},$$

the stochastic process $x(t)$ has the state space (SSP) $\mathbb{R}$ and evolves in continuous time. The probability that the process takes a value in a small interval at time t depends on its past history,

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x}+d\hat{x}]\Big] = \kappa\Big[\{x(\theta)\}_{0\leq\theta\leq t}\Big],$$

where κ represents the functional dependence on the trajectory $\{x(\theta)\}$ for $0 \leq \theta \leq t$.

Markov Process

A process is said to possess the **Markov property** if its future evolution depends solely on its present state rather than the entire past history. For the SDE above, the increment over an infinitesimal time interval dt can be written as

$$x(t+dt) = x(t) + a(x)\,dt + b(x)G_t\sqrt{dt},$$

where G_t is a Gaussian random variable with mean 0 and variance 1. Note that the update depends only on the current state $x(t)$, which exemplifies the Markov property.
To further illustrate this idea, consider a simple discrete deterministic process:

$$x_{t+1} = a\,x_t, \quad t \in \mathbb{N}_0.$$

Its solution is given by

$$x_t = a^t x_0.$$

Now, if we add a stochastic term to account for random fluctuations, we obtain

$$x_{t+1} = a\,x_t + \omega\nu_t,$$

where ν_t is a random variable representing noise. In this context, the distribution of x_t at time t, denoted by $\rho(x,t)$, evolves according to the stochastic dynamics. Often, this distribution can be expressed as

$$\rho(x,t) = L(x_t),$$

where $L(x_t)$ denotes the law governing the evolution of the process.
This example highlights that in a Markov process the next state is determined exclusively by the most recent state rather than by the full history of the process.

6 Lecture 21/03/2025

6.1 Evolution of the Probability Density Function: The Fokker-Planck Equation

Last time we examined Malthusian processes, where the dynamics of the state variable $x(t)$ are governed by the stochastic differential equation (SDE)

$$dx = f(x)\,dt + g(x)\,dW.$$

Over an infinitesimal time increment dt, the update can be written as

$$x(t+dt) = x(t) + \underbrace{f(x)\,dt}_{O(dt)} + \underbrace{g(x)\,dW}_{O(\sqrt{dt})}.$$

Since the increment dW is of order $\sqrt{dt}$ and the change in x is infinitesimal, the evolution of the probability density function (PDF) $\rho(x,t)$ for $x(t)$,

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x}+d\hat{x}]\Big] = \rho(\hat{x},t)d\hat{x},$$

depends only on the local properties of $\rho(x,t)$ (specifically its first and second derivatives). In other words, the future evolution of $\rho(x,t)$ is determined by its current state and the local changes, which leads to a Partial Differential Equation (PDE) for $\rho(x,t)$.
For instance, when the process $x(t)$ is a pure diffusion process—as is the case for a Wiener process—the PDF is given by

$$\rho(w,t) = A\,\exp\Big(-\frac{w^2}{2t}\Big),$$

where the normalization constant A ensures that the total probability is unity. In this case, one can derive that

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial w^2}.$$

This equation is a particular instance of the more general ***Fokker-Planck equation***, which describes how the probability density function of a stochastic process evolves over time. In the general case, the Fokker-Planck equation incorporates the contributions from both the drift term (related to $f(x)$) and the diffusion term (related to $g(x)$). It provides a powerful framework for understanding the dynamics of stochastic processes across various disciplines.

$$dx = a(x)dt + b(x)dW$$

$$y = \Psi(x) \quad \Rightarrow \quad d\Psi = [\Psi'(x)a(x) + Psi''(x)\frac{b^2(x)}{2}]dt + \Psi'(x)b(x)dW$$

$$\langle d\Psi\rangle = \langle \Psi'(x)a(x) + \Psi''(x)\frac{b^2(x)}{2}\rangle dt + \underbrace{\cancel{\langle \Psi'(x)b(x)\rangle dW}}_{\text{set to zero}}$$

The last term is oscillatory and averages to zero so we can ignore it. The first term is the drift term of the process $y(t)$. We have:

$$\frac{d}{dt}\langle \Psi\rangle = \langle \Psi'(x)a(x) + \Psi''(x)\frac{b^2(x)}{2}\rangle$$

The average is given by $\int_s z(x)\rho(x,t)dx$, where $z(x)$ is the function of x that we want to average. We have:

$$\frac{d}{dt}\int_{\mathbb{R}} \Psi(x)\rho(x,t)dx = \int_{\mathbb{R}} \left[\Psi'(x)a(x)\rho(x,t) + \Psi''(x)\frac{b^2(x)}{2}\rho(x,t)\right] dx$$

$$\int_{\mathbb{R}} \Psi(x)\frac{\partial \rho}{\partial t}(x,t)dx = \underbrace{\int_{\mathbb{R}} \Psi'(x)a(x)\rho(x,t)dx}_{I_1} + \underbrace{\int_{\mathbb{R}} \Psi''(x)\frac{b^2(x)}{2}\rho(x,t)dx}_{I_2}$$

Let's integrate by parts the first term on the right-hand side:

$$\begin{aligned}
I_1 = &\ \textstyle\int_{-\infty}^{+\infty} \Psi'(x)R(x,t)dx = \left|\Psi(x)R(x,t)\right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \Psi(x)\frac{\partial R}{\partial x}dx = 0 - \int_{-\infty}^{+\infty} \Psi(x)\frac{\partial R}{\partial x}dx \\
I_2 = &\ \textstyle\int_{-\infty}^{+\infty} \Psi''(x)z(x,t)dx = \left|\Psi'(x)z(x,t)\right| - \int_{-\infty}^{+\infty} \Psi'(x)\frac{\partial z}{\partial x}dx = -\int_{-\infty}^{+\infty} \Psi'(x)\frac{\partial z}{\partial x}dx \\
= &\ \textstyle -\left|\Psi(x)\frac{\partial z}{\partial x}\right|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \Psi(x)\frac{\partial^2 z}{\partial x^2}dx
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
&\int_{\mathbb{R}} \Psi(x)\frac{\partial \rho}{\partial t}(x,t)dx = -\int_{\mathbb{R}} \Psi(x)\left[-\frac{\partial R}{\partial x}\right]dx + \int_{\mathbb{R}} \Psi(x)\left[\frac{\partial^2 z}{\partial x^2}\right]dx = \\
&= \int_{\mathbb{R}} \Psi(x)\left\{-\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right]\right\}dx
\end{aligned}$$

$$\int_{-\infty}^{+\infty} \Psi(x)\frac{\partial \rho}{\partial t}dx = \int_{-\infty}^{+\infty} \left\{-\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right]\right\}\Psi(x)dx$$

$$\begin{cases}
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right] \\
\int_{-\infty}^{+\infty} \rho(x,t)dx = 1
\end{cases}$$

Example:

Let's consider the following SDE:

$$\dot{x} = f(x) + \omega\xi(t) \quad \rightarrow \quad m\ddot{x} = -\dot{x} + f(x) + \omega\xi(t)$$

if $m \ll 1$ we have $\dot{x} = f(x) + \omega\xi(t) \quad f = -\dfrac{\partial U}{\partial x}$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x}[f(x)\rho(x,t)] + \dfrac{\omega^2}{2} - \dfrac{\partial^2 \rho}{\partial x^2} \\ \int_{-\infty}^{+\infty} \rho(x,t)dx = 1 \end{cases}$$

ρ depends on x and t, P depends only on x:

$$\begin{cases} \dfrac{\omega^2}{2}\dfrac{d^2 P}{dx^2} - \dfrac{d}{dx}[f(x)P] = 0 \\ \int P(x)dx = 1 \end{cases}$$

$$\frac{\omega^2}{2}\frac{dP}{dx} = f(x)P \quad \Rightarrow \quad \frac{\omega^2}{2}\frac{dP}{x} = \frac{\partial U}{\partial x}P \quad \Rightarrow \quad \frac{dP}{P} = \frac{2}{\omega^2}\frac{\partial U}{\partial x}dx$$

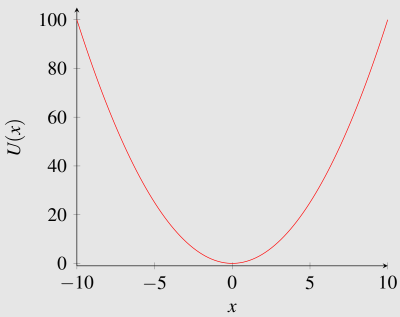
Now we can calculate C:

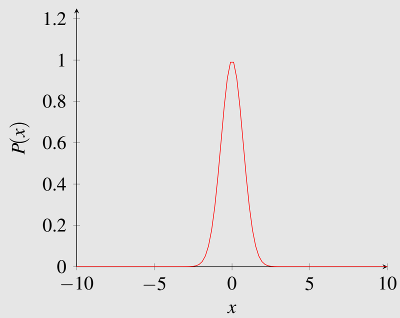
$$P(x) = Ce^{(2/\omega^2)U(x)}$$

...

$$C = \frac{1}{\displaystyle\int_{-\infty}^{+\infty} e^{(2/\omega^2)U(x)}dx}$$

...





...

6.2 Liouville Equation for Systems with Uncertain Initial Conditions

Consider a physical system governed by the ordinary differential equation

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0, \end{cases}$$

where the initial condition x_0 is not known exactly. Instead, we assume that x_0 is drawn from a probability distribution $\theta(x_0)$, so that the system is defined by

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0 \sim \theta(x_0). \end{cases}$$

Suppose there exists an equilibrium point x_e such that $a(x_e) = 0$, and that x_e is globally asymptotically stable (G.A.S.). This implies that, regardless of the uncertainty in the initial condition, the state $x(t)$ converges to x_e as $t \to \infty$. Consequently, the probability density function (PDF) $\rho(x,t)$ of $x(t)$ evolves towards a Dirac delta distribution centered at x_e:

$$\lim_{t\to\infty} \rho(x,t) = \delta(x - x_e).$$

The time evolution of $\rho(x,t)$ is governed by the **Liouville equation**. For the deterministic dynamics

$$dx = a(x)\, dt,$$

the Liouville equation is given by

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\Big[a(x)\, \rho(x,t)\Big].$$

This partial differential equation expresses the conservation of probability along the flow of the system. The term $-\frac{\partial}{\partial x}\left[a(x)\, \rho(x,t)\right]$ represents the net flux of probability density in the state space due to the vector field $a(x)$. As time evolves and the system converges to the stable equilibrium x_e, the density $\rho(x,t)$ becomes increasingly concentrated around x_e, reflecting the loss of uncertainty in the long-term behavior of the system.

$$m\dot{v} = -\gamma v + F_s(t) \tag{I}$$

$$Ri = -L\frac{di}{dt} - K\frac{dB_{ext}}{dt} \tag{II}$$

$$m\ddot{x} = -\hat{k}x - \gamma\dot{x} + \hat{F}(t) \tag{III}$$

…

$$\dot{z} = -\gamma z + \omega\xi(t) \leftrightarrow dz = .\gamma z dt + \omega\xi(t)dt \quad \Rightarrow \quad z = e^{-\gamma t}Q$$

$$-\gamma e^{\gamma t}Qdt + e^{-\gamma t}dQ = -\gamma e^{-\gamma t}Q + \omega dW \quad \Rightarrow \quad e^{-\gamma t}dz = \omega dW$$

So we have:

$$dQ = e^{\gamma t}\omega dW \quad = \quad Q(t) = z(0) + \omega\int_0^t e^{\gamma s}dW(s) \quad \Rightarrow \quad \boxed{z(t) = z_0 e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)}$$

$$\langle z(t)\rangle = \langle z_0\rangle e^{-\gamma t} + \Phi$$

…

$$z(t) = z(0)e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)$$

$$\begin{aligned}
\langle z^2(t)\rangle = &\ \left\langle \left(z_0 e^{-\gamma t}dW(s) + \textstyle\int_0^t e^{\gamma(s-t)}\xi(s)ds\right)\left(z_0 e^{-\gamma t} + \textstyle\int_0^t e^{\gamma(\theta-t)\xi(\theta)d\theta}\right)\right\rangle \\
= &\ \left\langle \left(z_0^2 e^{-2\gamma t} + z_0 e^{-\gamma t}\textstyle\int_0^t e^{\gamma(\theta-t)}\xi(\theta)d\theta + z_0 e^{-\gamma t}\textstyle\int_0^t e^{\gamma(s-t)}\xi(s)ds + J(t)\right)\right\rangle \\
= &\ \dots
\end{aligned}$$

$$\begin{aligned}
\langle J(t)\rangle = &\ \omega^2 \textstyle\int_0^t\int_0^t e^{\gamma(\theta+s-2t)}\delta(\theta-s)d\theta ds \\
= &\ \omega^2 e^{-2\gamma t}\textstyle\int_0^t \left\{\int_0^t e^{\gamma(\theta+s)}\delta(\theta-s)d\theta\right\}ds \\
= &\ \omega^2 e^{-2\gamma t}\textstyle\int_0^t e^{2\gamma s}ds \\
= &\ \omega^2 e^{-2\gamma t}\left[\frac{e^{2\gamma t}-1}{2\gamma}\right] \\
= &\ \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)
\end{aligned}$$

$$Var[z(t)] = \underbrace{\langle z_0^2\rangle e^{-2\gamma t} - (\langle z_0\rangle)^2 e^{-2\gamma t}}_{=\,Var(z_0)e^{-2\gamma t}} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right) = Var(z_0)e^{-2\gamma t} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$$

$$Var(z(t)) \to \frac{\omega}{2\gamma} = \sigma^2$$

$$z(t) = e^{-\gamma t} + \int_0^t e^{\gamma(s-t)}\xi(s)dt$$

…

$$z(0) = 0$$

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t \int_0^q e^{\gamma(\theta+s)} \delta(\theta - s) d\theta ds$$

We have 2 cases:

- $t < q$:

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t e^{2\gamma s} ds = \frac{\omega^2}{2\gamma} e^{-\gamma(t-q)} \dots$$

$$\langle z(t)z(q)\rangle = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|q-t|} - e^{-\gamma(q+t)} \right)$$

—

$$z_0 = 0 \quad \begin{cases} \langle z(t)\rangle = 0 \\ \langle z(q)\rangle = 0 \end{cases} \qquad C[\alpha, \beta] = \langle (\alpha - \hat{\alpha})(\beta - \hat{\beta})\rangle$$

$$\to C[z(t), z(q)]; \quad q = t + h \quad X[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|h|} - e^{-\gamma h} e^{-2\gamma t} \right)$$

$$R_z(h) = \lim_{t \to \infty} C[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} e^{-\gamma|h|}$$

- $t > q$:

Bibliography

[1] "2.080 Structural Mechanics Stability of Elastic Structures". In: 2014. URL: https://api.semanticscholar.org/CorpusID:30595920.