Automata Theory and Computability

Assignment 4 (CFGs, PDAs and Turing Machines)

(Total marks 75. Due on Thu 27th Mar 2025)

1. Consider the language $L = \{a^n b^{n^2} \mid n \geq 0\}$. Use Parikh's theorem or the Pumping Lemma for CFLs to show that L is not a CFL. (*10*)

2. Consider the CFG G below:

$$\begin{array}{lcl} S & \rightarrow & XC \mid AY \\ X & \rightarrow & aXb \mid ab \\ Y & \rightarrow & bYc \mid bc \\ A & \rightarrow & aA \mid a \\ C & \rightarrow & cC \mid c \end{array}$$

   (a) Describe the language accepted by G. (*5*)

   (b) Use the construction in Parikh's theorem to construct a semi-linear expression for $\psi(L(G))$. That is

      i. Identify the basic pumps for G. (*5*)

      ii. Identify the $\leq$-minimal parse trees. (*5*)

      iii. Use these to obtain an expression for $\psi(L(G))$. (*5*)

   (c) Use the semi-linear expression obtained above to give a regular expression that is letter-equivalent to $L(G)$. (*5*)

3. Give a PDA that accepts the language (*10*)

$$\{a, b\}^* - \{ww \mid w \in \{a, b\}^*\}.$$

4. Is the class of context-free languages closed under the prefix operation? Justify your answer. (*10*)

5. Consider a PDA with a single stack symbol say "1" (apart from the usual bottom-of-stack symbol "$\perp$"). This is also called a one-counter machine. Argue that a one-counter machine has less power (in terms of language recognition) than a full PDA. An informal argument is sufficient. (*10*)

6. Show that the following integer division function "*div*" is computable by a Turing Machine in the sense discussed in class. Give a complete description of the moves of the TM. (*10*)

   $div : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, where $div(m, n)$ is the largest integer less than or equal to m/n if $n > 0$, and 0 otherwise.