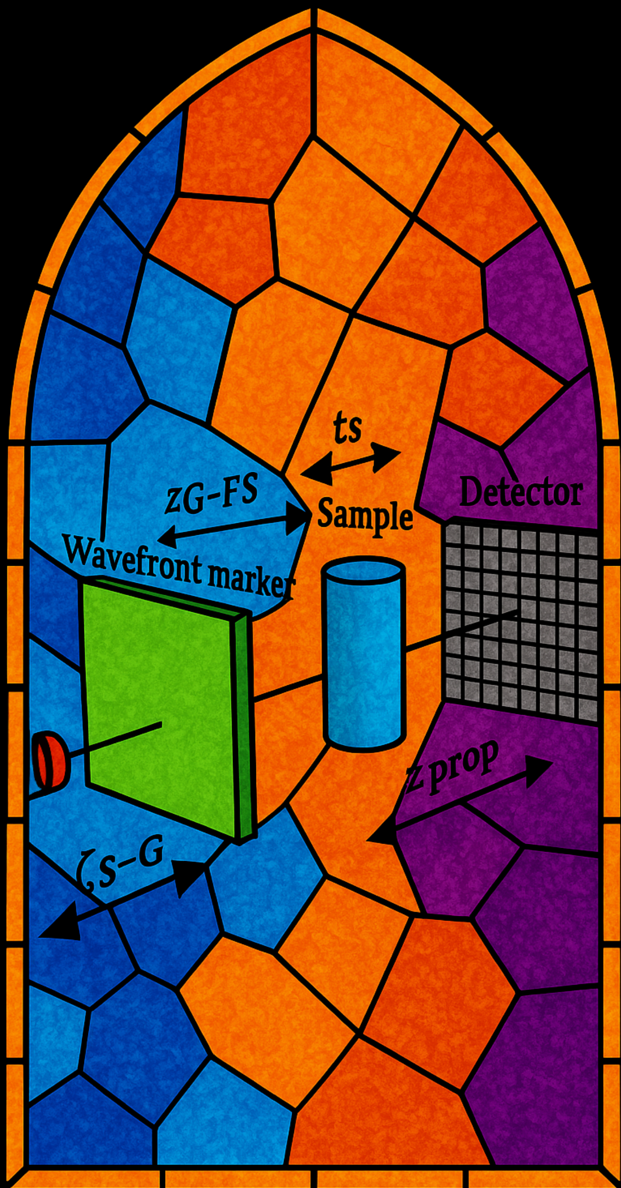
Brief Course on X-Ray Physics



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Chapter 2: Propagation of X-Rays

Derivation of Fresnel propagator

Let's begin with the wave equation (for the electric field associated to an X-ray beam propagating inside a uniform material)

$$\nabla^2 \vec{E}(\vec{r},t) = \frac{1}{v^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$= \frac{n^2(E)}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

where $\vec{E}(\vec{r},t)$ is the electric field, v is the speed of light within a uniform material with refractive index $n(E)$, E is the energy of the X-rays and $\vec{r}$ is the position vector.

Assuming monochromatic plane waves: $\vec{E}(\vec{r},t) = \psi(\vec{r})\, e^{-i\omega t}\, \hat{e}$,
where $\psi(\vec{r})$ is a scalar function, $\omega = \frac{E}{\hbar}$ and $\hat{e}$ is the unit vector that indicates the direction of oscillation of the electric field.
Then:

$$\nabla^2 \vec{E}(\vec{r},t) = \left(\nabla^2 \psi(\vec{r})\right) e^{-i\omega t}\, \hat{e}$$

$$\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = -\psi(\vec{r})\, \omega^2 e^{-i\omega t}\, \hat{e}$$

$$\Rightarrow \quad \left(\nabla^2 \psi(\vec r)\right) \cancel{e^{-i\omega t}} = -\frac{n^2(E)}{c^2}\,\omega^2 \psi(\vec r)\, \cancel{e^{-i\omega t}}$$

$$\Rightarrow \quad \nabla^2 \psi(\vec r) + \frac{n^2(E)}{c^2}\,\omega^2 \psi(\vec r) = 0$$

$$\Rightarrow \quad \nabla^2 \psi(\vec r) + n^2(E)\, k^2 \psi(\vec r) = 0$$

(Helmholtz Equation)

where $\quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

Since electric field was assumed a plane wave and if we assume the photon beam propagates along the z-axis:

$$\psi(\vec r) = \psi_0(\vec r)\, e^{in(E)kz}$$

where $\psi_0(\vec r)$ is a scalar function. Then:

$$\Rightarrow \nabla^2\left(\psi_0(\vec r)\, e^{in(E)kz}\right) + n^2(E)\, k^2 \psi_0(\vec r)\, e^{in(E)kz} = 0$$

$$\Rightarrow e^{in(E)kz}\frac{\partial^2\psi_0}{\partial x^2} + e^{in(E)kz}\frac{\partial^2\psi_0}{\partial y^2} + \frac{\partial^2}{\partial z^2}\left(\psi_0(\vec r)\, e^{in(E)kz}\right) + n^2(E)\, k^2 \psi_0(\vec r)\, e^{in(E)kz} = 0$$

$$\Rightarrow e^{in(E)kz}\underbrace{\left(\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2}\right)}_{\nabla_\perp^2 \psi_0(\vec r)} + \frac{\partial}{\partial z}\left(\frac{\partial\psi_0}{\partial z}\, e^{in(E)kz} + \psi_0\, in(E)k\, e^{in(E)kz}\right) + n^2(E)\, k^2 \psi_0\, e^{in(E)kz} = 0$$

$$\Rightarrow e^{in(\epsilon)kz}\nabla_\perp^2\psi_0(\vec{r}) + \frac{\partial^2\psi_0}{\partial z^2}e^{in(\epsilon)kz} + \frac{\partial\psi_0}{\partial z}in(\epsilon)k\,e^{in(\epsilon)kz} + \frac{\partial\psi_0}{\partial z}in(\epsilon)k\,e^{in(\epsilon)kz}$$

$$-\cancel{\psi_0 n^2(\epsilon)k^2 e^{in(\epsilon)kz}} + \cancel{n^2(\epsilon)k^2\psi_0 e^{in(\epsilon)kz}} = 0$$

$$\Rightarrow \cancel{e^{in(\epsilon)kz}}\nabla_\perp^2\psi_0(\vec{r}) + \frac{\partial^2\psi_0}{\partial z^2}\cancel{e^{in(\epsilon)kz}} + 2\frac{\partial\psi_0}{\partial z}in(\epsilon)k\,\cancel{e^{in(\epsilon)kz}} = 0$$

$$\Rightarrow \nabla_\perp^2\psi_0(\vec{r}) + \frac{\partial^2\psi_0}{\partial z^2} + 2\frac{\partial\psi_0}{\partial z}in(\epsilon)k = 0$$

In the near-field regime, the paraxial approximation is applied.

$$\left|\frac{\partial^2\psi_0}{\partial z^2}\right| \ll \left|2in(\epsilon)k\frac{\partial\psi_0}{\partial z}\right| \quad \text{(Slowly-varying } \psi_0\text{)}$$

$$\Rightarrow \nabla_\perp^2\psi_0(\vec{r}) + 2\frac{\partial\psi_0}{\partial z}in(\epsilon)k = 0 \quad \text{(Helmholtz paraxial Equation)}$$

Let's find a solution: Apply Fourier transform over the transverse plane (xy):

$$-(k_x^2+k_y^2)\underbrace{\mathcal{F}_{2D}\{\psi_0(\vec{r})\}}_{\tilde{\psi}_0(k_x,k_y,z)\,=\,\tilde{\psi}_0} + 2in(\epsilon)k\frac{\partial}{\partial z}\mathcal{F}_{2D}\{\psi_0(\vec{r})\} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial z}\tilde{\psi}_0 = \frac{(k_x^2 + k_y^2)}{2in(E)k}\tilde{\psi}_0$$

$$\Rightarrow \quad \tilde{\psi}_0(k_x, k_y, z) = \tilde{\psi}_0(k_x, k_y, z=0)\, e^{\frac{(k_x^2+k_y^2)}{2in(E)k} z}$$

Since $\mathcal{F}_{2D}\{\psi_0(\vec{r})\} = \tilde{\psi}_0(k_x, k_y, z) \Rightarrow \mathcal{F}_{2D}^{-1}\{\tilde{\psi}_0(k_x, k_y, z)\} = \psi_0(\vec{r})$

$$\Rightarrow \quad \mathcal{F}_{2D}^{-1}\{\tilde{\psi}_0(k_x, k_y, z)\} = \mathcal{F}_{2D}^{-1}\{\tilde{\psi}_0(k_x, k_y, z=0)\, e^{\frac{(k_x^2+k_y^2)}{2in(E)k} z}\}$$

$$\Rightarrow \quad \psi_0(\vec{r}) = \mathcal{F}_{2D}^{-1}\{\tilde{\psi}_0(k_x, k_y, z=0)\, e^{\frac{(k_x^2+k_y^2)}{2in(E)k} z}\}$$

Given that $\mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} \underbrace{\otimes}_{\text{Convolution}} \mathcal{F}\{g\}$

$$\Rightarrow \quad \psi_0(\vec{r}) = \mathcal{F}_{2D}^{-1}\{\tilde{\psi}_0(k_x, k_y, z=0)\} \otimes \mathcal{F}_{2D}^{-1}\{e^{\frac{(k_x^2+k_y^2)}{2in(E)k} z}\}$$

$$= \psi_0(x, y, z=0) \otimes \mathcal{F}_{2D}^{-1}\{e^{\frac{(k_x^2+k_y^2)}{2in(E)k} z}\}$$

$$= \psi_0(x, y, z=0) \otimes \mathcal{F}_{2D}^{-1}\{\underbrace{e^{\frac{k_x^2}{2in(E)k} z}}_{\text{Depends only on } k_x} \underbrace{e^{\frac{k_y^2}{2in(E)k} z}}_{\text{only on } k_y}\}$$

$$= \psi_0(x, y, z=0) \otimes \mathcal{F}_{1D}^{-1}\{e^{\frac{k_x^2}{2in(E)k} z}\}\, \mathcal{F}_{1D}^{-1}\{e^{\frac{k_y^2 z}{2in(E)k}}\}$$

$$= \psi_0(x, y, z=0) \otimes \mathcal{F}_{1D}^{-1}\{e^{\frac{-ik_x^2 z}{2n(E)k}}\}\, \mathcal{F}_{1D}^{-1}\{e^{\frac{-ik_y^2 z}{2n(E)k}}\}$$

Since

$$\mathcal{F}_{1D}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{k_x^2}{4a}}$$

$$\Rightarrow \mathcal{F}_{1D}^{-1}\{e^{-\frac{k_x^2}{4a}}\} = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$$

Identify: $\frac{1}{4a} = \frac{iz}{2n(\epsilon)k} \Rightarrow a = \frac{n(\epsilon)k}{2iz}$

$$\Rightarrow \psi_o(\vec{r}) = \psi_o(x,y,z=0) \otimes \left[\left(\sqrt{\frac{n(\epsilon)k}{2\pi i z}}\right) e^{-\frac{n(\epsilon)k}{2iz}x^2} \left(\sqrt{\frac{n(\epsilon)k}{2\pi i z}}\right) e^{-\frac{n(\epsilon)k}{2iz}y^2}\right]$$

$$= \psi_o(x,y,z=0) \otimes \left[\frac{n(\epsilon)k}{2\pi i z}\, e^{-\frac{n(\epsilon)k}{2iz}(x^2+y^2)}\right]$$

$$= \psi_o(x,y,z=0) \otimes \left[\frac{n(\epsilon)}{\cancel{2\pi} iz}\, \frac{\cancel{2\pi}}{\lambda}\, e^{\frac{in(\epsilon)k}{2z}(x^2+y^2)}\right]$$

$$= \psi_o(x,y,z=0) \otimes \underbrace{\left[\frac{n(\epsilon)}{i\lambda z}\, e^{\frac{in(\epsilon)k}{2z}(x^2+y^2)}\right]}_{H(x,y,z)\text{ : Fresnel propagator}}$$

$$\Rightarrow \psi_o(\vec{r}) = \frac{n(\epsilon)}{i\lambda z} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_o(x',y',z=0)\, e^{\frac{in(\epsilon)k}{2z}\left[(x-x')^2+(y-y')^2\right]}\, dx'dy'$$

Since $\psi(\vec{r}) = \psi_0(\vec{r})\, e^{in(E)kz}$ and hence $\psi(x,y,z=0) = \psi_0(x,y,z=0)$

$$\Rightarrow \psi(\vec{r}) = \left[\psi(x,y,z=0) \otimes \left[\frac{n(E)}{i\lambda z} e^{\frac{in(E)k}{2z}(x^2+y^2)}\right]\right] e^{in(E)kz}$$

$$\Rightarrow \psi(\vec{r}) = \frac{n(E)\, e^{in(E)kz}}{i\lambda z} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x',y',z=0)\, e^{\frac{in(E)k}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'dy'$$

(Fresnel Diffraction Integral)

In simple words: If you have an initial wavefield $\psi(x,y,z=0)$ and you want to propagate it across a material with refractive index $n(E)$ to a position z along the propagation axis, then the propagated wavefield $\psi(\vec{r})$ is $\boxed{\psi(\vec{r}) = \psi(x,y,z=0) \otimes H(x,y,z)}$

- In vacuum: $n(E) = 1$. In air $n(E) \approx 1$.

Transport of Intensity Equation (TIE):

Let's begin with the Helmholtz paraxial equation:

$$\nabla_\perp^2 \psi_0(\vec{r}) + 2\frac{\partial \psi_0}{\partial z} in(E)k = 0$$

To make things easy, let's consider $n(E) = 1$ (vacuum or air).

Since $\psi_0(\vec{r})$ is a complex number, then we can write it as

$$\psi_0(\vec{r}) = A(\vec{r})\, e^{i\phi(\vec{r})}.$$

$$\Rightarrow \nabla_\perp^2 \left(A(\vec{r})\, e^{i\phi(\vec{r})}\right) + 2\frac{\partial}{\partial z}\left(A(\vec{r})\, e^{i\phi(\vec{r})}\right) ik = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left(Ae^{i\phi}\right) + \frac{\partial^2}{\partial y^2}\left(Ae^{i\phi}\right) + 2\frac{\partial}{\partial z}\left(Ae^{i\phi}\right) ik = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x} e^{i\phi} + Ai\frac{\partial \phi}{\partial x} e^{i\phi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial y} e^{i\phi} + Ai\frac{\partial \phi}{\partial y} e^{i\phi}\right) + 2\frac{\partial A}{\partial z} e^{i\phi} ik - 2Ak\frac{\partial \phi}{\partial z} e^{i\phi} = 0$$

$$\Rightarrow \frac{\partial^2 A}{\partial x^2}\cancel{e^{i\phi}} + \frac{\partial A}{\partial x} i \frac{\partial \phi}{\partial x}\cancel{e^{i\phi}} + i\frac{\partial A}{\partial x}\frac{\partial \phi}{\partial x}\cancel{e^{i\phi}} + iA\frac{\partial^2 \phi}{\partial x^2}\cancel{e^{i\phi}} - A\left(\frac{\partial \phi}{\partial x}\right)^2 \cancel{e^{i\phi}} + \ldots$$

$$\ldots \frac{\partial^2 A}{\partial y^2}\cancel{e^{i\phi}} + \frac{\partial A}{\partial y} i \frac{\partial \phi}{\partial y}\cancel{e^{i\phi}} + i\frac{\partial A}{\partial y}\frac{\partial \phi}{\partial y}\cancel{e^{i\phi}} + iA\frac{\partial^2 \phi}{\partial y^2}\cancel{e^{i\phi}} - A\left(\frac{\partial \phi}{\partial y}\right)^2 \cancel{e^{i\phi}} + \ldots$$

$$\ldots\ 2\frac{\partial A}{\partial z}\cancel{e^{i\phi}} ik - 2Ak\frac{\partial \phi}{\partial z}\cancel{e^{i\phi}} = 0$$

$$\Rightarrow \frac{\partial^2 A}{\partial x^2} + 2i\frac{\partial A}{\partial x}\frac{\partial \phi}{\partial x} + iA\frac{\partial^2 \phi}{\partial x^2} - A\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial^2 A}{\partial y^2} + 2i\frac{\partial A}{\partial y}\frac{\partial \phi}{\partial y} + iA\frac{\partial^2 \phi}{\partial y^2} - A\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ 2ik\frac{\partial A}{\partial z} - 2Ak\frac{\partial \phi}{\partial z} = 0$$

Since

$$I(\vec{r}) = |\psi_0(\vec{r})|^2 = A^2(\vec{r}) \rightarrow A(\vec{r}) = \sqrt{I(\vec{r})}$$

$$\rightarrow \quad \frac{\partial I}{\partial x} = 2A\frac{\partial A}{\partial x} \ , \quad \frac{\partial I}{\partial y} = 2A\frac{\partial A}{\partial y} \ , \quad \frac{\partial I}{\partial z} = 2A\frac{\partial A}{\partial z}$$

$$\rightarrow \quad \frac{\partial^2 I}{\partial x^2} = 2\left(\frac{\partial A}{\partial x}\right)^2 + 2A\frac{\partial^2 A}{\partial x^2} \ , \quad \frac{\partial^2 I}{\partial y^2} = 2\left(\frac{\partial A}{\partial y}\right)^2 + 2A\frac{\partial^2 A}{\partial y^2}$$

Assuming slowly-varying Intensity:

$$\frac{\partial^2 A}{\partial x^2} \simeq \frac{\partial^2 A}{\partial y^2} \simeq \left(\frac{\partial A}{\partial x}\right)^2 \simeq \left(\frac{\partial A}{\partial y}\right)^2 \simeq 0$$

$$\Rightarrow \quad \frac{\partial^2 I}{\partial x^2} \simeq 0 \ , \quad \frac{\partial^2 I}{\partial y^2} \simeq 0$$

Then:

$$\cancelto{\simeq 0}{\frac{\partial^2 A}{\partial x^2}} + 2i\frac{\partial A}{\partial x}\frac{\partial \phi}{\partial x} + iA\frac{\partial^2 \phi}{\partial x^2} - A\left(\frac{\partial \phi}{\partial x}\right)^2 + \cancelto{\simeq 0}{\frac{\partial^2 A}{\partial y^2}} + 2i\frac{\partial A}{\partial y}\frac{\partial \phi}{\partial y} + iA\frac{\partial^2 \phi}{\partial y^2} - A\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ 2ik\frac{\partial A}{\partial z} - 2Ak\frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow 2i\frac{1}{2\sqrt{I}}\frac{\partial I}{\partial x}\frac{\partial \phi}{\partial x} + i\sqrt{I}\frac{\partial^2 \phi}{\partial x^2} - \sqrt{I}\left(\frac{\partial \phi}{\partial x}\right)^2 + 2i\frac{1}{2\sqrt{I}}\frac{\partial I}{\partial y}\frac{\partial \phi}{\partial y} + i\sqrt{I}\frac{\partial^2 \phi}{\partial y^2} - \sqrt{I}\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ 2ik\frac{1}{2\sqrt{I}}\frac{\partial I}{\partial z} - 2\sqrt{I}k\frac{\partial \phi}{\partial z} = 0$$

Multiplying by $\sqrt{I}$:

$$\Rightarrow i\frac{\partial I}{\partial x}\frac{\partial \phi}{\partial x} + iI\frac{\partial^2 \phi}{\partial x^2} - I\left(\frac{\partial \phi}{\partial x}\right)^2 + i\frac{\partial I}{\partial y}\frac{\partial \phi}{\partial y} + iI\frac{\partial^2 \phi}{\partial y^2} - I\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ ik\frac{\partial I}{\partial z} - 2Ik\frac{\partial \phi}{\partial z} = 0$$

Dividing by i:

$$\Rightarrow \frac{\partial I}{\partial x}\frac{\partial \phi}{\partial x} + I\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{i}I\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial I}{\partial y}\frac{\partial \phi}{\partial y} + I\frac{\partial^2 \phi}{\partial y^2} - \frac{I}{i}\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ k\frac{\partial I}{\partial z} - \frac{2}{i}Ik\frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow \frac{\partial I}{\partial x}\frac{\partial \phi}{\partial x} + I\frac{\partial^2 \phi}{\partial x^2} + iI\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial I}{\partial y}\frac{\partial \phi}{\partial y} + I\frac{\partial^2 \phi}{\partial y^2} + iI\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$+ k\frac{\partial I}{\partial z} + 2iIk\frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow -k\frac{\partial I}{\partial z} = \left[\frac{\partial I}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial I}{\partial y}\frac{\partial \phi}{\partial y} + I\frac{\partial^2 \phi}{\partial x^2} + I\frac{\partial^2 \phi}{\partial y^2}\right] + i\left[I\left(\frac{\partial \phi}{\partial x}\right)^2 + I\left(\frac{\partial \phi}{\partial y}\right)^2 + 2Ik\frac{\partial \phi}{\partial z}\right]$$

$$\Rightarrow -k\frac{\partial I}{\partial z} = \left[\left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right)\cdot\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) + I\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\right] + iI\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2k\frac{\partial \phi}{\partial z}\right]$$

$$\Rightarrow -k\frac{\partial I}{\partial z} = \left[\nabla_\perp I \cdot \nabla_\perp \phi + I\nabla_\perp^2 \phi\right] + iI\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2k\frac{\partial \phi}{\partial z}\right]$$

Since $k, I \in \mathbb{R}$, then $\mathrm{Im}\left(-k\frac{\partial I}{\partial z}\right) = 0$, Then

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2k\frac{\partial \phi}{\partial z} = 0$$

and:

$$-k\frac{\partial}{\partial z}I(\vec{r}) = I(\vec{r})\nabla_\perp^2\phi(\vec{r}) + \nabla_\perp I(\vec{r})\cdot\nabla_\perp\phi(\vec{r}) \quad (\mathrm{TIE})$$

For a pure-phase object it is assumed $I(\vec{r}) = I_0(z)$ (Constant with respect to (x,y))

Then:

$$-k\frac{dI_0(z)}{dz} = I_0(z)\,\nabla_\perp^2\phi(\vec{r})$$

Assuming $\phi(\vec{r}) = \phi(x,y)$ (Phase map):

$$-k\frac{dI_0(z)}{dz} = I_0(z)\,\nabla_\perp^2\phi(x,y)$$

$$\Rightarrow \int_{I_0(z=0)}^{I_0(z)} \frac{dI_0'(z)}{I_0'(z)} = -\frac{\nabla_\perp^2\phi(x,y)}{k}\int_{z=0}^{z} dz'$$

$$\Rightarrow \ln\left(\frac{I_0(z)}{I_0(z=0)}\right) = -\frac{\nabla_\perp^2\phi(x,y)}{k}\,z$$

$$\Rightarrow \quad I_0(z) = I_0(z=0)\, e^{-\frac{\nabla_\perp^2 \phi(x,y)\, z}{k}}$$

For X-rays: $\dfrac{\nabla_\perp^2 \phi(x,y)\, z}{k} \ll 1$

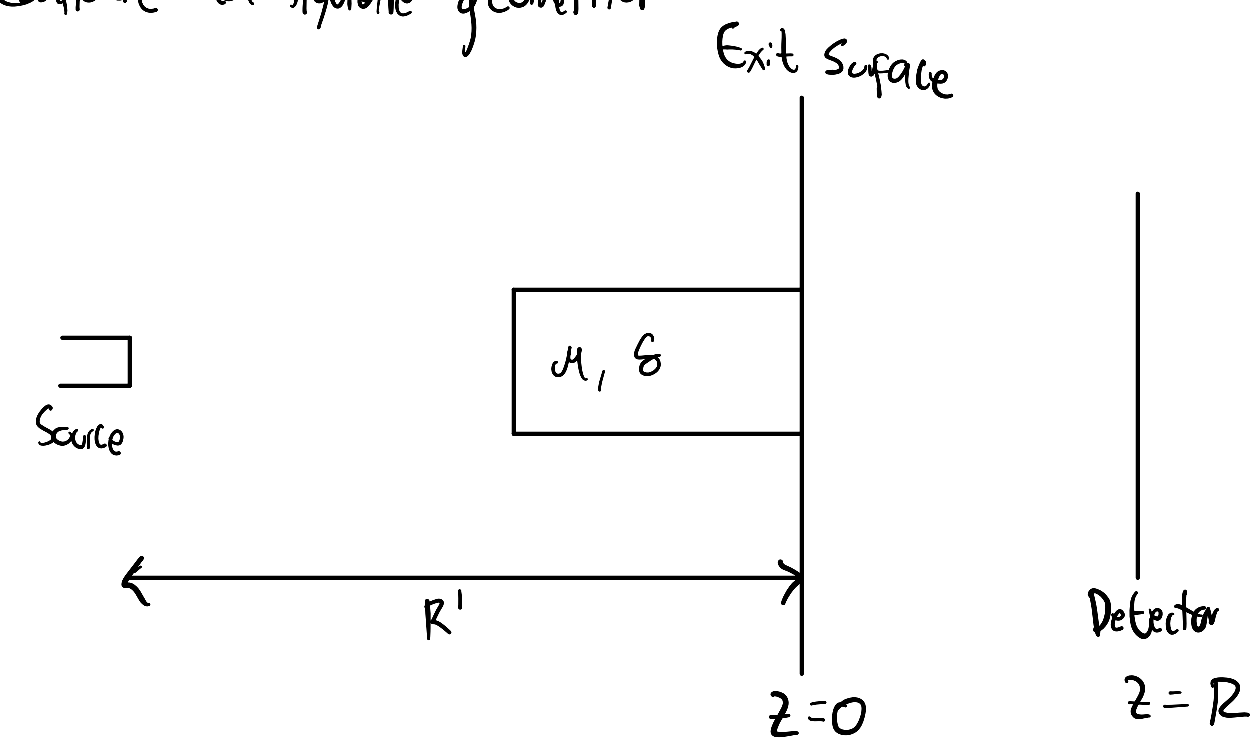
$$\Rightarrow \quad I_0(z) \approx I_0(z=0)\left[1 - \frac{z}{k}\nabla_\perp^2 \phi(x,y)\right]$$

Chapter 3: Phase Retrieval (Paganin)

De la última clase:

$$-k \frac{\partial I(\vec{r})}{\partial z} = I(\vec{r})\, \nabla_\perp^2 \phi(x,y) + \nabla_\perp I(\vec{r}) \cdot \nabla_\perp \phi(x,y)$$
$$= \nabla_\perp \cdot \left(I(\vec{r})\, \nabla_\perp \phi(x,y) \right)$$

Considere la siguiente geometría:



Dado que:

- $I(x,y,z=0) = I_{No\ samp}\, e^{\overbrace{-\mu t(x,y)}^{\text{No integral}}}$ → Projection – Approximation (Beer – Lambert)
- $\phi(x,y) = -\delta k t(x,y)$
- $\frac{\partial}{\partial z} I(x,y,z=0) \approx \frac{I(x,y,z=R) - I(x,y,z=0)}{R} = \frac{I(x,y,z=R) - I_{No\ samp}\, e^{-\mu t(x,y)}}{R}$

$$\Rightarrow \underset{\text{a } z=0}{(TIE)} \quad -k\frac{\partial}{\partial z} I(x,y,z=0) = \nabla_\perp \left(I(x,y,z=0)\, \nabla_\perp \phi(x,y) \right)$$

$$\Rightarrow \frac{\cancel{-k}\left(I(x,y,z=R) - I_{No\ samp}\, e^{-\mu t(x,y)} \right)}{R} = \nabla_\perp \left(I_{No\ samp}\, e^{-\mu t(x,y)} \, \nabla_\perp \left(\cancel{-\delta k}\, t(x,y) \right) \right)$$

$$\Rightarrow I(x,y,z=R) - I_{No\ samp}\, e^{-\mu t(x,y)} = \delta R I_{No\ samp} \nabla_\perp \cdot \left(e^{-\mu t(x,y)} \nabla_\perp t(x,y) \right)$$

$$= \delta R I_{No\ samp} \nabla_\perp \cdot \left(e^{-\mu t(x,y)} \left(-\frac{1}{\mu}\right) \nabla_\perp (-\mu t(x,y)) \right)$$

$$= -\frac{\delta}{\mu} R I_{No\ samp} \nabla_\perp \cdot \left(e^{-\mu t(x,y)} \nabla_\perp (-\mu t(x,y)) \right)$$

$$= -\frac{\delta}{\mu} R I_{No\ samp} \nabla_\perp \cdot \left(\nabla_\perp e^{-\mu t(x,y)} \right)$$

$$= -\frac{\delta}{\mu} R I_{No\ samp} \nabla_\perp^2 e^{-\mu t(x,y)}$$

$$\Rightarrow I(x,y,z=R) = I_{No\ samp} \left(e^{-\mu t(x,y)} - \frac{\delta R}{\mu} \nabla_\perp^2 e^{-\mu t(x,y)} \right)$$

$$\Rightarrow \frac{I(x,y,z=0)}{I_{no\ samp}} = e^{-\mu t(x,y)} - \frac{\delta R}{\mu} \nabla_\perp^2 e^{-\mu t(x,y)}$$

Necesitamos despejar $t(x,y)$!

$\Rightarrow$ Usar transformada de Fourier 2D sobre (x,y):

$$\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\} = \mathcal{F}_{2D}\left\{e^{-\mu t(x,y)}\right\} - \frac{\delta R}{\mu}\left(-k_x^2-k_y^2\right)\mathcal{F}_{2D}\left\{e^{-\mu t(x,y)}\right\}$$

$$= \left[1 + \frac{\delta}{\mu}R\left(k_x^2+k_y^2\right)\right]\mathcal{F}_{2D}\left\{e^{-\mu t(x,y)}\right\}$$

$$\Rightarrow \mathcal{F}_{2D}\left\{e^{-\mu t(x,y)}\right\} = \frac{1}{1+\frac{\delta}{\mu}R\underbrace{\left(k_x^2+k_y^2\right)}_{=k_\perp^2}}\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\}$$

$$= \frac{1}{1+\frac{\delta}{\mu}R\,k_\perp^2}\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\}$$

$$\Rightarrow e^{-\mu t(x,y)} = \mathcal{F}_{2D}^{-1}\left\{\frac{1}{1+\frac{\delta}{\mu}R\,k_\perp^2}\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\}\right\}$$

$$\Rightarrow t(x,y) = -\frac{1}{\mu}\ln\left[\mathcal{F}_{2D}^{-1}\left\{\frac{1}{1+\frac{\delta}{\mu}R\,k_\perp^2}\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\}\right\}\right]$$

$$\Rightarrow \phi(x,y) = \frac{\delta k}{\mu}\ln\left[\mathcal{F}_{2D}^{-1}\left\{\frac{1}{1+\frac{\delta}{\mu}R\,k_\perp^2}\mathcal{F}_{2D}\left\{\frac{I(x,y,z=0)}{I_{No\ Scnp}}\right\}\right\}\right]$$

Esta expresión es válida para rayos paralelos ($R' \to \infty$).

Si consideramos cone beam (efectos de magnificación), el Fresnel Scaling theorem indica:

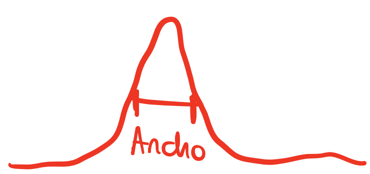
$$I_{mag}(x,y,z) = \frac{1}{M^2} I\left(\frac{x}{M},\frac{y}{M},\frac{z}{M}\right)$$

Con $M = \frac{R'+R}{R'}$ la magnificación. Luego:

$$\Rightarrow t(x,y) = -\frac{1}{\mu}\ln\left[\mathcal{F}_{2D}^{-1}\left\{\frac{1}{1+\frac{\delta}{\mu}\frac{R}{M}k_\perp^2}\,\mathcal{F}_{2D}\left\{\frac{M^2 I_{mag}(Mx,My,z=R)}{I_{no\ samp,\,mag}}\right\}\right\}\right]$$

Meaning:

$$t(x,y) = -\frac{1}{\mu}\ln\left[\mathcal{F}_{2D}^{-1}\left\{\underbrace{\frac{1}{1+\frac{\delta}{\mu}\frac{R}{M}k_\perp^2}}_{\text{Low-pass filter on Fourier domain}}\,\mathcal{F}_{2D}\left\{\underbrace{\frac{M^2 I_{mag}(Mx,My,z=R)}{I_{no\ samp,\,mag}}}_{\text{Flat-field corrected Image}}\right\}\right\}\right]$$

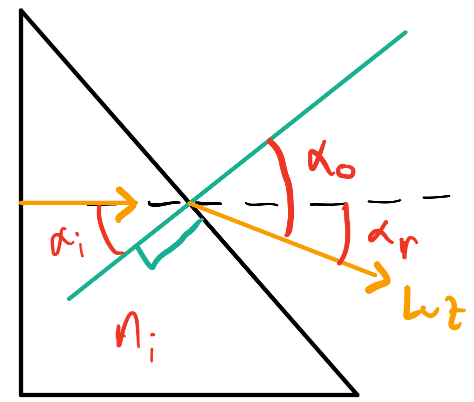


Ancho: Depende de $\frac{\delta}{\mu}\frac{R}{M}$

¿Por qué necesitamos distancias con detector grandes para notar efectos de fase?

Ley de Snell: $n_i \sin \underbrace{\alpha_i}_{\text{Ángulo de incidencia}} = n_o \sin \underbrace{\alpha_o}_{\text{ángulo de salida}}$ $(\underbrace{\alpha_r}_{\text{Ángulo de refracción.}} = \alpha_o - \alpha_i)$

n_o



Para ángulos pequeños:

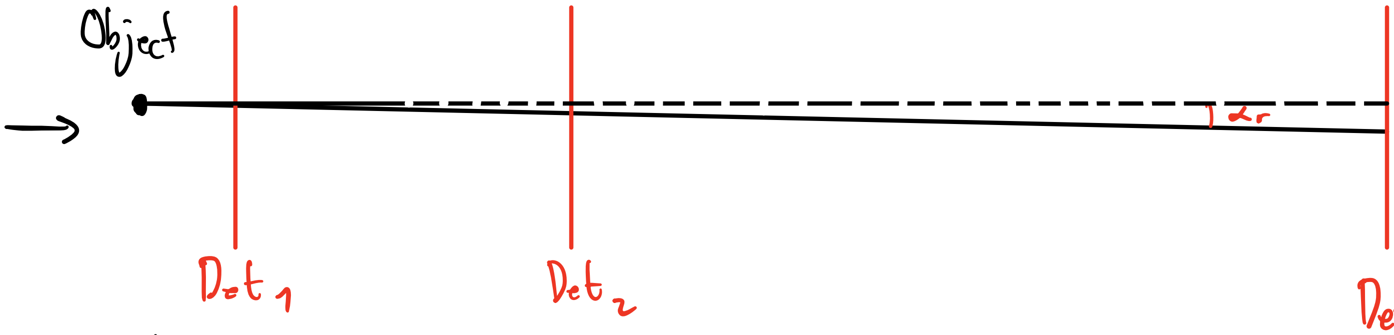
$$n_i \alpha_i \approx n_o \alpha_o$$

Supongamos $n_o = 1$ (Vacío, ~ aire)

$$n_i \alpha_i \approx \alpha_o$$

$$\rightarrow \alpha_r = \alpha_o - \alpha_i = n_i \alpha_i - \alpha_i = (n_i - 1)\alpha_i$$

Para rayos X $|n_i|$ es muy cercana a 1, luego $n_i - 1$ es muy cercano a 0 y α_r es MUY pequeño. En luz visible esto no es un problema $\longrightarrow$ Refracción (efectos de fase) pueden ser visibles fácilmente.



No podemos identificar muy bien α_r

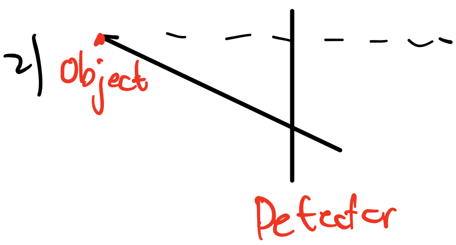
Casi, pero sigue siendo difícil

Acá la refracción es más fácilmente apreciable

1) $\alpha_r \sim \mu rad$

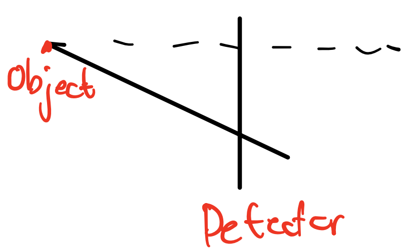
Propagation distance (objeto a detector) $= \dfrac{10^{-3}\,m}{\tan \alpha_r} \approx \dfrac{10^{-3}\,m}{\alpha_r} \simeq \dfrac{10^{-3}\,m}{10^{-6}\,rad} \sim 10^{3}\,m$

2)



$1\,mm = 10^{-3}\,m$

2.1)



$1\,\mu m = 10^{-6}\,m \longrightarrow$ prop. distance $\sim 1\,m$

¿Cómo medir α_r de forma fácil sin necesidad de tener distancias de propagación tan grandes?

Chapter 4: Fokker-Planck X-ray Equation (Path Integral Formulation)

A Path-Integral Way to Understand X-ray Multimodal X-ray Imaging

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Abstract

In this article, the path-integral formalism is introduced to explain the physics behind various X-ray multimodal imaging techniques using grating interferometry.

Keywords: Path integral, X-ray.

Nomenclature

k	Wave number
$I(x,z)$	Intensity
$\int \mathcal{D}x$	The functional integral. Also represented by $\int \mathcal{D}[x(z)]$

1. Introduction

2. Path-Integral from X-ray interactions

Consider the scenario shown in Fig. 1, where a non-polarized monochromatic X-ray plane wave interacts with a sample whose refractive index $n_s(x,z) = n_s$ is assumed to be constant. Also, assume that the X-ray wave travels in vacuum (where the refractive index is $n_v(x,y) = 1$, and at a certain point in time, a photon interacts with the sample at the point (x_0, z_0). Additionally, assume that the X-ray plane wave can be described by a scalar theory, allowing the X-ray photon interactions to be represented by a single scalar wave function $\psi(x,z)$.

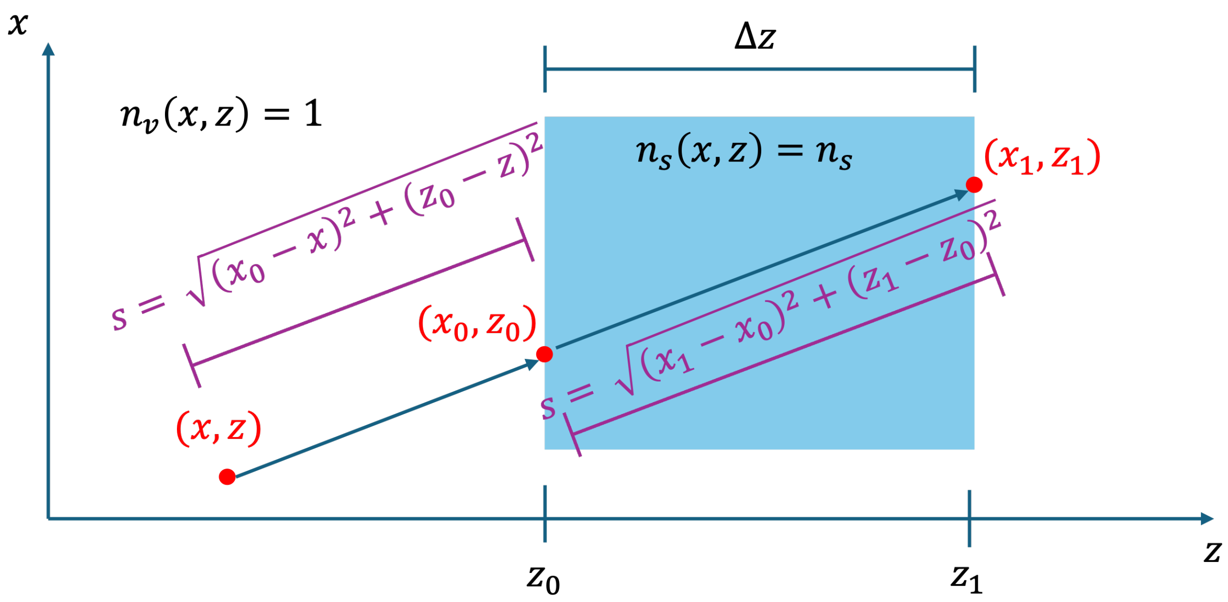

Figure 1: Propagation of a photon inside an homogeneous material of refractive index $n_s(x, y) = n_s$.

For $z \leq z_0^-$, the wave function can be modeled as shown in Eq. (1), where s is the geometrical length of the path that the photon traveled to reach the sample at the point (x_0, z_0) from another point (x, z) in vacuum, k is the wave number, and ψ_0 is a constant which represents the amplitude of the wave.

$$\begin{aligned}\psi(x, z) &= \psi_0 \exp\left(ik \int_{(x,z)}^{(x_0,z_0)} n_v(x, z)\mathrm{d}s\right) = \psi_0 \exp\left(ik \int_{(x,z)}^{(x_0,z_0)} \mathrm{d}s\right) \\ &= \psi_0 \exp\left(ik\sqrt{(x_0 - x)^2 + (z_0 - z)^2}\right).\end{aligned} \tag{1}$$

Within the material, i.e for $z_0^+ \leq z \leq z_1$, the wave function is described by Eq. 2, where s represents the geometrical length of the path that the photon traveled from the point (x_0, z_0) to any point (x, z) inside the material.

$$\begin{aligned}\psi(x, z) &= \psi_0 \exp\left(ik \int_{(x_0,z_0)}^{(x,z)} n_s(x, z)\mathrm{d}s\right) = \psi_0 \exp\left(ikn_s \int_{(x_0,z_0)}^{(x,z)} \mathrm{d}s\right) \\ &= \psi_0 \exp\left(ikn_s\sqrt{(x - x_0)^2 + (z - z_0)^2}\right).\end{aligned} \tag{2}$$

Note that $\psi(x_0, z_0^-) = \psi(x_0, z_0^+)$, ensuring continuity between both expressions for the wave function at (x_0, z_0). Consequently, the constant ψ_0 can be expressed as $\psi_0 = \psi(x_0, z_0)$, and hence, the wave function at (x_1, z_1) can be calculated as:

$$\psi(x_1, z_1) = \exp\left(ikn_s\sqrt{(x_1 - x_0)^2 + \Delta z^2}\right)\psi(x_0, z_0), \tag{3}$$

where $\Delta z = z_1 - z_0$ is the thickness of the material. However, photons interacting at a fixed point (x_0, z_0) are not the only ones that can propagate to the point (x_1, z_1). To find an expression for $\psi(x_1, z_1)$ that accounts for contributions from photons interacting at N_p different positions $\{x_{0,j}\}_{1\leq j\leq N_p}$ along the x-axis, we must consider **all possible propagation paths** that end at (x_1, z_1). Mathematically, this means that a more complete expression for $\psi(x_1, z_1)$ should be:

$$\psi(x_1, z_1) = \sum_{j=1}^{N_p} \exp\left(ikn_s\sqrt{(x_1 - x_{0,j})^2 + \Delta z^2}\right)\psi(x_{0,j}, z_0). \tag{4}$$

By considering not a discrete, but a continuum of possible interaction points, Eq. (4) turns into Eq. (5), where $K_s(x_1, z_1|x_0, z_0) = \exp\left(ikn_s\sqrt{(x_1 - x_0)^2 + \Delta z^2}\right)$ can be interpreted as an operator which propagates a photon to the point (x_1, z_1) given that such photon interacted with the sample at the point (x_0, z_0).

$$\psi(x_1, z_1) = \int_{-\infty}^{\infty} \mathrm{d}x_0 K_s(x_1, z_1|x_0, z_0)\psi(x_0, z_0). \tag{5}$$

Now consider the non-uniform sample shown in Fig. 2, described by a non-constant refractive index $n_s(x, z)$. The sample is divided into N parts of thickness $\Delta z = (x_b - x_a)/N$. Additionally, suppose that N is sufficiently large so that each slice can be considered a uniform sample (with a constant refractive index) along the z-axis. Since each slice is now considered a uniform sample, we can apply the result found in Eq. (5) to each slice.

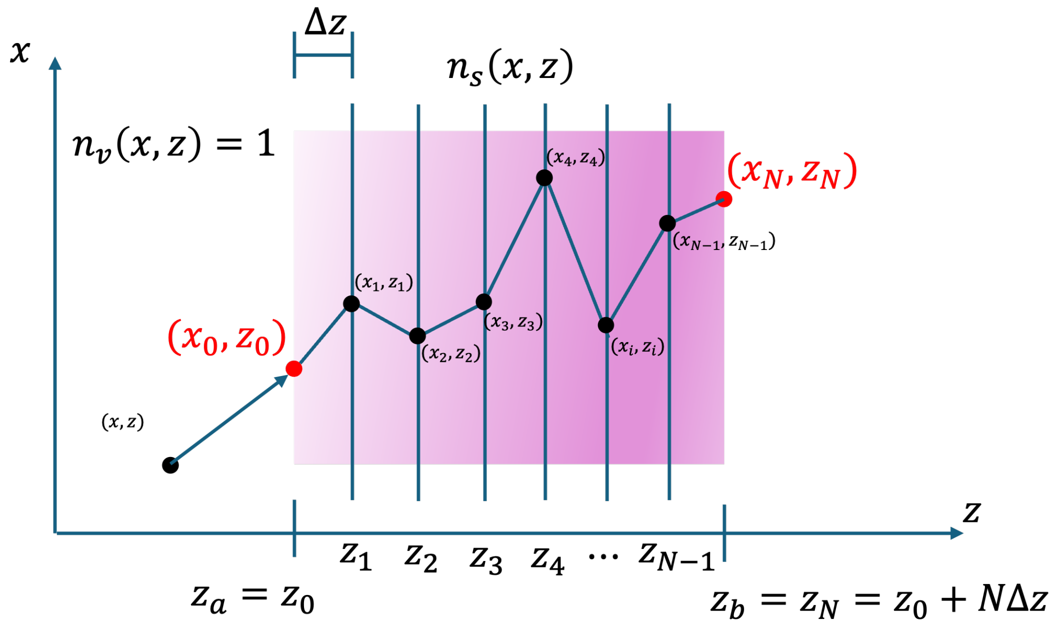

Figure 2: Propagation of a photon inside a non-homogeneous material of refractive index $n_s(x, y)$.

For the first slice, we know that:

$$\psi(x_1, z_1) = \int_{-\infty}^{\infty} \mathrm{d}x_0 K_s(x_1, z_1|x_0, z_0)\psi(x_0, z_0). \tag{6}$$

For the second slice:

$$\psi(x_2, z_2) = \int_{-\infty}^{\infty} \mathrm{d}x_1 K_s(x_2, z_2|x_1, z_1)\psi(x_1, z_1). \tag{7}$$

In consequence:

$$\begin{aligned}
\psi(x_2, z_2) &= \int_{-\infty}^{\infty} \mathrm{d}x_1 K_s(x_2, z_2|x_1, z_1)\left(\int_{-\infty}^{\infty} \mathrm{d}x_0 K_s(x_1, z_1|x_0, z_0)\psi(x_0, z_0)\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{-\infty}^{\infty} \mathrm{d}x_1 K_s(x_2, z_2|x_1, z_1)K_s(x_1, z_1|x_0, z_0)\psi(x_0, z_0).
\end{aligned}$$

By repeating the previous procedure iteratively for each slice, we find that:

$$\begin{aligned}
\psi(x_N, z_N) &= \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \prod_{i=1}^{N} K_s(x_i, z_i|x_{i-1}, z_{i-1})\psi(x_0, z_0) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0 K_N(x_N, z_N|x_0, z_0)\psi(x_0, z_0),
\end{aligned} \tag{8}$$

where:

$$\begin{aligned}
K_N(x_N, z_N|x_0, z_0) &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \prod_{i=1}^{N} K_s(x_i, z_i|x_{i-1}, z_{i-1}) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \prod_{i=1}^{N} \exp\left(ikn_s(x_i, z_i)\sqrt{(x_i - x_{i-1})^2 + \Delta z^2}\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \exp\left(ik\sum_{i=1}^{N} n_s(x_i, z_i)\sqrt{(x_i - x_{i-1})^2 + \Delta z^2}\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \exp\left(ik\sum_{i=1}^{N} n_s(x_1, z_i)\sqrt{1 + \left(\frac{x_i - x_{i-1}}{\Delta z}\right)^2}\Delta z\right)
\end{aligned}$$

is defined in analogy to $K_s(x_1, z_1|x_0, z_0)$ in Eq. (5). From a quantum-mechanics point of view, $K_N(x_N, z_N|x_0, z_0)$ represents the probability amplitude associated with a photon propagating to (x_N, z_N) given that it interacted with the sample at (x_0, z_0). From a practical point of view, this term describes the propagation of a photon from an initial fixed point (x_0, z_0) to a final fixed point (x_N, z_N). This is why the term is typically referred to as the **propagator**. To ensure the probabilistic nature of $K_N(x_N, z_N|x_0, z_0)$, a normalization constant $\mathcal{N}$ should be added to its definition, so that:

$$K_N(x_N, z_N|x_0, z_0) = \mathcal{N}\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \exp\left(ik \sum_{i=1}^{N} n_s(x_i, z_i)\sqrt{1 + \left(\frac{x_i - x_{i-1}}{\Delta z}\right)^2}\Delta z \right). \tag{9}$$

In the limit when $N \to \infty$, we must consider the next correspondences:

$$\begin{aligned} \sum_{i=1}^{N} n_s(x_i, z_i)\sqrt{1 + \left(\frac{x_i - x_{i-1}}{\Delta z}\right)^2}\Delta z &\to \int_{z_a}^{z_b} n_s(x, z)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2}\mathrm{d}z \\ \mathcal{N}\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} &\to \int \mathcal{D}[x(z)] \end{aligned} \tag{10}$$

and, in consequence:

$$\begin{aligned} K(x_b, z_b|x_a, z_a) &= \lim_{N\to\infty} K_N(x_N, z_N|x_0, z_0) \\ &= \int \mathcal{D}[x(z)] \exp\left(ik \int_{z_a}^{z_b} n_s(x, z)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2}\mathrm{d}z \right). \end{aligned} \tag{11}$$

The integral form of $K(x_b, z_b|x_a, z_a)$ in Eq. (11) is called a **path integral** or a **functional integral**. The general form of a path integral in physics is shown in Eq. (12), where the functional $S\left[x, \frac{\mathrm{d}x}{\mathrm{d}z}\right]$ is called the action. For the specific example in Fig. 2, the action is $S\left[x, \frac{\mathrm{d}x}{\mathrm{d}z}\right] = k\int_{z_a}^{z_b} n_s(x, z)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2}\mathrm{d}z$.

$$K(x, z|x', z') = \int \mathcal{D}[x(z)] \exp\left(iS\left[x, \frac{\mathrm{d}x}{\mathrm{d}z}\right] \right). \tag{12}$$

Typically, the action is expressed in terms of another function called Lagrangian $L\left(x, \frac{\mathrm{d}x}{\mathrm{d}z}\right)$, so that:

$$S\left[x, \frac{\mathrm{d}x}{\mathrm{d}z}\right] = \int_{z'}^{z} L\left(x, \frac{\mathrm{d}x}{\mathrm{d}\tilde{z}}\right)\mathrm{d}\tilde{z}. \tag{13}$$

For the specific example illustrated in Fig. 2, the Lagrangian is:

$$L\left(x, \frac{\mathrm{d}x}{\mathrm{d}z}\right) = kn_s(x, z)\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2}. \tag{14}$$

Considering that $N \to \infty$, Eq. (8) turns into Eq. (15).

$$\psi(x,z) = \int_{-\infty}^{\infty} \mathrm{d}x' K(x,z|x',z')\psi(x',z'). \tag{15}$$

As a side note, Eqs. (12) and (15) are equivalent to the Chapman-Kolmogorov equation and the Evolution equation respectively, which describes a Markov process. Similar expressions can be obtained if no wave functions but intensities are considered. In Appendix C it was found that the equivalents to Eqs. (12) and (15) in terms of intensities are:

$$I(x,z) = \int_{-\infty}^{\infty} \mathrm{d}x' K_I(x,z|x',z')I(x',z'), \tag{16}$$

and:

$$K_I(x,z|x',z') = \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x \exp\left(-2\,\mathrm{Im}\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]\right)\right), \tag{17}$$

respectively. For the example illustrated in Fig. 2, the propagator $K_I(x,z|x',z')$ is given by:

$$K_I(x,z|x',z') = \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x \exp\left(-\int_{z'}^{z} 2k\,\mathrm{Im}(n_s(x,\tilde{z}))\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}\tilde{z}}\right)^2}\,\mathrm{d}\tilde{z}\right). \tag{18}$$

The complex refractive index $n_s(x,z)$ is typically expressed as $n_s(x,z) = 1 - \delta(x,z) + i\beta(x,z)$, where $\delta(x,z)$ and $\beta(x,z)$ represent the dispersive and absorptive effects in X-ray interactions with matter. Since $\mathrm{Im}(n_s(x,z)) = \beta(x,z)$, then:

$$K_I(x,z|x',z') = \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x \exp\left(-\int_{z'}^{z} 2k\beta(x,\tilde{z})\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}\tilde{z}}\right)^2}\,\mathrm{d}\tilde{z}\right) \tag{19}$$

$$= \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x \exp\left(-\int_{z'}^{z} \mu(x,\tilde{z})\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}\tilde{z}}\right)^2}\,\mathrm{d}\tilde{z}\right), \tag{20}$$

where $\mu(x,z) = 2k\beta(x,z)$ is the linear attenuation coefficient.

3. Propagation using paraxial approximation

Consider the propagation of a photon inside a material with a constant refractive index $n_s(x,z) = n_s$. Additionally, assume we are in the near-field regime, where $\frac{\mathrm{d}x}{\mathrm{d}z}$ is considered to be small, so that:

$$\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2} \approx 1+\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2. \tag{21}$$

In consequence, Eq. (11) turns to be:

$$\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \int \mathcal{D}[x(z)] \exp\left(ik \int_{z_a}^{z_b} n_s(x,z) \left[1+\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2\right]\right) \mathrm{d}z \\
&= \int \mathcal{D}[x(z)] \exp\left(ikn_s \int_{z_a}^{z_b} \left[1+\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2\right]\right) \mathrm{d}z \\
&= e^{ik(z_b-z_a)} \int \mathcal{D}[x(z)] \exp\left(\frac{ik}{2} n_s \int_{z_a}^{z_b} \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2\right) \mathrm{d}z
\end{aligned} \tag{22}$$

The discretized form of the propagator is:

$$\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \mathcal{N} e^{ik(z_b-z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_1 \exp\left(\frac{ik}{2} n_s \sum_{i=1}^{N} \left(\frac{x_i - x_{i-1}}{\Delta z}\right)^2 \Delta z\right) \\
&= \mathcal{N} e^{ik(z_b-z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_1 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=1}^{N} (x_i - x_{i-1})^2\right) \\
&= \mathcal{N} e^{ik(z_b-z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_2 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=3}^{N} (x_i - x_{i-1})^2\right) \\
&\times \int \mathrm{d}x_1 \exp\left(\frac{ik}{2\Delta z} n_s [(x_1 - x_0)^2 + (x_2 - x_1)^2]\right),
\end{aligned} \tag{23}$$

where $x_0 = x_a$ and $x_N = x_b$. By using the property in Eq. (A3), we can find that:

$$
\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_2 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=3}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \sqrt{\frac{\pi}{-2\frac{ik}{2\Delta z} n_s}} \exp\left(\frac{\frac{ik}{2\Delta z} n_s}{2} (x_2 - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_2 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=3}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \exp\left(\frac{ik}{4\Delta z} n_s (x_2 - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_3 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=4}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \int \mathrm{d}x_2 \exp\left(\frac{ik}{2\Delta z} n_s (x_3 - x_2)^2 \right) \exp\left(\frac{ik}{4\Delta z} n_s (x_2 - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_3 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=4}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \int \mathrm{d}x_2 \exp\left(\frac{ik}{4\Delta z} n_s [2(x_3 - x_2)^2 + (x_2 - x_0)^2] \right),
\end{aligned} \tag{24}
$$

where the constant $\sqrt{\frac{\pi}{-2\frac{ik}{2\Delta z}n_s}}$ was absorbed into the normalization constant $\mathcal{N}$. From now on, all constants from Gaussian integrals will be absorbed into the constant $\mathcal{N}$. Using the property shown in Eq. (A2), the previous result can be expressed as: 85 86 87

$$
\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_3 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=4}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \sqrt{\frac{\pi}{-\frac{3}{4}\frac{ik}{\Delta z} n_s}} \exp\left(\frac{ik}{6\Delta z} n_s (x_3 - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_4 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=5}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \int \mathrm{d}x_3 \exp\left(\frac{ik}{2\Delta z} n_s (x_4 - x_3)^2 \right) \exp\left(\frac{ik}{6\Delta z} n_s (x_3 - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_4 \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=5}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \int \mathrm{d}x_3 \exp\left(\frac{ik}{6\Delta z} n_s [3(x_4 - x_3)^2 + (x_3 - x_0)^2] \right).
\end{aligned} \tag{25}
$$

If we keep using the property shown in Eq. (A2), we can arrive at the next generic expression: 88 89

$$
\begin{aligned}
K(x_b, z_b|x_a, z_a) = \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_{l+1} \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=l+2}^{N} (x_i - x_{i-1})^2 \right) \\
\times \int \mathrm{d}x_l \exp\left(\frac{ik}{2l\Delta z} n_s [l(x_{l+1} - x_l)^2 + (x_l - x_0)^2] \right)
\end{aligned} \tag{26}
$$

where l is a fixed index. For $l = N - 2$:

$$
\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \exp\left(\frac{ik}{2\Delta z} n_s \sum_{i=N}^{N} (x_i - x_{i-1})^2 \right) \\
&\times \int \mathrm{d}x_{N-2} \exp\left(\frac{ik}{2(N-2)\Delta z} n_s [(N-2)(x_{N-1} - x_{N-2})^2 + (x_{N-2} - x_0)^2] \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \exp\left(\frac{ik}{2\Delta z} n_s (x_N - x_{N-1})^2 \right) \\
&\times \int \mathrm{d}x_{N-2} \exp\left(\frac{ik}{2(N-2)\Delta z} n_s [(N-2)(x_{N-1} - x_{N-2})^2 + (x_{N-2} - x_0)^2] \right).
\end{aligned} \tag{27}
$$

Using property in Eq. (A2) once again:

$$
\begin{aligned}
K(x_b, z_b|x_a, z_a) &= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \exp\left(\frac{ik}{2\Delta z} n_s (x_N - x_{N-1})^2 \right) \\
&\times \sqrt{-\frac{\pi}{\frac{N-1}{2(N-2)} \frac{ik}{\Delta z} n_s}} \exp\left(\frac{ik}{2(N-1)\Delta z} n_s (x_{N-1} - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \exp\left(\frac{ik}{2\Delta z} n_s (x_N - x_{N-1})^2 \right) \\
&\times \exp\left(\frac{ik}{2(N-1)} n_s (x_{N-1} - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \int \mathrm{d}x_{N-1} \exp\left(\frac{ik}{2(N-1)\Delta z} n_s [(N-1)(x_N - x_{N-1})^2 + (x_{N-1} - x_0)^2] \right) \\
&= \mathcal{N} e^{ik(z_b - z_a))} \lim_{N\to\infty} \sqrt{-\frac{\pi}{\frac{N}{2(N-1)} \frac{ik}{\Delta z} n_s}} \exp\left(\frac{ik}{2N\Delta z} n_s (x_{N-1} - x_0)^2 \right) \\
&= \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \exp\left(\frac{ik}{2N\Delta z} n_s (x_N - x_0)^2 \right).
\end{aligned} \tag{28}
$$

By definition $N\Delta z = z_N - z_0 = z_b - z_a$. Then:

$$
K(x_b, z_b|x_a, z_a) = \mathcal{N} e^{ik(z_b - z_a)} \lim_{N\to\infty} \exp\left(\frac{ik}{2(z_b - z_a)} n_s (x_b - x_a)^2 \right). \tag{29}
$$

The expression above no longer depends on N. Then, we can simplify it as:

$$K(x,z|x',z') = \mathcal{N} e^{ik(z-z')} \exp\left(\frac{ik}{2(z-z')} n_s (x-x')^2\right). \tag{30}$$

Since the propagator represents a probability amplitude, it must obey the next normalisation condition:

$$\begin{aligned}
1 &= \left| \int_{-\infty}^{\infty} \mathrm{d}x K(x,z|x',z') \right|^2 \\
&= |\mathcal{N}|^2 |e^{ik(z-z')}|^2 \left| \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(\frac{ik}{2(z-z')} n_s (x-x')^2\right) \right|^2 \\
&= |\mathcal{N}|^2 \left| \sqrt{-\frac{\pi}{\frac{ik}{2(z-z')} n_s}} \right|^2 \\
&= |\mathcal{N}|^2 \left| \sqrt{\frac{\pi}{\frac{k}{2i(z-z')} n_s}} \right|^2 \\
\left| \sqrt{\frac{k n_s}{2\pi i (z-z')}} \right|^2 &= |\mathcal{N}|^2.
\end{aligned} \tag{31}$$

In consequence, we can deduce that:

$$\mathcal{N} = \sqrt{\frac{k n_s}{2\pi i (z-z')}}, \tag{32}$$

and:

$$\begin{aligned}
K(x,z|x',z') &= \sqrt{\frac{k n_s}{2\pi i (z-z')}} e^{ik(z-z')} \exp\left(\frac{ik n_s}{2(z-z')} (x-x')^2\right) \\
&= \sqrt{\frac{n_s}{i\lambda (z-z')}} e^{ik(z-z')} \exp\left(\frac{ik n_s}{2(z-z')} (x-x')^2\right),
\end{aligned} \tag{33}$$

which corresponds to the 1D Fresnel kernel or 1D Fresnel propagator in real space.

4. Conclusions

Some conclusions here.

Conflicts of Interest

The author declare no conflict of interest.

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Appendix

A. Gaussian integrals

The basic Gaussian integral is given by the next expression:

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-a(x-x')^2} = \sqrt{\frac{\pi}{a}}. \tag{A1}$$

An extended version of the basic Gaussian integral, found in the book of Feynman and Hibbs, is given by:

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-a(x-x')^2-b(x''-x)^2} = \sqrt{\frac{\pi}{a+b}} \exp\left(-\frac{(x'-x'')^2}{\frac{1}{a}+\frac{1}{b}}\right). \tag{A2}$$

For $a = b$:

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-a((x-x')^2+(x''-x)^2)} = \sqrt{\frac{\pi}{2a}} \exp\left(-\frac{a}{2}(x'-x'')^2\right) \tag{A3}$$

B. Dirac delta

The Dirac delta Fourier-functional form in terms of path integrals is given by:

$$\delta[f[x(z)] - g[x(z)]] = \int \mathcal{D}[J[x(z)] \exp\left(-2\pi i \int \mathrm{d}z J[x(z)](f[x(z)] - g[x(z)])\right) \tag{B1}$$

The product of a Dirac delta distribution and a function $f(x)$ is given by:

$$f(x)\delta(x-a) = f(a)\delta(x-a) \tag{B2}$$

C. Path Integral for Intensities

Taking into account the results in Eqs. (12) and (15), the intensity $I(x, z)$ is given by:

$$
\begin{aligned}
I(x,z) &= \psi(x,z)\psi^*(x,z) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \delta(x''-x') K(x,z|x',z') K^*(x,z|x'',z') \psi(x',z')\psi^*(x'',z') \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' K(x,z|x',z') K^*(x,z|x',z') I(x',z')
\end{aligned}
\tag{C1}
$$

where $I(x',z') = \psi(x',z')\psi^*(x',z')$, and the Dirac delta term $\delta(x''-x')$ was included to ensure that the paths considered in the path integrals defining $K(x,z|x',z')$ and $K(x,z|x'',z')$ begin from the same fixed point. The product of the propagators in Eq. (C1) is:

$$
\begin{aligned}
K(x,z|x',z')K^*(x,z|x',z') &= \left(\int_{x(z')=x'}^{x(z)=x} \mathcal{D}x e^{iS\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]}\right)\left(\int_{\tilde{x}(z')=x'}^{\tilde{x}(z)=x} \mathcal{D}\tilde{x} e^{-iS^*\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]}\right) \\
&= \int_{x(z')=x'}^{x(z)=x} \int_{\tilde{x}(z')=x'}^{\tilde{x}(z)=x} \mathcal{D}x\mathcal{D}\tilde{x} e^{i\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]-S^*\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]\right)}
\end{aligned}
\tag{C2}
$$

The initial conditions $x(z') = x'$ and $\tilde{x}(z') = x'$ are assumed to be included in $S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]$ and $S\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]$, respectively. However, the upper limits $x(z) = x$ and $\tilde{x}(z) = x$ can be imposed by using Dirac delta functions via:

$$
K(x,z|x',z')K^*(x,z|x',z') = \int\int \mathcal{D}x\mathcal{D}\tilde{x}\delta(x(z)-x)\delta(\tilde{x}(z)-x) e^{i\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]-S^*\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]\right)}. \tag{C3}
$$

Using the property , we can find that:

$$
\begin{aligned}
K(x,z|x',z')K^*(x,z|x',z') &= \int\int \mathcal{D}x\mathcal{D}\tilde{x}\delta(x(z)-x)\delta(\tilde{x}(z)-x(z)) e^{i\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]-S^*\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]\right)} \\
&= \int \mathcal{D}x\delta(x(z)-x) \int \mathcal{D}\tilde{x}\delta(\tilde{x}(z)-x(z)) e^{i\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]-S^*\left[\tilde{x},\frac{\mathrm{d}\tilde{x}}{\mathrm{d}z}\right]\right)} \\
&= \int \mathcal{D}x\delta(x(z)-x) e^{i\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]-S^*\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]\right)} \\
&= \int \mathcal{D}x\delta(x(z)-x) e^{-2\,\mathrm{Im}\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]\right)} \\
&= \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x e^{-2\,\mathrm{Im}\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]\right)}.
\end{aligned}
\tag{C4}
$$

In consequence:

$$I(x,z) = \int_{-\infty}^{\infty} \mathrm{d}x' K_I(x,z|x',z') I(x',z'), \tag{C5}$$

where the term $K_I(x,z|x',z')$ is given by:

$$K_I(x,z|x',z') = \int_{x(z')=x'}^{x(z)=x} \mathcal{D}x e^{-2\,\mathrm{Im}\left(S\left[x,\frac{\mathrm{d}x}{\mathrm{d}z}\right]\right)} \tag{C6}$$

$$I(x_b, z_b | x_a, z_a) = \int_{x(z_a)=x_a}^{x(z_b)=x_b} Dx \, P[x(z)] = K_I [x_b, z_b | x_a, z_a)$$

$$= \int_{\mathbb{R}} Dx \, \delta(x(z_b) - x_b) \, P[x(z)]$$

Since

$$\delta[f(x) - g(x)] = \int DJ' e^{i \int J'(x) (f(x) - g(x)) dx}$$

$$\delta[x(t_b) - x_b] = \int DJ' e^{2\pi i J'(x) (x(z_b) - x_b)} \qquad (J = 2\pi i J')$$

$$= \frac{1}{2\pi i} \int DJ \, e^{J(x) (x(z_b) - x_b)}$$

$$\rightarrow I(x_b, z_b | x_a, z_a) = \frac{1}{2\pi i} \int Dx \int DJ \, e^{J(x(z_b) - x_b)} \, P[x(z)]$$

$$= \frac{1}{2\pi i} \int Dx \int DJ \, e^{J(x(z_b) - x_b + x_a - x_a)} \, P[x(z)]$$

$$= \frac{1}{2\pi i} \int Dx \int DJ \, e^{J(x(z_b) - x_a)} e^{iJ(x_a - x_b)} \, P[x(z)]$$

$$= \frac{1}{2\pi i} \int Dx \int DJ \, e^{J(x(z_b) - x_a)} e^{-iJ(x_b - x_a)} \, P[x(z)]$$

$$= \frac{1}{2\pi i} \int DJ \, e^{-J(x_b - x_a)} \int Dx \, e^{J(x(z_b) - x_a)} \, P[x(z)]$$

$$= \frac{1}{2\pi i} \int DJ \, e^{-J(x_b - x_a)} \, Z[J]$$

Where $$Z[J] = \int Dx \, e^{J(x(z_b) - x_a)} P[x(z)]$$

$$= \int Dx \sum_{n=0}^{\infty} \frac{1}{n!} \left(J(x(z_b) - x_a) \right)^n P[x(z)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int Dx \, J^n (x(z_b) - x_a)^n \, P[x(z)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} J^n \int Dx \, (x(z_b) - x_a)^n P[x(z)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} J^n \langle (x(z_b) - x_a)^n \rangle_{x(z_a) = x_a}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} J^n \langle (x(z_b) - x_a)^n \rangle_{x(z_a) = x_a}$$

Then

$$I(x_b, z_b | x_a, z_a) = \frac{1}{2\pi i} \int DJ \, e^{-J(x_b - x_a)} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} J^n \langle (x(z_b) - x_a)^n \rangle \right.$$

$$= \frac{1}{2\pi i} \int DJ e^{-J(x_b - x_a)} + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n!} \int DJ e^{-J(x_b - x_a)} J^n \langle (x(z_b) - x_a)^n \rangle$$

$$= \delta[x_b - x_a] + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n!} \int DJ e^{-J(x_b - x_a)} J^n \langle (x(z_b) - x_a)^n \rangle$$

$$= \delta[x_b - x_a] + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n!} \langle (x(z_b) - x_a)^n \rangle \int DJ \, e^{-J(x_b - x_a)} J^n$$

Let $$Z_2 = \int DJ\, e^{-J(x_b - x_a)} = 2\pi i\, \delta[x_b - x_a]$$

Then

$$\frac{\partial Z_2}{\partial x_b} = \int DJ\, (-J)\, e^{-J(x_b - x_a)} = -\int DJ\, e^{-J(x_b - x_a)} J$$

$$\frac{\partial^2 Z_2}{\partial x_b^2} = \int DJ\, e^{-J(x_b - x_a)} J^2$$

$$\Rightarrow \frac{\partial^n Z_2}{\partial x_b^n} = (-1)^n \int DJ\, e^{-J(x_b - x_a)} J^n$$

$$\Rightarrow \int DJ\, e^{-J(x_b - x_a)} J^n = (-1)^n \frac{\partial^n Z_2}{\partial x_b^n} = (-1)^n \frac{\partial^n}{\partial x_b^n} 2\pi i\, \delta[x_b - x_a]$$

$$= 2\pi i \left(-\frac{\partial}{\partial x_b}\right)^n \delta[x_b - x_a]$$

In Consequence

$$I(x_b, z_b | x_a, z_a) = \delta[x_b - x_a] + \sum_{n=1}^{\infty} \frac{1}{n!} \langle (x(z_b) - x_a)^n \rangle \left(-\frac{\partial}{\partial x_b}\right)^n \delta[x_b - x_a]$$

$$= \delta[x_b - x_a] + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x_b}\right)^n \left(\langle (x(z_b) - x_a)^n \rangle \delta[x_b - x_a] \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x_b}\right)^n \left(\langle (x(z_b) - x_a)^n \rangle\, \delta[x_b - x_a] \right)$$

Evolution eq. tells that:

$$I(x_b, z_b) = \int dx_a \; I(x_b, z_b | x_a, z_a)\, I(x_a, z_a)$$

Let $x_b = x$, $z_b = z + \delta z$, $x_a = x - \delta x$, $z_a = z$

$$I(x, z+\delta z) = \int d(\delta x)\, I(x, z+\delta z | x - \delta x, z)\, I(x - \delta x, z)$$

$$= \int \frac{d(\delta x)}{n!} \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \left(\langle (x(z+\delta z) - x)^n \rangle\, \delta[\delta x - 0]\right) I(x - \delta x, z)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \langle (x(z+\delta z) - x)^n \rangle\, I(x, z)$$

$$I(x, z+\delta z) = I(x, z) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \left(\langle (x(z+\delta z) - x)^n \rangle\, I(x, z)\right)$$

$$\frac{I(x, z+\delta z) - I(x, z)}{\delta z} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \left(\frac{\langle (x(z+\delta z) - x)^n \rangle}{\delta z}\, I(x, z)\right)$$

Let $D_n(x) = \lim_{\delta z \to 0} \frac{\langle (x(z+\delta z) - x)^n \rangle}{\delta z}$, then, if $\delta z \to 0$:

$$\frac{\partial I}{\partial z} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \left(D_n(x)\, I(x, z)\right)$$

Let $D_1(x) = \frac{1}{k}\frac{\partial \phi}{\partial x}$ and $D_2(x) = D(x)$:

$$\underbrace{\frac{\partial I}{\partial z} = -\frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial \phi}{\partial x} I(x,z)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(D(x) I(x,z)\Big)}_{\text{X-ray Fokker-Planck eqt.}} + \sum_{n=3}^{\infty}\frac{1}{n!}\left(-\frac{\partial}{\partial x}\right)^n \left(D_n(x) I(x,z)\right)$$

Consider Langevin eqt!

$$\frac{dx}{dz} = \underbrace{\frac{1}{k}\frac{\partial \phi}{\partial x}}_{F(x)} + \underbrace{\sqrt{D(x)}}_{g(x)}\,\eta(z)$$

($\eta(z)$: Stochastic term
$\langle \eta(z)\rangle = 0$ and
$\langle \eta(z)\eta(z')\rangle = \delta(z-z')$)

$$\Rightarrow \quad dx = F(x)\,dz + \sqrt{D(x)}\,dW_z$$

(Ito stochastic diff. eqt)

$dW_z = \eta(z)\,dz$: Wiener process (white noise)

$$\underbrace{x_{i+1} - x_i = f_i(x_i)\,\delta z + \sqrt{D_i(x_i)}\,w_i\,\sqrt{\delta z}}_{\text{Ito discretization}}$$

w_i : Discrete random variable

$\langle w_i\rangle = 0$, $\langle w_i w_j\rangle = \delta_{ij}$

Then:

$$D_n = \lim_{\delta z \to 0}\frac{\langle (x(z+\delta z) - x(z))^n\rangle}{\delta z} = \lim_{\delta z\to 0}\frac{\langle (f_i(x_i)\,\delta z + \sqrt{D_i(x_i)}\,w_i\sqrt{\delta z})^n\rangle}{\delta z}$$

$$D_1 = \lim_{\delta z \to 0} \frac{\langle f_i(x_i)\,\delta z + \sqrt{D_i(x_i)}\;w_i\sqrt{\delta z}\rangle}{\delta z} = \lim_{\delta z \to 0} \left\langle f_i(x_i) + \cancelto{0}{\frac{\sqrt{D_i(x_i)}\;w_i}{\sqrt{\delta z}}} \right\rangle$$

$$= f(x) = \frac{1}{k}\frac{\partial \phi}{\partial x}$$

$$D_2 = \lim_{\delta z \to 0} \frac{\langle f_i^2(x_i)\,\delta z^2 + D_i(x_i)\,w_i^2\,\delta z + 2 f_i(x_i)\sqrt{D_i(x_i)}\,w_i\,\delta z\sqrt{\delta z}\rangle}{\delta z}$$

$$= \lim_{\delta z \to 0} \langle f_i^2(x_i)\,\delta z + D_i(x_i)\,w_i^2 + 2 f_i(x_i)\sqrt{D_i(x_i)}\,w_i\sqrt{\delta z}\rangle$$

$$= D(x)$$

$$D_3 = \lim_{\delta z \to 0} \frac{\langle O(\delta z^2)\rangle}{\delta z} = \lim_{\delta z \to 0} \langle O(\delta z)\rangle = 0$$

$D_n = 0 \quad (n \geq 3)$ (Pawula's theorem)