Continuous Functions

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Date: July 10th, 2024

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1 Continuous Functions

Text excerpts remixed from Vladimir A. Zorich - Mathematical Analysis I as well as Stephen Abbott - Understanding Analysis.

"Let f be a real-valued function defined in a neighborhood of a point $a \in \mathbb{R}$. In intuitive terms, the function f is continuous at a if its value $f(x)$ approaches the value $f(a)$ that it assumes at the point a itself as x gets nearer to a." (V. A. Zorich)

Definition 1 (Continuity of a function)**.** A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at the point* $a \in D$ if

$$\boxed{\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \quad \Big(|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon\Big)}$$

If f is continuous at every point in the domain D, then we say that f is *continuous on D*.

2 Examples

Here, we give a few examples of continuous functions alongside the respective proofs.

2.1 The constant function is continuous

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) \coloneqq c$, where $c \in \mathbb{R}$. That is, f is a constant function. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto c$ is continuous. Let $\varepsilon > 0$. We choose $\delta \coloneqq 1 > 0$. Let $x \in \mathbb{R}$. Then:

$$|f(x) - f(a)| = |c - c| = |0| = 0 < \varepsilon$$

With that, the implication also holds true since its conclusion is always true (as shown above) irregardless of the premise.

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

Therefore, f is continuous at a. □

2.2 Functions $x \mapsto mx + y_0$ are continuous

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) \coloneqq m \cdot x + y_0$, where $m, y_0 \in \mathbb{R}$. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto m \cdot x + y_0$ is continuous.

We first consider the simpler case where the slope is 0, that is $\mathbf{m = 0}$. Then our function is given by $f(x) = y_0$ for all $x \in \mathbb{R}$. This is a constant function and we have already shown that constant functions are continuous. Therefore, f is continuous at a when $m = 0$.

Now to the more interesting case where $\mathbf{m \neq 0}$. Let $\varepsilon > 0$. We choose $\delta \coloneqq \frac{\varepsilon}{|m|}$. Since $\epsilon > 0$ and $|m| > 0$, we have $\delta > 0$.

Let $x \in \mathbb{R}$ and $|x - a| < \delta$. Then:

$$\begin{aligned}
|f(x) - f(a)| &= \big|(m \cdot x + y_0) - (m \cdot a - y_0)\big| \\
&= |m \cdot x - m \cdot a| \\
&= |m \cdot (x - a)| \\
&= |m| \cdot |x - a| \\
&< |m| \cdot \delta = |m| \cdot \frac{\varepsilon}{|m|} = \varepsilon
\end{aligned}$$

In the last line, we have used the fact that $|x - a| < \delta$ and then used our definition of δ.

The argument shows that $|f(x) - f(a)| < \varepsilon$, which proves the continuity of f at a. □

2.3 The parabola is continuous

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) \coloneqq x^2$. That is, f is a parabola. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous.

Let $\epsilon > 0$. We choose $\delta \coloneqq \min\left(\frac{\epsilon}{2|a|+1}, 1\right)$. Since $\epsilon > 0$ and $2|a|+1 > 0$ and $1 > 0$, we have $\delta > 0$.

Let $x \in \mathbb{R}$ and $|x-a| < \delta$. Then:

$$
\begin{aligned}
|f(x) - f(a)| &= |x^2 - a^2| = |(x+a)(x-a)| \\
&= |x+a| \cdot |x-a| \\
&\leq \big(|x| + |a|\big) \cdot |x-a| && |x+a| \leq |x| + |a| \\
&\leq \big(|x| + |a|\big) \cdot \delta && |x-a| < \delta \\
&< \big((|a| + \delta) + |a|\big) \cdot \delta && |x| < |a| + \delta \textit{ (*)} \\
&= \big(2|a| + \delta\big) \cdot \delta \\
&\leq \big(2|a| + 1\big) \cdot \frac{\epsilon}{2|a|+1} && \delta \leq 1 \textit{ and } \delta \leq \frac{\epsilon}{2|a|+1} \\
&= \epsilon
\end{aligned}
$$

(*) is true because $|x| = |a + (x-a)| \leq |a| + |x-a| < |a| + \delta$. Therefore, $|x| < |a| + \delta$.

The argument shows that $|f(x) - f(a)| < \epsilon$, which proves the continuity of f at a. □

2.4 The hyperbola is continuous

Theorem 4. *Let f:* $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be defined by* $f(x) := \frac{1}{x}$*. Then* f *is continuous at every point* $a \in \mathbb{R} \setminus \{0\}$*.*

Proof. Let $a \in \mathbb{R} \setminus \{0\}$ be an arbitrary point where we want to show that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is continuous.

Let $\epsilon > 0$. We choose $\delta := \min\left(\frac{\epsilon|a|^2}{2}, \frac{|a|}{2}\right)$. Since $\epsilon > 0$ and $|a| > 0$ ($a \in \mathbb{R} \setminus \{0\}$), we have $\delta > 0$.

Let $x \in D := \mathbb{R} \setminus \{0\}$ and $|x - a| < \delta$. Then:

$$\begin{aligned}
|f(x) - f(a)| &= \left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| \\
&= \frac{|a-x|}{|ax|} = \frac{|a-x|}{|a||x|} && |a-x| = |x-a| < \delta \\
&< \frac{\delta}{|a||x|} && |x| > \frac{|a|}{2} \ \textit{(*)} \\
&< \frac{\delta}{|a|\frac{|a|}{2}} && \delta \le \frac{\epsilon|a|^2}{2} \\
&\le \frac{\epsilon|a|^2}{2} \frac{1}{|a|\frac{|a|}{2}} \\
&= \epsilon
\end{aligned}$$

(*) is true because

$$\begin{aligned}
|x| &= |a + (x-a)| && \textit{Reverse triangle inequality} \\
&\ge |a| - |x-a| && |x-a| < \delta \\
&> |a| - \delta && \delta \le \frac{|a|}{2} \\
&\ge |a| - \frac{|a|}{2} \\
&= \frac{|a|}{2} \quad \Rightarrow \quad |x| > \frac{|a|}{2}
\end{aligned}$$

The reverse triangle inequality used in the first step can also be derived. Let $m, n, u, v \in \mathbb{R}$.

$$\begin{aligned}
&|m+n| \le |m| + |n| \\
\Rightarrow\ &|m+n| - |n| \le |m| && m := u+v, \ n := -v \\
\Rightarrow\ &|u| - |v| \le |u+v|
\end{aligned}$$

The argument shows that $|f(x) - f(a)| < \epsilon$, which proves the continuity of f at a.

□

2.5 The exponential function is continuous

We assume that the exponential function is defined by its power series:

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := e^x$. Then f is continuous at $a = 0$.*

Proof. [1] We want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ is continuous at $a = 0$.

Let $\epsilon > 0$. We choose $\delta := \dfrac{\epsilon}{\epsilon + 1}$. Since $\epsilon > 0$ we have $0 < \delta < 1$.

Let $x \in \mathbb{R}$ and $|x - a| = |x - 0| = |x| < \delta$. Then:

$$\begin{aligned}
|f(x) - f(0)| &= |e^x - e^0| = |e^x - 1| \\
&= \left| \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 \right| && \textit{definition of } e^x \\
&= \left| \sum_{k=1}^{\infty} \frac{x^k}{k!} \right| \\
&\le \sum_{k=1}^{\infty} \frac{|x|^k}{k!} \\
&< \sum_{k=1}^{\infty} \frac{\delta^k}{k!} && |x| < \delta \\
&< \sum_{k=1}^{\infty} \delta^k && k > 0 \\
&= \left(\sum_{k=0}^{\infty} \delta^k \right) - 1 \\
&= \frac{1}{1 - \delta} - 1 && \textit{Geometric series, } 0 < \delta < 1 \\
&= \frac{1}{1 - \frac{\epsilon}{\epsilon+1}} - 1 && \delta := \frac{\epsilon}{\epsilon + 1} \\
&= \frac{1}{\frac{1}{\epsilon+1}} - 1 = \epsilon + 1 - 1 = \epsilon
\end{aligned}$$

The argument shows that $|f(x) - f(0)| < \epsilon$, which proves the continuity of f at 0. □

[1] This proof is adapted from this video.

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := e^x$. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. [2] Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x, \mapsto e^x$ is continuous.

Let $\epsilon > 0$. We let the choice of $\delta > 0$ open for now.

Let $x \in \mathbb{R}$ and $|x - a| < \delta$. Then:

$$\begin{aligned} |f(x) - f(a)| &= |e^x - e^a| && \\ &= \left|e^a \cdot (e^{x-a} - 1)\right| && e^{x'} e^{y'} = e^{x'+y'} \ \forall x', y' \in \mathbb{R} \\ &= e^a \cdot \left|e^{x-a} - 1\right| && e^{x'} > 0 \ \forall x' \in \mathbb{R} \\ &= e^a \cdot \left|e^{x-a} - e^0\right| && \\ &= e^a \cdot \left|e^z - e^0\right| && \end{aligned}$$

We substituted $z := x - a$ and therefore $|x - a| = |z| < \delta$ (and still $\delta > 0$). This was done to **reduce the problem**. Should $f : \mathbb{R} \to \mathbb{R}$, $z \mapsto e^z$ be continuous at 0, then by definition of continuity in 1, we know:

$$\forall \epsilon' > 0 \quad \exists \delta > 0 \quad \forall z \in \mathbb{R} : \quad \Big(|z - 0| < \delta \ \Rightarrow \ |f(z) - f(0)| = |e^z - e^0| < \epsilon'\Big)$$

And indeed $|z| < \delta$. Let's choose $\epsilon' := \dfrac{\epsilon}{e^a}$. Then continuing the above argument:

$$|f(x) - f(a)| = e^a \cdot \left|e^z - e^0\right| < e^a \cdot \epsilon' = e^a \cdot \frac{\epsilon}{e^a} = \epsilon$$

By Theorem 5, we know that $f : \mathbb{R} \to \mathbb{R}$, $z \mapsto e^z$ is continuous at 0. With the reasoning above, this implies that $|f(x) - f(a)| < \epsilon$ for all $x \in \mathbb{R}$ with $|x - a| < \delta$. Therefore, f is continuous at every point $a \in \mathbb{R}$.

□

[2]This proof is adapted from this video.

3 Sum and Product of Continuous Functions

Theorem 7 (The sum of continuous functions is continuous)**.** *If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f + g$ is continuous on D.*

Proof. Assume f and g are continuous on D. We need to show that $f + g$ is continuous at any $a \in D$. Let $a \in D$ and $\epsilon > 0$. By the continuity of f and g, we have:

- There $\exists \delta_1 > 0$ such that $\forall x \in D$, if $|x - a| < \delta_1$, then $|f(x) - f(a)| < \frac{\epsilon}{2}$.
- There $\exists \delta_2 > 0$ such that $\forall x \in D$, if $|x - a| < \delta_2$, then $|g(x) - g(a)| < \frac{\epsilon}{2}$.

Choose $\delta \coloneqq \min(\delta_1, \delta_2)$. Then $\delta > 0$ since both $\delta_1 > 0$ and $\delta_2 > 0$.

Let $x \in D$. If $|x - a| < \delta$, we have:

$$|f(x) - f(a)| < \frac{\epsilon}{2}$$
$$|g(x) - g(a)| < \frac{\epsilon}{2}$$

Putting it together:

$$\begin{aligned} |(f+g)(x) - (f+g)(a)| &= |f(x) + g(x) - f(a) - g(a)| \\ &= |f(x) - f(a) + g(x) - g(a)| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, $f + g$ is continuous at a. Since a was arbitrary, $f + g$ is continuous on D. □

Theorem 8 (The product of continuous functions is continuous)**.** *If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.*

Proof. Assume f and g are continuous on D. We need to show that $f \cdot g$ is continuous at any $a \in D$. Let $a \in D$ and $\epsilon > 0$. By the continuity of f and g, we have:

- There $\exists \delta_1 > 0$ such that $\forall x \in D$, if $|x - a| < \delta_1$, then $|f(x) - f(a)| < \dfrac{\epsilon}{2|g(a)| + 1}$.

  The 1 in the denominator is just to ensure we don't divide by 0.
- There $\exists \delta_2 > 0$ such that $\forall x \in D$, if $|x - a| < \delta_2$, then $|g(x) - g(a)| < \dfrac{\epsilon}{2(\epsilon + |f(a)|)}$.

Choose $\delta \coloneqq \min(\delta_1, \delta_2)$. Then $\delta > 0$ since both $\delta_1 > 0$ and $\delta_2 > 0$.

Let $x \in D$. If $|x - a| < \delta$, we have:

$$\begin{aligned} |f(x) - f(a)| &< \frac{\epsilon}{2|g(a)| + 1} \\ |g(a) - g(x)| &< \frac{\epsilon}{2(\epsilon + |f(a)|)} \end{aligned}$$

Putting it together:

$$\begin{aligned} |(f \cdot g)(x) - (f \cdot g)(a)| &= |f(x) \cdot g(x) - f(a) \cdot g(a)| \\ &= |f(x) \cdot g(x) - f(x) \cdot g(a) + f(x) \cdot g(a) - f(a) \cdot g(a)| \\ &= \big|f(x) \cdot \big(g(x) - g(a)\big) + \big(f(x) - f(a)\big) \cdot g(a)\big| \\ &\leq \big|f(x) \cdot (g(x) - g(a))\big| \; + \; \big|(f(x) - f(a)) \cdot g(a)\big| \\ &= |f(x)| \cdot |g(x) - g(a)| + |f(x) - f(a)| \cdot |g(a)| \\ &< |f(x)| \cdot \frac{\epsilon}{2(\epsilon + |f(a)|)} + \frac{\epsilon}{2|g(a)| + 1} \cdot |g(a)| \\ &= \frac{\epsilon}{2} \cdot \frac{|f(x)|}{\epsilon + |f(a)|} + \frac{\epsilon}{2} \frac{|g(a)|}{|g(a)| + 1} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

In the last step, we made use of the triangle inequality:

$$|f(x) - f(a)| < \frac{\epsilon}{2|g(a)| + 1} \leq \epsilon \quad \Rightarrow \quad |f(x)| \leq \epsilon + |f(a)| \tag{1}$$

Hence, $f \cdot g$ is continuous at a. Since a was arbitrary, $f \cdot g$ is continuous on D. $\square$

4 Left- and Right-Continuous Functions

Definition 2 (Definition of a Left Continuous Function)**.** A function $f : D \to \mathbb{R}$ is *right continuous at a point* $a \in D$ if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D, x > a \Rightarrow (|x - a| < \delta \to |f(x) - f(a)| < \epsilon).$$

If f is left continuous at every point in the domain D, then we say that f is *left continuous on* D.

Definition 3 (Definition of a Right Continuous Function)**.** A function $f : D \to \mathbb{R}$ is *right continuous at a point* $a \in D$ if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D, x < a \Rightarrow (|x - a| < \delta \to |f(x) - f(a)| < \epsilon).$$

If f is right continuous at every point in the domain D, then we say that f is *right continuous on* D.

Definition 4 (Definition of the Heaviside Function)**.** The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined as

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 9 (Heaviside Function is Right Continuous at 0)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be the Heaviside Function. Then f is right continuous in* 0*.*

Proof. Let $\epsilon > 0$. Choose $\delta = 1$. Then, for any $x > 0$, we have:

$$\begin{aligned} &\text{Since } x > 0, \text{we have } H(x) = 1. \\ &\text{So, } |H(x) - H(0)| = |1 - 1| = 0 < \epsilon. \end{aligned}$$

□

Theorem 10 (Heaviside Function is Not Continuous at 0)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be the Heaviside Function. Then f is not continuous in* 0*.*

Proof. Assume, for contradiction, that H is continuous at 0. Let $\epsilon = \frac{1}{2}$. Then there exists $\delta > 0$ such that for all $x \in D$, if $|x - 0| < \delta$, then $|H(x) - H(0)| < \epsilon$. Consider $x = -\frac{\delta}{2}$. We have:

$$\begin{aligned} &x < 0 \Rightarrow H(x) = 0, \\ &\text{and } |x| < \delta \Rightarrow |-\frac{\delta}{2}| < \delta, \\ &\text{So, } |H(x) - H(0)| = |0 - 1| = 1 \geq \frac{1}{2} = \epsilon. \end{aligned}$$

This is a contradiction, so H is not continuous at 0. □

Theorem 11 (Left and Right Continuous $\Leftrightarrow$ Continuous)**.** *A function $f : D \to \mathbb{R}$ is continuous at a point $x \in D$ if and only if it is both left continuous and right continuous at x.*

Proof.

($\Rightarrow$) Assume f is continuous at a.

- To show f is left continuous at a:

  $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, (|x - a| < \delta \to |f(x) - f(a)| < \epsilon)$.

  This implies the definition of left continuity.

- To show f is right continuous at a:

  $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, (|x - a| < \delta \to |f(x) - f(a)| < \epsilon)$.

  This implies the definition of right continuity.

($\Leftarrow$) Assume f is both left continuous and right continuous at a.

- We need to show $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, (|x - a| < \delta \to |f(x) - f(a)| < \epsilon)$.

  Since f is left continuous, $\exists \delta_1 > 0$ such that $\forall x \in D, (x < a \to |x - a| < \delta_1 \to |f(x) - f(a)| < \epsilon)$.

  Since f is right continuous, $\exists \delta_2 > 0$ such that $\forall x \in D, (x > a \to |x - a| < \delta_2 \to |f(x) - f(a)| < \epsilon)$.

  Let $\delta = \min(\delta_1, \delta_2)$. Then, for any $y \in D$,

  if $|x - a| < \delta,$ then either $a < x$ or $a > x$.

  If $a < x,$ then $|x - a| < \delta_1$ and $|f(x) - f(a)| < \epsilon$ by left continuity.

  If $a > x,$ then $|x - a| < \delta_2$ and $|f(x) - f(a)| < \epsilon$ by right continuity.

  Thus, f is continuous at a.

$\square$