**Due before lecture on Monday, December 9, 2019**

1. Let $A = \begin{bmatrix} 2 & -4 & -2 & 1 \\ 0 & -7 & 2 & 0 \\ 2 & 3 & -4 & 2 \end{bmatrix}$.

   (a) (*definition of $C(A^T)$ and $N(A^T)$ as column space and nullspace of $A^T$ respectivley*)

      i. Write down explicitly what the transpose $A^T$ of $A$ is.

      ii. Describe explicitly the left nullspace $N(A^T)$ by finding all solutions to a specific linear system.

      iii. Describe explicitly the row space $C(A^T)$ by finding out the conditions on $b_i$'s to ensure that $A^T\vec{x} = \vec{b}$ is solvable.

   (b) (*definition of $C(A^T)$ as span of vectors that are transposes of rows of $A$*)

      i. Explain why Gaussian eliminate steps on $A$ do change rows of $A$ but do NOT change the row space of $A$.

      ii. The upper echelon form for $A$ is $A = \begin{bmatrix} 2 & -4 & -2 & 1 \\ 0 & -7 & 2 & 0 \\ 2 & 3 & -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -4 & -2 & 1 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$. Verify that the non-zero row vectors of $U$ are linearly independent. As a consequence, they form a ______ for the row space $C(U^T)$ of $U$, which is the same as $C(A^T)$.

      iii. Verify that all basis vectors for $C(A^T)$ we found above satisfy the conditions on $b_i$ you found in part (a)iii.

The most efficient way to find basis for each of the four fundamental subspaces of a matrix turns out, once again, to be Gaussian elimination. Furthermore, we need Gaussian elimination on $A$ only. There is no need to do another elimination on $A^T$! To find basis for $C(A)$, $N(A)$, and $C(A^T)$, what we have discussed in lectures is enough. However, in order to find the basis for $N(A^T)$ we do need to know how to perform Gaussian elimination using elementary matrices, which is a topic in Math 146.

2. Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 10 & 13 \end{bmatrix}$. Gaussian elimination on $A$ leads to

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 10 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

   (a) (*nullspace, review*) Write down a basis for $N(A)$.

   (b) (*column space, review*) Write down a basis for $C(A)$.

   (c) (*row space*) Review part (b) of previous problem and write down a basis for $C(A^T)$.

   (*optional*)The Gaussian elimination on $A$ can be summarized using matrix multiplication as

$$L^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 10 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

   Multiplying $A$ from left by the matrix $L^{-1}$ has the exact same effect as performing Gaussian elimination on $A$.

   (d) ((*optional*) *left nullspace*) Think of the rows of $L^{-1}$ as transposed vectors $\vec{x}^T$. Which vector (vectors) $\vec{x}$ satisfies $\vec{x}^T A = \vec{0}^T$? Can we use this vector (these vectors) as a basis for $N(A^T)$ and why?

3. (*Strang §3.1 #7*) Find a vector $\vec{x}$ orthogonal to the row space of $A$, and a vector $\vec{y}$ orthogonal to the column space, and a vector $\vec{z}$ orthogonal to the nullspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

4. (*Strang §3.1 #30*) If $A\vec{x}$ is in the nullspace of $A^T$ then $A\vec{x} = \vec{0}$. Reason: $A\vec{x}$ is also in the ____________ of $A$ and the spaces are __________. As a consequence, the nullspaces $N(A^TA)$ ______ $N(A)$.

5. (*Strang §3.1 #35*) The floor and the wall are not orthogonal subspaces because they share a nonzero vector (along the line where they meet). Two planes in $\mathbb{R}^3$ cannot be orthogonal! Find a vector in both column spaces $C(A)$ and $C(B)$:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}.$$

(This will be a vector that can be written both as $A\vec{x}$ for some $\vec{x}$ and also as $B\vec{y}$ for some, probably different, $\vec{y}$. ) Hint: Think about using the 3×4 matrix $\begin{bmatrix} A & B \end{bmatrix}$.

6. (*Strang §3.1 #36*) Extend the previous problem to a $p$-dimensional subspace $V$ and a $q$-dimensional subspace $W$ of $\mathbb{R}^n$. What inequality on $p+q$ guarantees that $V$ intersects $W$ in a nonzero vector? These subspaces cannot be orthogonal as a consequence.

7. (*Strang §3.1 #49*) Why is each of these statements false?

   (a) $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is perpendicular to $\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^T$ so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal subspaces of $\mathbb{R}^3$.

   (b) The subspaces spanned by $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$ is the orthogonal complement of the subspace spanned by $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 2 & -2 & 3 & 4 & -4 \end{bmatrix}^T$.

   (c) Two subspaces that meet only in the zero vector are orthogonal.

8. (*Strang §3.1 #46*) Find $A^TA$ if the columns of $A$ are unit vectors, all mutually perpendicular.

9. (*Strang §3.1 #19*) Why are these statements false? Construct a counterexample for each case to show that it is false. $V$ and $W$ are subspaces of $\mathbb{R}^n$.

   (a) If $V$ is orthogonal to $W$, then $V^\perp$ is orthogonal to $W^\perp$.

   (b) $V$ orthogonal to $W$ and $W$ orthogonal to $Z$ makes $V$ orthogonal to $Z$.