Due before lecture on Monday, September 30, 2019

1. For pictures labeled "a" and "d" inn the exercise titled "Intersection Of Three Planes" done in lecture, write down a 3×3 system (that is, a system of 3 equations with 3 unknowns) that corresponds to that picture. Explain why you think your system is valid. In other words, use sentences, math notations and calculations, etc. (whatever is necessary), to show that the picture representing the system you wrote down indeed looks like the given picture.

2. Complete the following sentences about linear systems of equations by either choosing the correct words or filling in the blanks. Answer the question in part (c) too.

   (a) (*Strang* §*1.2 #14*) For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a _____.

   (b) (*Strang* §*1.2 #15*) For four linear equations in two unkwons x and y, the row picture shows four ____ in ____-dimensional space. The column picture is in ____-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of ___________.

   (c) (*Strang* §*1.2 #20 adapted*) Normally 4 "hyperplanes" in four-dimensional space meet at a _____. In order to determine what linear combination of $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ produces $\vec{b} = \begin{bmatrix}3\\3\\3\\2\end{bmatrix}$, which system of equations for x, y, z, t do you need to solve? Solve that system.

3. (*based on Strang* §*1.3 #9*) Consider the following system of equations.

$$\begin{aligned}3x_1 - 2x_2 &= b_1\\ 6x_1 - 4x_2 &= b_2\end{aligned}$$

   You will sketch ***two*** different graphs. You need relatively accurate graphs in order to answer part (f) which is the main point of this whole exercise.

   (a) If we write the above system using matrix notations $A\vec{x} = \vec{b}$, what is the coefficient matrix A, what is $\vec{x}$, and what is $\vec{b}$?

   (b) What is the definition of the nullspace of a given matrix? Describe the nullspace $N(A)$ of this particular A as a span.

   (c) What is the definition of the column space of a given matrix? Describe the column space $C(A)$ of this particular A by describing the condition(s) that the components of each vector in $C(A)$ have to satisfy.

   (d) Choose three concrete right-hand vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ for which the system is inconsistent. On your ***first graph***, sketch $N(A)$, $C(A)$, and $\vec{b}_1, \vec{b}_2, \vec{b}_3$.

   (e) Choose two concrete non-zero right-hand vectors $\vec{b}_4, \vec{b}_5$ for which the system is consistent. Find all solutions to the two systems for with $\vec{b}_4$ and $\vec{b}_5$ respectively. How many solutions are there for each system? On your ***second graph***, sketch $N(A)$, $C(A)$, $\vec{b}_4, \vec{b}_5$, and all solutions corresponding to $\vec{b}_4$ and $\vec{b}_5$.

   (f) Study your two graphs and notice any special relationship(s) among various objects. Complete the following sentences.

      i. In order for $A\vec{x} = \vec{b}$ to have a solution, the right-hand vectors $\vec{b}$ _______________.

      ii. In the case that $A\vec{x} = \vec{b}$ is solvable, all its solutions form _______________________.

4. (*cf. Strang* §*1.3 #18*) Suppose that $\vec{x}_0$ and $\vec{x}_1$ are two different solutions to the same linear system of equations $A\vec{x} = \vec{b}$, where A is the coefficient matrix, $\vec{x}$ the unknown, and $\vec{b}$ may or may not be the zero vector.

   (a) Write down, in terms of $\vec{x}_0$ and $\vec{x}_1$, a solution to $A\vec{x} = \vec{0}$. How many other solutions to $A\vec{x} = \vec{0}$ can you also write down?

   (b) Write down, in terms of $\vec{x}_0$ and $\vec{x}_1$, a third solution to $A\vec{x} = \vec{b}$. How many other solutions to $A\vec{x} = \vec{b}$ can you also write down?

   (c) (*my own*) Explain how this problem is related to the second sentence you need to complete in part (f) of problem 3.

5. A student is trying to figure out for what right-hand vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ a linear system of equations $A\vec{x} = \vec{b}$ is solvable. After she performed Gaussian elimination, the result, not the original augmented matrix, looks like

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right]$$

   (a) What is the rank of the matrix A? Which columns are pivot columns? Which variables are free variables?

   (b) What condition(s) do b_1, b_2, b_3 have to satisfy to make the system solvable?

   (c) According to the first sentence you need to complete in part (f) of problem 3, by describing what conditions b_1, b_2, b_3 have to satisfy for the system to be solvable, we are actually describing ________________ of the coefficient matrix A.

   (d) Construct a concrete $\vec{b}$ satisfying your conditions in part (b). Solve the system.

   (e) Find $N(A)$.

   (f) Note that we do not need to know what A is in order to answer all previous questions. However, we can find A. Find A.

6. (a) Does the Gaussian elimination process change the nullspace of a matrix? In order words, if we perform Gaussian elimination on a matrix A to get its upper echelon form U, are $N(A)$ and $N(U)$ different? Why or why not?

   (b) Repeat the discussion above for column space.

7. (a) (*Strang* §*2.2 #35*) What conditions on b_1, b_2, b_3, b_4 make each system solvable? Solve for $\vec{x}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

   (b) By describe what conditions b_1, b_2, b_3, b_4 have to satisfy so that the system is solvable, we are actually describing ________________ of the coefficient matrix. Explain why.

8. (***postponee to next week***) (*Strang* §*2.2 #39*) Explain why all these statements are all false (all statements are about solving linear systems $A\vec{x} = \vec{b}$):

   (a) The complete solution is any linear combination of $\vec{x}_{\text{particular}}$ and $\vec{x}_{\textbf{nullspace}}$.

(b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.

(c) The solution $\vec{x}_{\text{particular}}$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). (Find a 2×2 counterexample.)

(d) If A is invertible there is no solution $\vec{x}_{\text{nullspace}}$ in the nullspace. (Lei Yue's comment: you do not even need to know what it means to say a matrix is invertible.)