Energy in the Damped Harmonic Oscillator

Hywel Owen
University of Manchester
hywel.owen@manchester.ac.uk

January 21, 2020

Abstract

In this short note we consider the damped Simple Harmonic Oscillator, the expected variation of energy with time, and error in using Euler's Method compared with the Euler-Cromer Method.

1 The Damped Simple Harmonic Oscillator

Ignoring the possibility of an initial velocity $v(0)$, we may write an expression for damped SHM as

$$x(t) = Ae^{-bt/2m}\cos(\sqrt{(\frac{k}{m} - \frac{b^2}{4m^2})}t), \tag{1}$$

where k, b, m have their usual meaning and $x(0) = A$ is the initial amplitude of the particle. Obviously,

$$v(t) = \frac{dx(t)}{dt}, \tag{2}$$

and the total energy is just

$$E(t) = \frac{1}{2}kx^2(t) + \frac{1}{2}mv^2(t) \tag{3}$$

which doing the derivative and collecting terms is

$$E(t) = \frac{A^2 e^{-bt/m}}{8m}(4km + b^2 \cos(\frac{\sqrt{4km - b^2}t}{m}) + b\sqrt{4km - b^2} \sin(\frac{\sqrt{4km - b^2}t}{m})). \tag{4}$$

Notice that there is a monotonically-decreasing term, and two sinusoidal terms. The sinusoidal terms reflect the fact that the rate of energy reduction is not constant, but is proportional the velocity at a given time. Hence we expect the energy profile to have a superposed 'wobble'.

2 Fictitious Energy From Using Euler's Method

We already know that Euler's method may be expressed, for some time step h, and assuming no damping, as

$$x_{n+1} = x_n + hv_n \tag{5}$$

$$v_{n+1} = v_n - \frac{kh}{m}x_n \tag{6}$$

Let's calculate the energy after the step. Obviously,

$$E_{n+1} = \frac{1}{2}kx_{n+1}^2 + \frac{1}{2}mv_{n+1}^2. \tag{7}$$

Substituting in our expressions for x_{n+1} and v_{n+1} above, and expanding, we have

$$E_{n+1} = \frac{1}{2}kx_n^2 + \frac{1}{2}mv_n^2 + \frac{1}{2m}h^2k^2x_n^2 + \frac{1}{2}h^2kv_n^2. \tag{8}$$

Collecting terms, this can be re-written in terms of E_n as simply

$$E_{n+1} = E_n(1 + \frac{k}{m}h^2). \tag{9}$$

What does this mean? In the undamped case, know that the real physical system will have constant energy. However, our numerical method is giving us a fictitious energy increase at each step.

3 The Euler-Cromer Method

Can we improve on Euler's method? Let's try a little trick. Instead of updating x_{n+1} with v_n, let's use our value of v_{n+1} instead, in other words

$$
\begin{aligned}
x_{n+1} &= x_n + hv_{n+1} && (10) \\
v_{n+1} &= v_n - \frac{kh}{m}x_n && (11)
\end{aligned}
$$

This is the Euler-Cromer method. Note the change in the first of these two equations (look closely). We expand out our expression for E_{n+1} as before, to obtain this time

$$E_{n+1} = E_n - \frac{1}{2}h^2(\frac{k^2x_n^2}{m} - kv_n^2) - h^3\frac{k^2x_nv_n}{m} + h^4\frac{k^3x_n^2}{2m^2} \tag{12}$$

The second term (in h^2) averages out over one complete period of oscillation. The result is that the overall energy is conserved, although there are oscillations about this average. The Euler-Cromer method is one of a class of energy-preserving methods called Symplectic Integrators. Symplectic Integrators preserve the essence of the motion (the energy), whilst not preserving the exact detail of the motion. However, they are a lot better than non-symplectic methods.