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MATH 141 Linear Analysis Homework H5

1.) a.)

System of equation
$$\begin{cases} -x+y+z=1 \\ 5x-y+z=1 \\ x+y-z=3 \end{cases}$$
matrix form
$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 5 & -1 & 1 & 1 \\ 1 & 1 & -1 & 3 \end{array}\right]$$

Foword Gaussian Elimination:

$$\xrightarrow{R_2+5R_1=R_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & 4 & 6 & 6 \\ 1 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_3+R_1=R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & 4 & 6 & 6 \\ 0 & 2 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2=R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & 4 & 6 & 6 \\ 0 & 0 & -12 & -8 \end{array}\right]$$

• Since in the upper Echelon form it is full rank, we know the system of linear equations meet at one point as shown in the picture of the handout, making my chosen equation valid.

b.)

System of equation
$$\begin{cases} x-y+z=4 \\ x-y-z=-4 \\ -x-y-10z=12 \end{cases}$$
matrix form
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & -1 & -1 & -4 \\ -1 & -1 & -10 & 12 \end{array}\right]$$

Foward Gaussion Elimination:

$$\xrightarrow{R_2-R_1=R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 0 & -2 & -8 \\ -1 & -1 & -10 & 12 \end{array}\right] \xrightarrow{R_3+R_1=R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 0 & -2 & -8 \\ 0 & -2 & -9 & 16 \end{array}\right]$$

$$\xrightarrow{\text{Swap } R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & -2 & -9 & 16 \\ 0 & 0 & -2 & -8 \end{array}\right]$$

• Since in the upper echelon form it is in full rank. We know the system of linear equations meet at one point as shown in the given picture making my chosen equation valid.

2.) a.)

two; planes; three; two; line

b.)

line; two; four; the Column space of matrix A c.)

Plane ; you need to solve the following :

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] = \begin{aligned} x+y+z+w &= 3 \\ y+z+w &= 3 \\ z+w &= 3 \\ w &= 2 \end{aligned}$$

plugging in $w=2$; $x=0,\ y=0,\ z=1,\ w=2$

3.) Consider $\begin{cases} 3x_1 - 2x_2 = b_1 \\ 6x_1 - 4x_2 = b_2 \end{cases}$

a.) $A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

b.) $N(A) \Rightarrow A\vec{x} = 0$ for all possible $\vec{x}$:

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 6 & -4 & 0 \end{array}\right] \xrightarrow{R2-2R1=R2} \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} 3x_1 + 2x_2 &= 0 \\ -\tfrac{3}{2}x_1 &= x_2 \end{aligned}$$

pivot variable x_1, Free variable x_2, back substitution

$x_2 = 1 \Rightarrow 3x_1 + 2 = 0 \quad x_1 = -2/3 \quad x_2 = 1$

$\vec{x} = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$ one possible Solution

$N(A) = \left\{ \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} \right\}$

c.) Columns space is all possible linear Combinations of the columns of matrix A

$C(A) = \text{span}\left\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix}\right\} = x_1\begin{bmatrix} 3 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

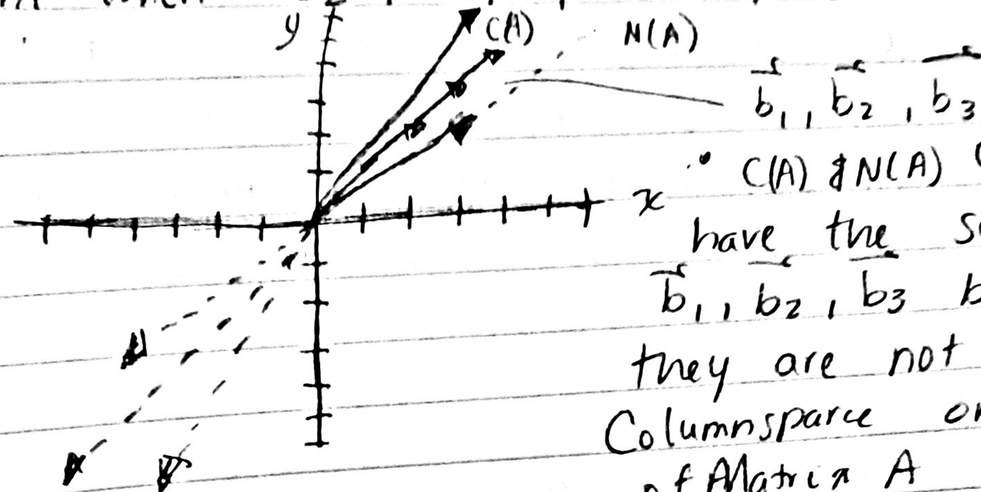
$$\left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 6 & -4 & b_2 \end{array}\right] \xrightarrow{R_2-2R_1=R_2} \left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 0 & 0 & b_2-2b_1 \end{array}\right]$$

$C(A) = \text{Span}\left\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix}\right\} = \left\{x_1\begin{bmatrix} 3 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ s.t. } b_2 - 2b_1 = 0 ;\ b_1, b_2 \in \mathbb{R} \right\}$

d.)

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

inconsistent when $b_2 \neq 2b_1$ & $\frac{3}{2}x_1 \neq x_2$



- C(A) & N(A) Can never have the solutions of $\vec{b}_1, \vec{b}_2, \vec{b}_3$ because they are not in the Columnsparce or nullspace of Matrix A

e.)

$$\vec{b}_4 = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \qquad \vec{b}_5 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

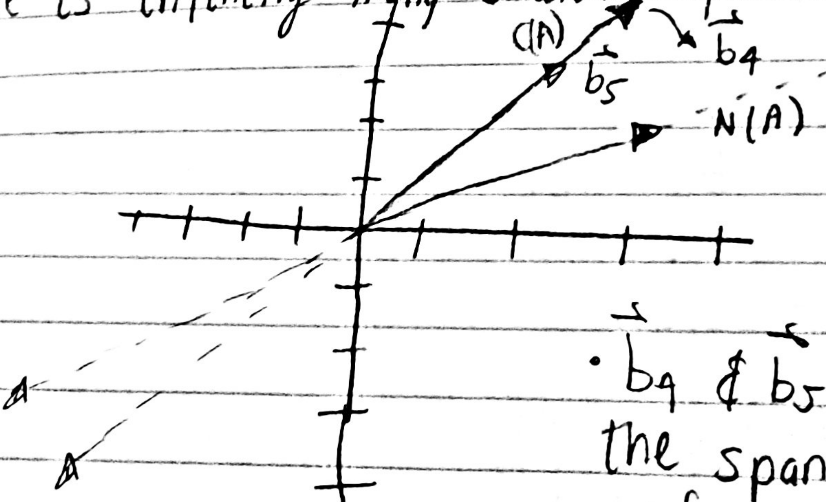
$$x_1 \begin{bmatrix} 3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 3 & -2 & 6 \\ 6 & -4 & 12 \end{array}\right] \xrightarrow{R_2 - 2R_1 = R_2}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 6 \\ 0 & 0 & 0 \end{array}\right] \qquad 3x_1 - 2x_2 = 6$$

Solution is $x_1 = \frac{6 + 2x_2}{3}$ for $\vec{b}_4$

Solution is $x_1 = \frac{2 + 2x_2}{3}$ for $\vec{b}_5$

There is infinetly many solutions for each system.



- $\vec{b}_4$ & $\vec{b}_5$ are within the span of the system of equation f.) i.) has to be in the column space of matrix A

ii.) The span of Matrix A

a.)

4.) • $\vec{x}_0 - \vec{x}_1 \parallel \vec{x}_1 - \vec{x}_0 = \vec{x}$ in $A\vec{x} = \vec{0}$

so $A\vec{x}_0 - A\vec{x}_1 = A(\vec{x}_0 - \vec{x}_1) = \vec{0}$

• you Can write infinetly many solutions as long as $\vec{x}$ is a Linear Combination of matrix A. Any you subtract the two vectors that are L.C. General CASE;

$A(\vec{x}_i - \vec{x}_j) = \vec{0}$ s.t. $\vec{x}_i, \vec{x}_j$ are differnt L.C. of matrix A

b.) $A\vec{x}_0 + A\vec{x}_i = A(x_0 + x_1) = \vec{b}$

- Infinetly many Solutions can be written as long as you have an $\vec{x}$ that already makes the Statement true, you Can add two L.C. of A and give a Solution for $\vec{b}$

c.)

All The Solutions Can be found within the span of A, & you Can find more Solutions from two given $\vec{x}$ for the equation $A\vec{x} = \vec{b}$

5.)

a.)

matrix has rank 2 ; pivot columns = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

Free variable is y

b.)

Conditions are;

$b_3 - 2b_1 = 0$

c.)

Column span

d.)

$\vec{b} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \leftrightarrow \vec{b} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & 3 & 6-3 \\ 0 & 0 & 0 & 6-2(3) \end{array}\right]$

d cont.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2y + 3z = 3$

$3z = 3 \quad z = 1$

$0 = 0$

$x + 2y = 0 \rightarrow x = -2y$

$x = 1,\ y = -1/2,\ z = 1$ One solution

Solution is $z = 1 \quad x = -2y$ for $x, y \in \mathbb{R}$

e.) $N(A) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ pivot variables x, z

free variable y

back substitution ; $3z = 0$

$y = 1,\ z = 0,\quad x + 2(1) + 3(0) = 0$

$x = -2$

One Solution is $x = -2,\ y = 1,\ z = 0$

$$N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

f.) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right] \xrightarrow{b_2 + b_1 = b_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 1 & 2 & 6 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right]$

$\xrightarrow{b_3 + 2b_1 = b_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 1 & 2 & 6 & b_2 \\ 2 & 4 & 6 & b_3 \end{array}\right] = A$

6.) a.)

it does not change the null space of a matrix because in order for us to determine the null space of A we first have to find the pivot variables and free variables in order for us to find them, the matrix has to be in upper echelon form. So this means that the $N(A) = N(U)$ Foward Gaussian elimination is the method used to find null space so no it doesnt change.

b.)

in order to find Column space we have to use foward Gaussion Elimination on the given matrix, so it does not change the column space.

$$C(A) = C(U)$$

7.) a.)

$$A = \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \xrightarrow{R_2 - 2R_1 = R_2} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \xrightarrow{R_3 - 2R_1 = R_3}$$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 3 & 9 & b_4 \end{array}\right] \xrightarrow{R4 - 3R1 = R4} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right] \begin{array}{c} \text{swap} \\ R_4 \leftrightarrow R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 3 & b_4 - 3b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right] \xrightarrow{R_3 - \frac{1}{3}R_2 = R_3} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 3 & b_4 - 3b_1 \\ 0 & 0 & (b_3 - 2b_1) - \frac{1}{3}(b_4 - 3b_1) \\ 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right] \xrightarrow{R_2 - 2R_1 = R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right] \xrightarrow{R_3 - 2R_1 = R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 3 & 9 & 12 & b_4 \end{array}\right] \xrightarrow{R_4 - 3R_1 = R_4} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right]$$

$$\begin{array}{c} \text{Swap} \\ R_4 \leftrightarrow R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] \xrightarrow{R_3 - \frac{1}{3}R_2 = R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 0 & 0 & (b_3 - 2b_1) - \frac{1}{3}(b_4 - 3b_1) \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

{Conditions on next page}

7a cont.)

• Condition for matrix A:

$(b_3 - 2b_1) - \frac{1}{3}(b_4 - 3b_1) = 0$ Simplified $\rightsquigarrow$

$\rightsquigarrow b_3 - b_1 - 1/3 b_4 = 0$

$\rightsquigarrow b_2 - 2b_1 = 0$

• Conditions for matrix B:

$(b_3 - 2b_1) - 1/3 (b_4 - 3b_1) = 0$ simplified $\rightsquigarrow$

$\rightsquigarrow b_3 - b_1 - 1/3 b_4 = 0$

$\rightsquigarrow b_2 - 2b_1 = 0$

b.) We are describing the column space of the matrices. We are solving $A\vec{x} = \vec{b}$ to be true which is the definition of column space.