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Second Order Asymptotics for Infinite-Time Ruin Probability in a Compound Renewal Risk Model

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Abstract
Consider a compound renewal risk model, in which a single accident may cause more than one claim. Under the condition that the common distribution of the individual claims is second order subexponential, we establish a second order asymptotic formula for the infinite-time ruin probability. Compared with the traditional ones, our second order asymptotic result is more precise and effective, which can be demonstrated by the numerical studies.

Keywords Compound renewal risk model · Infinite-time ruin probability · Second order asymptotic behavior · Second order subexponential distribution · Crude Monte-Carlo simulation

Mathematics Subject Classification (2010) 62E20 · 91B30 · 60G50

1 Introduction

In this paper we consider a compound renewal risk model, which is a natural extension of the classical one. In such a model, one accident may cause multiple claims, while in the ordinary renewal risk model, only one claim appears at each accident time. This is motivated by the fact that a severe accident/crisis, such as an earthquake, a tsunami or an economic cirsis, may trigger more than one claim. The compound renewal model is initially studied by Tang et al. (2001), and satisfies the following four requirements:

(a) The inter-arrival times of accidents $\{\theta_n, n \geq 1\}$ forms a sequence of independent and identically distributed (i.i.d.) nonnegative random variables (r.v.s) with a finite mean κ_1.

(b) The individual claims caused by the nth accident at time $\tau_n = \sum_{k=1}^{n} \theta_k$ are denoted by $\{X_k^{(n)}, k \geq 1\}$. Assume that $\left\{\{X_k^{(n)}, k \geq 1\}, n \geq 1\right\}$ are independent copies of a

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sequence of i.i.d. nonnegative r.v.s $\{X_k, k \geq 1\}$ with a common distribution F and a finite mean μ_1.

(c) The claim number caused by the nth accident is denoted by N_n. Assume that $\{N_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative integer-valued r.v.s with a finite mean ν_1.

(d) Assume that the three sequences $\{\theta_n, n \geq 1\}$, $\left\{\{X_k^{(n)}, k \geq 1\}, n \geq 1\right\}$ and $\{N_n, n \geq 1\}$ are mutually independent.

In the above compound renewal risk model, if the claim numbers $N_n, n \geq 1$, are all degenerate at 1, then the model reduces to the ordinary one.

Now, returning to a compound renewal risk model, the accident arrival times $\tau_n, n \geq 1$, with $\tau_0 = 0$, constitute a renewal counting process

$$\Theta(t) = \sup\{n \geq 0 : \tau_n \leq t\}, \qquad t \geq 0,$$

which represents the accident number occurring in the interval $[0, t]$. For each $n \geq 1$, the accumulated claims caused by the nth accident can be expressed by a random sum

$$Y_n = \sum_{k=1}^{N_n} X_k^{(n)},$$

with a finite mean $\nu_1\mu_1$. Clearly, $\{Y_n, n \geq 1\}$ is also a sequence of i.i.d. nonnegative r.v.s with a common distribution, denoted by G. Then, the insurer's surplus process can be defined as

$$U(t) = x - \sum_{n=1}^{\Theta(t)} (Y_n - c\theta_n), \qquad t \geq 0,$$

where $x \geq 0$ is the initial risk reserve of an insurer, and $c > 0$ is the constant premium rate. Under this setting, the ruin probability within time $[0, t]$, $t \geq 0$, and the infinite-time ruin probability can be defined, respectively, as

$$\psi_c(x; t) = P\left(\inf_{0 \leq s \leq t} U(s) < 0 \middle| U(0) = x\right),$$

and

$$\psi_c(x) = \lim_{t\to\infty} \psi_c(x; t) = P\left(\inf_{s \geq 0} U(s) < 0 \middle| U(0) = x\right). \tag{1.1}$$

Note that the infinite-time ruin probability (1.1) can be rewritten as

$$\psi_c(x) = P\left(\sup_{n \geq 0} \sum_{i=1}^{n} (Y_i - c\theta_i) > x\right), \tag{1.2}$$

where $\sum_{i=1}^{0} \cdot = 0$. Specially, in the ordinary renewal risk model (i.e., $N_n = 1$ for all $n \geq 1$), we denote the infinite-time ruin probability by $\psi_o(x) = P\left(\sup_{n\geq 0} \sum_{i=1}^{n} (X_i - c\theta_i) > x\right)$.

In the study of risk models with no interest rate, the relative safety loading condition is required to exclude the trivial case of $\psi_c(x) \equiv 1$, that is,

$$\rho := \frac{cE[\theta_1] - E[Y_1]}{E[Y_1]} = \frac{c\kappa_1 - \nu_1\mu_1}{\nu_1\mu_1} > 0, \tag{1.3}$$

which will be assumed throughout this paper.

In the insurance context, renewal risk models have been playing an important role, in which insurers and regulators measure all kinds of risks, such as ruin probabilities, deficit at ruin, surplus before ruin, etc. In recent decades, many earlier works have been devoted to the investigation of the asymptotic behavior for ruin probabilities in some (in)dependent ordinary renewal risk models. A classical result is established by Embrechts and Veraverbeke (1982): in the independent ordinary renewal risk model (i.e. the requirements (a)-(d) with $N_n = 1, n \geq 1$), if the equilibrium distribution of F, denoted by F_e, is subexponential, then, as $x \to \infty$,

$$\psi_o(x) \sim \frac{1}{\rho}\overline{F_e}(x), \tag{1.4}$$

where the symbol $\sim$ means that the quotient of both sides tends to 1, the relative safety loading coefficient $\rho > 0$ is definied in (1.3) with $\nu_1 = 1$, and

$$\overline{F_e}(x) = 1 - F_e(x) = \frac{1}{\mu_1}\int_x^\infty \overline{F}(t)dt. \tag{1.5}$$

Recently, Cang et al. (2020) considered a dependent ordinary renewal risk model, in which a general dependence structure exists between each pair of the individual claim and the claim inter-arrival time, and established an asymptotic formula for the finite-time ruin probability in the case of consistently varying tailed claims. Remark that this result holds with the uniformity for the time in some infinite regions, so it contains the asymptotics for the infinite-time ruin probability. Related extensions to some ordinary renewal risk models can be found in Leipus and Šiaulys (2007), Kočetova et al. (2009), Yang et al. (2011), Wang et al. (2012), among others.

The compound renewal risk model is introduced by Tang et al. (2001), who established a precise large deviation result on the aggregate claim process, under the condition that the individual claims are extended regularly varying tailed. Further, Kass and Tang (2005) considered a dependent compound renewal risk model in which the claims arrive in groups and the claim numbers in the groups may follow a certain negative dependence structure. A related reference is Yang et al. (2012), who considered a dependent compound renewal risk model with constant interest rate, and derived an asymptotic formula for the finite-time ruin probability in the case of subexponential claims. For more discussions on compound risk models, one can be referred to Aleškevičienė et al. (2008), Korshunov (2018), among many others.

We remark that all above approximations are established on the first order asymptotic behavior for ruin probabilities. With the development of the insurance industry, the investigation of the second (or higher) order estimates for ruin probabilities is needed to meet the demands of both insurers and regulators for the more accuracy of risk analysis. Baltrūnas (2005) considered an ordinary Poisson risk model and obtained the second order rough asymptotics for the infinite-time ruin probability in the case of subexponential claims. Recently, Lin (2012a) derived a second order precise result in an ordinary renewal risk model, under the condition that the equilibrium distribution of claims is second order subexponential. Lin (2021) further extended the corresponding result in Lin (2012a) to a delayed ordinary renewal risk model. For more discussions on the second order asymptotics, we refer the reader to Lin (2012b), Kortschak and Hashorva (2014), Lin (2014), Lin (2019), among others.

In this paper, we focus on the compound renewal risk model due to the recent frequent occurrence of catastrophes and severe accidents/crisises. A second order precise formula is established to estimate the infinite-time ruin probability, which will lend insurers and regulators some important insights on the control of insurance risks in a more precise and effective way. Besides, a Monte-Carlo simulation is also performed to illustrate the improvement of our result compared with that on the first order asymptotics.

The rest of this paper is organized as follows. Section 2 presents the main result after some preliminaries, and its proof is given in Section 3. Section 4 performs some numerical studies to check the accuracy of the obtained formula and conducts some sensitivity analysis on the parameters in the model. Section 5 makes some concluding remarks.

2 Preliminaries and Main Result

Throughout the paper, all limit relations are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $\lim \frac{f(x)}{g(x)} = 1$, write $f(x) = o(g(x))$ if $\lim \frac{f(x)}{g(x)} = 0$, write $f(x) = O(g(x))$ if $\limsup \frac{f(x)}{g(x)} < \infty$. In addition, C represents a positive constant, whose value may vary from place to place.

2.1 Heavy-Tailed Distributions

In this paper, we shall model the individual claims by some heavy-tailed distributions supported on $\mathbb{R}_+ = [0, \infty)$. A nonnegative function h is said to be long-tailed, denoted by $h \in \mathcal{L}$, if $h(x + y) \sim h(x)$ for any fixed $y > 0$. In particular, a distribution V on $\mathbb{R}$ is said to be long-tailed, denoted by $V \in \mathcal{L}$, if its tail distribution $\overline{V} = 1 - V$ is long-tailed. Chistyakov (1964) introduced the concept of subexponential distributions. By definition, a distribution V on $\mathbb{R}_+$ is said to be subexponential, denoted by $V \in \mathcal{S}$, if $\overline{V^{2*}}(x) \sim 2\overline{V}(x)$, where V^{2*} is the two-fold convolution of V. Further, Klüppolberg (1988) defined a subclass of subexponential distributions. A distribution V on $\mathbb{R}_+$ is said to belong to the class $\mathcal{S}^*$, if V has a finite mean μ_V and $\int_0^x \overline{V}(y)\overline{V}(x-y)dy \sim 2\mu_V \overline{V}(x)$. It is well-known that

$$\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}.$$

For more properties of subexponential distributions, one can be referred to Embrechts et al. (1997), Asmussen and Albrecher (2010), Foss et al. (2011).

The following is the local version of long-tailed distributions. For any $t \in (0, \infty]$, we write $\Delta(t) = (0, t]$, and $x + \Delta(t) = (x, x+t]$, further, for any distribution V on $\mathbb{R}$, we write

$$V(x + \Delta(t)) = V(x, x + t] = V(x + t) - V(x).$$

By definition, for any $t > 0$, a disbribution V on $\mathbb{R}$ is said to belong to the locally long-tailed distribution class $\mathcal{L}_{\Delta(t)}$, if, as $x \to \infty$,

$$V\,(x + y + \Delta(t)) \sim V\,(x + \Delta(t))$$

holds uniformly in $y \in [0, 1]$, and hence, it holds uniformly in any finite interval of y. Further, a distribution V on $\mathbb{R}$ is said to be locally long-tailed, deboted by $V \in \mathcal{L}_{loc}$, if $V \in \mathcal{L}_{\Delta(t)}$ for all $t \in (0, \infty)$. This locally long-tailed class was introduced by Asmussen et al. (2003). Recently, Lin (2012a, b) proposed the following second order subexponential distributions. A distribution V on $\mathbb{R}_+$ with finite mean μ_V is said to be second order subexponential, denoted by $V \in \mathcal{S}_2$, if $V \in \mathcal{L}_{loc}$ and

$$\overline{V^{2*}}(x) - 2\overline{V}(x) \sim 2\mu_V V(x, x + 1].$$

We remark that the inclusions

$$\mathcal{S}_2 \subset \mathcal{L}_{loc} \subset \mathcal{L} \text{ and } \mathcal{S}_2 \subset \mathcal{S}$$

hold, and that the class $\mathcal{S}_2$ includes many typical heavy-tailed distributions, such as the Pareto distribution with parameter larger than 1, the lognormal distribution, and the Weibull distribution with paremeter in (0,1).

2.2 Main Result

Assume that the common distribution F of individual claims is absolutely continuous with density function f, so its hazard function $Q(x) = -\ln \overline{F}(x)$ has a hazard rate $q(x) = Q'(x) = \frac{f(x)}{\overline{F}(x)}$. The following condition will be used in Theorem 2.1.

Condition 2.1 *Suppose that the hazard function $Q(x)$ and the hazard rate $q(x)$ of individual claims satisfy*

(1) $r := \limsup_{x\to\infty} \frac{xq(x)}{Q(x)} < 1$;
(2) $\beta := \liminf_{x\to\infty} xq(x) > \frac{2}{1-r}$.

Now we are ready to state the main result of this paper. Hereafter, assume that there exist the following three second moments of the individual claim $E\left[X_1^2\right] = \mu_2 < \infty$, the inter-arrival time $E[\theta_1^2] = \kappa_2 < \infty$, and the claim number $E\left[N_1^2\right] = \nu_2 < \infty$.

Theorem 2.1 *Consider the compound renewal risk model. Under Condition 2.1, if $f \in \mathcal{L}$, $q(xy) = O\left(q(x)\right)$ for any $y > 0$, and $E[(1+\epsilon)^{N_1}] < \infty$ for some $\epsilon > 0$, then*

$$\psi(x) = \frac{1}{\rho}\overline{F_e}(x) + \left(K\nu_1 + \frac{1}{\rho}\left(\frac{\nu_2}{\nu_1} - 1\right)\right)\overline{F}(x) + o\left(\overline{F}(x)\right), \tag{2.1}$$

where

$$\begin{aligned} K &= \frac{E\left[Y_1^2\right] + c^2\kappa_2 - 2c^2\kappa_1^2 + 2(c\kappa_1 - \mu_1\nu_1)C_\infty}{2(c\kappa_1 - \mu_1\nu_1)^2} \\ &= \frac{\left(\mu_2 - \mu_1^2\right)\nu_1 + \mu_1^2\nu_2 + c^2\kappa_2 - 2c^2\kappa_1^2 + 2(c\kappa_1 - \mu_1\nu_1)C_\infty}{2(c\kappa_1 - \mu_1\nu_1)^2}, \end{aligned}$$

and

$$C_\infty = E\left[\sup_{n\geq 0}\sum_{i=1}^{n}(Y_i - c\theta_i)\right].$$

Remark 2.1 Condition 2.1 is similar to but weaker than Condition B of Baltrūnas et al. (2004). By Lemma 3.8(a) of Baltrūnas et al. (2004), if $r < 1$ then $F \in \mathcal{S}$; and further if $\beta > \frac{2}{1-r}$ (implying $\beta > \frac{1}{2-2^r}$) for any $r < 1$ then $F \in \mathcal{S}^*$ by Lemmas 3.6(b) and 3.8(b) of Baltrūnas et al. (2004).

Remark 2.2 Lin (2012a) provided a sufficient and necessary condition on the second order expansion for the infinite-time ruin probability in an ordinary renewal risk model. Compared with Lin's result, we establish a similar second order asymptotic formula in a compound model.

Remark 2.3 Our obtained (2.1) is more precise than (1.4), regarded as the first order asymptotics, but the second term on the right-hand side of (2.1) is negligible compared with the first term because of $\overline{F}(x) = o\left(\overline{F_e}(x)\right)$ under the conditions of Theorem 2.1. This will be verified via the simulation study in Section 4.

By Theorem 2.1 and the similar discussions on Corollary 3.1 of Lin (2012a), we can derive the following corollary.

Corollary 2.1 *Consider the compound Poisson risk model, that is, the inter-arrival times of accidents are i.i.d. r.v.s with a common exponential distribution. Under the conditions of Theorem 2.1, relation* (2.1) *holds with* $K = \frac{(\mu_2-\mu_1^2)\nu_1+\mu_1^2\nu_2}{(c\kappa_1-\mu_1\nu_1)^2}$.

3 Proof of Theorem 2.1

Before the proof of our main result, we give a series of lemmas. The first lemma comes from Lin (2012b).

Lemma 3.1 *Assume that* $F \in \mathcal{S}_2$ *with a finite mean* μ_1.

(1) *For all* $n \geq 2$,
$$\overline{F^{n*}}(x) - n\overline{F}(x) \sim n(n-1)\mu_1 F(x, x+1].$$
(2) *For every fixed* $\varepsilon > 0$, *there exist two constants* $x_0 > 0$ *and* $C > 0$, *irrespective to* n, *such that for all* $n \geq 2$,
$$\sup_{x\geq x_0}\left|\frac{\overline{F^{n*}}(x) - n\overline{F}(x)}{F(x, x+1]}\right| \leq C(1+\varepsilon)^n.$$

Lemma 3.2 *If* $f \in \mathcal{L}$, *then* $F \in \mathcal{L}_{loc}$, *and*
$$F(x, x+1] \sim f(x). \tag{3.1}$$

Proof The proof can be found in Asmussen et al. (2003). Indeed, by $f \in \mathcal{L}$, for any $t > 0$ and $y > 0$ we have
$$\begin{aligned} F\left(x+y+\Delta(t)\right) &= \int_x^{x+t} f(u+y)du \\ &\sim \int_x^{x+t} f(u)du \\ &= F\left(x+\Delta(t)\right). \end{aligned}$$
Again by $f \in \mathcal{L}$,
$$F(x, x+1] = \int_0^1 f(x+u)du \sim f(x),$$
as claimed. □

The next lemma plays an important role in the proof of Theorem 2.1.

Lemma 3.3 *If* $f \in \mathcal{L}$, $r < 1$, $\beta > \frac{1}{2-2^r}$, $\mu_1 < \infty$, *and* $q(xy) = O\left(q(x)\right)$ *for all* $y > 0$, *then* $F \in \mathcal{S}_2$.

Proof By $r < 1$ and $\beta > \frac{1}{2-2^r}$, there exist a small $\delta > 0$ with $0 < r+\delta < r+2\delta < 1$ and a large $x_1 \geq 1$ such that for all $x \geq x_1$,
$$\frac{xq(x)}{Q(x)} \leq r+\delta \qquad \text{and} \qquad xq(x) > \frac{1}{2-2^{r+2\delta}}. \tag{3.2}$$
The first inequality in (3.2), together with Proposition 3.7(a) of Baltrūnas et al. (2004), implies that
$$Q(xy) \leq y^{r+\delta}Q(x)$$

holds for all $x \ge x_1$ and $y \ge 1$. Then, for $x \ge 2x_1$,

$$\begin{aligned} \frac{x\left(\overline{F}\left(\frac{x}{2}\right)\right)^2}{\overline{F}(x)} &= \exp\left\{-2Q\left(\frac{x}{2}\right)+Q(x)+\ln x\right\} \\ &\le \exp\left\{-(2-2^{r+\delta})Q\left(\frac{x}{2}\right)+\ln x\right\}. \end{aligned} \tag{3.3}$$

By integration, the second inequality in (3.2) gives that for all $x \ge 2x_1$,

$$Q\left(\frac{x}{2}\right)-Q(x_1) \ge \frac{1}{2-2^{r+2\delta}}\ln\left(\frac{x}{2x_1}\right). \tag{3.4}$$

Combining (3.3) and (3.4) leads to

$$\frac{x\left(\overline{F}\left(\frac{x}{2}\right)\right)^2}{\overline{F}(x)} \le \exp\left\{-\frac{2-2^{r+\delta}}{2-2^{r+2\delta}}\ln\left(\frac{x}{2x_1}\right)-(2-2^{r+\delta})Q(x_1)+\ln x\right\} \to 0. \tag{3.5}$$

It follows from (3.1), (3.5) and $\beta > \frac{1}{2-2^r}$ that

$$\lim_{x\to\infty}\frac{\left(\overline{F}\left(\frac{x}{2}\right)\right)^2}{F(x,x+1]} = \lim_{x\to\infty}\frac{\left(\overline{F}\left(\frac{x}{2}\right)\right)^2}{f(x)} = \lim_{x\to\infty}\frac{x\left(\overline{F}\left(\frac{x}{2}\right)\right)^2/\overline{F}(x)}{xq(x)} = 0. \tag{3.6}$$

In addition, by Remark 2.1 we have $F \in \mathcal{S}^*$. Therefore, we conclude $F \in \mathcal{S}_2$ from Lemma 3.2, (3.6) and by using Proposition 2.4 of Lin (2012b). □

Recently, Lin (2012a) introduced a new subclass $\mathcal{H}$ of the class $\mathcal{S}^*$. A distribution V on $\mathbb{R}_+$ is said to belong to $\mathcal{H}$ if

$$\lim_{x\to\infty}\frac{\int_0^{\frac{x}{2}} y\overline{V}(x-y)\overline{V}(y)dy}{\overline{V}(x)} = \frac{1}{2}\int_0^\infty y^2V(dy) < \infty.$$

As pointed out by Lin (2012a), the conclusions hold: $\mathcal{H} \subset \mathcal{S}^*$; if $F \in \mathcal{H}$, then $F_e \in \mathcal{S}_2$; and some common-used heavy-tailed distributions belong to $\mathcal{H}$, such as the Pareto distribution with parameter larger than 2, the lognormal distribution, and the Weibull distribution with paranmeter in (0,1).

Lemma 3.4 *If $r < 1$ and $\beta > \frac{2}{1-r}$, then $F \in \mathcal{H}$.*

Proof By $\beta > \frac{2}{1-r}$, similarly to (3.4), there exist a small $\delta > 0$ with $0 < r+2\delta < 1$, and a large $x_2 > 0$ such that for all $x \ge x_2$,

$$Q(x)-Q(x_2) \ge \frac{2}{1-r-2\delta}\ln\left(\frac{x}{x_2}\right).$$

Thus,

$$\begin{aligned} \int_{x_2}^\infty x\left(\overline{F}(x)\right)^{1-r-\delta}dx &= \int_{x_2}^\infty xe^{-(1-r-\delta)Q(x)}dx \\ &\le \int_{x_2}^\infty x\exp\left\{-\frac{2(1-r-\delta)}{1-r-2\delta}\ln\left(\frac{x}{x_2}\right)-(1-r-\delta)Q(x_2)\right\}dx \\ &= \left(\overline{F}(x_2)\right)^{1-r-\delta}x_2^{\frac{2(1-r-\delta)}{1-r-2\delta}}\int_{x_2}^\infty x^{-\frac{2(1-r-\delta)}{1-r-2\delta}+1}dx \\ &< \infty, \end{aligned}$$

which implies that $F \in \mathcal{H}$ by addressing Proposition 3.4 of Lin (2012a). □

Proof of Theorem 2.1. Note $\frac{2}{1-r} > \frac{1}{2-2^r}$ for any $r < 1$. Then, by Lemmas 3.3 and 3.4, we have $F \in \mathcal{S}_2 \cap \mathcal{H}$. Under the conditions of the theorem, Theorem 2.1 of Lin (2012b) can be applied to obtain

$$\overline{G}(x) - \nu_1 \overline{F}(x) \sim \mu_1(\nu_2 - \nu_1) F(x, x+1], \tag{3.7}$$

which further implies that

$$\overline{G}(x) \sim \nu_1 \overline{F}(x). \tag{3.8}$$

By $F \in \mathcal{H}$ we have $G \in \mathcal{H}$, and thus $G_e \in \mathcal{S}_2$, see Proposition 3.1 and Theorem 3.2 of Lin (2012a). In addition, by (3.7), Proposition 2.3 of Lin (2012b) gives $G \in \mathcal{S}_2$. So, Theorem 3.1 of Lin (2012a) yields

$$\psi(x) \sim \frac{1}{\rho} \overline{G_e}(x) + K\overline{G}(x) + o\left(\overline{G}(x)\right). \tag{3.9}$$

Now we deal with the expansion for $\overline{G_e}$. Note that

$$\frac{\overline{G_e}(x) - \overline{F_e}(x)}{\overline{F}(x)} = \frac{1}{\mu_1 \nu_1} \sum_{n=1}^{\infty} \left(\int_x^{\infty} \frac{\overline{F^{n*}}(t) - n\overline{F}(t)}{F(t, t+1]} \cdot \frac{F(t, t+1]}{\overline{F}(x)} dt \right) P(N_1 = n). \tag{3.10}$$

By $F \in \mathcal{S}_2$ and Lemma 3.1(2), for the given $0 < \epsilon < 1$, there exist some large $x_3 > 0$ and $C > 0$ such that for all $t \geq x \geq x_3$ and all $n \geq 1$,

$$\left| \frac{\overline{F^{n*}}(t) - n\overline{F}(t)}{F(t, t+1)} \right| \leq C(1+\epsilon)^n. \tag{3.11}$$

By using L'Hospital rule and Lemma 3.2, we have

$$\lim_{x\to\infty} \frac{\int_x^{\infty} F(t, t+1] dt}{\overline{F}(x)} = \lim_{x\to\infty} \frac{F(x, x+1]}{f(x)} = 1.$$

Then for the above $\epsilon > 0$, there exists some large $x_4 \geq x_3$ such that for all $x \geq x_4$,

$$\int_x^{\infty} F(t, t+1] dt \leq (1+\epsilon)\overline{F}(x). \tag{3.12}$$

Combining (3.11) and (3.12), we have that for all $x \geq x_4$ and all $n \geq 1$, the integral on the right-hand side of (3.10) is bounded by $C(1+\epsilon)^{n+1}$. By $E\left[(1+\epsilon)^{N_1}\right] < \infty$, the dominated convergence theorem can be used to obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{\overline{G_e}(x) - \overline{F_e}(x)}{\overline{F}(x)} &= \frac{1}{\mu_1 \nu_1} \sum_{n=1}^{\infty} \lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_x^{\infty} \overline{F^{n*}}(t) - n\overline{F}(t) dt \cdot P(N_1 = n) \\
&= \frac{1}{\mu_1 \nu_1} \sum_{n=1}^{\infty} \lim_{x\to\infty} \frac{\overline{F^{n*}}(x) - n\overline{F}(x)}{f(x)} \cdot P(N_1 = n) \\
&= \frac{1}{\nu_1} \sum_{n=1}^{\infty} n(n-1) \lim_{x\to\infty} \frac{F(x, x+1]}{f(x)} P(N_1 = n) \\
&= \frac{\nu_2}{\nu_1} - 1,
\end{aligned} \tag{3.13}$$

where we used L'Hospital rule in the second step, $F \in \mathcal{S}_2$ and Lemma 3.1(1) in the third step, and Lemma 3.2 in the last step. Plugging (3.8) and (3.13) into (3.9) leads to the desired relation (2.1). □

4 Simulation Studies

In this section, we perform some numerical simulations to check the accuracy of the asymptotic result for the infinite-time ruin probability $\psi_c(x)$ in Corollary 2.1, and illustrate the improvement of our result compared with that on the first order asymptotics via the crude Monte-Carlo method. We also conduct a sensitivity analysis on $\psi_c(x)$ to key model parameters, including the parameters of the individual claims, the inter-arrival times of accidents, and the claim numbers.

4.1 Accuracy of Asymptotic Result

We first check the accuracy of the approximation given by (2.1). Model specifications for the numerical studies are listed below:

- The individual claims $\{X_k, k \geq 1\}$ constitutes a sequence of i.i.d. nonnegative r.v.s with a common Pareto distribution of the form

$$F(x) = 1 - (1+x)^{-\alpha}, \qquad x \geq 0, \tag{4.1}$$

  for some $\alpha > 2$; or with a common Weibull distribution of the form

$$F(x) = 1 - \exp\{-x^a\}, \qquad x \geq 0, \tag{4.2}$$

  for some $0 < a < 1$. It can be proven that in both cases, the density function $f \in \mathcal{L}$ and $q(xy) = O(q(x))$ for any $y > 0$. In addition, it can be calculated that the indices $r = 0, \beta = \alpha$ in Condition 2.1 for the former, and $r = a < 1, \beta = \infty$ for the latter. Then, all conditions of Corollary 2.1 are satisfied for individual claims. Furthermore, comparing such two cases, the common tail of Pareto-distributed individual claims is much heavier than that of Weibull-distributed individual claims.
- The claim numbers $\{N_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative integer-valued r.v.s with a common Poisson distribution of the form

$$P(N_1 = l) = \frac{\lambda^l}{l!}e^{-\lambda}, \qquad l = 0, 1, 2, ..., \tag{4.3}$$

  for some $\lambda > 0$. Clearly, $E\left[(1+\epsilon)^{N_1}\right] < \infty$ for any $\epsilon > 0$.
- The inter-arrival times of accidents $\{\theta_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative r.v.s with a common exponential distribution of the form

$$P(\theta_1 \leq x) = 1 - e^{-\rho x}, \qquad x \geq 0, \tag{4.4}$$

  for some $\rho > 0$.

For the simulated value of $\psi_c(x)$, we first generate the three sequences of the accident inter-arrival times $\{\theta_n, n \geq 1\}$, the individual claims $\left\{\left\{X_k^{(n)}, k \geq 1\right\}, n \geq 1\right\}$, and the claim numbers $\{N_n, n \geq 1\}$, with a sample size m. For each sample $j = 1, ..., m$, denote the above three sequences by $\left\{\theta_n^{(j)}, n \geq 1\right\}$, $\left\{\left\{X_k^{(n,j)}, k \geq 1\right\}, n \geq 1\right\}$ and $\left\{N_n^{(j)}, n \geq 1\right\}$, respectively, and then calculate the quantity

$$L_n^{(j)} = \sum_{k=1}^{N_n^{(j)}} X_k^{(n,j)} - c\theta_n^{(j)},$$

Table 1 Accuracy of Corollary 2.1 in case of Pareto-distributed claims

x	62	114	322	687	1202
$\hat{\psi}_c(x; n_0)$	1.49×10^{-2}	7.71×10^{-3}	2.53×10^{-3}	1.09×10^{-3}	5.9×10^{-4}
$\psi_c^{(1)}(x)$	1.23×10^{-2}	6.53×10^{-3}	2.21×10^{-3}	1×10^{-3}	5.57×10^{-4}
	(2.6×10^{-3})	(1.18×10^{-3})	(3.2×10^{-4})	(9×10^{-5})	(3.3×10^{-5})
$\psi_c^{(2)}(x)$	1.37×10^{-2}	7.11×10^{-3}	2.40×10^{-3}	1.04×10^{-3}	5.71×10^{-4}
	(1.2×10^{-3})	$(6. \times 10^{-4})$	(1.3×10^{-4})	(5×10^{-5})	(1.9×10^{-5})
$\frac{\psi_c^{(1)}(x)}{\hat{\psi}_c(x;n_0)}$	0.826	0.847	0.874	0.917	0.944
$\frac{\psi_c^{(2)}(x)}{\hat{\psi}_c(x;n_0)}$	0.919	0.922	0.949	0.954	0.968
x	1333	1596	1838	1949	2000
$\hat{\psi}_c(x; n_0)$	5.25×10^{-4}	4.35×10^{-4}	3.75×10^{-4}	3.45×10^{-4}	3.35×10^{-4}
$\psi_c^{(1)}(x)$	4.99×10^{-4}	4.13×10^{-4}	3.56×10^{-4}	3.35×10^{-4}	3.26×10^{-4}
	(2.6×10^{-5})	(2.2×10^{-5})	(1.9×10^{-5})	(1.0×10^{-5})	(9.0×10^{-6})
$\psi_c^{(2)}(x)$	5.11×10^{-4}	4.22×10^{-4}	3.62×10^{-4}	3.41×10^{-4}	3.31×10^{-4}
	(1.4×10^{-5})	(1.3×10^{-5})	(1.2×10^{-5})	(4.0×10^{-6})	(4.0×10^{-6})
$\frac{\psi_c^{(1)}(x)}{\hat{\psi}_c(x;n_0)}$	0.950	0.949	0.949	0.971	0.973
$\frac{\psi_c^{(2)}(x)}{\hat{\psi}_c(x;n_0)}$	0.973	0.970	0.965	0.988	0.988

which represents the estimated value for the net loss caused by the nth accident. Choose a sufficiently large n_0, write the running maximum of $L_n^{(j)}$ as

$$M_{n_0}^{(j)} = \max_{1 \le n \le n_0} L_n^{(j)}.$$

Repeating the algorithm above m times, the value of $\psi_c(x)$ can be estimated by

$$\hat{\psi}_c(x; n_0) = \frac{1}{m} \sum_{j=1}^{m} 1_{\left\{M_{n_0}^{(j)} > x\right\}}, \tag{4.5}$$

according to (1.2), where 1_A is the indicator function of a set A. Although we estimate the infinite-time ruin probability $\psi_c(x)$, when simulating it we choose $\hat{\psi}_c(x; n_0)$ as the replacement for large but fixed n_0.

Set the large $n_0 = 10^5$ and the sample size $m = 10^6$ or $m = 10^7$ in the Pareto case or the Weibull case, respectively. We remark that the sample size in the latter is larger due to the much lighter tail of the Weibull-distributed individual claims. The various parameters are set to $c = 1, \lambda = 5$ in (4.3), $\rho = 0.1$ in (4.4), and $\alpha = 2.05$ in (4.1) for the Pareto-distributed individual claims or $c = 1, \lambda = 6$ in (4.3), $\rho = 0.2$ in (4.4), and $a = 0.335$ in (4.2) for the Weibull-distributed individual claims.

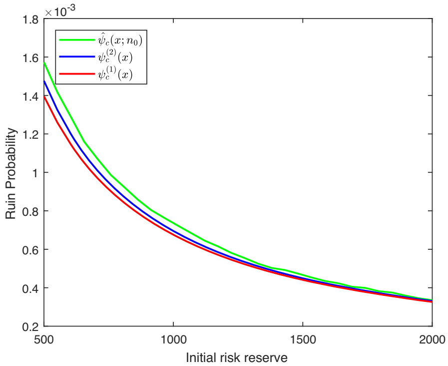
We consider both the first and the second order asymptotic values of $\psi_c(x)$. Denote by $\psi_c^{(1)}(x)$ the first order asymptotic value, i.e. the first term on the right-hand side of (2.1), and by $\psi_c^{(2)}(x)$ the second order one, i.e. the sum of the first two terms on the right-hand side of (2.1). Their deviations from $\hat{\psi}_c(x; n_0)$ are listed in the brackets after the corresponding asymptotic values.

Tables 1 and 2 summarize the simulated values $\hat{\psi}_c(x; n_0)$ in (4.5), the first and second order asymptotic values $\psi_c^{(1)}(x)$, $\psi_c^{(2)}(x)$, and the corresponding ratios with respect to different initial risk reserves x's, which are larger in Table 2 also due to the much lighter tail

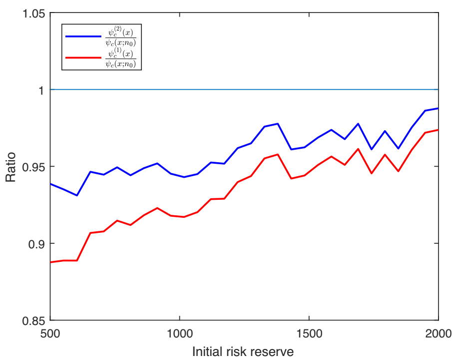
Table 2 Accuracy of Corollary 2.1 in case of Weibull-distributed claims

x	231	452	673	1558	2000
$\hat{\psi}_c(x; n_0)$	1.62×10^{-2}	4.83×10^{-3}	1.98×10^{-3}	1.73×10^{-4}	7.22×10^{-5}
$\psi_c^{(1)}(x)$	1.3×10^{-2}	3.98×10^{-3}	1.67×10^{-3}	1.56×10^{-4}	6.51×10^{-5}
	(3.2×10^{-3})	(8.5×10^{-4})	(3.1×10^{-4})	(1.7×10^{-5})	(7.1×10^{-6})
$\psi_c^{(2)}(x)$	1.49×10^{-2}	4.49×10^{-3}	1.85×10^{-3}	1.63×10^{-4}	6.94×10^{-5}
	(1.3×10^{-3})	(3.4×10^{-4})	(1.3×10^{-4})	(1×10^{-5})	(2.8×10^{-6})
$\frac{\psi_c^{(1)}(x)}{\hat{\psi}_c(x;n_0)}$	0.802	0.824	0.843	0.902	0.902
$\frac{\psi_c^{(2)}(x)}{\hat{\psi}_c(x;n_0)}$	0.919	0.930	0.934	0.942	0.961
x	2278	2456	2673	2814	3000
$\hat{\psi}_c(x; n_0)$	4.84×10^{-5}	3.31×10^{-5}	2.32×10^{-5}	1.65×10^{-5}	1.40×10^{-5}
$\psi_c^{(1)}(x)$	4.39×10^{-5}	3.03×10^{-5}	2.14×10^{-5}	1.53×10^{-5}	1.31×10^{-5}
	(4.5×10^{-6})	(2.8×10^{-6})	(1.8×10^{-6})	(1.2×10^{-6})	(9.0×10^{-7})
$\psi_c^{(2)}(x)$	4.67×10^{-5}	3.21×10^{-5}	2.25×10^{-5}	1.61×10^{-5}	1.37×10^{-5}
	(1.7×10^{-6})	(1.0×10^{-6})	(7.0×10^{-7})	(4.0×10^{-7})	(3.0×10^{-7})
$\frac{\psi_c^{(1)}(x)}{\hat{\psi}_c(x;n_0)}$	0.907	0.915	0.922	0.927	0.936
$\frac{\psi_c^{(2)}(x)}{\hat{\psi}_c(x;n_0)}$	0.965	0.970	0.970	0.976	0.979

of the Weibull-distributed individual claims. It can be seen that with the increase of x, the simulated and the two asymptotic values are closer and decrease gradually. Furthermore, the deviations originating from the second order estimate are slightly smaller than the corresponding ones from the first order estimate, and all deviations also decrease gradually with the increase of x. This indicates that our second order asymptotic estimate for the infinite-time ruin probability performs better than the traditional first order estimate, which is also shown by the ratios between the (first/second order) asymptotic and simulated values. Some further graphics of the accuracy are displayed in Figs. 1 and 2.



(a)



(b)

Fig. 1 Comparison between the simulated and asymptotic values of infinite-time ruin probability (left) and their ratios (right) in case of Pareto-distributed claims

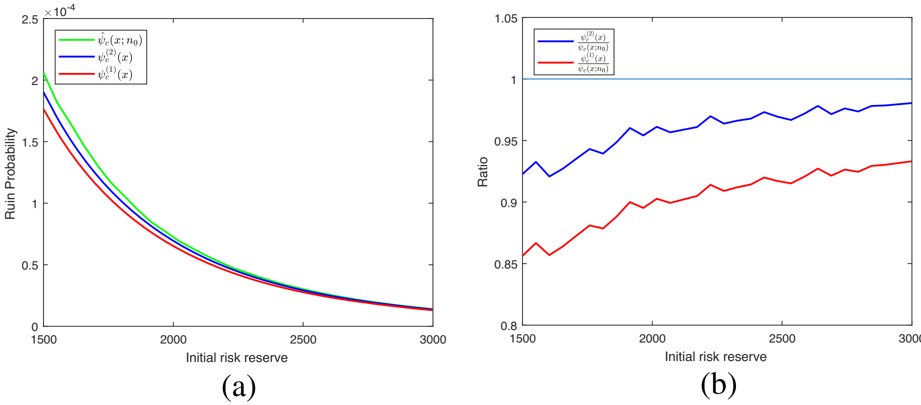


Fig. 2 Comparison between the simulated and asymptotic values of infinite-time ruin probability (left) and their ratio (right) in case of Weibull-distributed claims

To illustrate the stability and accuracy of simulation results, we further calculate the statistical measure, mean relative errors (MRE), for the simulated infinite-time ruin probability $\hat{\psi}_c(x; n_0)$ after repeating the Monte-Carlo simulation procedure by $M = 10$ times. By definition, the MRE for the simulated $\hat{\psi}_c(x; n_0)$ can be defined as

$$\frac{1}{M}\sum_{i=1}^{M}\frac{\hat{\psi}_c(x; n_0)^{i\text{th}}}{\psi_c^{(2)}(x)} - 1,$$

where the denominator $\psi_c^{(2)}(x)$ is the second order asymptotic value, and the numerator $\hat{\psi}_c(x; n_0)^{i\text{th}}$ is the simulated value in the ith repeating procedure. The MREs for the simulated infinite-time ruin probability in the cases of Pareto-distributed and Weibull-distributed claims are listed in Tables 3 and 4, respectively

According to Tables 3 and 4, it is obvious to see that all the MREs do not exceed 10%, indicating our theoretical result is reasonable.

4.2 Sensitivity Analysis

In this subsection, we conduct a sensitivity analysis on the simulated infinite-time ruin probability $\hat{\psi}_c(x; n_0)$ with respect to the adjusting parameters α, λ, ρ in the case of Pareto-distributed claims, and a, λ, ρ in the case of Weibull-distributed claims, respectively.

Table 3 MREs for simulated $\hat{\psi}_c(x; n_0)$ in case of Pareto-distributed claims

x	62	114	322	687	1202	1333	1596	1838	1949	2000
MRE	8.7%	8.4%	5.4%	4.8%	3.3%	2.7%	3.2%	3.6%	1.2%	1.1%

Table 4 MREs for simulated $\hat{\psi}_c(x; n_0)$ in case of Weibull-distributed claims

x	231	452	673	1558	2000	2278	2456	2673	2814	3000
MRE	8.7%	7.5%	7.0%	4.9%	4.1%	3.6%	3.1%	3.2%	2.5%	2.2%

Table 5 summarizes the percentage changes in $\hat{\psi}_c(x; n_0)$ with respect to the percentage changes in α, λ and ρ, for $x = 62, 687, 1333, 1838, 2000$, respectively. It shows that $\hat{\psi}_c(x; n_0)$ increases as α decreases, which is anticipated because a smaller value of α means larger individual claims the surplus process is exposed to, and hence the higher likelihood of ruin. It also shows that $\hat{\psi}_c(x; n_0)$ increases when λ or ρ increases, which is also reasonable because a larger value of λ or ρ means more claims caused by one accident or more frequent accidents, respectively, and hence the higher likelihood of ruin. We observe that although all three parameters change by the same percentage, $\hat{\psi}_c(x; n_0)$ is much more sensitive to α than λ and ρ, which is due to the dominance of the individual claims over the whole surplus process. This is consistent with the fact that the driving force for the ruin of an insurer is mostly brought from individual large claims, especially when economic is going downturn. In the same way, Table 6 shows that $\hat{\psi}_c(x; n_0)$ increases as a decreases, implying larger individual claims. It can be observed that $\hat{\psi}_c(x; n_0)$ is much more sensitive to a than λ and ρ, as shown in Table 6.

Table 5 Sensitivity testing for $\hat{\psi}_c(x; n_0)$ on α, λ, ρ in case of Pareto-distributed claims

Model parameters		$\hat{\psi}_c(x; n_0)$				
		$x = 62$	$x = 687$	$x = 1333$	$x = 1838$	$x = 2000$
% change in α	+1%	−9.76%	−14.10%	−17.14%	−17.33%	−17.26%
	+0.5%	−4.88%	−7.10%	−7.53%	−7.87%	−7.91%
	($\alpha = 2.05$)	(1.49×10^{-2})	(1.09×10^{-3})	(5.25×10^{-4})	(3.75×10^{-4})	(3.35×10^{-4})
	−0.5%	+4.88%	+7.0%	+8.13%	+8.44%	+8.46%
	−1%	+10.57%	+16.0%	+18.31%	+18.61%	+18.53%
% change in λ	+1%	+1.63%	+2.0%	+2.53%	+2.75%	+2.78%
	+0.5%	+0.81%	+1.0%	+0.73%	+0.89%	+0.82%
	($\lambda = 5$)	(1.49×10^{-2})	(1.09×10^{-3})	(5.25×10^{-4})	(3.75×10^{-4})	(3.35×10^{-4})
	−0.5%	−1.63%	−1.0%	−0.85%	−0.71%	−0.77%
	−1%	−1.95%	−2.0%	−1.42%	−1.48%	−1.39%
% change in ρ	+1%	+1.63%	+2.0%	+2.02%	+1.92%	+1.97%
	+0.5%	+0.81%	+1.0%	+1.02%	+0.90%	+0.98%
	($\rho = 0.1$)	(1.49×10^{-2})	(1.09×10^{-3})	(5.25×10^{-4})	(3.75×10^{-4})	(3.35×10^{-4})
	−0.5%	−0.81%	−0.90%	−1.22%	−1.28%	−1.25%
	−1%	−1.95%	−2.0%	−1.77%	−1.55%	−1.63%

Table 6 Sensitivity testing for $\hat{\psi}_c(x; n_0)$ on a, λ, ρ in case of Weibull-distributed claims

Model parameters		$\hat{\psi}_c(x; n_0)$				
		$x = 231$	$x = 1558$	$x = 2278$	$x = 2673$	$x = 3000$
% change in a	+1%	-13.85%	-26.54%	−33.13%	−34.80%	−36.21%
	+0.5%	−8.46%	−18.29%	-16.03%	−19.11%	−19.74%
	($a = 0.335$)	(1.62×10^{-2})	(1.73×10^{-4})	(4.84×10^{-5})	(2.32×10^{-5})	(1.40×10^{-5})
	−0.5%	+8.15%	+18.59%	+21.80%	+23.84%	+24.81%
	−1%	+16.92%	+39.74%	+48.11%	+52.43%	+55.47%
% change in λ	+1%	+0.77%	+1.28%	+1.29%	+1.51%	+1.46%
	+0.5%	+0.23%	+0.45%	+0.64%	+0.89%	+0.73%
	($\lambda = 6$)	(1.62×10^{-2})	(1.73×10^{-4})	(4.84×10^{-5})	(2.32×10^{-5})	(1.40×10^{-5})
	−0.5%	−1.00%	−0.64%	−0.64%	−0.44%	−0.74%
	−1%	−1.54%	−1.28%	−1.29%	−1.11%	−1.46%
% change in ρ	+1%	+0.90%	+1.09%	+1.29%	+1.31%	+1.27%
	+0.5%	+0.23%	+0.45%	+0.64%	+0.65%	+0.66%
	($\rho = 0.2$)	(1.62×10^{-2})	(1.73×10^{-4})	(4.84×10^{-5})	(2.32×10^{-5})	(1.40×10^{-5})
	−0.5%	−1.0%	−0.83%	−0.64%	−0.45%	−0.69%
	−1%	−1.62%	−1.41%	−1.29%	−1.18%	−1.28%

5 Concluding Remarks

In this paper, we establish a second order asymptotic formula for the infinite-time ruin probability in a compound renewal risk model, which is relevant and important in the insurance industry due to the recent frequent occurrence of catastrophes and severe accidents/crisises, especially during the post economic crisis era. Our main contribution is the establishment of a second order expansion for the infinite-time ruin probability, which is more precise than that of the traditional first order asymptotics. Another is that the individual claims in our model are second order subexponential, and so may be heavily heavy-tailed, containing such as the Pareto distributions, or may be moderately heavy-tailed, containing the heavy-tailed Weibull distributions and the lognormal distributions, etc. Some simulation studies are performed to check the accuracy of our obtained theoretical result, illustrating the improvement of our asymptotic formula compared with that on the first order one. Confined to the compound renewal risk model, the individual claims, the accident inter-arrival times, and the claim number caused by one accident play different roles in causing ruin of an insurer. By virtue of the sensitivity study, we quantitatively analyze and distinguish the different roles of the model parameters. Such a quantitative sensitivity analysis is fascinating and may provide regulators or insurers with some ideas on how to determine the proper effective factors in insurance risk management.

Several extensions of our work are worthy of pursuit in the future. First, our work considers a classical risk model, in which the effect of interest rates is excluded. In doing so, we have no way of knowing the true value of an insurer's assets in the future, while interest rates play an important role in estimating the discounted value of the future assets. Thus, it will be meaningful to extend the second order asymptotic study to such a situation. Second, the assumption of independence among the individual claims caused by one accident is insufficient when considering a compound renewal risk model. Actually, as complements, some dependence structures should be addressed in future studies. Finally, the assumption of the accident renewal counting process is inaccurate in some cases, which means that the accident inter-arrival times are independent of each other. For example, during the time of economic depression, there is a relatively strong relationship among the crisis/accident inter-arrival times. Introducing a quasi-renewal counting process into the model is significant.

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