Due before lecture on Monday, November 25, 2019

1. Find all eigenvectors and eigenvalues of the matrix $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 12 & 2 & 2 \end{bmatrix}$. Write 1-3 sentences that interpret these geometrically—in other words, what do the eigenvectors and eigenvalues tell you about the transformation geometrically (in terms of stretch factors and stretch directions)?

2. Construct an example for each of the following, or explain why such an example does not exist:

   (a) a 2×2 matrix that is invertible but not diagonalizable

   (b) a 2×2 non-diagonal matrix that is diagonalizable but not invertible

3. (*Strang §5.2 #11*) If all eigenvalues of A are 1, 1, and 2, which of the following are certain to be true? Give a reason if true or a counterexample if false.

   (a) A is invertible.

   (b) A is diagonalizable.

   (c) A is not diagonalizable.

4. (*Strang §5.2 #12*) Suppose the only eigenvectors of A are multiples of $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which of the following are certain to be true? Give a reason if true or a counterexample if false.

   (a) A is not invertible.

   (b) A has a repeated eigenvalue.

   (c) A is not diagonalizable.

5. (*Strang §5.1 #18*) Suppose a 3×3 matrix A has eigenvalues 0, 3, and 5 with associated eigenvectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ respectively.

   (a) Since the eigenvalues of A are all distinct, the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is ________________.

   (b) Write down a basis for the nullspace $N(A)$ and the column space $C(A)$.

   (c) Find one particular solution to $A\vec{x} = \vec{v} + \vec{w}$. Find all solutions to $A\vec{x} = \vec{v} + \vec{w}$.

   (d) Explain why $A\vec{x} = \vec{u}$ does not have a solution. (Hint: If there is a solution, then _______ is in $C(A)$. Explain why that is impossible.)

   (e) Is A invertible? Why or why not?

6. Let A be an n-by-n matrix. Suppose $A\vec{u} = 2\vec{u}$ and $A\vec{v} = 5\vec{v}$ for nonzero vectors $\vec{u}$ and $\vec{v}$. Complete the following proof that $\{\vec{u}, \vec{v}\}$ is linearly independent.

   Proof: In order for $\{\vec{u}, \vec{v}\}$ to be linearly independent, we need to show that

   the only solution to the vector equation $\quad x\vec{u} + y\vec{v} = \vec{0} \quad$ *is the trivial one.*

   Note that in this equation, $\vec{u}$ and $\vec{v}$ are known vectors, while x and y are unknown scalars. Now assume that $x = a$ and $y = b$ is some solution to the above equation. That is

$$a\vec{u} + b\vec{v} = \vec{0}. \tag{1}$$

Multiply both sides of equation (1) by matrix A from the left. Show your calculation details to explain why the following has to be true as well

$$2a\vec{u} + 5b\vec{v} = \vec{0}. \tag{2}$$

Explain in detail how combing vectors equations (1) and (2) leads to the conclusion that $a = 0 = b$. Therefore, the only solution to $x\vec{u} + y\vec{v} = \vec{0}$ is the trivial solution.

7. In one of the reflection questions, you saw that if A is 2×2 matrix then the product of its eigenvalues is equal to $\det(A)$ and the sum is equal to $\operatorname{trace}(A)$. The following shows that both claims still hold true for n×n matrices.

   Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues of an n×n matrix A. λ_i's are the roots of the polynomial $\det(A - \lambda I)$, which means that we have a factorization

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \tag{3}$$

   (a) (*Strang §5.1 #8*) By making a clever choice of the value for λ in equation (3), show that $\det(A)$ is equal to the product of eigenvalues.

   (b) (*Strang §5.1 #9*) Show that $\operatorname{trace}(A)$ is equal to the sum of eigenvalues in three steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the righthand side of equation (3). Next, find all terms on the righthand side of the following that involves $(-\lambda)^{n-1}$

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix},$$

   where a_{ij} are the entries of A. Add up all those terms to find the coefficient of $(-\lambda)^{n-1}$. Lastly, compare the coefficient of $(-\lambda)^{n-1}$ found in these two different ways.