MATH 191 Linear Analysis Homework #9

a.) • As we have solved the relationship of $\ker(T)$ is the same as finding the Null space in a previous homework. Yes $\ker(T)$ is a subspace for R^m. This is because for any $\vec{x} \in \ker(T)$ and $\vec{y} \in \ker(T)$, $T\vec{x} = 0 = T\vec{y}$

Also: $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0} + \vec{0} = \vec{0}$ for $\vec{x}+\vec{y} \in \ker(T)$

AND: For any $\vec{x} \in \ker(T)$ Scalar c $T(c\vec{x}) = cT(\vec{x})$ $= c\vec{0} = \vec{0}$ That is $c\vec{x} \in N(A)$

• To find a basis find all special solutions to $\ker(T)$

basis of $\ker(T)$ = span{ special Solutions }

b.) • As solved before in a previous homework range(T) is the same as Column space of A. As proved in The worksheet Basis for C(A); Since C(A) is a subspace and range(T) = Column space, range(T) is a subspace of R^m.

• To find a basis we have to find the pivot columns of the matrix T, those pivot columns make the basis of range(T).

2.)

One Concrete vector in $N(A)$:

$$\left\{ \begin{bmatrix} 0 \\ -5 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

This is because, This vector when matrix multiplied will give us column 3 and a negative version of column 3. manipulating the equation given to us, we know

$-5(\text{column } 2) + (\text{column-}4) - (\text{column-}3) = 0$

taking the coefficients and putting them on the respective $\vec{x}$ in $A\vec{x} = \vec{0}$ Then we get $\vec{0}$ as our solution as long as the other two columns are 0

3.) NO, V is not a subspace of R^m, in order for A subset to be a subspace it has to cross the origin, to do this we have to prove the $\vec{0}$ is in the subset. Since we have the Constraint $\vec{0} \neq b$, the $\vec{0}$ is not in the subspace so V is not a subspace

4.) a.)

i.) $f(g(x)) = 2(x^2 - 3x) + 4$

$g(f(x)) = (2x+4)^2 - 3(2x+4)$

ii.) $f(g(2)) = 4 - 6 = -2 \to f(-2) = \boxed{0}$

$g(f(2)) = 64 - 24 = \boxed{40}$

b.)

i.) $G\vec{x} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+8-1 \\ 0+6+2 \\ 5+0-1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 4 \end{bmatrix}$

$F\vec{y} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 9+0+8 \\ 18-8+0 \\ 0+24+16 \end{bmatrix} = \begin{bmatrix} 17 \\ 10 \\ 40 \end{bmatrix}$

ii.) $F\vec{x} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0-2 \\ 2-2+0 \\ 0+6-4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$G\vec{u} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 0+0-4 \\ -5+0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -3 \end{bmatrix}$

iii.)

$FG = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 3 \\ 4 & 5 & 4 \\ 20 & 9 & 2 \end{bmatrix}$

$GF = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 8 \\ 6 & -9 & -8 \\ 5 & 3 & 14 \end{bmatrix}$

4.) c.)

Composition of functions have the same rules as Matrix multiplication. There are both the same process in finding the resultant of multiple equation. Order matters for both or else they have different results

d.)

No matrix multiplication is not communitive because order in which you multiply matters, or else you get a different result

5.)

We have to make the column of matrix A equal to the rows of matrix B;

$$\underbrace{A_{m \times n} \cdot B_{m \times n}}_{=}$$ The new matrix will be the row of Matrix A and the column of matrix B

$$A_{n \times n} \cdot \underbrace{B_{m \times n}}_{\text{New matrix}} \quad m \times n$$

6.) a.) $AB = \underset{2\times 2}{\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}} \underset{2 \times 3}{\begin{bmatrix} 6 & -2 & 0 \\ 3 & 3 & 3 \end{bmatrix}} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 4 & 4 \end{bmatrix}$

- $\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ This is because $\vec{b}$ is in the Spa of $\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ Each column of product AB Can be found within the span of $\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$

b.)

Since 2 linearly independent vectors exist in both A & B then $C(A) = C(B)$, but since AB is within the column space of $C(A)$ then; $(C(A) = C(B)) \geq C(AB)$

7.) a.)

True, Since B columns are linearly dependent then $N(B)$ has solutions that are not the trivial solution. for $AB\vec{x} = 0$ we can still find a non trivial solution

EX1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ If we multiply by a linearly dependent set the out come will be a linearly dependent set.

b.)

False, as seen above if A is linearly independent, but B is linearly dependent the result will always be a linearly dependent matrix Ex2:

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 4 & 8 \end{bmatrix}$$

linearly dependent

8.) a.)

True, We are multiplying matrix A with the columns of B, so if 1 & 3 column of B is the same the columns of AB is the same as A time colum 1 & A times column 3.

b.)

False, they don't have to be square matrixes (A & B) Counter Example:

$$A_{3\times2} \quad B_{2\times3} = AB_{3\times3}$$

$$B_{2\times3} \quad A_{3\times2} = AB_{2\times2}$$

A & B are not square matrixes.

c.)

True, in order for AB & BA to exist then row of A = column of B & column of A = row of B so the result will always be a square matrix

8.) d.)

False, Counter Example

$A = \begin{bmatrix} 2 & 7 \\ 3 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 23 & 11 \\ 12 & 9 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 661 & 352 \\ 384 & 213 \end{bmatrix}$ $A^2 = \begin{bmatrix} 23 & 35 \\ 15 & 30 \end{bmatrix}$ $B^2 = \begin{bmatrix} 7 & 4 \\ 6 & 9 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 385 & 345 \\ 285 & 270 \end{bmatrix} \neq (AB)^2 \begin{bmatrix} 661 & 352 \\ 384 & 213 \end{bmatrix}$

e.)

False, Counter example

$A = \begin{bmatrix} 2 & 7 \\ 3 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $A + B = \begin{bmatrix} 3 & 9 \\ 6 & 4 \end{bmatrix}$ $(A+B)^2 = \begin{bmatrix} 63 & 63 \\ 42 & 70 \end{bmatrix}$

$A^2 = \begin{bmatrix} 23 & 35 \\ 15 & 30 \end{bmatrix}$ $B^2 = \begin{bmatrix} 7 & 4 \\ 6 & 9 \end{bmatrix}$ $A^2 + B^2 = \begin{bmatrix} 30 & 39 \\ 21 & 39 \end{bmatrix}$

$AB = \begin{bmatrix} 23 & 11 \\ 12 & 9 \end{bmatrix}$ $2AB = \begin{bmatrix} 46 & 22 \\ 24 & 18 \end{bmatrix}$ $A^2 + 2AB + B^2 = \begin{bmatrix} 76 & 61 \\ 45 & 57 \end{bmatrix}$

$(A+B)^2 = \begin{bmatrix} 63 & 63 \\ 42 & 70 \end{bmatrix} \neq A^2 + 2AB + B^2 \begin{bmatrix} 76 & 61 \\ 45 & 57 \end{bmatrix}$

f.) False Counter example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

9.) A.) $A(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $V_1 x_1 + V_2 x_2 + V_3 x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

B.) $A\vec{x} = \vec{b}$ $A\vec{x} = \begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ b_3 \end{bmatrix}$ row 1 + row 2 = row 3

inorder to make this true we need $\vec{b_1} + \vec{b_2} = \vec{b_3}$

C.) Since row 3 is a linear combination of row 1 & row 2 if foward Gaussian elimination is preformed it will turn to zero.

d.) In order to be invertable the transformation must match all ($R^m \to R^n$) n for the given m. If not then the inverse does not exist

10.) a.)

True, Since in order to be invertable the span has to be all of $\mathbb{R}^m$ so if linear transformation doesn't span all of R^m as well it cannot be invertable. So all pivot points must exist in both matrixes before & after transformation.

b.)

False, if two rows are equal even with the diggonals equalling to 1 it will not be invertable Counter Example!

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1 = R_3} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

C.)

True, because $(A^{-1})^T = (A^T)^{-1}$