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Stocasthic Modeling and Simulation

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Preface

As a student of Scientific and Data Intensive Computing, I've created these notes while attending the **Stochastic Modeling and Simulation** course.
The course covers a wide range of topics, including Stochastic Nonlinear Models in different application fields (as physics, biomedicine, mathematics, ...), Stochastic Differential Equations, and Stochastic Simulation.

The topics covered in these notes include:
- Recap of Deterministic Models
- Stochastic Differential Equations and White Noise
- Fokker Planck Equation
- Noise-induced Transitions
- Colored noises
- Bounded Stochastic Processes
- Spatio-temporal Stochastic Processes
- Parameter Estimation form Data
- Stochasticity …
- …
- Continuous state space-discrete time Stochastic Processes
- Discrete Time Markov Chains
- Continuous Time Markov Chains
- Mean Field Approximation

While these notes were primarily created for my personal study, they may serve as a valuable resource for fellow students and professionals interested in Stochastic Modeling and Simulation.

Contents

1 Introduction

Different fields as Epidemics spreading, Cancer growth, and many others, can be modeled using Stochastic Differential Equations (SDEs).
An example is SIR model for epidemics spreading, formulized during a huge cholera infection in early 19th century. The model is based on three compartments: Susceptible, Infected, and Recovered individuals. This is modeled using a system of SDEs.
Often stochastic models represents well the problem, but real data are noisy and chaotic. For instance often we have to deal with data varying spatially and temporally, and we have to deal with the problem of parameter estimation.
Example: Oscillating chemical system

$$A + Y \xrightarrow{k_1} X + P, \quad X \xrightarrow{k_2} 2P$$
$$A + X \xrightarrow{k_3} 2Z, \quad X + Z \xrightarrow{k_4} 2A$$

Other examples are preys and predators models,
…
Renewable energies introduces a high volatility and unpredictability in energy production
If the demand of energy exceedes the production, we need to activate standard plans, to reduce consumption by switching off devices (e.g. water boilers remotely controlled), or to activate additional production plants. All these scenarios are complex systems.

Definition: *Complex System*

A **complex system** is a system composed of interconnected parts that as a whole exhibit one or more properties (behavior among the possible properties) not obvious from the properties of the individual parts.

Emergent behavior is a property of complex systems, and it is not predictable from the behavior of the individual parts.

Definition: *Adaptivity and self-organization*

- **Adaption** meand achieving a fit between the system and its environment.
- **Self-organization** is the process where a system changes its structure spontaneously, in order to adapt to the environment.

An instance of self-organization is the formation of a flock of birds, where each bird follows simple rules, but the flock as a whole exhibits a complex behavior.

Observation: *Noise and Nonlinearities*

Noise and nonlinearities can (sometimes) favor the emergence of ”order”.

1.1 Modelling complex systems

Math for quantitatve models

We will be interersted in the temporal behavior of the system, and we will use some key ingredients for the maths:

- **Entities** can be modelled as *discrete* objects or *continuous* quantities.
- **...** ...
- **Time** can be *discrete* or *continuous*.

…

Data-Based vs Model-Based Approaches

Will data approaches make the kind of modelling obsolete?
Hybrid approacehs are possible:

1. Math models can be joined/hcybrized with machine learning models.
2. Deep Network to learn modules aor whole math models
   An exampleis *Physics-informed neural networks*: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations
3. ...

Definition: ***Dynamical System***

A **dynamical system** is a system whose state evolves over time according to a rule that depends on the current state.

[definition of differential equations]

Definition: ***Differential Equation***

A **differential equation** is an equation that relates one or more unknown functions and their derivatives.

In practice, differential equations are mathemathical instruments that describves the world around us. A note differential equation (maybe the first ever invented) is the Newton law:

$$F = ma \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = F(x(t))$$

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lecture 10/03/2025

2.1 Tumor Size over Time

Let's consider an example case of a tumor growth. Let's assume that $X(t)$ is the size of a tumor at time t.
The differential equation that describes the growth of the tumor is:

$$X(t+dt) = X(t) + \Phi X(t) - MX(t)$$

where Φ is the growth rate and M is the decay rate.
We can rewrite the equation as:

$$\frac{X(t+dt)-X(t)}{dt} = X(t)\frac{(\Phi - M)}{dt}$$

Let's rewrite Φ and M as:

$$\begin{matrix} \Phi = bdt + \cancel{O(dt^2)} \\ M = mdt + \cancel{O(dt^2)} \end{matrix} \qquad \text{(We neglect the higher order terms)}$$

We obtain the following differential equation:

$$\frac{X(t+dt)-X(t)}{dt} = X(t)(b-m) \quad \Rightarrow \quad \frac{dX}{dt} = X(b-m)$$

Often we have a starting condition $X(0) = X_0$. Defining $a = b - m$, the sistem becomes:

$$\begin{cases} \frac{dX}{dt} = aX \\ X(0) = X_0 \end{cases}$$

Since makes no sense to have a negative time or tumor size, we have the constraints:

$$\begin{cases} t \in \mathbb{R}^+ \cup \{0\} \\ X \in \mathbb{R}^+ \cup \{0\} \end{cases}$$

The solution is given by:

$$X(t) = X_0 e^{at}$$

$$X(t+dt) = X(t) + bdtX - mdtX - \theta dtX$$

$$\begin{cases} \dot{X} = (a-\theta)XX(0) = X_0 \end{cases}$$

dunque la soluzione è:

$$X(t) = X_0 e^{(a-\theta)t}$$

In questo caso, per $a > \theta$, il tumore cresce esponenzialmente, mentre per $a < \theta$, il tumore decresce esponenzialmente.
Nella realtà però il valore θ non è costante nel tempo, ma varia nel tempo. In tal caso il sistema diventa:

$$\begin{cases} \dot{X} = (a - \theta(t))X \\ X(0) = X_0 \end{cases}$$

e la soluzione è:

$$X(t) = X_0 e^{\int_0^t (a-\theta(s))ds}$$

Più in generale, un sistema del tipo:

$$z(y) = e^{G(y)} b$$

si ha

$$\frac{dz}{dy} = \frac{d}{dy} e^{G(y)} b = e^{G(y)} b \frac{dG}{dy} = G'(y) z(y)$$

Esempio:

$$\begin{cases} Z'(t) = \sin(t) Z(t) \\ Z(0) = Z_0 \end{cases}$$

Si ha:

$$\begin{cases} Z(t) = e^{-\cos(t)} B \\ Z_0 = e^{-1} B \end{cases} \quad \Rightarrow \quad Z(t) = e^{1-\cos(t)} z_0$$

2.2 Stabily and Eq. Points

2.2.1 Equilibrium Points

Given the system:

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}$$

An equilibrium point is a point x^* such that $f(x^*) = 0$.

Tip: ***Eq. points***

A system can have multiple equilibrium points.

We have three kinds of equilibrium:

- **Stable equilibrium**: if the system is in the neighborhood of the equilibrium point, it will remain there.
- **Neutral equilibrium**: if the system is in the neighborhood of the equilibrium point, it will remain there, but it will not return to it.
- **Unstable equilibrium**: if the system is in the neighborhood of the equilibrium point, it will move away from it.

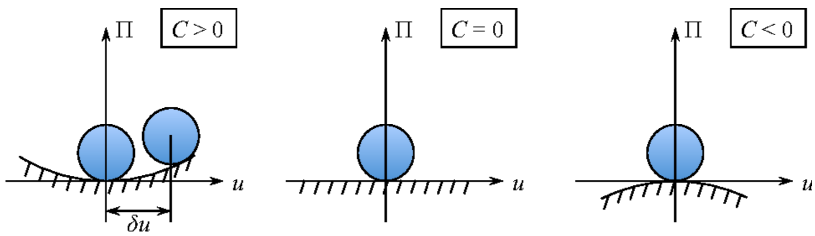

Figure 2.1: Stable, Natural and Unstable Equilibrium Points [1]

2.2.2 Stability

We say that a system is ***Globally Asymptotically Stable*** (or *Globally Attractive*) if it is stable and if it converges to the equilibrium point from any initial condition.

2.3 Local Analysis Near the Equilibrium Point

In this section, we study the system in the neighborhood of the equilibrium point. Let the initial condition be a small perturbation around the equilibrium:

$$X(0) = X_e + \varepsilon.$$

We introduce a deviation function $U(t)$ defined by

$$X(t) = X_e + U(t),$$

with the initial condition

$$U(0) = \varepsilon.$$

Thus, the evolution of the perturbation is governed by

$$\begin{cases} \dot{U} = f(X_e + U), \\ U(0) = \varepsilon. \end{cases}$$

Assume that the dynamics of the system are given by

$$\dot{X} = X(b(X) - m(X)).$$

Then the perturbed system becomes

$$\dot{U} = (b(X_e + U) - m(X_e + U))(X_e + U).$$

Expanding $b(X_e + U)$ and $m(X_e + U)$ in a Taylor series around X_e, we have:

$$b(X_e + U) \approx b(X_e) + b'(X_e)U,$$
$$m(X_e + U) \approx m(X_e) + m'(X_e)U.$$

Substituting these into the equation for $\dot{U}$, we get:

$$\begin{aligned} \dot{U} &= \big[b(X_e) + b'(X_e)U\big] - \big[m(X_e) + m'(X_e)U\big](X_e + U) \\ &= \big[b(X_e) + b'(X_e)U\big] - \big[m(X_e)X_e + m(X_e)U + m'(X_e)X_eU + m'(X_e)U^2\big]. \end{aligned}$$

Since the equilibrium condition implies that

$$(b(X_e) - m(X_e))X_e = 0,$$

the above expression simplifies (neglecting the higher-order term $m'(X_e)U^2$) to:

$$\dot{U} \approx [b'(X_e) - m(X_e) - m'(X_e)X_e]\,U.$$

Under the assumption that $b'(X_e)$ is negative, we can express this as

$$\dot{U} \approx -X_e\Big(|b'(X_e)| + m'(X_e)\Big)U.$$

The solution of this linearized differential equation is given by:

$$U(t) = U(0)\,e^{-X_e(|b'(X_e)| + m'(X_e))t}.$$

Hence, we identify the decay rate (or the inverse of the characteristic time constant) as

$$X_e\Big(|b'(X_e)|+m'(X_e)\Big),$$

and the characteristic time τ is:

$$\tau = \frac{1}{X_e\left(|b'(X_e)|+m'(X_e)\right)}.$$

This time constant represents the rate at which perturbations decay in the vicinity of the equilibrium point.

$$X(t) = X_e + U(t) \Rightarrow \dot{U} = f(X_e + U) = \underbrace{f(X_e)}_{=0} + f'(X_e)U + O(U^2)$$

$$\dot{U} = f'(X_e)U \Rightarrow U(t) = U(0)e^{f'(X_e)t}$$

So we have:

$$\begin{cases} f'(X_e) < 0 & \Rightarrow & X_e \text{ is Locally Asintotically stable} \\ f'(X_e) > 0 & \Rightarrow & X_e \text{ is Unstable} \end{cases}$$

2.4 Non-scalar Systems

Consider the system:

$$\begin{cases} \dot{x} = f(x) \\ x \in \{ \subseteq \mathbb{R}^n \end{cases}$$

As in the scalar case, we can linearize the system around the equilibrium point x_e:

$$f(X_e) = 0$$

$$X = X_e + U, \qquad |U| \ll 1$$

We have to consider the Jacobian matrix of f:

…

2.5 Exponential of a Matrix

Let's consider a linear system of the form:

$$\dot{x} = Ax$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix. The solution of this system is given by:

$$x(t) = e^{At}x(0)$$

where e^{At} is the exponential of the matrix A.

Definition: ***Exponential of a Matrix***

Given a matrix $A \in \mathbb{R}^{n\times n}$, the exponential of A is defined as:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

This comes from the Taylor series expansion of the exponential function.

$$\frac{d}{dt}e^{At} = \sum_{m=1}^{\infty} A^m m \frac{t^{m-1}}{m(m-1)!} = \sum_{k=0}^{\infty} AA^k \frac{t^k}{k!} = Ae^{At}$$

$$\begin{array}{rcl} A & = & H \cdot \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \cdot H^{-1} \\ A^2 & = & H \cdot \mathrm{Diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2) \cdot H^{-1} \\ A^m & = & H \cdot \mathrm{Diag}(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m) \cdot H^{-1} \end{array}$$

So we have:

$$e^{At} = \sum_{m=0}^{\infty} \frac{A^m t^m}{m!} =$$

An example of a matrix exponential is given by the Newton's law:

$$m\ddot{x} = F$$

Let's consider a more complex case with air resistance γ:

$$m\ddot{x} = -\gamma\dot{x} - F(x)$$

We can rewrite this system as:

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Last lecture we saw that we can approximate a non-linear system with a linear one, locally in an equilibrium point.
Let's consider an electric circuit, if the elecrical field is not static, it generates a variable magnetic field and viceversa.

$$\begin{cases} \Phi = Li \\ Ri = -\dfrac{d\Phi(\vec{B})}{dt} = -\dfrac{d(Li)}{dt} \end{cases}$$

We can write this system in the cauchy form:

$$\begin{cases} \dfrac{di}{dt} = -\dfrac{R}{L}i \\ i(0) = i_0 \end{cases}$$

The solution is:

$$i(t) = i_0 e^{-\frac{R}{t}}$$

$$L\frac{di}{dt} + Ri + V_c = 0$$

$$VC = Q$$

$$y = \begin{bmatrix} i \\ q \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{LC} \\ 1 & 0 \end{bmatrix}$$

$$m\ddot{x} = \gamma\dot{x} + kx$$

3.1 Dirak Delta

If ypu consider a foorball player that kicks a ball, the force is not constant, and it is not possible to model it with a constant force. We can model it with a Dirak Delta function.

$$\begin{cases} m\ddot{x} = m\dot{v} = F(t) \\ v(0) = 0 \end{cases} \quad \Rightarrow \quad mv_{after} = \int_0^a F(t)dt \quad \Rightarrow \quad \boxed{v_{after} = \frac{1}{m}\int_0^a F(t)dt}$$

So the Dirak Delta function is a function that is zero everywhere except in zero, where it is infinite. It is used to model impulses.

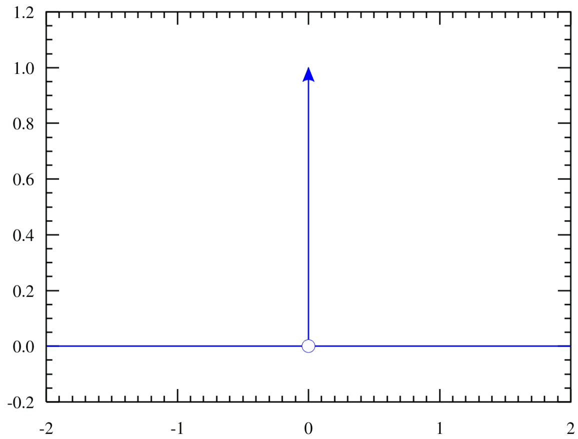

Figure 3.1: Dirak Delta Function

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Let's consider a function $f(t)$ such that $f(0) < \infty$ and $f'(0) < \infty$. We can calculate:

$$\int_{\mathbb{R}} \delta(t)f(t)dt = \int_{\mathbb{R}} \delta(t)[f(0) + f'(0)t]dt = \int_{\mathbb{R}} \delta(t)f(0)dt + \int_{\mathbb{R}} \delta(t)f'(0)tdt$$

Now, we use two key properties of the Dirac delta function:

1. **Sifting Property:**
$$\int_{\mathbb{R}} \delta(t)\,dt = 1.$$
2. **First Moment:**
$$\int_{\mathbb{R}} \delta(t)t\,dt = 0,$$
which follows because $t\delta(t)$ is an odd function.

Substituting these results, we obtain:

$$\int_{\mathbb{R}} \delta(t)f(t)\,dt = f(0)\cdot 1 + f'(0)\cdot 0 = f(0).$$

…

3.2 Random Processes

$$< x(t) >= \frac{1}{N}\sum_{i=1}^{N} x_{Ri}(t)$$

$$m_i\ddot{x}_i = -\gamma\dot{x}_i \quad \Rightarrow \quad m_i\dot{v}_i = -\gamma v_i \quad \Rightarrow \quad \dot{v}_i = -\frac{\gamma}{m_i}v_i \quad \Rightarrow \quad \boxed{v_i(t) = v_i(0)e^{-(\gamma/m_i)t}}$$

$$\boxed{m\dot{v} = -\gamma v + \hat{F}_s(t)}$$

$$\begin{gathered}
m\ddot{x} = -k\dot{x} + \hat{F}_p(x) + \hat{F}_s(t) \\
m\ddot{x} = -k\dot{x} + kf(x) + kf_s(t) \\
\frac{m}{k}\ddot{x} = -\dot{x} + f(x) + f_s(t) \\
\frac{m}{k}\ddot{x} \ll 1 \quad \Rightarrow \quad \frac{m}{k}\ddot{x} \approx 0 \\
\dot{x} \simeq f(x) + f_s(t) = f(x) + \omega\xi(t)
\end{gathered}$$

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$$m\dot{v} = -\gamma v + F_s(t)$$

if $m \ll 1$ then $\dfrac{m}{\gamma} \approx 0$

$$\dot{x} = f(x) + \omega\xi(t)$$

$$\frac{dx}{dt} = f(x) + g(x)\xi(t)$$

SI model:

$$\begin{cases} \dot{S} = -\beta SI + \theta I \\ \dot{I} = \beta SI - \theta I \end{cases}$$

where β is the infection rate and θ is the recovery rate.

$$\frac{dI}{dt} = \beta(1-I)I - \theta I$$

$\beta$is a stochastic variable, we can write it as:

$$\beta \to \beta + \omega\xi(t)$$

where ω is the amplitude of the noise and $\xi(t)$ is a white noise.
We can write the equation as:

$$\frac{dI}{dt} = (\beta + \omega\xi(t))(1-I)I - \theta I$$

Properties of $\xi(t)$

The noise process $\xi(t)$ is characterized by the following properties:

1. **White Noise:**
   $\xi(t)$ is a white noise process, meaning its values at different time instants are uncorrelated.
2. **Gaussian Noise:**
   $\xi(t)$ follows a Gaussian distribution, so all its finite-dimensional distributions are Gaussian.
3. **Zero Mean:**
   The expected value of the process is zero:
   $$\langle \xi(t) \rangle = 0.$$
4. **Temporal Uncorrelation:**
   For any two distinct time instants $t \neq q$, the noise is uncorrelated:
   $$\langle \xi(t)\xi(q) \rangle = 0.$$

5. **Delta-Correlated:**
   The autocorrelation function is given by the Dirac delta function:
$$\langle \xi(t)\xi(q)\rangle = \delta(t-q).$$

6. **Infinite Instantaneous Variance:**
   The variance at any fixed time is formally divergent:
$$\langle \xi^2(t)\rangle \gg 1,$$
   reflecting the idealized nature of white noise.

...

$$f(x) = 0$$
$$g(x) = \omega \neq 0$$
$$\dot{x} = \omega\xi(t)$$
$$x(t) = x(0) + \omega\int_0^t \xi(s)ds$$
$$< x(t) >= x(0) + \omega\int_0^t \underbrace{< \xi(s) >}_{=0} ds = x(0)$$
$$x(t)x(q) = \omega^2\int_0^t \xi(s)ds\int_0^q \xi(\theta)d\theta = \omega^2\int_0^t\int_0^q \xi(s)\xi(\theta)dsd\theta$$
$$< x(t)x(q) >= \omega^2\int_0^t\int_0^q \underbrace{< \xi(s)\xi(\theta) >}_{=\delta(s-\theta)} dsd\theta = \omega^2\int_0^t\int_0^q \delta(s-\theta)d\theta ds$$

We have $< x(t)x(q) >= \omega^2\min(t,q)$.

If $q \geq t$:

$$< x(t)x(q) >= \omega^2\int_0^t\int_0^t \delta(\theta-s)d\theta = \omega^2\int_0^t ds = \omega^2 t$$

Else if $0 < q < t$:

$$< x(t)x(q) >= \omega^2\int_0^q\left\{\int_0^q \delta(\theta-s)d\theta\right\}ds = \omega^2\int_0^q ds = \omega^2 q$$

...

$$\begin{aligned}\langle (x(t)-x(q))^2\rangle &= \langle x^2(t)\rangle + \langle x^2(q)\rangle - 2\langle x(t)x(q)\rangle = \omega^2(t+q-2\min(t,q))\\ &= \begin{cases} 0 & \text{if } t = q\\ \omega^2(t-q) & \text{if } t > q\\ \omega^2(q-t) & \text{if } t < q\end{cases} = \omega^2|t-q|\end{aligned}$$

…

$$\left\langle \left(\frac{x(t+h)-x(t)}{h} \right)^2 \right\rangle = \frac{\omega^2}{h}$$

This is an incremental ratio, so, if we take the limit as $h \to 0^+$ we get:

$$\lim_{h \to 0^+} \left\langle \left(\frac{x(t+h)-x(t)}{h} \right)^2 \right\rangle = +\infty$$

Wiener Process

The **Wiener Process** (also known as Brownian motion) is a continuous-time stochastic process widely used in physics and finance to model random behavior. It is named after Norbert Wiener, who introduced it in the 1920s. The process is defined by the stochastic differential equation

$$\begin{cases} \dfrac{dw}{dt} = \xi(t), \\ w(0) = 0, \end{cases}$$

where $\xi(t)$ represents a white Gaussian noise.

Properties of the Wiener Process:
The Wiener process is a Gaussian process with the following key properties:

1. **Zero Mean:**
   The expected value of the process is zero:

   $$\langle w(t) \rangle = 0.$$

2. **Gaussian Distribution:**
   For any fixed time t, $w(t)$ is normally distributed.

3. **Autocorrelation:**
   The autocorrelation function is given by

   $$\langle w(t)w(q) \rangle = \min(t,q).$$

4. **Independent Increments:**
   The increments of the process are independent. In particular,

   $$\langle w(q) - w(t) \rangle = 0,$$

   and these increments are also Gaussian.

5. **Increment Variance:**
   The variance of the increment over the interval $[t,q]$ is proportional to the time difference:

   $$\langle (w(q) - w(t))^2 \rangle = |q-t|.$$

6. **Increment Distribution:**
   More precisely, for $q > t$, the increment is distributed as

   $$w(q) - w(t) \sim \mathcal{N}\Big(0, |q-t|\Big),$$

   and in particular,

   $$w(t) - w(0) \sim \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{w^2}{2t}\Big).$$

Increment Analysis and the Derivative of the Wiener Process

Consider a small time increment defined as

$$q = t + dt, \qquad dt > 0.$$

The increment of the Wiener process over this interval is given by

$$dw = w(t+dt) - w(t).$$

Since the process is Gaussian, the increment satisfies

$$dw \sim G(\mu = 0, \sigma = dt),$$

which implies that the finite difference quotient behaves as

$$\frac{dw}{dt} \sim G\left(\mu = 0, \sigma = \frac{1}{dt}\right).$$

This relation highlights that, in the limit as $dt \to 0$, the notion of a derivative for the Wiener process becomes problematic due to the divergence in the standard deviation.
Furthermore, for a finite interval h, we consider the probability

$$\Pr\left(\left|\frac{w(t+h) - w(t)}{h}\right| > M\right) \quad \Rightarrow \quad \Pr\Big(|w(t+h) - w(t)| > hM\Big).$$

This formulation reinforces the scaling behavior of the process's increments and underscores the fact that the Wiener process has almost surely nowhere differentiable paths.

Euler-Maruyama Method

In stochastic differential equations, the dynamics of a system are often described by equations of the form

$$\boxed{dp = F\,dt},$$

where dp represents the infinitesimal change in momentum and F is the force.
Similarly, the evolution of a state variable x is given by

$$dx = f(x,t)\,dt + g(x,t)\,dw,$$

with $f(x,t)$ denoting the drift term, $g(x,t)$ the diffusion coefficient, and dw the stochastic increment. The stochastic increment is defined as

$$dw = G(t)\sqrt{dt},$$

so that the update of x over a small time interval dt can be written as

$$x(t+dt) = x(t) + f(x,t)\,dt + g(x,t)G(t)\sqrt{dt}.$$

When discretizing time, let t_j denote the j-th time step, and define

$$G_j = G(t_j) \sim \mathcal{N}(0,1).$$

An increment over a discrete time-step is then approximated by

$$dx \simeq x_{j+1} - x_j.$$

Thus, the discretized form of the stochastic differential equation becomes

$$\begin{cases} x_{j+1} = x_j + f(x_j,t_j)h + g(x_j,t_j)G_j\sqrt{h}, \\ x_0 = x(0), \end{cases}$$

where h is the time-step size.
This numerical scheme is known as the ***Euler-Maruyama method***.

$$\dot{x} = x(1-x)$$

$$dx = x(1-x)dt$$

$$\frac{dx}{x(1-x)} = dt$$

$$\frac{dx}{x} + \frac{dx}{1-x} = dt$$

$$d(\ln|x| - \ln|1-x|) = dt \quad \Rightarrow \quad \ln\frac{x}{1-x} = \ln\frac{x_0}{1-x_0}$$

...

$$dx = a(x)dt + b(x)dw$$

$$y = \Psi(x)$$

$$dy = \Psi(x+dx) - \Psi(x)$$

$$dy = \Psi(x + a(x)dt + b(x)dw) - \Psi(x) =$$

...

$$dy = \Psi'(x)[a(x)dt + b(x)dw] + \frac{1}{2}\Psi''(x)\left[b^2(x)dt^2 + \underbrace{\cancel{a^2(x)dt^2}}_{O(dt^2)} + \underbrace{\cancel{2a(x)b(x)dtdw}}_{O(dt^{2/3})}\right]$$

$$dy = \Psi'(x)a(x)dt + \Psi'(x)b(x)dw + \Psi''(x)\frac{b^2(x)}{2}(dw)^2$$

$$(dw)^2 = (dt + \Omega(t))$$
$$dw \sim \mathcal{N}(0, dt)$$

$$dy = \left[\frac{\partial\Psi}{\partial x}a(x) + \frac{b^2(x)}{2}\Psi''(x)\right] + \Psi'(x)b(x)dw + \cancel{O(dt^{2/3})}$$

...

Malthus model

$$\dot{x} = bx - mx = (b-m)x = rx$$

$$\begin{cases} b \to b + \text{fluctuations}(t) \\ m \to m + \text{fluctuations}(t) \end{cases}$$

$$\frac{dx}{dt} = (r + \omega\xi(t))x$$

$$dx = rxdt + \omega xdw$$

$$y = \Psi(x) = \ln x$$

$$\begin{cases} a(x) = rx \\ b(x) = \omega x \end{cases}$$

$$\Psi'(x) = \frac{1}{x}$$

$$\Psi''(x) = -\frac{1}{x^2}$$

$$dy = \left[\frac{1}{\cancel{x}} r\cancel{x} + \frac{1}{2}\omega^2 \cancel{x^2}\left(-\frac{1}{\cancel{x^2}}\right)\right] dt + \frac{1}{\cancel{x}}\omega\cancel{x}dw \quad = \quad \boxed{\left(r - \frac{\omega^2}{2}\right) dt + \omega dw}$$

$$y(t) = \underbrace{y(0)}_{= e^{x_0}} + \left(r - \frac{\omega^2}{2}\right) t + \omega^2 w(t)$$

...

$$\boxed{x(t) = e^{y(t)}} \quad \Rightarrow \quad x(t) \to 0$$

5 Lecture 17/03/2025

In this lecture we analyze a stochastic version of the Malthusian law. In the deterministic case, the Malthusian growth is given by

$$\dot{x} = rx,$$

which has the solution $x(t) = x(0)e^{rt}$. Here, we introduce a multiplicative noise term to account for random fluctuations, leading to the stochastic differential equation

$$\dot{x} = \Big(r + \omega\,\xi(t)\Big)x,$$

where $\xi(t)$ is a white noise process. By definition of white noise, we have

$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = \delta(t-t').$$

Since the noise has zero mean, the average growth rate remains r, so that

$$\langle r + \omega\,\xi(t) \rangle = r,$$

which implies

$$\langle x(t) \rangle = x(0)e^{rt}.$$

However, a paradox arises when comparing the average behavior with the typical (or almost sure) behavior of the system:

$$\begin{cases} x(t) \to 0 & \text{(typical behavior)}, \\ \langle x(t) \rangle = \infty & \text{(ensemble average)}. \end{cases}$$

This paradox is a consequence of the strong fluctuations induced by the multiplicative noise, which make the mean value unrepresentative of a typical realization.
To further analyze the dynamics, we perform a logarithmic transformation by setting

$$y(t) = \ln x(t).$$

Using Itô's calculus, the transformed variable satisfies

$$y(t) = y_0 + \left(r - \frac{\omega^2}{2}\right)t + \omega W(t),$$

where $W(t)$ is the Wiener process and $y_0 = \ln x(0)$. As $W(t)$ is normally distributed with mean 0 and variance t, it follows that

$$y(t) \sim \mathcal{N}\left(y_0 + \left(r - \frac{\omega^2}{2}\right)t,\, \omega^2 t\right).$$

Since $x(t) = e^{y(t)}$, the variable $x(t)$ is log-normally distributed. For a log-normal random variable, the mean is given by

$$\mu_{\log_N} = e^{\mu_G + \frac{\mathrm{Var}_G}{2}},$$

where μ_G and Var_G are the mean and variance of the corresponding Gaussian variable $y(t)$.
A useful side note is that if we define a new variable u with the density

$$\rho(u) = a\,e^{-au}H(u),$$

(where $H(u)$ is the Heaviside function) then its expected value is

$$\langle u \rangle = \frac{1}{a}.$$

This relation, although coming from a different context, similarly illustrates how averages may differ significantly from the most probable (median) value.
In fact, the median of a log-normally distributed variable is simply the exponential of the median of the underlying Gaussian distribution. Therefore, we have

$$\begin{cases} \text{Median}[x] = e^{\mu_{Gauss}}, \\ \text{Median}[x(t)] = e^{y_0 + \left(r - \frac{\omega^2}{2}\right)t}. \end{cases}$$

Notably, if $r - \frac{\omega^2}{2}$ is negative, the median of $x(t)$ decays to zero, even though the mean diverges. This discrepancy between the typical outcome and the ensemble average is a key feature of systems driven by multiplicative noise.

5.1 Linear Logistic Perturbed Model

We begin with the deterministic version of the logistic model in its linearized form:

$$\dot{x} = (b - m)x = rx, \qquad r > 0,$$

which implies that in the absence of density-dependent regulation, the solution grows exponentially and diverges as $x \to \infty$.
To incorporate environmental fluctuations, we introduce a stochastic perturbation into the model. The perturbed model is written as

$$\dot{x} = \big(r - \alpha x + \omega \xi(t)\big)x,$$

or equivalently, in differential form,

$$dx = (r_0 - \alpha x)x\,dt + \omega x \xi(t),$$

where r_0 represents the intrinsic growth rate, $\alpha > 0$ is the density-dependent regulation coefficient, ω quantifies the intensity of the noise, and $\xi(t)$ denotes a white noise process.
To simplify the analysis, it is useful to perform a logarithmic transformation by defining

$$y = \ln(x) \quad \Longleftrightarrow \quad x = e^y.$$

Applying Itô's formula to $y = \ln(x)$ (with the usual correction term due to the stochastic calculus), we obtain

$$dy = \left(r_0 - \frac{\omega^2}{2} - \alpha e^y\right) dt + \omega\, dW,$$

where dW is the Wiener process corresponding to $\xi(t)$.
This transformed stochastic differential equation can be formally integrated to yield

$$y(t) = y_0 + \left(r_0 - \frac{\omega^2}{2}\right) t + \omega W(t) - \alpha \int_0^t e^{y(s)} ds.$$

Notice that the first three terms,

$$y_0 + \left(r_0 - \frac{\omega^2}{2}\right) t + \omega W(t),$$

represent the contribution of the intrinsic growth and the noise, while the integral term

$$\alpha \int_0^t e^{y(s)} ds$$

captures the effect of density-dependent regulation.
If the noise intensity is sufficiently strong, specifically when

$$\frac{\omega^2}{2} > r_0,$$

then the combined effect of the noise and the regulation term drives $y(t)$ to $-\infty$ as $t \to \infty$. Consequently,

$$x(t) = e^{y(t)} \to 0^+,$$

which indicates that the population eventually goes extinct.
Let $\rho(x,t)$ denote the probability density function (PDF) of $x(t)$. As time progresses, the dynamics force the distribution to concentrate at $x = 0$, and one can show that

$$\lim_{t\to\infty} \rho(x,t) = \delta(x),$$

where $\delta(x)$ is the Dirac delta distribution. This result confirms that extinction is the almost sure outcome under strong stochastic perturbations.
It is also instructive to discuss the notion of an equilibrium in this stochastic context. For a deterministic system described by

$$\frac{dx}{dt} = f(x),$$

an equilibrium point x_e satisfies $f(x_e) = 0$, so that if $x(0) = x_e$, then $x(t) = x_e$ for all t. In contrast, for a stochastic differential equation of the form

$$dx = f(x)\,dt + g(x)\,dW,$$

a stochastic equilibrium (or steady state) x_{ES} is defined by the conditions

$$f(x_{ES}) = 0 \quad \text{and} \quad g(x_{ES}) = 0.$$

When these conditions hold, small deviations from equilibrium can be analyzed by setting

$$x = x_{ES} + U,$$

which leads to a linearized equation for the perturbation U:

$$dU = aU\,dt + bU\,dW.$$

In our logistic perturbed model, extinction ($x = 0$) acts as a stochastic equilibrium point. Linearizing the dynamics around $x = 0$, we find

$$dU = r_0 U\,dt + \omega U\,dW,$$

which describes the evolution of small perturbations near the extinct state.

In summary, the introduction of multiplicative noise in the logistic model not only modifies the dynamics but, under strong noise conditions, leads to extinction—even when the deterministic model predicts unbounded growth. The interplay between the intrinsic growth rate, the density-dependent term, and the noise intensity determines the long-term fate of the system.

5.2 Ito's Formula (Physical) Demonstration

We start by considering a stochastic differential equation (SDE) for a variable x:

$$dx = \underbrace{a(x)\,dt}_{O(dt)} + \underbrace{b(x)\,dW}_{O(\sqrt{dt})}.$$

Our goal is to derive the differential of a function $\Psi(x)$ using Ito's formula. Recall that if $\Psi(x)$ is twice differentiable, then

$$d\Psi = \Psi'(x)\,dx + \frac{1}{2}\Psi''(x)\,(dx)^2 + \dots.$$

Because dx contains a term of order $\sqrt{dt}$, the term $(dx)^2$ is of order dt. In particular, the properties of the Wiener process imply that

$$\boxed{\langle (dW)^2 \rangle = dt}.$$

To analyze the fluctuations in $(dW)^2$, we decompose it as follows:

$$(dW)^2 = dt + \left[(dW)^2 - dt\right] = dt + d\Omega,$$

where we define the random variable

$$y \equiv (dW)^2 - dt,$$

which satisfies $\langle y \rangle = 0$. Its variance is computed by

$$\mathrm{Var}(y) = \langle y^2 \rangle = \left\langle \left[(dW)^2 - dt\right]^2 \right\rangle.$$

Expanding the square, we have

$$\langle (dW)^4 - 2dt\,(dW)^2 + (dt)^2 \rangle.$$

Using the moment properties of the Wiener process:

$$\langle (dW)^2 \rangle = dt \quad \text{and} \quad \langle (dW)^4 \rangle = 3(dt)^2,$$

we obtain

$$3(dt)^2 - 2dt\,(dt) + (dt)^2 = 3(dt)^2 - 2(dt)^2 + (dt)^2 = 2(dt)^2.$$

Returning to the expansion for $d\Psi$, and substituting $dx = a(x)\,dt + b(x)\,dW$, we identify:

- The term $\Psi'(x)\,dx$ contributes a drift component and a stochastic component of order $O(\sqrt{dt})$.
- The term $\frac{1}{2}\Psi''(x)(dx)^2$ contributes an extra drift term of order $O(dt)$ due to the quadratic variation of dW.

Thus, the full expression for the differential of $\Psi(x)$ is given by

$$d\Psi = \left[\Psi'(x)a(x) + \frac{1}{2}\Psi''(x)b^2(x)\right]dt + \Psi'(x)b(x)dW.$$

This result is the celebrated Ito's formula. It shows that, unlike in ordinary calculus, the second derivative term multiplied by $\frac{1}{2}b^2(x)$ appears as a correction due to the non-negligible quadratic variation of the Wiener process. This additional term is what distinguishes stochastic calculus from its deterministic counterpart.

5.3 Probability Density Function and Markov Processes

Consider a stochastic process governed by the stochastic differential equation

$$dx = a(x)\,dt + b(x)\,dW.$$

Let $\rho(x,t)$ denote the probability density function (PDF) of $x(t)$, so that the probability of finding $x(t)$ in the interval $[\hat{x}, \hat{x} + d\hat{x}]$ is given by

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x} + d\hat{x}]\Big] = \rho(\hat{x},t)\,d\hat{x}.$$

In this way, the state of the system $x(t)$ is fully characterized by its PDF, $\rho(x,t)$.
More generally, if we consider

$$x \in \mathbb{R}, \quad t \in \mathbb{R},$$

the stochastic process $x(t)$ has the state space (SSP) $\mathbb{R}$ and evolves in continuous time. The probability that the process takes a value in a small interval at time t depends on its past history,

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x} + d\hat{x}]\Big] = \kappa\Big[\{x(\theta)\}_{0\le\theta\le t}\Big],$$

where κ represents the functional dependence on the trajectory $\{x(\theta)\}$ for $0 \le \theta \le t$.

Markov Process

A process is said to possess the **Markov property** if its future evolution depends solely on its present state rather than the entire past history. For the SDE above, the increment over an infinitesimal time interval dt can be written as

$$x(t+dt) = x(t) + a(x)\,dt + b(x)G_t\sqrt{dt},$$

where G_t is a Gaussian random variable with mean 0 and variance 1. Note that the update depends only on the current state $x(t)$, which exemplifies the Markov property.
To further illustrate this idea, consider a simple discrete deterministic process:

$$x_{t+1} = a\,x_t, \quad t \in \mathbb{N}_0.$$

Its solution is given by

$$x_t = a^t x_0.$$

Now, if we add a stochastic term to account for random fluctuations, we obtain

$$x_{t+1} = a\,x_t + \omega\,v_t,$$

where v_t is a random variable representing noise. In this context, the distribution of x_t at time t, denoted by $\rho(x,t)$, evolves according to the stochastic dynamics. Often, this distribution can be expressed as

$$\rho(x,t) = L(x_t),$$

where $L(x_t)$ denotes the law governing the evolution of the process.
This example highlights that in a Markov process the next state is determined exclusively by the most recent state rather than by the full history of the process.

6
Lecture 21/03/2025

6.1 Evolution of the Probability Density Function: The Fokker-Planck Equation

Last time we examined Malthusian processes, where the dynamics of the state variable $x(t)$ are governed by the stochastic differential equation (SDE)

$$dx = f(x)\,dt + g(x)\,dW.$$

Over an infinitesimal time increment dt, the update can be written as

$$x(t+dt) = x(t) + \underbrace{f(x)\,dt}_{O(dt)} + \underbrace{g(x)\,dW}_{O(\sqrt{dt})}.$$

Since the increment dW is of order $\sqrt{dt}$ and the change in x is infinitesimal, the evolution of the probability density function (PDF) $\rho(x,t)$ for $x(t)$,

$$\Pr\Big[x(t) \in [\hat{x}, \hat{x} + d\hat{x}]\Big] = \rho(\hat{x},t)d\hat{x},$$

depends only on the local properties of $\rho(x,t)$ (specifically its first and second derivatives). In other words, the future evolution of $\rho(x,t)$ is determined by its current state and the local changes, which leads to a Partial Differential Equation (PDE) for $\rho(x,t)$.
For instance, when the process $x(t)$ is a pure diffusion process—as is the case for a Wiener process—the PDF is given by

$$\rho(w,t) = A\,\exp\Big(-\frac{w^2}{2t}\Big),$$

where the normalization constant A ensures that the total probability is unity. In this case, one can derive that

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial w^2}.$$

This equation is a particular instance of the more general ***Fokker-Planck equation***, which describes how the probability density function of a stochastic process evolves over time. In the general case, the Fokker-Planck equation incorporates the contributions from both the drift term (related to $f(x)$) and the diffusion term (related to $g(x)$). It provides a powerful framework for understanding the dynamics of stochastic processes across various disciplines.

$$dx = a(x)dt + b(x)dW$$

$$y = \Psi(x) \quad \Rightarrow \quad d\Psi = [\Psi'(x)a(x) + Psi''(x)\frac{b^2(x)}{2}]dt + \Psi'(x)b(x)dW$$

$$\langle d\Psi\rangle = \langle \Psi'(x)a(x) + \Psi''(x)\frac{b^2(x)}{2}\rangle dt + \underbrace{\cancel{\langle \Psi'(x)b(x)\rangle dW}}_{\text{set to zero}}$$

The last term is oscillatory and averages to zero so we can ignore it. The first term is the drift term of the process $y(t)$. We have:

$$\frac{d}{dt}\langle\Psi\rangle = \langle \Psi'(x)a(x) + \Psi''(x)\frac{b^2(x)}{2}\rangle$$

The average is given by $\int_s z(x)\rho(x,t)dx$, where $z(x)$ is the function of x that we want to average. We have:

$$\frac{d}{dt}\int_{\mathbb{R}} \Psi(x)\rho(x,t)dx = \int_{\mathbb{R}} \left[\Psi'(x)a(x)\rho(x,t) + \Psi''(x)\frac{b^2(x)}{2}\rho(x,t)\right] dx$$

$$\int_{\mathbb{R}} \Psi(x)\frac{\partial\rho}{\partial t}(x,t)dx = \underbrace{\int_{\mathbb{R}} \Psi'(x)a(x)\rho(x,t)dx}_{I_1} + \underbrace{\int_{\mathbb{R}} \Psi''(x)\frac{b^2(x)}{2}\rho(x,t)dx}_{I_2}$$

Let's integrate by parts the first term on the right-hand side:

$$\begin{aligned}
I_1 =&\ \textstyle\int_{-\infty}^{+\infty} \Psi'(x)R(x,t)dx = \left|\Psi(x)R(x,t)\right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\Psi(x)\frac{\partial R}{\partial x}dx = 0 - \int_{-\infty}^{+\infty}\Psi(x)\frac{\partial R}{\partial x}dx \\
I_2 =&\ \textstyle\int_{-\infty}^{+\infty} \Psi''(x)z(x,t)dx = \left|\Psi'(x)z(x,t)\right| - \int_{-\infty}^{+\infty}\Psi'(x)\frac{\partial z}{\partial x}dx = -\int_{-\infty}^{+\infty}\Psi'(x)\frac{\partial z}{\partial x}dx \\
=&\ \textstyle -\left|\Psi(x)\frac{\partial z}{\partial x}\right|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\Psi(x)\frac{\partial^2 z}{\partial x^2}dx
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\int_{\mathbb{R}} \Psi(x)\frac{\partial\rho}{\partial t}(x,t)dx &= -\int_{\mathbb{R}}\Psi(x)\left[-\frac{\partial R}{\partial x}\right]dx + \int_{\mathbb{R}}\Psi(x)\left[\frac{\partial^2 z}{\partial x^2}\right]dx = \\
&= \int_{\mathbb{R}}\Psi(x)\left\{-\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right]\right\}dx
\end{aligned}$$

$$\int_{-\infty}^{+\infty}\Psi(x)\frac{\partial\rho}{\partial t}dx = \int_{-\infty}^{+\infty}\left\{-\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right]\right\}\Psi(x)dx$$

$$\begin{cases}
\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}[a(x)\rho(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{b^2(x)}{2}\rho(x,t)\right] \\
\int_{-\infty}^{+\infty}\rho(x,t)dx = 1
\end{cases}$$

Example:

Let's consider the following SDE:

$$\dot{x} = f(x) + \omega\xi(t) \quad \rightarrow \quad m\ddot{x} = -\dot{x} + f(x) + \omega\xi(t)$$

if $m \ll 1$ we have $\dot{x} = f(x) + \omega\xi(t) \quad f = -\dfrac{\partial U}{\partial x}$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x}[f(x)\rho(x,t)] + \dfrac{\omega^2}{2} - \dfrac{\partial^2 \rho}{\partial x^2} \\ \int_{-\infty}^{+\infty} \rho(x,t)dx = 1 \end{cases}$$

ρ depends on x and t, P depends only on x:

$$\begin{cases} \dfrac{\omega^2}{2}\dfrac{d^2 P}{dx^2} - \dfrac{d}{dx}[f(x)P] = 0 \\ \int P(x)dx = 1 \end{cases}$$

$$\frac{\omega^2}{2}\frac{dP}{dx} = f(x)P \quad \Rightarrow \quad \frac{\omega^2}{2}\frac{dP}{x} = \frac{\partial U}{\partial x}P \quad \Rightarrow \quad \frac{dP}{P} = \frac{2}{\omega^2}\frac{\partial U}{\partial x}dx$$

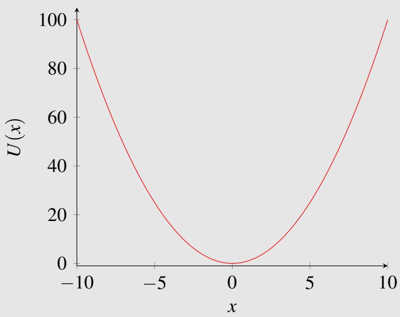
Now we can calculate C:

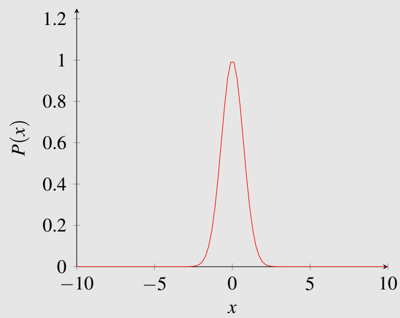
$$P(x) = Ce^{(2/\omega^2)U(x)}$$

…

$$C = \frac{1}{\int_{-\infty}^{+\infty} e^{(2/\omega^2)U(x)}dx}$$

…





…

6.2 Liouville Equation for Systems with Uncertain Initial Conditions

Consider a physical system governed by the ordinary differential equation

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0, \end{cases}$$

where the initial condition x_0 is not known exactly. Instead, we assume that x_0 is drawn from a probability distribution $\theta(x_0)$, so that the system is defined by

$$\begin{cases} \dfrac{dx}{dt} = a(x), \\ x(0) = x_0 \sim \theta(x_0). \end{cases}$$

Suppose there exists an equilibrium point x_e such that $a(x_e) = 0$, and that x_e is globally asymptotically stable (G.A.S.). This implies that, regardless of the uncertainty in the initial condition, the state $x(t)$ converges to x_e as $t \to \infty$. Consequently, the probability density function (PDF) $\rho(x,t)$ of $x(t)$ evolves towards a Dirac delta distribution centered at x_e:

$$\lim_{t\to\infty} \rho(x,t) = \delta(x - x_e).$$

The time evolution of $\rho(x,t)$ is governed by the **Liouville equation**. For the deterministic dynamics

$$dx = a(x)\,dt,$$

the Liouville equation is given by

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\Big[a(x)\,\rho(x,t)\Big].$$

This partial differential equation expresses the conservation of probability along the flow of the system. The term $-\frac{\partial}{\partial x}\left[a(x)\,\rho(x,t)\right]$ represents the net flux of probability density in the state space due to the vector field $a(x)$. As time evolves and the system converges to the stable equilibrium x_e, the density $\rho(x,t)$ becomes increasingly concentrated around x_e, reflecting the loss of uncertainty in the long-term behavior of the system.

$$m\dot{v} = -\gamma v + F_s(t) \tag{I}$$
$$Ri = -L\frac{di}{dt} - K\frac{dB_{ext}}{dt} \tag{II}$$
$$m\ddot{x} = -\hat{k}x - \gamma\dot{x} + \hat{F}(t) \tag{III}$$

...

$$\dot{z} = -\gamma z + \omega\xi(t) \leftrightarrow dz = .\gamma z dt + \omega\xi(t)dt \quad \Rightarrow \quad z = e^{-\gamma t}Q$$

$$-\gamma e^{\gamma t}Qdt + e^{-\gamma t}dQ = -\gamma e^{-\gamma t}Q + \omega dW \quad \Rightarrow \quad e^{-\gamma t}dz = \omega dW$$

So we have:

$$dQ = e^{\gamma t}\omega dW \quad = \quad Q(t) = z(0) + \omega\int_0^t e^{\gamma s}dW(s) \quad \Rightarrow \quad \boxed{z(t) = z_0 e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)}$$

$$\langle z(t)\rangle = \langle z_0\rangle e^{-\gamma t} + \Phi$$

...

$$z(t) = z(0)e^{-\gamma t} + \omega\int_0^t e^{\gamma(s-t)}dW(s)$$

$$\begin{aligned}
\langle z^2(t)\rangle = &\ \left\langle \left(z_0 e^{-\gamma t}dW(s) + \textstyle\int_0^t e^{\gamma(s-t)}\xi(s)ds\right)\left(z_0 e^{-\gamma t} + \textstyle\int_0^t e^{\gamma(\theta-t)\xi(\theta)d\theta}\right)\right\rangle \\
= &\ \left\langle \left(z_0^2 e^{-2\gamma t} + z_0 e^{-\gamma t}\textstyle\int_0^t e^{\gamma(\theta-t)}\xi(\theta)d\theta + z_0 e^{-\gamma t}\textstyle\int_0^t e^{\gamma(s-t)}\xi(s)ds + J(t)\right)\right\rangle \\
= &\ \ldots
\end{aligned}$$

$$\begin{aligned}
\langle J(t)\rangle = &\ \omega^2\textstyle\int_0^t\int_0^t e^{\gamma(\theta+s-2t)}\delta(\theta-s)d\theta ds \\
= &\ \omega^2 e^{-2\gamma t}\textstyle\int_0^t\left\{\int_0^t e^{\gamma(\theta+s)}\delta(\theta-s)d\theta\right\}ds \\
= &\ \omega^2 e^{-2\gamma t}\textstyle\int_0^t e^{2\gamma s}ds \\
= &\ \omega^2 e^{-2\gamma t}\left[\frac{e^{2\gamma t}-1}{2\gamma}\right] \\
= &\ \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)
\end{aligned}$$

$$Var[z(t)] = \underbrace{\langle z_0^2\rangle e^{-2\gamma t} - (\langle z_0\rangle)^2 e^{-2\gamma t}}_{=Var(z_0)e^{-2\gamma t}} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right) = Var(z_0)e^{-2\gamma t} + \frac{\omega^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$$

$$Var(z(t)) \to \frac{\omega}{2\gamma} = \sigma^2$$

$$z(t) = e^{-\gamma t} + \int_0^t e^{\gamma(s-t)}\xi(s)dt$$

...

$$z(0) = 0$$

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t \int_0^q e^{\gamma(\theta+s)} \delta(\theta - s) d\theta ds$$

We have 2 cases:

- $t < q$:

$$\langle z(t)z(q)\rangle = \omega^2 e^{-\gamma(t+q)} \int_0^t e^{2\gamma s} ds = \frac{\omega^2}{2\gamma} e^{-\gamma(t-q)} \dots$$

$$\langle z(t)z(q)\rangle = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|q-t|} - e^{-\gamma(q+t)} \right)$$

—

$$z_0 = 0 \quad \begin{cases} \langle z(t)\rangle = 0 \\ \langle z(q)\rangle = 0 \end{cases} \qquad C[\alpha, \beta] = \langle (\alpha - \hat{\alpha})(\beta - \hat{\beta})\rangle$$

$$\to C[z(t), z(q)]; \quad q = t + h \quad X[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} \left(e^{-\gamma|h|} - e^{-\gamma h} e^{-2\gamma t} \right)$$

$$R_z(h) = \lim_{t\to\infty} C[z(t), z(t+h)] = \frac{\omega^2}{2\gamma} e^{-\gamma|h|}$$

- $t > q$:

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$$\dot{z} = -\gamma z + \omega \xi(t) \leftrightarrow dz = -\gamma z dt + \omega dW$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial z}(\gamma z \rho) + \frac{\omega^2}{2}\frac{\partial^2 \rho}{\partial z^2}$$

…

Consider a system with a quadratic potential given by

$$U(z) = \frac{\gamma}{2} z^2.$$

In the absence of noise, the deterministic dynamics drive the system toward the minimum of the potential, so that

$$z(t) \to 0.$$

When we add a stochastic perturbation, the dynamics can be modeled by the Langevin equation

$$\dot{z} = -\gamma z + \omega \xi(t),$$

where ω quantifies the noise strength and $\xi(t)$ is a white noise process. (Note that the negative sign in front of γ ensures stability around $z = 0$.) In the stationary regime, the fluctuations of z are characterized by the variance

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Assuming that the system reaches a steady state, its stationary probability density function (PDF) is given by the Boltzmann distribution,

$$P_s(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right).$$

Substituting $\sigma^2 = \frac{\omega^2}{2\gamma}$, we obtain

$$P_s(z) = \frac{1}{\sqrt{2\pi \frac{\omega^2}{2\gamma}}} \exp\left(-\frac{z^2}{\omega^2/\gamma}\right) = \sqrt{\frac{\gamma}{\pi\omega^2}} \exp\left(-\frac{\gamma z^2}{\omega^2}\right).$$

This stationary PDF describes how the probability of finding the system at a value z is distributed. In particular:

- **For weak noise** ($|\omega| \ll 1$): The variance $\sigma^2 = \omega^2/(2\gamma)$ is very small, so the distribution $P_s(z)$ becomes sharply peaked around $z = 0$. In the limit of vanishing noise, $P_s(z)$ approaches a Dirac delta function, indicating that the system is almost surely at the equilibrium $z = 0$.
- **For strong noise** ($|\omega| \gg 1$): The variance is large, which results in a broad stationary PDF. The probability spreads over a wider range of z values, reflecting significant fluctuations around the equilibrium.

In summary, the behavior of the stationary PDF,

$$P_s(z) = \sqrt{\frac{\gamma}{\pi \omega^2}} \exp\left(-\frac{\gamma z^2}{\omega^2}\right),$$

is controlled by the noise intensity ω and the potential curvature γ. For small ω, the distribution is narrowly concentrated (nearly a Dirac delta), while for large ω, it becomes broad, indicating more pronounced stochastic fluctuations.

7.1 Stochastic Relaxation to Equilibrium

Suppose we wish to study the differential equation

$$\dot{y} = \gamma(b-y) + \omega\,\xi(t),$$

which describes a system relaxing toward an equilibrium value b with rate γ, perturbed by a stochastic force $\omega\,\xi(t)$ (with $\xi(t)$ being a white noise process).
To analyze the fluctuations around the equilibrium, we introduce the variable

$$z = y - b \quad \Longrightarrow \quad y = b + z.$$

Substituting this into the original equation yields

$$\dot{z} = -\gamma z + \omega\,\xi(t).$$

In the deterministic case (i.e., when $\omega = 0$), the equation for the deterministic component y_d is

$$\dot{y}_d = \gamma(b - y_d).$$

This equation has the solution

$$y_d(t) \to b \quad \text{as} \quad t \to \infty,$$

indicating that, in the absence of noise, the system relaxes exponentially to the equilibrium b.
When the stochastic term is present, the fluctuations z about b are governed by

$$\dot{z} = -\gamma z + \omega\,\xi(t).$$

At long times, the system reaches a stationary state in which z is a Gaussian random variable with zero mean and variance

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Thus, for large t the full solution behaves as

$$y(t) \sim b + \mathcal{N}\left(0, \frac{\omega^2}{2\gamma}\right),$$

or equivalently, in terms of the deviation z,

$$z_{\mathrm{cl}}(t) \sim \mathcal{N}\left(0, \frac{\omega^2}{2\gamma}\right) \quad \text{as} \quad t \to \infty.$$

This analysis shows that while the deterministic part drives the system to the equilibrium b, the stochastic fluctuations cause the state to be distributed around b according to a normal distribution with variance $\omega^2/(2\gamma)$.

7.2 Linearization around a Stable Equilibrium and Noise Scaling

Suppose x_e is an equilibrium point of the deterministic system

$$\dot{x} = F(x),$$

so that

$$F(x_e) = 0.$$

In the presence of stochastic perturbations, the dynamics are described by

$$\dot{x} = F(x) + \omega\,\xi(t),$$

where $\omega\,\xi(t)$ represents an additive noise term with $\xi(t)$ as white noise.
To study the behavior near the equilibrium, we expand $F(x)$ about x_e. Let y denote a variable so that

$$F(y) = F\left[x_e + (y - x_e)\right] = F(x_e) + F'(x_e)(y - x_e) + O((y - x_e)^2).$$

Since $F(x_e) = 0$, for small deviations we have

$$F(y) \approx F'(x_e)(y - x_e).$$

For a stable equilibrium we require

$$F'(x_e) < 0.$$

We define

$$F'(x_e) = -\gamma, \qquad \gamma > 0.$$

Thus, the linearized dynamics become

$$\dot{y} \simeq -\gamma(y - x_e) + \omega\,\xi(t).$$

It is convenient to introduce the deviation variable

$$z = y - x_e \quad \Longrightarrow \quad y = x_e + z.$$

Then the dynamics simplify to

$$\dot{z} = -\gamma z + \omega\,\xi(t), \qquad z(0) = 0.$$

The solution $z(t)$ is a Gaussian process with mean

$$\langle z(t) \rangle = 0,$$

and in the stationary state, its variance is given by

$$\sigma^2 = \frac{\omega^2}{2\gamma}.$$

Moreover, the autocorrelation function is

$$\langle z(t)\,z(t+h) \rangle = \frac{\omega^2}{2\gamma}\,e^{-\gamma|h|}.$$

This autocorrelation decays exponentially with a characteristic time $1/\gamma$. In the limit of rapid relaxation ($\gamma \to \infty$), the autocorrelation function becomes sharply peaked and tends toward a Dirac delta function.
To formalize this limit, we introduce a scaling relation between the noise intensity and the relaxation rate by setting

$$\boxed{\omega = c\,\gamma,}$$

with c a constant. Under this scaling, the SDE for z becomes

$$\dot{z} = -\gamma z + c\,\gamma\,\xi(t).$$

Alternatively, by defining $\delta = \frac{1}{\gamma}$, the equation can be written as

$$\dot{z} = -\frac{1}{\delta}z + \frac{c}{\delta}\,\xi(t).$$

The autocorrelation function for z then reads

$$R_z(h) = \frac{c^2}{2}\,\gamma e^{-\gamma|h|}, \qquad R_z(0) = \frac{c^2}{2}\,\gamma.$$

Its total area is given by

$$\int_{-\infty}^{+\infty} R_z(h)\,dh = \int_{-\infty}^{+\infty} \frac{c^2}{2}\,\gamma e^{-\gamma|h|}\,dh.$$

Since

$$\int_{-\infty}^{+\infty} e^{-\gamma|h|}\,dh = \frac{2}{\gamma},$$

it follows that

$$\int_{-\infty}^{+\infty} R_z(h)\,dh = \frac{c^2}{2}\,\gamma \cdot \frac{2}{\gamma} = c^2.$$

For small characteristic times ($\tau = 1/\gamma$ small), the autocorrelation function $R_z(h)$ approximates a white noise process:

$$\lim_{\gamma\to\infty} R_z(h;\gamma) = c^2\,\delta(h).$$

This derivation shows how, by choosing the scaling $\omega = c\,\gamma$, the fluctuations in the linearized dynamics around a stable equilibrium effectively become white noise in the fast relaxation limit.

The fourier transform

$$\frac{dz}{dt} = -\gamma z + \omega \xi(t)$$

let $f(t) : \mathbb{R} \to \mathbb{R}$ be a function s.t. $f^{(n)}(t)$ is continuous differentiable

$$\lim_{t \to \pm\infty} f(t) = 0$$

We have

$$\mathscr{F}[f(t)] = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \hat{f}(\omega)$$

$$\mathscr{F}[f'(t)] = \int_{-\infty}^{+\infty} f'(t) e^{-i\omega t} dt = \underbrace{\cancel{\left[f(t) e^{-i\omega t}\right]_{-\infty}^{+\infty}}}_{\to 0} - \int_{-\infty}^{+\infty} f(t)(-\omega) e^{-i\omega t} dt = i\omega \hat{f}(\omega)$$

so we have:

$$\boxed{\mathscr{F}[f'(t)] = i\omega \mathscr{F}[f(t)]}$$

$$f'(t) + f(t) = y(t) \quad \to \quad i\omega \hat{f} + \alpha \hat{f} = \hat{y}(\omega)$$

so:

$$\hat{f}(\omega) = \frac{y(\omega)}{i\omega + \alpha}$$

$$\left|\hat{f}(\omega)\right|^2 = \frac{|g(\omega)|^2}{\omega^2 + \alpha^2}$$

This is the principle of low-pass filter

$$\frac{dz}{dt} + \gamma z = \kappa \xi(t)$$
$$(i\omega + \gamma)\hat{z}(\omega) = \kappa \mathscr{F}[\xi(t)]$$
$$\hat{z}(\omega) = \frac{\kappa}{i\omega + \gamma} \mathscr{F}[\xi(t)]$$

Suppose we are able to calculate thepower spectrum.

$$R_\xi(h) = \lim_{t \to \infty} \langle \xi(t) \xi(t+h) \rangle = \lim_{t \to \infty} \delta(h) = \delta(h)$$

The spectrum is:

$$\mathscr{F}[\delta(h)] = \int_{-\infty}^{+\infty} \delta(h) e^{-i\omega h} dh = 1$$

$$R_{out}(h) = \frac{\kappa^2}{2\gamma} e^{-\gamma|h|}$$

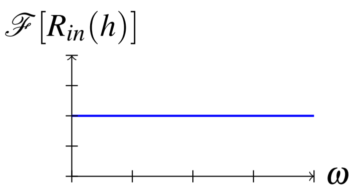
$$\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{2\gamma} \int_{-\infty}^{+\infty} e^{-\gamma|h|} e^{-i\omega|h|} dh = ...$$

We can split the integral in two parts:

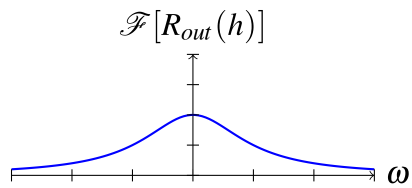
$$\begin{cases} \frac{\kappa^2}{2\gamma} \int_0^{+\infty} e^{-(\gamma+i\omega)h} = \left(\frac{\kappa^2}{2\omega}\right) \left[\frac{e^{-(\gamma+i\omega)h}}{-(\gamma+i\omega)}\right] = \left(\frac{\kappa^2}{2\gamma}\right) \frac{1}{\gamma+i\omega} \\ \frac{\kappa^2}{2\gamma} \int_{-\infty}^{0} e^{-(\gamma+i\omega)h} = ... = \left(\frac{\kappa^2}{2\gamma}\right) \frac{1}{\gamma-i\omega} \end{cases}$$

So the solution is:

$$\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{2\gamma} \left(\frac{1}{\gamma+i\omega} + \frac{1}{\gamma-i\omega}\right) = \frac{\kappa^2}{2\gamma} \frac{2\gamma}{\gamma^2+\omega^2} = \frac{\kappa^2}{\gamma^2+\omega^2}$$



Left: $\mathscr{F}[R_{in}(h)] = \kappa^2$.



Right: $\mathscr{F}[R_{out}(h)] = \frac{\kappa^2}{\gamma^2+\omega^2}$.

8 Lecture 28/03/2025

…

$$\frac{di}{dt} = -\frac{R}{L}i + \omega\xi_1(t) + \frac{K}{L}i\xi_2(t)$$

Suppose to have an equation $dx = a(x)dt + b_1(X)dW_1 + b_2(x)dW_2$
Then

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[a(x)P] + \frac{\partial^2}{\partial x^2}\left[\frac{b_1^2(x)b_2^2(x)}{2}P(x,t)\right]$$

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(0) \sim \theta(X_0) \end{cases} \quad \rightarrow \quad \begin{cases} dx = f(x)dt + OdW \\ \rho(x,0) = \theta(x) \end{cases}$$

$$\begin{cases} \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho] \\ \rho(x,0) = \theta(x) \\ \int_\Omega \rho(x,t)dx = 1 \end{cases}$$

Example:

Spring Let's consider the following SDE describing the motion of a particle moved by a spring force.

$$\begin{cases} \frac{dx}{dt} = -kx \\ x(0) \sim \mathcal{N}(\mu = 1.8, \sigma^2 = 0.05) \end{cases}$$

The solution of the SDE is given by:

$$x(t) = x_0 e^{-kt} \quad \Rightarrow \quad x(t) \rightarrow 0$$

Example: ***Stochastic Miultistability***

Let's consider the following SDE describing the motion of a particle in a double well potential.

$$\dot{x} = x - x^3 = x(1 - x^2)$$

This system has three equilibrium points:

$$\begin{cases} x_L = -1 & \Rightarrow LAS \\ x_C = 0 & \Rightarrow Unstable \\ x_R = +1 & \Rightarrow LAS \end{cases}$$

Where *LAS* means locally asymptotically stable.

$$\begin{cases} F = -\dfrac{dU}{dx} \\ U(x) = \dfrac{-x^2}{2} + \dfrac{x^4}{2} \end{cases}$$

We can calculate the limit of the probability density function $\rho(x,t)$ as $t \to \infty$.

$$\lim_{t\to\infty} \rho(x,t) = A_N \delta(x+1) + A_P \delta(x-1) + C\delta(x)$$

Where A_N and A_P are the normal and particular solutions of the system:

$$\begin{cases} A_P = \int_0^\infty \theta(x)dx = \Pr[x_0 > 0] \\ A_N = \int_{-\infty}^0 \theta(x)dx = \Pr[x_0 < 0] \\ C = \Pr[x_0 = 0] = 0 \end{cases}$$

...

$$\dot{x} = x - x^3 + \omega\xi(t)$$

$$\dot{x} = -\frac{dU}{dx} + \omega\xi(t)$$

$$P_s(x) = Ae^{-\dfrac{2}{\omega^2}U(x)}$$

Role of ω:

1. $\omega \ll 1$: we have small obscillations so we can use the linear approximation of the system. The system is stable and we have a single peak in the probability density function.
2. $\omega \gg 1$: we have large obscillations and in this case we can use the linear approximation of the system. The system is unstable and we have two peaks in the probability density function.

$$P_s(x) = A_s e^{-\frac{2}{\omega^2}} \approx 1$$

Observation:

Matematically it is possible to say that after a very long period of time, also the first case we could see that the system moves from one peak to the other. This is due to the fact that the system is not stable and we have a non-zero probability of moving from one peak to the other.

$$\frac{di}{dt} = -\frac{Ri + k\xi_1(t)}{L} + \omega\xi_2(t) = -\frac{R}{L} - \frac{K}{L}i\xi_2(t) + \omega\xi_1(t)$$

multiplicative noise

The general form of a SDE with multiplicative noise is given by:

$$dx = fxdt + g(x)dW$$

We already saw a case of multiplicative noise:

$$\dot{x} = (r_1 + \omega\xi(t))x - r_2x^2 \quad \rightarrow \quad dx = \underbrace{(r_1x - r_2x^2)}_{f(x)}dt + \underbrace{\omega x}_{g(x)}dW$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho] + \frac{\partial^2}{\partial x^2}\left[\frac{g^2(x)}{2}\rho\right]$$

$$\frac{d^2}{dx^2}\left[\frac{g^2(x)}{2}\rho\right] = -\frac{d}{dx}[f(x)P_s]$$

$$\begin{cases} \frac{d}{dx}\left[\frac{g^2(x)}{2}\rho\right] = -f(x)P \\ Q(x) = \frac{g^2(x)}{2}\rho \quad \Rightarrow \quad P = \frac{2}{g^2(x)}Q \end{cases}$$

$$\frac{dQ}{dx} = -f(x)\frac{2}{g^2(x)}Q$$

$$\frac{dy}{dx} = a(x)y \quad \Rightarrow \quad \begin{cases} y(x) = Ce^{A(x)} \\ A(x) = \int_\alpha^x a(s)ds \end{cases}$$

$$Q(x) = C\exp\left\{\int_\alpha^x -\frac{2f(s)}{g^2(s)}ds\right\} \quad \Rightarrow \quad P(x) = \frac{2}{g^2(x)}\exp\left\{\int_\alpha^x \frac{2f(z)}{g^2(z)}dz\right\}$$

Now we have to derivate the probability density function, before doing this, let's rewrite it in a way that simplifies calculus.
Considering that:

$$\frac{1}{g^2(x)} = e^{-\ln g^2(x)} = e^{-2\ln g(x)}$$

We can rewrite the probability density function as:

$$P(x) = 2C\exp\left\{-2\log(g(x)) + \int_\alpha^x \frac{2f(s)}{g^2(s)}ds\right\}$$

Now we can derivate:

$$P'(x) = 2C\exp\left\{-2\frac{g'(x)}{g(x)} + \frac{2f(x)}{g^2(x)}\right\}$$

…

$$\boxed{f(x) \geq g'(x)g(x)}$$

…

$$g(x) = \omega \Rightarrow P_s'(x) \geq 0 \equiv f(x) \geq 0 \qquad -\frac{dU}{dx} \geq 0$$

As results we have that:

1. Extrema of P_s are different from extrema of $U(x)$ (= equoilibrium point of the deterministic system).
2. The number of extrema of P_s is different from the numnber of equilibrium points.

Something biological idk why

Proteins in our cells follows a self-assembly model. The model is given by the following SDE:

$$\frac{dx}{dt} = \Pi_R(x) - \delta x$$

where $\Pi_R(x)$ is the production rate of the protein.
The production rate is given by the following equation:

$$\Pi_R(x) = R + \frac{kx^2}{k_0 + x^2}$$

The equilibrium points are given by:

$$\Pi_R(x) = \delta x$$

Let's now consider $\delta = \delta + \alpha\xi(t)$, where α is a constant and $\xi(t)$ is a white noise. We can rewrite the equation as:

$$\frac{dx}{dt} = \left(R + k\frac{x^2}{k_0 + x^2} - \delta x\right) + \alpha x \xi(t)$$

In this case we have a multiplicative noise, so we can use the previous results, but now we do not have to intercept Π_R with δ, but we have to intercept it with $\delta + \alpha^2$.

$$\begin{cases} R + k\dfrac{x^2}{k_0 + x^2} - \delta x \geq \alpha^2 x \\ R + k\dfrac{x^2}{k_0 + x^2} \geq (\delta + \alpha^2)x \end{cases}$$

$$\dot{z} = r_1 z - r_2 z, \qquad x = r_2 z \to z = \frac{x}{r_2}$$

$$\begin{cases} \dot{x} = r_1 x - x^2 \\ r_1 \to r_1 + \omega\xi(t) \end{cases} \quad \Rightarrow \quad dx = (r_1 x - x^2)dt + \omega x dW$$

$$f(x) = r_1 x - x^2, \qquad g(x) = \omega x$$

$$rx - x^2 \geq \omega^2 x \qquad \Rightarrow \qquad r_1 - x \geq \omega^2$$

in $\omega^2 = r_1$ we have a transition from an unimodal stationary distribution to a decreasing function.

$$P_s = \frac{2C}{\omega^2} x^{2r_1/\omega^2 - 2} \; e^{-(2/\omega^2)x}$$

which is not integrable

voglio anna a casa

$$dx = f(x)dt + g(x)dW$$

$$x(t+dt) = x(t) + f(x(t))dt + g(x(t))G_t\sqrt{dt}$$

We have that the probability of moving from s to a in a time dt is given by:

$$\Pr\left(x(t+dt)\text{"} = \text{"}a|x(t) = s\right) = \Omega(s,a)dt$$

While the probability of not moving is given by:

$$\Pr\left(x(t+dt)\text{"} = \text{"}s|x(t) = s\right) = 1 - \int \Omega(s,a)dt$$

9 Lecture: 31/03/2025

Last time we saw continuous state sspace and continuous time, but there are processes that are discrete in time and continuous in state space.

$$\Pr\left(x(t+dt)\text{''}=\text{''}a \mid x(t)=s\right) = \Omega(s,a)dt$$

where the probability of not moving is given by:

$$\Pr\left(x(x+dt)\text{''}=\text{''}s \mid x(t)=s\right) = 1 - dt\int \Omega(s,a)da$$

$$\Pr(x,t) = \Pr(x(t)=x)$$

$$\Pr(x(t+dt)=a \mid x(t)=s) = \Omega(s,a)dt$$

$$\Pr(x,t+dt) = \Pr(x,t)\left[1-dt\int\Omega(x,a)da\right] + dt\int \Pr(s,t)\Omega(s,x)ds$$

$$\Pr(x,t+dt) = \Pr(x,t) = -dt\Pr(x,t)\int\Omega(x,a)da + dt\int\Pr(s,t)\Omega(s,x)ds$$

dividing by dt we get:

$$\frac{\partial P}{\partial t} = \int \Pr(s,t)\Omega(s,x)ds - \Pr(x,t)\int\Omega(x,a)da$$

This is called the **master equation** and it is also a *Fokker-Plank equation*. The first term is the probability of moving to x from s, while the second term is the probability of moving away from x to a.

Tip: ***Compact Master Equation***

We can write the master equation in a more compact form:

$$\frac{\partial P}{\partial t}(x,t) = \int \left[\Pr(y,t)\Omega(y,x) - \Pr(x,t)\Omega(x,y)\right]dy$$

but this is not the preferred form by the teacher.

$$\Omega(y,x) = C\text{Heaviside}(\varepsilon - |x-y|)$$

$$\frac{\partial P}{\partial t} = C\int_{x-\varepsilon}^{x+\varepsilon}\Pr(s,t) - \Pr(x,t)\cdot C\cdot 2\cdot\varepsilon = \int_{x-\varepsilon}^{x+\varepsilon}\Pr(s,x)ds - 2\varepsilon C\Pr(x,t)$$

$$\frac{\partial P}{\partial t} = C\left\{\int_{x-\varepsilon}^{x+\varepsilon} \Pr(s,t)ds - 2\varepsilon \Pr(x,t)\right\} = C\left\{\int_{-\varepsilon}^{\varepsilon} \Pr(x+z,t)dz - 2\varepsilon \Pr(x,t)\right\}$$

where:

$$x-\varepsilon < s < x+\varepsilon; \quad z = s-x; \quad s = z+x; \quad -\varepsilon < z < \varepsilon$$

If we take the limit for $\varepsilon \to 0$ we can use the Taylor expansion:

$$\Pr(x+z,t) \approxeq \Pr(x,t) + \left.\frac{\partial P}{\partial s}\right|_{s=x} z + \frac{1}{2}\left.\frac{\partial^2 P}{\partial s^2}\right|_{s=x} z^2 = \Pr(x,t) + \frac{\partial P}{\partial x}z + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}z^2$$

$$\int_{-\varepsilon}^{\varepsilon} \Pr(x,t) + \frac{\partial P}{\partial x}z + \frac{\partial^2 P}{\partial z^2}\frac{z^2}{2}dz = 2\varepsilon P + 0 + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\left|\frac{z^3}{3}\right| = 2\varepsilon P + \frac{\varepsilon^3}{3}\frac{\partial^2 P}{\partial x^2}$$

$$\frac{\partial P}{\partial t} = C\left\{2\varepsilon P + \frac{\varepsilon^3}{3}\frac{\partial^2 P}{\partial x^2} - 2\varepsilon P\right\} \quad \Rightarrow \quad \frac{\partial P}{\partial t} = \left[\frac{C\varepsilon^3}{2}\right]\frac{\partial^2 P}{\partial x^2}$$

—

Another example:

$$\Omega(y,x) = A\delta\big(|x-y|-\varepsilon\big) = A\big[\delta(x-(y+\varepsilon)) + \delta(y-(x+\varepsilon))\big]$$

$$\partial_t P = \int \Pr(s,t)A\delta\big(|x-s|-\varepsilon\big)ds - 2A\Pr(x,t)$$

$$\partial_t P = A\left\{\underbrace{\Pr(x-\varepsilon,t)}_{I} + \underbrace{\Pr(x+\varepsilon,t)}_{II} - 2\Pr(x,t)\right\}$$

$$\partial_t P = A\left\{\underbrace{\Pr(x,t) - \varepsilon\frac{\partial P}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2}}_{I} + \underbrace{\Pr(x,t) + \varepsilon\frac{\partial P}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2} - 2\Pr(x,t)}_{II}\right\}$$

$$\frac{\partial P}{\partial t} = \left(\frac{A\varepsilon^2}{2}\right)\frac{\partial^2 P}{\partial x^2} \qquad \text{where} \quad \begin{cases} A = O\left(\dfrac{1}{\varepsilon^2}\right) \\ \ldots \end{cases}$$

—

$$x(t) \in \mathbb{Z}$$

$$\Omega(s,a) = A\delta\big(|a-s|-1\big)$$

$$\frac{\partial P}{\partial t} = \int \Pr(s,t)A\delta\big(|x-s|-1\big)ds - 2A\Pr(x,t)$$

$$\frac{\partial P}{\partial t} = A\Pr(x-1,t) + A\Pr(x+1,t) - 2A\Pr(x,t)$$

Which is the jump from $(x-1)$ to x, plus the jump backward from $(x+1)$ to x, minus the jump from x to $(x-1)$ and $(x+1)$.
So this is a Discrete Space and Continuous Time Markov Process (CTMC), becouse we have $x(0) \in \mathbb{Z}$.

10 Lecture: 04/04/2025

…

$$\frac{\partial P}{\partial t}(x,t) = \int_{S \in s} P(s,t)\Omega(s,x)ds - P(x,t)\int_{a \in s} \Omega(x,a)da$$

…

$$\frac{\partial P}{\partial t}(x,t) = \underbrace{rP(x+1,t)}_{backward} + \underbrace{rP(x-1,t)}_{forward} - 2rP(x,t)$$

…
More in general we can write

$$\Omega(s,a) = \sum_{j \in \mathbb{Z}} K_{s,a}\delta(s-a-j)$$

$$\Rightarrow \quad \frac{\partial P}{\partial t}(x,t) = \sum_{j \in \mathbb{Z}} \int P(s,t)K_{s,x}\delta(s-x-j)ds - P(x,t)\sum_{j \in \mathbb{Z}} \int K_{x,a}\delta(x-a-j)da$$

$$\frac{\partial P(x,t)}{\partial t} = \left(\sum_{j \in \mathbb{Z}} K_{x,x+j}K_{x+j,x}P(x+j,t)\right) - P(x,t)\left(\sum_{j \in \mathbb{Z}} K_{x,x-j}\right)$$

…

$$\begin{cases} S' = -\beta \frac{I}{N} S \\ I' = \beta \frac{I}{N} S - \gamma I \end{cases}$$

This system represents the dynamics of a population of individuals that can be in one of two states: susceptible (S) or infected (I). The parameter β represents the rate at which susceptible individuals become infected, while γ represents the rate at which infected individuals recover.
We have $(R = N - S - I)$ and:

$$I(t) \geq 0; \quad S(t) \geq 0; \quad R(t) \geq 0$$

$$(t, t+dt)X(t) = (S(t), I(t)) \in \mathbb{R}^2$$

The ”removal” of individuals accounts for the recovery of infected individuals, its probability is given by:

$$\Pr\left[\big(S(t+dt), I(t+dt)\big) = \big(S(t), I(t)-1\big) \;\Big|\; \big(s(t), I(t)\big)\right] = \gamma dt$$

…

$$\underbrace{\begin{pmatrix} S \\ I \end{pmatrix}}_{t} \to \underbrace{\begin{pmatrix} S \\ I-1 \end{pmatrix} = \begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{t+dt}$$

The contagion process instead is given by:

$$\underbrace{\begin{pmatrix} S \\ I \end{pmatrix}}_{t} \to \underbrace{\begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{t+dt}$$

…

We can think of σ and α as the states of the system at time t and $t+dt$, respectively:

$$\sigma = \begin{pmatrix} S_\sigma \\ I_\sigma \end{pmatrix}, \qquad \alpha = \begin{pmatrix} S_\alpha \\ I_\alpha \end{pmatrix}$$

…

$$\Omega(\sigma,\alpha) = \gamma I_\sigma \delta\left(\sigma - \alpha - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) + \beta \frac{I_\sigma}{N} S_\sigma \Omega\left(\sigma - \alpha - \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$

…

$$\frac{\partial P(S,I,t)}{\partial t} = \beta \frac{(I-1)}{N}(S+1)P(S+1,I-1,t) + \gamma(I+1)P(S,I+1,t) - \left(\gamma I + \beta \frac{I}{N} S\right) P(S,I,t)$$

Your starting point is an integer position; whatever jump you take, x + dt must be another integer position. In fact, the probability sitribution is non-zero only for integer positions.

10.1 Discrete time, Discrete state space

State s is discrete
Time $t \subseteq \mathbb{N} \cup \{0\}$

$$P\{x(t+1) = \alpha \big| x(t) = \sigma\} = \theta_{\sigma\alpha} \in [0,1]$$

$$P\{x(t) = \omega\} = P_\omega(t)$$

The probability of beiing in a state α at time $t+1$ is given by the probability of being in state σ at time t multiplied by the transition probability from σ to α:

$$\boxed{P_\alpha(t+1) = \sum_{\sigma \in S} \Pr_\sigma(t) \theta_{\sigma\alpha}}$$

$$P_\alpha(t+1) = P_\alpha(t)\theta_{\alpha\alpha} + \sum_{\sigma \in S \setminus \{\alpha\}} P_\sigma(t)\theta_{\sigma\alpha}$$

$$\theta_{\alpha,\alpha} = 1 - \sum_{\beta \in S \setminus \{\alpha\}}$$

$$P_\alpha(t+1) = P_\alpha(t)\left[1 - \sum_{\beta \in S \setminus \{\alpha\}} \theta_{\alpha,\beta}\right] + \sum_{\sigma \in S \setminus \{\alpha\}} P_\sigma(t)\theta_{\sigma,\alpha}$$

$$P_\alpha(t+1) = P_\alpha(t) + \sum_\sigma P_\sigma(t)\theta_{\sigma,\alpha} - P_\alpha(t) \sum_{\beta \in S\setminus\{\alpha\}} \theta_{\alpha,\beta}$$
$$P_\alpha(t+1) - P_\alpha(t) = \sum_\sigma P_\sigma(t)\theta_{\sigma,\alpha} - P_\alpha(t) \sum_{\beta \in S\setminus\{\alpha\}} \theta_{\alpha,\beta}$$
$$\frac{P_\alpha(t+1) - P_\alpha(t)}{U} = \sum_\sigma P_\sigma(t)\frac{\theta_{\sigma,\alpha}}{U} - P_\alpha(t) \sum_\beta \frac{\theta_{\alpha,\beta}}{U}$$

…

10.2 SIS model

A SIS model is a simple model of disease spread in a population. In this model, individuals can be in one of two states: susceptible (S) or infected (I). The dynamics of the system are governed by two parameters: the infection rate β and the recovery rate γ. The model assumes that individuals can move between these two states, with susceptible individuals becoming infected at a rate proportional to the number of infected individuals they come into contact with, and infected individuals recovering at a constant rate.
Moreover, μ is the natural death rate of the population, which is assumed to be the same as the birth rate. This means that the population size remains constant over time, and the total number of individuals in the population is given by $N = S + I$.

$$\begin{cases} S' = \mu - \mu S - \beta I S \\ I' = \beta I S - (\mu + \gamma) I \end{cases}, \qquad \begin{cases} \mu \to \mu + \omega_\mu \xi_\mu \\ \beta \to \beta + \omega_\beta \xi_\beta \end{cases}, \qquad \xi(t) = \begin{pmatrix} \xi_\mu \\ \xi_\beta \end{pmatrix}.$$

This is actually a stochastic model of disease spread, where the parameters μ, β, are subject to random fluctuations.

…

[a lot of stuff missing]

…

Case whit indipendent noise and ...

$$x_j = x(jh)$$
$$x_{j+1} = x_j + \alpha(x_j)h + \beta(x_j)\begin{bmatrix} G_j \\ G_{j+1} \end{bmatrix}\sqrt{h}$$

…

10.3 ma che cazzo ne so

Suppose we are in a region of R^N and all the points follows the following law:

$$\dot{x}_i = f(x_i)$$
$$n(x,0) = \tilde{O}(x)$$
$$\inf n(x,t)dx = N$$
$$\int n(x,0)dx = \int \tilde{O}(x)dx = N$$
$$c = [a,b]$$

$$N_c(t) = \int_a^b n(x,t)dx$$

$$P[x(t) \in [a,b]] \simeq \frac{N_c(t)}{n}$$

...

10.4 ma che cazzo ne so pt.2

Let's consider an interval $[t, t+dt]$ and some particles that follows the law:

$$\dot{x} = f(x) \equiv v(x)$$

Then, the number of particles that enter and exit the interval $[x, x+dx]$ at time t is given by:

$$\left[\begin{array}{ll} Enter: & n(x,t)v(x)dt \\ Exit: & n(x+dt,t)v(x+dt)dt \end{array}\right.$$

...

[missing a lot of stuff]

...

the product $n \cdot v$ is called ***current*** density and is denoted by $J(x,t)$:

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial x} J(x,t) = 0 \\ J(x,t) = n(x,t)v(x) \end{cases} \quad , \qquad \frac{\partial n}{\partial t} + \mathrm{div} J(x,t) = 0$$

...

... probabiulity current ...

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…
Ito:

$$dx = a(x)dt + b(x)dW \quad \Rightarrow \quad x(t+dt) = x(t) + a(x(t))dt + b(x(t))dW$$

Stratonovich:

$$dx = a(x)dt + b(x)\circ dW \quad \Rightarrow \quad x(t+dt) = x(t) + a(x(t))dt + b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW$$

$$x\left(t+\tfrac{dt}{2}\right) = x(t) + a(x(t))dt + b\left(x(t)\right)d\hat{W} \quad \text{where } d\hat{W} = W\left(t+\tfrac{dt}{2}\right) - W(t)$$

$$b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW = b(x(t))dW + b'(x(t))\left(a(x)dt + b\left(t+\tfrac{dt}{2}\right)d\hat{W}\right)dW + b'(x(t))b\left(x\left(t+\tfrac{dt}{2}\right)\right)d\hat{W}dW$$

$$\left\langle\left(W\left(t+\tfrac{dt}{2}\right) - W(t)\right)\left(W(t+dt) - W(t)\right)\right\rangle =$$
$$= \left\langle\left(W\left(t+\tfrac{dt}{2}\right)W(t+dt)\right) - W\left(t+\tfrac{dt}{2}\right)W(t) - W(t)W(t+dt) + W^2(t)\right\rangle$$

$$= t + \tfrac{dt}{2} - t - t + t = \tfrac{dt}{2} \quad \Rightarrow \quad b(x(t+dt))dW = \left(\tfrac{dt}{2}\right)b'(x(t))b\left(x\left(t+\tfrac{dt}{2}\right)\right) =$$

$$= \tfrac{dt}{2}b'(x(t))\left[b(x(t)) + O(\sqrt{dt})\right] = dt\frac{b'(x(t))b(x(t))}{2}$$

$$dx = a(x)dt + b\left(x\left(t+\tfrac{dt}{2}\right)\right)dW = a(x)dt + b(x)dW + \frac{b'(x)b(x)}{2}dt$$

$$dx = a(x)dt + b(x)\circ dW \quad \Rightarrow \quad \left[a(x) + \frac{b'(x)b(x)}{2}\right]dt + b(x)dW = dx$$

So the Stratonovich formula is equivalent to the Ito where instead of $a(x)$ we have: $a(x) + \dfrac{b'(x)b(x)}{2}$
…

$$I = \sum_i f(x(t_i + \tfrac{\Delta}{2}))(W(t_i+\Delta) - W(t_i)) \quad \Rightarrow \quad I = \sum_i \frac{f(x(t_i)) + f(x(t_i+\Delta))}{2}(W(t_i+\Delta) - W(t_i))$$

$$b(x(t+dt))dW = [b(x(t))b'(x(t))[a(x)dt + b(x)dW]]\,dW \quad \Rightarrow \quad b(x)dW + \frac{1}{2}b'(x)b(x)dt$$

so

$$x(t+dt) = x(t) + a(x(t)) + dW\frac{b(x)+b(x)}{2} + \frac{b'(x)b(x)}{2}dt = \left\{a(x) + \frac{b'(x)b(x)}{2}\right\}dt + b(x)dW$$

—

$$dx = a(x)dt + b(x) \circ dW$$

$$dx = \left[a(x) + \frac{b'(x)b(x)}{2}\right] dt + b(x)dW$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[a(x)\rho + \frac{b'(x)b(x)}{2}\rho\right] + \frac{\partial^2}{\partial x^2}\left[b(x)^2\rho\right] = -\frac{\partial}{\partial x}\left[a(x)\rho\right] + \frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\partial}{\partial x}\left(b(x)^2\right)\rho\right] + \frac{\partial^2}{\partial x^2}\left[b(x)^2\rho\right]$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}[a(x)\rho] + \frac{1}{2}\frac{\partial}{\partial x}\left[b(x)\frac{\partial}{\partial x}\{b(x)\rho\}\right]$$

$$dx = \left[a(x) + \frac{b'(x)b(x)}{2}\right] dt + b(x)dW$$

—

$$\dot{x} = \alpha(x) + ph(x) \qquad p \to p + \omega\xi(t)$$

$$\dot{x} = [\alpha(x) + ph(x)] + \begin{cases} \omega h(x)\xi(t) & \text{Ito} \\ \omega h(x) \circ \xi(t) & \text{Stratonovich} \end{cases}$$

If we choose the Ito interpretation, we have:

…

Else, if we choose the Stratonovich interpretation, we have:

$$dx = \left[\alpha(x) + \frac{b'(x)b(x)}{2}\right] dt + b(x)dW$$

…

$$a(x) + \frac{b'(x)b(x)}{2} = b'(x)b(x) \quad \Rightarrow \quad a(x) = \frac{b'(x)b(x)}{2}$$

—

Suppose we have a population

$$\dot{X} = Rx, \qquad R > 0$$

if we have $R \to R + \omega\xi(t)$

The Ito interpretation gives us:

$$R < \frac{\omega^2}{2} \to x(t) \to 0$$

Instead, the Stratonovich interpretation gives us:

$$dx = \underbrace{Rx}_{a(x)} dt + \underbrace{\omega x}_{b(x)} \circ dW \quad \Rightarrow \quad dx = \left[R + \frac{\omega^2}{2}\right] xdt + \omega x dW$$

let's consider the transformation $\quad y = \ln x \quad \to \quad x = e^y$:

$$dy = \left[R + \cancel{\frac{\omega^2}{2}} - \cancel{\frac{\omega^2}{2}}\right] dt + \omega dW$$

so

$$dy = Rdt + \omega dW \quad \Rightarrow \quad y(t) = y_0 + Rt + \omega W(t)$$

—

$$\dot{x} = a(x) + b(x)\eta_h(t)$$

$$\langle \eta_h(t) \rangle = 0$$

$\mathbb{R}_h$ is a function with a peak at $|z| < h$ and $\mathbb{R}_h(z) = 0$ for $|z| > h$.

If the limit $\lim_{h \to 0^+} \mathbb{R}_h(z) = \delta(\tau)$, then:

$$\dot{x}_h = a(x_h) + b(x_h)\eta_h(t) \quad \to \quad \dot{x} = a(x) + b(x) \circ \xi(t)$$

—

Let's consider a population and two opinions:

$$x + y = 1$$

People can change their opinion with a rate R and the ratio of changing from x to y is θ, and from y to x is k.

$$\begin{cases} \dot{x} = +\theta xy - kyx - \varepsilon x + \varepsilon y \\ \dot{y} = -\theta xy + kyx + \varepsilon x - \varepsilon y \end{cases}$$

where ε is the rate of changing opinion. (?)
Substituting $y = 1 - x$ we can rewrite the first equation as:

$$\dot{x} = x(1-x)(\theta - k) + \varepsilon(1-x) - \varepsilon x$$

$$\frac{dx}{dt} = \lambda x(1-x) + 1 - 2x, \quad \lambda \to \lambda + \alpha\xi(t)$$

$$dx = \underbrace{\{\lambda x(1-x) + 1 - 2x\}}_{a(x)} dt + \underbrace{\alpha x(1-x)}_{b(x)} \circ dW$$

…

$$dx = \{1 - 2x\} dt + \alpha x(1-x) \circ dW$$

$$a(x) = \tfrac{1}{2} b'(x) b(x), \qquad b(x) = \alpha(x - x^2), \qquad b'(x) = \alpha(1 - 2x)$$

so

$$1 - 2x = \frac{\alpha^2}{2} x(1-x)(1-2x)$$

which has two equilibrium points:

$$x_1 = \frac{1}{2}, \qquad x_2 : 1 = \frac{\alpha^2}{2} x(1-x) \ \ (?)$$

...

—

$$m\ddot{x} = -\gamma_T \dot{x} + F_T(x) + \omega_T \xi(t)$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{\gamma_T}{m} v + \frac{F_T(x)}{m} + \frac{\omega_T}{m} \xi(t) \end{cases}$$

to simplify the notation we can write:

$$\gamma = \frac{\gamma_T}{m}, \qquad F = \frac{F_T(x)}{m}, \qquad \omega = \frac{\omega_T}{m}, \qquad \underbrace{U = \int F_T(x)dx, \qquad U' = \frac{dU}{dx} = F_T(x)}_{?}$$

we get:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\gamma v + F(x) + \omega \xi(t) \end{cases}$$

$$\frac{\partial p}{\partial t} = -v \frac{\partial p}{\partial x} - \frac{\partial}{\partial v} \left[(F(x) - \gamma v) p \right] + \frac{\partial^2}{\partial v^2} \frac{\omega^2}{2}$$

$$\frac{\partial p}{\partial t} = -v \frac{\partial p}{\partial x} - F(x) \frac{\partial p}{\partial v} + \gamma \frac{\partial}{\partial v}(vp) + \frac{\partial^2}{\partial v^2} \frac{\omega^2}{2}$$

$$0 = -v \frac{\partial p}{\partial x} + U'(x) \frac{\partial p}{\partial v} + \gamma p + \gamma v \frac{\partial p}{\partial v} + \frac{\partial^2}{\partial v^2} \frac{\omega^2}{2}$$

$$p(x,v) = A(x)B(v)$$

$$-vA'(x)B(v) + U'(x)A(x)B'(v) + \gamma A(x)B(v) + \gamma v A(x)B'(v) + \frac{\omega^2}{2} B''(v) = 0$$

$$\underbrace{\overbrace{-v\frac{A'(x)}{A(x)} + U'(x)\frac{B'(v)}{B(v)}}^{1^{st}\text{ term}}}_{=0} + 1\,\overbrace{\gamma + \gamma v \frac{B'(v)}{B(v)} + \frac{\omega^2}{2} \frac{B''(v)}{B(v)}}^{2^{nd}\text{ term}} = 0$$

We have to options:

1. set the second term to zero
2. boh

Let's consider the first option and let's define some "test" variables B_T and B'_T. We have:

$$\frac{B'_T(v)}{B_T(v)} = -\eta v \quad \Rightarrow \quad B'_T(v) = -\eta v B_T(v) \quad \Rightarrow \quad B(v) = Ce^{-\eta v^2/2}$$

we have

$$B'(v) = -\eta v B(v), \qquad B''(v) = -\eta v B'(v) = -\eta v(-\eta v B(v))$$

MISSING: boh

$$P_s = \frac{1}{z} e^{-\frac{\gamma}{\omega^2}v^2 - \frac{2\gamma}{\omega^2}U(x)} = \frac{1}{z} e^{-\frac{2\gamma}{\omega^2}\left[\frac{v^2}{2}+U(x)\right]}$$

Applying back the transformation we have:

$$p(x,v) = \frac{1}{z} e^{-\frac{2\gamma_T}{\omega_T}\left[\frac{mv^2}{2}+U_T(x)\right]}$$

so:

$$\iint p_s(x,v)dxdv = 1, \qquad \frac{1}{z}\iint e^{-\frac{2\gamma_T}{\omega_T^2}E_T(x,v)}dxdv = 1$$

TODO: check if this is correct

MISSING: fishes example ?

$$dx = f(x)dt - \underbrace{(cxdt + \omega x dW)}_{\text{\#fishes killed in } (t,\, t+dt)}$$

We want the number of fishes to be positive.

MISSING: end of the lecture

12 Lecture: 05/05/2025

...

... if there is no linearity, ...

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{(\theta \int zP(z,t)\mathrm{d}z + (1-\theta)x - x^3)P\right\} + \frac{\omega^2}{2}\frac{\partial^2 P}{\partial x^2} \qquad N \gg 1$$

$$M(t) = \int_{\mathbb{R}} xP(x,t)dx$$

$$P_s(x, M_s) = C(M_s)\exp\{\theta M_s x + ...\}$$

MISSING: end of the formula above

This solution is not actually so "usable"

$$M_s = \int_{\mathbb{R}} xP_s(x; M_s)dx \quad \Rightarrow \quad M_s = \Psi(M_s)$$

$$M_s = \Psi(M_s) \quad \rightarrow \quad \text{"unique solution"}$$

There are more interesting cases, for instance when $\Psi(M_s)$ has more than one solution:
In this case, our system has more than one steady states. It means that we loose the unicity of the solution (so there is no more global actractiveness)

E.g:

$$\dot{X}_i = f(x_i, <x>) + g(x_i)\xi_i \quad N \gg 1$$

$$\dot{x} = f(x, M(t)) + g(x)\xi(t)$$

$$M(t) = \int zP(z,t)dt$$

The Fokker-Plank equation will be:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x, M(t))] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2(x)P]$$

The steady state solutions will be the solutions of the following equation:

$$\begin{cases} 0 = -\frac{\mathrm{d}}{\mathrm{d}x}[t(x, M_s)P] + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{g^2(x)}{2}P\right] \\ M_s = \int zP_s(z; M_s)dx \end{cases}$$

$$\boxed{M_s = \Psi(M_s)}$$

TODO: add linking sentence

$$P(x, M_s, \theta) = C(M, \theta) \exp\left[\frac{2}{\omega}\left(\theta M_s x + (1-\theta)\frac{x^2}{2} - \frac{x^4}{4}\right)\right]$$

$$\boxed{M_s = 0}$$

…(?)
$0 < \theta < \theta_c$
…(?)
So we have two solutions:

$$\begin{aligned} M_s &= a \\ M_s &= -a \end{aligned}$$

Example:

$$M_s = \Psi(M_s; \theta)$$

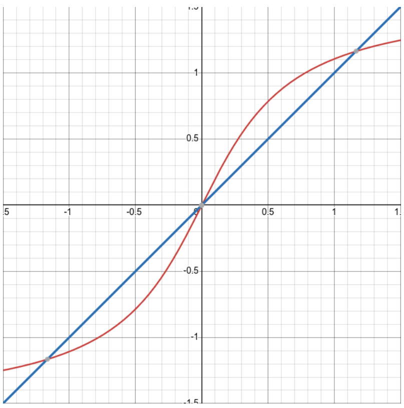
$$\begin{cases} y = M_s \\ y = \Psi(M_s; \theta) \end{cases} \quad \Rightarrow \quad \begin{aligned} &P_s(x; M_1) \\ &P_s(x; M_2) \\ &P_s(x; M_3) \end{aligned}$$

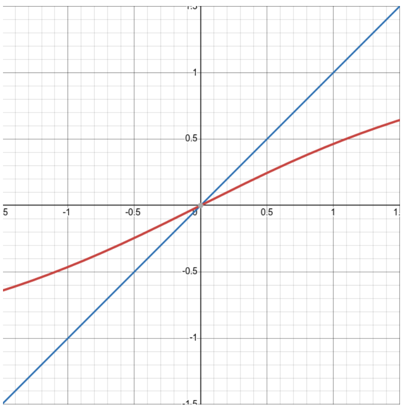
so for $\theta = \theta_1$ we have ***multistability***, while for $\theta = \theta_2$ we have ***monostability***.

Theorem 1.

$$\left| \left.\frac{\mathrm{d}\Psi}{\mathrm{d}M_s}\right|_{M_s = M_c} \right| < 1 \quad \Rightarrow \quad P_s(x; M_1, \theta^1) \textit{ is locally stable}$$

$$\left| \left.\frac{\mathrm{d}\Psi}{\mathrm{d}M_s}\right|_{M_s = M_c} \right| > 1 \quad \Rightarrow \quad P_s(x; M_2, \theta^2) \textit{ is locally unstable}$$





…

$$\dot{x} = (ax + x^3 - x^5) - D(x - M(t)) + \alpha(1 + x^2) \odot \xi(t)$$

$$M = \Psi(M; D, \alpha)$$

we have that for small D and α we have a unique solution, while for large D and α we have 5 different solutions.

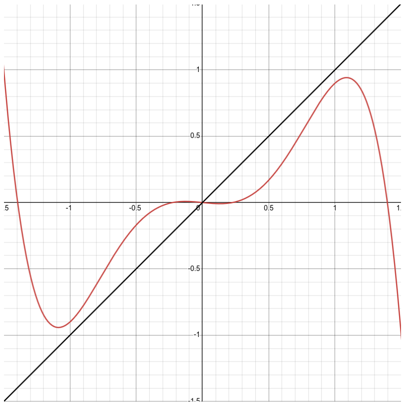


Figure 12.1: 1 solution

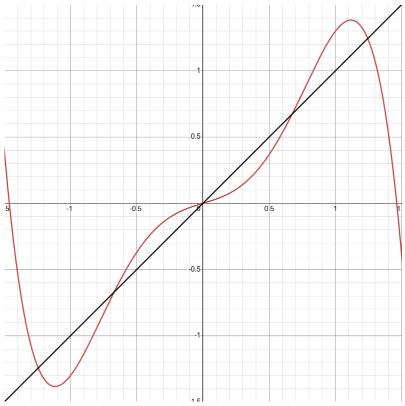


Figure 12.2: 5 solutions

So 0 is always a solution, and from a certain value of D we have 5 solutions, 3 of which are stable and 2 are unstable.

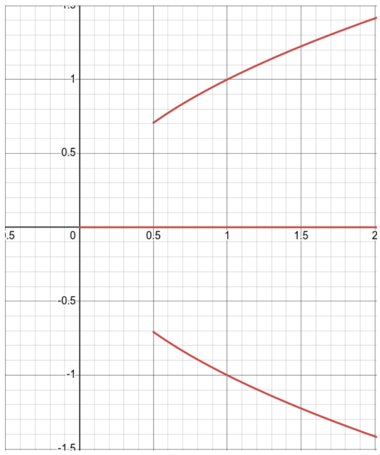


Figure 12.3: Stable solutions of the system

…

$$\dot{x} = F(x_i, < x >) + g(x_i, < x >)\xi_i(t)$$

An example iis the movement of a guitar string that vibrates.

$$m_i \ddot{x}_i = -\gamma \dot{x}_i$$

If the deviation is big, we have a function in the copmplex space, but if the deviation is small, we can use the linear approximation.

$$m_j \ddot{z}_j = -\gamma \dot{z}_j - k(z_j - z_{j-1}) - k(z_j - z_{j+1}) = -\gamma \dot{z}_j - k(z_{j-1} - 2z_j + z_{j+1})$$

we have now a discretization of the position of the string $z(t,x)$:

$$m_j \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}(t,x_j) = -\gamma \frac{\mathrm{d}z}{\mathrm{d}t}(t,x_j) + k[z(t,x_j+D) - 2z(t,x_j) + z(t,x_j-D)]$$

$$\mu \frac{\partial^2 z}{\partial t^2} = -\gamma \frac{\partial z}{\partial t} + c\frac{\partial^2 z}{\partial x^2} + \hat{\omega}\xi(x,t)$$

We can see the stochastic term as the wind that moves the string.

13 Lecture: 09/05/2025

13.1 Spatiotemporal noisy model

$$\frac{\partial \phi}{\partial t} = f(\phi) + g(\phi)\xi_m(r,t) + D\mathscr{L}[\phi] + h(\phi)F(t) + \xi_a(r,t)$$

- $f(\phi)$: deterministic part
- $g(\phi)$: multiplicative noise
- $D\mathscr{L}[\phi]$: linear part
- $h(\phi)$: additive noise

L[ϕ] is a Laplacian or a integral operator
Examples:

- $$\mathscr{L}[\phi] = \nabla^2\phi$$
- $$\mathscr{L}[\phi] = -a_0\nabla^2\phi - \nabla^4\phi$$
- $$\mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi = -(K_0^2 + 2k_0\nabla^2 + \nabla^4)\phi$$

Observation:

If we apply the fourier transform of:

$$\mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi = -(K_0^2 + 2k_0\nabla^2 + \nabla^4)\phi$$

we get:

$$...F(\phi)$$

$$\mathscr{L}[\phi(r)] = \int \phi(r')\omega(r-r')\mathrm{d}r'$$

How do we simulate this equation? We can simply obtain the domain and discretize it.
Lattice.based Approximation:

...

Field coupling approximation:

$$l(\phi_i,\phi_j) = w_i\phi_i + \sum_{j\in nn(i)} w_j\phi_j$$

For example:

$$\mathscr{L}[\phi] = \nabla^2\phi \approx l(\phi_i, \phi_j) = \frac{1}{\Delta^2} \sum_{j \in nn(i)} (\phi_j - \phi_i)$$

If we have a stochastic process which is discrete in time and space, we have:

$$\langle \xi(r,t)\xi(r',t') \rangle = sC\left(\frac{|r-r'|}{d}, \frac{|t-t'|}{\tau_c}\right)$$

As in the purely temporal noise, τ_c is a measure of the temporal memory of the noise, d is the spatial memory of the noise.
The spatiotemporal brother of the ornstein-uhlenbeck noise is ”Ojalvo e al” noise.

$$\frac{\partial\phi}{\partial t} = a\phi + D\nabla^2\phi + \xi_{gn}$$

Observation: ***Ojalvo e al and the Ornstein-Uhlenbeck process***

If we set $D = 0$ we have a series of Ornstein-Uhlenbeck processes at each point of the domain.

$$\frac{\partial\phi}{\partial t} = a\phi + \xi_{gn}$$

Noise induced pattenrs §

$$\frac{\partial\phi}{\partial t} = f(\phi) + g(\phi)\xi_m(r,t) + D\mathscr{L}[\phi] + ...$$

…
Perturbed Swift-Hohenberg model:

$$\frac{\partial\phi}{\partial t} = a\phi + D\mathscr{L}[\phi] + \xi_{gn}...$$

$$\frac{\partial\phi}{\partial t} = f(\phi) + D\mathscr{L}[\phi] = a\phi - D(\nabla^2 + k_0^2)^2\phi$$

Transitory pattern that disappear.
Additive noise generate patenrs

$$\frac{\partial\phi}{\partial t} = a\phi + D\mathscr{L}[\phi] + \xi_{gn}, \qquad \mathscr{L}[\phi] = -(\nabla^2 + k_0^2)^2\phi$$

Permanent patterning: details change in time.
we can distinguish two cases:

- $a < 0$
- $a > 0$

with multiplicative noise it can induce bimodality in the pdf of ϕ:

$$\frac{\partial\phi}{\partial t} = a\phi - phi^3 + \phi\xi_{gn} + D\mathscr{L}[\phi]$$

—

A bad model of glaciations

$$\mathrm{d}x = [x(a - x^2) + A\cos\Omega t]\mathrm{d}t$$

where
- x: is the (normalized) Earth's temperature
- $A\cos(\Omega t)$: small periodic variations of the solar irradiation.

We have that if A is small, $x(t)$ fluctuates around $+\sqrt(a)$.
The model fails.
Including stochastic noise:

$$\mathrm{d}x = [x(a-x^2) + A\cos\Omega t]\mathrm{d}t + \varepsilon \mathrm{d}W$$

where ε is the noise intensity.
This time we have a white noise, according to:
- ε is small: the noise is negligible
- ε is large: the noise is dominant

they finally managed to model the glaciations.
—

Spatial Stochastic Resonance

$$\frac{\partial \phi}{\partial t} = a\phi - \phi^3 + D\frac{\partial^2 \phi}{\partial x^2} + F(t) + \varepsilon \xi_{gn}$$

TODO: check the formula
…

14
Discrete Time Markov Chains

We consider the case where the time is a subset of the integers ($t \subset \mathbb{Z}$ or $t \subset \mathbb{N}_0$)
The state space is a finite set $S = \{s_1, s_2, \ldots, s_N\}$

$$P\{x(t+1)|x(0), x(1), \cdots, x(t)\} = P\{x(t+1)|x(t)\}$$

The probability that the process in $t+1$ is σ is the sum of the probability that the process in t is δ and the probability that the process in $t+1$ is σ given that the process in t is δ.

$$P\{x(t+1) = \sigma\} = \sum_{\delta \in S} P\{x(t+1) = \sigma | x(t) = \delta\} P\{x(t) = \delta\} = \sum_{\delta \in S} P\{x(t) = \delta\} \theta_{\delta\sigma}$$

with $\theta_{\delta\sigma} \in [0,1]$.
So we have:

$$P_\sigma(t+1) = \sum_{\delta \in S} P_\delta(t) \theta_{\delta\sigma} \quad \Rightarrow \quad P(t+1) = P(t)\Theta(t)$$

Where $P(t) = [P_1(t), P_2(t), \cdots, P_N(t)]$ is the probability vector at time t and $\Theta(t)$ is the transition matrix at time t.
We have:

$$P(1) = P(0)\Theta, \quad P(2) = P(1)\Theta = P(0)\Theta^2, \quad P(3) = P(2)\Theta = P(0)\Theta^3, \quad \cdots$$

We can write:

$$P(t) = P(0)\Theta^t = P(0) \prod_{q=0}^{t-1} \Theta(q)$$

We have two properties:

$$\sum_{\sigma \in S} \theta_{\delta\sigma} = 1, \qquad \sum_{\delta \in S} P_\delta(t) = 1$$

So we have:

$$P_\sigma(t+1) = \sum_{\delta \in S} P_\delta(t) \theta_{\delta\sigma} \quad \Rightarrow \quad \sum_{\sigma \in S} P_\sigma(t+1) = \sum_\sigma \sum_\delta P_\delta(t) \theta_{\delta\sigma} = \sum_\delta P_\delta(t) \sum_\sigma \theta_{\delta\sigma} = 1$$

So
MISSING: something

—

Let's consider again the discrete time Markov chain.

$$P_\sigma(t+1) = P(t)\Theta \quad \Rightarrow \quad P(t) = P(0)\Theta^t$$

…

$$P^\infty = P^\infty \Theta$$

Let's study the eigenvalues of Θ:

$$\Theta v = \lambda v$$

$$\sum_{\sigma} \theta_{\delta\sigma} = 1$$

(... non ho capito perchè ma l'autovettore di Θ è formato da tutti 1 ...)
If the multiplicity of the eigenvalue is more than 1, we have multiple solutions to our system.

Tip:

Sometimes this equation is written as:

$$P_a(t+1) = \sum_{s \in S} W_{as} P_s(t)$$

where W_{as} ...

MISSING: something

—

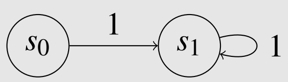
Example: ***Simple markov chain***

The simplest markov chain is the one with only one state, with a transition with probability 1 from the state to itself.

$$P(0) = [1], \qquad P(t) = [1]$$

—

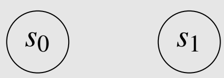
Another simple markov chain is the one with two states, with a transition with probability 1 from the state s_0 to the state s_1 and with probability 1 from the state s_1 to the state s_1 itself.



We have the situation below:

$$P(0) = [1, 0], \qquad P(t) = [0, 1]$$

A more complex example is the following:



TODO: add the transitions in the figure

$$P_0(t+1) = \theta_{00} P_0(t) + P_1(t)\theta_{10}$$

$$P_1(t) = 1 - P_0(t) \quad \Rightarrow \quad P_0(t+1) = \theta_{00} P_0(t) + \theta_{10}(1 - P_0(t))$$

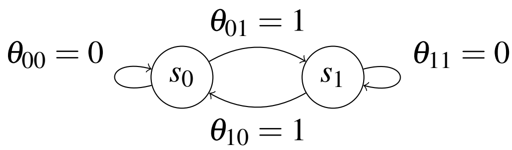
$$\boxed{P_0(t+1) = (\theta_{00} - \theta_{10}) P_0(t) + \theta_{10}}$$

If we have a case like:

$$x(t+1) = ax + b$$

then we have a term b that don't allow us to use the resolution formula we are used to $(x(t) = x(0)a^t)$, but ...

...



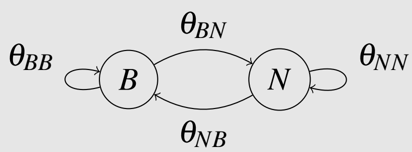
$$\begin{aligned}
P(0) &= [1,0] = x(0) = 0 \\
P(1) &= [0,1] = x(1) = 1 \\
P(2) &= [1,0] = x(2) = 0 \\
&\vdots
\end{aligned}$$

Which is periodic.

...

Example: ***Bora example***

Let's consider the case of Bora in Trieste. We have two states: B and N.

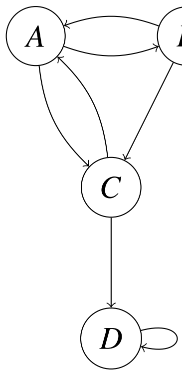


We have the following transition probabilities:

$$\theta_{BB} \approxeq \frac{NT^{BB}}{N_B} \qquad \theta_{BN} \approxeq \frac{NT^{BN}}{N_B}$$
$$\theta_{NB} \approxeq \frac{NT^{NB}}{N_N} \qquad \theta_{NN} \approxeq \frac{NT^{NN}}{N_N}$$

This model is actually too artificious, because the transition probabilities are not independent, and also depends on other parameters, like the temperature.

The most natural representatiin of Markov Chains are oriened graphs.
Let's consider a 4 states model:



...boh...

15
Lecture: 12/05/2025

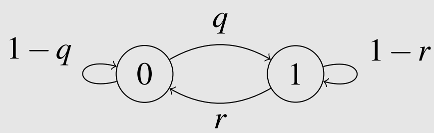
$$P(t+1) = P(t)\theta = ...$$

$$P(t) = P(0)\theta^t = P(0)...$$

???????

Example: ***2 States Markov Chain***

Let's consider the following example (similar to the previous one):



We have two states (0 and 1), and a probability q of going from 0 to 1, and a probability r of going from 1 to 0. Let $P_0(t)$ be the probability that the system is in state 0 at time t, and $P_1(t)$ the probability of being in state 1 at time t. The evolution of these probabilities is given by:

$$\begin{cases} P_0(t+1) = (1-q)P_0(t) + rP_1(t) \\ P_1(t+1) = qP_0(t) + (1-r)P_1(t) \end{cases}$$

Since there are only two states, the probabilities must sum to 1:

$$P_1(t) + P_0(t) = 1$$

So, $P_1(t) = 1 - P_0(t)$.

Stationary (Equilibrium) Distribution: In the long run, the probabilities reach a steady state (stationary distribution), where $P_0(t+1) = P_0(t) = P_0^e$ and $P_1(t+1) = P_1(t) = P_1^e$. Setting the evolution equations to equilibrium, we get:

$$\begin{cases} P_0^e = (1-q)P_0^e + rP_1^e \\ P_1^e = qP_0^e + (1-r)P_1^e \end{cases}$$

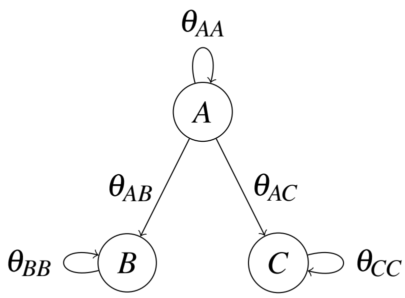
Using $P_1^e = 1 - P_0^e$, substitute into the first equation, we obtain:

$$P_0^e = \frac{r}{q+r}$$

Similarly:

$$P_1^e = \frac{q}{q+r} = 1 - P_0^e$$

Let's consider a Markov chain with three states: A, B, and C. The transitions are as follows:



Let $P_A(t)$, $P_B(t)$, $P_C(t)$ be the probabilities of being in states A, B, C at time t. The initial state is:

$$P(0) = (P_A(0), P_B(0), P_C(0))$$

Evolution Equations:

$$P_A(t+1) = \theta_{AA} P_A(t)...$$

…

$$P_B(t+1) = P_B(t) + \theta_{AB} P_A(t) = P_B(t) + \theta_{AB} P_A(0) \theta^t$$

$$P_B(1) = P_B(0) + \theta_{AB} P_A(0)$$

$$P_B(2) = P_B(1) + \theta_{AB} P_A(1) = P_B(0) + \theta_{AB} P_A(0) + \theta_{AB} P_A(1)$$

$$P_B(t) = P_B(0) + \theta_{AB} P_A(0) \left[1 + \theta_{AA} + \theta_{AA}^2 + ... + \theta_{AA}^{t-1}\right]$$

…

$$P_B^{eq} = P_B(0) + P_A(0) \frac{\theta_{AB}}{1 - \theta_{AA}}$$

$$P_C^{eq} = P_C(0) + P_A(0) \frac{\theta_{AC}}{1 - \theta_{AA}}$$

$$det(\theta - \lambda z) = \begin{vmatrix} \theta_{AA} - \lambda & \theta_{AB} & \theta_{AC} \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

...

$$P^e = H \text{ Diag}[1, 0, ..., 0] H^{-1} P(0)$$

and we have:

$$P^e = W P^e$$

Properties of W:

1. At least $\lambda_1 = 1$
2. $|\lambda_j| \leq 1$ for all j
3. $\sum_{j=1}^{n} V_j = 0$

where V_j is the eigenvector of W associated with λ_j.
…

$$WV_j = \lambda_j V_j$$

$$\sum_{C=1}^{n} W_{RC}(V_j)_C = \lambda_j (V_j)_R$$

$$\sum_{R=1}^{n}\sum_{C=1}^{n} W_{RC}(V_j)_C = \lambda_j \left(\sum_{R=1}^{n}(V_j)_R\right)$$

$$\underbrace{\sum_{C=1}^{n}(V_j)_C}_{S} = \lambda_j \underbrace{\left(\sum_{R=1}^{n}(V_j)_R\right)}_{S} \quad \Rightarrow \quad (1-\lambda_j)S = 0$$

…

$$P^{eq} = H\ \mathrm{Diag}[1,0,...]P(0)$$

$$P(0) = C_2V_1 + \sum_{j=2}^{N} C_jV_j$$

…

$$P(0) = U_1 + \sum_{j=2}^{N} C_jV_j$$

Where U_1 is the normalized vector
…

$$P(0) = C_1V_1 + C_2V_2 + \sum_{j=3}^{N} C_jV_j$$

$$1 = C_1\|V_1\|_1 + C_2\|V_2\|_1 \quad \Rightarrow \quad C_2 = \frac{1-C_1\|V_1\|_1}{\|V_2\|_1}$$

$$P(0) = C_1U_1 + (1-C_1)U_2 + \sum_{j=3}^{N} C_jU_j$$

$$P(t) = W^tP(0) = C_1U_1 + (1-C_1)U_2 + \sum_{j=3}^{N} C_j\lambda_j^tV_j$$

So C_1 depends on the initial state $P(0)$:

$$\boxed{C_1 = P(P(0))}$$

…

$$P(t) = C_1P(0)U_1 + (1-C_1(P(0)))U_2 + \sum_{j} C_j\lambda_j^tV_j$$

$$P^e = C_1 U_1 + (1 - C_1) U_2$$

—

If we have a problem and we want to study the time of remaining in a state A, we can simply consider all the transitions from A to other states as a single transition from A to a new state B (which represents the rest of the world), and it is given by the sum of all the initial transitions.
So we have now only 2 transitions: the loop from A to itself, and the new AB transition.
We have:

$$x(0) = A, \qquad P(0) = (1,0)$$

$$P_A(t) = \theta_{AA}^t$$

…

$$P_A(T) = \theta_{AB} = \theta_{AB}\theta_{AA}^t = (1-\theta_{AA})\theta_{AA}$$

So the average value of T is:

$$\langle T \rangle = \sum T \theta_{AB}\theta_{AA}^t = \theta_{AB}\sum_{t=0}^{\infty} t\theta_{AA}^t$$

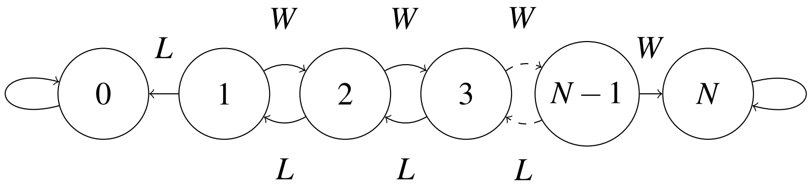
$$= \theta_{AB}\left[0 + \theta_{AA}^1 + 2\theta_{AA}^2 + 3\theta_{AA}^3\right] = \theta_{AB}\theta_{AA}\left[1 + 2\theta_{AA} + 3\theta_{AA}^2 + ...\right]$$

$$= \theta_{AB}\theta_{AA}\frac{\mathrm{d}}{\mathrm{d}\theta_{AA}}\left[-1 + 1 + \theta_{AA} + \theta_{AA}^2 + ...\right]$$

$$= \theta_{AB}\theta_{AA}\frac{\mathrm{d}}{\mathrm{d}\theta_{AA}}\left[\frac{1}{1-\theta_{AA}}\right]$$

16 Lecture: 16/05/2025

Consider a game, you have a probability p of winning, and a probability $1-p$ of losing.
You can bet 1 euro each time, and if you win you gain 1 euro, if you lose you lose 1 euro, if you run out of money you stop playing.



We can find that $L < W$.
…

$$P(t) = W^t P(0)$$

…

$$(\theta^t)_{AB} > 0$$

…
0 and N are ***absorbing*** states.
We have that a state σ is ***transient*** if (for $t \gg 1$) $P_\sigma(t) = 0$.
Another important property is the ***ergodicity***: All the states are visited during the process lifetime, and there is no periodicity.
…
We can approximate the probability distribution of the transitions as follows:

$$P_A^{eq} = \frac{\#(x(t) = A)}{T}, \qquad P_B^{eq} = \frac{\#(x(t) = B)}{T}$$

Which is *the number of times the process is in state A at time t divided by the total number of transitions T* (Same for B).
…
The ***return time*** is the time it takes for a process to return to a given state. For a state σ, we can define the return time T_σ as:

$$T_\sigma = \min\{t > 0 : x(t) = \sigma | x(0) = \sigma\}$$

The ***mean return time*** $\langle T_\sigma \rangle$ is the average time it takes for the process to return to state σ after leaving it. For an ergodic Markov chain, the mean return time is related to the equilibrium probability by:

$$\langle T_\sigma \rangle = \frac{1}{P_\sigma^\infty}$$

This makes intuitive sense: if a state has a high equilibrium probability, it will be visited frequently, leading to a short mean return time. Conversely, states with low equilibrium probabilities will have longer mean return times.

All the problems continuous in time but that involves a discrete state space can be solved using a Contionuous Time Markov Chain (CTMC), which are a subset of all the stochastic processes with discrete state space and continuous time.

$$\Pr\left\{x(t) = \sigma \mid x(\theta), \quad \theta \in [0,t]\right\}$$

If we consider an infinitesimal time interval $\mathrm{d}t$, we can write:

$$\Pr\left\{x(t+\mathrm{d}t) = \alpha \mid x(t) = \sigma\right\}$$

So the probability of eving more than one transition in the interval is:

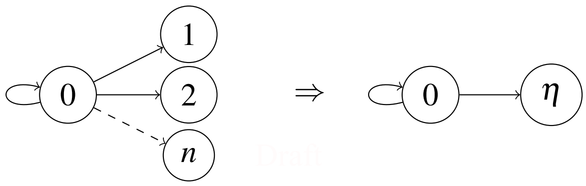
$$\begin{array}{lll} \Pr\left\{\geq 2 \text{ events} \quad (t, t+\mathrm{d}t)\right\} & = 0 \\ \Pr\left\{1 \text{ event} \quad (t, t+\mathrm{d}t)\right\} & = O(\mathrm{d}t) \\ \Pr\left\{0 \text{ events} \quad (t, t+\mathrm{d}t)\right\} & = 1 - O(\mathrm{d}t) \simeq 1 \end{array}$$

We can define the ***transition rate*** as:

$$\Pr\left\{x(t+\mathrm{d}t) = \alpha \mid x(t) = \sigma\right\} = W_{\alpha\sigma}\mathrm{d}t$$

$W_{\alpha\sigma}$ is always positive and can be $>$ or $<$ than 1.

...



To compute the probability of remaining in the state 0, we can collapse all the other states into a single state η.

...

$$x(t) = \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$$

With: $(S,I) \to (S-1, I+1)$,

$$S(t+\mathrm{d}t) = S(t) - 1, I(t+\mathrm{d}t) = I(t) + 1$$

$$x(t+\mathrm{d}t) = x(t) + (-1, 1)$$

...

$$\begin{cases} \Pr(Cont) = \Pr\left\{x(t+\mathrm{d}t) = x(t) + \binom{-1}{1}\right\} = \beta\dfrac{I(t)}{N}S(t)\mathrm{d}t \\ \Pr(Rec) = \Pr\left\{x(t+\mathrm{d}t) = x(t) + \binom{0}{-1}\right\} = \gamma I(t)\mathrm{d}t \end{cases} \Rightarrow \begin{cases} W_{Cont} = \beta\dfrac{I(t)}{N}S(t) \\ W_{Rec} = \gamma I(t) \end{cases}$$

...

$$W_{\eta\sigma} = \sum_{\alpha \in \mathscr{S}\setminus\{\sigma\}} W_{\alpha\sigma}$$

$$A \xrightarrow{W} B \qquad\qquad \mathscr{S} = \{A, B\}$$

We have that:

$$\begin{cases} \Pr\{x(t+\mathrm{d}t) = B | x(t) = A\} = W_{AB}\mathrm{d}t \\ \Pr\{x(t+\mathrm{d}t) = B | x(t) = B\} = 1 \end{cases}$$

$$P_A(t+\mathrm{d}t) = (1 - W_{AB}\mathrm{d}t)P_A(t) \quad \Rightarrow \quad P_A(t+\mathrm{d}t) = P_A(t) - W P_A(t)\mathrm{d}t$$

$$P_A'(t) = -W P_A(t)$$

$$\boxed{P_A(t) = P_A(0)e^{-Wt}}$$

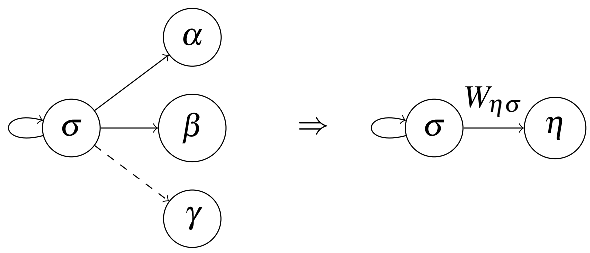
...
Let's define T the time of remaining in the state A:
...

$$\mathscr{P}_I(T) = We^{-WT}$$

Since this is an exponential distribution, we can compute the mean return time as:

$$\langle T \rangle = \frac{1}{W}$$

...
—



If we want to compute the probability of remaining in the state σ for a time T, we can write:

$$P\big\{x(T+\mathrm{d}t) \in \mathscr{S} \setminus \{\sigma\} | x(T) = \sigma\big\}$$

Also in this case we can collapse the states into a single state η.

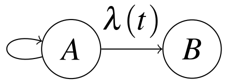
$$W_{\eta\sigma} = \sum_{\alpha \neq \sigma} W_{\alpha\sigma} = W_{sum}$$

So we can say that:

$$\begin{cases} t_\eta \quad x(t_\eta) = \sigma \\ T \sim \exp(\lambda = W_{sum}) \end{cases} \quad \Rightarrow \quad t_{next} = t_\eta + T$$

$$\begin{aligned} &\Pr\{x(t_\eta + T + \mathrm{d}t) = \alpha | x(t+T) = \sigma\} = W_{\alpha\sigma} \cdot 1 \cdot \mathrm{d}t, \\ &\Pr\{x(t_\eta + T + \mathrm{d}t) = \beta | x(t+T) = \sigma\} = W_{\beta\sigma} \cdot 1 \cdot \mathrm{d}t, \\ &\dots \end{aligned}$$

...



$$P_A(t+\mathrm{d}t) = (1-\lambda(t)\mathrm{d}t)P_A(t)$$

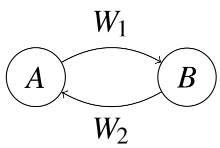
$$P_A'(t) = -\lambda(t)P_A(t) = P_A(t) = P_A(0)e^{-L(t)}, \qquad L(t) = \int_0^t \lambda(s)\mathrm{d}s$$

$$\mathscr{P}(T) = \lambda(T)e^{-L(t)}$$

$$T \sim \mathscr{P}(T) = \lambda(T)e^{-L(t)}$$

$$P(\sigma \to \alpha) = \frac{\lambda_\sigma(T)}{L_{sum}(t)}$$

—



$$\begin{cases} \dot{P}_1(t) = W_2 P_B(t) - W_1 + P_A(t) \\ \dot{P}_2(t) = W_1 P_A(t) - W_2 P_B(t) \end{cases} \quad \Rightarrow \quad \begin{array}{l} \dot{P}_A + \dot{P}_B = 0 \\ P_A + P_B = 1 \end{array}$$

$$\begin{cases} \dot{P}_A = \sum (W_{Ay}P_y - W_{yA}P_A) \\ \sum_{A\in\mathscr{S}} P_A(t) = 1 \end{cases}$$

And we have that:

$$\dot{P}_A = -W_1 P_A + W_2(1-P_A) = W_2 - (W_1+W_2)P_A$$

So, for an equilibrium we have that:

$$0 = W_2 - (W_1+W_2)P_A^{eq}$$

$$P_A^{eq} = \frac{W_2}{W_1+W_2}, \qquad P_B^{eq} = \frac{W_1}{W_1+W_2}$$

We can write our transition matrix:

$$W = \begin{bmatrix} -W_1 & W_2 \\ W_1 & -W_2 \end{bmatrix}$$

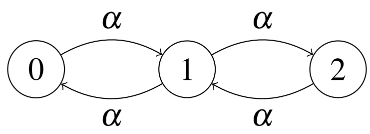
Which is singular, so one of the eigenvalue is 0.

…

$$
\begin{cases}
\dot{P}_\sigma = \sum(W_{\sigma y}P_y - W_{y\sigma}P_\sigma) \\
\underline{\dot{P}} = A\underline{P} \\
\sum_\sigma \dot{P}_\sigma(t) = 0
\end{cases}
$$

$$
0 = Ap^{eq}
$$

…



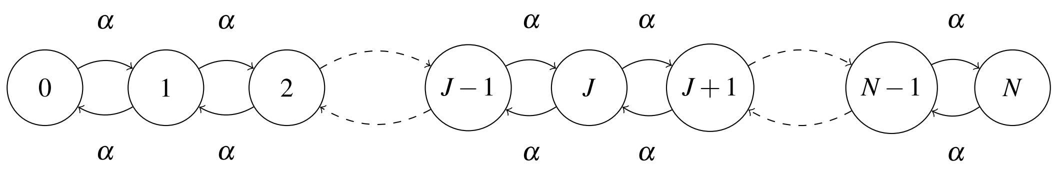
The steady state is given by:

$$
\begin{aligned}
\dot{P}_0 = &\quad -\alpha P_0 \quad +\alpha P_1 \\
\dot{P}_1 = &\quad \alpha P_0 \quad -2\alpha P_1 + \quad \alpha P_2 \\
\dot{P}_2 = &\quad \qquad \alpha P_1 - \quad \alpha P_2
\end{aligned}
$$

…

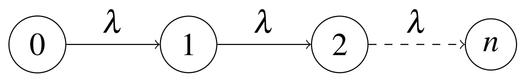
$$
3\phi = 1 \quad \Rightarrow \quad \boxed{\phi = \frac{1}{3}}
$$

we can now generalize this to a system with N states:



…

Let's consider an unidirectional chain, with a single transition rate λ:



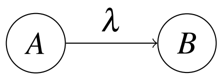
$$
\begin{aligned}
&\dot{P}_0 = -\lambda P_0 \\
&\dot{P}_1 = \lambda P_0 - \lambda P_1 \\
&\dots \\
&\dot{P}_\sigma = \lambda P_\sigma - \lambda P_{\sigma-1}
\end{aligned}
$$

…

The probability distribution $P_n(t) = \Pr\{N(t) = n\}$ has a simmetry property:

$$
P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}
$$

17 Lecture 19/05/2025



We have two properties:

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

$$\lim_{t\to\infty} P_n(t) = 0 \quad \forall n \geq 0$$

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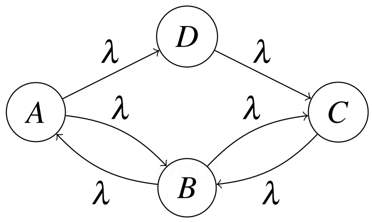


To solve a general case, we need to solve the following system of equations:

$$\dot{P} = A(t)P$$

we have:

$$A_{ij} = \lambda(t)\alpha_{ij}$$

$$\mathrm{d}\theta = \lambda(t)\mathrm{d}t$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(t)$$

$$\theta = \int_0^t \lambda(s)\mathrm{d}s = L(t)$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda(t)\alpha P$$

So:

$$P(t) = e^{\lambda(t)\alpha} P(0)$$

—

Let’s consider now a case where we have a periodic

$$A(t) = \begin{cases} A_1 & 0 < \mathrm{mod}(t,T) < Q \\ A_2 & Q < \mathrm{mod}(t,T) < T \end{cases}$$

$$t = 0 \quad \Rightarrow \quad P(0) = P_0$$

$$0 < t < Q \quad \Rightarrow \quad \dot{P} = A_1 P \quad \Rightarrow \quad P(t) = e^{A_1 t} P_0$$

$$Q < t < T \quad \Rightarrow \quad \dot{P} = A_2 P \quad \Rightarrow \quad P(t) = e^{A_2(t-Q)} P(Q)$$

$$T < t < T+Q \quad \Rightarrow \quad \dot{P} = A_1 P \quad \Rightarrow \quad P(t) = \underbrace{e^{A_2(T-Q)} e^{A_1 Q}}_{B(T,Q)} P(0)$$

so

$$\begin{aligned} P(T) &= B(T,Q)P(0) \\ P(2T) &= B^2(T,Q)P(0) \\ P(nT) &= B^n(T,Q)P(0) \end{aligned}$$

—

CTMC model:

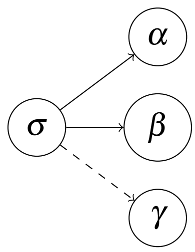
$$f = \{\sigma_1, \dots, \sigma_n\}$$

$$\dot{P} = A(t)P$$

$$P(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = e_y$$

—

Gillespie algorithm:



$$X(0) = \sigma_0$$

We want to find when where will be the jump and which will be the next state.
Also in this case we can collapse the states in a single one; the transition rate is given by the sum of the transition rates of the original states.

$$\sigma \xrightarrow{W_\Sigma} \eta$$

So we can write:

$$\mathscr{P}(T) = W_\Sigma e^{-W_\Sigma T}$$

And we have:

$$t_{n+1} = t_n + T_n, \quad \text{where} \quad T_n \sim \mathscr{P}(T)$$

$$P(\sigma_i) = \Pr(Event(\sigma \to \sigma_i) ara\ t_n + T_n | oneE \sim t_n + T_n)$$

TODO: Che cazzo ha scritto il prof? ↑↑↑↑↑

$$\Pr\{\sigma \to \sigma_i \ \ (T, T + \mathrm{d}t)\} \ = \ (W_x \mathrm{d}t) e^{-W_\Sigma T} \ = \ \boxed{P(\sigma_i)(W_\Sigma e^{W_\Sigma T} \mathrm{d}t)}$$

MISSING: qualcosa

…

1. $$T_n \sim W_\Sigma e^{-W_\Sigma T} \qquad \Rightarrow \qquad t_{n+1} = t_n + T_n$$
2. $$P(\sigma_i) = \frac{W_{\sigma_i}}{W_\Sigma} \qquad t_n \to n$$

A first application of the Gillespie algorithm is the following:

Example: ***Gillespie Algorithm: Contagion and Recovery (SIR Model)***

Consider a system with two possible events:

- **Contagion**: $(S, I) \to (S-1, I+1)$, with rate $W_{con} = \beta \frac{I}{N} S$
- **Recovery**: $(S, I) \to (S, I-1)$, with rate $W_{rec} = \gamma I$

The Gillespie algorithm proceeds as follows:

1. **Initialize**: $t_0 = 0$, $x_0 = (S_0, I_0)$
2. **Compute the rates**:
   - $W_{con} = \beta \frac{I_n}{N} S_n$
   - $W_{rec} = \gamma I_n$
   - $W_\Sigma = W_{con} + W_{rec}$
3. **Draw two random numbers** $U_1, U_2 \sim \mathscr{U}[0, 1]$
4. **Determine the time to the next event**:
   The waiting time T_n is exponentially distributed:
   $$T_n = \frac{-\ln U_1}{W_\Sigma}$$
   Update the time: $t_{n+1} = t_n + T_n$

5. **Determine which event occurs**:
   Compute the probabilities:
$$\Pr(\text{contagion}) = \frac{W_{con}}{W_\Sigma}, \qquad \Pr(\text{recovery}) = \frac{W_{rec}}{W_\Sigma}$$
   If $U_2 < \Pr(\text{contagion})$, a contagion event occurs; otherwise, a recovery event occurs.
6. **Update the state**:
   - If contagion: $(S_{n+1}, I_{n+1}) = (S_n - 1, I_n + 1)$
   - If recovery: $(S_{n+1}, I_{n+1}) = (S_n, I_n - 1)$
7. **Repeat** from step 2 until $I_n = 0$ or another stopping criterion is met.

Note: The formula for T_n comes from inverting the CDF of the exponential distribution:
$$CDF(T) = 1 - e^{-W_\Sigma T} \implies T = \frac{-\ln(1-U_1)}{W_\Sigma}$$
Since U_1 is uniformly distributed, so is $1 - U_1$, and it is common to write $T = \frac{-\ln U_1}{W_\Sigma}$.

$$\sigma \xrightarrow{W_\Sigma} \eta$$

$X(t_n) = \sigma$

$$\mathscr{P}(T_n) = W_\Sigma(t_n + T)e^{-\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z}$$

$$CDF(T_n) = 1 - e^{-\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z}$$

…
calling $\Psi(T) = \int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z$

$$1 - e^{-\Psi(T_n)} = U_n \quad \Rightarrow \quad e^{-\Psi(T_n)} = 1 - U_n \quad \Rightarrow \quad \Psi(T_n) = -\ln(1-U_n)$$

…

$$\beta(t) = \beta_n(1 + \delta\cos(\omega t))$$

$$x(t_n) = (S_n, I_n)$$

$$\int_{t_n}^{t_n+T_n} W_\Sigma(z)\mathrm{d}z = -\ln(1-U_n)$$

$$\int_{t_n}^{t_n+T_n} \left[\gamma I_n + \beta_n \frac{I_n}{N} S_n(1 + \delta\cos(\omega z))\right] \mathrm{d}z = -\ln(1-U_n)$$

$$\left[\gamma I_n + \beta_n \frac{I_n}{N} S_n\right] T_n + \beta_n \frac{I_n}{N} S_n \frac{1}{\omega}\left[\sin(t_n + T_n) - \sin(t_n)\right] = -\ln(1-U_n)$$

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MISSING: everything

19 Lecture 30/05/2025

19.1 Random Walks

Let's consider a particle that moves on a line.

$$\rho(x,t) = \Pr[\text{Particle is at location } (x, x+\mathrm{d}x) \text{ at time } t]$$

we have:

$$\frac{\partial \rho}{\partial t} = \int [\rho(y,t)\Omega(y,x) - \rho(x,t)\Omega(x,y)]\,\mathrm{d}y$$

Let's consider now the probability that the particle just jumped in the location $(x, x+\mathrm{d}x)$ at time t:

$$p(x,t) = \Pr[\text{Particle just jumped in the location } (x, x+\mathrm{d}x) \text{ at time } t]$$

We define a jump weight as:

$$\eta(y)$$

so:

$$p(x,t) = \int_0^t \int_{-\infty}^{+\infty} \mathrm{d}y \eta(y) p(x-y, t-T)\omega(t)\mathrm{d}t + \delta(x)\delta(t)$$

Definition: ***Fourier and Laplace Transforms***

- **Fourier transform**
  Let's recall the Fourier transform:

$$\mathscr{F}_x[f(x)] = \int_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x$$

  we have:

$$\mathscr{F}_x[f(x) \star g(x)] = \hat{f}(k)\hat{g}(k)$$

- **Laplace transform**
  Let's recall the Laplace transform:

$$\mathscr{L}_x[f(x)] = \int_0^{\infty} f(x)e^{-st}\mathrm{d}t$$

  we have:

$$\mathscr{L}_x[f(t) \star g(t)] = \tilde{f}(s)\tilde{g}(s)$$

$$p(x,t) = [p(x,t) \star_s \eta(x)] \star_t \omega(t)$$

Let's call $\bar{p}(k,s) = \hat{\tilde{p}}(x,t)$, then we can write:

$$\bar{p}(k,s) = \hat{\eta}(k)\tilde{\omega}(s)\bar{p}(k,s) + 1 \qquad \Rightarrow \qquad \bar{p}(k,s)\{1 - \hat{\eta}(k)\tilde{\omega}(s)\} = 1$$

$$\bar{p}(k,s) = \frac{1}{1 - \hat{\eta}(k)\tilde{\omega}(s)}$$

$$\Psi(T) = \int_T^{+\infty} \omega(q)\mathrm{d}q = 1 - \int_0^T \omega(q)\mathrm{d}q$$

and we can rewrite $\rho(x,t)$ as:

$$\rho(x,t) = \int_0^t p(x,t-T)\Psi(T)\mathrm{d}T$$

The Laplace transform of $\Psi(T)$ is:

$$\mathscr{L}[\Psi(T)] = \mathscr{L}[1 - \int_0^T \omega(q)\mathrm{d}q] = \frac{1}{s} - \frac{1}{s}\tilde{\omega}(s)$$

$$\bar{\rho}(k,s) = \left(\frac{1-\tilde{\omega}(s)}{s}\right)\left(\frac{1}{1-\hat{\eta}(k)\tilde{\omega}(s)}\right)$$

Cauchy distribution (???)

$$\omega(T) \sim \frac{1}{T^{1+\alpha}} \qquad T \gg 1$$

$$\omega(T) = \frac{A}{1 + B\,T^2} \qquad T \gg 1 \qquad \Rightarrow \qquad \omega(T) \sim \frac{A}{B}\frac{1}{T^2}$$

Then its integral is infinite (cause compares a logarithm).

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \qquad \dot{x} = c\xi(t)$$

$$\langle x^2(t)\rangle \equiv t^\alpha$$

for $0 < \alpha < 1$ we have subdiffusion, for $1 < \alpha < 2$ we have superdiffusion.

Bibliography

[1] "2.080 Structural Mechanics Stability of Elastic Structures". In: 2014. URL: https://api.semanticscholar.org/CorpusID:30595920.