

## Research Article

# Asymptotic Behavior of Ruin Probabilities in an Insurance Risk Model with Quasi-Asymptotically Independent or Bivariate Regularly Varying-Tailed Main Claim and By-Claim

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This paper considers a by-claim risk model under the asymptotical independence or asymptotical dependence structure between each main claim and its by-claim. In the presence of heavy-tailed main claims and by-claims, we derive some asymptotic behavior for ruin probabilities.

## 1. Introduction

Consider a continuous-time risk model with no interest rate in which each severe accident leads to two kinds of claims: one is caused by the accident immediately, called the main claim, and the other is caused as a compensation occurring after a period of time, called the by-claim. The surplus process can be described as

$$U(t) = x + ct - \sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{\infty} Y_i 1_{\{\tau_i + D_i \leq t\}}, \quad t \geq 0, \quad (1)$$

where  $1_A$  stands for the indicator function of a set  $A$ . In relation (1),  $x \geq 0$  is interpreted as the initial reserve of an insurance company,  $c \geq 0$  is the constant premium rate,  $X_i$  is the size of the  $i$ th main claim occurring at the arrival time  $\tau_i$ , the counting process  $N(t) = \sum_{k=1}^{\infty} 1_{\{\tau_k \leq t\}}$  represents the number of the accidents (equal to that of the main claims) before time  $t \geq 0$ ,  $Y_i$  is the size of the by-claim corresponding to the  $i$ th main claim, and  $D_i$  denotes the uncertain delay time after the accident arrival time  $\tau_i$ . In this setup, the finite-time and infinite-time ruin probabilities can be defined, respectively, as

$$\psi(x, t) = P\left(\inf_{0 \leq s \leq t} U(s) < 0 \mid U(0) = x\right), \quad t \geq 0, \quad (2)$$

$$\begin{aligned} \psi(x) &= \lim_{t \rightarrow \infty} \psi(x, t) \\ &= P\left(\inf_{s \geq 0} U(s) < 0 \mid U(0) = x\right). \end{aligned} \quad (3)$$

Such a by-claim risk model is initially studied by Yuen et al. [1] and may be of practical use in insurance. For instance, a traffic accident will cause different kinds of claims, including an immediate payoff for the car damage and another delayed medical claim for injured drivers which may need a random period of time to be settled.

We are concerned with the asymptotic behavior for the finite-time and infinite-time ruin probabilities, since, except for few cases under ideal distributional assumptions, a closed-form expression for the ruin probability  $\psi(x, t)$  or  $\psi(x)$  is hardly available. Thus, the mainstream of the study focuses on characterizing the asymptotic behavior for ruin probabilities. Some earlier works have been achieved with light-tailed and mutually independent claims; that is, both  $\{X_i; i \in \mathbb{N}\}$  and  $\{Y_i; i \in \mathbb{N}\}$  are sequences of independent and identically distributed (i.i.d.) and light-tailed nonnegative random variables, and they are mutually independent too; see Yuen and Guo [2], Xiao and Guo [3], Wu and Li [4], and Li and Wu (2015).

Clearly, modeling total surplus should address extreme risks, which result from the marginal tails of and the tail dependence between claims. In the past decade, more and

more attention has been paid to heavy-tailed claims and the dependence between them. In the presence of heavy-tailed claim sizes, Tang [5], Leipus and Šiaulyš [6, 7], Yang et al. [8], Wang et al. [9], Liu et al. [10], Yang and Yuen [11], Yang et al. [12, 13], and Chen et al. [14] investigated some independent or dependent risk models with no by-claims. Li [15] considered a dependent by-claim risk model with positive interest rate and extendedly varying tailed main claims and by-claims under the pairwise quasi-asymptotical independence structure (see the definition below). Fu and Li [16] further generalized Li's result by allowing the insurance company to invest its surplus into a portfolio consisting of risk-free and risky assets. Recently, Li [17] studied a by-claim risk model with no interest rate under the setting that each pair of the main claim and by-claim follows the asymptotical independence structure or possesses a bivariate regularly varying tail (hence, follows the asymptotical dependence structure).

In this paper we continue to consider the above by-claim risk model (1). Precisely speaking, let  $\{(X_i, Y_i); i \in \mathbb{N}\}$  be a sequence of nonnegative and i.i.d. random vectors, representing the main claims and by-claims, with generic random vector  $(X, Y)$  having marginal distributions  $F, G$  and finite means  $\mu_F, \mu_G$ . Assume that the counting process, not necessarily the renewal one,  $\{N(t); t \geq 0\}$  is generated by the identically distributed and nonnegative interarrival times  $\{\theta_i = \tau_i - \tau_{i-1}; i \in \mathbb{N}\}$ , with finite mean function  $\lambda(t) = EN(t)$ . The delay times  $\{D_i; i \in \mathbb{N}\}$  are a sequence of nonnegative (possibly degenerate at zero) random variables. In addition, we assume that  $\{(X_i, Y_i); i \in \mathbb{N}\}$  and  $\{N(t); t \geq 0\}$  are mutually independent.

Inspired by the work of Li [17], our goal is to derive sharp asymptotics for the finite-time and infinite-time ruin probabilities in some dependent by-claim risk models. We make some meaningful adjustments and extensions on his model. First, we allow some certain dependence structure among the interarrival times  $\{\theta_i; i \in \mathbb{N}\}$ ; that is,  $\{N(t); t \geq 0\}$  is not necessarily a renewal counting process. Second, when investigating the infinite-time ruin probability, we extend both distributions  $F$  and  $G$  to be consistently varying tailed in the case that the main claim  $X$  and the by-claim  $Y$  are asymptotically independent. We also complement another case that  $X$  and  $Y$  are arbitrarily dependent when  $X$  dominates  $Y$ .

The rest of the paper is organized as follows. Section 2 prepares some preliminaries including some classes of heavy-tailed distributions and some concepts of dependence structures. Section 3 exhibits our main results. The proofs of the main results as well as several lemmas needed for the proofs are relegated to Section 4.

## 2. Preliminaries

Throughout the paper, all limit relations are according to  $x \rightarrow \infty$  unless otherwise stated. For two positive functions  $f(x)$  and  $g(x)$ , we write  $f(x) \sim g(x)$  if  $\lim(f(x)/g(x)) = 1$ , write  $f(x) \leq g(x)$  or  $f(x) \geq g(x)$  if  $\limsup(f(x)/g(x)) \leq 1$ , and write  $f(x) = o(g(x))$  if  $\lim(f(x)/g(x)) = 0$ . Furthermore, for two positive bivariate functions  $f(x, t)$  and  $g(x, t)$ , we write  $f(x, t) \sim g(x, t)$  uniformly for  $t \in A$  if

$\limsup_{t \in A} |f(x, t)/g(x, t) - 1| = 0$ . For any  $x, y \in \mathbb{R}$ , we write  $x \vee y = \max\{x, y\}$  and  $x \wedge y = \min\{x, y\}$ .

In this paper, we shall restrict the claim distributions to some classes of heavy-tailed distributions supported on  $\mathbb{R}_+ = [0, \infty)$ . A commonly used class is the class  $\mathcal{C}$  of consistently varying tailed distributions. A distribution  $V$  belongs to the class  $\mathcal{C}$ , if  $\lim_{y \downarrow 1} \liminf_{x \rightarrow \infty} (\bar{V}(xy)/\bar{V}(x)) = 1$ , where  $\bar{V}(x) = 1 - V(x) > 0$  for all  $x \geq 0$ . Closely related is a wider class  $\mathcal{L}$  of long-tailed distributions. A distribution  $V$  belongs to the class  $\mathcal{L}$ , if  $\bar{V}(x + y) \sim \bar{V}(x)$  for any  $y > 0$ . An important subclass of  $\mathcal{C}$  is the class  $\mathcal{R}$  of regularly varying tailed specified by  $\bar{V}(xy) \sim y^{-\alpha} \bar{V}(x)$  for any  $y > 0$  and some  $\alpha > 0$ , denoted by  $V \in \mathcal{R}_{-\alpha}$ . The reader is referred to Embrechts et al. [18] and Foss et al. [19] for related discussions on the properties of the subclasses of heavy-tailed distributions.

Modeling a practical risk model must carefully address asymptotical (in)dependence between each pair of the main claim and its corresponding by-claim or among the interarrival times. Asymptotical dependence represents that the probability of two components being simultaneously large cannot be negligible compared with one component being large. Generally, two random variables  $\xi_1$  and  $\xi_2$  are said to be asymptotically dependent if they have a positive coefficient of (upper) tail dependence, defined by

$$\kappa = \lim P(\xi_1 > x \mid \xi_2 > x); \quad (4)$$

see, e.g., McNeil et al. [20]. The following concept of bivariate regular variation exhibits the asymptotic dependence of two random variables. A random pair  $(\xi_1, \xi_2)$  taking values in  $\mathbb{R}_+^2$  is said to follow a distribution with a bivariate regularly varying (BRV) tail if there exist a distribution  $V$  and a nondegenerate (i.e., not identically 0) limit measure  $\nu$  such that the following vague convergence holds:

$$\frac{1}{\bar{V}(x)} P\left(\frac{(\xi_1, \xi_2)}{x} \in \cdot\right) \xrightarrow{\nu} \nu(\cdot) \quad \text{on } [0, \infty] \setminus \{0\}. \quad (5)$$

Necessarily,  $\bar{V}$  is regularly varying. Assume that  $V \in \mathcal{R}_{-\alpha}$  for some  $\alpha > 0$ , for which case we write  $(\xi_1, \xi_2) \in \text{BRV}_{-\alpha}(\nu, V)$ . By definition, for a random pair  $(\xi_1, \xi_2) \in \text{BRV}_{-\alpha}(\nu, V)$ , if  $\nu(1, \infty] > 0$ , then its marginal tails satisfy

$$\begin{aligned} \lim \frac{P(\xi_1 > x)}{\bar{V}(x)} &= \nu((1, 0), \infty] > 0, \\ \lim \frac{P(\xi_2 > x)}{\bar{V}(x)} &= \nu((0, 1), \infty] > 0, \end{aligned} \quad (6)$$

$$\lim \frac{P(\xi_1 > x, \xi_2 > x)}{\bar{V}(x)} = \nu(1, \infty] > 0,$$

which imply  $\kappa = \nu(1, \infty]/\nu((0, 1), \infty] > 0$  in (4) showing that  $\xi_1$  and  $\xi_2$  are asymptotically dependent; see, e.g., Tang et al. [21] and Tang and Yang [22]. If  $\kappa = 0$  in (4) then  $\xi_1$  and  $\xi_2$  are asymptotically independent. A natural extension is the

concept of quasi-asymptotical independence (QAI) proposed by Chen and Yuen [23] and defined by

$$\lim_{x \rightarrow \infty} \frac{P(\xi_1 > x, \xi_2 > x)}{P(\xi_1 > x) + P(\xi_2 > x)} = 0. \quad (7)$$

In our main results we shall use the structure of BRV or QAI to model the main claim and its corresponding by-claim and capture their tail dependence simultaneously. The following concept of dependence structure is a special case of asymptotical independence, which will be used to describe the inter-arrival times. A sequence of random variables  $\{\xi_i; i \in \mathbb{N}\}$  is said to be widely upper orthant dependent (WUOD), if there exists a finite real sequence  $\{g_U(n); n \in \mathbb{N}\}$  satisfying, for each  $n \in \mathbb{N}$  and for all  $x_i \in \mathbb{R}$ ,

$$P\left(\bigcap_{i=1}^n (\xi_i > x_i)\right) \leq g_U(n) \prod_{i=1}^n P(\xi_i > x_i), \quad (8)$$

and it is said to be widely lower orthant dependent (WLOD), if there exists a finite real sequence  $\{g_L(n); n \in \mathbb{N}\}$  satisfying, for each  $n \in \mathbb{N}$  and for all  $x_i \in \mathbb{R}$ ,

$$P\left(\bigcap_{i=1}^n (\xi_i \leq x_i)\right) \leq g_L(n) \prod_{i=1}^n P(\xi_i \leq x_i). \quad (9)$$

The sequence of  $\{\xi_i; i \in \mathbb{N}\}$  is said to be widely orthant dependent (WOD), if both (8) and (9) hold. Here,  $\{g_U(n), g_L(n); n \in \mathbb{N}\}$  are called dominating coefficients. Such dependence structures are introduced by Wang et al. [24]. Specially, when  $g_U(n) = g_L(n) \equiv 1$  for any  $n \in \mathbb{N}$  in (8) and (9), the sequence of random variables  $\{\xi_i; i \in \mathbb{N}\}$  is said to be upper negatively dependent (UND) and lower negatively dependent (LND), respectively. The sequence is said to be negatively dependent (ND), if it is both UND and LND. See Ebrahimi and Ghosh [25]. Note that the ND structure is weaker than the well-known negative association; see Alam and Saxena [26] and Joag-Dev and Proschan [27], among others.

### 3. Main Results

In this section, we firstly introduce some assumptions on the by-claim risk model (1). Throughout this paper, let  $\{(X_i, Y_i); i \in \mathbb{N}\}$  be a sequence of i.i.d. random pairs with generic random vector  $(X, Y)$  having marginal distributions  $F$  and  $G$  both on  $\mathbb{R}_+$  and finite means  $\mu_F$  and  $\mu_G$ , respectively; let  $\{\theta_i; i \in \mathbb{N}\}$  be a sequence of nonnegative and dependent random variables with finite mean  $\lambda^{-1}$ , which are independent of  $\{(X_i, Y_i); i \in \mathbb{N}\}$ ; and let  $\{D_i; i \in \mathbb{N}\}$  be a sequence of nonnegative and upper bounded random variables; that is, there exists a positive constant  $M$  such that  $D_i \leq M$  for all  $i \in \mathbb{N}$ . In addition, as usual in the risk model with no interest rate we require the safety loading condition:

$$\rho := \frac{\lambda}{c} (\mu_F + \mu_G) < 1. \quad (10)$$

We remark that the reasonability for the upper bounded  $D_i$ 's is that by-claims should occur before the termination date due

to the insurance policy, and the safety loading condition (10) excludes the trivial case  $\psi(x) \equiv 1$ .

The following is our first main result, which investigates the asymptotics for infinite-time ruin probability under three kinds of the dependence structures between  $X$  and  $Y$ .

**Theorem 1.** Consider the by-claim risk model (1) with WLOD interarrival times  $\{\theta_i; i \in \mathbb{N}\}$  satisfying

$$\lim_{n \rightarrow \infty} g_L(n) n^{-p} = 0, \quad (11)$$

for some  $p > 0$ . Assume further that either of the following four conditions is satisfied.

*Condition 1.*  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of LND random variables.

*Condition 2.*  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of WOD random variables, and there exists a positive and nondecreasing function  $g(x)$  such that  $g(x) \rightarrow \infty$ ,  $x^{-k}g(x) \downarrow$  for some  $0 < k < 1$ ,  $E\theta_1 g(\theta_1) < \infty$  and  $g_U(n) \vee g_L(n) \leq g(n)$ ; here,  $x^{-k}g(x) \downarrow$  means that there exists some  $C > 0$  such that

$$x_1^{-k} g(x_1) \geq C x_2^{-k} g(x_2) \quad \text{for all } x_2 > x_1 \geq 0. \quad (12)$$

*Condition 3.*  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of WOD random variables with  $E\theta_1^r < \infty$  for some  $r \geq 2$ , and there exists a constant  $p_1 > 0$  such that

$$\lim_{n \rightarrow \infty} (g_U(n) \vee g_L(n)) n^{-p_1} = 0. \quad (13)$$

*Condition 4.*  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of WOD random variables with  $Ee^{s\theta_1} < \infty$  for some  $s > 0$ , and, for any  $p_2 > 0$ ,

$$\lim_{n \rightarrow \infty} (g_U(n) \vee g_L(n)) e^{-p_2 n} = 0. \quad (14)$$

(i) Assume that  $X$  and  $Y$  are QAI. If  $F \in \mathcal{C}$  and  $G \in \mathcal{C}$ , then

$$\psi(x) \sim \frac{\lambda}{c(1-\rho)} \int_x^\infty (\bar{F}(u) + \bar{G}(u)) du. \quad (15)$$

(ii) If  $(X, Y) \in \text{BRV}_{-\alpha}(\nu, H)$ , for some  $\alpha > 1$  and  $\nu(1, \infty) > 0$ , then

$$\psi(x) \sim \frac{\lambda \nu(A)}{c(1-\rho)} \int_x^\infty \bar{H}(u) du, \quad (16)$$

where  $A = \{(x, y) \in \mathbb{R}_+^2 : x + y > 1\}$ .

(iii) Assume that  $X$  and  $Y$  are arbitrarily dependent. If  $F \in \mathcal{C}$  and  $\bar{G}(x) = o(\bar{F}(x))$ , then

$$\psi(x) \sim \frac{\lambda}{c(1-\rho)} \int_x^\infty \bar{F}(u) du. \quad (17)$$

We remark that if Condition 1 is satisfied, that is,  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of LND random variables, then (11) holds automatically.

Our second result is concerned with the asymptotic behavior of the finite-time ruin probability. Comparing with Theorem 2 of Li [17], we allow  $\{N(t); t \geq 0\}$  to be a quasi-renewal counting process generated by the WOD and identically distributed inter-arrival times  $\{\theta_i; i \in \mathbb{N}\}$ , whereas it is required to be a Poisson process in Li [17].

**Theorem 2.** *Under the conditions of Theorem 1, assume that  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of WOD and identically distributed nonnegative random variables such that (11) and Condition 2 in Theorem 1 are satisfied. Then, for every  $t > 0$ ,*

$$\lim_{x \rightarrow \infty} \frac{\psi(x, xt)}{\psi(x)} = 1 - (1 + c(1 - \rho)t)^{-\gamma+1} \quad (18)$$

holds if either of the following holds:

- (i)  $X$  and  $Y$  are QAI, and  $F \in \mathcal{R}_{-\alpha}, G \in \mathcal{R}_{-\beta}$  for some  $\alpha \wedge \beta > 1$ . In this case,  $\gamma = \alpha \wedge \beta$ .
- (ii)  $(X, Y) \in \text{BRV}_{-\alpha}(\nu, H)$  for some  $\alpha > 1$ . In this case,  $\gamma = \alpha$ .
- (iii)  $X$  and  $Y$  are arbitrarily dependent,  $F \in \mathcal{R}_{-\alpha}$  for some  $\alpha > 1$ , and  $\bar{G}(x) = o(\bar{F}(x))$ . In this case,  $\gamma = \alpha$ .

As pointed by Li [17], Theorem 2 shows that, given that the ruin occurs, the ruin time divided by  $x$  converges in distribution to a Pareto random variable of type II. In addition, if  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of ND random variables, then (11) and Condition 2 in Theorem 1 are both satisfied automatically.

## 4. Proofs of Main Results

Before the proofs of our two main results, we firstly cite a series of lemmas. Consider a nonstandard risk model

$$U^*(t) = x + ct - \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0, \quad (19)$$

where  $\{Z_i = X_i + Y_i; i \in \mathbb{N}\}$  is a sequence of i.i.d. nonnegative random variables with generic random variable  $Z$ . As (2) and (3), define the infinite-time and finite-time ruin probabilities of risk model (19) with  $U(\cdot)$  replaced by  $U^*(\cdot)$ , denoting them by  $\psi^*(x, t)$  and  $\psi^*(x)$ , respectively. The following first lemma comes from Corollary of Wang et al. [9]; see some similar results in Yang et al. [8].

**Lemma 3.** *Consider a nonstandard risk model (19), in which the claims  $\{Z_i; i \in \mathbb{N}\}$  form a sequence of i.i.d. nonnegative random variables with common distribution  $F_Z$  and finite mean  $EZ$ ; the interarrival times  $\{\theta_i; i \in \mathbb{N}\}$  are a sequence of nonnegative, NLOD, and identically distributed random variables with finite positive mean  $E\theta_1 = \lambda^{-1}$  such that (11) is satisfied; and  $\{Z_i; i \in \mathbb{N}\}$  and  $\{\theta_i; i \in \mathbb{N}\}$  are mutually independent. Assume further that either of Conditions 1–4 in Theorem 1 is satisfied. If  $F_Z \in \mathcal{C}$  and (10) holds, then, for any  $T$  such that  $\lambda(T) = EN(T) > 0$ ,*

$$\psi^*(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{F}_Z(u) du \quad (20)$$

holds uniformly for all  $t \in [T, \infty)$ , where  $\mu = c\lambda^{-1} - EZ$ .

We remark that in Lemma 3 the uniformity of (20) implies

$$\psi^*(x) \sim \frac{1}{\mu} \int_x^\infty \bar{F}_Z(u) du. \quad (21)$$

We point out that Corollary 2.1 of Wang et al. [9] gave some more situations when discussing the uniform asymptotics for  $\psi^*(x, t)$ , in which they allowed the distribution  $F_Z$  to be strongly subexponential.

The second lemma can be found in Yang et al. [12].

**Lemma 4.** *Let  $(X, Y)$  be a random vector with marginal distributions  $F$  and  $G$  both on  $\mathbb{R}_+$ , respectively, but  $X$  and  $Y$  are arbitrarily dependent. If  $F \in \mathcal{C}$  and  $\bar{G}(x) = o(\bar{F}(x))$ , then*

$$P(X + Y > x) \sim \bar{F}(x). \quad (22)$$

The third lemma gives the elementary renewal theorem for WOD random variables, which is due to Theorem 1.4 of Wang and Cheng [28].

**Lemma 5.** *Let  $\{\theta_i; i \in \mathbb{N}\}$  be a sequence of nonnegative, WOD, and identically distributed random variables with finite positive mean  $E\theta_1 = \lambda^{-1}$ , which constitutes a quasi-renewal counting process  $N(t)$  with mean function  $\lambda(t) = EN(t)$ . If Condition 2 in Theorem 1 is satisfied, then  $\lambda(t) \sim \lambda t$  as  $t \rightarrow \infty$ .*

As pointed out by Tang [5], it is easy to see that, under the conditions of Lemma 3, by using Lemma 5, relation (20) holds with  $\lambda(t)$  replaced by  $\lambda t$ , but the range of the uniformity for  $t$  becomes smaller.

**Lemma 6.** *Under the conditions of Lemma 3, if, in addition,  $\{\theta_i; i \in \mathbb{N}\}$  is a sequence of nonnegative, WOD, and identically distributed random variables with finite positive mean  $E\theta_1 = \lambda^{-1}$ , such that (11) and Condition 2 in Theorem 1 are satisfied, then*

$$\psi^*(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda t} \bar{F}_Z(u) du \quad (23)$$

holds uniformly for all  $t \in [f(x), \infty)$ , where  $f(x)$  is an arbitrary infinitely increasing function and  $\mu = c\lambda^{-1} - EZ$ .

*Proof.* By Lemmas 3 and 5, for any  $0 < \epsilon < 1/2$  and all  $t \in [f(x), \infty)$ , when  $x$  is sufficiently large, we have

$$\begin{aligned} \frac{1}{\mu} \int_x^{x+(1-\epsilon)\mu\lambda t} \bar{F}_Z(u) du &\leq \psi^*(x, t) \\ &\leq \frac{1}{\mu} \int_x^{x+(1+\epsilon)\mu\lambda t} \bar{F}_Z(u) du. \end{aligned} \quad (24)$$



For the upper bound, by the right-hand side of (24) we have that for sufficiently large  $x$  and all  $t \in [f(x), \infty)$ ,

$$\begin{aligned} \psi^*(x, t) &\leq \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{F_Z}(u) du \\ &\quad \cdot \left( 1 + \frac{\int_{x+\mu\lambda t}^{x+(1+\epsilon)\mu\lambda t} \overline{F_Z}(u) du}{\int_x^{x+\mu\lambda t} \overline{F_Z}(u) du} \right) \\ &\leq \frac{1+\epsilon}{\mu} \int_x^{x+\mu\lambda t} \overline{F_Z}(u) du, \end{aligned} \quad (25)$$

which concludes the upper bound by the arbitrariness of  $\epsilon > 0$ . We next consider the lower bound. Similarly, by the left-hand side of (24) we have that for sufficiently large  $x$  and all  $t \in [f(x), \infty)$ ,

$$\begin{aligned} \psi^*(x, t) &\geq \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{F_Z}(u) du \\ &\quad \cdot \left( 1 - \frac{\int_{x+(1-\epsilon)\mu\lambda t}^{x+\mu\lambda t} \overline{F_Z}(u) du}{\int_x^{x+\mu\lambda t} \overline{F_Z}(u) du} \right) \\ &\geq \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{F_Z}(u) du \\ &\quad \cdot \left( 1 - \epsilon \cdot \frac{\overline{F_Z}(x + (1-\epsilon)\mu\lambda t)}{\overline{F_Z}(x + \mu\lambda t)} \right) \\ &\geq \frac{1-c_0\epsilon}{\mu} \int_x^{x+\mu\lambda t} \overline{F_Z}(u) du, \end{aligned} \quad (26)$$

for some  $c_0 > 0$ , where the last step holds because, by  $0 < \epsilon < 1/2$  and  $F_Z \in \mathcal{C}$ ,

$$\begin{aligned} \limsup_{x \rightarrow \infty} \sup_{t \geq f(x)} \frac{\overline{F_Z}(x + (1-\epsilon)\mu\lambda t)}{\overline{F_Z}(x + \mu\lambda t)} \\ \leq \limsup_{x \rightarrow \infty} \sup_{t \geq f(x)} \frac{\overline{F_Z}((1/2)(x + \mu\lambda t))}{\overline{F_Z}(x + \mu\lambda t)} < \infty. \end{aligned} \quad (27)$$

This completes the proof of the lemma.  $\square$

We are now ready for the proofs of the two main results.

*Proof of Theorem 1.* We firstly consider the tail probability of  $Z = X + Y$ . In case (i), since  $X$  and  $Y$  are QAI and  $F \in \mathcal{C}$ ,  $G \in \mathcal{C}$ , by using Theorem 3.1 of Chen and Yuen [23] we have

$$P(Z > x) \sim \overline{F}(x) + \overline{G}(x), \quad (28)$$

implying  $F_Z \in \mathcal{C}$ . In case (ii), by  $(X, Y) \in \text{BRV}_{-\alpha}(\gamma, H)$  we have

$$P(Z > x) = P\left(\frac{(X, Y)}{x} \in A\right) \sim \nu(A) \overline{H}(x), \quad (29)$$

implying  $F_Z \in \mathcal{R}_{-\alpha} \subset \mathcal{C}$ . In case (iii), applying Lemma 4 gives

$$P(Z > x) \sim \overline{F}(x), \quad (30)$$

also implying  $F_Z \in \mathcal{C}$ . The above three relations, together with (21), lead to

$$\begin{aligned} \psi^*(x) &\sim \frac{\lambda}{c(1-\rho)} \int_x^\infty \overline{F_Z}(u) du \\ &\sim \begin{cases} \frac{\lambda}{c(1-\rho)} \int_x^\infty (\overline{F}(u) + \overline{G}(u)) du, & \text{in case (i)} \\ \frac{\lambda \nu(A)}{c(1-\rho)} \int_x^\infty \overline{H}(u) du, & \text{in case (ii)} \\ \frac{\lambda}{c(1-\rho)} \int_x^\infty \overline{F}(u) du. & \text{in case (iii)} \end{cases} \end{aligned} \quad (31)$$

By the first equivalence of (31), we have  $\psi^*(x+y) \sim \psi^*(x)$  for any fixed  $y > 0$  because of  $F_Z \in \mathcal{C} \subset \mathcal{L}$ . Then, we can follow the same line of Li [17] to verify

$$\psi(x) \sim \psi^*(x). \quad (32)$$

Therefore, the desired relations (15)–(17) hold from (31) and (32).  $\square$

*Proof of Theorem 2.* Under the conditions of Theorem 2, by (28)–(30) we have  $F_Z \in \mathcal{R}_{-\gamma}$  with  $\gamma > 1$  specified in Theorem 2. By Lemma 6 we have that, for every  $t > 0$ ,

$$\psi^*(x, xt) \sim \frac{\lambda}{c(1-\rho)} \int_x^{x(1+c(1-\rho)t)} \overline{F_Z}(u) du; \quad (33)$$

by this and (31), according to Karamata's theorem, we derive

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\psi^*(x, xt)}{\psi^*(x)} &= \lim_{x \rightarrow \infty} \frac{\int_x^{x(1+c(1-\rho)t)} \overline{F_Z}(u) du}{\int_x^\infty \overline{F_Z}(u) du} \\ &= 1 - (1 + c(1-\rho)t)^{-\gamma+1}. \end{aligned} \quad (34)$$

Similarly to (32), Li [17] proved

$$\psi(x, xt) \sim \psi^*(x, xt). \quad (35)$$

Thus, the desired relation (18) follows from (32), (34), and (35). This completes the proof of the theorem.  $\square$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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