Systems of Linear Equations

Assume we have the following system of linear equations:

$$\begin{aligned} 2x + 4y - 2z &= 4 \\ x + y + z &= 6 \\ x + 3y - 2z &= 1 \end{aligned}$$

In algebra classes, we learned how to solve for x, y, z using the substitution method. Today we're going to learn how to solve these systems using matrices. This will allow us to create computer programs to solve for the variables for us. This comes in handy when solving very large systems (10+ variables).

Before we start discussing matrices, notice that we can do any of the following to our system of equations and it will still produce the same values for x, y, z:

(1) Multiply an equation by a constant
(2) Switch the order of equations
(3) Add one equation to another

Matrices

Instead of using equation format, we're going to use matrix format.

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 4y + -2z \\ 1x + 1y + 1z \\ 1x + 3y + -2z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$$

Notice that the first matrix above contains all of the coefficients of the equations above. This is known as the coefficient matrix. Next to that, we have a column array containing only the variables.

When a matrix is multiplied by a column array, you multiply the elements of the first row to the elements of the column and add them together.

To condense the amount we write, we're going to write the matrix equation as an augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & 3 & -2 & 1 \end{array}\right]$$

Solving Matrices

Reflect back onto our systems of equations. If you solve these equations using substitution, you'll know you've arrived at the answer when you have the following equations:

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

The equivalent in our matrix format is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This special matrix where all of the diagonal numbers are 1 and every other value is 0 is known as the identity matrix. Once you reach this form, you know the values of x, y, z.
In order to manipulate our original matrix into the identity form, we must follow the following (familiar) rules:

(1) Multiply a row by a constant
(2) Switch the order of rows
(3) Add one row to another

To make the process simpler to program we're also going to introduce the idea of Gaussian elimination, partial pivoting, and back substitution.

Example

Below I describe what I'm doing to each row with an equation that looks something like $R_1 + R_2 \rightarrow R_2$. This can be read as "add row 1 to row 2 and put it in the position of row 2".
The general outline of this process is:

(1) Partial Pivot (if needed)
(2) Divide by the row by the value on the diagonal
(3) Cancel out every element below the diagonal element

(1) First Row

Partial Pivot Check: Compare the 1st element in the 1st row (2) to the 1st element in all rows below (1 and 1). If the 1st element is bigger in any rows below, swap the rows. But it isn't, so we don't need to do anything in this step.

Divide by Diagonal Term: Divide the entire row by the number in the diagonal position (2).

$$1/2R_1 \rightarrow R_1$$
$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & 4 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 6 \\ 1 & 4 & -2 & 3 \end{array}\right]$$

Cancel the Element: Add a multiple times the first row from the rows below it such that the 1st element becomes 0 in all following rows.

$$-R_1 + R_2 \rightarrow R_2$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 6 \\ 1 & 4 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 1 & 4 & -2 & 3 \end{array}\right]$$

$$-R_1 + R_3 \rightarrow R_3$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 1 & 4 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

(2) Second Row

Partial Pivot Check: Compare the 2nd element in the 2nd row (-1) to the 2nd element in all rows below (2). If the 2nd element is bigger in any rows below, swap the rows. It is let's swap them.

$$R_2 \longleftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & 2 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & -1 & 2 & 4 \end{array}\right]$$

Divide by Diagonal Term: Divide the entire row by the number in the diagonal position (2).

$$1/2R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & -1 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & -1 & 2 & 4 \end{array}\right]$$

Cancel the Element: Add a multiple times the second row from the rows below it such that the 2nd element becomes 0 in all following rows.

$$R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & -1 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & 0 & 1.5 & 4.5 \end{array}\right]$$

(3) Third Row

Partial Pivot Check: This is the last row, so we don't need to pivot.
Divide by Diagonal Term: Divide the entire row by the number in the diagonal position (1.5).

$$2/3R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & 0 & 1.5 & 4.5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Cancel the Element: This is the last row, so we don't need to do anything for this step.

(4) Back-substitution Now we have what's known as an "upper triangular" matrix. This is a matrix with 1s along the diagonal. Everything below the diagonal is a 0.
The next step is to start at the last row, and add some multiple of the row to each row above it:

$$0.5R3 + R2 \rightarrow R2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -0.5 & 0.5 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_3 + R_1 \rightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$-2R_2 + R_1 \rightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

...and we're done!