NCERT-9.5.13

EE24BTECH11065 - Spoorthi yellamanchali

Question:
Find the solution of the following differential equation:

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+x\,dy=0;\quad y=\frac{\pi}{4}\text{ when }x=1$$

Theoretical Solution:
From the question,

$$\frac{dy}{dx}=\frac{y}{x}-\sin^2\frac{y}{x}; \tag{0.1}$$

Let $t=\frac{y}{x}$,then,

$$\frac{dy}{dx}=x\frac{dt}{dx}+t \tag{0.2}$$

On substituting the value of $\frac{dy}{dx}$ in the equation (0.1),we get,

$$x\frac{dt}{dx}=-\sin^2 t;-\operatorname{cosec}^2 t\ dt=\frac{dx}{x}; \tag{0.3}$$

On integrating on both sides,

$$\int-\operatorname{cosec}^2 t\ dt=\int\frac{dx}{x}; \tag{0.4}$$

$$\cot t=\ln x+c; \tag{0.5}$$

$$\cot\frac{y}{x}=\ln x+c \tag{0.6}$$

On Substituting given initial conditions in the equation (0.6)
we get $c=1$.

$$\therefore y=x\cot^{-1}(\ln x+1) \tag{0.7}$$

Solution by the method of finite differences:
The finite difference method is a numerical technique for solving differential equations by approximating derivatives with differences.
The first forward difference approximation of the derivative of $f(x)$ at x is given by:

$$\frac{dy}{dx}=\frac{f(x+h)-f(x)}{h} \tag{0.8}$$

On taking the given point on the curve as the initial conditions (x_0, y_0),we can get ,

$$x_1 = x_0 + h; \tag{0.9}$$

And from the above equation,we can get

$$y_1 = y_0 + h\left(\frac{dy}{dx}|_{x=x_0}\right) \tag{0.10}$$

we know that our derivative is given by:

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}; \tag{0.11}$$

On substituting the expression of the derivative in equation (0.11), we get

$$y_1 = y_0 + h\left(\frac{y_0}{x_0} - \sin^2\frac{y_0}{x_0}\right) \tag{0.12}$$

On assuming a value for h which is close to zero and by substituting the values of x_0 and y_0 in the above equations we get the point (x_1, y_1).
what we have essentially done above is, obtaining a point which is very close to the initial point along the direction of derivative at that point.
similarly we get,

$$x_n = x_{n-1} + h; \tag{0.13}$$

$$y_n = y_{n-1} + h\left(\frac{y_{n-1}}{x_{n-1}} - \sin^2\frac{y_{n-1}}{x_{n-1}}\right) \tag{0.14}$$

we can obtain points on the curve by using the above expressions for y_n and x_n
$\therefore$ we can plot the curve by the points obtained.