Pulsar Glitches: Insights from Brownian Motion, State-Dependent Poisson Processes, and Hybrid Model

A Comparative Study of Glitch Mechanisms

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Introduction

Pulsar Glitches

Pulsars: rapidly rotating, highly magnetized neutron stars that slowly spin down. Occasionally, they exhibit sudden spin-up events called **glitches**.

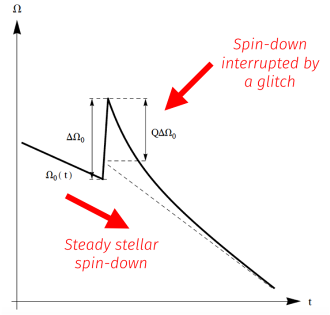


Figure 1: Glitch and spin-down [3]

Glitches likely result from:

- sudden unpinning of superfluid vortices
- crustal failures

Both release stored angular momentum, causing the spin-up.

Role of Stochastic Processes

Stochastic processes can be used to model glitches in a microphysics-agnostic approach.

In particular:

- **Brownian Process** - *Carlin & Melatos (2020)* [1]
  - Internal stress builds up stochastically as a Brownian process between glitch events.
  - A glitch occurs when the stress exceeds a critical ***threshold***.
- **State-dependent Poisson Process** - *Fulgenzi, & Melatos (2017)* [2]
  - The glitch rate is a function of the stress.
  - The stress increases linearly between glitch events.

Brownian Process

Brownian Process: Equation of Motion

We model the globally averaged stress X in the system as a stochastic variable.

Stochastic evolution: Between glitches, $X(t)$ evolves as a Wiener process, described by the Langevin (Itô) equation:

$$\frac{dX(t)}{dt} = \xi + \sigma B(t) \tag{1}$$

where:

- ξ is the drift (mean stress accumulation rate)
- σ is the diffusion coefficient (random fluctuations)
- $B(t)$ is white noise (zero mean, unit variance)

Brownian Process: Fokker-Planck Equation

The **Fokker-Planck equation** for $p = p(X, t \mid X_0)$ is:

$$\frac{\partial p}{\partial t} = -\xi \frac{\partial p}{\partial X} + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial X^2} \tag{2}$$

Boundary conditions:

- *Initial:* $p(X, 0 \mid X_0) = \delta(X - X_0)$
- *Absorbing at threshold:* $p(X = X_c, t \mid X_0) = 0$
- *Reflecting at zero:* $\left.\frac{\partial p}{\partial X}\right|_{X=0} = \frac{2\xi}{\sigma^2} p(0, t \mid X_0)$

Brownian Process: How to sample glitches?

Trigger Mechanism:

A glitch is triggered **deterministically** when the accumulated stress $X(t)$ reaches a predefined **critical threshold,** X_c.

$$X(t) \geq X_c \implies \text{Glitch!}$$

Glitch Size ($\Delta X_{release}$):

After a glitch, the stress decreases. The stress drop $\Delta X_{release}$ is drawn from a probability distribution $\eta(\Delta X)$, with several forms for considered by Carlin & Melatos.

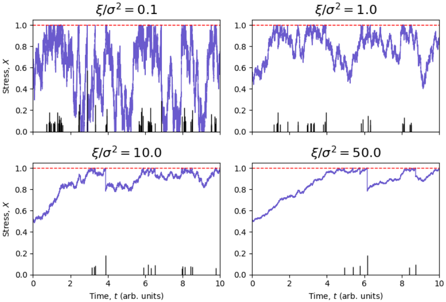
- **Power-law**: $\eta(f) \propto f^{-\delta}$ for $\beta \leq f \leq 1$, where $f = \Delta X / X_c$.
- **Gaussian**: Truncated Gaussian for f.
- **Lognormal**: Truncated Lognormal for f.

Brownian Process: key parameter

The key parameter is the ratio $\mu = \xi X_c / \sigma^2$:

- High μ: drift dominates, stress grows steadily
- Low μ: diffusion dominates, large random fluctuations

The **power-law** distribution seems to best fit the data.



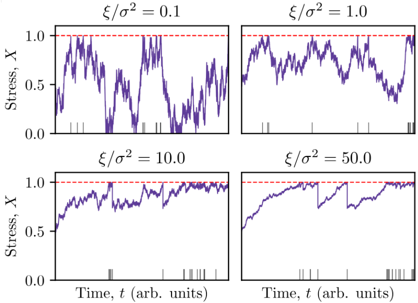


Figure 2: Stress varying μ, glitches $\sim$ power-law distribution.
Left: my model, **Right**: Carlin & Melatos (2020)

Brownian Process: other glitch sampling distribution

The conclusions regarding μ apply regardless of the chosen glitch size distribution.

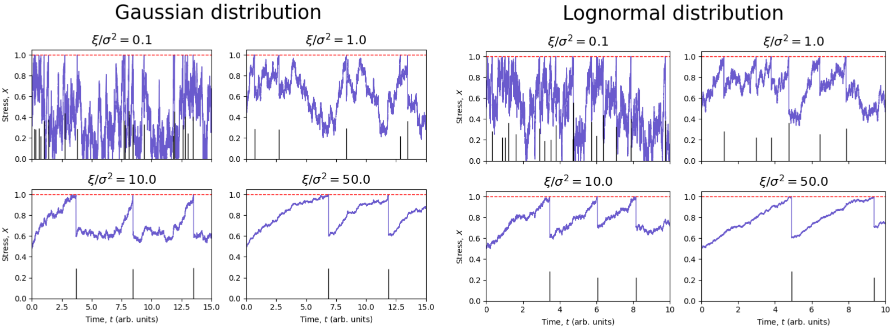


Figure 3: Stress trajectories with different glitch size distributions: **Left**: truncated Gaussian, **Right**: truncated Lognormal.

Brownian Process: Waiting Time Distribution

Since the internal stress $X(t)$ is not directly observable, we focus on the statistics of glitch events:

- Waiting times between glitches, Δt.
- Sizes of the glitches, $\Delta X_{release}$.

The distribution of waiting times, $p(\Delta t)$, depends on two factors:

1. **The drift/diffusion ratio** μ, which controls how quickly the stress evolves towards the threshold X_c.
2. **The stress release distribution** $\eta(\Delta X)$, which sets the initial stress X_0 for the next interval ($X_0 \approx X_c - \Delta X_{release}$).

Power-law glitch sizes generate a **power-law-like** waiting time distribution for low μ, transitioning to **exponential-like** for high μ.

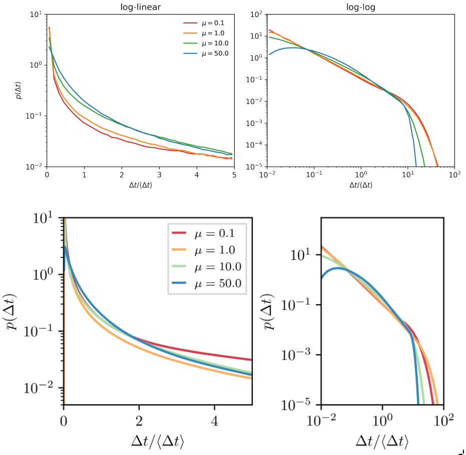


Figure 4: $p(\Delta t)$ for glitches with power-law size distribution, varying μ.
Top: my model, **Bottom**: Carlin & Melatos (2020)

Brownian: Waiting Time Distribution (Gaussian)

- **High** μ: $p(\Delta t)$ is unimodal and Gaussian-like, set by the previous glitch size.
- **Low** μ: $p(\Delta t)$ is exponential-like, dominated by random fluctuations.

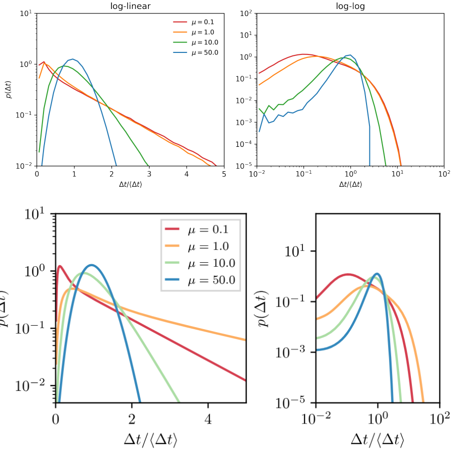


Figure 5: $p(\Delta t)$ for glitches with Gaussian size distribution, varying μ.
Top: my model, **Bottom**: Carlin & Melatos (2020)

Brownian: Waiting Time Distribution (Lognormal)

- **High** μ: $p(\Delta t)$ is unimodal and Gaussian-like, set by the previous glitch size.
- **Low** μ: $p(\Delta t)$ is exponential-like, dominated by random fluctuations.

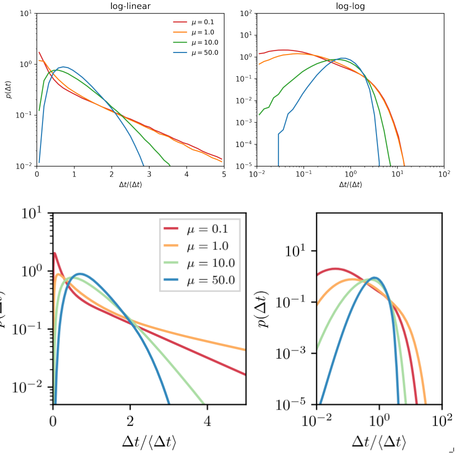


Figure 6: $p(\Delta t)$ for glitches with lognormal size distribution, varying μ. **Top**: my model, **Bottom**: Carlin & Melatos (2020)

State-dependent Poisson Process

State-dependent Poisson Process

This model, introduced by *Fulgenzi, Melatos & Hughes (2017)*, provides an alternative framework for pulsar glitches.

Core Idea:

- Between glitches, the stress $X(t)$ increases **deterministically**.
- Glitches do not occur at a fixed threshold. Instead, the **instantaneous glitch rate,** $\lambda(X)$, increases as the stress $X(t)$ grows.

This means glitches become more likely as the system approaches a critical stress level, but can occur at any time.

SDP: The Master Equation

The SDP model is a **Continuous-Time Markov Chain (CTMC)** where the "state" is the continuous stress level X.

The evolution of $p(x,t)$ is governed by the **Master Equation**:

$$X(t) = X(0) + f \cdot t - \sum_{i=1}^{N(t)} \Delta X^{(i)} \tag{3}$$

- f: Deterministic stress loading rate ($\dot{X}_{sd}$).
- $\sum \Delta X^{(i)}$: Sum of stochastic stress drops from glitches.

The **glitch rate** $\lambda(X)$ depends on the current stress:

$$\lambda(X) = \frac{\alpha \cdot f}{X_{peak} - X} \tag{4}$$

This rate diverges as $X \to X_{peak}$, making a glitch almost certain.

SDP: How to Sample Glitches?

Unlike the step-based Brownian model, the SDP model is **event-driven**. This simulation method is a form of the **Gillespie Algorithm** adapted for a time-varying rate.

Sample Waiting Time (Δt):

- Given X_0, we solve the Gillespie integral $\int_0^{\Delta t} \lambda(X_0 + f\tau)d\tau = -\ln(U)$.
- For this model, it yields the analytical solution [2]:

$$\Delta t = \frac{X_c - X_0}{f}\left(1 - U^{f/\alpha}\right), \quad U \sim \mathrm{Uniform}(0,1) \tag{5}$$

Sample Glitch Size ($\Delta X_{release}$):

- The stress just before the glitch is $X_p = X_0 + f \cdot \Delta t$.
- The stress drop $\Delta X_{release}$ is then sampled from a distribution $\eta(\Delta X|X_p)$ that is **conditional** on the pre-glitch stress X_p.

SDP: Trajectory

The deterministic, linear increase in stress between stochastic glitches results in a characteristic **sawtooth pattern**.

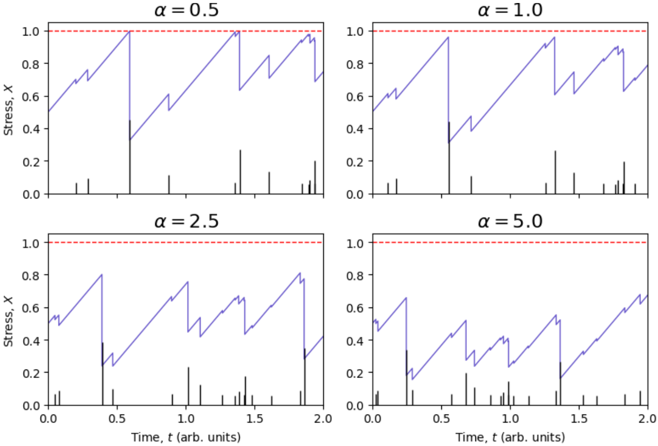


Figure 7: A typical stress trajectory for the SDP model.

SDP: Waiting Time Distribution (Power Law)

A **power-law** glitch size distribution generates a **power-law-like** waiting time. Higher α values lead to a steeper cutoff, reducing the probability of long waiting times and making glitches more regular.

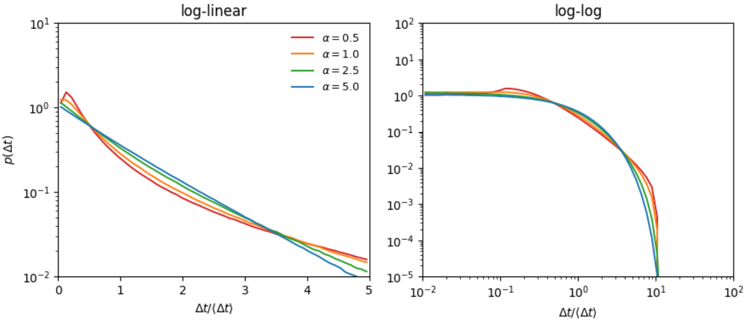


Figure 8: $p(\Delta t)$ for SDP model, power-law $\eta(\Delta X)$.

SDP: Waiting Time Distribution (Gaussian)

- **High** α: $p(\Delta t)$ is exponential-like, characteristic of random, uncorrelated events.
- **Low** α: $p(\Delta t)$ is unimodal and narrow, implying predictable, quasi-periodic glitches.

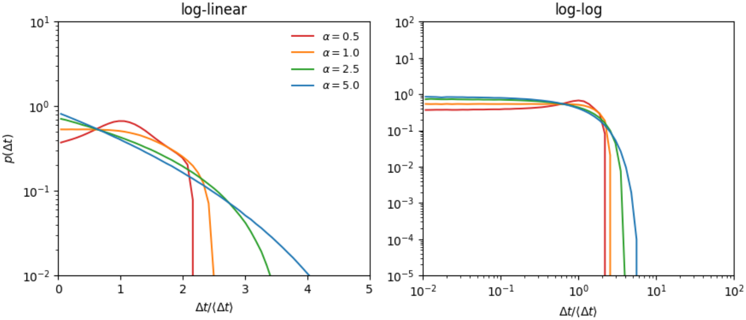


Figure 9: $p(\Delta t)$ for SDP model, Gaussian $\eta(\Delta X)$.

SDP: Waiting Time Distribution (Log-normal)

- **High** α: $p(\Delta t)$ is exponential-like, dominated by random glitch triggers.
- **Low** α: $p(\Delta t)$ is unimodal and narrow, similar to the Gaussian case.

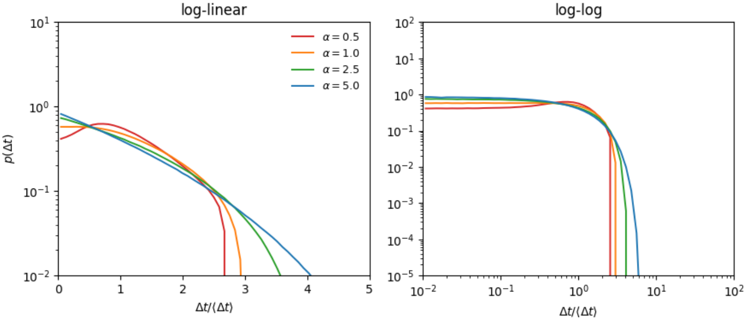


Figure 10: $p(\Delta t)$ for SDP model, Lognormal $\eta(\Delta X)$.

Hybrid Model

The Hybrid Model: A Synthesis

This project introduces a **novel hybrid model**, combining features from both the Brownian and SDP frameworks.

Core Idea:

- The stress $X(t)$ evolves stochastically as a **Brownian process** between glitches (like in Carlin & Melatos).
- Glitches are triggered **stochastically** with a state-dependent rate $\lambda(X)$ (like in Fulgenzi et al.).

This approach models a system where stress accumulation is noisy, and the failure mechanism is probabilistic rather than a hard threshold.

Hybrid Model: Simulation Method

The hybrid model requires a **step-based simulation**, similar to the Brownian model, to handle both the continuous evolution and the probabilistic events.

Algorithm at each step Δt:

1. **Evolve Stress**: Update the stress level X using the Langevin equation:
$$\Delta X = \xi \Delta t + \sigma \sqrt{\Delta t} \cdot N(0,1)$$
2. **Calculate Glitch Probability**: Determine the instantaneous glitch rate $\lambda(X)$ based on the new stress level. The probability of a glitch in this step is:
$$P_{\text{glitch}} \approx \lambda(X) \Delta t$$
3. **Trigger Glitch?**: Draw a random number $U \sim \text{Uniform}(0,1)$. If $U < P_{\text{glitch}}$, a glitch occurs.

Hybrid Model: The Master Equation

Because it combines a continuous diffusion and state-dependent jumps, its evolution equation is a **Fokker-Planck equation with a non-local jump term**.

For the probability density $p(x,t)$, the equation is:

$$\underbrace{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\xi p) + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}}_{\text{Drift-Diffusion (Brownian)}} \underbrace{- \lambda(x)p(x,t) + \int \lambda(y)p(y,t)\eta(x|y)dy}_{\text{Jump Process (SDP)}} \quad (6)$$

This equation is complex to solve analytically, making numerical simulations essential to explore the model's behavior.

Hybrid Model: Stress Trajectory

The trajectory shows a **stochastic ramp-up** due to the Brownian nature, followed by stochastic drops, combining features of both parent models.

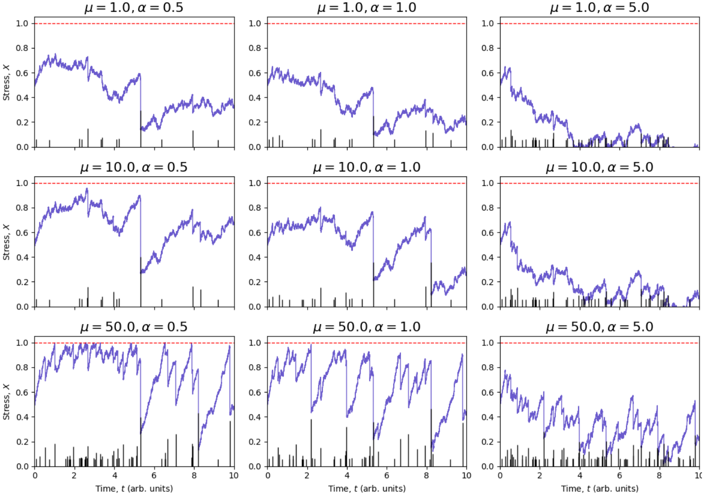


Figure 11: A typical stress trajectory for the Hybrid model.

Hybrid Model: Waiting Time Distribution (Power Law)

- At **high** μ and **high** α (drift-dominated, low intrinsic rate), the waiting times are **exponential-like**.
- As μ **or** α **decrease** (more noise or higher rate), the distribution shifts towards a **power-law**, but with a smoother shape than the pure SDP model.

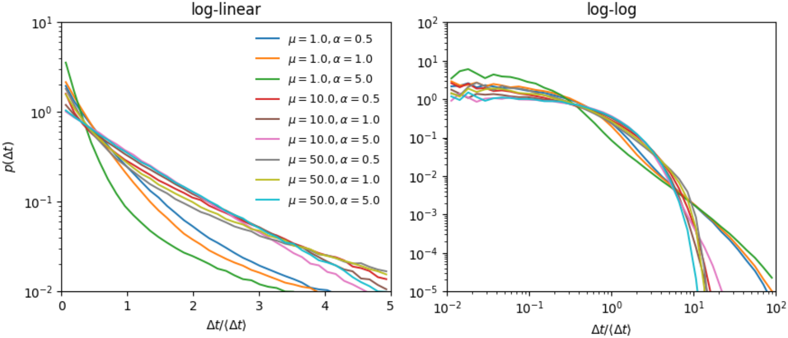


Figure 12: $p(\Delta t)$ for hybrid model, power-law $\eta(\Delta X)$, varying μ and α.

Hybrid Model: Waiting Time Distribution (Gaussian)

Using a **Gaussian** glitch size distribution introduces unimodal behavior, similar to the pure Brownian model in the high-drift regime.

- The distribution $p(\Delta t)$ is generally **unimodal**.
- The **width and skewness** of the distribution depend on the complex interplay between the Brownian noise (σ) and the SDP rate (α).

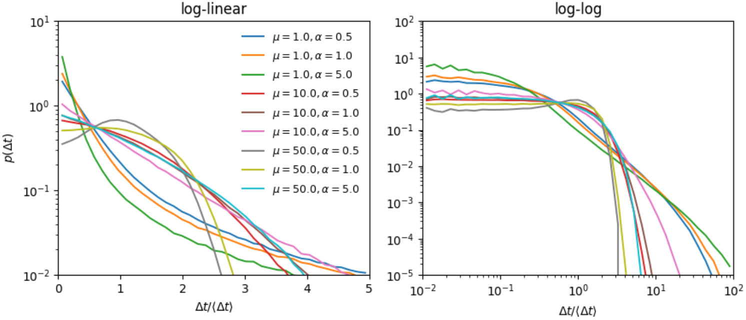


Figure 13: $p(\Delta t)$ for hybrid model, Gaussian $\eta(\Delta X)$.

Hybrid Model: Waiting Time Distribution (Lognormal)

The Lognormal reset distribution confirms the trend observed with the Gaussian case, yielding **unimodal** waiting time distributions.

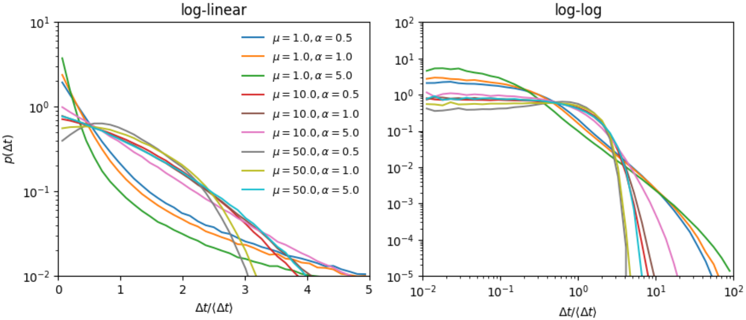
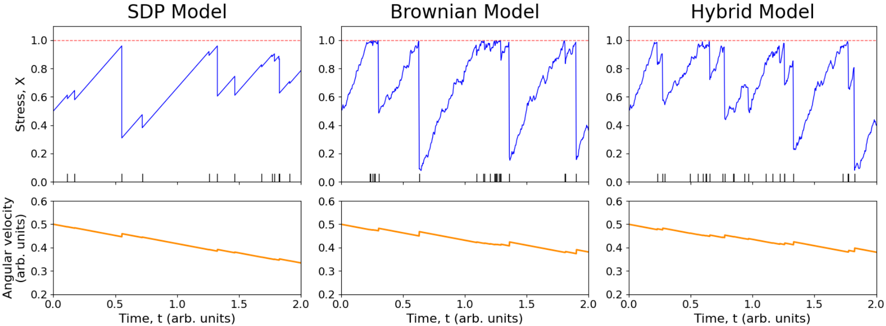


Figure 14: $p(\Delta t)$ for hybrid model, Lognormal $\eta(\Delta X)$.

Model Comparison & Conclusions

Model Comparison: Trajectories

A visual comparison of the stress trajectories and the resulting spin velocity reveals the fundamental differences between the models.



- **SDP Model:** Deterministic, linear ramps with probabilistic triggers.
- **Brownian Model:** Stochastic ramps up to a hard threshold.
- **Hybrid Model:** Stochastic ramps with probabilistic triggers, showing intermediate behavior.

Model Comparison: Key Predictions

The three models offer distinct predictions for observable glitch statistics.

Feature	SDP Model	Brownian Model	Hybrid Model
Stress Evolution	Deterministic	Stochastic (SDE)	Stochastic (SDE)
Glitch Trigger	Stochastic Rate $\lambda(X)$	Deterministic Threshold X_c	Stochastic Rate $\lambda(X)$
Waiting Time PDF	Power-law (low α) or Exponential (high α)	Log-Normal / Gaussian (high μ) or Exponential (low μ)	Complex interplay of all parameters
Correlations	Predicts backward correlations for some $\eta(\Delta X)$	Predicts strong forward correlations for high μ	Potentially rich correlation structure

The hybrid model bridges the gap, allowing for a more nuanced exploration of the underlying physics.

Conclusions

- The **Brownian** and **SDP** models offer two contrasting approaches: noisy stress buildup vs. probabilistic failure.
- Both explain some observed statistics, but the stress release law $\eta(\Delta X)$ is especially important for the Brownian model.
- The **Hybrid Model** unifies both ideas, combining noise and probabilistic triggering for greater flexibility in modeling real data.

Future Work:

- A systematic analysis of the Hybrid model's parameter space.
- Direct comparison of all three models' predictions (e.g., correlation functions) with observational data from pulsar glitch catalogs.

J. B. Carlin and A. Melatos.
Long-term statistics of pulsar glitches triggered by a brownian stress accumulation process.
Monthly Notices of the Royal Astronomical Society, 2020.

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Thank You!