

# Continuous Functions

Felix Lentze & Dominic Plein

Date: July 16th, 2024

---

## Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Continuous Functions</b>                                    | <b>2</b>  |
| <b>2</b> | <b>Examples</b>  | <b>3</b>  |
| 2.1      | The constant function is continuous . . . . .                  | 3         |
| 2.2      | Functions $x \mapsto mx + y_0$ are continuous . . . . .        | 3         |
| 2.3      | The parabola is continuous . . . . .                           | 4         |
| 2.4      | The hyperbola is continuous . . . . .                          | 5         |
| 2.5      | The exponential function is continuous . . . . .               | 6         |
| <b>3</b> | <b>Sum and Product of Continuous Functions</b>                 | <b>8</b>  |
| <b>4</b> | <b>Left- and Right-Continuous Functions</b>                    | <b>10</b> |
| 4.1      | Definitions . . . . .  | 10        |
| 4.2      | Heaviside function as example . . . . .                        | 10        |
| 4.3      | Left- and right-continuous is the same as continuous . . . . . | 11        |

## 1 Continuous Functions

Let  $f$  be a real-valued function defined in a neighborhood of a point  $a \in \mathbb{R}$ . In intuitive terms, the function  $f$  is continuous at  $a$  if its value  $f(x)$  approaches the value  $f(a)$  that it assumes at the point  $a$  itself as  $x$  gets nearer to  $a$ .

---

V. Zorich in Mathematical Analysis I

**Definition 1** (Continuity of a function). A function  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is *continuous at the point*  $a \in D$  if

$$\boxed{\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \quad (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)}$$

If  $f$  is continuous at every point in the domain  $D$ , then we say that  $f$  is *continuous on*  $D$ .

## 2 Examples

Here, we give a few examples of continuous functions alongside the respective proofs.

### 2.1 The constant function is continuous

**Theorem 1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function given by  $f(x) := c$ , where  $c \in \mathbb{R}$ . That is,  $f$  is a constant function. Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto c$  is continuous. Let  $\varepsilon > 0$ . We choose  $\delta := 1 > 0$ . Let  $x \in \mathbb{R}$ . Then:

$$|f(x) - f(a)| = |c - c| = |0| = 0 < \varepsilon$$

With that, the implication also holds true since its conclusion is always true (as shown above) irregardless of the premise.

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

Therefore,  $f$  is continuous at  $a$ . □

### 2.2 Functions $x \mapsto mx + y_0$ are continuous

**Theorem 2.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function given by  $f(x) := m \cdot x + y_0$ , where  $m, y_0 \in \mathbb{R}$ . Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto m \cdot x + y_0$  is continuous.

We first consider the simpler case where the slope is 0, that is  $\mathbf{m} = \mathbf{0}$ . Then our function is given by  $f(x) = y_0$  for all  $x \in \mathbb{R}$ . This is a constant function and we have already shown that constant functions are continuous. Therefore,  $f$  is continuous at  $a$  when  $m = 0$ .

Now to the more interesting case where  $\mathbf{m} \neq \mathbf{0}$ . Let  $\varepsilon > 0$ . We choose  $\delta := \frac{\varepsilon}{|m|}$ . Since  $\varepsilon > 0$  and  $|m| > 0$ , we have  $\delta > 0$ .

Let  $x \in \mathbb{R}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned} |f(x) - f(a)| &= |(m \cdot x + y_0) - (m \cdot a + y_0)| \\ &= |m \cdot x - m \cdot a| \\ &= |m \cdot (x - a)| \\ &= |m| \cdot |x - a| \\ &< |m| \cdot \delta = |m| \cdot \frac{\varepsilon}{|m|} = \varepsilon \end{aligned}$$

In the last line, we have used the fact that  $|x - a| < \delta$  and then used our definition of  $\delta$ .

The argument shows that  $|f(x) - f(a)| < \varepsilon$ , which proves the continuity of  $f$  at  $a$ . □

### 2.3 The parabola is continuous

**Theorem 3.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) := x^2$ . That is,  $f$  is a parabola. Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x^2$  is continuous.

Let  $\epsilon > 0$ . We choose  $\delta := \min\left(\frac{\epsilon}{2|a|+1}, 1\right)$ . Since  $\epsilon > 0$  and  $2|a|+1 > 0$  and  $1 > 0$ , we have  $\delta > 0$ .

Let  $x \in \mathbb{R}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned}
 |f(x) - f(a)| &= |x^2 - a^2| = |(x+a)(x-a)| \\
 &= |x+a| \cdot |x-a| && \left. \begin{array}{l} \downarrow |x+a| \leq |x|+|a| \\ \downarrow |x-a| < \delta \\ \downarrow |x| < |a|+\delta \text{ (*)} \end{array} \right\} \\
 &\leq (|x|+|a|) \cdot |x-a| \\
 &\leq (|x|+|a|) \cdot \delta \\
 &< ((|a|+\delta)+|a|) \cdot \delta \\
 &= (2|a|+\delta) \cdot \delta && \left. \begin{array}{l} \downarrow \delta \leq 1 \text{ and } \delta \leq \frac{\epsilon}{2|a|+1} \end{array} \right\} \\
 &\leq (2|a|+1) \cdot \frac{\epsilon}{2|a|+1} \\
 &= \epsilon
 \end{aligned}$$

(\*) is true because  $|x| = |a + (x - a)| \leq |a| + |x - a| < |a| + \delta$ . Therefore,  $|x| < |a| + \delta$ .

The argument shows that  $|f(x) - f(a)| < \epsilon$ , which proves the continuity of  $f$  at  $a$ .  $\square$

## 2.4 The hyperbola is continuous

**Theorem 4.** Let  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be defined by  $f(x) := \frac{1}{x}$ . Then  $f$  is continuous at every point  $a \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Let  $a \in \mathbb{R} \setminus \{0\}$  be an arbitrary point where we want to show that  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ ,  $x \mapsto \frac{1}{x}$  is continuous.

Let  $\epsilon > 0$ . We choose  $\delta := \min\left(\frac{\epsilon|a|^2}{2}, \frac{|a|}{2}\right)$ . Since  $\epsilon > 0$  and  $|a| > 0$  ( $a \in \mathbb{R} \setminus \{0\}$ ), we have  $\delta > 0$ .

Let  $x \in D := \mathbb{R} \setminus \{0\}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned}
 |f(x) - f(a)| &= \left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a - x}{ax} \right| \\
 &= \frac{|a - x|}{|ax|} = \frac{|a - x|}{|a||x|} \quad \left. \begin{array}{l} |a - x| = |x - a| < \delta \\ |x| > \frac{|a|}{2} \quad (*) \end{array} \right\} \\
 &< \frac{\delta}{|a||x|} \\
 &< \frac{\delta}{|a|\frac{|a|}{2}} \\
 &\leq \frac{\epsilon|a|^2}{2} \frac{1}{|a|\frac{|a|}{2}} \quad \left. \begin{array}{l} \delta \leq \frac{\epsilon|a|^2}{2} \end{array} \right\} \\
 &= \epsilon
 \end{aligned}$$

(\*) is true because

$$\begin{aligned}
 |x| &= |a + (x - a)| \\
 &\geq |a| - |x - a| \quad \left. \begin{array}{l} \text{Reverse triangle inequality} \\ |x - a| < \delta \end{array} \right\} \\
 &> |a| - \delta \\
 &\geq |a| - \frac{|a|}{2} \quad \left. \begin{array}{l} \delta \leq \frac{|a|}{2} \end{array} \right\} \\
 &= \frac{|a|}{2} \Rightarrow |x| > \frac{|a|}{2}
 \end{aligned}$$

The reverse triangle inequality used in the first step can also be derived. Let  $m, n, u, v \in \mathbb{R}$ .

$$\begin{aligned}
 |m + n| &\leq |m| + |n| \\
 \Rightarrow |m + n| - |n| &\leq |m| \\
 \Rightarrow |u| - |v| &\leq |u + v| \quad \left. \begin{array}{l} m := u + v, \quad n := -v \end{array} \right\}
 \end{aligned}$$

The argument shows that  $|f(x) - f(a)| < \epsilon$ , which proves the continuity of  $f$  at  $a$ .

□

## 2.5 The exponential function is continuous

We assume that the exponential function is defined by its power series:

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

**Theorem 5.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) := e^x$ . Then  $f$  is continuous at  $a = 0$ .*

*Proof.* <sup>1</sup> We want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto e^x$  is continuous at  $a = 0$ .

Let  $\epsilon > 0$ . We choose  $\delta := \frac{\epsilon}{\epsilon + 1}$ . Since  $\epsilon > 0$  we have  $0 < \delta < 1$ .

Let  $x \in \mathbb{R}$  and  $|x - a| = |x - 0| = |x| < \delta$ . Then:

$$\begin{aligned}
 |f(x) - f(0)| &= |e^x - e^0| = |e^x - 1| && \left. \begin{array}{l} \\ \end{array} \right\} \text{definition of } e^x \\
 &= \left| \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 \right| \\
 &= \left| \sum_{k=1}^{\infty} \frac{x^k}{k!} \right| \\
 &\leq \sum_{k=1}^{\infty} \frac{|x|^k}{k!} && \left. \begin{array}{l} \\ \end{array} \right\} |x| < \delta \\
 &< \sum_{k=1}^{\infty} \frac{\delta^k}{k!} && \left. \begin{array}{l} \\ \end{array} \right\} k > 0 \\
 &< \sum_{k=1}^{\infty} \delta^k \\
 &= \left( \sum_{k=0}^{\infty} \delta^k \right) - 1 && \left. \begin{array}{l} \\ \end{array} \right\} \text{Geometric series, } 0 < \delta < 1 \\
 &= \frac{1}{1 - \delta} - 1 \\
 &= \frac{1}{1 - \frac{\epsilon}{\epsilon + 1}} - 1 && \left. \begin{array}{l} \\ \end{array} \right\} \delta := \frac{\epsilon}{\epsilon + 1} \\
 &= \frac{1}{\frac{1}{\epsilon + 1}} - 1 = \epsilon + 1 - 1 = \epsilon
 \end{aligned}$$

The argument shows that  $|f(x) - f(0)| < \epsilon$ , which proves the continuity of  $f$  at 0.  $\square$

---

<sup>1</sup>This proof is adapted from [this video](#).

**Theorem 6.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) := e^x$ . Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .

*Proof.*<sup>2</sup> Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^x$  is continuous.

Let  $\epsilon > 0$ . We let the choice of  $\delta > 0$  open for now.

Let  $x \in \mathbb{R}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned}
 |f(x) - f(a)| &= |e^x - e^a| \\
 &= |e^a \cdot (e^{x-a} - 1)| \quad \left. \begin{array}{l} e^{x'} e^{y'} = e^{x'+y'} \quad \forall x', y' \in \mathbb{R} \\ e^{x'} > 0 \quad \forall x' \in \mathbb{R} \end{array} \right\} \\
 &= e^a \cdot |e^{x-a} - 1| \\
 &= e^a \cdot |e^{x-a} - e^0| \\
 &= e^a \cdot |e^z - e^0|
 \end{aligned}$$

We substituted  $z := x - a$  and therefore  $|x - a| = |z| < \delta$  (and still  $\delta > 0$ ). This was done to **reduce the problem**. Should  $f : \mathbb{R} \rightarrow \mathbb{R}, z \mapsto e^z$  be continuous at 0, then by definition of continuity in [1](#), we know:

$$\forall \epsilon' > 0 \quad \exists \delta > 0 \quad \forall z \in \mathbb{R} : \left( |z - 0| < \delta \Rightarrow |f(z) - f(0)| = |e^z - e^0| < \epsilon' \right)$$

And indeed  $|z| < \delta$ . Let's choose  $\epsilon' := \frac{\epsilon}{e^a}$ . Then continuing the above argument:

$$|f(x) - f(a)| = e^a \cdot |e^z - e^0| < e^a \cdot \epsilon' = e^a \cdot \frac{\epsilon}{e^a} = \epsilon$$

By [Theorem 5](#), we know that  $f : \mathbb{R} \rightarrow \mathbb{R}, z \mapsto e^z$  is continuous at 0. With the reasoning above, this implies that  $|f(x) - f(a)| < \epsilon$  for all  $x \in \mathbb{R}$  with  $|x - a| < \delta$ . Therefore,  $f$  is continuous at every point  $a \in \mathbb{R}$ .

□

---

<sup>2</sup>This proof is adapted from [this video](#).

### 3 Sum and Product of Continuous Functions

**Theorem 7.** *If  $f$  and  $g$  are continuous functions on a set  $D \subseteq \mathbb{R}$ , then  $f + g$  is continuous on  $D$ .*

*Proof.* Assume  $f$  and  $g$  are continuous on  $D$ . We need to show that  $f + g$  is continuous at any  $a \in D$ . Let  $a \in D$  and  $\epsilon > 0$ . By the continuity of  $f$  and  $g$ , we have:

- There  $\exists \delta_1 > 0$  such that  $\forall x \in D$ , if  $|x - a| < \delta_1$ , then  $|f(x) - f(a)| < \frac{\epsilon}{2}$ .
- There  $\exists \delta_2 > 0$  such that  $\forall x \in D$ , if  $|x - a| < \delta_2$ , then  $|g(x) - g(a)| < \frac{\epsilon}{2}$ .

Choose  $\delta := \min(\delta_1, \delta_2)$ . Then  $\delta > 0$  since both  $\delta_1 > 0$  and  $\delta_2 > 0$ .

Let  $x \in D$ . If  $|x - a| < \delta$ , we have:

$$\begin{aligned} |f(x) - f(a)| &< \frac{\epsilon}{2} \\ |g(x) - g(a)| &< \frac{\epsilon}{2} \end{aligned}$$

Putting it together:

$$\begin{aligned} |(f + g)(x) - (f + g)(a)| &= |f(x) + g(x) - f(a) - g(a)| \\ &= |f(x) - f(a) + g(x) - g(a)| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence,  $f + g$  is continuous at  $a$ . Since  $a$  was arbitrary,  $f + g$  is continuous on  $D$ .  $\square$



**Theorem 8.** *If  $f$  and  $g$  are continuous functions on a set  $D \subseteq \mathbb{R}$ , then  $f \cdot g$  is continuous on  $D$ .*

*Proof.* Assume  $f$  and  $g$  are continuous on  $D$ . We need to show that  $f \cdot g$  is continuous at any  $a \in D$ . Let  $a \in D$  and  $\epsilon > 0$ . By the continuity of  $f$  and  $g$ , we have:

- There  $\exists \delta_1 > 0$  such that  $\forall x \in D$ , if  $|x - a| < \delta_1$ , then  $|f(x) - f(a)| < \frac{\epsilon}{2|g(a)| + 1}$ .

The 1 in the denominator is just to ensure we don't divide by 0.

- There  $\exists \delta_2 > 0$  such that  $\forall x \in D$ , if  $|x - a| < \delta_2$ , then  $|g(x) - g(a)| < \frac{\epsilon}{2(\epsilon + |f(a)|)}$ .

Choose  $\delta := \min(\delta_1, \delta_2)$ . Then  $\delta > 0$  since both  $\delta_1 > 0$  and  $\delta_2 > 0$ .

Let  $x \in D$ . If  $|x - a| < \delta$ , we have:

$$\begin{aligned} |f(x) - f(a)| &< \frac{\epsilon}{2|g(a)| + 1} \\ |g(x) - g(a)| &< \frac{\epsilon}{2(\epsilon + |f(a)|)} \end{aligned}$$

Putting it together:

$$\begin{aligned} |(f \cdot g)(x) - (f \cdot g)(a)| &= |f(x) \cdot g(x) - f(a) \cdot g(a)| \\ &= |f(x) \cdot g(x) - f(x) \cdot g(a) + f(x) \cdot g(a) - f(a) \cdot g(a)| \\ &= |f(x) \cdot (g(x) - g(a)) + (f(x) - f(a)) \cdot g(a)| \\ &\leq |f(x) \cdot (g(x) - g(a))| + |(f(x) - f(a)) \cdot g(a)| \\ &= |f(x)| \cdot |g(x) - g(a)| + |f(x) - f(a)| \cdot |g(a)| \\ &\leq |f(x)| \cdot \frac{\epsilon}{2(\epsilon + |f(a)|)} + \frac{\epsilon}{2|g(a)| + 1} \cdot |g(a)| \\ &\stackrel{(1)}{<} \frac{\epsilon}{2} \cdot \frac{\epsilon + |f(a)|}{\epsilon + |f(a)|} + \frac{\epsilon}{2} \cdot \frac{|g(a)|}{|g(a)| + 1/2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

In step (1), we made use of the triangle inequality:

$$|f(x) - f(a)| < \frac{\epsilon}{2|g(a)| + 1} \leq \epsilon \quad \Rightarrow \quad |f(x)| < \epsilon + |f(a)| \quad (1)$$

Hence,  $f \cdot g$  is continuous at  $a$ . Since  $a$  was arbitrary,  $f \cdot g$  is continuous on  $D$ .  $\square$

## 4 Left- and Right-Continuous Functions

### 4.1 Definitions

**Definition 2.** A function  $f : D \rightarrow \mathbb{R}$  is *left-continuous at a point*  $a \in D$  if

$$\boxed{\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D, x < a : \left( |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)}$$

If  $f$  is left-continuous at every point in  $D$ , then we say that  $f$  is *left-continuous on*  $D$ .

**Definition 3.** A function  $f : D \rightarrow \mathbb{R}$  is *right-continuous at a point*  $a \in D$  if

$$\boxed{\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D, x > a : \left( |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)}$$

If  $f$  is right-continuous at every point in  $D$ , then we say that  $f$  is *right-continuous on*  $D$ .

### 4.2 Heaviside function as example

**Definition 4.** The Heaviside function  $\Theta : \mathbb{R} \rightarrow \{0, 1\}$  is defined as

$$\Theta(x) := \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

**Theorem 9.** Let  $\Theta : \mathbb{R} \rightarrow \mathbb{R}$  denote the Heaviside function.  $\Theta$  is right-continuous at 0.

*Proof.* We consider the point 0. Let  $\epsilon > 0$ . Choose  $\delta := 1 > 0$ .

Let  $x \in \mathbb{R}$ ,  $x > 0$  and  $|x - 0| < \delta$ , i.e.  $|x| < 1$ . In summary,  $x \in (0, 1)$ . Then  $\Theta(x) = 1$ .

We find:  $|\Theta(x) - \Theta(0)| = |1 - 1| = 0 < \epsilon$ . This proves that  $\Theta$  is right-continuous at 0.  $\square$

**Theorem 10.** Let  $\Theta : \mathbb{R} \rightarrow \mathbb{R}$  denote the Heaviside function.  $\Theta$  is not continuous at 0.

*Proof.* Assume, for contradiction, that  $\Theta$  is continuous at 0. Let  $\epsilon = 1/2$ . Then there  $\exists \delta > 0$ , such that  $\forall x \in D$ , if  $|x - 0| < \delta$ , then  $|\Theta(x) - \Theta(0)| < \epsilon = 1/2$ . Consider  $x = -\delta/2$ , which fulfills  $|x| < \delta$  since  $|\delta/2| = \delta/2 < \delta$ .

For our choice of  $x$ , we find  $\Theta(x) = 0$  since  $x < 0$ . Therefore:

$$|\Theta(x) - \Theta(0)| = |0 - 1| = 1$$

This is a contradiction to  $|\Theta(x) - \Theta(0)| < 1/2$ . Therefore,  $\Theta$  is not continuous at 0.  $\square$

### 4.3 Equivalence of continuity and left- and right-continuity

**Theorem 11.** *A function  $f : D \rightarrow \mathbb{R}$  is continuous at a point  $a \in D$  if and only if it is both left-continuous and right-continuous at  $a$ .*

*Proof.* Let  $a \in D$  and consider a function  $f : D \rightarrow \mathbb{R}$ . That is,

( $\Rightarrow$ ) Assume  $f$  is continuous at  $a$ .

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \left( |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)$$

This implies both left- and right-continuity at  $a$  due to  $x < a$  and  $x > a$  being stronger conditions than  $x \in D$ .

( $\Leftarrow$ ) Assume  $f$  is both left- and right-continuous at  $a$ .

Since  $f$  is left-continuous:  $\exists \delta_1 > 0 \quad \forall x \in D, x < a : (|x - a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon)$

Since  $f$  is right-continuous:  $\exists \delta_2 > 0 \quad \forall x \in D, x > a : (|x - a| < \delta_2 \Rightarrow |f(x) - f(a)| < \epsilon)$

Choose  $\delta := \min(\delta_1, \delta_2)$ . Let  $x \in D$  and  $|x - a| < \delta$ . Due to our choice of  $\delta$ , we also have  $|x - a| < \delta_1$  and  $|x - a| < \delta_2$ .

Now consider these cases:

- If  $x < a$ , then  $|f(x) - f(a)| < \epsilon$  by left-continuity.
- If  $x > a$ , then  $|f(x) - f(a)| < \epsilon$  by right-continuity.
- If  $x = a$ , then  $|f(x) - f(a)| = 0 < \epsilon$ .

Hence,  $f$  is continuous at  $a$ .

□