ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 1
Due: January 30, 2025

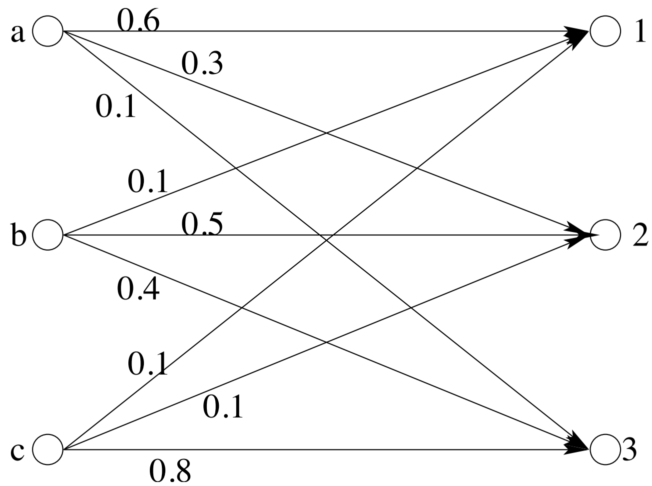
Instructions Submit your original solution to this homework in class (prefered) or through Blackboard (if you cannot attend class in person). You are encouraged to collaborate on this homework but each student must submit an original solution.

Reading Madhow: Appendix A and Section 3.1.

Note: the material in sections A.1 and A.2 has been covered in ECE 528 and you are expected to be familiar and comfortable with that material.

Problems Most of these problems are review problems for probability and random variables.

1. A noisy discrete communication channel is available. Once each second one letter from the three-letter alphabet $\{a, b, c\}$ can be transmitted and one letter from the three-letter alphabet $\{1, 2, 3\}$ is received. The conditional probabilities of the various received letters, given the various transmitted letters are specified by the diagram in the accompanying diagram.



The source sends a, b, and c with the following probabilities:

$$\begin{aligned} \mathrm{P}[a] &= 0.3 \\ \mathrm{P}[b] &= 0.5 \\ \mathrm{P}[c] &= 0.2 \end{aligned}$$

(a) Compute all (nine) conditional probabilities of the form $P(X|Y)$ for $X \in \{a, b, c\}$ and $Y \in \{1, 2, 3\}$.

(b) Compute all (nine) joint probabilities of the form $P(X, Y)$ for $X \in \{a, b, c\}$ and $Y \in \{1, 2, 3\}$.

(c) A receiver makes decisions as follows:
- If 1 is received, decide a was sent.
- If 2 is received, decide b was sent.
- If 3 is received, decide c was sent.

What is the probability that this receiver makes a wrong decision? (I.e., its decision is different from what was actually sent.)

(d) What is the best receiver decision rule (assignment from 1, 2, 3 to a, b, c)?

(e) What is the resulting probability of error?

2. Consider a random variable X having a double-exponential (Laplacian) density,
$$p_X(x) = a\mathrm{e}^{-b|x|}, -\infty < x < \infty$$
where a and b are positive constants.
   (a) Determine the relationship between a and b such that $p_X(x)$ is a valid density function.
   (b) Determine the corresponding probability distribution function $P_X(x)$.
   (c) Find the probability that the random variable lies between 2 and 3.
   (d) What is the probability that X lies between 2 and 3 given that the magnitude of X is less than 3.

3. Let x_1, x_2, …, x_N be a set of N identically distributed statistically independent random variables, each with density function p_x and distribution function F_x. These variables are applied to a system that selects as its output, y_N, the *largest* of the $\{x_i\}$, i.e., $y_N = \max\{x_1, x_2, \dots, x_N\}$. Clearly, y_N is a random variable.
   (a) Express p_{y_N} in terms of N, p_x, and F_x.
   (b) Assume now that the x_i are exponentially distributed random variables:
$$p_x(\alpha) = \begin{cases} e^{-\alpha} & \alpha \geq 0, \\ 0 & \alpha < 0. \end{cases}$$
   Calculate the expectation $\mathbf{E}[y_N]$ for $N = 1, 2$.

4. **Path Loss and SNR** Friis transmission equation
$$L_P = \frac{P_r}{P_t} = \left(\frac{c}{4\pi f_c d}\right)^2$$
describes the path loss L_p under line-of-sight propagation conditions as a signal travels from transmitter to receiver.
   (a) Convert the path loss expression above to a logarithmic scale (i.e., to dB) by taking $10\log_{10}(\cdot)$ of both sides of the relationship.
   (b) The transmitter of a communication system sends signals with the following parameters:
      - transmit power $P_t = 10\,\mathrm{dBm}$
      - bandwidth $W = 10\,\mathrm{MHz}$
      - carrier frequency $f_c = 1\,\mathrm{GHz}$

   Compute the received power P_r, as a function of the distance d between transmitter and receiver. Express P_r in dBm, i.e., compute $10\log_{10}(\frac{P_r}{1\,\mathrm{mW}})$.

(c) The communication system is impaired by thermal noise and is designed so that a signal-to-noise ratio $\frac{P_r}{P_N}$ of at least 10 dB is required for successful operation. What is the maximum distance d for which the system will work?