CSE 180 – Introduction to Robotics
Homework N. 3 – Spring 2019

Important Information

1. This homework is due on April 25, 2019 at 4:15pm. The deadline is strict and will be enforced by CatCourses. **No exceptions.**

2. Your solution must be submitted in electronic form through CatCourses. Paper submissions will not be accepted. **No exceptions.**

3. If you write your solution by hand, write clearly. Unreadable documents will not be graded.

4. The method you follow to determine the solution is more important than the final results. Clearly illustrate and explain the intermediate steps. If you just write the final result you will get not credit.

5. **THIS HOMEWORK MUST BE SOLVED INDIVIDUALLY. NO EXCEPTIONS**.

1 Probability spaces

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{A} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4, 5, 6\}\}$ and $P : \mathcal{A} \to \mathbb{R}$ be defined as follows: $P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\{1, 2\}) = 0.1$, $P(\{3, 4, 5, 6\}) = 0.9$. Is $(\Omega, \mathcal{A}, P)$ a probability space or not? Explain your answer (simply answering Yes/No will give you no credit.)

Answer: A Probability Space must satisfy the following properties:

1. Ω is a set

2. $\mathcal{A}$ is an algebra on Ω

3. $P : \mathcal{A} \to \mathbb{R}$ is a function subject to the following three constraints:

   - $P(A) \geq 0$ for each $A \in \mathcal{A}$
   - $P(\Omega) = 1$
   - if $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

To prove the FIRST property we can take the definition of set as a finite or infinite collection of objects. In this case, Ω is a collection of the positive integers, or objects, from 1 to 6.

For the SECOND property we must look at the definition of an algebra. For $\mathcal{A}$ to be an algebra of Ω is must satisfy the following conditions:

- $\emptyset \in \mathcal{A}$

- $A \in \mathcal{A} \to \bar{A} \in \mathcal{A}$: closed under complement

- $A, B \in \mathcal{A} \to A \cup B \in \mathcal{A}$: closed under finite union

This must be true for any event A or B in $\mathcal{A}$. The first condition can be quickly checked. The next condition can be listed for all events in $\mathcal{A}$:

- If $A = \emptyset$, then $\bar{A} = \Omega \in \mathcal{A}$
- If $A = \Omega$, then $\bar{A} = \emptyset \in \mathcal{A}$
- If $A = \{1, 2\}$, then $\bar{A} = \{3, 4, 5, 6\} \in \mathcal{A}$
- If $A = \{3, 4, 5, 6\}$, then $\bar{A} = \{1, 2\} \in \mathcal{A}$

Finally the last condition can be proven in a similar way a few examples are listed below:

- If $A = \emptyset, B = \Omega$, then $A \cup B = \Omega \in \mathcal{A}$
- If $A = \emptyset, B = \{3, 4, 5, 6\}$, then $A \cup B = \{3, 4, 5, 6\} \in \mathcal{A}$
- If $A = \{1, 2\}, B = \{3, 4, 5, 6\}$, then $A \cup B = \Omega \in \mathcal{A}$
- If $A = \{3, 4, 5, 6\}, B = \Omega$, then $A \cup B = \Omega \in \mathcal{A}$

The THIRD property deals with the probability function. The first two constraints can be verified easily. While the third we can take a look at the events A $= \{1, 2\}$ and B $= \{3, 4, 5, 6\}$ with corresponding probabilities $P(A) = 0.1$ and $P(B) = 0.9$. $A \cup B = \Omega \to P(A \cup B) = P(\Omega) = 1.0 = 0.1 + 0.9 = P(A) + P(B)$. This is true for any two event in $\mathcal{A}$

2 Roulette Playing

Consider the game of roulette, where there are 37 pockets on the wheel (numbered from 0 to 36) and the wheel is fair, i.e., all pockets have the same probability. The 0 pocket is green, even numbers are black, and odd numbers are red (this is not true in reality, but it makes things simpler.) Compute the following probabilities:

- probability that the outcome is divisible by three, given that it is black.
- probability that the outcome is divisible by three or four, given that a number is in the range 1 to 12.

Trivia: you may find interesting to read the wikipedia page describing the roulette game to numerically assess why the game is designed to ensure the house has an advantage
`https://en.wikipedia.org/wiki/Roulette`

Answer:

1. If we set event A to be outcome is divisible by three and event B to be the outcome is black, then we can say we are looking for $P(A|B)$ which can be expressed as $\frac{P(A \cap B)}{P(B)}$.
   - A $= \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$
   - B $= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36\}$
   - $A \cap B = \{6, 12, 18, 24, 30, 36\}$
   - $P(B) = \frac{18}{37}$, $P(A \cap B) = \frac{6}{37} \to P(A|B) = \frac{1}{3}$
2. Event A is the outcome is divisible by three or four and event B is the outcome is in range 1 to 12, then:
   - A $= \{0, 3, 4, 6, 8, 9, 12, 15, 16, 18, 20, 21, 24, 27, 28, 30, 32, 33, 36\}$
   - B $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
   - $A \cap B = \{3, 4, 6, 8, 9, 12\}$
   - $P(B) = \frac{12}{37}$, $P(A \cap B) = \frac{6}{37} \to P(A|B) = \frac{1}{2}$

3 Bayes Rule

Recall Bayes rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

5 Let us now apply this rule to the scenario where a robot moves along a corridor where there is a door and is equipped with a sensor to determine if the robot is in front of the door or not (for example, a computer vision algorithm processing images coming from a camera on top of the robot.) To simplify things, assume that there are only two locations for the robot, i.e., the robot is either in front of the door or not. These two possibilities are modeled by these two events:

- A: the robot is in front of the door;
- $\bar{A}$: the robot is not in front of the door.

The image processing algorithm returns a binary quantity, i.e., it either determines that the robot is in front of the door or it determines that the robot is not in front of the door. This is modeled by these two events:

- B: the sensor determines that the robot is in front of the door;
- $\bar{B}$: the sensor determines that the robot is not in front of the door.

Finally, the sensor is not perfect but it rather makes errors. More specifically, it is possible that the robot is in front of the door, but the algorithm fails to recognize it. But it is also possible that the robot is not in front of the door, and the algorithm instead believes the robot is in front of the door. These errors are modeled by the following probabilities:

$$P(\bar{B}|A) = 0.1 \qquad P(B|\bar{A}) = 0.2$$

Assume that initially we do not know where the robot is, and we assign equal probability to both events. The robot then queries the sensor and the sensor returns B, i.e., the robot is in front of the door. Based on this unique sensor reading, what is $P(A|B)$, i.e., the probability that the robot is indeed in front of the door?

Hint: you may think that you miss the value of $P(B)$, but you can compute that (recall what we saw in class and carefully study the lecture notes.)

Answer:

We are looking for the value $P(A|B)$ which is given by $\frac{P(B|A)P(A)}{P(B)}$ according to Bayes Rule. We give equal probability to initial location of the robot, thus, $P(A) = P(\bar{A}) = 0.5$. So to compute $P(A|B)$ we need to compute the values $P(B|A)$ and $P(B)$

- Using the give probability above, $P(B|A) = 1 - P(\bar{B}|A)$ by the conditional complement rule. Hence, $P(B|A) = 1 - 0.1 = 0.9$
- $P(B)$, on the other hand, can be computed using the total probability theorem. $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) = (0.9)(0.5) + (0.2)(0.5) = 0.55$
- Finally $P(A|B) = \frac{(0.9)(0.5)}{(0.55)} \approx 0.81$