ECE 630: Statistical Communication Theory

Dr. B.-P. Paris
Dept. Electrical and Comp. Engineering
George Mason University

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Part I

Introduction

Elements of a Digital Communications System

Source: produces a sequence of information symbols b.

Transmitter: maps symbol sequence to analog signal $s(t)$.

Channel: models corruption of transmitted signal $s(t)$.

Receiver: produces reconstructed sequence of information symbols $\hat{b}$ from observed signal $R(t)$.

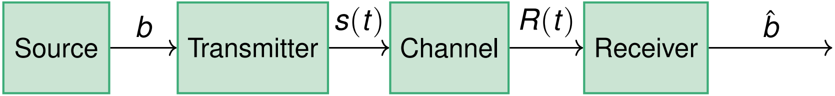
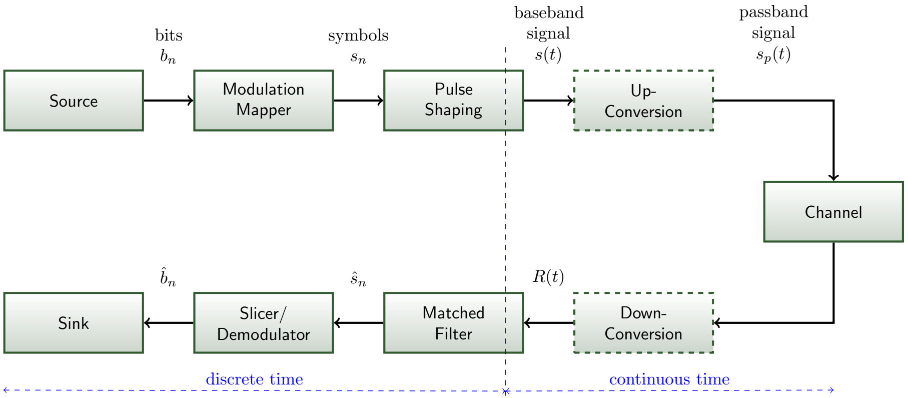


Figure: Block Diagram of a Generic Digital Communications System

Detail: Linear Digital Modulation



- Transmitted signal: $s(t) = \sum_n s_n p(t - nT)$.

The Source

- The source models the statistical properties of the digital information source.
- Three main parameters:
  - **Source Alphabet:** list of the possible information symbols the source produces; also called *Signal Constellation*
    - Example: $\mathcal{A} = \{0, 1\}$; symbols are called bits.
    - Alphabet for a source with M (typically, a power of 2) symbols: e.g., $\mathcal{A} = \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$.
    - Alphabet with positive and negative symbols is often more convenient.
    - Symbols may be complex valued; e.g., $\mathcal{A} = \{\pm 1, \pm j\}$.

- **A priori Probability:** relative frequencies with which the source produces each of the symbols.
  - Example: a binary source that produces (on average) equal numbers of 0 and 1 bits has $\pi_0 = \pi_1 = \frac{1}{2}$.
  - Notation: π_n denotes the probability of observing the n-th symbol.
  - Typically, a-priori probabilities are all equal, i.e., $\pi_n = \frac{1}{M}$.
  - A source with M symbols is called an M-ary source.
    - binary ($M = 2$)
    - quaternary ($M = 4$)

Bit 1	Bit 2	Symbol
0	0	-3
0	1	-1
1	1	$+1$
1	0	$+3$

Table: Example: Representing two bits in one quaternary symbol.

- **Symbol Rate:** The number of information symbols the source produces per second. Also called the baud rate R.
  - Related: information rate R_b, indicates number of bits source produces per second.
  - Relationship: $R_b = R \cdot \log_2(M)$.
  - Also, $T = 1/R$ is the symbol period.
  - Note: for most communication systems, the bandwidth W occupied by the transmitted signal is approximately equal to the baud rate R,

$$W \approx R$$

Remarks

- This view of the source is simplified.
- We have omitted important functionality normally found in the source, including
  - error correction coding and interleaving, and
  - Usually, a block that maps bits to symbols is broken out separately.
- This simplified view is sufficient for our initial discussions.
- Missing functionality will be revisited when needed.

The Transmitter

- The transmitter translates the information symbols at its input into signals that are “appropriate” for the channel, e.g.,
  - meet bandwidth requirements due to regulatory or propagation considerations,
  - provide good receiver performance in the face of channel impairments:
    - noise,
    - distortion (i.e., undesired linear filtering),
    - interference.
- A digital communication system transmits only a discrete set of information symbols.
  - Correspondingly, only a discrete set of possible signals is employed by the transmitter.
  - The transmitted signal is an analog (continuous-time, continuous amplitude) signal.

Illustrative Example

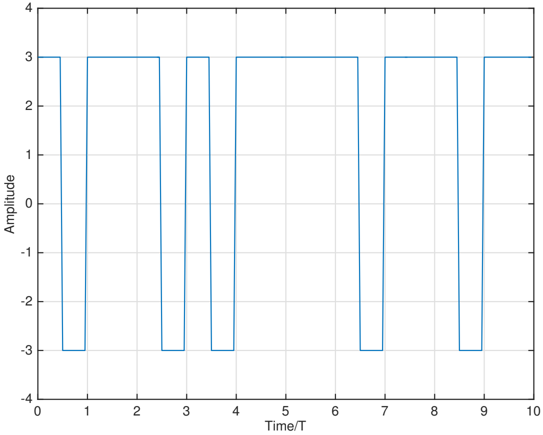
- The sources produces symbols from the alphabet $\mathcal{A} = \{0, 1\}$.
- The transmitter uses the following rule to map symbols to signals:
  - If the n-th symbol is $b_n = 0$, then the transmitter sends the signal

$$s_0(t) = \begin{cases} A & \text{for } (n-1)T \leq t < nT \\ 0 & \text{else.} \end{cases}$$

  - If the n-th symbol is $b_n = 1$, then the transmitter sends the signal

$$s_1(t) = \begin{cases} A & \text{for } (n-1)T \leq t < (n-\frac{1}{2})T \\ -A & \text{for } (n-\frac{1}{2})T \leq t < nT \\ 0 & \text{else.} \end{cases}$$

Symbol Sequence $b = \{1, 0, 1, 1, 0, 0, 1, 0, 1, 0\}$



The Communications Channel

- The communications channel models the degradation the transmitted signal experiences on its way to the receiver.
- For wireless communications systems, we are concerned primarily with:
  - **Noise:** random signal added to received signal.
    - Mainly due to thermal noise from electronic components in the receiver.
    - Can also model interference from other emitters in the vicinity of the receiver.
    - Statistical model is used to describe noise.
  - **Distortion:** undesired filtering during propagation.
    - Mainly due to multi-path propagation.
    - Both deterministic and statistical models are appropriate depending on time-scale of interest.
    - Nature and dynamics of distortion is a key difference between wireless and wired systems.

Thermal Noise

- At temperatures above absolute zero, electrons move randomly in a conducting medium, including the electronic components in the front-end of a receiver.
- This leads to a random waveform.
  - The power of the random waveform equals $P_N = kT_0 W$.
    - k: Boltzmann's constant (1.38×10^{-23} W s/K).
    - T_0: temperature in degrees Kelvin (room temperature $\approx$ 290 K).
    - For bandwidth W equal to 1 Hz, $P_N \approx 4 \times 10^{-21}$ W (-174 dBm).
- Noise power is small, but power of received signal decreases rapidly with distance from transmitter.
  - Noise provides a fundamental limit to the range and/or rate at which communication is possible.

Exercise: Path Loss and Signal-to-Noise Ratio

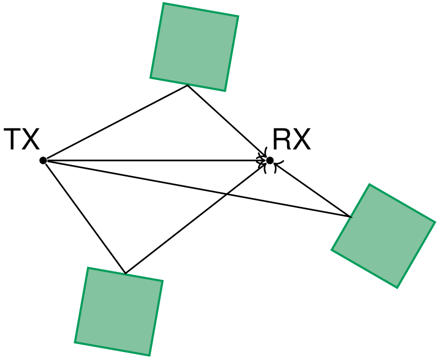
- A transmitter emits a signal with:
  - bandwidth $W = 1\,\text{MHz}$
  - transmitted power $P_t = 1\,\text{mW}$
  - carrier frequency $f_c = 1\,\text{GHz}$
- During propagation from transmitter to receiver, the signal's power decreases; the received power follows Friis law:

$$P_r = P_t \cdot \left(\frac{c}{4\pi f_c d}\right)^2$$

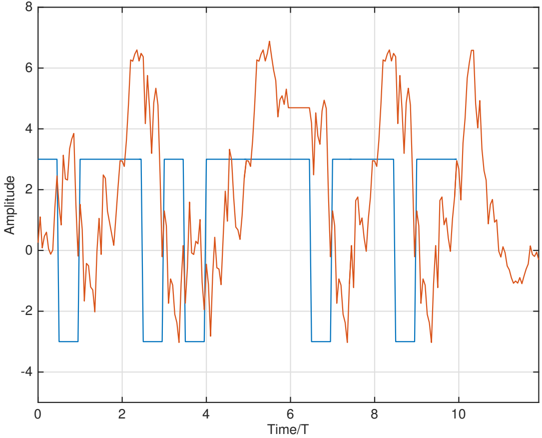
  where $c = 3 \times 10^8\,\text{m/s}$ is the speed of light and d is the distance between transmitter and receiver (in meters).
- Find:
  - the power of the received signal P_r for $d = 10\,\text{km}$
  - the noise power P_N in the bandwidth W occupied by the transmitted signal
  - the ratio $\frac{P_r}{P_N}$; this is called the signal-to-noise ratio (SNR)

Multi-Path

- In a multi-path environment, the receiver sees the combination of multiple scaled and delayed versions of the transmitted signal.



Distortion from Multi-Path



- Received signal "looks" very different from transmitted signal.
- Inter-symbol interference (ISI).
- Multi-path is a very serious problem for wireless systems.

The Receiver

- The receiver is designed to reconstruct the original information sequence b.
- Towards this objective, the receiver uses
  - the received signal $R(t)$,
  - knowledge about how the transmitter works,
    - Specifically, the receiver knows how symbols are mapped to signals.
  - the a-priori probability and rate of the source.
- The transmitted signal typically contains information that allows the receiver to gain information about the channel, including
  - training sequences to estimate the impulse response of the channel,
  - synchronization preambles to determine symbol locations and adjust amplifier gains.

The Receiver

- The receiver input is an analog signal and it's output is a sequence of discrete information symbols.
  - Consequently, the receiver must perform analog-to-digital conversion (sampling).
- Correspondingly, the receiver can be divided into an analog front-end followed by digital processing.
  - Many receivers have (relatively) simple front-ends and sophisticated digital processing stages.
  - Digital processing is performed on standard digital hardware (from ASICs to general purpose processors).
  - Moore's law can be relied on to boost the performance of digital communications systems.

Measures of Performance

- The receiver is expected to perform its function optimally.
- **Question:** optimal in what sense?
  - Measure of performance must be statistical in nature.
    - observed signal is random, and
    - transmitted symbol sequence is random.
  - Metric must reflect the reliability with which information is reconstructed at the receiver.
- **Objective:** Design the receiver that minimizes the probability of a symbol error.
  - Also referred to as symbol error rate.
  - Closely related to bit error rate (BER).

Learning Objectives

1. Understand the mathematical foundations that lead to the design of optimal receivers in AWGN channels.
   - statistical hypothesis testing
   - signal spaces
2. Understand the principles of digital information transmission.
   - baseband and passband transmission
   - relationship between data rate and bandwidth
3. Apply receiver design principles to communication systems with additional channel impairments
   - random amplitude or phase
   - linear distortion (e.g., multi-path)

Course Outline

- Mathematical Prerequisites
  - Basics of Gaussian Random Variables and Random Processes
  - Signal space concepts
- Principles of Receiver Design
  - Optimal decision: statistical hypothesis testing
  - Receiver frontend: the matched filter
- Signal design and modulation
  - Baseband and passband
  - Linear modulation
  - Bandwidth considerations
- Advanced topics
  - Rayleigh Fading and Receiver Diversity