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ORIGINAL PAPER

Asymptotic finite-time ruin probabilities in a dependent bidimensional renewal risk model with subexponential claims

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Abstract

This paper considers a bidimensional continuous-time renewal risk model, in which the two components of each pair of claim sizes are linked via the strongly asymptotic independence structure and the two claim-number processes from different lines of business are (almost) arbitrarily dependent. Precise asymptotic formulas for three kinds of finite-time ruin probabilities are established when the claim sizes have heavy-tailed tails.

Keywords Bidimensional renewal risk model · Ruin probability · Subexponential distribution · Strongly asymptotic independence

Mathematics Subject Classification Primary 62P05 · Secondary 62E10

1 Introduction

Consider a bidimensional continuous-time risk model, in which the discounted value of the surplus process is described as

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$$\begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \int_{0-}^{t} e^{-rs} C_1(ds) \\ \int_{0-}^{t} e^{-rs} C_2(ds) \end{pmatrix} - \begin{pmatrix} \int_{0-}^{t} e^{-rs} D_1(ds) \\ \int_{0-}^{t} e^{-rs} D_2(ds) \end{pmatrix} - \begin{pmatrix} \sum_{k=1}^{N_1(t)} X_k e^{-r\tau_k^{(1)}} \\ \sum_{k=1}^{N_2(t)} Y_k e^{-r\tau_k^{(2)}} \end{pmatrix}, \ t \geq 0, \tag{1.1}$$

where (x, y) denotes the vector of the initial surpluses; $r \geq 0$ is the constant interest rate; $\{(C_1(t), C_2(t)); t \geq 0\}$ represents the premium accumulation process with the nondecreasing and right-continuous paths satisfying $(C_1(0), C_2(0)) = (0, 0)$; $\{(X_k, Y_k); k \geq 1\}$ constitutes a sequence of independent and identically distributed (i.i.d.) claim-size vectors with generic vector (X, Y) following the marginal distributions F_1 and F_2 both on $[0, \infty)$, respectively; $\{(\tau_k^{(1)}, \tau_k^{(2)}); k \geq 1\}$ is a sequence of claim-arrival time vectors with $\tau_k^{(i)} = \sum_{l=1}^{k} \theta_l^{(i)}, i = 1, 2,$ which constitutes the renewal claim-number vector process $\{(N_1(t), N_2(t)); t \geq 0\}$; and $\{(D_1(t), D_2(t)); t \geq 0\}$ is a real-valued stochastic vector process reflecting the additional net losses of the two lines of business (equal to the total additional expenses minus income not including claims and premiums) satisfying $(D_1(0), D_2(0)) = (0, 0)$. Assume that $\{(X_k, Y_k); k \geq 1\}$, $\{(C_1(t), C_2(t)); t \geq 0\}$, $\{D_1(t); t \geq 0\}$, $\{D_2(t); t \geq 0\}$, $(\theta_1^{(1)}, \theta_1^{(2)})$ and $\{(\theta_k^{(1)}, \theta_k^{(2)}); k \geq 2\}$ are mutually independent; in addition, $\theta_1^{(1)}$ and $\theta_1^{(2)}$ are independent of each other. Denote the two finite renewal functions by $\lambda_i(t) = E[N_i(t)] = \sum_{k=1}^{\infty} P(\tau_k^{(i)} \leq t), \ i = 1, 2.$

In the above continuous-time bidimensional risk model (1.1), for any $T \geq 0$, define the three kinds of finite-time ruin probabilities by

$$\begin{aligned} \psi_{\text{sim}}(x, y) &= P\left(\inf_{0 \leq t \leq T} \{R_1(t) \vee R_2(t)\} < 0 \,\middle|\, (R_1(0), R_2(0)) = (x, y) \right) \\ &= P\left(R_i(t) < 0, i = 1, 2, \ \text{ for some } 0 \leq t \leq T \,\middle|\, (R_1(0), R_2(0)) = (x, y) \right), \\ \psi_{\text{and}}(x, y) &= P\left(\inf_{0 \leq t \leq T} R_1(t) < 0 \ \text{ and } \inf_{0 \leq t \leq T} R_2(t) < 0 \,\middle|\, (R_1(0), R_2(0)) = (x, y) \right), \\ \psi_{\text{or}}(x, y) &= P\left(\inf_{0 \leq t \leq T} R_1(t) < 0 \ \text{ or } \inf_{0 \leq t \leq T} R_2(t) < 0 \,\middle|\, (R_1(0), R_2(0)) = (x, y) \right), \end{aligned}$$

where $R_1(t) \vee R_2(t) = \max\{R_1(t), R_2(t)\}$. We remark that in such three definitions, $\psi_{\text{sim}}(x, y)$ represents the probability that ruin occurs in all business lines simultaneously over the time horizon $[0, T]$; $\psi_{\text{and}}(x, y)$ is the one that ruin occurs in all business lines but not necessarily at the same time; while $\psi_{\text{or}}(x, y)$ reflects that ruin occurs in at least one business line. Clearly, for any $T \geq 0$,

$$\psi_{\text{sim}}(x, y) \leq \psi_{\text{and}}(x, y) \leq \psi_{\text{or}}(x, y). \tag{1.2}$$

Ruin probabilities have been extensively studied in the past two decades, since it plays an important role in helping insurers to assess the solvency ability more accurately. In the case of unidimensional risk models, Tang [21], Leipus and Šiaulys [12, 13], Yang et al. [28] and Wang et al. [25] investigated the finite-time ruin probability in some independent or dependent risk models with no interest rate, whereas [24] considered the model with constant interest rate, and Li et al. [14, 18], Yang et al. [29], Wang et al. [23], Cheng and Cheng [6], Yang et al. [33] and Tang and Yang [22] further studied the models with stochastic return. In recent years, more attention has been paid to multi-dimensional risk models, especially bidimensional ones, where insurers operate two kinds of business at the same time. See, for instance, Li et al. [17], Zhang and Wang [35], Yang and Li [26], Jiang et al. [9], Li and Yang [19], Yang and Yuen [32], Yang and Li [27], Cheng and Yu [4], Li [16], Chen and Yang [1], among many others.

In most of the existing literature, the additional net loss processes $D_i(t), i = 1, 2$, are not considered, and a typical assumption is that the two lines of business share a common claim-number process, namely, $N_1(t) = N_2(t) = N(t)$ for all $t \geq 0$. In this case, Yang and Li [26] established a precise asymptotic result for the finite-time ruin probability $\psi_{\text{sim}}(x, y)$ under the assumption that (X, Y) is linked via the Farlie–Gumbel–Morgenstern (FGM) copula and has subexponential marginal distributions. By definition, a distribution V on $[0, \infty)$ is said to be subexponential, written as $V \in \mathcal{S}$, if $\overline{V}(x) = 1 - V(x) > 0$ for all $x \geq 0$ and

$$\overline{V^{*n}}(x) \sim n\overline{V}(x) \qquad \text{as } x \to \infty,$$

where V^{*n} is the n-fold convolution of V and the symbol $\sim$ means that the quotient of the both sides tends to 1. If $V \in \mathcal{S}$, then for any $\varepsilon > 0$,

$$e^{-\varepsilon x} = o(\overline{V}(x)) \qquad \text{as } x \to \infty, \tag{1.3}$$

see, e.g., Lemma 1.3.5 (b) of Embrechts et al. [8]. Later, Yang and Yuen [32] generalized Yang and Li's result by extending the dependence structure between X and Y from the FGM distribution to the Sarmanov distribution (see the definition in Remark 1.1 below). Recently, Li [16] further considered a more general dependence structure between X and Y, namely, the strongly asymptotic independence.

Assumption A *Two random variables X and Y are said to be strongly asymptotically independent (SAI), if there exists some $\rho > 0$ such that*

$$P(X > x, Y > y) \sim \rho\overline{F_1}(x)\overline{F_2}(y) \quad \text{as } (x, y) \to (\infty, \infty). \tag{1.4}$$

Remark 1.1 Assumption A is a mild condition. A random vector (X, Y) satisfies (1.4), if it follows a bivariate Sarmanov distribution of the form

$$P(X \in dx, Y \in dy) = \big(1 + \omega\phi_1(x)\phi_2(y)\big)P(X \in dx)P(Y \in dy),$$

satisfying $\omega \lim_{x\to\infty} \phi_1(x) \lim_{y\to\infty} \phi_2(y) > -1$, where ϕ_1, ϕ_2 are two kernels and ω is a real parameter such that $E[\phi_1(X)] = E[\phi_2(Y)] = 0$ and $1 + \omega\phi_1(x)\phi_2(y) \geq 0$, $(x, y) \in \mathbb{R}^2$. For more discussions on Sarmanov distributions, one can be referred to Kotz et al. [11] and Yang and Wang [31].

In terms of copulas, Assumption A can be rewritten as follows. Let $C(\cdot, \cdot)$ be the copula of (X, Y). There is some $\rho > 0$ such that

$$\hat{C}(u,v) \sim \rho uv, \quad \text{as } (u,v) \to (0+,0+), \tag{1.5}$$

where $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ is the survival copula of $C(\cdot,\cdot)$. As pointed out by Li [16], many commonly-used bivariate copulas satisfy (1.5) and so Assumption A, such as the Johnson-Kotz iterated FGM copula (see [10])

$$C(u,v) = uv + (\kappa + \gamma uv)uv(1-u)(1-v) \tag{1.6}$$

with $\rho = 1 + \kappa + \gamma$ for some parameters $\kappa \in (-1,1]$ and $\gamma \in (-1-\kappa, \frac{1}{2}(3-\kappa+\sqrt{9-6\kappa-3\kappa^2}))$. In particular, if $\gamma = 0$, then (1.6) reduces to the classical FGM copula with $\kappa \in (-1,1]$, for which case $\rho = 1+\kappa$.

Remark 1.2 If a random vector (X, Y) is two-dimensional normally distributed with the joint probability density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left\{ \frac{1}{2(1-\gamma^2)}(x^2 - 2\gamma xy + y^2) \right\}$$

for some parameter $\gamma \neq 0$, then Assumption A is not satisfied for $(|X|, |Y|)$.

Under the SAI dependence structure, Li [16] obtained the asymptotic behavior for the finite-time ruin probability $\psi_{\text{sim}}(x,y)$. However, as pointed out by Yang and Li [27], the assumption that the two lines of the business share a common claim-number process is far away from practical situations. For instance, a car accident may cause one claim for vehicle damage immediately and more than one medical claims for injures of both the drivers and passengers in the subsequent periods. Then the claim-number processes of the car insurance business and the medical insurance business are neither independent nor the same. Yang and Li [27] considered the case that the two different claim-number processes are arbitrarily dependent, but the claims from different lines of business are independent and the two premium processes are both deterministic. Related discussions can be found in Chen et al. [1, 3, 5] and Yang et al. [30] and so on.

Theorem 1.A (Theorem 2.1 of Yang and Li [27]) *Consider the bidimensional risk model* (1.1) *with* $F_1 \in \mathcal{S}, F_2 \in \mathcal{S}, (C_1(t), C_2(t)) = (c_1 t, c_2 t)$ *and* $(D_1(t), D_2(t)) \equiv (0,0)$ *for some* $c_1 > 0, c_2 > 0$ *and all* $t \geq 0$. *Assume that* $\{X_k; k \geq 1\}$, $\{Y_k; k \geq 1\}$ *and* $\{(N_1(t), N_2(t)); t \geq 0\}$ *are mutually independent. Then, regardless of arbitrary dependence between* $N_1(t)$ *and* $N_2(t)$, *for any* $T > 0$ *with* $E[N_1(T)N_2(T)] > 0$,

$$\psi_{\text{sim}}(x,y) \sim \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})dE[N_1(s)N_2(t)] \quad \text{as } (x,y) \to (\infty,\infty). \tag{1.7}$$

Inspired by all the above-mentioned work, this paper aims to study asymptotic behavior of the finite-time ruin probabilities $\psi_{\text{sim}}(x,y)$, $\psi_{\min}(x,y)$ and $\psi_{\text{and}}(x,y)$ in a more general bidimensional risk model (1.1), in which the two different claim-number processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are (almost) arbitrarily dependent;

the generic claim-size vector (X, Y) satisfies the SAI dependence structure (1.4); the premium processes $\{C_1(t);t \geq 0\}$ and $\{C_2(t);t \geq 0\}$ are also arbitrarily dependent; but $\{(X_k, Y_k);k \geq 1\}$, $\{(C_1(t), C_2(t));t \geq 0\}$, $\{D_1(t);t \geq 0\}$, $\{D_2(t);t \geq 0\}$, $(\theta_1^{(1)}, \theta_1^{(2)})$ and $\{(\theta_k^{(1)}, \theta_k^{(2)});k \geq 2\}$ are mutually independent; and $\theta_1^{(1)}$ and $\theta_1^{(2)}$ are independent of each other.

As suggested by one of the referees, in the future work we shall consider a more complicated case that the claim vectors $\{(X_k, Y_k);k \geq 1\}$ are not necessarily independent. The m-dependence and the absolutely regular mixing condition may be the alternatives for modelling $\{(X_k, Y_k);k \geq 1\}$, see [34].

The rest of the paper is organized as follows. Section 2 presents the main results of the paper, and Sect. 3 prepares some lemmas before the proofs of the main results given in Sect. 4.

2 Main results

In the sequel, we assume that all limit relations hold as $x \wedge y = \min\{x, y\} \to \infty$ unless otherwise stated. For two positive bivariate functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$, we write $f \sim g$ if $\lim f/g = 1$, write $f \lesssim g$ if $\limsup f/g \leq 1$, write $f = O(g)$ if $\limsup f/g < \infty$, and write $f = o(g)$ if $\lim f/g = 0$.

Now we state the main results of this paper. The first result investigates asymptotics for $\psi_{\text{sim}}(x, y)$ and $\psi_{\text{and}}(x, y)$ in the case of subexponential claims.

Theorem 2.1 *Consider the bidimensional risk model (1.1) with $F_1 \in \mathcal{S}$ and $F_2 \in \mathcal{S}$. Assume that (1.4) holds for some $\rho > 0$ and $E[N_1(T)N_2(T)] > 0$ for some $T > 0$. If*

$$\begin{cases} P\left(\sup\limits_{0\leq t\leq T} \int_{0-}^{t} e^{-rs} D_i(ds) > xe^{-rT}\right) = o(\overline{F_i}(x)) \\ P\left(\inf\limits_{0\leq t\leq T} \int_{0-}^{t} e^{-rs} D_i(ds) > xe^{-rT}\right) = o(\overline{F_i}(x)) \end{cases}, \quad i = 1, 2, \tag{2.1}$$

then it holds that

$$\begin{aligned} \psi_{\text{sim}}(x, y) \sim \psi_{\text{and}}(x, y) \sim & \int_{0-}^{T} \int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})\hat{\lambda}(ds, dt) \\ & + (\rho - 1) \int_{0-}^{T} \int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})\tilde{\lambda}(ds, dt), \end{aligned} \tag{2.2}$$

where $\hat{\lambda}(s, t) = E[N_1(s)N_2(t)] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(\tau_i^{(1)} \leq s, \tau_j^{(2)} \leq t)$ *and* $\tilde{\lambda}(s, t) = \sum_{i=1}^{\infty} P(\tau_i^{(1)} \leq s, \tau_i^{(2)} \leq t)$.

Remark 2.1 If $(D_1(t), D_2(t)) \equiv (0, 0)$ in the risk model (1.1), then Theorem 2.1 holds for arbitrarily dependent renewal claim-number processes $\{N_1(t);t \geq 0\}$ and $\{N_2(t);t \geq 0\}$, which is ensured by Remark 3.1 and the proof of Theorem 2.1 in Sects. 3 and 4.

Remark 2.2 Clearly, if X and Y are independent, then $\rho = 1$ in (1.4), and thus Theorem 1.A can be retrieved from Remark 2.1.

Remark 2.3 Condition (2.1) is satisfied due to (1.3) and Lemma 2.7 of Wang et al. [23], if $D_i(t) = \delta_i B_i(t), i = 1, 2$, where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions and $\delta_i \geq 0, i = 1, 2$, are two constant volatility factors.

An immediate corollary can be obtained from Theorem 2.1 and Remark 2.2.

Corollary 2.1 *Under the conditions of Theorem 2.1, it holds that for each* $i = 1, 2$,

$$P\left(\inf_{0\leq t\leq T} R_i(t) < 0 \middle| R_i(0) = x\right) \sim \int_{0-}^{T} \overline{F_i}(xe^{rs})\lambda_i(ds), \quad \text{as} \quad x \to \infty.$$

The second result establishes an asymptotic formula for $\psi_{\text{or}}(x, y)$ in the case of long-tailed and dominatedly-varying-tailed claims. It is well-known that if $V \in \mathcal{S}$, then $V \in \mathcal{L}$, which stands for the class of long-tailed distributions characterized by $\overline{V}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{V}(x-1)}{\overline{V}(x)} = 1.$$

Related is the class $\mathcal{D}$ of dominatedly-varying-tailed distributions, defined by

$$\limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty$$

for any $0 < y < 1$. Note that $\mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$, see, e.g., Cline and Samorodnitsky [7], Embrechts et al. [8], and the references therein.

Theorem 2.2 *Under the conditions of Theorem 2.1, if further* $F_1 \in \mathcal{L} \cap \mathcal{D}$ *and* $F_2 \in \mathcal{L} \cap \mathcal{D}$, *then it holds that*

$$\psi_{\text{or}}(x, y) \sim \int_{0-}^{T} \overline{F_1}(xe^{rs})\lambda_1(ds) + \int_{0-}^{T} \overline{F_2}(ye^{rt})\lambda_2(dt).$$

We end this section by specializing the formulas in Theorems 2.1 and 2.2 to much more transparent forms. One of the most important subclasses of $\mathcal{L} \cap \mathcal{D}$ (hence, of $\mathcal{S}$) is the class of regularly-varying-tailed distributions. By definition, a distribution V is said to be regularly-varying-tailed, denoted by $V \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, if $\overline{V}(x) > 0$ for all $x \geq 0$ and for any fixed $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = y^{-\alpha}. \tag{2.3}$$

By the uniformity of relation (2.3) on each $[a, b]$ for any $0 < a \le b < \infty$, we have the following corollaries for regularly-varying-tailed F_1 and F_2 immediately.

Corollary 2.2 *Under the conditions of Theorem 2.1, assume further that* $F_1 \in \mathcal{R}_{-\alpha_1}$ *and* $F_2 \in \mathcal{R}_{-\alpha_2}$ *for some* $\alpha_1 \ge 0$ *and* $\alpha_2 \ge 0$.

(1) *If* $N_1(t) = N_2(t) = N(t)$ *for all* $t \ge 0$ *with finite* $\lambda(t) = E[N(t)]$, *then*

$$\psi_{\mathrm{sim}}(x, y) \sim \psi_{\mathrm{and}}(x, y) \sim \Bigg(\int_{0-}^{T}\int_{0-}^{T-s} e^{-r(\alpha_1+\alpha_2)s}(e^{-r\alpha_1 t} + e^{-r\alpha_2 t})\lambda(dt)\lambda(ds) + \rho\int_{0-}^{T} e^{-r(\alpha_1+\alpha_2)s}\lambda(ds)\Bigg)\overline{F_1}(x)\overline{F_2}(y).$$

(2) *If* $\{N_1(t);t \ge 0\}$ *and* $\{N_2(t);t \ge 0\}$ *are mutually independent, then*

$$\psi_{\mathrm{sim}}(x, y) \sim \psi_{\mathrm{and}}(x, y) \sim \Bigg(\int_{0-}^{T} e^{-\alpha_1 rt}\lambda_1(dt)\cdot\int_{0-}^{T} e^{-\alpha_2 rt}\lambda_2(dt) + (\rho - 1)\sum_{i=1}^{\infty}\int_{0-}^{T} e^{-r\alpha_1 t}P(\tau_i^{(1)} \in dt)\cdot\int_{0-}^{T} e^{-r\alpha_2 t}P(\tau_i^{(2)} \in dt)\Bigg)\overline{F_1}(x)\overline{F_2}(y),$$

where 1_A *is the indicator function of an event* A.

Corollary 2.3 *Under the conditions of Theorem 2.1, if further* $F_1 \in \mathcal{R}_{-\alpha_1}$ *and* $F_2 \in \mathcal{R}_{-\alpha_2}$ *for some* $\alpha_1 \ge 0$ *and* $\alpha_2 \ge 0$, *then it holds that*

$$\psi_{\mathrm{or}}(x, y) \sim \overline{F_1}(x)\int_{0-}^{T} e^{-r\alpha_1 t}\lambda_1(dt) + \overline{F_2}(y)\int_{0-}^{T} e^{-r\alpha_2 t}\lambda_2(dt).$$

Below are two examples based on Corollaries 2.2 and 2.3.

Example 2.1 Under the conditions of Theorem 2.1, assume further that $F_1 \in \mathcal{R}_{-\alpha_1}$ and $F_2 \in \mathcal{R}_{-\alpha_2}$ for some $\alpha_1 > 0$ and $\alpha_2 > 0$. If $\{N_1(t);t \ge 0\}$ and $\{N_2(t);t \ge 0\}$ are two homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then

$$\psi_{\mathrm{or}}(x, y) \sim \frac{\lambda_1(1 - e^{-r\alpha_1 T})}{r\alpha_1}\overline{F_1}(x) + \frac{\lambda_1(1 - e^{-r\alpha_2 T})}{r\alpha_2}\overline{F_2}(y).$$

If further $N_1(t) = N_2(t) = N(t)$ with the same intensity $\lambda > 0$, then

$$\psi_{\mathrm{sim}}(x, y) \sim \psi_{\mathrm{and}}(x, y) \sim \left(\frac{\lambda^2(1 + e^{-r(\alpha_1+\alpha_2)T} - e^{-r\alpha_1 T} - e^{-r\alpha_2 T})}{r^2\alpha_1\alpha_2} + \frac{\rho\lambda(1 - e^{-r(\alpha_1+\alpha_2)T})}{r(\alpha_1 + \alpha_2)}\right)\overline{F_1}(x)\overline{F_2}(y).$$

3 Some lemmas

In this section, we prepare some lemmas. Hereafter, ξ and η are assumed to be two independent real-valued random variables with distributions F_ξ and F_η, respectively, and they are assumed to be independent of all other sources of randomness.

We begin with two existing results for subexponential distributions, which can be found in Lemma 3.2 of Li [15] and Lemma A.1 of Yang and Li [26].

Lemma 3.1 *Let $\{Z_k;k \geq 1\}$ be a sequence of independent real-valued random variables with distributions $V_k, k \geq 1$, respectively. Assume that there exists a distribution $V \in \mathcal{S}$ such that $\overline{V_k}(x) \sim l_k \overline{V}(x)$ for some $l_k > 0, k \geq 1$. If $\overline{F_\xi}(x) = o(\overline{V}(\frac{x}{a}))$ for some $a > 0$, then for each $n \geq 1$ and any fixed $b \geq a$, it holds uniformly for all $\overline{c_n} := (c_1, \cdots, c_n) \in [a,b]^n$ that*

$$P\left(\sum_{k=1}^{n} c_k Z_k + \xi > x\right) \sim \sum_{k=1}^{n} \overline{V_k}\left(\frac{x}{c_k}\right),$$

where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{\overline{c_n}\in[a,b]^n} \left| \frac{P\left(\sum_{k=1}^{n} c_k Z_k + \xi > x\right)}{\sum_{k=1}^{n} \overline{V_k}\left(\frac{x}{c_k}\right)} - 1 \right| = 0.$$

Lemma 3.2 *If $V \in \mathcal{L}$, then there exists a function $h(\cdot) : (0,\infty) \mapsto (0,\infty)$ such that $h(x) = o(x)$, $h(x) \uparrow \infty$, $h(zx) \sim h(x)$ for any fixed $z > 0$, and*

$$\overline{V}(c_1 x \pm c_2 h(x)) \sim \overline{V}(c_1 x) \tag{3.1}$$

holds uniformly for all $(c_1, c_2) \in [a,b]^2$, $0 < a \leq b < \infty$.

The third lemma gives a bidimensional version of the Kesten-type bound for random vectors satisfying a dependence structure weaker than (1.4).

Lemma 3.3 *Let $\{(X,Y),(X_k,Y_k);k \geq 1\}$ be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions $F_1 \in \mathcal{S}$ and $F_2 \in \mathcal{S}$, respectively. If*

$$P(X > x, Y > y) = O(1)\overline{F_1}(x)\overline{F_2}(y), \tag{3.2}$$

and $\overline{F_\xi}(x) = O(\overline{F_1}(x))$, $\overline{F_\eta}(y) = O(\overline{F_2}(y))$, then for any $\varepsilon > 0$, there is a positive constant C such that for all $x \geq 0, y \geq 0$ and $n \geq 1, m \geq 1$,

$$P\left(\sum_{i=1}^{n} X_i + \xi > x, \sum_{j=1}^{m} Y_j + \eta > y\right) \leq C(1+\varepsilon)^{n+m}\overline{F_1}(x)\overline{F_2}(y).$$

Proof By (3.2), there exist two large numbers $x_0 > 0$ and $C_1 > 0$ such that

$$\frac{P(X > x, Y > y)}{\overline{F_1}(x)\overline{F_2}(y)} \leq C_1$$

holds for all $x \geq x_0$ and $y \geq x_0$. If $0 \leq x < x_0$ or $0 \leq y < x_0$, then we have, respectively,

$$\frac{P(X > x, Y > y)}{\overline{F_1}(x)\overline{F_2}(y)} \leq \frac{1}{\overline{F_1}(x_0)} \quad \text{or} \quad \frac{P(X > x, Y > y)}{\overline{F_1}(x)\overline{F_2}(y)} \leq \frac{1}{\overline{F_2}(x_0)}.$$

Take $C_2 = \max\left\{C_1, \frac{1}{\overline{F_1}(x_0)}, \frac{1}{\overline{F_2}(x_0)}\right\}$, then it holds that for all $x \geq 0$ and $y \geq 0$,

$$P(X > x, Y > y) \leq C_2\overline{F_1}(x)\overline{F_2}(y).$$

Construct two independent nonnegative random variables X' and Y' with survival functions

$$\overline{F_1'}(x) = 1 \wedge \sqrt{C_2\overline{F_1}(x)} \quad \text{and} \quad \overline{F_2'}(y) = 1 \wedge \sqrt{C_2\overline{F_2}(y)}.$$

Clearly,

$$\overline{F_1'}(x) \sim \sqrt{C_2\overline{F_1}(x)} \quad \text{and} \quad \overline{F_2'}(y) \sim \sqrt{C_2\overline{F_2}(y)}, \tag{3.3}$$

and for all $x \geq 0$ and $y \geq 0$,

$$P(X > x, Y > y) \leq P(X' > x, Y' > y) = (1 \wedge \sqrt{C_2\overline{F_1}(x)})(1 \wedge \sqrt{C_2\overline{F_2}(y)}). \tag{3.4}$$

It is easy to check that $F_1' \in \mathcal{S}$ and $F_2' \in \mathcal{S}$ by Lemma A 3.15 of Embrechts et al. [8], and (X, Y) is stochastically dominated by (X', Y') by (3.4). We denote the latter relation by $(X, Y) \leq_{st} (X', Y')$, which further implies $X \leq_{st} X'$ and $Y \leq_{st} Y'$. Let $\{(X', Y'), (X_k', Y_k'); k \geq 1\}$ be a sequence of i.i.d. random vectors with independent components, which are also independent of all other sources of randomness. By $F_1' \in \mathcal{S}, F_2' \in \mathcal{S}$, (3.3) and Lemma 2.6 of Wang et al. [23], for any $\varepsilon > 0$, there exists some large $C > 0$ such that for all $x \geq 0, y \geq 0$ and $n \geq 1, m \geq 1$,

$$P\left(\sum_{i=1}^{n} X_i' + \xi > x\right) \leq \sqrt{C}(1+\varepsilon)^n\overline{F_1}(x),$$
$$P\left(\sum_{j=1}^{m} Y_j' + \eta > y\right) \leq \sqrt{C}(1+\varepsilon)^m\overline{F_2}(y).$$

By using $(X, Y) \leq_{st} (X', Y')$ and the above two inequalities, we have that for all $x \geq 0, y \geq 0$ and $n \geq 1, m \geq 1$,

$$\begin{aligned} &P\left(\sum_{i=1}^{n} X_i + \xi > x, \sum_{j=1}^{m} Y_j + \eta > y\right) \\ &\leq P\left(\sum_{i=1}^{n} X_i' + \xi > x\right)P\left(\sum_{j=1}^{m} Y_j' + \eta > y\right) \\ &\leq C(1+\varepsilon)^{n+m}\overline{F_1}(x)\overline{F_2}(y), \end{aligned}$$

as claimed. □

The next lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.4 *Let* $\{(X,Y),(X_k,Y_k);k\ge 1\}$ *be a sequence of i.i.d. nonnegative random vectors with generic marginal distributions* $F_1\in\mathcal{S}$ *and* $F_2\in\mathcal{S}$, *respectively. Assume that (1.4) holds for some* $\rho>0$. *If for any fixed* $0<a\le b<\infty$,

$$\overline{F_\xi}(x)=o(1)\overline{F_1}\left(\frac{x}{a}\right)\quad\text{and}\quad\overline{F_\eta}(y)=o(1)\overline{F_2}\left(\frac{y}{a}\right),\tag{3.5}$$

then for any fixed $n\ge 1$ *and* $m\ge 1$, *it holds uniformly for all* $\overline{c_n}\in[a,b]^n$ *and* $\overline{d_m}:=(d_1,\cdots,d_m)\in[a,b]^m$ *that*

$$\begin{aligned}&P\left(\sum_{i=1}^n c_iX_i+\xi>x,\sum_{j=1}^m d_jY_j+\eta>y\right)\\&\sim\sum_{i=1}^n\sum_{1\le j\ne i\le m}\overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_j}\right)+\rho\sum_{i=1}^{m\wedge n}\overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_i}\right)\end{aligned}\tag{3.6}$$

$$\sim\sum_{i=1}^n\sum_{j=1}^m P\left(X_i>\frac{x}{c_i},Y_j>\frac{y}{d_j}\right).\tag{3.7}$$

Proof Since relation (3.7) is obvious by considering the SAI dependence structure (1.4), we focus on the proof of (3.6). In the following, we shall divide the procedure into three steps.

Step 1 In this step, we aim to prove (3.6) in the case of $n=m=1$, namely

$$P(cX+\xi>x,dY+\eta>y)\sim\rho\overline{F_1}\left(\frac{x}{c}\right)\overline{F_2}\left(\frac{y}{d}\right)\tag{3.8}$$

holds uniformly for all $(c,d)\in[a,b]^2$.

We first deal with the upper bound of (3.8). Let $h_1(\cdot)$ and $h_2(\cdot)$ be the two functions specified in Lemma 3.2 for F_1 and F_2, respectively. Take $h(\cdot)=h_1(\cdot)\wedge h_2(\cdot)$, then (3.1) holds with V replaced by F_1 or F_2. According to the value of (ξ,η) belonging to $(-\infty,h(x)]\times(-\infty,h(y)]$, $(h(x),\infty)\times\mathbb{R}$ and $\mathbb{R}\times(h(y),\infty)$, we split the probability on the left-hand side of (3.8) into three parts, denoted by I_k, $k=1,2,3$, respectively. Clearly,

$$P(cX+\xi>x,dY+\eta>y)\le\sum_{k=1}^3 I_k(x,y;c,d).\tag{3.9}$$

It is important to note that every asymptotic relationship appearing in the proof holds uniformly for the coefficients of the random variables involved. Taking the asymptotic relationship in this step as an example, the uniformity is for all $(c,d)\in[a,b]^2$. By (1.4) and Lemma 3.2 of this paper,

$$
\begin{aligned}
I_1(x, y; c, d) &\le P(cX > x - h(x), dY > y - h(y)) \\
&\sim \rho \overline{F_1}\left(\frac{x}{c}\right)\overline{F_2}\left(\frac{y}{d}\right).
\end{aligned}
\tag{3.10}
$$

Recall the dominating random vector (X', Y') constructed in the proof of Lemma 3.3. By $(X, Y) \le_{st} (X', Y')$,

$$
I_2(x, y; c, d) \le P\big(cX' + \xi > x, \xi > h(x)\big) P(dY' + \eta > y).
$$

By (3.5), (3.3) and Lemmas 3.1 and 3.2, we have

$$
\begin{aligned}
&P\big(cX' + \xi > x, \xi > h(x)\big) \\
&\quad \le P\big(cX' + \xi > x\big) - P(cX' + \xi > x, -h(x) < \xi \le h(x)) \\
&\quad \le P\big(cX' + \xi > x\big) - P(cX' > x + h(x)) P(-h(x) < \xi \le h(x)) \\
&\quad = o(1)\overline{F_1}\left(\frac{x}{c}\right),
\end{aligned}
$$

and by Lemma 3.1 and (3.3),

$$
P\big(dY' + \eta > y\big) \sim \sqrt{C_2 \overline{F_2}}\left(\frac{y}{d}\right).
$$

A combination of the above three estimates implies

$$
I_2(x, y; c, d) = o(1)\overline{F_1}\left(\frac{x}{c}\right)\overline{F_2}\left(\frac{y}{d}\right).
\tag{3.11}
$$

Symmetrically,

$$
I_3(x, y; c, d) = o(1)\overline{F_1}\left(\frac{x}{c}\right)\overline{F_2}\left(\frac{y}{d}\right).
\tag{3.12}
$$

Plugging (3.10)–(3.12) into (3.9) gives the upper bound of (3.8).

As for the lower bound of (3.8), we obtain

$$
\begin{aligned}
&P(cX + \xi > x, dY + \eta > y) \\
&\quad \ge P(cX + \xi > x, -h(x) < \xi \le h(x), dY + \eta > y, -h(y) < \eta \le h(y)) \\
&\quad \ge P(cX > x + h(x), dY > y + h(y)) P(-h(x) < \xi \le h(x)) P(-h(y) < \eta \le h(y)) \\
&\quad \sim \rho \overline{F_1}\left(\frac{x}{c}\right)\overline{F_2}\left(\frac{y}{d}\right),
\end{aligned}
$$

where in the last step we used the same idea as in the derivation of (3.10).

Step 2 In this step, we consider (3.6) in the case of $n = m > 1$ by the induction method. Assuming by induction that (3.6) holds for $n = m = l$, we are to prove uniformly for all $\overline{c_{l+1}} \in [a, b]^{l+1}$ and $\overline{d_{l+1}} \in [a, b]^{l+1}$,

$$
\begin{aligned}
&P\left(\sum_{i=1}^{l+1} c_i X_i + \xi > x, \sum_{j=1}^{l+1} d_j Y_j + \eta > y\right) \\
&\quad \sim \sum_{i=1}^{l+1} \sum_{1\le j\ne i\le l+1} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_j}\right) + \rho \sum_{i=1}^{l+1} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_i}\right).
\end{aligned} \tag{3.13}
$$

According to the value of $(c_{l+1}X_{l+1}, d_{l+1}Y_{l+1})$ belonging to $[0, h(x)] \times [0, h(y)]$, $[0, h(x)] \times (y - h(y), \infty)$, $(x - h(x), \infty) \times [0, h(y)]$, $(x - h(x), \infty) \times (y - h(y), \infty)$, $(h(x), x - h(x)] \times [0, \infty)$ and $[0, \infty) \times (h(y), y - h(y)]$, we split the probability on the left-hand side of relation (3.13) into six parts and denote the obtained probabilities by $L_k(x, y; \overline{c_{l+1}}, \overline{d_{l+1}})$, $1 \le k \le 6$, respectively. It is clear that

$$
P\left(\sum_{i=1}^{l+1} c_i X_i + \xi > x, \sum_{j=1}^{l+1} d_j Y_j + \eta > y\right) \le \sum_{k=1}^{6} L_k(x, y; \overline{c_{l+1}}, \overline{d_{l+1}}).
$$

By applying Lemmas 3.1, 3.2, using the induction hypothesis for $n = m = l$, and mimicking the proof of Lemma 3.5 of Li [16] with some minor modification, we can prove that relation (3.13) holds uniformly for all $\overline{c_{l+1}} \in [a, b]^{l+1}$ and $\overline{d_{l+1}} \in [a, b]^{l+1}$. We omit the details.

Step 3 We finally prove (3.6) in the case of $n \ne m$. Without loss of generality, we assume that $n < m$. According to the three disjoint events $(\sum_{j=n+1}^{m} d_j Y_j \le h(y))$, $(\sum_{j=1}^{n} d_j Y_j + \eta \le h(y), \sum_{j=n+1}^{m} d_j Y_j > h(y))$ and $(\sum_{j=1}^{n} d_j Y_j + \eta > h(y), \sum_{j=n+1}^{m} d_j Y_j > h(y))$, we divide the joint tail probability in (3.6) into three parts, denoted by $J_k(x, y; \overline{c_n}, \overline{d_m})$, $k = 1, 2, 3$, respectively. Clearly,

$$
P\left(\sum_{i=1}^{n} c_i X_i + \xi > x, \sum_{j=1}^{m} d_j Y_j + \eta > y\right) = \sum_{k=1}^{3} J_k(x, y; \overline{c_n}, \overline{d_m}). \tag{3.14}
$$

Since (3.6) holds for $n = m$ by step 2, by using Lemma 3.2 and the dominated convergence theorem, we have

$$
\begin{aligned}
&J_1(x, y; \overline{c_n}, \overline{d_m}) \\
&\quad = \int_{0-}^{h(y)} P\left(\sum_{i=1}^{n} c_i X_i + \xi > x, \sum_{j=1}^{n} d_j Y_j + \eta > y - u\right) P\left(\sum_{j=n+1}^{m} d_j Y_j \in du\right) \\
&\quad \sim \int_{0-}^{h(y)} \left(\sum_{i=1}^{n} \sum_{1\le j\ne i\le n} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y-u}{d_j}\right) + \rho \sum_{i=1}^{n} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y-u}{d_i}\right)\right) P\left(\sum_{j=n+1}^{m} d_j Y_j \in du\right) \\
&\quad \sim \sum_{i=1}^{n} \sum_{1\le j\ne i\le n} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_j}\right) + \rho \sum_{i=1}^{n} \overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_i}\right).
\end{aligned} \tag{3.15}
$$

As for $J_2(x, y; \overline{c_n}, \overline{d_m})$, for sufficiently large y, by Lemma 3.1,

$$
\begin{aligned}
J_2(x, y; \overline{c_n}, \overline{d_m}) &= \int_{-\infty}^{h(y)} P\left(\sum_{j=n+1}^{m} d_j Y_j > y - u\right) P\left(\sum_{j=1}^{n} d_j Y_j + \eta \in du, \sum_{i=1}^{n} c_i X_i + \xi > x\right) \\
&\sim \int_{-\infty}^{h(y)} \sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y-u}{d_j}\right) P\left(\sum_{j=1}^{n} d_j Y_j + \eta \in du, \sum_{i=1}^{n} c_i X_i + \xi > x\right).
\end{aligned}
$$

Notice that by Lemma 3.2,

$$
\frac{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y-u}{d_j}\right)}{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)} 1_{(u \le h(y))} \le \frac{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y-h(y)}{d_j}\right)}{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)} \le 2,
$$

which is integrable with respect to $P\left(\sum_{j=1}^{n} d_j Y_j + \eta \in du, \sum_{i=1}^{n} c_i X_i + \xi > x\right)$. Thus by the dominated convergence theorem and $F_2 \in \mathcal{S} \subset \mathcal{L}$, we have

$$
\begin{aligned}
&\lim_{y \to \infty} \frac{J_2(x, y; \overline{c_n}, \overline{d_m})}{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)} \\
&\quad = \int_{-\infty}^{\infty} \lim_{y \to \infty} \frac{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y-u}{d_j}\right)}{\sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)} 1_{(u \le h(y))} P\left(\sum_{j=1}^{n} d_j Y_j + \eta \in du, \sum_{i=1}^{n} c_i X_i + \xi > x\right) \\
&\quad = P\left(\sum_{i=1}^{n} c_i X_i + \xi > x\right),
\end{aligned}
$$

which, together with Lemma 3.1, further leads to

$$
J_2(x, y; \overline{c_n}, \overline{d_m}) \sim \sum_{i=1}^{n} \overline{F_1}\left(\frac{x}{c_i}\right) \sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right). \tag{3.16}
$$

Now we show that $J_3(x, y; \overline{c_n}, \overline{d_m})$ is negligible. Indeed, by noting $(X, Y) \le_{st} (X', Y')$, as done in the proof of Lemma 3.3,

$$
\begin{aligned}
&J_3(x, y; \overline{c_n}, \overline{d_m}) \\
&\quad \le P\left(\sum_{i=1}^{n} c_i X_i' + \xi > x\right) \\
&\qquad \times P\left(\sum_{j=1}^{n} d_j Y_j' + \eta + \sum_{j=n+1}^{m} d_j Y_j > y, \sum_{j=1}^{n} d_j Y_j' + \eta > h(y), \sum_{j=n+1}^{m} d_j Y_j > h(y)\right) \\
&\quad =: J_{31}(x, y; \overline{c_n}, \overline{d_m}) \times J_{32}(x, y; \overline{c_n}, \overline{d_m}).
\end{aligned} \tag{3.17}
$$

By Lemma 3.1 and (3.3),

$$J_{31}(x, y; \overline{c_n}, \overline{d_m}) \sim \sum_{i=1}^{n} \overline{F'_1}\left(\frac{x}{c_i}\right) \sim \sqrt{C_2} \sum_{i=1}^{n} \overline{F_1}\left(\frac{x}{c_i}\right). \tag{3.18}$$

As for $J_{32}(x, y; \overline{c_n}, \overline{d_m})$, we have for sufficiently large y,

$$\begin{aligned} J_{32}(x, y; \overline{c_n}, \overline{d_m}) \le{}& P\left(\sum_{j=1}^{n} d_j Y'_j + \eta + \sum_{j=n+1}^{m} d_j Y_j > y\right) \\ &- P\left(\sum_{j=1}^{n} d_j Y'_j + \eta + \sum_{j=n+1}^{m} d_j Y_j > y, -h(y) < \sum_{j=1}^{n} d_j Y'_j + \eta \le h(y)\right) \\ &- P\left(\sum_{j=1}^{n} d_j Y'_j + \eta + \sum_{j=n+1}^{m} d_j Y_j > y, -h(y) < \sum_{j=n+1}^{m} d_j Y_j \le h(y)\right) \\ =:{}& J_{321}(x, y; \overline{c_n}, \overline{d_m}) - J_{322}(x, y; \overline{c_n}, \overline{d_m}) - J_{323}(x, y; \overline{c_n}, \overline{d_m}). \end{aligned} \tag{3.19}$$

Similarly to (3.18), we have

$$J_{321}(x, y; \overline{c_n}, \overline{d_m}) \sim \sqrt{C_2} \sum_{j=1}^{n} \overline{F_2}\left(\frac{y}{d_j}\right) + \sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right). \tag{3.20}$$

Again by Lemmas 3.1 and 3.2,

$$\begin{aligned} J_{322}(x, y; \overline{c_n}, \overline{d_m}) \ge{}& P\left(\sum_{j=n+1}^{m} d_j Y_j > y + h(y)\right) P\left(-h(y) < \sum_{j=1}^{n} d_j Y'_j + \eta \le h(y)\right) \\ \sim{}& \sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right). \end{aligned} \tag{3.21}$$

In the same way,

$$J_{323}(x, y; \overline{c_n}, \overline{d_m}) \gtrsim \sum_{j=1}^{n} \overline{F'_2}\left(\frac{y}{d_j}\right) \sim \sqrt{C_2} \sum_{j=1}^{n} \overline{F_2}\left(\frac{y}{d_j}\right). \tag{3.22}$$

Combining (3.19)–(3.22) gives

$$\begin{aligned} J_{32}(x, y; \overline{c_n}, \overline{d_m}) &= o(1)\left(\sqrt{C_2} \sum_{j=1}^{n} \overline{F_2}\left(\frac{y}{d_j}\right) + \sum_{j=n+1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)\right) \\ &= o(1)\left(\sum_{j=1}^{m} \overline{F_2}\left(\frac{y}{d_j}\right)\right). \end{aligned} \tag{3.23}$$

It follows from (3.17), (3.18) and (3.23) that

$$J_3(x,y;\overline{c_n},\overline{d_m}) = o(1)\left(\sum_{i=1}^{n}\sum_{1\le j\ne i\le m}\overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_j}\right) + \rho\sum_{i=1}^{n}\overline{F_1}\left(\frac{x}{c_i}\right)\overline{F_2}\left(\frac{y}{d_i}\right)\right). \tag{3.24}$$

Plugging (3.15), (3.16) and (3.24) into (3.14) yields the desired (3.6) in the case of $n < m$.

This completes the proof of Lemma 3.4. □

For simplicity, we denote by $\phi(x,y)$ the right-hand side of relation (2.2).

Lemma 3.5 *Consider the bidimensional risk model (1.1) with $F_1 \in \mathcal{S}$ and $F_2 \in \mathcal{S}$. Assume that (1.4) holds for some $\rho > 0$ and $E[N_1(T)N_2(T)] > 0$ for some $T > 0$. Let ξ and η be two independent real-valued random variables satisfying*

$$\overline{F_\xi}(x) = o(1)\overline{F_1}\left(xe^{rT}\right) \text{ and } \overline{F_\eta}(y) = o(1)\overline{F_2}\left(ye^{rT}\right). \tag{3.25}$$

Then it holds that

$$P\left(\sum_{i=1}^{N_1(T)} X_i e^{-r\tau_i^{(1)}} + \xi > x, \sum_{j=1}^{N_2(T)} Y_j e^{-r\tau_j^{(2)}} + \eta > y\right) \sim \phi(x,y). \tag{3.26}$$

Proof Rewrite the tail probability in the left-hand side of (3.26) as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i^{(1)}} + \xi > x, \sum_{j=1}^{m} Y_j e^{-r\tau_j^{(2)}} + \eta > y, N_1(T) = n, N_2(T) = m\right)\\
&\quad + P(\xi > x, \eta > y, N_1(T) = 0, N_2(T) = 0)\\
&\quad + P(\xi > x)\sum_{m=1}^{\infty} P\left(\sum_{j=1}^{m} Y_j e^{-r\tau_j^{(2)}} + \eta > y, N_1(T) = 0, N_2(T) = m\right)\\
&\quad + P(\eta > y)\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i^{(1)}} + \xi > x, N_1(T) = n, N_2(T) = 0\right)\\
&=: \sum_{k=1}^{4} K_k(x,y).
\end{aligned} \tag{3.27}$$

Note that $\{N_i(t); t \ge 0\}, i = 1, 2$, are two renewal claim-number processes, whose moment generating functions are analytic in a neighborhood of 0, see, e.g., Stein [20]. Thus, we can choose some sufficiently small $\varepsilon > 0$ such that

$$E\left[(1+\varepsilon)^{2N_i(T)}\right] < \infty, \qquad i = 1, 2,$$

which further implies $E[(N_i(T))^p] < \infty, i = 1, 2,$ for any $p > 0$. By using Lemmas 3.3 and 3.4 and following the same lines of the proof of Lemma 4.4 of Yang and Li [27], we obtain

$$
\begin{aligned}
K_1(x,y) &\sim \sum_{i=1}^{\infty} \sum_{1\le j\ne i<\infty} \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_j^{(2)}\in dt\Big) \\
&\quad + \rho \sum_{i=1}^{\infty} \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_i^{(2)}\in dt\Big) \\
&= \phi(x,y).
\end{aligned}
\tag{3.28}
$$

Since $\theta_1^{(1)}$ is independent of all the other sources of randomness, it is easy to check from (3.28) that

$$
\begin{aligned}
\phi(x,y) &\ge (\rho\wedge 1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_j^{(2)}\in dt\Big) \\
&\ge (\rho\wedge 1)P\Big(\theta_1^{(1)}\le T\Big)\overline{F_1}\big(xe^{rT}\big)\int_{0-}^{T}\overline{F_2}(ye^{rt})\lambda_2(dt)
\end{aligned}
\tag{3.29}
$$

$$
\ge (\rho\wedge 1)P\Big(\theta_1^{(1)}\le T\Big)P\Big(\theta_1^{(2)}\le T\Big)\overline{F_1}\big(xe^{rT}\big)\overline{F_2}(ye^{rT}). \tag{3.30}
$$

Thus by the independence between ξ and η, (3.30) and (3.25) we have

$$
K_2(x,y) = o(1)\phi(x,y). \tag{3.31}
$$

As for $K_3(x,y)$, similarly to the proof of (3.28) and by using (3.25) and (3.29), we can obtain

$$
\begin{aligned}
K_3(x,y) &= o(1)P(\xi > x)\int_{0-}^{T}\overline{F_2}(ye^{rt})\lambda_2(dt) \\
&= o(1)\phi(x,y).
\end{aligned}
\tag{3.32}
$$

In the same way, we have

$$
K_4(x,y) = o(1)\phi(x,y). \tag{3.33}
$$

Plugging (3.28), (3.31), (3.32) and (3.33) into (3.27) yields relation (3.26). □

Remark 3.1 It can be easily seen from the proof of Lemma 3.5 that if $\xi=\eta=0$ a.s., then, regardless of arbitrary dependence between $N_1(t)$ and $N_2(t)$, (3.26) still holds because in this case $K_2(x,y)=K_3(x,y)=K_4(x,y)=0$ in (3.27).

4 Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1 Construct two independent random variables

$$
\xi = \sup_{0\le t\le T}\int_{0-}^{t} e^{-rs}D_1(ds) \ \text{ and } \ \eta = \sup_{0\le t\le T}\int_{0-}^{t} e^{-rs}D_2(ds),
$$

which are independent of all other sources of randomness. Then (3.25) is satisfied due to (2.1). On the one hand, by using Lemma 3.5 we have

$$\begin{aligned}\psi_{\text{and}}(x,y) &\le P\left(\sum_{i=1}^{N_1(T)} X_i e^{-r\tau_i^{(1)}} + \xi > x, \sum_{j=1}^{N_2(T)} Y_j e^{-r\tau_j^{(2)}} + \eta > y\right) \\ &\sim \phi(x,y).\end{aligned} \tag{4.1}$$

On the other hand, let

$$\xi' = \inf_{0\le t\le T}\int_{0-}^{t} e^{-rs}D_1(ds) \ \text{ and } \ \eta' = \inf_{0\le t\le T}\int_{0-}^{t} e^{-rs}D_2(ds).$$

Again by Lemma 3.5, we have

$$\begin{aligned}&\psi_{\text{sim}}(x,y)\\ &\ge P\left(\sum_{i=1}^{N_1(T)} X_i e^{-r\tau_i^{(1)}} + \xi' - \int_{0-}^{T} e^{-rs}C_1(ds) > x, \sum_{j=1}^{N_2(T)} Y_j e^{-r\tau_j^{(2)}} + \eta' - \int_{0-}^{T} e^{-rt}C_2(dt) > y\right)\\ &= \int_{0-}^{\infty}\int_{0-}^{\infty} P\left(\sum_{i=1}^{N_1(T)} X_i e^{-r\tau_i^{(1)}} + \xi' > x+u, \sum_{j=1}^{N_2(T)} Y_j e^{-r\tau_j^{(2)}} + \eta' > y+v\right) H(du,dv)\\ &\sim \int_{0-}^{\infty}\int_{0-}^{\infty} \phi(x+u,y+v)H(du,dv)\\ &\sim \phi(x,y),\end{aligned} \tag{4.2}$$

where $H(\cdot,\cdot)$ is the joint distribution of $(\int_{0-}^{T} e^{-rs}C_1(ds), \int_{0-}^{T} e^{-rt}C_2(dt))$, and in the last step we used the same arguments in the proof of Theorem 2.1 of Yang and Li [26]. Thus, (2.2) follows from (4.1), (4.2) and (1.2). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 Clearly, for any $T > 0$,

$$\begin{aligned}\psi_{\text{or}}(x,y) &= P\left(\inf_{0\le t\le T} R_1(t) < 0 \middle| R_1(0) = x\right) + P\left(\inf_{0\le t\le T} R_2(t) < 0 \middle| R_2(0) = y\right)\\ &\quad - \psi_{\text{and}}(x,y).\end{aligned} \tag{4.3}$$

Note that

$$\int_{0-}^{T} \overline{F_1}(xe^{rs})\lambda_1(ds) \ge \overline{F_1}(xe^{rT})\lambda_1(T). \tag{4.4}$$

By using Theorem 2.1, (4.4) and $F_1 \in \mathcal{D}$, we have

$$
\begin{aligned}
\psi_{\text{and}}(x,y) &\sim \phi(x,y) \\
&\sim \sum_{i=1}^{\infty} \sum_{1\le j\ne i<\infty} \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_j^{(2)}\in dt\Big) \\
&\quad + \rho \sum_{i=1}^{\infty} \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_i^{(2)}\in dt\Big) \\
&\le (1\vee\rho) \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{0-}^{T}\int_{0-}^{T} \overline{F_1}(xe^{rs})\overline{F_2}(ye^{rt})P\Big(\tau_i^{(1)}\in ds, \tau_j^{(2)}\in dt\Big) \\
&\le (1\vee\rho)E[N_1(T)N_2(T)]\overline{F_1}(x)\overline{F_2}(y) \\
&= o(1)\int_{0-}^{T} \overline{F_1}(xe^{rs})\lambda_1(ds).
\end{aligned}
\tag{4.5}
$$

Therefore, the desired relation follows from (4.3), (4.5) and Corollary 2.1. □

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