ECE 630: Statistical Communication Theory

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Part I

Mathematical Prerequisites

Gaussian Random Variables — Why we Care

- Gaussian random variables play a critical role in modeling many random phenomena.
  - By central limit theorem, Gaussian random variables arise from the superposition (sum) of many random phenomena.
    - Pertinent example: random movement of very many electrons in conducting material.
    - Result: thermal noise is well modeled as Gaussian.
  - Gaussian random variables are mathematically tractable.
    - In particular: any linear (more precisely, affine) transformation of Gaussians produces a Gaussian random variable.
- Noise added by channel is modeled as being Gaussian.
  - Channel noise is the most fundamental impairment in a communication system.

Gaussian Random Variables

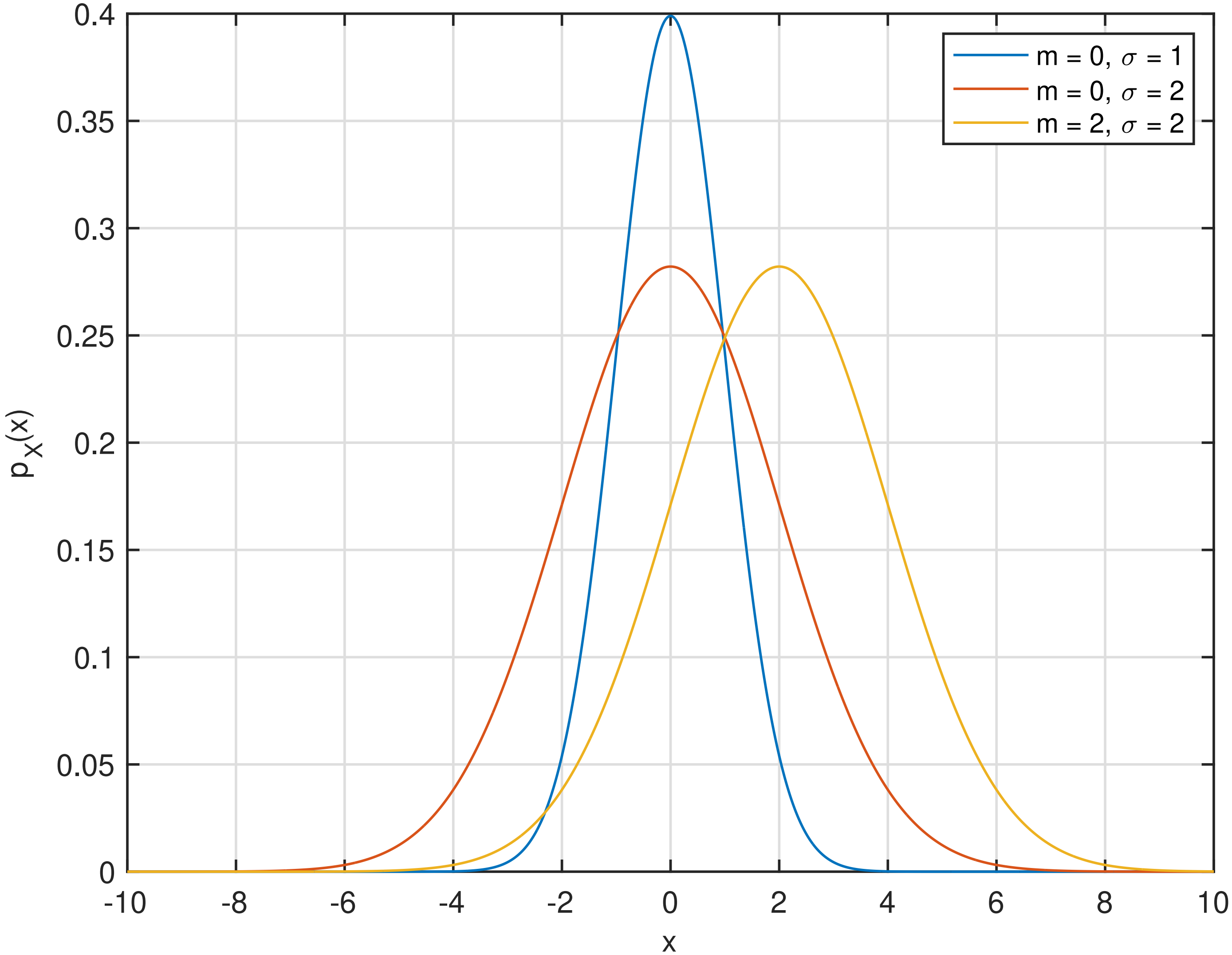
- A random variable X is said to be Gaussian (or Normal) if its pdf is of the form

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

- All properties of a Gaussian are determined by the two parameters m and σ^2.
- **Notation:** $X \sim \mathcal{N}(m, \sigma^2)$.
- **Moments:**

$$\begin{aligned} \mathbf{E}[X] &= \int_{-\infty}^{\infty} x \cdot p_X(x)\, dx = m \\ \mathbf{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot p_X(x)\, dx = m^2 + \sigma^2. \end{aligned}$$

Plot of Gaussian pdf's



The Gaussian Error Integral — Q(x)

- We are often interested in $\Pr\{X > x\}$ for Gaussian random variables X.
- These probabilities cannot be computed in closed form since the integral over the Gaussian pdf does not have a closed form expression.
- Instead, these probabilities are expressed in terms of the Gaussian error integral

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz.$$

The Gaussian Error Integral — $Q(x)$

- **Example:** Suppose $X \sim \mathcal{N}(1,4)$, what is $\Pr\{X>5\}$?

$$\Pr\{X>5\} = \int_5^\infty \frac{1}{\sqrt{2\pi\cdot 2^2}} e^{-\frac{(x-1)^2}{2\cdot 2^2}}\,dx \qquad \text{substitute } z=\frac{x-1}{2}$$

$$= \int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = Q(2)$$

Exercises

- Let $X \sim \mathcal{N}(-3, 4)$, find expressions in terms of $Q(\cdot)$ for the following probabilities:
  1. $\Pr\{X > 5\}$?
  2. $\Pr\{X < -1\}$?
  3. $\Pr\{X^2 + X > 2\}$?
- Matlab does not provide a function to compute $Q(x)$. Instead, Matlab provides the *error function* `erf` and its complement `erfc`. The complementary error function `erfc` is defined as

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.$$

  Express $Q(x)$ in terms of $\mathrm{erfc}(x)$.

Bounds for the Q-function

- Since no closed form expression is available for $Q(x)$, bounds and approximations to the Q-function are of interest.
- The following bounds are tight for large values of x:

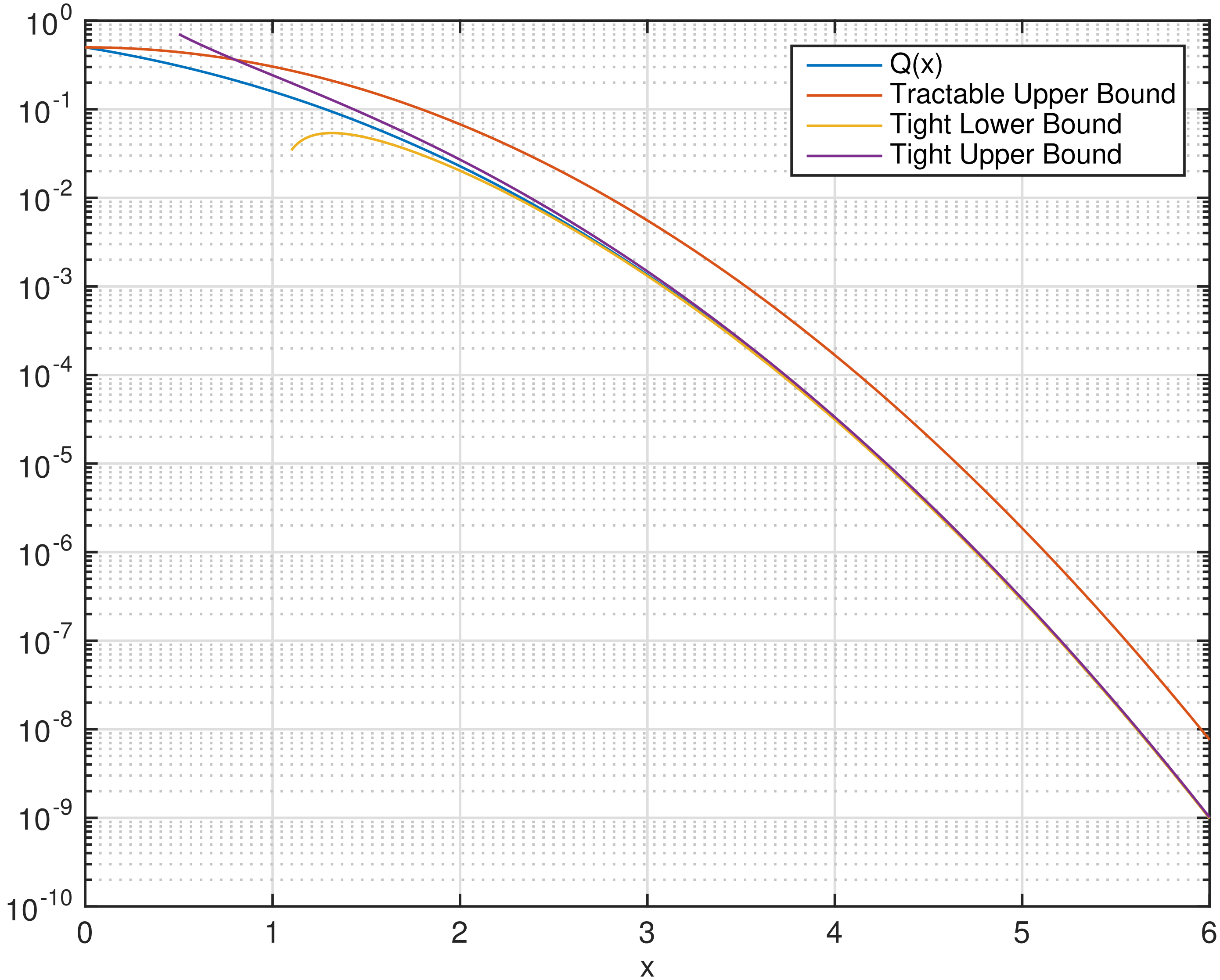
$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}.$$

- The following bound is not as quite as tight but very useful for analysis

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}.$$

- Note that all three bounds are dominated by the term $e^{-\frac{x^2}{2}}$; this term determines the asymptotic behaviour of $Q(x)$.

Plot of $Q(x)$ and Bounds



Exercise: Chernoff Bound

- For a random variable X, the Chernoff Bound provides a tight upper bound on the probability $\Pr\{X > x\}$.
- The Chernoff bound is given by

$$\Pr\{X > x\} \leq \min_{t>0} \frac{\mathbf{E}[e^{tX}]}{e^{tx}}.$$

- Let $X \sim \mathcal{N}(0,1)$; use the Chernoff bound to show that

$$\Pr\{X > x\} = \mathrm{Q}(x) \leq e^{-x^2/2}$$

Gaussian Random Vectors

- A length N random vector $\vec{X}$ is said to be Gaussian if its pdf is given by

$$p_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2}|K|^{1/2}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{m})^T K^{-1}(\vec{x}-\vec{m})\right).$$

- **Notation:** $\vec{X} \sim \mathcal{N}(\vec{m}, K)$.
- Mean vector

$$\vec{m} = \mathbf{E}[\vec{X}] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \vec{x} p_{\vec{X}}(\vec{x})\, d\vec{x}.$$

- Covariance matrix

$$K = \mathbf{E}[(\vec{X}-\vec{m})(\vec{X}-\vec{m})^T] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (\vec{x}-\vec{m})(\vec{x}-\vec{m})^T p_{\vec{X}}(\vec{x})\, c$$

  - $|K|$ denotes the determinant of K.
  - K must be positive definite, i.e., $\vec{z}^T K \vec{z} > 0$ for all $\vec{z}$.

Exercise: Important Special Case: N=2

- Consider a length-2 Gaussian random vector with

$$\vec{m} = \vec{0} \text{ and } K = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \text{ with } |\rho| \leq 1.$$

- Find the pdf of $\vec{X}$.

Exercise: Important Special Case: N=2

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- Find the pdf of $\vec{X}$.
- Answer:

$$p_{\vec{X}}(\vec{x}) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2\sigma^2(1-\rho^2)}\right)$$

Important Properties of Gaussian Random Vectors

1. If the N Gaussian random variables X_n comprising the random vector $\vec{X}$ are uncorrelated ($\text{Cov}[X_i, X_j] = 0$, for $i \neq j$), then they are statistically independent.
2. Any affine transformation of a Gaussian random vector is also a Gaussian random vector.
   - Let $\vec{X} \sim \mathcal{N}(\vec{m}, K)$
   - Affine transformation: $\vec{Y} = A\vec{X} + \vec{b}$
   - Then, $\vec{Y} \sim \mathcal{N}(A\vec{m} + \vec{b}, AKA^T)$

Exercise: Generating Correlated Gaussian Random Variables

- Let $\vec{X} \sim \mathcal{N}(\vec{m}, K)$, with

$$\vec{m} = \vec{0} \text{ and } K = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  - The elements of $\vec{X}$ are uncorrelated.
- Transform $\vec{Y} = A\vec{X}$, with

$$A = \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix}$$

- Find the pdf of $\vec{Y}$.