Continuous Functions

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1 Continuous Functions

Text excerpts remixed from Vladimir A. Zorich - Mathematical Analysis I as well as Stephen Abbott - Understanding Analysis.

Let f be a real-valued function defined in a neighborhood of a point $a \in \mathbb{R}$. In intuitive terms, the function f is continuous at a if its value $f(x)$ approaches the value $f(a)$ that it assumes at the point a itself as x gets nearer to a.

Definition 1 (Continuous at a point)**.** A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at the point* $a \in D$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \quad \Big(|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon \Big) \tag{1}$$

If f is continuous at every point in the domain D, then we say that f is *continuous on D*.

2 Examples

Here, we give a few examples of continuous functions alongside the respective proofs.

2.1 The constant function is continuous

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) \coloneqq c$, where $c \in \mathbb{R}$. That is, f is a constant function. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto c$ is continuous. Let $\varepsilon > 0$. We choose $\delta \coloneqq 1 > 0$. Let $x \in \mathbb{R}$. Then:

$$|f(x) - f(a)| = |c - c| = |0| = 0 < \varepsilon$$

With that, the implication also holds true since its conclusion is always true (as shown above) irregardless of the premise.

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

Therefore, f is continuous at a. □

2.2 Functions $x \mapsto mx + y_0$ are continuous

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) \coloneqq m \cdot x + y_0$, where $m, y_0 \in \mathbb{R}$. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto m \cdot x + y_0$ is continuous.

We first consider the simpler case where the slope is 0, that is $\mathbf{m = 0}$. Then our function is given by $f(x) = y_0$ for all $x \in \mathbb{R}$. This is a constant function and we have already shown that constant functions are continuous. Therefore, f is continuous at a when $m = 0$.

Now to the more interesting case where $\mathbf{m \neq 0}$. Let $\varepsilon > 0$. We choose $\delta \coloneqq \frac{\varepsilon}{|m|}$. Since $\epsilon > 0$ and $|m| > 0$, we have $\delta > 0$.

Let $x \in \mathbb{R}$ and $|x - a| < \delta$. Then:

$$\begin{aligned} |f(x) - f(a)| &= \big|(m \cdot x + y_0) - (m \cdot a - y_0)\big| \\ &= |m \cdot x - m \cdot a| \\ &= |m \cdot (x - a)| \\ &= |m| \cdot |x - a| \\ &< |m| \cdot \delta = |m| \cdot \frac{\varepsilon}{|m|} = \varepsilon \end{aligned}$$

In the last line, we have used the fact that $|x - a| < \delta$ and then used our definition of δ.

The argument shows that $|f(x) - f(a)| < \varepsilon$, which proves the continuity of f at a. □

2.3 The parabola is continuous

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) \coloneqq x^2$. That is, f is a parabola. Then f is continuous at every point $a \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$ be an arbitrary point where we want to show that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous.

Let $\epsilon > 0$. We choose $\delta \coloneqq \min\left(\frac{\epsilon}{2|x|+1}, 1\right)$. Since $\epsilon > 0$ and $2|x|+1 > 0$ and $1 > 0$, we have $\delta > 0$.

Let $x \in \mathbb{R}$ and $|x - a| < \delta$. Then:

$$
\begin{aligned}
|f(x) - f(a)| &= |x^2 - a^2| = |(x+a)(x-a)| \\
&= |x+a| \cdot |x-a| \\
&\leq \big(|x| + |a|\big) \cdot |x-a| && |x+a| \leq |x| + |a| \\
&\leq \big(|x| + |a|\big) \cdot \delta && |x-a| < \delta \\
&< \big(|x| + (|x| + \delta)\big) \cdot \delta && |a| < |x| + \delta \text{ (*)} \\
&= \big(2|x| + \delta\big) \cdot \delta \\
&\leq \big(2|x| + 1\big) \cdot \tfrac{\epsilon}{2|x|+1} && \delta \leq 1 \textit{ and } \delta \leq \frac{\epsilon}{2|x|+1} \\
&= \epsilon
\end{aligned}
$$

(*) is true because $|a| = |x + (a - x)| \leq |x| + |a - x| = |x| + |x - a| < |x| + \delta$. Therefore, $|a| < |x| + \delta$.

The argument shows that $|f(x) - f(a)| < \epsilon$, which proves the continuity of f at a. $\square$

2.4 The hyperbola is continuous

Proof. Sei $x \in \mathbb{R}$ mit $x \neq 0$ und $\epsilon > 0$ gegeben. Wir müssen ein $\delta > 0$ finden, so dass für alle y mit $0 < |y - x| < \delta$ gilt, dass $\left|\frac{1}{y} - \frac{1}{x}\right| < \epsilon$.

Setze $\delta = \min\left(\frac{\epsilon|x|^2}{2}, \frac{|x|}{2}\right)$.

- Da $\epsilon > 0$ und $|x| > 0$, ist $\delta > 0$.

Sei nun y mit $y \neq 0$ und $|y - x| < \delta$ gegeben.

- Zuerst zeigen wir, dass $\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right|$:

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{y}\right| &= \left|\frac{y-x}{xy}\right| \\ &= \frac{|y-x|}{|x||y|}\end{aligned}$$

- Da $|y - x| < \delta \leq \frac{|x|}{2}$, folgt $|y| > \frac{|x|}{2}$:

$$\begin{aligned}|y| &= |x + (y - x)| \\ &\geq |x| - |y - x| \\ &> |x| - \frac{|x|}{2} \\ &= \frac{|x|}{2}\end{aligned}$$

- Da $\delta \leq \frac{\epsilon|x|^2}{2}$, folgt:

$$\begin{aligned}\frac{|x-y|}{|x||y|} &< \frac{\delta}{|x| \cdot \frac{|x|}{2}} \\ &= \frac{\delta}{\frac{|x|^2}{2}} \\ &\leq \frac{\frac{\epsilon|x|^2}{2}}{\frac{|x|^2}{2}} \\ &= \epsilon\end{aligned}$$

Somit haben wir gezeigt, dass für alle y mit $y \neq 0$ und $|y - x| < \delta$ gilt, dass $\left|\frac{1}{y} - \frac{1}{x}\right| < \epsilon$. Daher ist f stetig an x. □