Due before lecture on Monday, October 7, 2019

1. (***if you have not done this problem from last week***) (*Strang* §2.2 #39) Explain why all these statements are all false (all statements are about solving linear systems $A\vec{x} = \vec{b}$):
   (a) The complete solution is any linear combination of $\vec{x}_{\text{particular}}$ and $\vec{x}_{\textbf{nullspace}}$.
   (b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.
   (c) The solution $\vec{x}_{\text{particular}}$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). (Find a 2×2 counterexample.)
   (d) If A is invertible there is no solution $\vec{x}_{\text{nullspace}}$ in the nullspace. (Lei Yue's comment: you do not even need to know what it means to say a matrix is invertible.)

2. (*Making connections of different perspectives of the same idea*)
   (a) Write equivalent statements of the sentence:
   $$A\vec{x} = \vec{0} \textit{ has only the } \vec{x} = \vec{0} \textit{ solution.}$$
   Explain in each case why your statement is equivalent.
      i. in term of $N(A)$ or $C(A)$;
      ii. in terms of pivots of A;
      iii. in terms of the column vectors of A;
      iv. in terms of the existence and/or uniqueness of solutions to $A\vec{x} = \vec{b}$ for other $\vec{b}$'s.
   (b) Write equivalent statements of (in other words, necessary and sufficient conditions to) the sentence:
   $$A\vec{x} = \vec{b} \textit{ is solvable for any } \vec{b}.$$
   Explain in each case why your statement is equivalent.
      i. in term of $N(A)$ or $C(A)$;
      ii. in terms of pivots of A;
      iii. in terms of the column vectors of A;

3. Complete the worksheet titled "Existence and Uniqueness of Solutions". Study your examples, and summarize the method to come up with examples satisfying each pair of criteria twice:
   (a) once in terms of pivots of the matrix A, and
   (b) another time in terms of values of m, n, and r, where m is the number of rows of A, n the number of columns, and $r = \mathrm{rank}(A)$. Recall that $\mathrm{rank}(A)$ is, by definition, the number of pivots of A.

4. Do you think the set of all special solutions to $A\vec{x} = \vec{0}$ are linearly dependent, independent. or cannot be decided (meaning that special solutions to certain homogeneous systems are dependent while to others are independent)? Explain your reasoning.

5. A is a 3-by-4 matrix and its upper echelon form is $U = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 0 & 0 & -7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Determine the following statements true or false. Explain your reasoning.
   (a) The first and third columns of U are linearly independent.
   (b) The second column of U is a linear combination of its first and third columns. So is the fourth column of U.

(c) The first and third columns of the original matrix A are linearly independent.

(d) The second column of the original matrix A is a linear combination of its first and third column. So is the fourth column of A.

(e) A and U have the same column space. That is, $C(A) = C(U)$.

6. Let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ and denote the function it defines as L_A. That is, $L_A : \mathbb{R}^n \to \mathbb{R}^m$, $L_A(\vec{x}) = A\vec{x}$. Answer the following questions about this particular L_A.

   (a) What are the values of m and n?

   (b) $\ker(L_A)$ is another name for ________ of matrix A. Find $\ker(L_A)$.

   (c) $\mathrm{range}(L_A)$ is another name for ____________ of matrix A. Describe $\mathrm{range}(L_A)$.

   (d) Find the image under L_A of $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Find all vectors $\vec{x}$'s who have the same $L_A(\vec{u})$ as its image.

7. Let $A_{m\times n}$ by an m-by-n matrix and $L_A : \mathbb{R}^n \to \mathbb{R}^m$ the function it defines. Complete the following sentences and explain your reasoning.

   (a) L_A is onto if and only if $\mathrm{range}(L_A)$__________.

   (b) L_A is one-to-one if and only if $\ker(L_A)$__________. Hint: You may find problem#4 of Homework05 helpful.

   (c) For the A and L_A from the previous problem, is L_A one-to-one? Is L_A onto?

8. (*making connections*) Use the previous two problems as hint to write down the more general statements in this problem.

   Let A be an m×n matrix and define $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by $L_A(\vec{x}) = A\vec{x}$.

   (a) Write down equivalent statements to

   "L_A is one-to-one"

   i. in terms of the existence and/or uniqueness of solutions;
   ii. in term of nullspace or column space of A;
   iii. in terms of the column vectors of A;
   iv. in terms of pivots in A.

   (b) Write down equivalent statements to

   "L_A is onto"

   i. in terms of the existence and/or uniqueness of solutions;
   ii. in term of nullspace or column space of A;
   iii. in terms of the column vectors of A;
   iv. in terms of pivots in A.

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