ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 2
Due: February 6, 2025

Reading Madhow: Appendix A, especially section A.3, and Section 3.1.

Problems

1. Madhow: Problem 3.2
2. Madhow: Problem 3.4
3. Let X and Y be independent Gaussian random variables with mean $m = 0$ and variance $\sigma^2 = 1$.
   (a) Sketch a two-dimensional coordinate system with axes X and Y. Indicate the region $R_1 = \{X > \alpha \text{ and } Y > \alpha\}$ in that coordinate system; assume that $\alpha \geq 0$.
   (b) Show that $\Pr\{X > \alpha, Y > \alpha\} = Q^2(\alpha)$. Note that this is the probability that a point (X, Y) falls in the region R_1.
   (c) Now, add the region $R_2 = \{X, Y \geq 0 \text{ and } X^2 + Y^2 > 2\alpha^2\}$ to your diagram. How does the region R_2 compare to R_1 from part (a)?
   (d) Show that $\Pr\{X, Y \geq 0, X^2 + Y^2 > 2\alpha^2\} = \frac{1}{4}\exp(-\alpha^2)$. Note that this is the probability that a point (X, Y) falls in the region R_2.
   (e) From the above, show that we can conclude the well known bound
   $$Q(\alpha) \leq \frac{1}{2}\exp(-\frac{\alpha^2}{2}).$$
4. Let $\vec{X}$ be a zero mean Gaussian random vector with covariance matrix K.
   $$K = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$
   (a) Give an expression for the density function $f_{\vec{X}}(x)$.
   (b) If $Y = X_1 + 2X_2 - X_3$, find $f_Y(y)$.
   (c) If the vector $\vec{Z}$ has components defined by
   $$\begin{aligned} Z_1 &= 5X_1 - 3X_2 - X_3 \\ Z_2 &= -X_1 + 3X_2 - X_3 \\ Z_3 &= X_1 + X_3 \end{aligned}$$
   determine $f_{\vec{Z}}(\vec{z})$. What are the properties of the new random vector?
   (d) Determine $f_{X_1|X_2}(x_1|x_2 = \beta)$