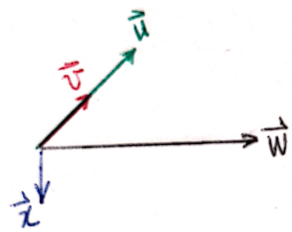
Due before lecture on Monday, September 16, 2019

1. Consider the vectors given in the picture below. Sketch (or explain what you want to sketch) what each of the following span looks like, and explain your reasoning in words and pictures.



   (a) $\mathrm{span}\{\vec{u}\}$, $\mathrm{span}\{\vec{v}\}$, $\mathrm{span}\{\vec{w}\}$, $\mathrm{span}\{\vec{x}\}$.
   (b) $\mathrm{span}\{\vec{u}, \vec{v}\}$, $\mathrm{span}\{\vec{u}, \vec{w}\}$, $\mathrm{span}\{\vec{w}, \vec{x}\}$.
   (c) $\mathrm{span}\{\vec{u}, \vec{v}, \vec{w}\}$, $\mathrm{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$

2. (*I phrased this problem entirely in formal mathematical language. If you are not sure what the questions are asking, try to rephrase them in terms of modes of transportation and finding Gauss*)

   Consider three vectors in $\mathbb{R}^2$: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

   (a) Write down a vector equation that you need to solve in order to determine whether or not $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ is a linear combination of the three given vectors.
   (b) Write a system of equations that has the same solution as the vector equation you wrote in part (a).
   (c) Which one of the following best describes the solutions to your equations in part (a) and (b)? Choose one please.
      i. There is a unique solution.
      ii. There are more than one solutions.
      iii. There is no solution.

      Explain your choice. Give an example of one or more solutions if possible. Otherwise, make clear why it is not possible to solve the equations.

3. Construct a set of vectors in $\mathbb{R}^2$ to satisfy each of the following requirement. If such a set does not exist, say so and explain why. Use concrete vectors whose components are given numbers. You may use the previous problem as a hint to some parts.

   (a) The set contains three vectors and its span is the entire $\mathbb{R}^2$.
   (b) The set contains three vectors but its span is not the entire $\mathbb{R}^2$.
   (c) The set contains two vectors and its span is the entire $\mathbb{R}^2$.
   (d) The set contains only one vector and its span is the entire $\mathbb{R}^2$.
   (e) The span of the set contains only one vector in $\mathbb{R}^2$.

4. (a) If a set of vectors is linearly dependent and spans $\mathbb{R}^2$, how may vectors must it have?
   (b) If a set of vectors is linearly independent but does not span $\mathbb{R}^2$, how may vectors must it have?
   (c) If a set of vectors is both linearly independent and spans $\mathbb{R}^2$, how may vectors must it have?

5. For this problem, you will need a 3D plotting tool. You may use any software of your choice. If you do not have one, use this free online tool

   `https://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/`

   To plot a vector using the online tool, click the radio button next to "Add to graph:" on the panel to the left of the graphing area, and select "Vector: $< a, b, c >$". You may then enter components of the vector you want to plot. You may rotate the plot by clicking in the graphing area and drag your pointer around.

   (a) Use a 3D plotting tool of your choice.Plot the three given vectors from the 3D magic carpet problem

   $$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}.$$

   And plot at least 5 more different linear combinations of these three vectors, some of them involving negative scalars for some vectors.

   (b) Rotate your plot, and make an educated guess to answer questions 2 & 3 on the back side of "*The Carpet Ride Problem: Getting Back Home*".

6. (a) Name one vector in $\mathbb{R}^3$ that is in $\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (b) Does there exist a vector in $\mathbb{R}^3$ that is NOT in $\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$? Explain your reasoning.

7. Recall the three modes of transportation in the 3D Magic Carpet problem:

   $$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

   Determine whether the following claims is true or false. Explain your reasoning.

   (a) $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{v_1, v_2\}$
   (b) $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{v_2, v_3\}$
   (c) $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{v_1, v_3\}$

8. Suppose $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$.

   (a) For what value(s) of h is $\vec{v}_3 \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ (this notation means that $\vec{v}_3$ is a vector in $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$)? How do you know?

(b) For what value(s) of h is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent? How do you know?

9. (*Strang, §2,3, #8*) If $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are linearly independent vectors, show that the sums

$$\vec{v}_1 = \vec{w}_2 + \vec{w}_3, \quad \vec{v}_2 = \vec{w}_1 + \vec{w}_3, \quad \vec{v}_3 = \vec{w}_1 + \vec{w}_2$$

are also *independent*. (Write down the vector equation you need to solve in order to test whether or not $\vec{v}_i$'s are independent or dependent. Write that same vector equation in terms of the $\vec{w}_i$'s. Solve the equation.)

10. True or false: If $\{\vec{x}, \vec{y}\}$ is linearly independent but $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent, then $\vec{z} \in \operatorname{span}\{\vec{x}, \vec{y}\}$. If true, explain why. If false, provide a counterexample (that is, an example to show that the statement is false).