Computational & Data Sciences, Indian Institute of Science, Bengaluru 560012
DS290. Modeling and Simulation. Aug-Dec 2025
MidTerm Examination 1. September 2025. Time allowed: 90 minutes. Full Marks: 25.

Please use your own A4 size sheets to submit the answers. Please write your name and SR number on the top at the beginning of your answer sheet. You can submit the answers digitally at the end of the exam on the TEAMS Assignments. A Queuing theory summary of results is appended to the question paper.

1. (a) If a discrete random variable counts Bernoulli trials as its event values and has the memoryless property, then derive its probability mass function *uniquely* regardless of whether the random variable counts the trials needed to get one success or counts the number of failures before the first success. (2 marks)

(b) If a continuous random variable has the memoryless property, then derive its probability density function uniquely. (2 marks)

(c) Let X be distributed normally as $N(\mu,\sigma^2)$. If $f \in C_b^1$ is an at least once differentiable continuous function with bounded values and $f:\mathbb{R}\rightarrow\mathbb{R}$, then find the expectation $E(\sigma^2 \frac{d}{dX} f(X) - (X-\mu)(f(X) - X + \mu))$. (1 mark)

2. Let X be a continuous random variable taking real values in [0,3] and is distributed as $f(x)=(x-floor(x))^2$. If we use $g(x)=\frac{2}{3}(x-floor(x))$ as a simpler distribution to generate the random numbers distributed as *f(x)*, then,
(a) What would be an algorithm for it, which uses the inverse transform method for the simpler proposal distribution and an acceptance-rejection for the target distribution *f(x) ?* (2 marks)
(b) What is the acceptance rate of the random variates for the target distribution ? (1 mark)

3. For a linear congruential generator with $c=0$ and $X_{i+1}=(a X_i + c)$ *modulo* m, find K_1 as a function of *n and a* such that $X_{i+n}=K_1 X_i \, modulo \, m$. Also find a K_2 in terms of *n, a,* and *m* such that $X_{i+n}=K_2 X_i \, modulo \, m$ (2 marks)

4. Let there be two queues, Q1 and Q2. At Q1, the arrival and service rates are 6 and 12 per hour, respectively. At Q2 arrival and service rates are 12 and 24 respectively. Assume that the arrivals occur in Poisson processes and that service times have exponential distributions for both the queues and that the queues have infinite calling population and capacity.
(a) If the two queues operate independently of each other, compare the two queues in terms of the the mean waiting time for a customer in the line for service. (2 marks)
(b) If Q1 and Q2 are merged into a single queue Q0 with arrival rate 18 per hour and service rate 18 per hour with Poisson arrivals and exponential service times as before, then find out if Q0 will have lesser or more waiting time than each of Q1 and Q2. (1 mark)

5. Find the condition on *a* with respect to *x* for which the Poisson distribution $\frac{a^x e^{-a}}{x!}$ can be approximated by a Normal distribution with the same mean and variance. It may be helpful to use the approximation $x! \approx x^x e^{-x}\sqrt{2\pi x}$ for large enough integer *x*. (2 marks)

For the next 3 questions, a summary of queuing theory results appears on the last page.

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6. A repair and inspection facility consists of two stations: a repair station with two technicians, and an inspection station with 1 inspector. Each repair technician works at the rate of 3 items per hour; the inspector can inspect 8 items per hour. Approximately 10% of all items fail inspection and are sent back to the repair station. (This percentage holds even for items that have been

repaired two or more times.) If items arrive at the rate of 5 per hour, what is the long-run expected delay that items experience at each of the two stations, assuming a Poisson arrival process and exponentially distributed service times? What is the maximum arrival rate that the system can handle without adding personnel? (2 marks)

7. A tool crib with one attendant serves a group of 10 mechanics. Mechanics work for an exponentially distributed amount of time with mean 20 minutes, then go to the crib to request a special tool. Service times by the attendant are exponentially distributed, with mean 3 minutes. If the attendant costs 6 units of resources per hour and the mechanic costs 10 units of resources per hour, would it be advisable to have a second attendant? (2 marks)

8. Let X be distributed as uniform on [-1,1]. Let Y given X, that is, the random variable $Y|X$, be distributed as uniform on $[x^2-½, x^2+½]$.

(a) Find the cumulative distribution function of Y. (2 Marks)

(b) Find the Covariance(X,Y). (2 Marks)

9. Let $X_i,\ i=1,\ldots,m$ be independently and uniformly distributed random variables over [0,1]. Find the variance of the random variable $Y:=\sum_{i=1}^{m-1} X_i X_{i+1}$. (2 marks)

Table 2 Queueing Notation for Parallel Server Systems

P_n	Steady-state probability of having n customers in system
$P_n(t)$	Probability of n customers in system at time t
λ	Arrival rate
λ_e	Effective arrival rate
μ	Service rate of one server
ρ	Server utilization
A_n	Interarrival time between customers $n-1$ and n
S_n	Service time of the nth arriving customer
W_n	Total time spent in system by the nth arriving customer
W_n^Q	Total time spent waiting in queue by customer n
$L(t)$	The number of customers in system at time t
$L_Q(t)$	The number of customers in queue at time t
L	Long-run time-average number of customers in system
L_Q	Long-run time-average number of customers in queue
w	Long-run average time spent in system per customer
w_Q	Long-run average time spent in queue per customer

Table 3 Steady-State Parameters of the $M/G/1$ Queue

ρ	$\dfrac{\lambda}{\mu}$
L	$\rho + \dfrac{\lambda^2(1/\mu^2+\sigma^2)}{2(1-\rho)} = \rho + \dfrac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$
w	$\dfrac{1}{\mu} + \dfrac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}$
w_Q	$\dfrac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}$
L_Q	$\dfrac{\lambda^2(1/\mu^2+\sigma^2)}{2(1-\rho)} = \dfrac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$
P_0	$1-\rho$

Table 4 Steady-State Parameters of the $M/M/1$ Queue

L	$\dfrac{\lambda}{\mu-\lambda} = \dfrac{\rho}{1-\rho}$
w	$\dfrac{1}{\mu-\lambda} = \dfrac{1}{\mu(1-\rho)}$
w_Q	$\dfrac{\lambda}{\mu(\mu-\lambda)} = \dfrac{\rho}{\mu(1-\rho)}$
L_Q	$\dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\rho^2}{1-\rho}$
P_n	$\left(1-\dfrac{\lambda}{\mu}\right)\left(\dfrac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n$

Table 5 Steady-State Parameters for the $M/M/c$ Queue

ρ	$\dfrac{\lambda}{c\mu}$
P_0	$\left\{\left[\sum_{n=0}^{c-1}\dfrac{(\lambda/\mu)^n}{n!}\right] + \left[\left(\dfrac{\lambda}{\mu}\right)^c\left(\dfrac{1}{c!}\right)\left(\dfrac{c\mu}{c\mu-\lambda}\right)\right]\right\}^{-1}$
	$= \left\{\left[\sum_{n=0}^{c-1}\dfrac{(c\rho)^n}{n!}\right] + \left[(c\rho)^c\left(\dfrac{1}{c!}\right)\dfrac{1}{1-\rho}\right]\right\}^{-1}$
$P(L(\infty)\geq c)$	$\dfrac{(\lambda/\mu)^c P_0}{c!(1-\lambda/c\mu)} = \dfrac{(c\rho)^c P_0}{c!(1-\rho)}$
L	$c\rho + \dfrac{(c\rho)^{c+1}P_0}{c(c!)(1-\rho)^2} = c\rho + \dfrac{\rho P(L(\infty)\geq c)}{1-\rho}$
w	$\dfrac{L}{\lambda}$
w_Q	$w - \dfrac{1}{\mu}$
L_Q	$\lambda w_Q = \dfrac{(c\rho)^{c+1}P_0}{c(c!)(1-\rho)^2} = \dfrac{\rho P(L(\infty)\geq c)}{1-\rho}$
$L - L_Q$	$\dfrac{\lambda}{\mu} = c\rho$

Table 6 Steady-State Parameters for the $M/G/\infty$ Queue

P_0	$e^{-\lambda/\mu}$
w	$\dfrac{1}{\mu}$
w_Q	0
L	$\dfrac{\lambda}{\mu}$
L_Q	0
P_n	$\dfrac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!}, n = 0, 1, \ldots$

Table 7 Steady-State Parameters for the $M/M/c/N$ Queue
(N = System Capacity, $a = \lambda/\mu$, $\rho = \lambda/(c\mu)$)

P_0	$\left[1 + \sum_{n=1}^{c}\dfrac{a^n}{n!} + \dfrac{a^c}{c!}\sum_{n=c+1}^{N}\rho^{n-c}\right]^{-1}$
P_N	$\dfrac{a^N}{c!c^{N-c}}P_0$
L_Q	$\dfrac{P_0 a^c \rho}{c!(1-\rho)^2}\left[1-\rho^{N-c}-(N-c)\rho^{N-c}(1-\rho)\right]$
λ_e	$\lambda(1-P_N)$
w_Q	$\dfrac{L_Q}{\lambda_e}$
w	$w_Q + \dfrac{1}{\mu}$
L	$\lambda_e w$

Table 8 Steady-State Parameters for the $M/M/c/K/K$ Queue

P_0	$\left[\sum_{n=0}^{c-1}\dbinom{K}{n}\left(\dfrac{\lambda}{\mu}\right)^n + \sum_{n=c}^{K}\dfrac{K!}{(K-n)!c!c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n\right]^{-1}$
P_n	$\begin{cases}\dbinom{K}{n}\left(\dfrac{\lambda}{\mu}\right)^n P_0, & n = 0, 1, \ldots, c-1\\ \dfrac{K!}{(K-n)!c!c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n P_0, & n = c, c+1, \ldots, K\end{cases}$
L	$\sum_{n=0}^{K} nP_n$
L_Q	$\sum_{n=c+1}^{K}(n-c)P_n$
λ_e	$\sum_{n=0}^{K}(K-n)\lambda P_n$
w	L/λ_e
w_Q	L_Q/λ_e
ρ	$\dfrac{L-L_Q}{c} = \dfrac{\lambda_e}{c\mu}$