

# Continuous Functions

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## 1 Continuous Functions

*Text excerpts remixed from Vladimir A. Zorich - Mathematical Analysis I as well as Stephen Abbott - Understanding Analysis.*

Let  $f$  be a real-valued function defined in a neighborhood of a point  $a \in \mathbb{R}$ . In intuitive terms, the function  $f$  is continuous at  $a$  if its value  $f(x)$  approaches the value  $f(a)$  that it assumes at the point  $a$  itself as  $x$  gets nearer to  $a$ .

**Definition 1** (Continuous at a point). A function  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is *continuous at the point*  $a \in D$  if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \quad \left( |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right) \quad (1)$$

If  $f$  is continuous at every point in the domain  $D$ , then we say that  $f$  is *continuous on*  $D$ .

## 2 Examples

Here, we give a few examples of continuous functions alongside the respective proofs.

### 2.1 The constant function is continuous

**Theorem 1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function given by  $f(x) := c$ , where  $c \in \mathbb{R}$ . That is,  $f$  is a constant function. Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto c$  is continuous. Let  $\varepsilon > 0$ . We choose  $\delta := 1 > 0$ . Let  $x \in \mathbb{R}$ . Then:

$$|f(x) - f(a)| = |c - c| = |0| = 0 < \varepsilon$$

With that, the implication also holds true since its conclusion is always true (as shown above) irregardless of the premise.

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

Therefore,  $f$  is continuous at  $a$ . □

### 2.2 Functions $x \mapsto mx + y_0$ are continuous

**Theorem 2.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function given by  $f(x) := m \cdot x + y_0$ , where  $m, y_0 \in \mathbb{R}$ . Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto m \cdot x + y_0$  is continuous.

We first consider the simpler case where the slope is 0, that is  $\mathbf{m} = \mathbf{0}$ . Then our function is given by  $f(x) = y_0$  for all  $x \in \mathbb{R}$ . This is a constant function and we have already shown that constant functions are continuous. Therefore,  $f$  is continuous at  $a$  when  $m = 0$ .

Now to the more interesting case where  $\mathbf{m} \neq \mathbf{0}$ . Let  $\varepsilon > 0$ . We choose  $\delta := \frac{\varepsilon}{|m|}$ . Since  $\varepsilon > 0$  and  $|m| > 0$ , we have  $\delta > 0$ .

Let  $x \in \mathbb{R}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned} |f(x) - f(a)| &= |(m \cdot x + y_0) - (m \cdot a + y_0)| \\ &= |m \cdot x - m \cdot a| \\ &= |m \cdot (x - a)| \\ &= |m| \cdot |x - a| \\ &< |m| \cdot \delta = |m| \cdot \frac{\varepsilon}{|m|} = \varepsilon \end{aligned}$$

In the last line, we have used the fact that  $|x - a| < \delta$  and then used our definition of  $\delta$ .

The argument shows that  $|f(x) - f(a)| < \varepsilon$ , which proves the continuity of  $f$  at  $a$ . □

### 2.3 The parabola is continuous

**Theorem 3.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) := x^2$ . That is,  $f$  is a parabola. Then  $f$  is continuous at every point  $a \in \mathbb{R}$ .*

*Proof.* Let  $a \in \mathbb{R}$  be an arbitrary point where we want to show that  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$  is continuous.

Let  $\epsilon > 0$ . We choose  $\delta := \min\left(\frac{\epsilon}{2|x|+1}, 1\right)$ . Since  $\epsilon > 0$  and  $2|x|+1 > 0$  and  $1 > 0$ , we have  $\delta > 0$ .

Let  $x \in \mathbb{R}$  and  $|x - a| < \delta$ . Then:

$$\begin{aligned}
 |f(x) - f(a)| &= |x^2 - a^2| = |(x+a)(x-a)| \\
 &= |x+a| \cdot |x-a| && \left. \begin{array}{l} \downarrow |x+a| \leq |x|+|a| \\ \downarrow |x-a| < \delta \text{ and } |a| < |x|+\delta \text{ (*)} \end{array} \right\} \\
 &\leq (|x|+|a|) \cdot |x-a| \\
 &< (|x|+(|x|+\delta)) \cdot \delta \\
 &= (2|x|+\delta) \cdot \delta && \left. \downarrow \delta \leq 1 \text{ and } \delta \leq \frac{\epsilon}{2|x|+1} \right\} \\
 &\leq (2|x|+1) \cdot \frac{\epsilon}{2|x|+1} \\
 &= \epsilon
 \end{aligned}$$

(\*) is true because  $|a| = |x + (a - x)| \leq |x| + |a - x| = |x| + |x - a| < |x| + \delta$ . Therefore,  $|a| < |x| + \delta$ .

The argument shows that  $|f(x) - f(a)| < \epsilon$ , which proves the continuity of  $f$  at  $a$ .  $\square$

## 2.4 The hyperbola is continuous

*Proof.* Sei  $x \in \mathbb{R}$  mit  $x \neq 0$  und  $\epsilon > 0$  gegeben. Wir müssen ein  $\delta > 0$  finden, so dass für alle  $y$  mit  $0 < |y - x| < \delta$  gilt, dass  $\left| \frac{1}{y} - \frac{1}{x} \right| < \epsilon$ .

Setze  $\delta = \min\left(\frac{\epsilon|x|^2}{2}, \frac{|x|}{2}\right)$ .

- Da  $\epsilon > 0$  und  $|x| > 0$ , ist  $\delta > 0$ .

Sei nun  $y$  mit  $y \neq 0$  und  $|y - x| < \delta$  gegeben.

- Zuerst zeigen wir, dass  $\left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right|$ :
$$\begin{aligned} \left| \frac{1}{x} - \frac{1}{y} \right| &= \left| \frac{y-x}{xy} \right| \\ &= \frac{|y-x|}{|x||y|} \end{aligned}$$

- Da  $|y - x| < \delta \leq \frac{|x|}{2}$ , folgt  $|y| > \frac{|x|}{2}$ :

$$\begin{aligned} |y| &= |x + (y - x)| \\ &\geq |x| - |y - x| \\ &> |x| - \frac{|x|}{2} \\ &= \frac{|x|}{2} \end{aligned}$$

- Da  $\delta \leq \frac{\epsilon|x|^2}{2}$ , folgt:

$$\begin{aligned} \frac{|x-y|}{|x||y|} &< \frac{\delta}{|x| \cdot \frac{|x|}{2}} \\ &= \frac{\delta}{\frac{|x|^2}{2}} \\ &\leq \frac{\frac{\epsilon|x|^2}{2}}{\frac{|x|^2}{2}} \\ &= \epsilon \end{aligned}$$

Somit haben wir gezeigt, dass für alle  $y$  mit  $y \neq 0$  und  $|y - x| < \delta$  gilt, dass  $\left| \frac{1}{y} - \frac{1}{x} \right| < \epsilon$ .  
Daher ist  $f$  stetig an  $x$ .  $\square$