CSE 180 – Introduction to Robotics
Homework N. 1 – Spring 2019 SOLUTIONS

Important Information

1. This homework is due on February 28, 2019 at 4:30pm. The deadline is strict and will be enforced by CatCourses. **No exceptions.**
2. Your solution must be submitted in electronic form through CatCourses. Paper submissions will not be accepted. **No exceptions.**
3. If you write your solution by hand, write clearly. Unreadable documents will not be graded.
4. The method you follow to determine the solution is more important than the final results. Clearly illustrate and explain the intermediate steps. If you just write the final result you will get not credit.
5. Each question is worth the same amount of points.
6. **This homework must be solved individually**.

1 Composite Rotations

Two frames A and B are initially coincident. Frame B then undergoes the following sequence of transformations:

1. a rotation of $\pi/4$ about the $\mathbf{y}$ axis (fixed);
2. a rotation of $\pi/2$ about the $\mathbf{x}$ axis (fixed);
3. a rotation of $\pi/6$ about the $\mathbf{z}$ axis (moving);
4. a rotation of $\pi/3$ about the $\mathbf{x}$ axis (fixed);
5. a rotation of $\pi/3$ about the $\mathbf{y}$ axis (moving).

Write the final rotation matrix ${}^A_B\mathbf{R}$ describing the orientation of B with respect to A.
Note: you do not need to compute the final matrix by performing all intermediate multiplications. All that matters here is the order, so you can leave matrices in their symbolic form (as long as it is correct).

Answer: Using the notation introduced in the lecture notes, the overall rotation matrix can be written as follows:

$${}^A_B\mathbf{R} = [\mathbf{R_x}(\pi/3)[[\mathbf{R_x}(\pi/2)\mathbf{R_y}(\pi/4)]\mathbf{R_z}(\pi/6)]]\mathbf{R_y}(\pi/3)$$

This expression is obtained by pre-multiplying when the rotation is about the fixed axis and post-multiplying when the rotation is about the moving axis. So the rotation in step 2 is pre-multiplied to the rotation in step 1, then the rotation in step 3 is post-multiplied to the previous result, and so on. The parentheses in the expression evidence how the result was put together.

2 Transformation Matrices

Two frames A and B are initially coincident. Frame B then undergoes the following transformations:

1. a rotation of $\pi/2$ about the $\mathbf{x}$ axis;
2. a translation of 3 units about the $\mathbf{y}$;
3. a rotation of $\pi/2$ about the $\mathbf{z}$ axis (fixed frame).

Write the transformation matrices ${}^A_B\mathbf{T}$ and ${}^B_A\mathbf{T}$.

Answer: To answer this question we interpret transformation matrices as operators to transform transformation matrices (section 3.7.4 in the lecture notes), whereby we pre-multiply each transformation matrix to the previous one. The first transformation matrix is the identity matrix (because A and B are coincident) and is therefore omitted from the calculation. First, let us compute ${}^A_B\mathbf{T}$, as B moves and A remains stationary:

$$ {}^A_B\mathbf{T} = \mathbf{T}(\mathbf{z}, \pi/2)\mathbf{T}(0,3,0)\mathbf{T}(\mathbf{x}, \pi/2) $$

Substituting the values we get

$$ {}^A_B\mathbf{T} = \begin{bmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The inverse ${}^B_A\mathbf{T}$ can be computed using formula 3.26 in the lectured notes.

$$ {}^B_A\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Note that the rotation part is obtained by transposing the rotation sub matrix in ${}^A_B\mathbf{T}$, while the origin is obtained using the formula $-{}^B_A\mathbf{R}{}^AO'$

3 Quaternions to Rotations

Let $\mathbf{q} = a + bi + cj + dk$ be a unit quaternion. In the lecture notes it is stated that its associated rotation matrix is

$$ \mathbf{R} = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}. $$

Show that $\mathbf{R}$ is a rotation matrix.

Answer: First recall that in a unit quaternion we have $\sqrt{a^2+b^2+c^s+d^2} = 1$. Next, for the statement to be proved, we have to show that the three properties characterizing rotation matrices hold. These are the following.

1. The length of all columns is 1, i.e., by taking the dot product of each column with itself we must get 1. To this end, let us write the dot product of the first column by itself:

$$[2(a^2+b^2)-1]^2 + [2(bc+ad)]^2 + [2(bd-ac)]^2$$

After developing the quadratic terms and doing some algebra, we get to the following expression:

$$4a^2(a^2-1) + 4b^2(b^2-1) + 4b^2(c^2+d^2) + 4a^2(c^2+d^2) + 8a^2b^2 + 1$$

This can be rewritten as

$$4a^2(a^2+c^2+d^2-1) + 4(b^2+c^2+d^2-1) + 8a^2b^2 + 1$$

At this point we exploit the hypothesis that the quaternion is a unit quaternion, so $a^2+c^2+d^2-1 = -b^2$ and $b^2+c^2+d^2-1 = -a^2$. Substituting in the last expression, we get

$$-4a^2b^2 - 4a^2b^2 + 8a^2b^2 + 1$$

and the result therefore follows. Equivalent derivations can be written for the second and the third column.

2. All columns in the matrix must be mutually orthogonal, i.e., their dot product must be 0. To prove this, let us start writing the dot product between the first and the second column:

$$[2(a^2+b^2)-1]2(bc-ad) + 2(bc+ad)[2(a^2+c^2)-1] + 2(bd-ac)2(cd+ab)$$

After some algebra, all terms cancel out showing that the result is 0. Equivalent derivations can be written when multiplying the first and the third column, as well as the second and the third column.

3. The determinant of the matrix is 1. This is again an algebraic exercise where we have to write the determinant, and use the hypothesis that we are dealing with a unit quaternion to simplify the expression and cancel out term. One option is to write the determinant by developing it along the first column:

$$\begin{aligned} det(\mathbf{R}) =& [2(a^2+b^2)-1][(2(a^2+c^2)-1)(2(a^2+d^2)-1) - 4(cd-ab)(cd+ab)] + \\ & - 2(bc+ad)[2(bc-ad)(2(a^2+d^2)-1) - 4(bd+ac)(cd+ab)] \\ & + 2(bd-ac)[4(bc-ad)(cd-ab) - (2(a^2+c^2)-1)2(bd+ac)] \end{aligned}$$

After quite some algebra, using the hypothesis that a, b, c, d are the components of a unit quaternion, we can show that this expression simplifies to 1, thus completing the proof.

4 Change of Coordinates

Three robots are operating in a shared space. Let A, B, and C the three frames attached to the robots, and let W be a world frame. Assume that the following transformation matrices are known: ${}^{B}_{A}\mathbf{T}$, ${}^{C}_{W}\mathbf{T}$, ${}^{B}_{C}\mathbf{T}$, ${}^{W}_{B}\mathbf{T}$ Assume robot A perceives a point of interest whose coordinates are ${}^{A}\mathbf{p}$. Can you determine any of the following: ${}^{B}\mathbf{p}$, ${}^{C}\mathbf{p}$. ${}^{W}\mathbf{p}$? For each of the required points, if the answer is positive, show how it can be computed, and if the answer is negative explain why it cannot be computed.

Answer: We are given four frames (A, B, C, W) and four transformation matrices. If we draw the transformation graph as per definition 3.4 in the lecture notes, we see that the graph is connected. So the answer is that we can compute the required change of coordinates irrespective of the reference frame. The fact that we have a graph rather than a tree means that we may have more than one solution in some instances. The required change of coordinates are as follows:

- ${}^B\mathbf{p} = {}^B_A\mathbf{T}\,{}^A\mathbf{p}$ where we used the basic formula and ${}^B_A\mathbf{T}$ is given.
- ${}^C\mathbf{p} = {}^C_B\mathbf{T}\,{}^B\mathbf{p}$ where we used the previous result and obtained ${}^C_B\mathbf{T}$ by inverting the matrix ${}^B_C\mathbf{T}$ that was given.
- ${}^W\mathbf{p} = {}^W_B\mathbf{T}\,{}^B\mathbf{p}$ where we again used the basic formula for the change of coordinate, as well as the provided transformation matrix ${}^W_B\mathbf{T}$ and the result from the first question.

5 Quaternions

Quaternions can be multiplied following rules similar to those we follow for complex numbers. A fundamental thing to remember is that **quaternions product is not commutative**. When multiplying two quaternions, keep in mind the following definitions about products between their imaginary coefficients i, j, k:

- $i^2 = j^2 = k^2 = ijk = -1$
- $ij = k$, $ji = -k$
- $jk = i$, $kj = -i$
- $ki = j$, $ik = -j$

Consider the following two quaternions:

$$\mathbf{p} = 1 + 2i - 3k$$

$$\mathbf{q} = 5 + 4j + 2k.$$

Compute:

1. the product $\mathbf{pq}$.
2. the norm of the product $\mathbf{pq}$.

Note: show the intermediate steps; if you just write the result, you will not get any point.

Answer: The first question can be answered by multiplying the quaternions as polynomials and then using the rules given above every time one of the given terms appears. Let us first multiply the two quaternions as polynomials:

$$\mathbf{pq} = (1 + 2i - 3k)(5 + 4j + 2k) = 5 + 4j + 2k + 10i + 8ij + 4ik - 15k - 12kj - 6k^2$$

Next, we substitute the terms ij, ik, kj and k^2 as per the above rules, thus getting:

$$\mathbf{pq} = 5 + 4j + 2k + 10i + 8k - 4j - 15k + 12i + 6$$

The result then follows by adding together the terms:

$$\mathbf{pq} = 11 + 22i - 5k$$

The norm of the product is then obtained by the formula given in the definition:

$$|\mathbf{pq}| = \sqrt{a^{+}b^2 + c^2 + d^1} = \sqrt{11^2 + 22^2 + 5^2} \approx 25.0998$$