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UMC-205

Assignment No-02

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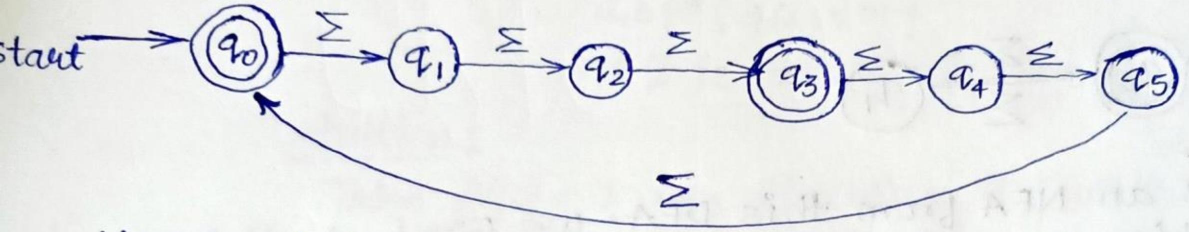
Question No. 1

Let L be a regular language over the alphabet A. Which of the following languages over A are regular? Justify your Answer.

(a) mid-thirds(L) = $\{v \mid \exists u, w : |u| = |v| = |w| \text{ and } uvw \in L\}$.

First we construct a DFA whose string with length divisible by 3 by taking the intersection of DFA for L with the following:



Now we do the following to make the NFA. First we no. out the states, giving label q_0 to s and rest of the states number in range i in Q. Then keep a white pebble on the start state and one blue and black coloured pebble on each state. Thenstring x. we will duplicate the white coloured pebble as we move on. For a symbol a in x: white pebble goes to the possible to the possible states it can transition to, the blue coloured pebbles moves according to DFA rule and black coloured pebbles move like white pebbles. After have parsed the whole string x, we need to make a set of the no. on which the white pebble are, let this set be W and then for each i in W, we make another set B that contains the no. of the states at which blue pebble of q_i is located. Then for f in B, if q_f is a final state we can say that x belongs to mid-third (L) and Hence this language is also regular. Construct $Q' = Q \times Q \times Q$

Therefore

$$\Delta'((S_1, S_2, S_3), x) = (g(S_1, x), \delta(S_2, x), g(S_3, x))$$

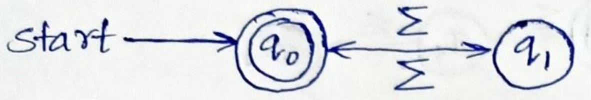
$$g(S, x) = \bigcup x \mid \delta(x, \Sigma) = S$$

$F'(S_1, S_2, S_3)$ if there is k in S1, such that q_K in S2. is on position j such that $q_j \in F$.

(b) The language

$$half(L) = \{w \mid ww \in L\}.$$

we can prove that regular by taking an NFA for this language. For the NFA, we can get the corresponding DFA using subset. First we move string to odd length from L. we make intersection of DFA for L with DFA that only accepts even length strings. Here $a \in \Sigma$



we now make an NFA from this DFA. we First of all keep numbered pebbles on each state and assign the same no. to that state. We keep a white pebble on the start state of DFA that is unique and for each other state we keep a no. black pebble. Now for each pebble we traverse a string x the same way we do for a DFA, then after each pebble is traversed to its state according to $\delta(x)$, then we make a set of all no. of pebbles that are in final state of the DFA and if the no. of state at which the white pebble is in this set, we can say that $x \in half(L)$.

$Q' = \{q_j, i \mid j, i \in [|Q|]\}$ where q_0 is S and all other states are assigned a no.

$s' = Q'$

$\delta'(S, x) = \bigcup_{s \in S} \delta(s, x)$

A string x is accepted if $\hat{\delta}(q_0, x)$ in G for $G = \{i : q_i \text{ in } F\}$.

Question No- 02

Describe the equivalence classes of the Myhill- Nerode relation $\equiv_L$ for the language $L = \{x \in \{a,b\}^* \mid \#_a(x) = \#_b(x)\}$. Depict the canonical DA for this language.

Recall: The tree notation for depictions the free DA for L:

This accepts $L = \{x \in \{a,b\}^* \mid \#_a(x) = \#_b(x)\}$.

Let $d(w) = \#_a(x) - \#_b(x)$.

$A_n = \{x \in \{a,b\}^* \mid d(x) = n\}$

$n \in \mathbb{Z}$ is the description of equivalence classes.

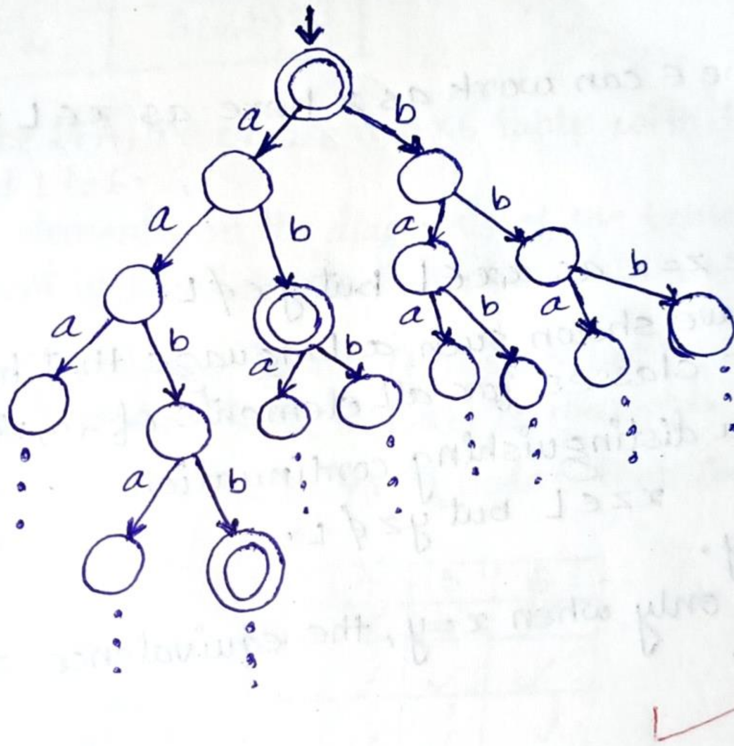
Intuition:

Take $d(w) = 1$ i.e. $\#_a(w) = \#_b(w) + 1$

Ex:- aab aba

If $z = b$ it is accepted

$z = ab$ it is not.



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Question No-03

Find a language L for which the canonical Myhill-Nerode relation for, $\equiv_L$, has all singleton equivalence classes.

$x \sim_L y \Leftrightarrow \forall z \in \{0,1\}^*, (xz \in L \Leftrightarrow yz \in L)$

we will show an example of a language L over the alphabet $\{0,1\}$ for which the MN relation. Has very strong property that every equivalence classes are singleton, that means for all x and $y \in \{0,1\}$, $\exists z \in \{0,1\}^*$ such that exactly one of xz or yz is in L.

So for each case we need to find such z. Consider $L = \{ww : w \in \{0,1\}^*\}$

Cases : → $x, y \notin L$

- Since $x \neq y$, we can say that $xx \in L$.
  Hence $y \notin |x|$ and vice versa.

→ $x \in L\ y \notin L$

- we can say the ϵ can work as z here as $x \in L$.
  but $y \notin L$.

→ $x, y \in L$

- we can take $z = x$ as $xx \in L$ but $yx \notin L$.
  Hence we have shown such a language that has Singleton equivalence classes for all elements of $\{0,1\}^*$

Thus, $z = x$ is a distinguishing continuation

$xz \in L$ but $yz \notin L$.

So, $\not\equiv_L y$.

Since $x \equiv_L y$ only when $x = y$, the equivalence classes of $\equiv_L$ are all singletons.

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Question No. 4

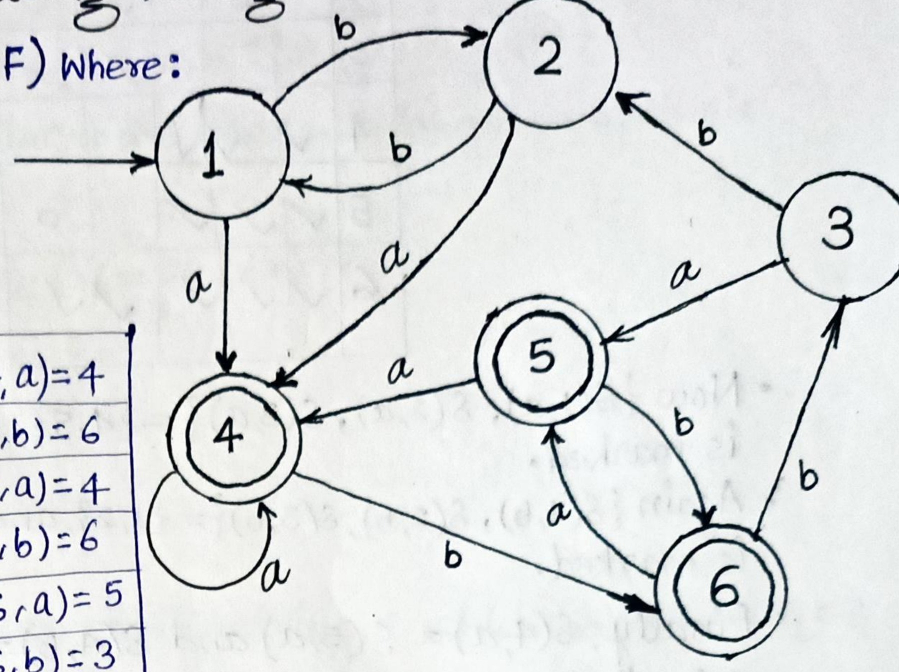
→ Minimize the DFA below using the algorithm done in class:

Let the DFA be $A = (Q, S, \delta, F)$ where:

$Q = \{1, 2, 3, 4, 5, 6\}$

$S = 1$

$F = \{4, 5, 6\}$ then,



The 'δ' Function is given by

$\delta(1,a) = 4$	$\delta(4,a) = 4$
$\delta(1,b) = 2$	$\delta(4,b) = 6$
$\delta(2,a) = 4$	$\delta(5,a) = 4$
$\delta(2,b) = 1$	$\delta(5,b) = 6$
$\delta(3,a) = 5$	$\delta(6,a) = 5$
$\delta(3,b) = 2$	$\delta(6,b) = 3$

To minimize the DFA, we create a 6X6 table with the rows and Columns labelled 1 to 6.

(i) Ignore all the elements on the diagonals of the table by setting them

(ii) Mark all element of (P,q), where one of them is a final state and other is not.

(iii) Now, iteratively mark the states (P,q) such that (P,q) is unmarked and $\exists a \in A$ such that $(\delta(P,a), \delta(q,a))$ is marked.

→ We start with marking all pairs of nodes that contain a final state and a non-final state.

	1	2	3	4	5	6
1				✓	✓	✓
2				✓	✓	✓
3				✓	✓	✓
4	✓	✓	✓			
5	✓	✓	✓			
6	✓	✓	✓			

Now $\delta(4,6) = \delta(5,b) = 6$, but $\delta(6,b) = 3$. Thus we can mark $(4,6)$ and $(5,6)$.

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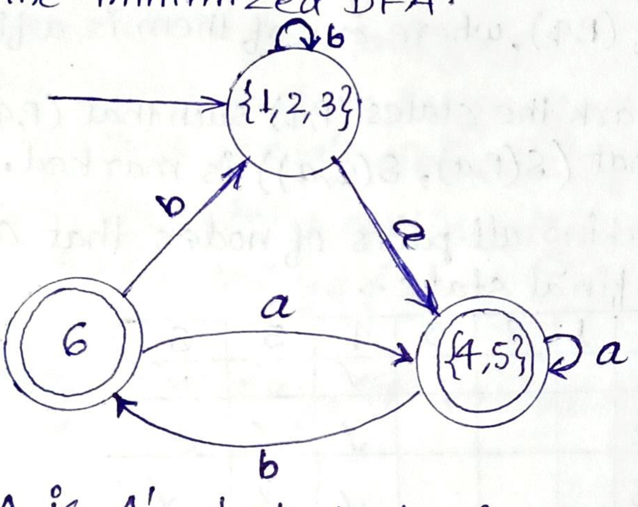
	1	2	3	4	5	6
1	o			✓	✓	✓
2		o		✓	✓	✓
3			o	✓	✓	✓
4	✓	✓	✓	o		✓
5	✓	✓	✓		o	✓
6	✓	✓	✓	✓	✓	o

- Now $\{\delta(1,a), \delta(2,a), \delta(3,a)\} = \{4,5\}$, but no pair from $\{4,5\}$ is marked.
- Again $\{\delta(1,b), \delta(2,b), \delta(3,b)\} = \{1,2\}$, and again no pair from $\{1,2\}$ is marked.
- Finally, $\delta(4,a) = \delta(5,a)$ and $\delta(4,b) = \delta(5,b)$, but obviously no Pair (q,q) is ever marked.

Thus there are no more pairs to mark, and we get equivalences classes. $\{1,2,3\}$ $\{4,5\}$ $\{6\}$.

This gives the minimized DFA.



Now the DFA is $A' = \{Q', s', \delta', F'\}$ where:

$Q' = \{\{1,2,3\}, \{4,5\}, 6\}$

$s' = \{1,2,3\}$

$F' = \{\{4,5\}, 6\}$

The δ' function is given by:

$\delta'(\{1,2,3\}, a) = \{4,5\}$

$\delta'(\{1,2,3\}, b) = \{1,2,3\}$

$\delta'(\{4,5\}, a) = \{4,5\}$

$\delta'(\{4,5\}, b) = 6.$

$\delta'(6, a) = \{4,5\}$

$\delta'(6, b) = \{1,2,3\}.$

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Question No-05

Give a Context-free grammar for the following language. Prove that your grammar is Correct:

"Equal a's and b's" – i.e. $\{x \in \{a,b\}^* \mid \#_a(x) = \#_b(x)\}$.

Hint: Give a grammar similar to the one for balanced parenthesis

The language L is defined as:

$$L = \{x \in \{a,b\}^* \mid \#_a(x) = \#_b(x)\}.$$

Let the context-free grammar G be:

$$S \rightarrow aSb \mid bSa \mid SS \mid \epsilon$$

Proof: $L(G) \subseteq L$.

By using induction

Base Case: $S \rightarrow \epsilon$ generates empty string ϵ which has $\#a = \#b = 0$

We Assume $\alpha S \beta$ has $\#a = \#b$

- If $S \rightarrow aSb$ then $\alpha S \beta \rightarrow \alpha aSb \beta$ Where

  $\#a' = \#a+1$

  $\#b' = \#b+1$

  $\#a' = \#b'$

- If $S \rightarrow bSa$, then $\alpha S \beta \longrightarrow \alpha bSa \beta$

  $\#a'' = \#a+1$

  $\#b'' = \#b+1$

  $\#a'' = \#b''$

- If $S \rightarrow SS$, then still $\#a = \#b$

Hence By induction $L(G)$ gives strings only in L.

Therefore,

$$L(G) \subseteq L$$

Part (b)

Proof: $L \subseteq L(G)$

We prove by using induction on no. of a's in the string.

if $n=0$, $w=\epsilon$ we can derived by CFG.

Let us Assume that every balanced parenthesis string of length $\leq 2n$ $(\#a = \#b)$ can be $\#a = \#b \leq n$

- ~~By~~ Derived by Context given Grammar.

If w has no. of a's $= n+1 = \#b$

Case (i) w starts with 'a' & ends with 'b' by the removing of 1st 'a' & last ... leaves w'.

Therefore

$\#a(w') = \#_b(w')$ [it has n a's & b's W can be obtained by aSb.]

Hence by induction hypothesis we derived w by using G

Case (ii) W starts with b & ends with a

Similar to above argument

W can be obtained by $S \rightarrow bSa$ & then the w' can be derived

Case (iii) splitting w into w_1 and w_2.

Such that $\#a(w_1) = \#_b(w_2) < n$

$\#a(w_2) = \#_b(w_2) < n$

By inductive step both w_1 & w_2 can be derived so that

Start with $S \rightarrow SS$ & derive w_1 & w_2

Hence w can be derived. so by induction $L \subseteq L(G)$

Therefore,

$L \subseteq L(G)$ & $L(G) \subseteq L$ (by previous proof)

$\Rightarrow L = L(G)$

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Question No-6

Give a Context grammar for the language
$$L_2 = a^* b^* c^* - \{a^n b^n c^n \mid n \geq 0\}.$$

$L_2 \rightarrow$ represents the set of strings in $a^* b^* c^*$ such that no. of a's, b's and c's are equal. Let $x \in a^* b^* c^*$

Case(i) $\#_a(x) > \#_b(x)$ & $\#_c(x) \geq 0$.

Clearly $x \in L_2$.

we generate x !

$x \rightarrow PD$
$P \rightarrow aPb \mid A$
$A \rightarrow a \mid aA$
$D \rightarrow CD \mid \varepsilon$.

Case(ii) $\#_a(x) < \#_b(x)$ & $\#_c(x) \geq 0$.

$X \rightarrow QD$
$Q \rightarrow aQb \mid B$
$B \rightarrow bB \mid b$
$D \rightarrow CD \mid \epsilon$

Case(iii) $\#_b(x) > \#_c(x)$, $\#_a(x) \geq 0$

$Y \rightarrow ER$
$R \rightarrow bRC \mid B$
$B \rightarrow bB \mid b$
$E \rightarrow aE \mid \epsilon$

Case(iv) $\#_b(x) < \#_c(x)$, $\#_a(x) \geq 0$.

$Z \rightarrow ES$
$S \rightarrow bSC \mid C$
$C \rightarrow cC \mid c$
$E \rightarrow aE \mid \varepsilon$

we have 2 remaining cases i.e. $\#_a(x) \neq \#_b(x)$ or $\#_b(x)$ is arbitrary. we claim these 2 are covered already in the 4 cases so far.

Proof: Let $\#_a(x) > \#_c(x)$ By WLOG. Now we have three cases,

C1: $\#_b(x) < \#_c(x) < \#_a(x)$ Covered in case (i)

C2: $\#_b(x) > \#_a(x) > \#_c(x)$ Covered in " (ii)

C3: $\#_a(x) > \#_b(x) > \#_c(x)$ covered in

We have P:

$T \rightarrow W \mid X \mid Y \mid Z$
$W \rightarrow PD$
$X \rightarrow QD$

$Y \rightarrow ER$
$Z \rightarrow ES$ then,

$P \rightarrow aPb \mid A$
$Q \rightarrow aQb \mid B$
$R \rightarrow bRC \mid B$
$S \rightarrow bSC \mid C$
$A \rightarrow aA \mid a$
$B \rightarrow bB \mid b$
$C \rightarrow cC \mid c$
$D \rightarrow cD \mid \varepsilon$
$E \rightarrow aE \mid \epsilon$

The CFG generated is $G = (N, A, S, P)$ where:

$N = \{T, W, X, Y, Z, P, Q, R, S, A, B, C, D, E\}$
$A = \{a, b, c\}$
$S = T.$

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stion No-7

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ive an equivalent grammar in chomsky Normal Form for the following CFG:

$$S \to aSbb \mid T,$$
$$T \to bTaa \mid S \mid \epsilon.$$

Let $G = \{N, A, S, P\}$ then therefore P:-

$S \to aSbb \mid T,$
$T \to bTaa \mid S \mid \epsilon.$

1.) Create a new set of production by adding all original productions:

P':-

$S \to aSbb \mid T$
$T \to bTaa \mid S \mid \epsilon.$

2.) Let $A \to a$
$B \to b$
$A' \to aa$
$B' \to bb$

$\}$ we will add them to P'

3.) we have

$T \to BTA'$ & $T \to \epsilon$ $\Rightarrow$ Add the production $T \to BA'$

$S \to T$ & $T \to \epsilon$ $\Rightarrow$ $S \to \epsilon$

$S \to ASB'$ & $S \to \epsilon$ $\Rightarrow$ $S \to AB'$

As $S \to T$, add all productions of T in S and as $T \to S$ replace all instance of T with S.

we have:

$S \to ASB' \mid S \mid \epsilon \mid AB' \mid BA' \mid BSA'$
$A' \to AA$
$B' \to BB$
$B \to b$
$A \to a$

4.) Add $X \to AS$, $Y \to BS$ to the productions.

Then P':

$S \to XB' \mid S \mid \epsilon \mid AB' \mid BA' \mid YA'$
$X \to AS$
$Y \to BS$
$A' \to AA$
$B' \to BB$
$A \to a$
$B \to b.$

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5.) In order to convert into CFG, drop all ϵ & unit productions we have:

$S \rightarrow XB' \mid AB' \mid BA' \mid YA'$

$X \rightarrow AS$

$Y \rightarrow BS$

$A' \rightarrow AA$

$B' \rightarrow BB$

$A \rightarrow a$

$B \rightarrow b$

where Return $G' = (N', A', S', P')$ then:

$N' = \{S, X, Y, A', B', A, B\}$

$A' = \{a, b\}$

$S' = S$.

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