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# Asymptotics for value at risk and conditional tail expectation of a portfolio loss

**Xiaonan Su** | **Xinzhi Wang** | **Yang Yang**

School of Statistics Mathematics, Nanjing Audit University, Nanjing, China

**Correspondence**
Yang Yang, School of Statistics Mathematics, Nanjing Audit University, Nanjing 211815, China.
Email: yangyangmath@163.com

Consider a risk model in which $X_1, \ldots, X_n$ are $n$ potential losses from different risky assets at the terminal time, and $\theta_1, \ldots, \theta_n$ are $n$ discount factors over the period. In this paper, we establish some asymptotic formulas for the value at risk and conditional tail expectation of the total discounted loss $S_n = \sum_{i=1}^{n} \theta_i X_i$ of an investment portfolio. We also demonstrate our obtained results through Monte Carlo simulations with asymptotics.

**KEYWORDS**

asymptotics, conditional tail expectation, heavy tailed distribution, quasi-asymptotic independence, value at risk

**MOS SUBJECT CLASSIFICATION**

MSC: 62P05; 91B30

## 1 | INTRODUCTION

Consider a risk model in which the wealth is invested in $n$ risky assets over one period. For each $i = 1, \ldots, n$, the real-valued random variable $X_i$ represents the potential loss from the $i$th risky asset at the terminal time, and the nonnegative random variable $\theta_i$ is the discount factor over the period. Then, the total discounted loss of the investment portfolio can be expressed by a randomly weighted sum

$$S_n = \sum_{i=1}^{n} \theta_i X_i. \tag{1}$$

Except for few cases under ideal distributional assumptions, a closed-form expression for the tail probability of $S_n$ in (1) is not available. Thus, one mainstream of the study focuses on characterizing the asymptotic tail behavior of $S_n$, which has become an increasingly interesting topic in insurance, finance and risk management. See Tang and Tsitsiashvili, Goovaerts et al, Chen and Yuen, Tang and Yuan, Chen,[1-6] to name a few that are closely related to the current study. In this paper, we are interested in the asymptotics for some risk measures of $S_n$, which can be dealt with through investigating its tail probability.

Since risky portfolio may incur potential losses, regulators require financial institutions to hold a certain amount of risk capital to withstand large losses. One of the most important tasks in risk management is to determine a proper amount of the risk capital or to measure risk. The choice of a risk measure is subjective, but value at risk (VaR) and conditional tail expectation (CTE) are the two risk measures commonly used in insurance regulation. It is well known that the Solvency II defines an insurance regulatory environment in the European Union based on VaR, and the Swiss Solvency Test does it in Switzerland based on CTE. The former was firstly proposed by J.P. Morgan to meet the needs of banking risk. It refers to the maximum potential loss of a portfolio in a given holding period under normal market conditions and a given confidence level. Let $X$ be a real-valued random variable with distribution $F$, representing the potential loss in an investment of risky asset, then its VaR can be defined by

$$VaR_q(X) = \inf\{x : F(x) \geq q\} = F^{\leftarrow}(q),$$

where $q \in (0,1)$ is the confidence level, and $F^{\leftarrow}$ is the (left continuous) generalized inverse of $F$. However, with the development of the study of risk measures, VaR is criticized for its lack of sub-additivity, see Asimit et al[7] for details. In fact, sub-additivity is the property of risk reduction by diversification. This means that when VaR is used to measure risk, a portfolio can be riskier than the sum of its components. In addition, VaR is insensitive to the events with low probability but high severity. Hence, Artzner et al[8] introduced the concept of CTE,

$$CTE_q(X) = E[X|X > VaR_q(X)],$$

for $q \in (0,1)$, which captures the expectation of the risk beyond VaR and satisfies monotonicity, positive homogeneity, translation invariance, and sub-additivity. Thus, CTE is a coherent risk measure, whereas VaR is not, as stated in Zhu and Li.[9] Some further discussions on the advantages and disadvantages of the two risk measures can be found in Emmer et al,[10] but there is no evidence for global advantage of one risk measure against the other.

Over the past decade, there has been considerable attention paid to the asymptotics for the two risk measures. Asimit et al[11] considered the asymptotics for risk capital allocations based on CTE under some asymptotically dependent models. Joe and Li[12] analyzed the tail risk for the case of multivariate regularly varying-tailed losses. Hua and Joe[13] conducted some asymptotic analysis on CTE under the second order regular variation. Yang and Hashorva[14] obtained some asymptotic results for both the sum and the product of two risks, as well as some applications to actuarial mathematics. Yang et al[15] further studied the asymptotics for the CTE of a randomly weighted sum under the bivariate Sarmanov dependence structure. Recently, Asimit and Li[16] established some asymptotic formulas for some coherent risk measures of conditional distributions around extreme regions. Xing et al[17] investigated the CTE under the Farlie-Gumbel-Morgenstern (FGM) copula. Some related works can be referred to Harshorva and Li, Xing and Gan, Yang et al,[18-20] among many others.

In this paper, motivated by Xing et al,[17] we are interested in the asymptotics for the VaR and CTE of the total discounted loss $S_n$ defined in (1). Note that in good times or under normal market conditions, the potential losses may be weakly dependent. Throughout the paper, we assume that the potential losses $X_1, \ldots, X_n$ are $n$ arbitrarily dependent, or pairwise quasi-asymptotically independent, or independent real-valued random variables with distributions $F_1, \ldots, F_n$, respectively; the discount factors $\theta_1, \ldots, \theta_n$ are $n$ nonnegative and nondegenerate at 0 random variables; and $\theta_1, \ldots, \theta_n$, independent of $X_1, \ldots, X_n$, are arbitrarily dependent on each other, due to their coming from the same period. We also consider some extensions in the case allowing some certain dependence between the potential losses and the discount factors.

The rest of the paper is organized as follows. Section 2 states our main results after preparing some preliminaries on the asymptotic independence structure and heavy tailed distributions. Section 3 gives the proofs of the main results. In Section 4, we conduct some simulation studies. Finally, Section 5 concludes the paper.

## 2 | PRELIMINARIES AND MAIN RESULTS

Throughout the paper, all limit relations are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, write $f(x) \sim g(x)$ if $\lim \frac{f(x)}{g(x)} = 1$, write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $lim\ sup \frac{f(x)}{g(x)} \leq 1$, write $f(x) = o(g(x))$ if $\lim \frac{f(x)}{g(x)} = 0$, and write $f(x) \asymp g(x)$ if $0 < \liminf \frac{f(x)}{g(x)} \leq \limsup \frac{f(x)}{g(x)} < \infty$. For any $x, y \in \mathbb{R}$ and any set $A$, denote by $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and by $1_A$ the indicator function of $A$. Denote by $F_\xi$ the distribution of a random variable $\xi$, letting the notation speak for itself.

### 2.1 | Heavy tailed distributions and dependence structure

In this paper, we shall use heavy tailed distributions to model potential losses. A distribution $V$ on $\mathbb{R}$ is said to be subexponential, denoted by $V \in \mathcal{S}$, if $\overline{V}(x+y) \sim \overline{V}(x)$ and $\overline{V^{2*}}(x) \sim 2\overline{V}(x)$ for any $y \in \mathbb{R}$, where $\overline{V}(x) = 1 - V(x) > 0$ for all $x \geq 0$

and $V^{2^*}$ is the twofold convolution of $V$. One of the most useful subclass of subexponential distributions is the class of regularly varying tailed distributions, whose tails behave like a power function. A distribution $V$ on $\mathbb{R}$ is said to be regularly varying tailed with index $-\alpha$, denoted by $V \in \mathcal{R}_{-\alpha}$, if $\overline{V}(xy) \sim y^{-\alpha}\overline{V}(x)$ for any $y>0$ and some $\alpha > 0$. Closely related is the class of rapidly varying tailed distributions, which complements the class $\mathcal{R}$ with an extreme case of $\alpha = \infty$. A distribution $V$ on $\mathbb{R}$ is said to be rapidly varying tailed, denoted by $V \in \mathcal{R}_{-\infty}$, if $\overline{V}(xy) = o(\overline{V}(x))$ for any $y > 1$. Note that the class $\mathcal{R}_{-\infty}$ contains some heavy- and light tailed distributions. For detailed discussions on heavy tailed and rapidly varying tailed distributions, one can be referred to Embrechts et al[21] and Foss et al.[22]

Although during a steady economic period, the potential losses from different risky assets are weakly dependent, the dependence among them still needs to be taken into account. One of the commonly-used weak dependence structures is the so-called quasi-asymptotic independence proposed by Chen and Yuen.[3] Two nonnegative random variables $\xi_1$ and $\xi_2$ are said to be quasi-asymptotically independent (QAI), if

$$\lim \frac{P(\xi_1 > x, \xi_2 > x)}{P(\xi_1 > x) + P(\xi_2 > x)} = 0. \tag{2}$$

More generally, two real-valued random variables $\xi_1$ and $\xi_2$ are still said to be QAI, if relation (2) holds with $(\xi_1, \xi_2)$ in the numerator replaced by $(\xi_1^+, \xi_2^+)$, $(\xi_1^+, \xi_2^-)$ and $(\xi_1^-, \xi_2^+)$, where $x^+ = x \vee 0$ and $x^- = -x \wedge 0$ denote the positive and negative parts of $x \in \mathbb{R}$, respectively.

Remark that the following two dependence structures are typical examples of (pairwise) quasi-asymptotic independence. The first one is introduced by Asimit and Badescu.[23]

**Assumption 1.** For a nonnegative random vector $(\xi_1, \xi_2)$, there exists a measurable function $h{:}[0,\infty) \mapsto (0,\infty)$ such that

$$P(\xi_1 > x | \xi_2 = t) \sim h(t)P(\xi_1 > x),$$

holds uniformly for all $t \in [0,\infty)$, that is,

$$\lim_{x\to\infty} \sup_{t\in[0,\infty)} \left| \frac{P(\xi_1 > x|\xi_2 = t)}{h(t)P(\xi_1 > x)} - 1 \right| = 0.$$

When $t$ is not a possible value of $\xi_2$, the conditional probability $P(\xi_1 > x|\xi_2 = t)$ is understood as the unconditional one and therefore $h(t) = 1$ for such $t$. In addition, Assumption 1 implies $E[h(\xi_2)] = 1$ if $P(\xi_1 > x) > 0$ for all $x \in [0,\infty)$. Thus, if $(\xi_1, \xi_2)$ satisfies Assumption 1, then

$$\begin{aligned} P(\xi_1 > x, \xi_2 > x) &= \int_x^\infty P(\xi_1 > x|\xi_2 = t)P(\xi_2 \in dt) \\ &\sim P(\xi_1 > x)E[h(\xi_2)1_{(\xi_2 > x)}] \\ &= o(P(\xi_1 > x)), \end{aligned}$$

implying that the two nonnegative $\xi_1$ and $\xi_2$ are QAI. Another concept of Sarmanov distributions is proposed by Sarmanov.[24] A real-valued random vector $(\xi_1, \dots, \xi_n)$ is said to follow a multivariate Sarmanov distribution if

$$P(\bigcap_{i=1}^{n} (\xi_i \in dx_i)) = \left(1 + \sum_{1\le i<j\le n} \omega_{ij}\phi_i(x_i)\phi_j(x_j)\right) \prod_{i=1}^{n} P(\xi_i \in dx_i), \tag{3}$$

where $\phi_1, \dots, \phi_n$ are real-valued kernel functions and $\omega_{ij}, 1 \le i < j \le n$, are real-valued parameters such that $E[\phi_i(\xi_i)] = 0$, $i = 1, \dots, n$, and for every $I \subset \{1, \dots, n\}$, $1 + \sum_{i<j\in I} \omega_{ij}\phi_i(x_i)\phi_j(x_j) \ge 0$ for all $(x_i, x_j) \in \text{supp}(\xi_i) \times \text{supp}(\xi_j)$. It is easy to see that, for every $I \subset \{1, \dots, n\}$ with $|I| \ge 2$, the random vector $(\xi_i, i \in I)$ still inherits the Sarmanov distribution, that is,

$$P\left(\bigcap_{i\in I} (\xi_i \in dx_i)\right) = \left(1 + \sum_{i<j\in I} \omega_{ij}\phi_i(x_i)\phi_j(x_j)\right) \prod_{i\in I} P(\xi_i \in dx_i).$$

Hence, if a real-valued random vector $(\xi_1, \dots, \xi_n)$ follows a Sarmanov distribution of form (3), then $\xi_1, \dots, \xi_n$ are pairwise QAI. Indeed, for each $1 \le i < j \le n$,

$$\begin{aligned} P(\xi_i > x, \xi_j > x) &= \int_x^\infty \int_x^\infty (1 + \omega_{ij}\phi_i(x_i)\phi_j(x_j))P(\xi_i \in dx_i)P(\xi_j \in dx_j) \\ &\le CP(\xi_i > x)P(\xi_j > x) \\ &= o(1)(P(\xi_i > x) + P(\xi_j > x)), \end{aligned} \tag{4}$$

for some $C > 0$, where the second step holds due to the fact that $\phi_k$ is bounded on $\mathrm{supp}(\xi_k)$, for each $k = 1, \dots, n$, see lemma 3.2 of Wei and Yuan[25] or proposition 1.1 of Yang and Wang.[26] Similarly, (4) still holds for $(\xi_i^+, \xi_j^-)$ and $(\xi_i^-, \xi_j^+)$. Furthermore, for a nonnegative random vector $(\xi_1, \xi_2)$ following a bivariate Sarmanov distribution, Assumption 1 is satisfied under some additional conditions, see lemma 3.5 of Yang et al.[15] For more discussions on the dependence Assumption 1 and Sarmanov distributions, the reader can be referred to Li et al, Lee, Vernic,[27-29] and among others.

## 2.2 | Main results

In this subsection, we investigate the asymptotics for the VaR and CTE of the portfolio loss $S_n$ in (1) as $q \uparrow 1$. The first two results consider extremely heavy-tailed potential losses. The following one allows for arbitrarily dependent potential losses.

**Theorem 1.** *Consider risk model (1), in which $X_1, \dots, X_n$ are n arbitrarily dependent random variables with distributions $F_1, \dots, F_n$, respectively; $\theta_1, \dots, \theta_n$ are n arbitrarily dependent, nondegenerate at 0 and nonnegative random variables, independent of $X_1, \dots, X_n$. If $F_1 \in \mathcal{R}_{-\alpha_1}$ for some $\alpha_1 > 0$, $P(|X_i| > x) = o(\overline{F_1}(x))$, $i = 2, \dots, n$, and $E[\theta_i^{\alpha_1+\varepsilon}] < \infty$ for some $\varepsilon > 0$, $i = 1, \dots, n$, then it holds that as $q \uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim (E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}} \mathrm{VaR}_q(X_1),$$

*if, further, $\alpha_1 > 1$, then*

$$\mathrm{CTE}_q(S_n) \sim \frac{\alpha_1}{\alpha_1 - 1}(E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}} \mathrm{VaR}_q(X_1). \tag{5}$$

Our second result is concerned with the pairwise QAI tail-equivalent potential losses with regularly varying tails.

**Theorem 2.** *Consider risk model (1), in which $X_1, \dots, X_n$ are n pairwise QAI real-valued random variables with distributions $F_1, \dots, F_n$, respectively; $\theta_1, \dots, \theta_n$ are n arbitrarily dependent, nondegenerate at 0, and nonnegative random variables, independent of $X_1, \dots, X_n$. If $F_1 \in \mathcal{R}_{-\alpha_1}$ for some $\alpha_1 > 0$, $E[\theta_i^{\alpha_1+\varepsilon}] < \infty$ for some $\varepsilon > 0$, and*

$$\lim \frac{\overline{F_i}(x)}{\overline{F_1}(x)} = c_i, \tag{6}$$

*for some constants $c_i \ge 0$, $i = 1, \dots, n$, then it holds that as $q \uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim \left( \sum_{i=1}^n c_i E[\theta_i^{\alpha_1}] \right)^{\frac{1}{\alpha_1}} \mathrm{VaR}_q(X_1),$$

*if, further, $\alpha_1 > 1$, then*

$$\mathrm{CTE}_q(S_n) \sim \frac{\alpha_1}{\alpha_1 - 1} \left( \sum_{i=1}^n c_i E[\theta_i^{\alpha_1}] \right)^{\frac{1}{\alpha_1}} \mathrm{VaR}_q(X_1). \tag{7}$$

The third result is established for moderately heavy tailed but independent and identically distributed (i.i.d.) potential losses.

**Theorem 3.** *Consider risk model (1), in which $X_1, \dots, X_n$ are n i.i.d. real-valued random variables with common distribution $F_1$; $\theta_1, \dots, \theta_n$ are n arbitrarily dependent, nondegenerate at 0, identically distributed, and nonnegative random variables, which are bounded above and independent of $X_1, \dots, X_n$. If $F_1 \in \mathcal{S} \cap \mathcal{R}_{-\infty}$, then it holds that as $q\uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim \mathrm{VaR}_{1-\frac{1-q}{n}}(\theta_1 X_1). \tag{8}$$

If further, $\theta_1 = \dots = \theta_n = e^{-\delta T}$, where $\delta > 0$ is the constant force of interest, and $T > 0$ is the length of the period, then we can derive the following corollary.

**Corollary 1.** *Under the conditions of Theorem 3, if $\theta_1 = \dots = \theta_n = e^{-\delta T}$ for some $\delta > 0$ and $T > 0$, then it holds that as $q\uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{n}}(X_1). \tag{9}$$

## 2.3 | Some extensions

In this subsection, we further consider the case allowing some certain dependence between the potential losses and the discount factors. For simplicity, assume that the losses $X_1, \dots, X_n$ are identically distributed.

**Theorem 4.** *Consider risk model (1), in which $(X_1, \dots, X_n, \theta_1, \dots, \theta_n) := (\xi_1, \dots, \xi_{2n})$ follows a 2n-dimensional Sarmanov distribution of form (3) with $d_i = \lim \phi_i(x) < \infty, i = 1, \dots, n$. Assume that $X_1, \dots, X_n$ are n identically distributed and real-valued random variables with common distribution $F_1 \in \mathcal{R}_{-\alpha_1}$ for some $\alpha_1 > 0$; and $\theta_1, \dots, \theta_n$ are n nondegenerate at 0 and nonnegative random variables with $E[\theta_i^{\alpha_1+\varepsilon}] < \infty$ for some $\varepsilon > 0$, $i = 1, \dots, n$. Then, it holds that as $q\uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim \left(\sum_{i=1}^{n}(E[\theta_i^{\alpha_1}] + \omega_{i(n+i)}d_iE[\phi_{n+i}(\theta_i)\theta_i^{\alpha_1}])\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1),$$

*if, further, $\alpha_1 > 1$, then*

$$\mathrm{CTE}_q(S_n) \sim \frac{\alpha_1}{\alpha_1 - 1}\left(\sum_{i=1}^{n}(E[\theta_i^{\alpha_1}] + \omega_{i(n+i)}d_iE[\phi_{n+i}(\theta_i)\theta_i^{\alpha_1}])\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1). \tag{10}$$

**Theorem 5.** *Consider risk model (1), in which $(X_1, \theta_1), \dots, (X_n, \theta_n)$ are independent but Assumption 1 is satisfied for each $(X_i^+, \theta_i)$ with a measurable function $h_i$, $i = 1, \dots, n$. Assume that $X_1, \dots, X_n$ are n identically distributed and real-valued random variables with common distribution $F_1 \in \mathcal{R}_{-\alpha_1}$ for some $\alpha_1 > 0$; and $\theta_1, \dots, \theta_n$ are n nondegenerate at 0 and nonnegative random variables with $E[\theta_i^{\alpha_1+\varepsilon}] \vee E[h_i(\theta_i)\theta_i^{\alpha_1+\varepsilon}] < \infty$ for some $\varepsilon > 0$, $i = 1, \dots, n$. Then, it holds that as $q\uparrow 1$,*

$$\mathrm{VaR}_q(S_n) \sim \left(\sum_{i=1}^{n}E[h_i(\theta_i)\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1),$$

*if, further, $\alpha_1 > 1$, then*

$$\mathrm{CTE}_q(S_n) \sim \frac{\alpha_1}{\alpha_1 - 1}\left(\sum_{i=1}^{n}E[h_i(\theta_i)\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1). \tag{11}$$

As stated in Section 2.1, although the conditions of Theorem 4, together with some mild others, can ensure Assumption 1 for each $(X_i^+, \theta_i)$, $i = 1, \dots, n$, Theorem 5 also requires the mutual independence among $(X_1, \theta_1), \dots, (X_n, \theta_n)$.

# 3 | PROOFS OF MAIN RESULTS

Before proving our main results, we firstly cite a series of lemmas. The first lemma comes from lemma 3.3 of Yang et al.[30]

**Lemma 1.** *Let $(X_1, X_2)$ be a real-valued random vector with arbitrarily dependent components and marginal distributions $F_1$ and $F_2$, respectively. If $F_1 \in \mathcal{R}_{-\alpha_1}$ for some $\alpha_1 > 0$ and $P(|X_2| > x) = o(\overline{F_1}(x))$, then*

$$P(X_1 + X_2 > x) \sim \overline{F_1}(x).$$

The second lemma investigates the asymptotic tail behavior of a randomly weighted sum, which is due to theorem 3.2 of Chen and Yuen[3] and theorem 1 of Tang and Yuan.[4]

**Lemma 2.** *(1) Let $X_1, \dots, X_n$ be n pairwise QAI real-valued random variables with distributions $F_1, \dots, F_n$, respectively; and $\theta_1, \dots, \theta_n$ be n arbitrarily dependent, nondegenerate at 0, and nonnegative random variables, independent of $X_1, \dots, X_n$. If $F_i \in \mathcal{R}_{-\alpha_i}$ for some $\alpha_i > 0$ and $E[\theta_i^{\alpha_i+\epsilon}] < \infty$ for some $\epsilon > 0$, $i = 1, \dots, n$, then it holds that*

$$P\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} P(\theta_i X_i > x). \tag{12}$$

*(2) Let $X_1, \dots, X_n$ be n i.i.d. real-valued random variables with common distribution $F_1$; and $\theta_1, \dots, \theta_n$ be n arbitrarily dependent, nondegenerate at 0, and nonnegative random variables, which are bounded above and independent of $X_1, \dots, X_n$. If $F_1 \in \mathcal{S}$, then (12) holds.*

The third lemma can be found in Böcker and Klüppelberg.[31]

**Lemma 3.** *Let $F_1$ and $F_2$ be two distribution functions satisfying $\overline{F_1}(x) \sim \overline{F_2}(x)$. If $F_1 \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha \leq \infty$, then*

$$\left(\frac{1}{\overline{F_1}}\right)^{\leftarrow}(x) \sim \left(\frac{1}{\overline{F_2}}\right)^{\leftarrow}(x).$$

Now we are ready for the proofs of our main results.

*Proof of Theorem* 1.. We firstly aim to prove that

$$P(S_n > x) \sim P(\theta_1 X_1 > x) \sim E[\theta_1^{\alpha_1}]\overline{F_1}(x). \tag{13}$$

Note that the second relation in (13) holds by $F_1 \in \mathcal{R}_{-\alpha_1}$ and Breiman's theorem. Hence, we mainly prove the first relation in (13). Without loss of generality, we assume $n = 2$. For some sufficiently small $\epsilon > 0$ with $(1-\epsilon)(\alpha_1+\epsilon) > \alpha_1$, by $P(|X_2| > x) = o(\overline{F_1}(x))$ and Markov's inequality,

$$\begin{aligned}
P(\theta_2|X_2| > x) &\leq \int_0^{x^{1-\epsilon}} P\left(|X_2| > \frac{x}{u}\right) P(\theta_2 \in du) + P(\theta_2 > x^{1-\epsilon}) \\
&\leq o(1) \int_0^{x^{1-\epsilon}} \overline{F_1}\left(\frac{x}{u}\right) P(\theta_2 \in du) + x^{-(1-\epsilon)(\alpha_1+\epsilon)} E[\theta_2^{\alpha_1+\epsilon}] \\
&= o(1)P(\theta_2 X_1 > x) + o(\overline{F_1}(x)) \\
&= o(\overline{F_1}(x)) \\
&= o(1)P(\theta_1 X_1 > x),
\end{aligned} \tag{14}$$

where in the third step we used lemma 3.5 of Tang and Tsitsiashvili[1] and in the last two steps we used Breiman's theorem twice. In addition, by corollary of Embrechts and Goldie,[32] we have $F_{\theta_1 X_1} \in \mathcal{R}_{-\alpha_1}$. Then, by using (14) and Lemma , we can obtain that (13) holds with $n = 2$, which implies $F_{S_n} \in \mathcal{R}_{-\alpha_1}$. By using (13) and Lemma 3, it holds that as $q \uparrow 1$,

$$\begin{aligned}
\mathrm{VaR}_q(S_n) &= \left(\frac{1}{\overline{F_{S_n}}}\right)^{\leftarrow}\left(\frac{1}{1-q}\right) \\
&\sim \left(\frac{1}{\overline{F_1}}\right)^{\leftarrow}\left(\frac{E[\theta_1^{\alpha_1}]}{1-q}\right) \\
&\sim (E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}} \mathrm{VaR}_q(X_1),
\end{aligned} \tag{15}$$

where in the last step we used the fact that $\left(\frac{1}{\overline{F_1}}\right)^{\leftarrow}$ is regularly varying with index $\frac{1}{\alpha_1}$.

Now we turn to the estimate of $\mathrm{CTE}_q(S_n)$. Clearly,

$$\mathrm{CTE}_q(S_n) = \mathrm{VaR}_q(S_n) + \frac{\int_{\mathrm{VaR}_q(S_n)}^{\infty} P(S_n > x)dx}{P(S_n > \mathrm{VaR}_q(S_n))}.$$

Then, from $F_{S_n} \in \mathcal{R}_{-\alpha_1}$, Karamata's theorem, and (15), we can obtain the desired relation (5). This completes the proof of Theorem 1. ■

*Proof of Theorem* 2.. By Breiman's theorem or similarly done in (14), it holds that for each $i = 1, \ldots, n$,

$$\lim \frac{P(\theta_i X_i > x)}{\overline{F_1}(x)} = c_i E[\theta_i^{\alpha_1}],$$

under (6). Similarly to the proof of lemma 3.1 of Chen and Yuen,[3] under the QAI assumption,

$$\lim \frac{P(\theta_i X_i^{\pm} > x, \theta_j X_j > x)}{\overline{F_1}(x)} = 0,$$

holds for each $1 \le i \neq j \le n$. Then, by mimicking the proof of theorem 3.1 of Chen and Yuen[3] and applying Breiman's theorem, we can obtain

$$P(S_n > x) \sim \sum_{i=1}^{n} c_i E[\theta_i^{\alpha_1}]\overline{F_1}(x).$$

Following the same line of the proof of Theorem 1, we derive the desired relations. ■

*Proof of Theorem* 3.. By Lemma 2(2), we have

$$P(S_n > x) \sim \sum_{i=1}^{n} P(\theta_i X_i > x) = nP(\theta_1 X_1 > x). \tag{16}$$

Since $\theta_1$ is bounded above, denoting by $b > 0$ its upper bound, by $F_1 \in \mathcal{S}$ and theorem 1.1 of Tang,[33] we have $F_{\theta_1 X_1} \in \mathcal{S}$. In addition, by $F_1 \in \mathcal{R}_{-\infty}$,

$$\begin{aligned} P(\theta_1 X_1 > xy) &= \int_0^b \frac{\overline{F_1}\left(\frac{xy}{u}\right)}{\overline{F_1}\left(\frac{x}{u}\right)} \overline{F_1}\left(\frac{x}{u}\right) P(\theta_1 \in du) \\ &= o(1) \int_0^b \overline{F_1}\left(\frac{x}{u}\right) P(\theta_1 \in du) \\ &= o(1) P(\theta_1 X_1 > x), \end{aligned}$$

for any $y > 1$, which implies $F_{\theta_1 X_1} \in \mathcal{R}_{-\infty}$. By (16) and Lemma 3, we have that as $q \uparrow 1$,

$$\begin{aligned} \mathrm{VaR}_q(S_n) &= \left(\frac{1}{\overline{F_{S_n}}}\right)^{\leftarrow} \left(\frac{1}{1-q}\right) \\ &\sim \inf\left\{x : \frac{1}{P(\theta_1 X_1 > x)} \ge \frac{n}{1-q}\right\} \\ &= \mathrm{VaR}_{1-\frac{1-q}{n}}(\theta_1 X_1), \end{aligned}$$

as claimed. ■

*Proof of Corollary* 1.. By (8), as $q \uparrow 1$, it holds that

$$\begin{aligned}\mathrm{VaR}_q(S_n) &\sim \mathrm{VaR}_{1-\frac{1-q}{n}}(e^{-\delta T}X_1)\\ &= e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{n}}(X_1),\end{aligned}$$

which ends the proof. ■

*Proof of Theorem* 4.. Since $(\xi_1, \dots, \xi_{2n}) = (X_1, \dots, X_n, \theta_1, \dots, \theta_n)$ follows a $2n$-dimensional Sarmanov distribution of form (3), we have that, for each $i = 1, \dots, n$, $(\xi_i, \xi_{n+i}) = (X_i, \theta_i)$ follows the bivariate Sarmanov distribution with the kernel functions $\phi_i, \phi_{n+i}$ and the parameter $\omega_{i(n+i)}$. By theorem 2.1 of Yang and Wang,[26] for each $i = 1, \dots, n$,

$$P(\theta_i X_i > x) \sim (E[\theta_i^{\alpha_1}] + \omega_{i(n+i)} d_i E[\phi_{n+i}(\theta_i)\theta_i^{\alpha_1}])\overline{F_1}(x), \tag{17}$$

which implies $F_{\theta_i X_i} \in \mathcal{R}_{-\alpha_1}$. As done in (4),

$$\begin{aligned}P(\theta_1 X_1 > x, \theta_2 X_2 > x) &= \int \cdots \int_{x_1x_3>x, x_2x_4>x} \left(1 + \sum_{1\le i<j\le 4} \omega_{ij}\phi_i(x_i)\phi_j(x_j)\right)\\ &\quad P(X_1 \in dx_1)P(X_2 \in dx_2)P(\theta_1 \in dx_3)P(\theta_2 \in dx_4)\\ &\le CP(\theta_1^{\perp}X_1^{\perp} > x)P(\theta_2^{\perp}X_2^{\perp} > x)\\ &= o(P(\theta_1^{\perp}X_1^{\perp} > x) + P(\theta_2^{\perp}X_2^{\perp} > x))\\ &= o(\overline{F_1}(x) + \overline{F_2}(x))\\ &= o(P(\theta_1 X_1 > x) + P(\theta_2 X_2 > x)),\end{aligned} \tag{18}$$

for some $C > 0$, where $(X_1^{\perp}, X_2^{\perp}, \theta_1^{\perp}, \theta_2^{\perp})$ has the independent components and the same marginal distributions as those of $(X_1, X_2, \theta_1, \theta_2)$, and in the second step we used lemma 3.2 of Wei and Yuan,[25] the last two steps are due to Breiman's theorem and (17), respectively. In the same way, (18) also holds for $(X_1^+, X_2^-)$ and $(X_1^-, X_2^+)$. This ensures that $\theta_1 X_1, \dots, \theta_n X_n$ are pairwise QAI. Then, by using Lemma 2(1) with all weights equal to 1 and (17) we have

$$\begin{aligned}P(S_n > x) &\sim \sum_{i=1}^{n} P(\theta_i X_i > x)\\ &\sim \overline{F_1}(x)\sum_{i=1}^{n}(E[\theta_i^{\alpha_1}] + \omega_{i(n+i)} d_i E[\phi_{n+i}(\theta_i)\theta_i^{\alpha_1}]).\end{aligned}$$

Therefore, the desired relations follow along the line of the proof of Theorem 1. ■

*Proof of Theorem* 5. Consider the tail probability of the product $\theta_i X_i$, $i = 1, \dots, n$. For each $i = 1, \dots, n$ and any $0 < \epsilon < \frac{\varepsilon}{\alpha_1+\varepsilon}$, we divide $P(\theta_i X_i > x)$ into three parts according to $\theta_i$ in $[0,1]$, $(1, x^{1-\epsilon}]$ and $(x^{1-\epsilon}, \infty)$, denoted by $I_1$, $I_2$ and $I_3$, respectively. Clearly, by Markov's inequality, $F_1 \in \mathcal{R}_{-\alpha_1}$ and $E[\theta_i^{\alpha_1+\varepsilon}] < \infty$ we have

$$\begin{aligned}I_3 &\le P(\theta_i > x^{1-\epsilon})\\ &\le x^{-(1-\epsilon)(\alpha_1+\varepsilon)}E[\theta_i^{\alpha_1+\varepsilon}]\\ &= o(\overline{F_1}(x)).\end{aligned}$$

By Potter's bound (see proposition 2.2.1 of Bingham et al)[34] and $E[h(\theta_i)\theta_i^{\alpha_1+\varepsilon}] < \infty$, we know that the dominated convergence theorem can be applied to derive

$$\begin{aligned}I_2 &= \int_1^{x^{1-\epsilon}} P\left(X_i > \frac{x}{u}\,\middle|\, \theta_i = u\right) P(\theta_i \in du)\\ &\sim \int_1^{x^{1-\epsilon}} h_i(u)\overline{F_1}\left(\frac{x}{u}\right) P(\theta_i \in du)\\ &\sim \overline{F_1}(x)E[h_i(\theta_i)\theta_i^{\alpha_1}1_{(\theta_i>1)}].\end{aligned}$$

Similarly, again by the dominated convergence theorem and $E[h_i(\theta_i)] = 1$, we have

$$I_1 \sim \overline{F_1}(x)E[h_i(\theta_i)\theta_i^{\alpha_1}1_{(\theta_i\le 1)}].$$

Combining the above three estimates yields that for each $i = 1, \dots, n$,

$$P(\theta_i X_i > x) \sim \overline{F_1}(x)E[h_i(\theta_i)\theta_i^{\alpha_1}],$$

from which and theorem 1 of Yang et al,[35] it follows that

$$\begin{aligned} P(S_n > x) &\sim \sum_{i=1}^{n} P(\theta_i X_i > x) \\ &\sim \overline{F_1}(x)\sum_{i=1}^{n} E[h_i(\theta_i)\theta_i^{\alpha_1}]. \end{aligned}$$

As done in the proofs of the previous theorems, we conclude the desired relations. ∎

## 4 | SIMULATION STUDIES

In this section, we perform some simulation studies through the Monte Carlo method to check the accuracy of our obtained results by using software Python.

Consider a portfolio of two potential losses $X_1$ and $X_2$, which is dependent according to the Clayton copula of the form

$$C(u, v) = ((u^{-\beta} + v^{-\beta} - 1) \vee 0)^{-\frac{1}{\beta}}, \qquad (u, v) \in [0, 1]^2, \tag{19}$$

for some parameter $\beta \in [-1, 0) \cup (0, \infty)$, see, for example, Nelsen.[36] It can be checked that if $\beta = 1$, then $X_1$ and $X_2$ are QAI.

We firstly consider Situation (1), in which $X_1$ and $X_2$ follow the Pareto distributions

$$F_i(x) = 1 - \left(1 + \frac{x}{\sigma_i}\right)^{-\alpha_i}, \qquad x \ge 0,\ i = 1, 2, \tag{20}$$

respectively, for some $\alpha_2 > \alpha_1 > 1$ and $\sigma_1 > 0$, $\sigma_2 > 0$. Clearly, $F_i \in \mathcal{R}_{-\alpha_i}$, $i = 1, 2$, and $\overline{F_2}(x) = o(\overline{F_1}(x))$. The two discount factors $\theta_1 = Y_1$ and $\theta_2 = Y_1Y_2$, where $Y_1$ and $Y_2$ are two independent uniform random variables on (0,1), which are independent of $X_1$ and $X_2$. Then, all conditions in Theorem 1 are satisfied. Now we are ready to perform some Monte Carlo simulations to illustrate the asymptotic relation (5).

The following algorithm is used to generate $N$ pairs of dependent random vector $(X_1, X_2)$:

Step a: generate $N$ pairs of i.i.d. random variables $u_i$ and $v_i$ following the uniform distribution on (0,1), $i = 1, \dots, N$;
Step b: set $x_{1,i} = \sigma_1\left((1 - u_i)^{-\frac{1}{\alpha_1}} - 1\right)$, $i = 1, \dots, N$;
Step c: set $w_i = \left(1 + \left(v_i^{-\frac{\beta}{1+\beta}} - 1\right)u_i^{-\beta}\right)^{-\frac{1}{\beta}}$ and $x_{2,i} = \sigma_2\left((1 - w_i)^{-\frac{1}{\alpha_2}} - 1\right)$, $i = 1, \dots, N$.

Then, for each $i = 1, \dots, N$, the obtained $(x_{1,i}, x_{2,i})$ returns the outcome of a random vector following the Clayton copula (19) and marginal Pareto distributions defined in (20).

The computation procedure of the estimate of $\mathrm{VaR}_q(X_1)$ and $\mathrm{CTE}_q(S_2)$ is listed here:

Step 1: generate $N$ pairs of i.i.d. random variables $y_{1,i}$ and $y_{2,i}$ following the uniform distribution on (0,1), and calculate $\theta_{1,i} = y_{1,i}$ and $\theta_{2,i} = y_{1,i}y_{2,i}$. Then $s_{2,i} = \theta_{1,i}x_{1,i} + \theta_{2,i}x_{2,i}$, $i = 1, \dots, N$;

Step 2: set the order statistics $x_{1,(1)} \le \dots \le x_{1,(N)}$, and choose $x_{1,(\lceil Nq\rceil)}$ as $\mathrm{VaR}_q(X_1)$, here $\lceil x\rceil$ is the smallest integer not less than $x$;

Step 3: set the order statistics $s_{2,(1)} \le \dots \le s_{2,(N)}$, and choose $s_{2,(\lceil Nq\rceil)}$ as $\mathrm{VaR}_q(S_2)$, then calculate the average value of $s_{2,1}, \dots, s_{2,N}$ larger than $\mathrm{VaR}_q(S_2)$ as $\mathrm{CTE}_q(S_2)$.

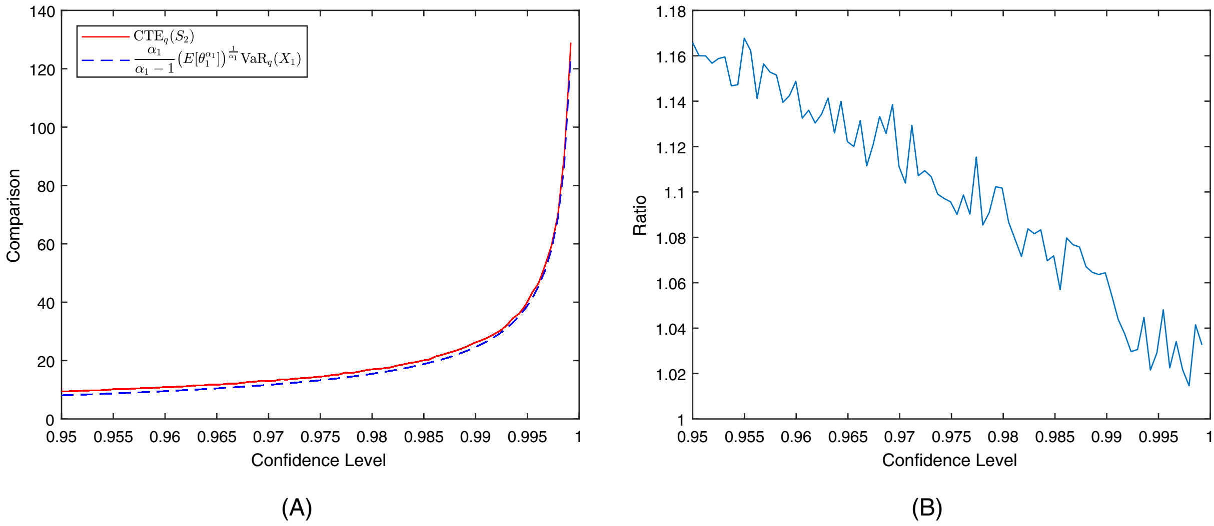


**FIGURE 1** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}(E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ (A) and their ratio (B) via the Clayton copula with $\alpha_1 = 1.6, \alpha_2 = 2.5, \sigma_1 = \sigma_2 = 1, \beta = 1$, and $N = 3 \times 10^7$, in Theorem 1 [Color figure can be viewed at wileyonlinelibrary.com]

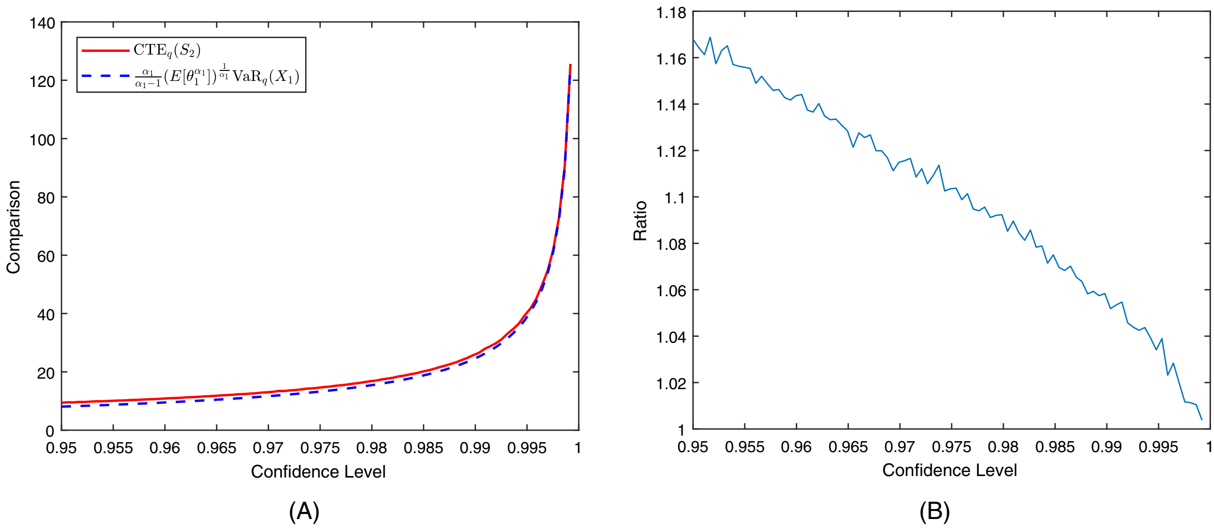


**FIGURE 2** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}(E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ (A) and their ratio (B) via the Clayton copula with $\alpha_1 = 1.6, \alpha_2 = 2.5, \sigma_1 = \sigma_2 = 1, \beta = 1$, and $N = 4 \times 10^8$, in Theorem 1 [Color figure can be viewed at wileyonlinelibrary.com]

Choose the sample size $N = 3 \times 10^7$ and change the confidence level $q$ from 0.95 to 0.9995. The various parameters are set to $\alpha_1 = 1.6, \alpha_2 = 2.5, \sigma_1 = \sigma_2 = 1$ and $\beta = 1$. Then the simulation results in Situation (1) are shown in Figure 1.

In Figure 1A, we compare the estimates of $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}(E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ and present their ratio in Figure 1B. From Figure 1A we can see that with the increase of the confidence level, the estimates increase quickly and the two lines get closer. Despite the fluctuation in Figure 1(b), the ratio of $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}(E[\theta_1^{\alpha_1}])^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ is close to 1 with the confidence level up to 1.

In fact, the fluctuation to some extent is due to the poor performance of the Monte Carlo method, which requires a sufficiently large sample size to meet the demands of high accuracy. In order to eliminate some inaccuracy, we repeat the simulation with the sample size $N$ increasing up to $N = 4 \times 10^8$, and the corresponding results are shown in Figure 2. We can see from Figure 2 that there is a significant improvement for the convergence of the ratio, which exhibits much stable and falls into the interval [0.95,1.05]. Therefore, our obtained asymptotic relationship between $\mathrm{CTE}_q(S_2)$ and $\mathrm{VaR}_q(X_1)$ in Theorem 1 is reasonable.

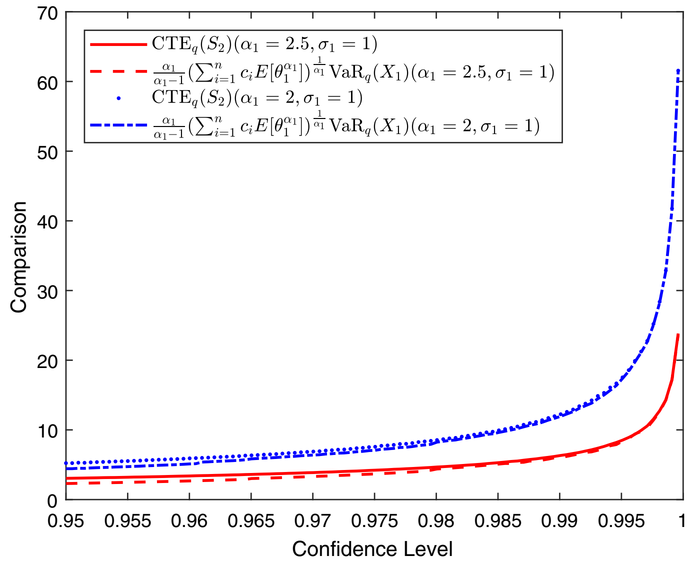


**FIGURE 3** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^2 c_iE[\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ via the Clayton copula with $\beta=1,\sigma_1=1,\alpha_1=2.5$ (or 2), and $N=3\times10^7$, in Theorem 2 [Color figure can be viewed at wileyonlinelibrary.com]

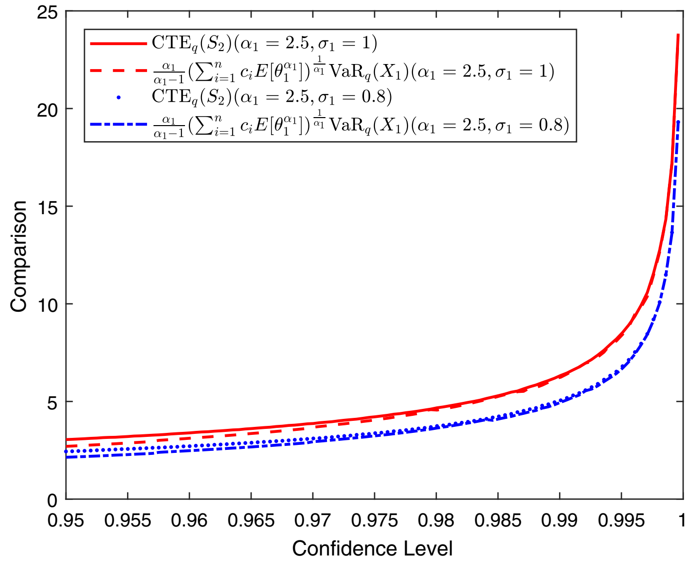


**FIGURE 4** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^2 c_iE[\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ via the Clayton copula with $\beta=1,\alpha_1=2.5,\sigma_1=1$ (or 0.8), and $N=3\times10^7$, in Theorem 2 [Color figure can be viewed at wileyonlinelibrary.com]

Next, we consider Situation (2) which is based on Theorem 2. In Theorem 2, assume that the distribution $F_1$ is of the form (20) with parameters $\alpha_1$ and $\sigma_1$; the second potential loss $X_2$ has its tail distribution $\overline{F_2}$ proportional to $\overline{F_1}$ with coefficient $c_2$; and $\theta_1$ and $\theta_2$ are the same as those in Situation (1). Then all conditions in Theorem 2 are satisfied. In the second simulation, we aim to check the accuracy of relation (7), the sensitivity of parameters $\alpha_1$, $\sigma_1$, and the influence of different dependence structures.

In Situation (2), we consider four cases to check the influence of different parameters $\alpha_1$, $\sigma_1$ and different dependence structures. In cases (1) to (3), the potential losses $X_1$ and $X_2$ are dependent via the Clayton copula (19), and the various parameters are set to $\alpha_1=2.5,\sigma_1=1,\beta=1,c_2=0.5$ in case (1); $\alpha_1=2,\sigma_1=1,\beta=1,c_2=0.5$ in case (2); and $\alpha_1=2.5,\sigma_1=0.8,\beta=1,c_2=0.5$ in case (3). In case (4), we compare the case of Clayton dependent $(X_1,X_2)$ with that of $(X_1,X_2)$ following the Frank copula

$$C(u,v)=-\frac{1}{\gamma}\log\left(1+\frac{(e^{-\gamma u}-1)(e^{-\gamma v}-1)}{e^{-\gamma}-1}\right),\qquad (u,v)\in[0,1]^2, \tag{21}$$

for some parameter $\gamma\in(-\infty,0)\cup(0,\infty)$. It can be checked that if $\gamma=1$, then $X_1$ and $X_2$ are QAI. The parameter is set to $\gamma=1$ and some others are the same as those in case (1). We compare cases (1) and (2) in Figure 3, cases (1) and (3) in Figure 4, and cases (1) and (4) in Figure 5, where the sample size is chosen as $N=3\times10^7$ and the confidence level $q$ varies from 0.95 to 0.9995.

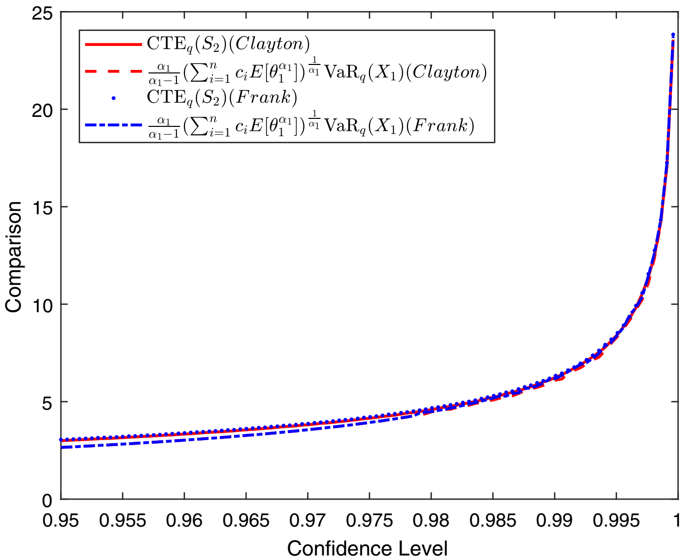


**FIGURE 5** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^2 c_i E[\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ via the Clayton and Frank copulas with $\beta = 1, \gamma = 1, \alpha_1 = 2.5, \sigma_1 = 1$, and $N = 3\times 10^7$, in Theorem 2 [Color figure can be viewed at wileyonlinelibrary.com]

We firstly check the accuracy of relation (7) according to Figures 3 and 4. From Figure 3 we can see that either $\alpha_1 = 2.5$ or $\alpha_1 = 2$, the estimates increase quickly and the lines of $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^n c_i E[\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ get closer with the increase of confidence level, so does Figure 4, in which $\sigma_1$ is reduced from 1 to 0.8. Secondly, we analyze the sensitivity of $\alpha_1$ and $\sigma_1$. In Figure 3, the two estimates increase almost three times when fixing $\sigma_1$ and reducing $\alpha_1$ from 2.5 to 2; whereas in Figure 4 the two estimates vary a little (twenty percent or so) when fixing $\alpha_1$ but reducing $\sigma_1$ from 1 to 0.8. It seems that changing the Pareto parameter $\sigma_1$ makes no significant influence on the asymptotic behavior of the two estimates. Actually, this is because the tail of the Pareto distribution is mainly decided by $\alpha_1$. As a result, parameter $\alpha_1$ is more sensitive than $\sigma_1$. Finally, we investigate the influence of different dependence structures in Figure 5, in which parameters $\alpha_1, \sigma_1$ and $c_2$ are the same as those in case (1), but the Frank copula is considered instead of the Clayton one in case (1). Despite some subtle difference, the curves under different copulas almost coincide, so the convergence is robust with respect to different QAI copulas. This indicates that under our setting, the VaR and CTE of the portfolio loss are mainly affected by the heavy tailedness of the potential losses rather than their quasi-asymptotic independence structure, which is reasonable and consistent with our obtained theoretical results.

We now turn to Situation (3) based on Theorem 3. We simplify the simulation according to Corollary 1, in which the potential losses $X_1$ and $X_2$ are two i.i.d. random variables with a common Weibull distribution of the form

$$F_1(x) = 1 - e^{-cx^{\tau}}, \qquad x \geq 0,$$

for some $c > 0$ and $0 < \tau < 1$, implying $F_1 \in \mathcal{S} \cap \mathcal{R}_{-\infty}$. The discount factors $\theta_1$ and $\theta_2$ are specialized to a constant $e^{-\delta T}$ for some $\delta > 0$ and $T > 0$. Then, all conditions in Corollary 1 are satisfied. As above, we shall check the accuracy of relation (9) and the sensitivity of parameters $\tau$ and $c$.

In this situation, the various parameters are set to $\tau = 0.5$ (or $\frac{1}{3}$), $c = 1$ (or $\frac{2}{3}$), $\delta = 1$, $T = 1$, and the sample size is chosen as $N = 3\times 10^7$ and the confidence level $q$ varies from 0.95 to 0.9995. The comparisons between $\mathrm{VaR}_q(S_2)$ and $e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{2}}(X_1)$ are shown in Figures 6 and 7.

Figures 6 and 7 both show that Corollary 1 is reasonable for moderately heavy tailed potential losses. Moreover, the curves of $\mathrm{VaR}_q(S_2)$ and $e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{2}}(X_1)$ exhibit more difference with different parameter $\tau$ than that with different $c$, which indicates that the two estimates are more sensitive with respect to parameter $\tau$ than $c$. This is due to the tail of the Weibull distribution is dominated by $\tau$.

At the end of this section, we consider Situations (4) and (5) based on Theorems 4 and 5, respectively. As for Situation (4), assume that the two losses $X_1, X_2$ and the two discount factors $\theta_1, \theta_2$ are dependent according to a four-dimensional FGM copula

$$C(\mathbf{u}) = \prod_{k=1}^{4} u_k \left(1 + \sum_{1\leq i<j\leq 4} \omega_{ij}(1-u_i)(1-u_j)\right), \qquad \mathbf{u} \in [0,1]^4,$$

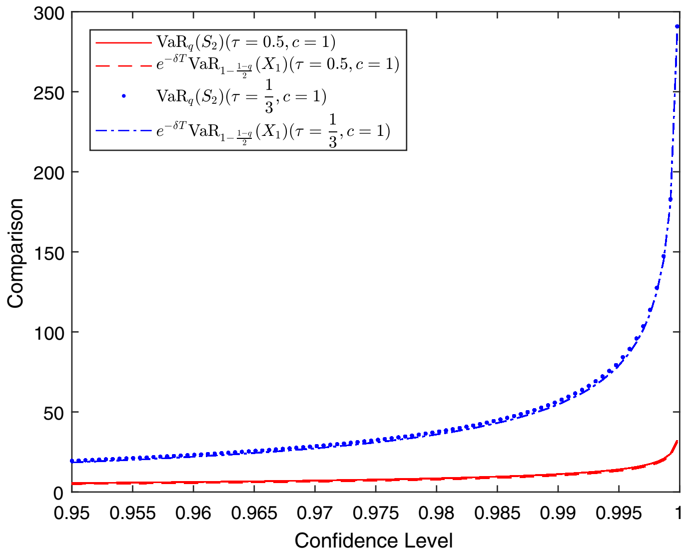


**FIGURE 6** Comparison between $\mathrm{VaR}_q(S_2)$ and $e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{2}}(X_1)$ under the independence structure with $c = 1, \tau = 0.5$ (or $\frac{1}{3}$), and $N = 3 \times 10^7$, in Corollary 1 [Color figure can be viewed at wileyonlinelibrary.com]

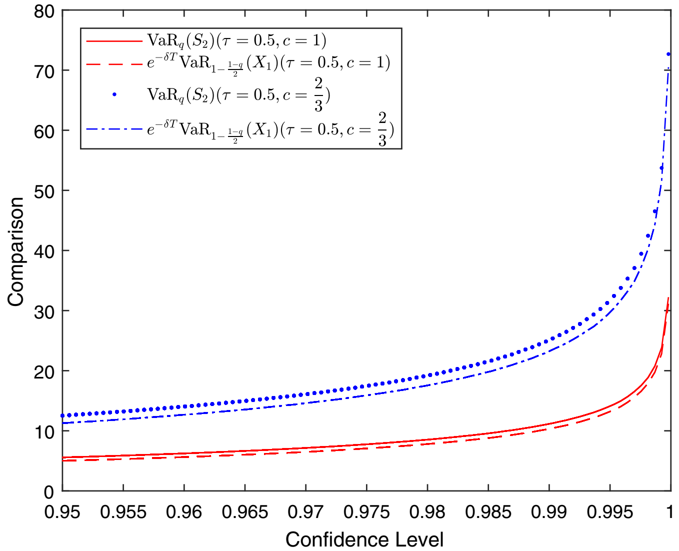


**FIGURE 7** Comparison between $\mathrm{VaR}_q(S_2)$ and $e^{-\delta T}\mathrm{VaR}_{1-\frac{1-q}{2}}(X_1)$ under the independence structure with $\tau = 0.5, c = 1$ (or $\frac{2}{3}$), and $N = 3 \times 10^7$, in Corollary 1 [Color figure can be viewed at wileyonlinelibrary.com]

with parameters $\omega_{ij} \in [-1, 1], 1 \le i < j \le 4$. Clearly, $(X_1, X_2, \theta_1, \theta_2)$ follows the four-dimensional Sarmanov distribution, see Wei and Yuan.[25] Assume further that $X_1$ and $X_2$ both follow the Pareto distribution (20) with parameters $\alpha_1 > 0$ and $\sigma_1 > 0$, and $\theta_1$ and $\theta_2$ both follow the Exponential distribution with intensity $\lambda_1 > 0$. Then, all conditions in Theorem 4 are satisfied with $d_1 = d_2 = -1$ and $\phi_3(x) = \phi_4(x) = 2e^{-\lambda_1 x} - 1$. In this simulation, we consider the case of heavier-tailed loss distribution and higher confidence levels by reducing parameter $\alpha_1$ and increasing $q$. The various parameters are set to $\alpha_1 = 1.24, \sigma_1 = 1, \lambda_1 = 1, \omega_{ij} = 0.5, 1 \le i < j \le 4$, the sample size is chosen as $N = 3 \times 10^7$, and the confidence level $q$ varies from 0.9995 to 0.99999.

It can be seen that, both simulated and asymptotic values of CTE in Figure 8, compared with the previous ones, are extraordinarily larger due to the smaller $\alpha_1 > 1$ and the $q$ closer to 1, meaning the heavier tailed loss distribution and the higher confidence levels.

As for Situation (5), assume that $(X_1, \theta_1)$ and $(X_2, \theta_2)$ are two i.i.d. random vectors, and for each $i = 1,2$, $X_i$ and $\theta_i$ are dependent according to the Frank copula of the form (21) with parameter $\gamma \ne 0$. The distributions of $X_1$,$X_2$ and $\theta_1, \theta_2$ are the same as those in Situation (4). Then, all conditions in Theorem 5 are satisfied with $h_1(x) = h_2(x) = \frac{\gamma e^{\gamma(1-e^{-\lambda_1 x})}}{e^{\gamma}-1}$, see example 3.3 of Li et al.[27] The various parameters are set to $\alpha_1 = 1.28, \sigma_1 = 1, \lambda_1 = 1, \gamma = 0.5$, the sample size is chosen as $N = 3 \times 10^7$, and the confidence level $q$ varies from 0.9995 to 0.99999. We plot the simulated and asymptotic values of $\mathrm{CTE}_q(S_2)$ in Figure 9.

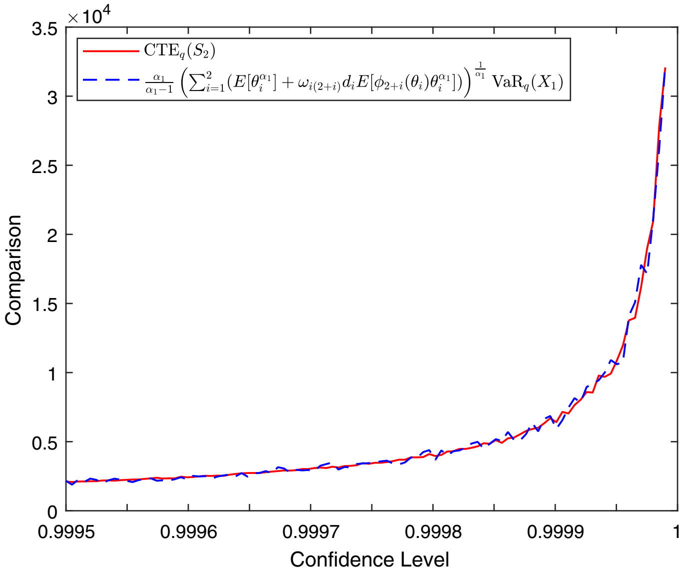


**FIGURE 8** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^{2}(E[\theta_i^{\alpha_1}]+\omega_{i(2+i)}d_iE[\phi_{2+i}(\theta_i)\theta_i^{\alpha_1}])\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ via the FGM copula with $\alpha_1 = 1.24, \sigma_1 = 1, \lambda_1 = 1, \omega_{ij} = 0.5$, and $N = 3\times10^7$, in Theorem 4 [Color figure can be viewed at wileyonlinelibrary.com]

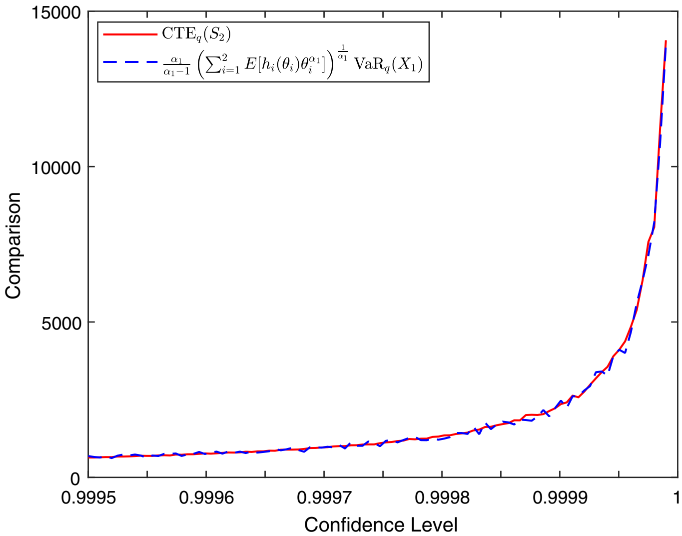


**FIGURE 9** Comparison between $\mathrm{CTE}_q(S_2)$ and $\frac{\alpha_1}{\alpha_1-1}\left(\sum_{i=1}^{2}(E[h_i(\theta_i)\theta_i^{\alpha_1}]\right)^{\frac{1}{\alpha_1}}\mathrm{VaR}_q(X_1)$ via the Frank copula with $\alpha_1 = 1.28, \sigma_1 = 1, \lambda_1 = 1, \gamma = 0.5$, and $N = 3\times10^7$, in Theorem 5 [Color figure can be viewed at wileyonlinelibrary.com]

## 5 | CONCLUSIONS

In this paper, we investigate the asymptotics for the VaR and CTE of a portfolio loss, which is described as the sum of the products of potential losses and discount factors. We allow arbitrary dependence, or some weak dependence (QAI), or independence existing among the extremely or moderately heavy tailed potential losses, while the discount factors are arbitrarily dependent; and we complement the case allowing some certain dependence between the potential losses and the discount factors. Some simulations studies are performed to check the accuracy of our obtained theoretical results. We also analyze the sensitivity of different parameters in the potential losses' distributions to the VaR and CTE of the portfolio loss, and take account of the influence with respect to different dependence structures among potential losses. This may provide regulators with some ideas on how to determine the proper effective factors in insurance or financial risk management.

Our results are suitable for some steady economic periods or normal market conditions. In the future, we are going to consider some other cases in which the economy goes down or undergoes a crisis. In such circumstances, some asymptotic dependence structures can be the candidates for modeling potential losses. We shall further capture the tail dependence resulting from potential losses and discount factors. In addition, more moderately heavy-tailed even light-tailed potential losses can be further considered.

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## ORCID

*Yang Yang* https://orcid.org/0000-0002-1080-8658


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