Due before lecture on Monday, October 28, 2019

1. (*if you did not finish this last week*) $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

   (a) Is $\ker(T)$ a subspace of $\mathbb{R}^n$?. Explain your reasoning. If yes, how can you find a basis for $\ker(T)$?

   (b) Is $\text{range}(T)$ a subspace of $\mathbb{R}^m$?. Explain your reasoning. If yes, how can you find a basis for $\text{range}(T)$?

   (Hint: Connect $\ker(T)$ and $\text{range}(T)$ to column space and nullspace of some matrix.)

2. $A_{7\times 5}$ is matrix with 7 rows and 5 columns. The columns of A satisfy

$$(column\text{-}3) = -5(column\text{-}2) + (column\text{-}4).$$

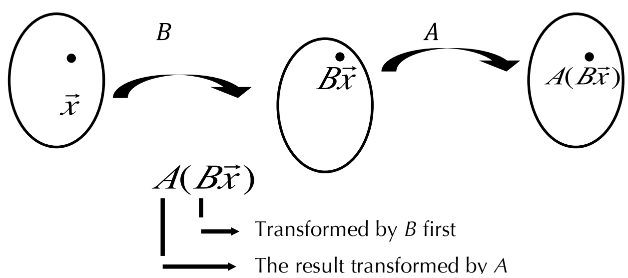
   Write down one concrete vector in $N(A)$. Explain your reasoning.

3. In class, we agreed that $N(A)$, the set of all solutions to $A\vec{x} = \vec{0}$, is a vector subspace. What about all solutions to an inhomogeneous system? More precisely, given a fixed matrix $A_{m\times n}$ and fixed right hand side vector $\vec{0} \neq \vec{b}$ in $\mathbb{R}^m$, define

$$V = \{\text{all solutions to } A\vec{x} = \vec{b}\}.$$

   Is V a vector subspace of $\mathbb{R}^n$?

4. In class, we discussed to think of matrix multiplication as composition of functions. We thought about how, if you have $(AB)\vec{x}$, where A and B are matrices and $\vec{x}$ is a vector, $(AB)\vec{x} = A(B\vec{x})$ could be thought of as B transforming $\vec{x}$ first, and then A transforming the result of $B\vec{x}$.



   This exercise reinforces that connection.

   (a) Given functions f and g below

$$f(x) = 2x + 4 \qquad g(x) = x^2 - 3x$$

   compute

   i. $f(g(x))$ and $g(f(x))$

   ii. $f(g(2))$ and $g(f(2))$

   (b) Let the matrices F and G be defined as below. Answer the following questions accordingly.

$$F = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \qquad G = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix}$$

   i. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and let $G\vec{x} = \vec{y}$. Compute $G\vec{x}$ and compute $F\vec{y}$.

ii. Let $\vec{x}$ be the same vector as in i., and let $F\vec{x} = \vec{u}$. Compute $F\vec{x}$ and compute $G\vec{u}$.

iii. Compute FG and GF.

(c) Summarize, in words, the similarities between matrix multiplication and composition of functions. Point out the equivalence, in terms of compositions of functions f and g, of the various quantities in part (b): $G\vec{x}$, $F\vec{y}$, $F\vec{x}$, $G\vec{u}$, FG and GF. All notations have the same meaning as in parts (a) and (b).

(d) Is matrix multiplication commutative (That is, $AB = BA$ for any matrices A and B)? Why or why not?

When we solved the Italicizing N Task 1 problem in class, some groups have written all the input vectors side by side into a matrix and all the output vectors the same way:

$$\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 4 & 4 \end{bmatrix}$$

We used that as an example to introduce one interpretation of matrix multiplication—-each column of the product matrix AB is the product of matrix A with the corresponding column vector of matrix B. Keep this interpretation in mind when answering following questions.

5. In order for us to be able to multiply two matrices A and B together, what conditions do we have to put on the shapes of A and B? What is the shape of the product matrix AB?

6. (a) Fill in the blanks and explain your reasoning: Each column vector of the product matrix AB is a linear combination of ________________, and so each column vector of AB is in the span of ________________.

   (b) As a consequence of part (a), what can you say about the relation among three column spaces $C(AB)$, $C(A)$, and $C(B)$?

7. Assume that AB is defined. Determine the following statements true or false. If true, provide a justification. If false, provide a counterexample.
   Hint: You may start by applying the definition of (in)dependence to columns of A or B and then try to multiply the equation by the other matrix.

   (a) If the columns of B are linearly dependent, then so are the columns of AB.

   (b) If the columns of A are linearly independent, then so are the columns of AB.

8. True of false? If true, explain why. If false, provide a counterexample. A and B are matrices of appropriate shape so that each addition or multiplication is defined.

   (a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

   (b) If AB and BA are defined then A and B are square.

   (c) If AB and BA are defined then AB and BA are square.

   (d) $(AB)^2 = A^2B^2$.

   (e) $(A + B)^2 = A^2 + 2AB + B^2$

   (f) If $AB = B$ then $A = I$.

9. (*Strang, §1.6, #25*) Suppose that A is a 3×3 matrix with (Row-1)+(Row-2)=(Row-3).

   (a) Explain why $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot have a solution.

(b) Which right-hand sides $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ might allow a solution to $A\vec{x} = \vec{b}$?

(c) What happens to Row-3 if we perform forward elimination on A?

(d) Explain why each of the above three situations leads to the conclusion that A is not invertible? (Hint: Think in terms of the linear transformation L_A that A defines.)

10. (*Strang, §1.6, #40*) True or False. If true, explain why. If false, ***show a concrete counterexample***. (Hint: Use the fact that A is invertible if and only if the linear transformation L_A which it defines is invertible.)

 (a) A 4×4 matrix with a row of zeros is not invertible.

 (b) A matrix with 1's down the main diagonal is invertible.

 (c) If A is invertible, then A^{-1} is invertible.