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1 Vector Shift Formula Derivation

To derive the formula for a vector shift, we need to use time-dependent perturbation theory.

The energy levels for this will be all the states $|J, I, F, M\rangle$ which form an orthonormal basis, $\langle J', I, F', M'|J, I, F, M\rangle = \delta_{J'J}\delta_{F'F}\delta_{M'M}$

The perturbed hamiltonian is $H = H_0 + H'(t)$

where $H'(t)$ is the interaction of the atoms with the trapping light,

$$H'(t) = e\mathbf{E}(t) \cdot \mathbf{r}$$

where

$$\mathbf{E}(t) = \frac{E_0}{2}(\hat{\epsilon}e^{-i\omega t} + \hat{\epsilon}^* e^{i\omega t})$$

2 Computing Vector Shift

Want to make sense of:

$$\Delta E = \frac{-e^2 E_0^2}{4\hbar}\Sigma_{J',F',m'}[\frac{\langle J', I, F', M'|\epsilon \cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon^* \cdot r|J', I, F', M'\rangle}{\omega_{J'} - \omega}$$

$$+\frac{\langle J', I, F', M'|\epsilon^* \cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon \cdot r|J', I, F', M'\rangle}{\omega_{J'} + \omega}]$$

Using the formalism of spherical tensors,

$$\langle J', I, F', M'|\epsilon \cdot r|J, I, F, M\rangle =$$

$$\Sigma_\rho(-1)^{\rho+F'-M'+J'+I+F+1}\epsilon_\rho\langle J||r||J\rangle\sqrt{(2F+1)(2F'+1)}\begin{pmatrix} F' & 1 & F \\ -M' & -\rho & M \end{pmatrix}\begin{Bmatrix} J' & F' & I \\ F & J & 1 \end{Bmatrix}$$

So,

$$\langle J', I, F', M'|\epsilon \cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon^* \cdot r|J', I, F', M'\rangle =$$

$$(2F+1)(2F'+1)\langle J'||r||J\rangle\langle J||r||J'\rangle \begin{Bmatrix} J' & F' & I \\ F & J & 1 \end{Bmatrix} \begin{Bmatrix} J & F & I \\ F' & J' & 1 \end{Bmatrix} \times$$

$$\Sigma_{\rho,\rho'}(-1)^{\rho+\rho'+2F'+2F-M'-M+J'+J+2I+2}\epsilon_\rho\epsilon^*_{\rho'} \begin{pmatrix} F' & 1 & F \\ -M' & -\rho & M \end{pmatrix} \begin{pmatrix} F & 1 & F' \\ -M & -\rho' & M' \end{pmatrix}$$

Note: Both $\begin{pmatrix} F' & 1 & F \\ -M' & -\rho & M \end{pmatrix}$ and $\begin{pmatrix} F & 1 & F' \\ -M & -\rho' & M' \end{pmatrix}$ have to be nonzero to contribute to the sum. By the rules of Wigner 3j symbols,

$$-M' - \rho + M = 0$$

and

$$-M - \rho' + M' = 0$$

Therefore,

$$\rho = M - M' = -\rho'$$

Furthermore, by the triangle inequality rules, $\rho = 0, \pm 1$
Therefore, $M' = M-1, 0, M+1$ if the term is to be nonzero.
Now, consider the epsilons:

$$\hat{\epsilon} = \epsilon_L \frac{-\hat{x} - i\hat{y}}{\sqrt{2}} + \epsilon_R \frac{\hat{x} - i\hat{y}}{\sqrt{2}}$$

Thus,

$$\hat{\epsilon} = \epsilon_L \hat{e}_{+1} + \epsilon_R \hat{e}_{-1}$$

and,

$$\hat{\epsilon}^* = -\epsilon_R \hat{e}_{+1} - \epsilon_L \hat{e}_{-1}$$

So,

$\epsilon_{+1} = \epsilon_L \quad \epsilon^*_{+1} = -\epsilon_R$
$\epsilon_0 = 0 \quad \epsilon^*_0 = 0$
$\epsilon_{-1} = \epsilon_R \quad \epsilon^*_{-1} = -\epsilon_L$

Therefore, the term is also zero for $\rho = 0$. Thus, the two M' needed to be summed over are $M+1, M-1$

Consider now $\Sigma_{J',F',M'}\langle J', I, F', M'|\epsilon \cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon^* \cdot r|J', I, F', M'\rangle$

$$= \Sigma_{J',F'}(2F+1)(2F'+1)\langle J'||r||J\rangle\langle J||r||J'\rangle \begin{Bmatrix} J' & F' & I \\ F & J & 1 \end{Bmatrix} \begin{Bmatrix} J & F & I \\ F' & J' & 1 \end{Bmatrix} \times$$

$$\Sigma_{M'=M\pm1}(-1)^{2F'+2F-M'-M+J'+J+2I+2}\epsilon_{M-M'}\epsilon^*_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}\begin{pmatrix}F & 1 & F'\\ -M & M-M' & M'\end{pmatrix}$$

Note: by the rules of wigner 3J symbols,

$$\begin{pmatrix}F & 1 & F'\\ -M & M-M' & M'\end{pmatrix} = (-1)^{F+F'+1}\begin{pmatrix}F' & 1 & F\\ M' & M-M' & -M\end{pmatrix} = (-1)^{2(F+F')}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}$$

Therefore,

$$\Sigma_{J',F',M'}\langle J', I, F', M'|\epsilon\cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon^*\cdot r|J', I, F', M'\rangle$$

$$= \Sigma_{J',F'}(2F+1)(2F'+1)\langle J'||r||J\rangle\langle J||r||J'\rangle\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}\begin{Bmatrix}J & F & I\\ F' & J' & 1\end{Bmatrix}\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon_{M-M'}\epsilon^*_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2$$

Furthermore,

$$\langle J||r||J'\rangle = (-1)^{J-J'}\langle J'||r||J\rangle^*$$

and, by the rules of Wigner 6-j symbols,

$$\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix} = \begin{Bmatrix}F & J & I\\ J' & F' & 1\end{Bmatrix} = \begin{Bmatrix}J & F & I\\ F' & J' & 1\end{Bmatrix}$$

So,

$$\Sigma_{J',F',M'}\langle J', I, F', M'|\epsilon\cdot r|J, I, F, M\rangle\langle J, I, F, M|\epsilon^*\cdot r|J', I, F', M'\rangle$$

$$= \Sigma_{J',F'}(2F+1)(2F'+1)(-1)^{J-J'}|\langle J'||r||J\rangle|^2\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}^2\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon_{M-M'}\epsilon^*_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2$$

Now,

$$|\langle J'||r||J\rangle|^2 = \frac{3\hbar(2J+1)}{2m\omega_{JJ'}}f_{JJ'}$$

where $f_{JJ'}$ is the oscillator strength
Therefore,

$$\Sigma_{J',F',M'}\langle J',I,F',M'|\epsilon\cdot r|J,I,F,M\rangle\langle J,I,F,M|\epsilon^*\cdot r|J',I,F',M'\rangle$$

$$=\Sigma_{J',F'}(2F+1)(2F'+1)(-1)^{J-J'}\frac{3\hbar(2J+1)f_{JJ'}}{2m\omega_{JJ'}}\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}^2\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon_{M-M'}\epsilon^*_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2$$

Likewise,

$$\Sigma_{J',F',M'}\langle J',I,F',M'|\epsilon^*\cdot r|J,I,F,M\rangle\langle J,I,F,M|\epsilon\cdot r|J',I,F',M'\rangle$$

$$=\Sigma_{J',F'}(2F+1)(2F'+1)(-1)^{J-J'}\frac{3\hbar(2J+1)f_{JJ'}}{2m\omega_{JJ'}}\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}^2\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon^*_{M-M'}\epsilon_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2$$

Therefore,

$$\Delta E=\frac{-e^2E_0^2}{4\hbar}\Sigma_{J',F'}[(2F+1)(2F'+1)(-1)^{J-J'}\frac{3\hbar(2J+1)f_{JJ'}}{2m\omega_{JJ'}(\omega_{JJ'}-\omega)}\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}^2\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon_{M-M'}\epsilon^*_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2+$$

$$(2F+1)(2F'+1)(-1)^{J-J'}\frac{3\hbar(2J+1)f_{JJ'}}{2m\omega_{JJ'}(\omega_{JJ'}+\omega)}\begin{Bmatrix}J' & F' & I\\ F & J & 1\end{Bmatrix}^2\times$$

$$\Sigma_{M'=M\pm1}(-1)^{4F'+4F-M'-M+J'+J+2I+2}\epsilon^*_{M-M'}\epsilon_{M'-M}\begin{pmatrix}F' & 1 & F\\ -M' & M'-M & M\end{pmatrix}^2]$$