

NCERT-9.5.13

EE24BTECH11065 - Spoorthi yellamanchali

Question:

Find the solution of the following differential equation:

$$\left[x \sin^2 \left(\frac{y}{x} \right) - y \right] dx + x dy = 0; \quad y = \frac{\pi}{4} \text{ when } x = 1$$

Theoretical Solution:

From the question,

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x};$$

Let $t = \frac{y}{x}$, then,

$$\frac{dy}{dx} = x \frac{dt}{dx} + t$$

On substituting the value of $\frac{dy}{dx}$ in the above equation,

$$\begin{aligned} x \frac{dt}{dx} &= -\sin^2 t; \\ -\operatorname{cosec}^2 t \, dt &= \frac{dx}{x}; \end{aligned}$$

On integrating on both sides,

$$\begin{aligned} \int -\operatorname{cosec}^2 t \, dt &= \int \frac{dx}{x}; \\ \cot t &= \ln x + c; \\ \cot \frac{y}{x} &= \ln x + c \end{aligned}$$

On Substituting given initial conditions in the above equation, we get ,

$$c = 1$$

$$\therefore y = x \cot^{-1} (\ln x + 1)$$

Solution by the method of finite differences:

The finite difference method is a numerical technique for solving differential equations by approximating derivatives with differences.

The first forward difference approximation of the derivative of $f(x)$ at x is given by:

$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h} \quad (1)$$

On taking the given point on the curve as the initial conditions (x_0, y_0) , we can get ,

$$x_1 = x_0 + h;$$

And from the above equation, we can get

$$y_1 = y_0 + h \left(\frac{dy}{dx} \Big|_{x=x_0} \right)$$

we know that our derivative is given by:

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x};$$

On substituting the expression of the derivative in equation (1), we get

$$y_1 = y_0 + h \left(\frac{y_0}{x_0} - \sin^2 \frac{y_0}{x_0} \right)$$

On assuming a value for h which is close to zero and by substituting the values of x_0 and y_0 in the above equations we get the point (x_1, y_1) .

what we have essentially done above is, obtaining a point which is very close to the initial point along the direction of derivative at that point.

similarly we get,

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h \left(\frac{y_{n-1}}{x_{n-1}} - \sin^2 \frac{y_{n-1}}{x_{n-1}} \right)$$

we can obtain points on the curve by using the above expressions for y_n and x_n

\therefore we can plot the curve by the points obtained.

